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FEM simulation of thixo-viscoplastic flow problems: Error analysis

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This note is concerned with the essential part of Finite Element Methods (FEM) approximation of error analysis for quasi-Newtonian modelling of thixo-viscoplastic (TVP) flow problems. The developed FEM settings for thixotropic generalized Navier-Stokes equations is based on a constrained monotonicity and continuity for the coupled system, which is a cornerstone for an efficient monolithic Newton-multigrid solver. The manifested coarseness in the energy inequality by means of proportional dependency of its constants on regularization, nonoptimal estimate for microstructure, and extra regularity requirement for velocity, is due to the weak coercivity of microstructure operator on one hand and the modelling approach on the other hand, which we dealt with stabilized higher order FEM. Furthermore, we show the importance of taking into consideration the thixotropy inhabited in material by presenting the numerical solutions of TVP flow problems in a 4:1 contraction configuration.

1 Introduction

FEM approximation of thixo-viscoplastic flow problems using quasi-Newtonian modelling approach is a straightforward way to generalize the FEM standard setting of Navier-Stokes equations, as a well standing tool for simulation of incompressible flow problems [10]. In this context, the extended viscosity, $\mu(\cdot,\cdot)$, is dependent on the internal material microstructure parameter, λ, beside the shear rate, $\|\mathbf{D}\|$, for the generalized Navier-Stokes equations [12]. The well defined approximation for the term $\|\mathbf{D}\|^{-1}$, as for instance Papanastasiuo approximation [13], is used to deal with the singularity of the modelling,

$$\frac{1}{\sqrt{D_{\mathrm{II},r}}} := \frac{1}{\sqrt{D_{\mathrm{II}}}}\left(1 - e^{-k\sqrt{D_{\mathrm{II}}}}\right), \tag{1}$$

$D_{\mathrm{II}} = \frac{1}{2}\big(\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{u})\big)$ denotes the second invariant of the strain rate tensor, and k is the regularization parameter. Then, the viscosity in the generalized Navier-Stokes equations is given as follows

$$\mu(D_{\mathrm{II},r},\lambda) = \eta(D_{\mathrm{II}},\lambda) + \tau(\lambda)\frac{\sqrt{2}}{2}\frac{1}{\sqrt{D_{\mathrm{II},r}}}, \tag{2}$$

and the full set of equations for thixo-viscoplastic problems reads

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + u\cdot\nabla\right)\boldsymbol{u} - \boldsymbol{\nabla}\cdot\Big(2\mu(D_{\mathrm{II},r},\lambda)\mathbf{D}(\boldsymbol{u})\Big) + \nabla p = \boldsymbol{f_u}, \\ \nabla\cdot\boldsymbol{u} = 0, \\ \left(\dfrac{\partial}{\partial t} + u\cdot\nabla\right)\lambda + \mathcal{M}(D_{\mathrm{II}},\lambda) = f_\lambda, \end{cases} \tag{3}$$

in Ω, with external forces $\boldsymbol{f_u}$, and f_λ. $\boldsymbol{u}$, p, and λ denote velocity, pressure, and microstructure, respectively. The supplemented evolution equation for the microstructure to generalized Navier-Stokes equations in (3) induces the time-dependent process of competition between the breakdown, $\mathcal{G}$, and the buildup, $\mathcal{F}$, inhabited in the material. A collection of thixotropic models with various choices of plastic viscosity, η, yield stress, τ, buildup function, $\mathcal{F}$, and breakdown function, $\mathcal{G}$, is given in Table 1. We briefly define the thixotropic model as

$$\mathcal{M} := \mathcal{G} - \mathcal{F}. \tag{4}$$

The paper is organized as follows. In section §2, we show the wellposedness of continuous problem followed by the best approximation for the discrete one. Next in section §3, we present the numerical simulations of TVP flow problems in a 4:1 curved contraction configuration showing the importance of not ignoring the thixotropy inhabited in material. In summary section §4, we outline the effect of weak coercivity of microstructure operator and regularization parameter on energy inequality, beside the importance of taking into account the thixotropy inhabited in material giving a way to a stabilized higher order FEM and better understanding of TVP flow characteristics.

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Table 1: Thixotropic models

	η	τ	$\mathcal{F}$	$\mathcal{G}$
Worrall et al. [15]	$\lambda\,\eta_0$	τ_0	$\mathcal{M}_a(1-\lambda)\,\|\mathbf{D}\|$	$\mathcal{M}_b\lambda\,\|\mathbf{D}\|$
Coussot et al. [5]	$\lambda^g\,\eta_0$		$\mathcal{M}_a$	$\mathcal{M}_b\lambda\,\|\mathbf{D}\|$
Houška [7]	$(\eta_0+\eta_\infty\lambda)\,\|\mathbf{D}\|^{n-1}$	$(\tau_0+\tau_\infty\lambda)$	$\mathcal{M}_a(1-\lambda)$	$\mathcal{M}_b\lambda^m\,\|\mathbf{D}\|$
Mujumbar et al. [9]	$(\eta_0+\eta_\infty\lambda)\,\|\mathbf{D}\|^{n-1}$	$\lambda^{g+1}G_0\Lambda_c$	$\mathcal{M}_a(1-\lambda)$	$\mathcal{M}_b\lambda\,\|\mathbf{D}\|$

Here η_0 and τ_0 are initial plastic viscosity and yield stress, respectively, in the absence of any thixotropic phenomena. η_∞ and τ_∞ are thixotropic plastic viscosity and yield stress. Λ_c is the critical elastic strain, and G_0 is the elastic modulus of unyielded material. $\mathcal{M}_a$ and $\mathcal{M}_b$ are buildup and breakage constants, and g, p, m, n are rate indices.

2 Finite element approximations

For FEM approximations, we start by deriving the variational form for thixo-viscoplastic flow problems, followed by the wellposedness results of continuous problem, then we show the best approximation for the discrete problem.

Let's consider $\mathbb{T} := H^1_{\Gamma^-}(\Omega)$, $\mathbb{V} := (H^1_0(\Omega))^2$, $\mathbb{W} := \mathbb{T}\times\mathbb{V}$, and $\mathbb{Q} := L^2_0(\Omega)$ spaces associated with corresponding norms H^1-norm $\|\cdot\|_1$ and L^2-norm $\|\cdot\|_0$, respectively, [1]. We set $\tilde{\boldsymbol{u}} = (\lambda, \boldsymbol{u})$, $\tilde{\boldsymbol{v}} = (\xi, \boldsymbol{v})$, and define on $\mathbb{W}\times\mathbb{W}$

$$a_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) = a_\lambda(\tilde{\boldsymbol{u}})(\lambda, \xi) + a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{v}) \quad \forall\,(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) \in \mathbb{W}\times\mathbb{W}. \tag{5}$$

The weak formulation for TVP flow problems (3) reads: *Find* $(\tilde{\boldsymbol{u}}, p) \in \mathbb{W}\times\mathbb{Q}$ *s. t.*

$$a_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) + b(\boldsymbol{v}, p) - b(\boldsymbol{u}, q) = l(\tilde{\boldsymbol{v}}), \quad \forall(\tilde{\boldsymbol{v}}, q) \in \mathbb{W}\times\mathbb{Q}, \tag{6}$$

where operators $a_\lambda(\tilde{\boldsymbol{u}})(\cdot,\cdot)$, $a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $l(\cdot)$ are given as follows

$$a_\lambda(\tilde{\boldsymbol{u}})(\lambda, \xi) = \int_\Omega \Big(-\mathcal{F}(D_{\mathrm{II}}, \lambda) + \mathcal{G}(D_{\mathrm{II}}, \lambda)\Big)\xi\, d\Omega + \int_\Omega \boldsymbol{u}\cdot\nabla\lambda\,\xi\, d\Omega, \tag{7}$$

$$a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega 2\mu(D_{\mathrm{II}}, \lambda)\mathbf{D}(\boldsymbol{u}) : \mathbf{D}(\boldsymbol{v})\, d\Omega + \int_\Omega \boldsymbol{u}\cdot\nabla\boldsymbol{u}\,\boldsymbol{v}\, d\Omega, \tag{8}$$

$$b(\boldsymbol{v}, q) = -\int_\Omega \nabla\cdot\boldsymbol{v}\, q\, d\Omega, \tag{9}$$

$$l(\tilde{\boldsymbol{v}}) = \Big(f_\lambda, \xi\Big) + \Big(\boldsymbol{f}_{\boldsymbol{u}}, \boldsymbol{v}\Big). \tag{10}$$

The wellposedness results are stated in theorem 2.1 with the following conditions

$$\eta_0\mathcal{C}_K - \mathcal{C}_1\,|\boldsymbol{u}|_1 > 0, \tag{11}$$

$$\mathcal{M}_a - \mathcal{C}_2\mathcal{M}_b\,|\boldsymbol{u}|_{1,\infty} > 0, \tag{12}$$

where, $\mathcal{C}_1$ is the continuity constant of convective term of momentum equation in $((H^1(\Omega))^d)^3$ due to the embedding of $((H^1(\Omega))^d)^3$ in $((L^4(\Omega))^d)^3$ for $d \le 4$ [8]. And $\mathcal{C}_2$ is the maximum constant of continuity constants of thixotropy buildup tri-linear form and convective term of microstructure equation due to Hölder inequality (L^∞, L^2, L^2).

Theorem 2.1 (Begum et. al 2022 [1]: Wellposedness) *Let* $\boldsymbol{f}_{\boldsymbol{u}} \in (L^2(\Omega))^2$ *and* $f_\lambda \in L^2(\Omega)$*, and assume conditions (11) and (12) are satisfied. Then, the thixo-viscoplastic problem (6) has a unique solution* $(\tilde{\boldsymbol{u}}, p) = (\lambda, \boldsymbol{u}, p) \in \mathbb{W}\times\mathbb{Q}$ *with the following bounds of the solution on data*

$$\|\boldsymbol{u}\|_1 \le \frac{1}{\eta_0\mathcal{C}_K}\,\|\boldsymbol{f}_{\boldsymbol{u}}\|_0 \tag{13}$$

$$\|p\|_0 \le \frac{1}{\beta}\Big(1 + \frac{2\,(\eta_\infty + k\tau_\infty) + \|\boldsymbol{u}\|_\infty}{\eta_0\mathcal{C}_K}\Big)\,\|\boldsymbol{f}_{\boldsymbol{u}}\|_0 \tag{14}$$

$$\mathcal{M}_a\,\|\lambda\|_0^2 + \frac{1}{2}\langle\lambda\rangle^2 \le \frac{1}{\mathcal{M}_a}\,\|f_\lambda\|_0^2 \tag{15}$$

where $\mathcal{C}_K$ *denotes the Korn's inequality constant, and* β *is the Ladyzhenskaya-Babuška-Brezzi (LBB) constant.*

Remark 2.2 If the body force in the pressure bound (14) tends towards zero, the limit for pressure is not necessarily zero due to regularization parameter, means that the pressure is underdetermined in rigid zones. Moreover, higher order derivatives of microstructure are not controlled, i.e. it is only bounded with L^2-norm and boundary norm (15).

The approximation of TVP problem, in its general abstract form using conforming framework, is to seek an approximated solution $(\tilde{\boldsymbol{u}}_h, p_h) \in \mathbb{W}_h\times\mathbb{Q}_h$ s. t.

$$a_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}_h)(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + b(\boldsymbol{v}_h, p_h) - b(\boldsymbol{u}_h, q_h) = l(\tilde{\boldsymbol{v}}_h), \quad \forall(\tilde{\boldsymbol{v}}_h, q_h) \in \mathbb{W}_h\times\mathbb{Q}_h, \tag{16}$$

where, $\mathbb{T}_h \subset \mathbb{T}, \mathbb{V}_h \subset \mathbb{V}, \mathbb{W}_h \subset \mathbb{W}$, and $\mathbb{Q}_h \subset \mathbb{Q}$ are finite dimensional subspaces with the subscript h being a parameter dependent on the mesh spacing. The problems that we have to solve here are the existence and uniqueness of the solution $(\tilde{\boldsymbol{u}}_h, p_h)$ and the estimation $\|\lambda - \lambda_h\|_0$, and $\|\boldsymbol{u} - \boldsymbol{u}_h\|_0$. We assume that the inf-sup condition for the pair $(\mathbb{V}_h, \mathbb{Q}_h)$ is satisfied i.e.

$$\exists \beta > 0 \, \text{s.t.} \sup_{\boldsymbol{v}_h \in \mathbb{V}_h} \frac{b(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|} \geq \beta \, \|q_h\|_{\mathbb{Q}/\ker \mathcal{B}_h^{\mathrm{T}}} \quad \forall q_h \in \mathbb{Q}_h, \tag{17}$$

where, β is independent of h, $(\mathcal{B}_h \boldsymbol{v}_h, q_h) := b(\boldsymbol{v}_h, q_h)$, and $\ker \mathcal{B}_h = \{\boldsymbol{v}_h \in \mathbb{V}_h \,|\, b(\boldsymbol{v}_h, q_h) = 0, \ \forall q_h \in \mathbb{Q}_h\}$. Clearly, the inclusion $\ker \mathcal{B}_h \subset \ker \mathcal{B}$ is not true in general. Nevertheless, results of theorem 2.1 concerning existence, uniqueness, and boundedness of the solution with data are directly applied here. Indeed, the necessary properties of $a_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}})(\cdot,\cdot)$ are satisfied in whole space $\mathbb{W}$. Similar to the continuous problem, we assume the following conditions

$$\mathcal{M}_a - \mathcal{C}_2 \mathcal{M}_b \, |\boldsymbol{u}_h|_{1,\infty} > 0 \tag{18}$$

$$\eta_0 \mathcal{C}_K - \mathcal{C}_1 \, |\boldsymbol{u}_h|_1 > 0 \tag{19}$$

Now, we move to the essential part of FEM approximation of comparing the discrete solution $(\lambda_h, \boldsymbol{u}_h, p_h)$ of the approximated TVP problem (16) to the exact solution $(\lambda, \boldsymbol{u}, p)$ of the continuous TVP problem (6). The straightforward way is to use monotonicity combined with the continuity for the coupled operator $a_{\tilde{\boldsymbol{u}}}(\cdot)(\cdot,\cdot)$ which is not true in this case. So, we use a constrained monotonicity Proposition (2.4) together with the continuity Proposition (2.3) to establish our results.

Proposition 2.3 (Continuity) *For all $\tilde{\boldsymbol{u}} = (\lambda, \boldsymbol{u}), \tilde{\boldsymbol{v}} = (\xi, \boldsymbol{v}), \tilde{\boldsymbol{\eta}} = (\zeta, \boldsymbol{\eta}) \in \mathbb{W}_0$, we have*

$$a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{\eta}) - a_{\boldsymbol{u}}(\tilde{\boldsymbol{v}})(\boldsymbol{v}, \boldsymbol{\eta}) \leq (2\eta_\infty + 2\tau_\infty k + \mathcal{C}_1 |\boldsymbol{u}|_1 + \mathcal{C}_1 |\boldsymbol{v}|_1) \|\boldsymbol{u} - \boldsymbol{v}\|_1 \|\boldsymbol{\eta}\|_1 + 2(\eta_\infty |\boldsymbol{v}|_1 + \tau_\infty) \|\lambda - \xi\|_1 \|\boldsymbol{\eta}\|_1 \tag{20}$$

$$a_\lambda(\tilde{\boldsymbol{u}})(\lambda, \zeta) - a_\lambda(\tilde{\boldsymbol{v}})(\xi, \zeta) \leq (\mathcal{M}_a + (2\mathcal{C}_1 + \mathcal{C}_2 \mathcal{M}_b) |u|_1) \|\lambda - \xi\|_0 \|\zeta\|_1 + (2\mathcal{C}_1 + \mathcal{C}_2 \mathcal{M}_b) \|\xi\|_1 \|\boldsymbol{u} - \boldsymbol{v}\|_1 \|\zeta\|_1 \tag{21}$$

Proposition 2.4 (Constrained monotonicity) *Let $\tilde{\boldsymbol{u}}$ be the solution of TVP problem, for all $\tilde{\boldsymbol{u}} = (\lambda, \boldsymbol{u}), \tilde{\boldsymbol{v}} = (\xi, \boldsymbol{v}) \in \mathbb{W}_0$, and set $(\zeta, \boldsymbol{\eta}) = (\lambda - \xi, \boldsymbol{u} - \boldsymbol{v})$. We have*

$$a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{\eta}) - a_{\boldsymbol{u}}(\boldsymbol{v})(\boldsymbol{v}, \boldsymbol{\eta}) \geq \ (\eta_0 \mathcal{C}_K - \mathcal{C}_1 |\boldsymbol{u}|_1) \|\boldsymbol{\eta}\|_1^2 - (\tau_\infty + 2\eta_\infty |\boldsymbol{u}|_1) \|\zeta\|_0 \|\boldsymbol{\eta}\|_1 \tag{22}$$

$$\begin{aligned} a_\lambda(\tilde{\boldsymbol{u}})(\lambda, \zeta) - a_\lambda(\tilde{\boldsymbol{v}})(\xi, \zeta) \geq & \ (\mathcal{M}_a - \mathcal{C}_2 \mathcal{M}_b |\boldsymbol{u}|_{1,\infty}) \|\zeta\|_0^2 + \frac{1}{2} \langle |\boldsymbol{u} \cdot \boldsymbol{n}| \, \zeta, \zeta \rangle - (\mathcal{C}_2 |\boldsymbol{\eta}|_{0,\infty} + \mathcal{C}_2 \mathcal{M}_b |\boldsymbol{\eta}|_{1,\infty}) \|\xi\|_0 \|\zeta\|_1 \\ & - \langle |\boldsymbol{u} \cdot \boldsymbol{n}| \, \xi \rangle_+ \langle |\boldsymbol{u} \cdot \boldsymbol{n}| \, \zeta \rangle_+ - \langle |\boldsymbol{v} \cdot \boldsymbol{n}| \, \xi \rangle_+ \langle |\boldsymbol{v} \cdot \boldsymbol{n}| \, \zeta \rangle_+ \end{aligned} \tag{23}$$

P r o o f. By straightforward calculations (see [1]), inequalities in **Proposition 2.3** and **Proposition 2.4** hold. □

Theorem 2.5 *Let $\boldsymbol{f}_{\boldsymbol{u}} \in (L^2(\Omega))^2$ and $f_\lambda \in L^2(\Omega)$, assume in addition that the conditions (18) and (19) are satisfied. Then, the approximate thixo-viscoplastic problem (16) has a unique solution $(\tilde{\boldsymbol{u}}_h, p_h) = (\lambda_h, \boldsymbol{u}_h, p_h) \in \mathbb{W}_h \times \mathbb{Q}_h$ with the following best approximation*

$$\|\lambda - \lambda_h\|_0^2 \leq \ (2 + 2\tilde{\mathcal{C}}_{\lambda,\lambda}) \inf_{\xi_h \in \mathbb{T}_h} \|\lambda - \xi_h\|_1^2 + \tilde{\mathcal{C}}_{\lambda,\boldsymbol{u}} \inf_{\boldsymbol{v}_h \in \mathbb{V}_h} |\boldsymbol{u} - \boldsymbol{v}_h|_{1,\infty}^2 \tag{24}$$

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,\infty}^2 \leq \tilde{\mathcal{C}}_{\boldsymbol{u},\lambda} \inf_{\xi_h \in \mathbb{T}_h} \|\lambda - \xi_h\|_1^2 + (2 + 2\tilde{\mathcal{C}}_{\boldsymbol{u},\boldsymbol{u}}) \inf_{\boldsymbol{v}_h \in \mathbb{V}_h} |\boldsymbol{u} - \boldsymbol{v}_h|_{1,\infty}^2 + \mathcal{C}_{\boldsymbol{u},p} \inf_{q_h \in \mathbb{Q}_h} \|p - p_h\|_0^2 \tag{25}$$

where, $\tilde{\mathcal{C}}_{\lambda,\lambda}, \tilde{\mathcal{C}}_{\lambda,\boldsymbol{u}}, \tilde{\mathcal{C}}_{\boldsymbol{u},\boldsymbol{u}}$, and $\tilde{\mathcal{C}}_{\boldsymbol{u},\lambda}$ are a constants depending only on $\mathcal{M}_a, \mathcal{M}_b, \eta_0, \eta_\infty, \tau_\infty, k, \beta, \mathcal{C}_K, d$, $\|\xi_h\|_0$, $|\boldsymbol{u}|_1$, $|\boldsymbol{u}|_{1,\infty}$, $|\boldsymbol{u}|_{0,\infty}$, $|\boldsymbol{u}_h|_{1,\infty}$, and $|\boldsymbol{v}_h|_{1,\infty}$.

Remark 2.6 The energy inequality for microstructure (24) states that the error for the approximation of microstructure in L^2-norm is bounded by the error of best approximation of the solution in H^1-norm, which is not optimal. In contrast, the energy inequality (25) states that the error estimate for velocity approximation in H^1-norm is bounded by the best approximation of the solution in H^1-norm as well which is optimal modulo the regularity requirement. These coarseness, i.e. the extra regularity requirement for velocity on one hand and the non-optimality of the estimate for microstructure on the other hand, is due to the weak coercivity of $a_\lambda(\cdot)(\cdot,\cdot)$ i.e. coercivity only in L^2-norm and boundary norm.

P r o o f. To derive the error we subtract the approximated TVP (16) problem from the exact (6) TVP problem

$$a_\lambda(\tilde{\boldsymbol{u}})(\lambda, \xi_h) - a_\lambda(\tilde{\boldsymbol{u}}_h)(\lambda_h, \xi_h) = 0, \qquad \forall \xi_h \in \mathbb{T}_h \tag{26}$$

$$a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{v}_h) - a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}}_h)(\boldsymbol{u}_h, \boldsymbol{v}_h) = b(\boldsymbol{v}_h, p - p_h), \qquad \forall \boldsymbol{v}_h \in \mathbb{V}_h \tag{27}$$

Let ζ_h and $\boldsymbol{\eta}_h$, $\zeta_h := \xi_h - \lambda_h$ $(\zeta_h \in \mathbb{T}_h)$, $\boldsymbol{\eta}_h := \boldsymbol{v}_h - \boldsymbol{u}_h$ $(\boldsymbol{\eta}_h \in \mathbb{V}_h)$, be test functions and add respectively on both side of (26) and (27) terms $a_\lambda(\tilde{\boldsymbol{v}}_h)(\xi_h, \zeta_h) - a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\lambda, \zeta_h)$ and $a_{\boldsymbol{u}}(\tilde{\boldsymbol{v}}_h)(\boldsymbol{v}_h, \boldsymbol{\eta}_h) - a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{\eta}_h)$, we get

$$a_\lambda(\tilde{\boldsymbol{v}}_h)(\xi_h, \zeta_h) - a_\lambda(\tilde{\boldsymbol{u}}_h)(\lambda_h, \zeta_h) = a_\lambda(\tilde{\boldsymbol{v}}_h)(\xi_h, \zeta_h) - a_\lambda(\tilde{\boldsymbol{u}})(\lambda, \zeta_h) \tag{28}$$

$$a_{\boldsymbol{u}}(\tilde{\boldsymbol{v}}_h)(\boldsymbol{v}_h, \boldsymbol{\eta}_h) - a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}}_h)(\boldsymbol{u}_h, \boldsymbol{\eta}_h) = b(\boldsymbol{\eta}_h, p - p_h) + a_{\boldsymbol{u}}(\tilde{\boldsymbol{v}}_h)(\boldsymbol{v}_h, \boldsymbol{\eta}_h) - a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u}, \boldsymbol{\eta}_h) \tag{29}$$

We apply monotonicity and continuity of $a_\lambda(\cdot)(\cdot,\cdot)$ and $a_{\boldsymbol{u}}(\cdot)(\cdot,\cdot)$ on left hand side and right hand side of (28) and (29), respectively, to have,

$$(\mathcal{M}_a - \mathcal{C}_2\mathcal{M}_b \left|\boldsymbol{u}_h\right|_{1,\infty}) \left\|\zeta_h\right\|_0^2 \leq a_\lambda(\tilde{\boldsymbol{v}}_h)(\xi_h,\zeta_h) - a_\lambda(\tilde{\boldsymbol{u}}_h)(\lambda_h,\zeta_h) + (2\mathcal{C}_1 + \mathcal{C}_2\mathcal{M}_b)\left|\boldsymbol{u}_h - \boldsymbol{v}_h\right|_{1,\infty}\left\|\lambda_h\right\|_0\left\|\zeta_h\right\|_1 \tag{30}$$

$$(\eta_0\mathcal{C}_K - \mathcal{C}_1\left|\boldsymbol{u}_h\right|_1)\left\|\boldsymbol{\eta}_h\right\|_1^2 \leq b(\boldsymbol{\eta}_h, p-p_h) + a_{\boldsymbol{u}}(\tilde{\boldsymbol{v}}_h)(\boldsymbol{v}_h,\boldsymbol{\eta}_h) - a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u},\boldsymbol{\eta}_h) + (\tau_\infty + 2\eta_\infty\left|\boldsymbol{u}_h\right|_1)\left\|\lambda_h - \xi_h\right\|_0\left\|\boldsymbol{\eta}_h\right\|_1 \tag{31}$$

and

$$a_\lambda(\tilde{\boldsymbol{v}}_h)(\xi_h,\zeta_h) - a_\lambda(\tilde{\boldsymbol{u}})(\lambda,\zeta_h) \leq (\mathcal{M}_a + (2\mathcal{C}_1 + \mathcal{C}_2\mathcal{M}_b)\left|u\right|_1)\left\|\lambda-\xi_h\right\|_0\left\|\zeta_h\right\|_1 + (2\mathcal{C}_1 + \mathcal{C}_2\mathcal{M}_b)\left|\boldsymbol{u}-\boldsymbol{v}_h\right|_1\left\|\xi_h\right\|_0\left\|\zeta_h\right\|_1 \tag{32}$$

$$\begin{aligned} a_{\boldsymbol{u}}(\tilde{\boldsymbol{v}}_h)(\boldsymbol{v}_h,\boldsymbol{\eta}_h) - a_{\boldsymbol{u}}(\tilde{\boldsymbol{u}})(\boldsymbol{u},\boldsymbol{\eta}_h) \leq 2\eta_\infty\left(\left|\boldsymbol{v}_h - \boldsymbol{u}\right|_1 + \left|\boldsymbol{u}\right|_1\left\|\lambda-\xi_h\right\|_0\right)\left|\boldsymbol{\eta}_h\right|_1 + \tau_\infty\left(2k\left|\boldsymbol{v}_h-\boldsymbol{u}\right|_1 + \left\|\lambda-\xi_h\right\|_0\right)\left|\boldsymbol{\eta}_h\right|_1 \\ + \left(\mathcal{C}_1\left|\boldsymbol{v}_h\right|_1 + \mathcal{C}_1\left|\boldsymbol{u}\right|_1\right)\left|\boldsymbol{v}_h - \boldsymbol{u}\right|_1\left|\boldsymbol{\eta}_h\right|_1 \end{aligned} \tag{33}$$

beside the continuity of $b(\cdot,\cdot)$ on right hand side of (29)

$$b(\boldsymbol{\eta}_h, p-p_h) \leq \sqrt{2d}\left\|p-p_h\right\|_0\left|\boldsymbol{\eta}_h\right|_1, \tag{34}$$

to conclude

$$\left\|\zeta_h\right\|_0^2 \leq \mathcal{C}_{\lambda,\lambda}\left\|\lambda-\xi_h\right\|_0^2 + \mathcal{C}_{\lambda,\boldsymbol{u}}\left|\boldsymbol{u}-\boldsymbol{v}_h\right|_{1,\infty}^2 + \mathcal{C}_{\lambda,\boldsymbol{u}}\left|\boldsymbol{u}_h - \boldsymbol{v}_h\right|_{1,\infty}^2 \tag{35}$$

$$\left|\boldsymbol{\eta}_h\right|_1^2 \leq \mathcal{C}_{\boldsymbol{u},\boldsymbol{u}}\left|\boldsymbol{u}-\boldsymbol{v}_h\right|_1^2 + \mathcal{C}_{\boldsymbol{u},\lambda}\left|\lambda-\xi_h\right|_1^2 + \mathcal{C}_{\boldsymbol{u},\lambda}\left\|\lambda_h-\xi_h\right\|_0^2 + \mathcal{C}_{\boldsymbol{u},p}\left\|p-p_h\right\|_0^2 \tag{36}$$

where $\mathcal{C}_{\lambda,\lambda}, \mathcal{C}_{\lambda,\boldsymbol{u}}, \mathcal{C}_{\boldsymbol{u},\lambda}$, and $\mathcal{C}_{\boldsymbol{u},\boldsymbol{u}}$ are given as follows

$$\mathcal{C}_{\lambda,\lambda}(\mathcal{M}_a,\mathcal{M}_b,\left|\boldsymbol{u}\right|_{0,\infty},\left|\boldsymbol{u}\right|_{1,\infty},\left|\boldsymbol{u}_h\right|_{0,\infty}) = \frac{6(\mathcal{M}_a^2 + (4\mathcal{C}_1^2 + \mathcal{C}_2^2\mathcal{M}_b^2)\left|\boldsymbol{u}\right|_{1,\infty}^2)}{(M_a - \mathcal{C}_2\mathcal{M}_b\left|\boldsymbol{u}_h\right|_{1,\infty})^2} \tag{37}$$

$$C_{\lambda,\boldsymbol{u}}(\mathcal{M}_a,\mathcal{M}_b,\left\|\xi_h\right\|_0,\left|\boldsymbol{u}_h\right|_{0,\infty}) = \frac{6\mathcal{C}_2^2\mathcal{M}_b^2\left\|\xi_h\right\|_0^2}{(M_a - \mathcal{C}_2\mathcal{M}_b\left|\boldsymbol{u}_h\right|_{1,\infty})^2} \tag{38}$$

$$\mathcal{C}_{\boldsymbol{u},\boldsymbol{u}}(\eta_0,\eta_\infty,\tau_\infty k,\mathcal{C}_K,\left|\boldsymbol{u}\right|_1) = \frac{4(2\eta_\infty + 2\tau_\infty k + \mathcal{C}_1\left|\boldsymbol{v}_h\right|_1 + \mathcal{C}_1\left|\boldsymbol{u}\right|_1)^2}{(\eta_0\mathcal{C}_K - \mathcal{C}_1\left|\boldsymbol{u}_h\right|_1)^2} \tag{39}$$

$$\mathcal{C}_{\boldsymbol{u},\lambda}(\eta_0,\eta_\infty,\tau_\infty,\mathcal{C}_K,\left|\boldsymbol{u}\right|_1) = \frac{4(2\eta_\infty\left|\boldsymbol{u}\right|_1 + \tau_\infty)^2}{(\eta_0\mathcal{C}_K - \mathcal{C}_1\left|\boldsymbol{u}_h\right|_1)^2} \tag{40}$$

$$\mathcal{C}_{\boldsymbol{u},p}(d,\eta_0,\mathcal{C}_K) = \frac{4(\sqrt{2d})^2}{(\eta_0\mathcal{C}_K - \mathcal{C}_1\left|\boldsymbol{u}_h\right|_1)^2} \tag{41}$$

Then,

$$\left\|\xi_h - \lambda_h\right\|_0^2 \leq \tilde{\mathcal{C}}_{\lambda,\lambda}\left\|\lambda-\xi_h\right\|_1^2 + \tilde{\mathcal{C}}_{\lambda,\boldsymbol{u}}\left|\boldsymbol{u}-\boldsymbol{v}_h\right|_{1,\infty}^2 \tag{42}$$

$$\left|\boldsymbol{v}_h - \boldsymbol{u}_h\right|_{1,\infty}^2 \leq \tilde{\mathcal{C}}_{\boldsymbol{u},\lambda}\left\|\lambda-\xi_h\right\|_1^2 + \tilde{\mathcal{C}}_{\boldsymbol{u},\boldsymbol{u}}\left|\boldsymbol{u}-\boldsymbol{v}_h\right|_{1,\infty}^2 + \mathcal{C}_{\boldsymbol{u},p}\left\|p-p_h\right\|_0^2 \tag{43}$$

where $\tilde{\mathcal{C}}_{\lambda,\lambda}, \tilde{\mathcal{C}}_{\lambda,\boldsymbol{u}}, \tilde{\mathcal{C}}_{\boldsymbol{u},\lambda}$, and $\tilde{\mathcal{C}}_{\boldsymbol{u},\boldsymbol{u}}$ are dependent on $\mathcal{C}_{\lambda,\lambda}, \mathcal{C}_{\lambda,\boldsymbol{u}}, \mathcal{C}_{\boldsymbol{u},\lambda}$, and $\mathcal{C}_{\boldsymbol{u},\boldsymbol{u}}$. Thus, using triangular inequalities

$$\left\|\lambda-\lambda_h\right\|_0 \leq \left\|\lambda-\xi_h\right\|_0 + \left\|\xi_h-\lambda_h\right\|_0, \tag{44}$$

$$\left|\boldsymbol{u}-\boldsymbol{u}_h\right|_{1,\infty} \leq \left|\boldsymbol{u}-\boldsymbol{v}_h\right|_{1,\infty} + \left|\boldsymbol{v}_h - \boldsymbol{u}_h\right|_{1,\infty}, \tag{45}$$

we conclude the proof. □

The FEM approximations of problem (6) have to take care of its saddle point character, due to the bilinear form (9), the weak coercivity of $a_\lambda(\cdot)(\cdot,\cdot)$, and the dependency of solution on regularization parameter k. We opt for higher order stable FEM pair biquadratic for velocity and piecewise linear discontinuous for pressure, Q_2/P_1^{disc}, and higher order quadratic for microstructure, Q_2, with an appropriate stabilization terms [12, 14]. On one hand, higher order choice for velocity counterbalances the regularization impact and stabilization on the other hand enhances the coercivity to match the complete norm of the microstructure space $\mathbb{T}$ equivalently as H_1-norm i.e.

$$\left\|\!\left|\xi_h\right|\!\right\|^2 = \left\|\xi_h\right\|_0^2 + j_\lambda(\xi_h,\xi_h) \tag{46}$$

where, $j_\lambda(\xi_h,\xi_h)$ is bilinear form supplementing the microstructure equation. Indeed, let the domain Ω be partitioned by a grid with $K\in\mathcal{T}_h$ which are assumed to be quadrilaterals such that $\overline{\Omega} = \left(\bigcup_{k\in\mathcal{T}_h} K\right)$. For an element $K\in\mathcal{T}_h$, we denote by $\mathcal{E}(K)$ the set of all 1-dimensional edges of K. Let $\mathcal{E}_i := \left\{\bigcup_{k\in\mathcal{T}_h}\mathcal{E}(K)\right\}\backslash\partial\overline{\Omega}$ be set of all interior element edges of the grid $\mathcal{T}_h$. We define the conforming finite element spaces $\mathbb{T}_h\subset\mathbb{T}$, $\mathbb{V}_h\subset\mathbb{V}$, $\mathbb{W}_h := \mathbb{T}_h\times\mathbb{V}_h$, and $\mathbb{Q}_h\subset\mathbb{Q}$ such that:

$$\mathbb{W}_h\times\mathbb{Q}_h = \left\{\tilde{\boldsymbol{v}}_h = (\xi_h,\boldsymbol{v}_h)\in\mathbb{W}, q_h\in\mathbb{Q}_h \mid \tilde{\boldsymbol{v}}_{h|K}\in(Q_r(K))^3, q_{h|K}\in P_{r-1}^{\text{disc}}(K); r\geq 2, \forall\, K\in\mathcal{T}_h,\, \boldsymbol{v}_h = 0 \text{ on } \partial\Omega_h\right\}, \tag{47}$$

where Q_r and P_r are polynomials with maximum power in each coordinate less or equal r, and total power less or equal r, respectively.

The stabilized approximate problem reads: *Find* $(\tilde{\boldsymbol{u}}_h, p_h) \in \mathbb{W}_h \times \mathbb{Q}_h$ *s. t.*

$$a_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + j_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) + b(\boldsymbol{v}_h, p_h) - b(\boldsymbol{u}_h, q_h) = 0, \quad \forall\, (\tilde{\boldsymbol{v}}_h, q_h) \in \mathbb{W}_h \times \mathbb{Q}_h. \tag{48}$$

The stabilization term $j_{\tilde{\boldsymbol{u}}}(\cdot,\cdot)$ is given as follows [11, 14]

$$\begin{aligned} j_{\tilde{\boldsymbol{u}}}(\tilde{\boldsymbol{u}}_h, \tilde{\boldsymbol{v}}_h) &:= j_\lambda(\lambda_h, \xi_h) + j_{\boldsymbol{u}}(\boldsymbol{u}_h, \boldsymbol{v}_h), \\ j_{\boldsymbol{u}}(\boldsymbol{u}_h, \boldsymbol{v}_h) &= \sum_{E\in\mathcal{E}_i} \gamma_{\boldsymbol{u}} |E|^2 \int_E [\nabla \boldsymbol{u}_h]\,[\nabla \boldsymbol{v}_h]\, d\sigma, \quad \text{and} \quad j_\lambda(\lambda_h, \xi_h) = \sum_{E\in\mathcal{E}_i} \gamma_\lambda |E|^2 \int_E [\nabla \lambda_h]\,[\nabla \xi_h]\, d\sigma. \end{aligned} \tag{49}$$

The stabilization (49) is consistent and it is expected to recover the optimal order of convergence. The detailed corresponding analysis goes beyond the goal of this note and will be reported in a separate work.

3 Numerical simulations

We investigate numerical solutions of Houška's [7] thixo-viscoplastic material in a 4:1 curved contraction configuration. The fully-developed flow conditions according to Houška thixotropic model are imposed at entry, Γ^- , together with no-slip on walls of reservoir, Γ.

The numerical solutions are obtained using a monolithic Newton-multigrid FEM solver. On one hand, we are using an adaptive discrete Newton method to linearize the discrete nonlinear TVP problem, where the adaptive discrete Newton method is based on step-length in divided difference for the Jacobian calculation. The adaptive strategy is exclusively due to the current convergence rate of residual (for numerical tests see [6]). On other hand, the linearized systems inside outer Newton loops are solved using a monolithic geometrical multigrid solver based on local pressure Schur complement schemes, which are simple iterative relaxation methods solving directly on element level and performing an outer block Gauss-Seidel iteration. The local character of this procedure together with a global defect-correction mechanism on one hand, and the choice of discontinuous FE approximations for pressure (P_1^{disc}) on the other hand, results in an efficient solver for TVP problems. For details, we refer to [2–4].

Our emphasis is to revisit flow characteristics by not ignoring thixotropy inhabited in a material in pipelines, which is a typical industrial application in transportation of waxy crude oils. Figure 1 illustrates the impact of breakdown parameter $\mathcal{M}_b$ on the flow in the vicinity of walls. By a simple increase in breakdown parameter, we induce more breakdown layers close to walls of downstream section.

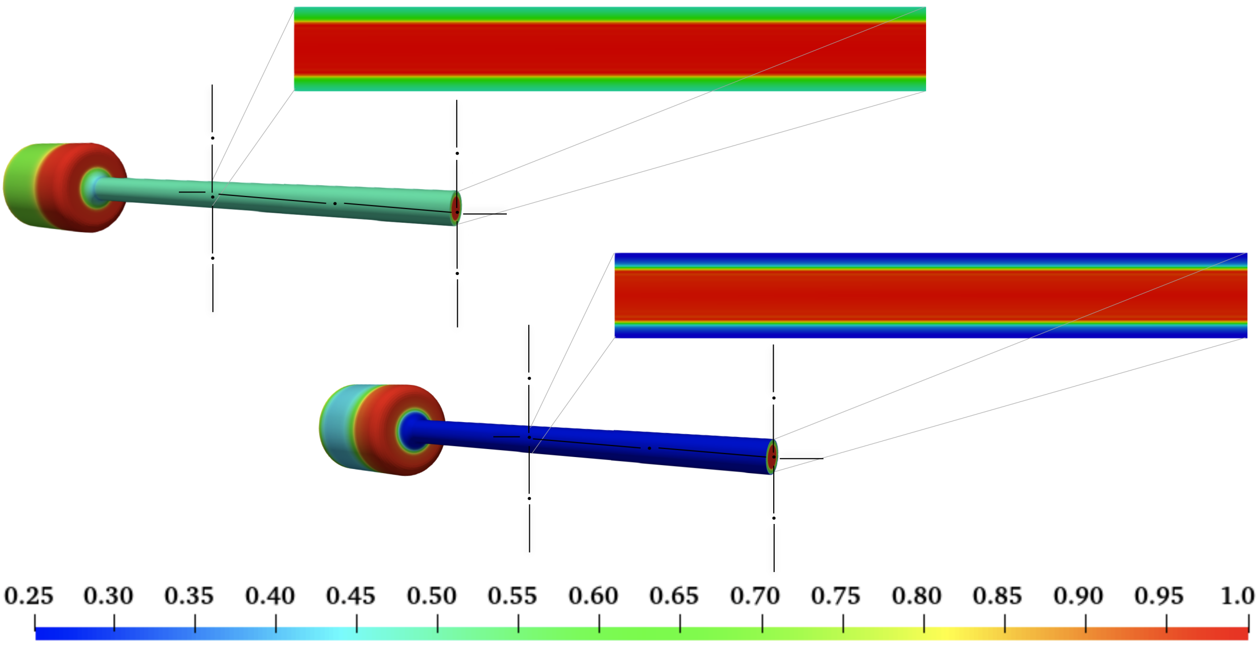

Fig. 1: Thixo-viscoplastic flows in contractions: The structuring level of material λ for thixotropic flows in 4:1 contractions w.r.t. breakdown parameters, b, for two different values $\mathcal{M}_b = 1.0$ (TOP) and $\mathcal{M}_b = 2.0$ (BOTTOM), while the other parameters are set to constants $\eta_0 = \eta_1 = 1.0$, $\tau_0 = 0.0$, $\mathcal{M}_a = 1.0$,$\tau_1 = 2.0$, and $k = 10^4$.

Clearly, more breakdown layers prevent the material from resting along pipelines, which circumvent the need for extra lubrication and highlight the restart pressure to an optimal settings.

4 Summary

We investigated the essential part of error analysis of FEM approximations for the quasi-Newtonian modelling approach of thixo-viscoplastic flow problems. In this regard, the standard FEM settings of Navier-Stokes equations are adapted to deal with the new thixo-viscoplastic generalized Navier-Stokes equations. The wellposedness results beside the boundedness of the solutions with the data are used to set a constrained monotonicity of the coupled problem, which serves beside the continuity to elegantly elaborate the energy inequality of the best approximation.

On one hand, the energy inequality for the microstructure shows that the estimation of the error for the approximation of microstructure in zero norm is bounded by the error of best approximation of the solution in one norm, causing the loss of one order of convergence. On other hand, the energy inequality for velocity shows that the error estimate for velocity approximation in one norm is bounded by the error estimate of the best approximation of the solution in one norm modulo an extra regularity requirement which is a clear manifestation of the weak coercivity of microstructure operator in zero norm and boundary norm only. Moreover, constants in energy inequalities are proportionally dependent on the regularization parameter. We dealt with these coarseness, the proportional dependency of constants on the regularization parameter and the weak coercivity of microstructure operator, by opting for stabilized higher order FEM. The higher order FEM choice counterbalances the regularization effect, while the stabilization enhances the coercivity to an equivalent one norm.

We analysed numerically solutions of Houška's [7] thixo-viscoplastic material in a 4:1 curved contraction configuration using monolithic Newton-multigrid FEM solver. We investigated the impact of thixotropy breakdown parameter $\mathcal{M}_b$ on material microstructuring level λ. In fact, increasing the breakdown parameter induces more breakdown layers in vicinity of walls of downstream channel preventing the material from rest along pipelines, which circumvents the need for extra lubrication and puts forward the restart pressure settings issue.

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