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Preprint 2009-07 Juni 2009

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On optimal metrics preventing mass transfer

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June 19, 2009

Abstract: A Riemannian metric a in the plane together with a point $A \subset \mathbb{R}^2$ induces a distance function $d_a(A, \cdot)$. We investigate optimization problems that seek for a scalar metric a maximizing the distance between A and a set B. We find necessary conditions for optimal metrics which help to determine solutions a. In the case that the set B is a single point, we determine the optimal metric explicitly.

1 Introduction

One of the classical problems in real analysis concerns geodesics. To be specific, let us assume that we are given a metric $a : \mathbb{R}^N \to \mathbb{R}_+$ on the set $\mathbb{R}^N$, $N \geq 1$. This metric, which we consider as scalar for simplicity, measures the lengths of infinitesimal paths. To every Lipschitz curve $\gamma : [0, l] \to \mathbb{R}^N$ we associate the length

$$L_a(\gamma) := \int_0^l a(\gamma(t))|\gamma'(t)|\, dt. \tag{1.1}$$

The problem of finding geodesics can now be formulated as follows: Given two points $x, y \in \mathbb{R}^N$, find the shortest path that connects x and y. The shortest path-length induces a distance function on $\mathbb{R}^N$ by

$$d_a(x, y) := \inf \left\{ L_a(\gamma) : \gamma \in \mathrm{Lip}([0, l], \mathbb{R}^2), \gamma(0) = x, \gamma(l) = y \right\}. \tag{1.2}$$

This concept received recently considerable attention in the framework of Wasserstein distances in spaces of probability measures, [16, 13, 5, 3, 9, 6]. For an overview on mass transportation problems we refer to [17].

Having described the *forward problem* of finding geodesics, we can now consider the corresponding *optimization problem*. Given a point $A \in \mathbb{R}^N$ (without loss of generality the origin) and a closed set $B \subset \mathbb{R}^N$, we search for the metric a that makes the distance

$$d_a(A, B) := \inf \{d_a(A, y) : y \in B\} \tag{1.3}$$

as large as possible, in some admissibility class specified below. We therefore try to design a geometry that prevents mass from being transported from A to B, and

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one might think of the problem of finding an optimal insulation. We note that the opposite optimization problem, namely to minimize the distance, always has the trivial solution of making a small on a straight path from A to B. We remark that most of our results can also be applied to integrals over the distance $d_a(A, \cdot)$, which correspond to maximizing a Wasserstein distance. Precisely, one can prescribe a probability measure μ on B, and seek a metric a such that

$$\int_B d_a(A, y)\, d\mu(y) \tag{1.4}$$

is maximal. In this paper we focus, however, on (1.3).

In order for the integral (1.1) to exist for all Lipschitz curves we assume the metric a to be Borel measurable, and in order to have non-trivial solutions we investigate the problem with a mass constraint; following [7] the metric a is assumed to have fixed lower and upper bounds $0 < \alpha < \beta < \infty$. Precisely, we assume that we are given a number $m \in \mathbb{R}_+$, and consider the metrics a such that

$$a : \mathbb{R}^2 \to \mathbb{R} \text{ Borel measurable, } \; a(x) \in [\alpha, \beta]\ \forall x \in \mathbb{R}^2, \quad \int_{\mathbb{R}^2} (a - \alpha) \le m. \tag{1.5}$$

We say that a is an *admissible metric* if the conditions (1.5) are satisfied. We call an admissible metric *optimal* if it maximizes $d_a(A, B)$ of (1.3). We say that a is a *non-trivial optimal metric*, if the value of the functional in the optimum is below the value for the (non-admissible) metric $\tilde{a} \equiv \beta$. We refer to a region with $a = \beta$ as *black*, to a region with $a = \alpha$ as *white*, to a region with $a \in (\alpha, \beta)$ as *gray*, and denote an optimal a that uses only its extreme values as a black-white metric. A related optimization problem with a quadratic energy that admits only measure-valued solutions was considered in [4].

Our analysis continues the work of Buttazzo, Davini, Fragalà, and Macià [7]. Their main result is the existence of a metric a that solves the optimization problem. In their proof, the easier part is to establish the existence of an optimal Finsler structure on $\mathbb{R}^N$, where the length of a path γ is measured with an integrand $\varphi(\gamma(t), \gamma'(t))$. The main part of the proof is to replace $\varphi(x, \xi)$ with a function $\tilde{\varphi}(x, \xi) = a(x)|\xi|$ which is conforming to the constraints.

We start our contribution with a thorough analysis of the relation between two dual formulations of the problem. In the primary problem we seek a metric a, in the dual problem we seek a Lipschitz function $u : \mathbb{R}^2 \to [0, \infty)$. For solutions, u can be compared to the distance function $d_a(A, \cdot)$, see Lemma 2.2. Our second preparation regards rays: one should imagine them as shortest paths from A to a point in B. The precise definition (Def. 2.3) demands that rays realize distances.

Our main result is the derivation of Euler-Lagrange equations of the above optimization problem, i.e., local conditions on the metric, the distance function and the rays which are necessarily satisfied by solutions of the optimization problem. Theorem 1.1 and Proposition 4.1 show that, loosely speaking, rays pass through interfaces without changing their direction. Moreover, rays can pass only in normal direction through interfaces. Corollary 1.2 is the following: If a is a black-white solution of the optimization problem (with moderate regularity), then a consists of a black disk with center A, with white outside. This shape is optimal e.g. if B is a circle with center A. Instead, for generic sets B, our results imply that the optimal metric can be black-white only if it has a very low regularity. Theorem 1.3 investigates the distance function in gray regions where a has values in (α, β). In gray regions, the distance function has isolines that are straight lines.

Our final result is a complete description of the optimal metric in the case that B is a single point (or a straight line, which, by symmetry, leads to an equivalent problem). In this case, the Euler-Lagrange equations of the optimization problem can be used to characterize rays and isolines with explicit formulas. With Theorem 5.1 we include the proof that the construction provides us indeed with the optimal metric.

The optimization problem in terms of curves

Let a be an admissible metric in the sense of (1.5). It is sometimes convenient to think of $V(x) = 1/a(x)$ as the speed of a particle in the neighborhood of a point x. Then, the integral $\int a(\gamma)|\gamma'|$ stands for the time that a particle needs to travel along the path γ. We are looking for paths γ such that particles starting from A reach B as quick as possible.

An optimal metric with intermediate values might be interpreted as an homogenization effect, where black and white are distributed in a fine mixture to result in gray. The problem relates to works of [1] and [8], where the distance functional (1.2) is considered in an homogenization context. For a family of metrics a_ε with periodic oscillations on the scale ε, the Γ-limit of the functionals d_{a_ε} is investigated.

Main results

Theorem 1.1 (Rays at black-white interfaces)**.** *Let a be a non-trivial optimal metric, $\omega \subset \mathbb{R}^2 \setminus B$ an open Lipschitz set. Assume $x \in \partial\omega \setminus B \setminus \{A\}$ and $\rho > 0$ are given such that $a = \alpha$ on $B_\rho(x) \cap \omega$, and $a = \beta$ on $B_\rho(x) \setminus \overline{\omega}$. Assume $\partial\omega$ to be differentiable in x, and let γ be a ray that passes through x non-tangentially. Then γ is a straight line that passes the boundary in normal direction.*

Theorem 1.1 is shown in Subsection 3.2, where the condition of *non-tangential* passing is defined. It is for example satisfied in the case that γ has a derivative on each side of x, which is not in the tangent space to $\partial\omega$ in x. Proposition 4.1 generalizes the result to gray metrics.

We emphasize that a domain with Lipschitz boundary Ω is a set which locally lies on one side of the graph of a Lipschitz function. This is a stronger assumption than stating that the boundary $\partial\Omega$ is a union of graphs of Lipschitz functions.

Proof. Follows immediately from Proposition 3.2 below. □

Corollary 1.2 (Optimal black-white metrics)**.** *Let a be a non-trivial optimal black-white metric such that the white region has a Lipschitz boundary, and A is in the interior of the black region. Then all rays are straight lines starting in A. The black region is a disk with center A.*

This Corollary is proven in Section 3.3.

Our second theorem concerns gray domains, its proof is given in Section 4.2.

Theorem 1.3 (Parallel in gray regions)**.** *Let a be a non-trivial optimal metric with distance function $u(x) = d_a(A, x)$. Assume that u is continuously differentiable in an open set $\omega \subset \mathbb{R}^2 \setminus \{A\} \setminus B$, and that, for some $p > 1$,*

$$a \in C^0(\omega; (\alpha, \beta)) \cap W^{1,p}(\omega)\,.$$

Then in ω the level curves of u are the union of disjoint segments with endpoints in $\partial\omega$.

In the case that B is a point, we can determine the optimal metric. Explicit formulas are provided in Section 5.

Theorem 1.4. *Let B be a point with $|A - B| = 2$, and let $m > 0$ define a mass constraint. Then there exists an optimal metric with the distance*

$$d_a(A, B) = \min\left\{2\beta, 2\alpha + 2\frac{m^{1/2}(\beta-\alpha)^{1/4}\alpha^{1/4}\cosh(\eta\pi)}{(\sinh(2\eta\pi)^{1/2}}\right\}\,, \tag{1.6}$$

where $\eta = \alpha^{1/2}(\beta - \alpha)^{-1/2}$. The optimal metric is given in (5.10) below (after isometries and symmetric extension to the other half-plane). It is piecewise smooth in each of the white, gray, and black regions; the boundaries are smooth curves except for four points (distinct from A and B).

Proof. After an isometry we can assume $A = 0$, $B = (2, 0)$. Let $d := d_a(A, B)$. Clearly $d \le 2\beta$. If the inequality is strict, the result follows from Theorem 5.1 (Section 5) applied to $B' := \{1\} \times \mathbb{R}$. □

Underlying geometric ideas.

We informally present the main geometric ideas that lead to the above results.

Orthogonal at interfaces. Let us consider one of the shortest paths in the optimal metric and let us focus on a point x where the path leaves the black domain and enters the white metric, illustrated in Figure 2. We can improve the metric a as follows: In a thin strip we replace black and white by gray, i.e. with the metric $a = (\alpha + \beta)/2$. In this way, we make the indicated path longer (it is longer in the new metric than in the old one), since the old path spent less time in the black part and more time in the white part. The argument has to be improved in two respects: One regards the fact that the old path is no longer the shortest path in the new metric, the other regards paths that travel through edges of the strip. Both improvements can be performed for C^2 boundaries.

We observe that the improvement of the metric is not possible when the path crosses the interface orthogonally. This is the point in Proposition 3.1: Rays hit interfaces of optimal metrics orthogonally.

Parallel in gray. The rays will in general be curved in the gray domain. A useful notion of *parallel rays in gray* is to say that isolines of the distance function (which are orthogonal to the rays) are straight lines. This is the result of Theorem 1.3 which uses the direction of rays, $m = \nabla u/|\nabla u|$, and the optimality condition $\operatorname{div} m = 0$. A geometric argument why rays must be parallel is shown in Figure 3.

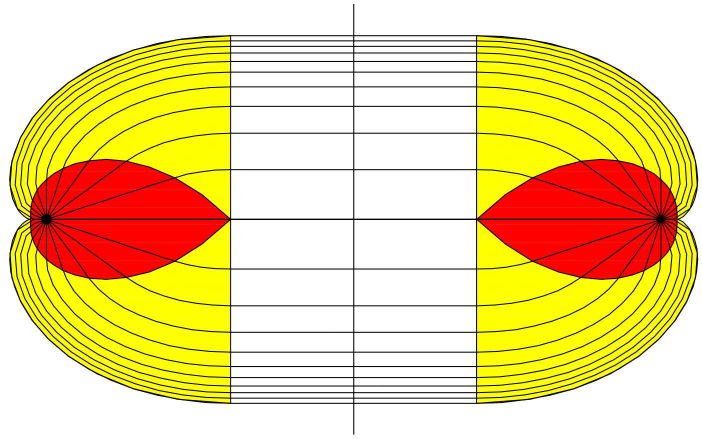

Figure 1: The optimal metric a preventing mass transport between two points A and B. The figure indicates regions of black ($a = \beta$, plotted red/dark gray) and gray ($a \in (\alpha, \beta)$, plotted yellow/light gray) around A. The curves joining A and B are rays.

2 Preliminaries

2.1 The dual problem

We can formulate the problem as a minimization over functions $u : \mathbb{R}^2 \to \mathbb{R}$ which stand for the distance from the point A. We call a function $u : \mathbb{R}^2 \to \mathbb{R}$ *admissible for the dual problem* (or *admissible distance function*) if it satisfies

$$u \in \mathrm{Lip}_\beta(\mathbb{R}^2, \mathbb{R}), \quad u(A) = 0, \quad \int_{\mathbb{R}^2} \max\{|\nabla u| - \alpha, 0\} \leq m. \tag{2.1}$$

We search for an admissible function u which solves one of the following.

$$\inf_{y \in B} u(y) = \sup \left\{ \inf_{y \in B} \tilde{u}(y) : \tilde{u} \text{ admissible} \right\}. \tag{2.2}$$

Before we compare the original problem (1.3) with the dual problem (2.2), we provide some general statements concerning distance functions.

Lemma 2.1 (Distance functions)**.** *Let a be an admissible metric and let $u = d_a(A, \cdot)$ be the corresponding distance function. Then*

1. *$u(.)$ is Lipschitz continuous with constant β. In particular, u is differentiable almost everywhere.*

2. *Let $x \in \mathbb{R}^2 \setminus \{A\}$ be such that a is continuous in x and u is differentiable in x. Then $|\nabla u(x)| = a(x)$.*

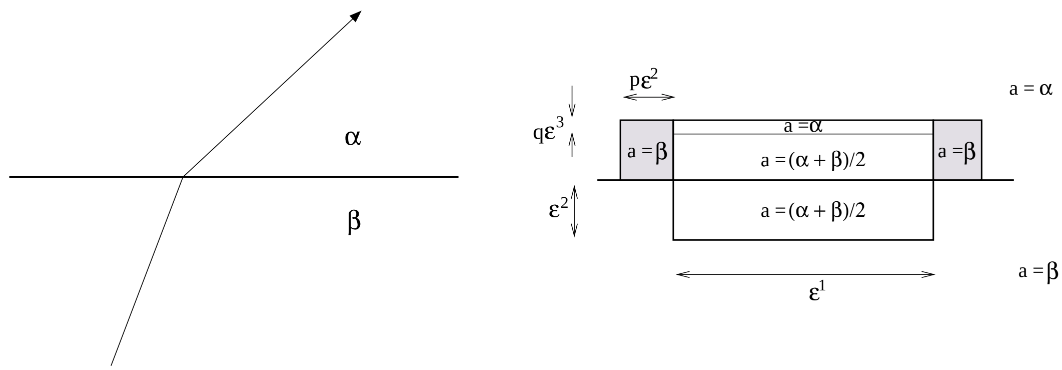

Figure 2: Orthogonality argument. If an optimal path crosses a smooth black-white interface non-orthogonally, then the metric can be improved: There is a gray metric that uses less mass and all paths are longer in the new metric. The new metric is sketched on the right, q and p are chosen suitably, $\varepsilon > 0$ small, the small strip on top is responsible for the saving of mass.

3. *let $x \in \mathbb{R}^2 \setminus \{A\}$ be such that x is a Lebesgue point for a and u is differentiable in x. Then*

$$|\nabla u(x)| \leq a(x)\,. \tag{2.3}$$

4. *The distance function u is admissible for the dual problem.*

Vice versa, given an admissible u for the dual problem, the metric defined by

$$a(x) = \max\left\{\alpha, \limsup_{y\to x} \frac{|u(y)-u(x)|}{|y-x|}\right\} \tag{2.4}$$

is admissible for the primal problem.

We emphasize that the inequality $|\nabla u(x)| \geq a(x)$ is false for general metrics a satisfying the assumptions of (3.). To see this, it suffices to consider the metric $a(x_1, x_2) = \alpha$ for x_1 and x_2/x_1 rational, and $a(x_1, x_2) = \beta$ else. The corresponding distance is $d_a(x, y) = \alpha|x - y|$, since a rational line can be found arbitrarily close to the segment $[x, y]$. Accordingly, we have $u(x) = \alpha|x|$. The derivative satisfies $|\nabla u| = \alpha$ and is, at almost every point, strictly smaller than a.

Proof. Item 1. is immediate, it suffices to consider, for two arbitrary points, the straight line between the two points and its length. Regarding 2., the inequality $|\nabla u(x)| \leq a(x)$ follows as in 1. To derive the opposite inequality, fix $\varepsilon > 0$, and let $\delta \in (0, \varepsilon)$ be such that $A \notin B_\delta(x)$, $|a(y) - a(x)| \leq \varepsilon$ for all $y \in B_\delta(x)$. By the definition of u there is $\gamma \in \mathrm{Lip}([0, 1], \mathbb{R}^2)$ such that $\gamma(0) = A$, $\gamma(1) = x$, and $L_a(\gamma) \leq u(x) + \varepsilon\delta$. Let $s = \inf\{t \in (0, 1) : \gamma((t, 1)) \subset B_\delta(x)\}$. Then

$$\begin{aligned} u(x) &\geq \int_0^1 a(\gamma(t))|\gamma'(t)|dt - \varepsilon\delta \\ &= \int_0^s a(\gamma(t))|\gamma'(t)|dt + \int_s^1 a(\gamma(t))|\gamma'(t)|dt - \varepsilon\delta \end{aligned}$$

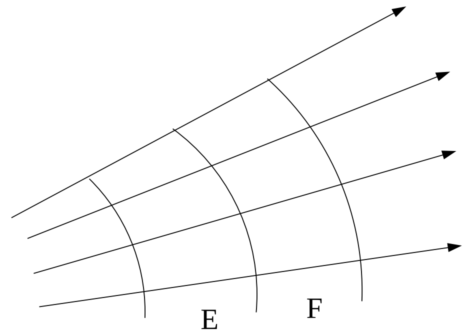

Figure 3: Parallel in gray. Assume that a takes values strictly between α and β. In the indicated situation of non-parallel rays we can shift mass from region F to region E, i.e. increase the metric in E and decrease it in F. This construction yields a new metric which leaves the length of the rays unchanged, but uses less mass — a contradiction to optimality.

$$\geq u(\gamma(s)) + (a(x) - \varepsilon)|\gamma(s) - x| - \varepsilon\delta\,.$$

Therefore we have found a point $y = \gamma(s) \in \partial B_\delta(x)$ such that

$$\frac{|u(x) - u(y)|}{|x - y|} \geq a(x) - 2\varepsilon\,.$$

Since u is differentiable in x the conclusion follows.

To prove item 3. we assume the contrary and find $\delta > 0$ and a sequence of points $x_n \to x$ such that

$$|x_n - x|(a(x) + \delta) \leq u(x_n) - u(x)\,.$$

For every curve γ connecting x and x_n one has

$$u(x_n) - u(x) \leq d_a(x_n, x) \leq \int_0^1 a(\gamma(t))\,|\gamma'(t)|\,dt.$$

The idea is now to consider a specific collection of curves γ. Firstly, we define a family of curves connecting $(0,0)$ with $(1,0)$ in the plane. We fix $0 < r < 1/2$ and define, for $s > 0$, the curve $\tilde{\gamma}_s : [0,1] \to \mathbb{R}^2$ as the curve that connects the points $(0,0)$, (r,s), $(1-r,s)$, and $(1,0)$ with straight lines. We choose a parametrization with $\tilde{\gamma}_s(r) = (r,s)$ and $\tilde{\gamma}_s(1-r) = (1-r,s)$. A rigid motion and scaling applied to these curves yields a family of curves γ_s such that each γ_s connects x with x_n.

We now choose $r > 0$ and $s_0 > 0$ small, the size will be specified below. Inserting the family γ_s, $s \in (-s_0, s_0)$ of curves in the above estimate and averaging over s gives

$$\begin{aligned}|x_n - x|(a(x) + \delta) &\leq \fint_{-s_0}^{s_0} \int_0^1 a(\gamma_s(t))\,|\gamma_s'(t)|\,dt\,ds \\ &\leq 2r\beta\frac{\sqrt{r^2 + s_0^2}}{r}|x - x_n| + \fint_{-s_0}^{s_0} \int_r^{1-r} a(\gamma_s(t))\,|\gamma_s'(t)|\,dt\,ds\end{aligned}$$

$$\leq 2\beta\sqrt{r^2+s_0^2}|x-x_n| + (1-2r)|x-x_n| \fint_{-s_0}^{s_0}\fint_r^{1-r} a(\gamma_s(t))\,dt\,ds.$$

We therefore have found a rectangle $R = \{\gamma_s(t) : t \in (r, 1-r), s \in (-s_0, s_0)\}$ of volume $(1-2r)\cdot 2s_0 \cdot |x_n - x|^2$ such that

$$\fint_R a \geq \frac{1}{1-2r}\left[a(x) + \delta - 2\beta\sqrt{r^2+s_0^2}\right] \geq a(x) + \frac{1}{4}\delta,$$

for all $r, s_0 \leq \delta/(4\beta)$. This is in contradiction with the fact that $a(x)$ is the Lebesgue value of a in x. We have thus shown (2.3).

The admissibility of u follows immediately from (2.3).

It remains to show that the metric a constructed in (2.4) is admissible. It is clear that $a(x) \in [\alpha, \beta]$ and that $(a-\alpha)_+ = (|\nabla u| - \alpha)_+$ almost everywhere. It only remains to show that it is Borel measurable. Since u is Lipschitz-continuous, there is a sequence $\rho_i \to 0$ such that

$$\limsup_{y\to x} \frac{|u(y)-u(x)|}{|y-x|} = \limsup_{i\to\infty} \frac{|u(x+\rho_i)-u(x)|}{|\rho_i|}$$

for all x. The pointwise lim sup of continuous functions is Borel measurable, and the same holds for the maximum of two measurable functions. □

We now make precise in which sense the original problem and the dual problem are equivalent.

Lemma 2.2 (The dual problem)**.** *The values of primary and dual problem coincide,*

$$\begin{aligned} S_a &:= \sup\{d_{\tilde a}(A,B) : \tilde a \text{ admissible metric}\} \\ &= \sup\left\{\inf_B \tilde u : \tilde u \text{ admissible distance function}\right\} =: S_u. \end{aligned} \tag{2.5}$$

If a solves the primary problem, then the corresponding distance function $d_a(A,.)$ solves the dual problem. Vice versa, if u solves the dual problem, then the metric defined in (2.4) solves the primal problem.

If a is optimal and non-trivial, then

$$\int_{\mathbb{R}^2}(a-\alpha) = m.$$

Furthermore, let $u(\cdot) = d_a(A,\cdot)$, and let $\tilde a$ be the corresponding metric, defined as in (2.4). Then $\tilde a = a$ almost everywhere.

Proof. **Step 1: Equivalence of the dual problem.** Let a be optimal and u be the corresponding distance function, the admissibility of u was already checked. Trivially, $\inf_B u = d_a(A,B) = S_a$, hence $S_u \geq S_a$.

Vice versa, let u be optimal and a be the corresponding (admissible) metric. We calculate, for an arbitrary Lipschitz curve $\gamma : [0,l] \to \mathbb{R}^2$ joining A and the a-closest point $b \in B$

$$L_a(\gamma) = \int_0^l a(\gamma(t))|\gamma'(t)|\,dt \geq \int_0^l |\partial_t[u\circ\gamma](t)|\;dt \geq u(b) = S_u.$$

The inequality among the integrals is checked pointwise based on the definition (2.4) and on the fact that γ is almost everywhere differentiable. Since γ was arbitrary, we find $d_a(A,B) \geq S_u$. In particular, $S_a \geq S_u$. This shows equality of the values. In particular, the above constructions yield solutions to the respective dual problems.

Step 2: Non-trivial metrics. We denote by $c_0 := d_1(A,B)$ the Euclidean distance between A and B. Non-triviality of a implies that $d_a(A,B) = \beta c_0 - 4c_0 l_0$ for some positive l_0. We assume that $\int_{\mathbb{R}^2}(a-\alpha) = m - \delta_0$ for some positive δ_0. Our aim is to find a contradiction to the optimality of a.

Let $\gamma : [0,1] \to \mathbb{R}^2$ be a Lipschitz path connecting A and $b \in B$. In the following we will, w.l.o.g., always assume that $|\gamma'(t)| = c$ for some $c > 0$ and almost all $t \in [0,1]$. We note that necessarily $c \geq c_0$, since b has at least the Euclidean distance c_0 from A. For $l := \min\{(\beta - \alpha)/2, \beta/2, l_0\}$ and paths with $L_a(\gamma) \leq (\beta - 2l)c_0$ we calculate the quantity

$$M := |\{t \in [0,1] : a(\gamma(t)) \leq \beta - l\}|.$$

We find

$$(\beta - 2l)c \geq (\beta - 2l)c_0 \geq L_a(\gamma) = c\int_0^1 a(\gamma(t))\,dt \geq Mc\alpha + (1-M)c(\beta - l).$$

Dividing by c and subtracting $\beta - l$ we find

$$M(\beta - \alpha - l) \geq l.$$

We have therefore a quantitative result stating that good paths see a-values away from β on a set M with uniform lower bound.

We now define a comparison metric $\tilde{a}$ by

$$\tilde{a}(x) = \min\{\beta, a(x) + \delta\chi_{B_R(A)}(x)\}.$$

In this definition we choose first $R > 0$ large such that $|\gamma - A| \leq R$ for all γ as above (this is possible because of $\alpha > 0$), then we choose $\delta > 0$ small in dependence of δ_0 and R, such that $\tilde{a}$ is admissible. We furthermore demand $\delta \leq l$. We can now calculate

$$L_{\tilde{a}}(\gamma) \geq L_a(\gamma) + \delta c M \geq L_a(\gamma) + \frac{\delta c l}{\beta - \alpha - l}$$

for all γ as above. On the other hand, for paths γ with $L_a(\gamma) \geq (\beta - 2l)c_0$ we find

$$L_{\tilde{a}}(\gamma) \geq L_a(\gamma) \geq (\beta - 2l)c_0 \geq \beta c_0 - 2l_0 c_0.$$

Taking the infimum over all γ yields

$$d_{\tilde{a}}(A,B) \geq \min\left\{\beta c_0 - 2l_0 c_0, d_a(A,B) + \frac{\delta c l}{\beta - \alpha - l}\right\} > d_a(A,B),$$

in contradiction to optimality of a.

Concerning the last statement, we note that u is a solution of the dual problem, and, accordingly, $\tilde{a}$ is a solution of the primal problem. Since it is necessarily non-trivial as a, it satisfies $\int(\tilde{a} - \alpha) = m$. On the other hand, u satisfies $|\nabla u| \leq a$ almost everywhere by (2.3), whence we have $\tilde{a} \leq a$. The equality of the integrals, $\int(\tilde{a} - \alpha) = m = \int(a - \alpha)$ provides equality almost everywhere. □

Regarding the last point of the lemma we include the following warning on the dual procedure: let u be optimal, let a be constructed from u, and define $\tilde{u} = d_a(A,.)$. Then we cannot expect the equality $\tilde{u} = u$. We illustrate this fact with an example. For $A = (0,0)$ and $B = \{(1,1)\}$ we study $u(y) = |y_1| + |y_2|$. Then $u(A) = 0$ and $u(B) = 2$, and the corresponding metric is $a := |\nabla u| = \sqrt{2}$ almost everywhere. The distance function for this metric is $\tilde{u}(y) = \sqrt{2}|y - A|$. It coincides with u on the line connecting A and B, but not away from this line. Somehow, u is optimized for the point B, but not for all points of the plane.

We also mention that $|\nabla u| \geq \alpha$ is not true under general circumstances (because for any solution u of the dual problem the function $\tilde{u} = \min\{u, \inf_B u\}$ is also a solution).

2.2 Rays

An important object in the analysis of optimal metrics is the notion of optimal paths. A technical problem regards the fact that there need not exist curves that realize the distance between two points. An elementary example is a metric a with $a(x) = \beta$ for $x \in [0,1] \times [0,1]$ and $a = \alpha$ outside the rectange. The "ray" connecting $A = (0,0)$ and $B = (1,0)$ according to our definition will be the straight line even though the length of this line is β. Nevertheless, the ray coincides with the intuitive idea of the shortest path.

Definition 2.3 (Rays). *Given a metric a with $d_a(A,B) = l$, we call a continuous curve $\gamma : [0,l] \to \mathbb{R}^2$ with $\gamma(0) = A$ and $\gamma(l) \in B$ a* ray*, if for all $t_1, t_2 \in [0,l]$ holds*

$$d_a(\gamma(t_2), \gamma(t_1)) = |t_2 - t_1|.$$

In particular, we demand $d_a(A, \gamma(t)) = t$.

As a consequence of the definition, rays are always embedded and are Lipschitz continuous.

Lemma 2.4 (Properties of rays). *Let a be a non-trivial optimal metric. Then the following holds.*

1. *Let $\gamma_n : [0,l] \to \mathbb{R}^2$ be a family of Lipschitz curves with $\gamma_n(0) = A$ and $\gamma_n(l) \in B$ and $L_a(\gamma_n|_{(0,t)}) \to t$ for all $t \in [0,l]$. Then a uniform limit γ of the family γ_n is a ray. Similarly, every uniform limit of rays is again a ray.*

2. *Through every point $x \in \mathbb{R}^2$ with $d_a(A,x) + d_a(x,B) = d_a(A,B)$ passes a ray. Let x be a Lebesgue point of a with $a(x) > \alpha$. Then there is a ray γ passing through x.*

For the following we assume additionally that γ is a ray, $x = \gamma(t) \notin \{A\} \cup B$, and that a is constant in a ball $B_\varepsilon(x)$ disjoint from $\{A\} \cup B$, for some $\varepsilon > 0$.

3. *$W := \gamma([0,l]) \cap B_\varepsilon(x)$ is a straight segment.*

4. *There exists no ray $\tilde{\gamma}$ passing through x with $\tilde{\gamma}([0,1]) \cap B_\varepsilon(x) \neq W$.*

5. *Assume that $u = d_a(A,\cdot)$ is differentiable in x. Then $\nabla u(\gamma(t)) \mid\mid \gamma'(t)$.*

Proof. Item 1. For $t_1 < t_2$ we find

$$t_2 \leftarrow L_a(\gamma_n|_{(0,t_2)}) = L_a(\gamma_n|_{(0,t_1)}) + L_a(\gamma_n|_{(t_1,t_2)}).$$

The fact that $L_a(\gamma_n|_{(0,t_1)}) \to t_1$ implies $L_a(\gamma_n|_{(t_1,t_2)}) \to t_2 - t_1$. Similarly, we find $L_a(\gamma_n|_{(t_2,l)}) \to l - t_2$. Since γ_n can be used as a competitor, this implies for the distances $\limsup_n d_a(A, \gamma_n(t_1)) \le t_1$ etc. Since, on the other hand

$$d_a(A, \gamma_n(t_1)) + d_a(\gamma_n(t_1), \gamma_n(t_2)) + d_a(\gamma_n(t_2), B) \ge l,$$

we find $\lim_n d_a(A, \gamma_n(t_1)) = t_1$ and $\lim_n d_a(\gamma_n(t_1), \gamma_n(t_2)) = t_2 - t_1$. The continuity of d_a implies the claim.

The second statement follows similarly from the continuity of d_a.

Item 2. For the first statement it suffices to consider curves that realize, up to an error $1/n$, the distances $d_a(A,x)$ and $d_a(x,B)$. Connecting the two curves with appropriate parametrization and taking the limit $n \to \infty$, we find a ray through x by item 1.

Concerning Lebesgue-points of a we first claim that, for every $\varepsilon > 0$, there exists a ray passing the ball $B_\varepsilon(x)$. To prove this, we assume the contrary and consider $\varepsilon \in (0, 1/2)$ such that no ray passes $B_\varepsilon(x)$. By the first part of item 2 and the continuity of d_a we have, for some $\delta > 0$,

$$\inf_{\xi \in B_{\varepsilon/2}(x)} d_a(A,\xi) + d_a(\xi, B) = d_a(A,B) + \delta.$$

Let us now consider the comparison metric $\tilde{a}$ with $\tilde{a} = a$ on $\mathbb{R}^2 \setminus B_{\varepsilon/2}(x)$ and $\tilde{a} = \max\{\alpha, a - \delta\}$ on $B_{\varepsilon/2}(x)$. Then $d_{\tilde{a}}(A,B) = d_a(A,B)$, since each curve γ passing through $B_{\varepsilon/2}(x)$ has an $\tilde{a}$-length of at least $L_{\tilde{a}}(\gamma) \ge d_a(A,B) + \delta - 2\varepsilon\delta \ge d_a(A,B)$. We can therefore decrease the used mass without making shortest curves longer. This is in contradiction with Lemma 2.2.

By now we have shown the existence of a sequence of points $x_n \to x$ such that rays γ_n are passing through x_n. The family γ_n has a bounded Lipschitz constant and we can pass to a uniformly convergent subsequence. By item 1 this yields a ray passing through x.

Item 3. We consider $s_1 = \inf\{t' > t : \gamma(t') \notin B_\varepsilon(x)\}$ and $s_2 = \sup\{t' < t : \gamma(t') \notin B_\varepsilon(x)\}$. Since $s_1 < t < s_2$, we obtain $L_a(\gamma|_{(s_1,s_2)}) \ge 2\varepsilon a(x)$, with equality only if $\gamma([s_1,t])$ and $\gamma([t,s_2])$ are segments. Let now $t_1 = \inf\{t : \gamma(t) \in B_\varepsilon(x)\}$ and $t_2 = \sup\{t : \gamma(t) \in B_\varepsilon(x)\}$, and consider the Lipschitz curve $\tilde{\gamma}$ which equals γ outside $[t_1,t_2]$, and is the affine interpolation between $\gamma(t_1)$ and $\gamma(t_2)$ inside. Then

$$L_a(\tilde{\gamma}) = L_a(\gamma|_{(0,t_1)}) + a(x)|\gamma(t_1) - \gamma(t_2)| + L_a(\gamma|_{(0,t_1)}) \le L_a(\gamma)\,.$$

Since γ was a minimizer, equality holds throughout. Therefore $|\gamma(t_1) - \gamma(t_2)| = 2\varepsilon$, i.e., they are diametrically opposite. Further, $L_a(\gamma_{(t_1,s_1)}) = 0$, i.e., $\gamma([t_1,s_1])$ is a point (contained in $\partial B_\varepsilon(x)$), analogously for $[s_2,t_2]$.

Item 4. Let $\tilde{\gamma}$ be a ray passing through x which is different from γ. Then $\tilde{\gamma} \cap B_\varepsilon(x)$ is again a straight segment. Since the two segments are different, they form an angle at x. Starting from this situation we can construct a new ray by following γ until x and $\tilde{\gamma}$ beginning at x. This new ray forms an angle and contradicts item 3.

Item 5. By the same argument used in proving 2 of Lemma 2.1, applied to the ray γ, we obtain a sequence $t_n \to t$ such that

$$\lim_{n\to\infty} \frac{|u(\gamma(t_n)) - u(\gamma(t))|}{|\gamma(t_n) - \gamma(t)|} = a(x) .$$

Since $|\nabla u(\gamma(t)| \le a(x)$ the conclusion follows. □

3 Local properties at interfaces

3.1 Wedges of optimal distance functions

With Proposition 3.1, in this subsection we show the key result concerning interfaces of the metric.

Given two vectors $\xi^+, \xi^- \in \mathbb{R}^2$ and orthogonal unit vectors $e_1, e_2 \in \mathbb{R}^2$, we define the function $b(y) := \xi^+$ for $e_2 \cdot y > 0$ and $b(y) := \xi^-$ for $e_2 \cdot y < 0$. Given, additionally, a point $x_0 \in \mathbb{R}^2$ and a value $u_0 \in \mathbb{R}$, we consider the following piecewise affine function $\bar{u}$.

$$\bar{u}(x_0 + y) := u_0 + b(y) \cdot y = \begin{cases} u_0 + \xi^+ \cdot y & \text{for } e_2 \cdot y \ge 0, \\ u_0 + \xi^- \cdot y & \text{for } e_2 \cdot y < 0. \end{cases} \tag{3.1}$$

The function $\bar{u}$ is continuous whenever $(\xi^+ - \xi^-) \cdot e_1 = 0$.

Proposition 3.1 (Wedges of optimal distance functions)**.** *Let u be an optimal distance function, x_0 a point in $\mathbb{R}^2$ and $e_1, e_2 \in \mathbb{R}^2$ an orthonormal basis. Furthermore, let $u_0 \in \mathbb{R}$ and $\xi^+, \xi^- \in \mathbb{R}^2$ with $|\xi^\pm| \in [\alpha, \beta]$, $\xi^+ \ne \xi^-$, $\max\{|\xi^+|, |\xi^-|\} > \alpha$ and $(\xi^+ - \xi^-) \cdot e_1 = 0$.*

We assume that $\bar{u}$ defined as in (3.1) is an approximation of u in the vicinity of x_0 in the following sense: For every $\eta > 0$ there exists $\varepsilon > 0$ such that, on $B_\varepsilon = B_\varepsilon(x_0)$

$$\|u - \bar{u}\|_{L^\infty(B_\varepsilon)} \le \eta\, \varepsilon, \tag{3.2}$$

$$\|\, |\nabla u| - |\nabla \bar{u}|\, \|_{L^1(B_\varepsilon)} \le \eta\, \varepsilon^2. \tag{3.3}$$

Then, necessarily, ξ^+, ξ^-, and e_2 are parallel.

Proof. We assume that $c = (\xi^+ - \xi^-) \cdot e_2 > 0$, otherwise we can swap the sign of e_2. After a change of variables we can assume without loss of generality that $x_0 = 0$ and e_1 and e_2 are the canonical basis of $\mathbb{R}^2$. We have to show that the tangential component $\xi^+ \cdot e_1 = \xi^- \cdot e_1$ vanishes.

We argue by contradiction and assume from now on that $\xi_1^+ = \xi_1^- \ne 0$. Our construction is based on the fact that convex combinations of non-parallel vectors shorten the length. To abbreviate notation we introduce the convex function

$$|\,.\,|_\alpha : \mathbb{R}^2 \to \mathbb{R}, \quad |\zeta|_\alpha := \max\{\alpha, |\zeta|\}.$$

Various small numbers appear in the following. The order is as follows: We choose first δ, then $h = h(\delta)$, then $\eta = \eta(\delta, h)$, and then $\varepsilon = \varepsilon(\delta, h, \eta)$.

Step 1. Construction of a new distance function. The averaged gradient shortens the α-length,

$$\bar{\xi} := \frac{\xi^+ + \xi^-}{2}, \qquad q := \frac{|\xi^+|_\alpha + |\xi^-|_\alpha}{2} - |\bar{\xi}|_\alpha > 0$$

by the convexity of the norm and since $\xi_2^+ \neq \xi_2^-$ excludes that both vectors $\xi^\pm$ are parallel. We recall that $\bar{\xi}_1 = \xi_1^+ = \xi_1^-$ and $\bar{\xi}_2 = \xi_2^+ - c/2 = \xi_2^- + c/2$.

The basic building block of our comparison function is the tilted hat function

$$w(y) = u_0 + \bar{\xi} \cdot y - \delta|y_1| - \delta|y_2| + \frac{c}{2}\varepsilon h\,,$$

where $\delta = q/4$. A simple computation shows that

$$|\nabla w|_\alpha \leq |\bar{\xi}| + 2\delta \leq \frac{|\xi^+|_\alpha + |\xi^-|_\alpha}{2} - \frac{1}{2}q. \tag{3.4}$$

We define a thin rectangle by $R_\varepsilon := \{y : |y_1| < \varepsilon, |y_2| < h\varepsilon\}$ and our comparison distance function $\tilde{u}$ by

$$\tilde{u}(y) := \begin{cases} u(y) & \text{for } y \notin R_\varepsilon, \\ \max\{u, w\}(y) & \text{for } y \in R_\varepsilon. \end{cases} \tag{3.5}$$

Our aim is to show that $\tilde{u}$ is a competitor of u which uses less mass.

Step 2. Choice of h and ε, continuity of $\tilde{u}$. We have to verify that $\tilde{u}$ is Lipschitz continuous. To this end it suffices to make sure that $w \leq u$ on ∂R_ε. By (3.2) the function $v(y) = u(y) - \bar{u}(y)$ is bounded uniformly by $|v(y)| \leq \eta\varepsilon$. From the definition of $\bar{\xi}$ we have $\bar{u}(y) = u_0 + \bar{\xi} \cdot y + \frac{c}{2}|y_2|$. Therefore

$$(w - u)(y) = (w - \bar{u} - v)(y) = -\delta|y_1| - \delta|y_2| + \frac{c}{2}(\varepsilon h - |y_2|) - v(y). \tag{3.6}$$

Therefore, on the upper and lower boundaries of the rectangle, i.e. $y_2 = \pm\varepsilon h$,

$$(w - u)(y_1, \pm\varepsilon h) \leq -\delta\,|y_1| - \delta\varepsilon h + \eta\varepsilon \leq -\delta\,|y_1| \leq 0,$$

provided that η has been chosen such that $\eta \leq \delta h$. Analogously, for the lateral boundaries of the rectangle and $y_2 \in (-h\varepsilon, h\varepsilon)$,

$$(w - u)(\pm\varepsilon, y_2) \leq -\delta\varepsilon + \frac{c}{2}\varepsilon h + \eta\varepsilon \leq 0,$$

provided h and η have been chosen with $h < \delta/c$ and $\eta < \delta/2$. We conclude that $\tilde{u}$ is Lipschitz continuous.

Step 3. Saving in mass. We consider points in R_ε with $w \geq u$, i.e.

$$D_\varepsilon := \{y \in R_\varepsilon : \tilde{u}(y) = w(y)\}.$$

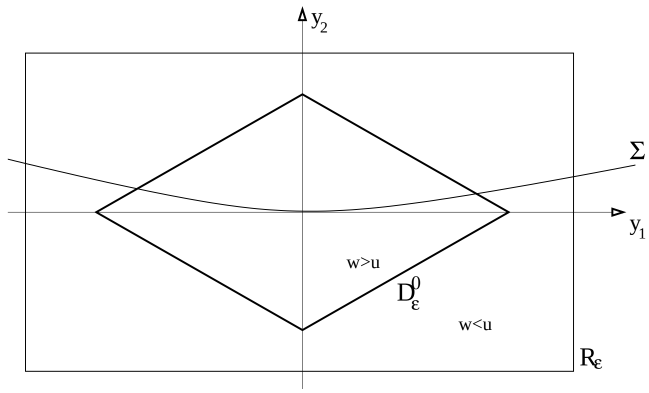

Figure 4: Within the diamond shaped region, the distance function u is replaced by w. The new distance function is admissible and uses less mass. Indicated is, additionally, a possible interface Σ between two regions with different values of $a = |\nabla u|$.

With the hat function

$$F(y) := -\delta|y_1| - \left(\frac{c}{2} + \delta\right)|y_2| + \frac{c}{2}\varepsilon h,$$

equation (3.6) implies that $w \geq u$ if and only if $F - v \geq 0$, hence

$$D_\varepsilon = \{y \in R_\varepsilon : v(y) \leq F(y)\}.$$

This shows that with D_ε we consider a small perturbation of the diamond shaped region

$$D_\varepsilon^0 := \{y \in R_\varepsilon : F(y) > 0\}.$$

Since $|\nabla F| \geq \delta$ a.e. and $|v| \leq \eta\varepsilon$ everywhere, the symmetric difference of the two domains can be estimated by

$$|D_\varepsilon \setminus D_\varepsilon^0| + |D_\varepsilon^0 \setminus D_\varepsilon| \leq C(h,\delta)\eta\varepsilon^2$$

where the constant depends on h and δ. This estimate is also valid separately on the two sides of the interface. We write $D_\varepsilon^\pm := \{y \in D_\varepsilon : \pm y_2 > 0\}$ for the upper and lower part of it, and estimate

$$\left| |D_\varepsilon^\pm| - \frac{1}{2}|D_\varepsilon^0| \right| \leq C_1(h,\delta)\eta\varepsilon^2.$$

Using first (3.3) and then this comparison of the domains, we can estimate

$$\begin{aligned} \int_{D_\varepsilon} |\nabla u|_\alpha &\geq \int_{D_\varepsilon} |\nabla \bar{u}|_\alpha - \eta\varepsilon^2 = |D_\varepsilon^+||\xi^+| + |D_\varepsilon^-||\xi^-| - \eta\varepsilon^2 \\ &\geq |D_\varepsilon^0|\frac{|\xi^+| + |\xi^-|}{2} - C(h,\delta)\eta\varepsilon^2. \end{aligned} \tag{3.7}$$

We can now compare the mass related to the original metric $a_u = |\nabla u|_\alpha$ with the mass related to the comparison metric $a_w = |\nabla w|_\alpha$, exploiting (3.4).

$$\begin{aligned}\int_{D_\varepsilon} |\nabla w|_\alpha &\le |D_\varepsilon| \left(\frac{|\xi^+|_\alpha + |\xi^-|_\alpha}{2} - \frac{1}{2} q \right) \\ &\le \left(|D_\varepsilon^0| + C\eta\varepsilon^2 \right) \left(\frac{|\xi^+|_\alpha + |\xi^-|_\alpha}{2} - \frac{1}{2} q \right) \\ &\le |D_\varepsilon^0| \frac{|\xi^+|_\alpha + |\xi^-|_\alpha}{2} - \frac{1}{2} q |D_\varepsilon^0| + C\eta\varepsilon^2 \\ &\le \int_{D_\varepsilon} |\nabla u|_\alpha + C\eta\varepsilon^2 - qC_2(\delta, h)\varepsilon^2,\end{aligned}$$

where we abbreviate $|D_\varepsilon^0| = C_2(\delta, h)\varepsilon^2$. Choosing η sufficiently small we achieve

$$\int_{D_\varepsilon} |\nabla w|_\alpha < \int_{D_\varepsilon} |\nabla u|_\alpha.$$

This is in contradiction with the optimality of u and concludes the proof. □

3.2 Black and white metrics

In this subsection we apply Proposition 3.1 to non-trivial optimal metrics a which takes only the values α and β. Our aim is to show that, loosely speaking, at a Lipschitz black-white interface all rays are straight lines. This result, made precise in Proposition 3.2 below, implies immediately Theorem 1.1. The key idea in the proof of the Proposition is to study comparison paths consisting of two or three segments such that long segments lie almost completely in one of the two phases. Optimality permits to derive the refraction law of geometrical optics and to conclude sharp bounds for the values of u at the endpoints. In a vectorial context, a similar idea was used in proving rigidity estimates in [11, 10].

Proposition 3.2 (No changes of direction at black-white interfaces)**.** *Let a be a non-trivial optimal metric, $\omega \subset \mathbb{R}^2 \setminus B$ a Lipschitz set. Assume $x_0 \in \partial\omega \setminus B \setminus \{A\}$ and $\rho > 0$ are given such that $a = \alpha$ on $B_\rho(x_0) \cap \omega$, and $a = \beta$ on $B_\rho(x_0) \setminus \overline{\omega}$. Assume $\partial\omega$ to be differentiable in x_0 with normal ν, and let $\gamma \in \mathrm{Lip}([0, l]; \mathbb{R}^2)$ be a ray that crosses tangentially in x_0, in the sense that there are $t_1 < t_0 < t_2 \in (0, l)$ such that $\gamma(t_0) = x_0$, one of the two sets $\gamma((t_1, t_0))$, $\gamma((t_0, t_1))$ lies in ω, and the other is disjoint from $\overline{\omega}$. Then, in a neighbourhood of x_0, the ray γ is a parametrization of $x_0 + \mathbb{R}\nu$.*

Let us introduce some further notation. The assumption on x_0 implies that, possibly after making ρ smaller, there are two orthogonal vectors μ_1, μ_2 and Lipschitz function $f : (-\rho, \rho) \to \mathbb{R}$, with $f(0) = 0$, such that

$$\omega \cap B_\rho(x_0) = \{x \in B_\rho(x_0) : (x - x_0) \cdot \mu_2 < f((x - x_0) \cdot \mu_1)\}$$

and the function f is differentiable in 0. The outer normal to $\partial\omega$ in x is then

$$\nu = \frac{\mu_2 - f'(0)\mu_1}{|\mu_2 - f'(0)\mu_1|}.$$

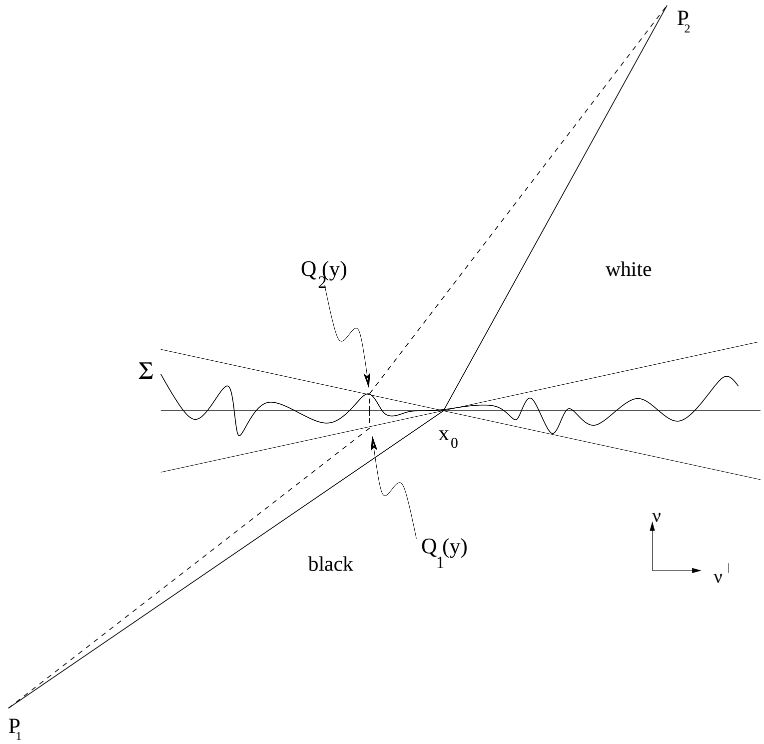

Figure 5: Sketch of the construction used in the proof of Proposition 3.2. The point x_0 is at the center, the points P_j are far from x_0, the points $Q_j(y)$ are much closer to x_0. The wedge delimits the region where $\partial\omega$ lies.

Proof. Assume for definiteness that the ray runs from white to black, i.e., that $\gamma((t_1, t_0)) \subset \omega$ (in the other case one only needs to swap a few indices). From Lemma 2.4(3.) it follows that $\gamma((t_1, t_0))$ and $\gamma((t_0, t_1))$ are segments, affinely parameterized by γ.

Step 1. Refraction law. This proof uses geometrical considerations and the construction of competitors for γ to prove that the orientation of the segments $\gamma((t_1, t_0))$ and $\gamma((t_0, t_1))$ satisfies the usual refraction law of geometrical optics.

We define for $y \in \mathbb{R}$ a point on the tangent to $\partial\omega$ by

$$Q(y) := x_0 + y\nu^\perp.$$

Let $\delta > 0$, and consider the points

$$Q_1(y) := x_0 + y\nu^\perp - \delta|y|\nu\,,$$
$$Q_2(y) := x_0 + y\nu^\perp + \delta|y|\nu$$

(see Fig. 5). Since f is differentiable in 0, for every $\delta > 0$ there is $\varepsilon > 0$ such that for all $y \in (-\varepsilon, \varepsilon)$ we have $Q_1(y) \in \omega$, $Q_2(y) \notin \omega$.

We set $P_1 = \gamma(t_1)$, $P_2 = \gamma(t_2)$. Possibly replacing t_1 and t_2 by values closer to t_0, we may ensure that for all y sufficiently small the two segments $[P_1Q_1]$ and $[P_2Q_2]$ do not intersect $\partial\omega$. We stress that the resulting P_1 and P_2 are fixed, and do not depend on the parameters ε and δ. In the following we always consider three points

Q, Q_1, Q_2, generated from the same y, which remains a free parameter. The points P_1, x_0, and P_2 are considered to be far apart, since their distances remains finite for $y \to 0$, whereas the points Q, Q_1, Q_2, x_0 are close together since their distances are of order y.

We study the piecewise affine path joining P_1 with Q_1, then with Q_2, and then with P_2. Since $|Q_1 - Q_2| = 2\delta|y|$ we can calculate

$$\begin{aligned}\alpha|x_0 - P_1| + \beta|P_2 - x_0| &\overset{(1)}{=} d_a(P_1, P_2)\\ &\overset{(2)}{\leq} d_a(P_1, Q_1) + d_a(Q_1, Q_2) + d_a(Q_2, P_2)\\ &\overset{(3)}{\leq} \alpha|Q_1 - P_1| + \beta|Q_2 - P_2| + 2\delta\beta|y|,\end{aligned}$$

where we used in (1) that the ray realizes distances, passes x_0, and is a straight line on both sides, in (2) the triangle inequality, and in (3) that between Q_1 and P_1 there is no black.

We linearize this inequality in the limit $y \to 0$. In particular, the length of $Q_j - P_j = (x_0 - P_j) + (Q_j - x_0)$ can be expressed as

$$|Q_j - P_j| = |x_0 - P_j| + r_j \cdot (Q_j - x_0) + o(y)\,,$$

where

$$r_1 := \frac{x_0 - P_1}{|x_0 - P_1|} \text{ and } r_2 := \frac{x_0 - P_2}{|x_0 - P_2|}$$

are unit vectors along the two segments $[x_0 P_j]$. Recalling that $|Q - Q_j| = \delta|y|$, inserting above and subtracting $\alpha|P_1 - x_0| + \beta|P_2 - x_0|$ we find

$$0 \leq \alpha r_1 \cdot (Q - x_0) + \beta r_2 \cdot (Q - x_0) + 4\beta\delta|y| + o(y)$$

for all y sufficiently small, and therefore

$$\left|(\alpha r_1 + \beta r_2) \cdot \nu^\perp\right| \leq 4\beta\delta\,.$$

Since δ was arbitrary, we obtain the desired refraction law,

$$\alpha r_1 \cdot \nu^\perp + \beta r_2 \cdot \nu^\perp = 0\,. \tag{3.8}$$

At this point it is important to note that the vectors r_j do not change when we restrict to a smaller neighborhood of x_0 in order to have a smaller $\delta > 0$.

Step 3. Comparison distance function. We study the distance function $u(\cdot) = d_a(A, \cdot)$ corresponding to the optimal metric a. Our aim is to apply Proposition 3.1. We set $u_0 = d_a(A, x_0) = u(x_0)$, $e_1 = -\nu^\perp$, $e_2 = \nu$, $\xi^- = \alpha r_1$, $\xi^+ = \beta r_2$, and define the function $\bar{u}$ as in (3.1). We shall now verify the assumptions of Proposition 3.1 and then conclude from that that $r_1 \| | r_2 \| | \nu$, which concludes the proof.

The refraction law (3.8) implies $(\xi^+ - \xi^-) \cdot e_1 = 0$, which is equivalent to continuity of $\bar{u}$. The fact that $\partial\omega$ is differentiable in x_0 implies (3.3). It remains to show (3.2). We argue by contradiction and assume that, for some $\eta > 0$, there exists a sequence $x^k = (x_1^k, x_2^k) \to x_0$ such that

$$|u(x^k) - \bar{u}(x^k)| \geq \eta|x^k - x_0| \quad \forall k \in \mathbb{N}. \tag{3.9}$$

In the following, we will show that (3.9) is in contradiction with the refraction law. The key of the argument is that *all* the paths P_1QP_2 of Step 2 give the optimal length up to a small error $\delta|y|$, and we can choose δ small compared to η.

The refraction law (3.8) implies that

$$\alpha|x_0 - P_1| + \beta|P_2 - x_0| = \alpha|Q(y) - P_1| + \beta|Q(y) - P_2| + o(y).$$

We therefore consider the comparison path γ_y that connects P_1 with $Q(y)$ with P_2 by two straight segments. Since at most a length $C\delta|y|$ of ther segment $P_2Q(y)$ is outside ω, we find

$$L_a(\gamma_y) \le d_a(P_2, P_1) + C(\beta - \alpha)\delta|y| + o(y).$$

Therefore, the path γ_y realizes the distance up to small errors. Then the path realizes the distance, up to the same error, on every point. We can choose Q in order to hit x^k, i.e. $x^k = \gamma_y(t) \in [P_1, Q] \cup [Q, P_2]$. We conclude that

$$\begin{aligned} C(\beta - \alpha)\delta|y| + o(y) &\ge L_a(\gamma_y|_{(0,t)}) - d_a(x^k, P_1) \\ &\ge [\bar{u}(x^k) - \bar{u}(P_1)] - [u(x^k) - u(P_1)] \\ &\ge \bar{u}(x^k) - u(x^k) \ge 0. \end{aligned}$$

For an appropriate choice of δ, using $y = O(x^k - x_0)$, this is the desired contradiction with (3.9). □

3.3 Global situation

Proof of Corollary 1.2. Let Σ be the interface between black and white region, a one dimensional Lipschitz-continuous object. By the lemma of Sard, almost all straight half-lines starting in A hit Σ non-tangentially. Furthermore, since Σ is differentiable in almost all points, almost all (with respect to the angle variable) straight lines hit Σ in a point of differentiability. We conclude that almost all (and hence all) rays coincide with a straight line. Since Σ hits rays normally, it must coincide with circles around A. Since a white ring between black regions is not optimal, the best metric consists of a single black disk. □

4 Local properties for gray regions

4.1 Interfaces

Our next result can be seen as a version of Proposition 3.2 for the case of interfaces of which one side is gray. It also follows from Proposition 3.1.

Proposition 4.1 (No changes of direction at gray interfaces)**.** *Let a be a non-trivial optimal metric, $u = d_a(A, \cdot)$. Let ω_1 and ω_2 be two disjoint open sets with Lipschitz boundary, and $x \in \partial\omega_1$ and $\varepsilon > 0$ be such that $B_\varepsilon(x_0) \subset \bar{\omega}_1 \cup \bar{\omega}_2$, with both boundaries differentiable in x_0, with normal $\pm\nu$.*

We assume that ∇u and a have continuous extensions to $\bar{\omega}_j$ for $j = 1, 2$. Let $\xi^\pm$ be the two limits of ∇u in x_0 from the two sides and assume that $\min\{|\xi^+|, |\xi^-|\} > \alpha$*. Then*

1. $\nabla u/|\nabla u|$ *is continuous in* x_0.

2. If $|\nabla u|$ *is not continuous in* x_0*, then* ∇u *is orthogonal to* Σ.

Proof. A piecewise affine function $\bar{u}$ can be constructed from $\xi^\pm$ and $u(x_0)$ as in (3.1). The approximation estimates (3.2) and (3.3) follow from the continuity of ∇u on both sides of the interface. If $\max\{|\xi^+|, |\xi^-|\} > \alpha$ then Proposition 3.1 yields 1 and 2, since ξ^+ and ξ^- are parallel and, if $\xi^+ \neq \xi^-$, then they are orthogonal to the interface. □

4.2 Parallel in gray regions

Our aim here is to prove Theorem 1.3, which states that level sets of u in the gray domain are straight lines.

Proof of Theorem 1.3. We are given a non-trivial optimal metric a, its distance function $u(.) = d_a(A, .)$ and an open subset $\omega \subset \mathbb{R}^2 \setminus B$ on which a and u obey the stated regularity properties.

Step 1. The regularity of a and u together with Lemma 2.1 yields $|\nabla u(x)| = a(x)$ in ω. The distance function u is a minimizer of the functional

$$\int_\omega \min\{|\nabla u|, \alpha\}\, dx$$

with respect to its own boundary values in the class of all β-Lipschitz functions. Let $\varphi \in C_c^\infty(\omega)$ be fixed. Since $\alpha < |\nabla u| < \beta$ on the compact support of φ, by continuity of $a = |\nabla u|$, for small $\varepsilon > 0$ we have $\alpha < |\nabla(u + \varepsilon\varphi)| < \beta$ and hence the function $u + \varepsilon\varphi$ is a possible competitor. We conclude

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \int_\omega |\nabla u + \varepsilon \nabla\varphi| dx = \int_\omega \frac{\nabla u}{|\nabla u|} \cdot \nabla\varphi\, dx,$$

which is the weak form of the equation

$$\operatorname{div}\left(\frac{1}{a}\nabla u\right) = 0\,. \tag{4.1}$$

Since a is continuous and has a weak derivative in L^p, by standard elliptic regularity we obtain that $u \in W^{2,p}(\tilde{\omega})$ for every compactly contained subset $\tilde{\omega} \subset \omega$.

Therefore the normalized gradient $n = \nabla u/a$ belongs to $W^{1,p}(\tilde{\omega}; S^1)$ for all compactly contained subsets $\tilde{\omega} \subset \omega$, and has zero distributional divergence.

Step 2a. Equation (4.1) implies the theorem by a result on the Eikonal equation by Jabin, Otto and Perthame [14]. We explain this conclusion in Lemma 4.2.

Step 2b. For the convenience of the reader we present an elementary self-contained argument in the case $p > 2$. The vector field n obeys

$$\operatorname{Tr} \nabla n = 0\,, \qquad n \cdot \nabla n = 0 \;\; a.e.$$

The first condition corresponds to $\operatorname{div} n = 0$, the second one comes from the fact that $0 = \nabla 1 = \nabla n^2 = 2n\nabla n$. Both are valid a.e. in $\tilde{\omega}$ for any $\tilde{\omega} \subset\subset \omega$, hence a.e. in ω.

We now observe that for any $F \in \mathbb{R}^{2\times 2}$ and $v \in \mathbb{R}^2 \setminus \{0\}$ one has

$$\operatorname{Tr} F = 0 \text{ and } F^T v = 0 \text{ implies } F v^\perp = 0\,,$$

where $v^\perp = (-v_2, v_1)$ (to see this, it suffices to express F in the basis $(v, v^\perp)$). Applying this observation to $F = \nabla n(x)$ and $v = n(x)$ we obtain

$$\nabla n(x) n^\perp(x) = 0 \text{ a.e. in } \omega. \tag{4.2}$$

We shall first present a simple formal calculation for the rest of the argument, and then give the proof. One key fact used below is that, since $p > 2$, the condition $u \in W^{2,p}(\tilde{\omega})$ implies $u \in C^{1,\alpha}(\tilde{\omega})$, for some $\alpha > 0$.

Formal calculation. Pick a point $x \in \tilde{\omega}$. We can use the implicit function theorem to find an isoline of u, more specifically, a path $\sigma \in C^{1,\alpha}((-\rho, \rho); \tilde{\omega})$ such that $|\sigma'| = 1$, $\sigma(0) = x$, and $u(\sigma(t)) = u(x)$ for all t. We compute

$$0 = \frac{d}{dt} u(\sigma(t)) = \nabla u(\sigma(t))\, \sigma'(t) = a n \cdot \sigma'(t)$$

which implies $\sigma'(t) = \pm n^\perp(\sigma(t))$, by continuity only one sign is used, say $+$. A formal computation using (4.2) reads

$$\frac{d}{dt}(\sigma'(t)) = \nabla n^\perp \circ \sigma \sigma' = \nabla n^\perp \cdot n^\perp = 0.$$

This result would prove that σ is affine. This is however incorrect, since we cannot assume globally $\sigma \in W^{1,p}$, at least not for all points x.

Proof. Pick $x \in \tilde{\omega}$, and assume $n_1(x) \neq 0$. Consider the function $U(y) := (u(y), y_2)$, which is $C^{1,\alpha}$ around x. Since $\det \nabla U(x) = \partial_1 u(x) \neq 0$ we can apply the implicit function theorem, and obtain open neighborhoods I of x and J of $U(x) = (u(x), x_2)$ such that U is a diffeomorphism of I onto J. Let $\psi = U^{-1} \in C^{1,\alpha}(J; I)$ be the inverse. We note that ψ has the special form $\psi(z) = (\psi_1(z), z_2)$ and inherits the regularity of U. The formula for the inverse of a 2×2-matrix shows that

$$\nabla U(y) = \begin{pmatrix} a(y) n(y) \\ e_2 \end{pmatrix}, \qquad \nabla \psi(U(y)) = \begin{pmatrix} 1/(a n_1)(y) \quad - n_2/n_1(y) \\ e_2 \end{pmatrix}.$$

Since U is bilipschitz on I, this implies $\nabla \psi \circ U \in W^{1,p}(I; \mathbb{R}^2)$, and analogously $\nabla \psi \in W^{1,p}(J; \mathbb{R}^2)$. The above expressions show that for all $z \in J$

$$\frac{\partial}{\partial z_2} \psi_1(z) = -\frac{n_2}{n_1}(\psi(z))\,.$$

With the function $f : \mathbb{R}^2 \setminus (\{0\} \times \mathbb{R}) \to \mathbb{R}$, $f(m) = -m_2/m_1$, and abbreviating its gradient by $F := \nabla f$, we compute, in the sense of L^p functions,

$$\begin{aligned} \frac{\partial^2}{\partial z_2^2} \psi_1(z) &= \frac{\partial}{\partial z_2}[f \circ n \circ \psi(z)] = \nabla f(n(\psi(z))) \cdot \nabla n(\psi(z)) \cdot \frac{\partial}{\partial z_2} \psi(z) \\ &= F \cdot \nabla n \cdot \left(-\frac{n_2}{n_1}(\psi(z))\ ,\ 1 \right) = \frac{1}{n_1} F \cdot \nabla n \cdot n^\perp = 0 \end{aligned}$$

by (4.2).

We found that ψ is a $W^{2,p}$ function with a vanishing second derivative. Therefore for a.e. value of z_1 the expression $\psi(z_1,\cdot)$ is in $W^{2,p}$ of an interval, and its second derivative is zero, which means that $\psi_1(z) = q(z_1) + z_2 r(z_1)$ (for a.e. z_1). But since ψ is continuous, we conclude that the same holds for all z_1. By $u(\psi(z)) = z_1$, for any fixed z_1 the function

$$z_2 \mapsto u(q(z_1) + z_2 r(z_1), z_2) = z_1$$

is constant, which is the thesis. □

We now come to the statement that divergence-free vector fields with values in S^1 are orthogonal to straight lines. This result depends crucially on the regularity of the vector field. A proof for the case $m \in W^{1,2}$ was given by Jabin, Otto and Perthame in [14], writing the divergence-free condition as a kinetic equation and considering suitable entropies. Their argument builds upon previous results on the eikonal functional and on two-dimensional models in micromagnetics [2, 15, 12] and holds without significant changes also for $W^{1,1}$ functions, as we now show.

Lemma 4.2. *Let $\omega \subset \mathbb{R}^2$ be open, $m \in W^{1,1}(\omega;\mathbb{R}^2)$ obey $|m| = 1$ almost everywhere and $\operatorname{div} m = 0$ almost everywhere. Then m is constant along segments orthogonal to m contained in ω. More precisely, let $x + (-a,b)m(x)^\perp \subset \omega$ for some $a, b > 0$, then $m(y) = m(x)$ for all $y \in x + (-a,b)m(x)^\perp$.*

Proof. Let $\Phi, \Psi \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and $\alpha \in C_0^\infty(\mathbb{R}^2,\mathbb{R})$ be related by

$$D\Phi(z) = -\Psi(z) \otimes z + \alpha(z)\mathrm{Id} \ .$$

Since $m \in W^{1,1}$ we obtain $\Phi \circ m \in W^{1,1}$, and the chain rule gives, in the sense of L^1-functions,

$$\operatorname{div}(\Phi \circ m) = \operatorname{tr}[(D\Phi)(m) \cdot \nabla m] = \operatorname{tr}[-\Psi(m) \otimes m \cdot \nabla m] + \alpha(m)\operatorname{div} m.$$

This expression vanishes, since $\operatorname{div} m = 0$ almost everywhere and $m^2 = 1$ implies $m \cdot \nabla m = 0$. We conclude that all vector fields $\Phi \circ m$ are divergence-free.

The arguments in [12] (Lemma 2.5) yield the following. Let $\xi \in S^1$ be arbitrary and $\chi(x,\xi)$ be one if $\xi \cdot m(x) > 0$ and 0 else. Then $\xi \cdot \nabla\chi(\cdot,\xi)) = 0$ distributionally in ω. Lemma 3.1 and Proposition 3.2 in [14] show that m is constant along segments orthogonal to m contained in ω, in the sense specified in the statement. □

We next show that also the metric has a very special form in the gray domain.

Corollary 4.3 (The metric in the gray region)**.** *Let a, u and ω be as in Theorem 1.3 and let Γ be a connected component of an isoline of u contained in ω (which is a segment by Theorem 1.3). Let Γ_∞ be the straight line containing Γ. Then there exist $a_\infty \in \mathbb{R}$ and $x_\infty \in \Gamma_\infty$ such that*

$$\begin{aligned} &\textit{either} \quad a(\cdot) = a_\infty \quad \textit{on } \Gamma \\ &\textit{or} \quad a(y) = a_\infty |y - x_\infty|^{-1} \quad \forall y \in \Gamma. \end{aligned}$$

Proof. Lemma 2.2 implies $a = |\nabla u|$. We have to show a purely geometrical fact: If all isolines are straight lines, then $|\nabla u|$ is necessarily of the asserted form.

Let Γ be an isoline segment with normal n. For the rest of the proof we fix $x \in \Gamma$ and choose a sequence $\omega \ni x_i \to x$ with $(x_i - x) \cdot n = |x_i - x|$. We write Γ_i for the isoline segment through x_i and denote the tangent of the angle between Γ_i and Γ by t_i. For an appropriate subsequence and some limit value $B_x \in \mathbb{R}$ we find

$$\lim_{i\to\infty} \frac{t_i}{|u(x_i) - u(x)|} = B_x.$$

The boundedness of the sequence on the left hand side follows from the fact that isolines can not intersect.

Let now $y \in \Gamma$ be another point on the isoline. We consider the points $y_i \in \Gamma_i$ with $(y_i - y) \cdot n = |y_i - y|$. Exploiting that u is differentiable and that $\nabla u || n$, we can calculate, with $d = (x - y) \cdot n^\perp$,

$$\frac{1}{a(y)} = \frac{1}{\nabla u(y) \cdot n} = \lim_{i\to\infty} \frac{n \cdot (y_i - y)}{|u(y_i) - u(y)|} = \lim_{i\to\infty} \frac{n \cdot (x_i - x) + t_i \cdot d}{|u(x_i) - u(x)|} = \frac{1}{a(x)} + B_x\, d.$$

In the case $B_x = 0$ we find that a is constant and we have derived the first case of the claim. In order to recover the geometric expression that appears in our claim for $B_x \neq 0$, we write

$$a(y) = \frac{1}{a(x)^{-1} + B_x\,(x - y) \cdot n^\perp} = \frac{B_x^{-1}}{|x + a(x)^{-1} B_x^{-1} n^\perp - y|}.$$

Since y was arbitrary on Γ, this provides the claim. □

5 The optimal metric for two points

With the following Theorem we solve the optimization problem explicitly in the case that $B \subset \mathbb{R}^2$ is a straight line. By symmetry, the solution is equivalent to the solution of the two-point problem.

Theorem 5.1. *Let $A = (0,0)$, $B = \{(1,0)\} \times \mathbb{R}$, $0 < \alpha < \beta$, $d \in (\alpha, \beta]$. Assume that $a : \Omega \to [0, \beta]$ is Borel-measurable and generates distance at least d, i.e.,*

$$\int_0^L a(\gamma(t))|\gamma'(t)|\, dt \geq d \tag{5.1}$$

for all $\gamma \in \mathrm{Lip}([0,L], \mathbb{R}^2)$ with $\gamma(0) = A$ and $\gamma(L) \in B$. Then the minimal used mass is given by

$$\int_\Omega (a - \alpha)_+ \geq \frac{(d - \alpha)^2 \sinh(2\eta\pi)}{2(\beta - \alpha)^{1/2} \alpha^{1/2} (\cosh(\eta\pi))^2}\,, \tag{5.2}$$

with $\eta = \alpha^{1/2}/(\beta - \alpha)^{1/2}$. Furthermore, the metric a_0 defined in (5.9)-(5.10) satisfies (5.1) and renders (5.2) an equality.

Before giving the proof we sketch the main ideas and illustrate how the explicit expressions can be found. In the entire argument we assume enough regularity so that the statements of the previous sections can be applied. The final expression we derive justifies this assumption.

The key idea is to parametrize the boundary of the black region Ω_b in polar coordinates by a function $f : [-\pi, \pi] \to (0, \infty)$, so that a point $\rho e_\theta \in \Omega_b$ whenever $\rho \in [0, f(\theta)]$, see Figure 6. In this region rays are segments starting from A and $u(x) = \beta|x|$. The angle θ is taken with respect to the *negative* horizontal axis, so that $e_\theta = (-\cos\theta, \sin\theta)$ (this choice renders f' positive in the computations below). We shall focus in the heuristics on the set $\{x_2 > 0\}$, the construction in the lower half-plane is symmetric.

The next layer contains a gray region, along which level sets of u are segments (by Theorem 1.3). Since on $\partial\Omega_b$ the gradient of u does not change direction (Prop. 4.1), every $x = f(\theta)e_\theta \in \partial\Omega_b$ is contained in a level set parallel to $e_\theta^\perp$. Let $g(\theta)$ be the width of the gray region along this direction (see Figure 6). For all $s \in [0, g(\theta)]$ we have

$$u(f(\theta)e_\theta + se_\theta^\perp) = u(f(\theta)e_\theta) = \beta f(\theta). \tag{5.3}$$

This defines u in the gray region. Rays in gray are then automatically defined as integral curves of ∇u (an expression is given in (5.9) is given below). It remains to determine the boundary between gray and white, i.e., to fix g. By Proposition 4.1 an interface which is not perpendicular to the ray is only possible if $|\nabla u|$ is continuous; therefore we determine g by locating the point where $|\nabla u|$ (which is completely determined by (5.3), once f is given) reaches the value α.

A direct computation shows that

$$\frac{\partial}{\partial\theta}(f(\theta)e_\theta + se_\theta^\perp) = (f'(\theta) + s)e_\theta - f(\theta)e_\theta^\perp\,,$$

and

$$\nabla u(f(\theta)e_\theta + se_\theta^\perp) = \beta\frac{f'(\theta)}{f'(\theta) + s}e_\theta.$$

Hence demanding $|\nabla u| \geq \alpha$ is equivalent (for $f' \geq 0$) to

$$s \leq g(\theta) := \frac{\beta - \alpha}{\alpha}f'(\theta)\,. \tag{5.4}$$

Therefore the gray region is $\bigcup_{\theta\in[0,\pi]} f(\theta)e_\theta + [0, g(\theta)]e_\theta^\perp$ (plus the symmetric part in the lower half-plane); the white region Ω_w will be the rest of $\mathbb{R}^2$. The condition that a equals $|\nabla u|$ on Ω_g completely defines the metric for each f.

In the above construction the level set corresponding to $\theta = \pm\pi$ is parallel to $e_\pi^\perp$, hence a vertical line through the point $(f(\pi), 0)$. The value of u on this line is $\beta f(\pi)$. For larger x the function u is affine, with gradient $(\alpha, 0)$. This implies that the line $\mathbb{R} \times \{0\}$ contains no gray, and computing along that line, we obtain $u(B) = \beta f(\pi) + \alpha(1 - f(\pi))$. Therefore we can determine $f(\pi) \in (0, 1]$ by

$$\beta f(\pi) + \alpha(1 - f(\pi)) = d\,. \tag{5.5}$$

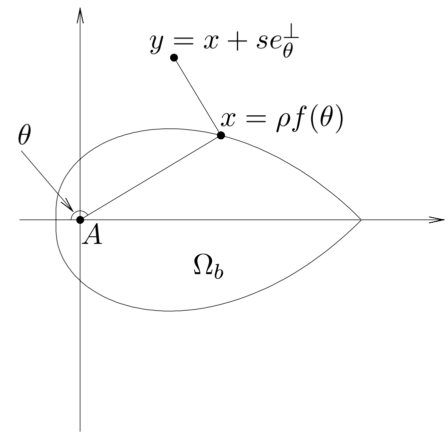

Figure 6: The construction of the optimal metric. Indicated is the boundary of the black region where rays are straight lines. Outside the central shape is a gray region where level sets of u are straight lines. The two are orthogonal to each other on the interface.

It remains to determine the function f. This should, given this constaint, minimize the mass, which corresponds to the integral of $(a-\alpha)_+$. If f is even and increasing on $(0,\pi)$,

$$\int_\Omega (a-\alpha)_+ = (\beta-\alpha)|\Omega_b| + 2\int_0^\pi \int_0^{g(\theta)} \left(\beta\frac{f'(\theta)}{f'(\theta)+s} - \alpha\right)(f'(\theta)+s)\,ds\,d\theta\,.$$

After some rearrangement this gives

$$\int_\Omega (a-\alpha)_+ = (\beta-\alpha)\int_0^\pi \left(f^2 + \eta^{-2}(f')^2\right)\,d\theta\,, \tag{5.6}$$

where $\eta = (\alpha/(\beta-\alpha))^{1/2}$. Explicit minimization (with even f) gives

$$f(\theta) = c\cosh(\eta\theta)\,,$$

where c is determined by the boundary condition (5.5). Together with relation (5.4) for g we have determined all unknowns.

In order to show that the construction provides indeed an optimal metric, a different approach is more convenient, which focusses on the rays instead of u. We shall consider a one-parameter family of rays, each ray parameterized by ℓ which coincides with the value of u [i.e., $u(\gamma(\theta,\ell)) = \ell$]; the family shall be parameterized by the angle θ in the initial black region.

Proof of Theorem 5.1. **Step 1. Construction of the family of rays γ and of the metric a_0.** We define

$$\eta := \left(\frac{\alpha}{\beta-\alpha}\right)^{1/2} \tag{5.7}$$

and

$$c := \frac{d-\alpha}{(\beta-\alpha)\cosh(\eta\pi)}\,, \tag{5.8}$$

so that the function $f(\theta) = c\cosh(\eta\theta)$ obeys (5.5). The condition $d \in (\alpha, \beta]$ ensures that $0 < c\cosh(\eta\pi) \le 1$. We define $\gamma : (-\pi, \pi] \times (0, d) \to \Omega$ by

$$\gamma(\theta,\ell) := \begin{cases} \frac{1}{\beta}\ell e_\theta & \text{if } 0 < \ell < c\beta\cosh(\eta\theta)\,, \\ \frac{1}{\beta}\ell e_t + \frac{c}{\eta}(\sinh(\eta t) - \sinh(\eta\theta))e_t^\perp & \text{if } c\beta\cosh(\eta\theta) \le \ell < u_0\,, \\ \left[\frac{u_0}{\beta} + \frac{\ell-u_0}{\alpha}\right] e_\pi + \frac{c}{\eta}(\sinh(\eta\pi) - \sinh(\eta\theta))e_\pi^\perp & \text{if } \ell \ge u_0 \end{cases} \tag{5.9}$$

where $t = t(\ell)$ is defined by $c\beta\cosh(\eta t) = \ell$, $e_\theta = (-\cos\theta, \sin\theta)$, the perpendicular unit vector is $e_\theta^\perp = (-\sin\theta, -\cos\theta) = -\partial_\theta e_\theta$, and $u_0 = c\beta\cosh(\eta\pi)$. From the definition of c one can check that $u_0 \le d$. The three cases correspond to intervals where $\gamma(\theta, \cdot)$ is in the black, gray and white region (see Figure 1).

The map γ is continuous and injective, we set $\omega = \gamma((-\pi, \pi] \times (0, d))$ (to check injectivity one can e.g. verify that $\det\nabla\gamma > 0$ and consider the behavior of γ on the boundary of $(-\pi, \pi) \times (\varepsilon, d)$, for some $\varepsilon \in (0, c\beta)$). We define

$$a_0 := \frac{1}{|\partial_\ell\gamma|} \circ \gamma^{-1} \tag{5.10}$$

in ω, and and $a_0 = \alpha$ outside ω. We compute, for $c\beta\cosh(\eta\theta) < \ell < u_0$, the derivatives $\partial_\ell t = 1/(c\beta\eta\sinh(\eta t))$,

$$\partial_\theta\gamma = -c\cosh(\eta\theta)e_t^\perp \tag{5.11}$$

$$\partial_\ell\gamma = \left[\frac{1}{\beta} + \frac{c}{\eta}\frac{\sinh(\eta t) - \sinh(\eta\theta)}{c\beta\eta\sinh(\eta t)}\right] e_t = \left[\frac{1}{\alpha} - \frac{1}{\beta\eta^2}\frac{\sinh(\eta\theta)}{\sinh(\eta t)}\right] e_t\,. \tag{5.12}$$

We emphasize that γ is constructed such that no component along $e_t^\perp$ arises in $\partial_\ell\gamma$ (this motivated by the form of the isolinies discussed above). In particular, (5.12) implies $\frac{1}{\beta} \le |\partial_\ell\gamma| \le \frac{1}{\alpha}$, hence it is admissible. We observe that for each $\theta \in (-\pi, \pi]$ the curve $\ell \mapsto \gamma(\theta, \ell)$ is a Lipschitz curve with $\gamma(\theta, 0) = A$ and $\gamma(\theta, d) \in B$. Along these curves condition (5.1) is an equality for a_0 by definition of a_0. Furthermore, $\partial_\ell\gamma$ and $\partial_\theta\gamma$ are orthogonal and hence $|\det\nabla\gamma| = |\partial_\ell\gamma|\,|\partial_\theta\gamma|$.

Step 2. Estimates for an arbitrary metric a. Let a be any admissible metric with induced distance at least d. Let γ be as in (5.9). Since γ is injective we have

$$\begin{aligned} \int_\Omega (a-\alpha)_+\,dx \ge \int_\omega (a-\alpha)_+\,dx &= \int_{-\pi}^{\pi}\int_0^d (a\circ\gamma - \alpha)_+ |\det\nabla\gamma|\,d\ell\,d\theta \\ &= \int_{-\pi}^{\pi}\int_0^d (a\circ\gamma-\alpha)_+ |\partial_\ell\gamma||\partial_\theta\gamma|\,d\ell d\theta\,. \end{aligned}$$

Replacing a by $\max\{a, \alpha\}$ we can assume without loss of generality $a \ge \alpha$.

The idea of the proof is to show that the factor $|\partial_\theta\gamma|$ in the last integral can, in the relevant region, be replaced by a constant. Then the remaining integral in ℓ

has the form of the a-length of the curve $\ell \mapsto \gamma(\theta, \ell)$, which we know to be bounded from below by d.

The factor $|\partial_\theta \gamma|$ obeys, by the definition of γ,

$$|\partial_\theta \gamma|(\theta, \ell) \leq c \cosh(\eta\theta)\,, \tag{5.13}$$

with a strict inequality in the first (black) region, i.e., in the region corresponding to $a_0 = \beta$. The precise estimate will be made comparing the expression containing a with the corresponding expression containing the optimal metric a_0. For each value of θ we can estimate via Lemma 5.2, with $\Omega = (0, d)$, $f = |\partial_\ell \gamma|$, $g = |\det \nabla\gamma|$, and $\lambda = c \cosh \eta\theta$,

$$\int_0^d (a \circ \gamma - \alpha)|\partial_\ell \gamma||\partial_\theta \gamma|\, d\ell \geq \int_{(0,d)_+} (\beta - \alpha)(g - \lambda f)\, d\ell + \lambda \int_0^d (a \circ \gamma - \alpha)|\partial_\ell \gamma|\, d\ell\,,$$

where $(0, d)_+ = \{s \in (0, d) : g < \lambda f\}$ (and, by (5.13), $(0, d)_- = \emptyset$). The first integral does not depend on a (also the set $(0, d)_+$ only depends on γ). The second one contains one term which corresponds to the a-length of the curve $\gamma(\theta, \cdot)$, and which is estimated by $\int_0^d a \circ \gamma|\partial_\ell \gamma| \geq d$, and another one which does not depend on a.

At the same time, by (5.13) and the choice of λ the corresponding estimate obtained from Lemma 5.2 in the case of a_0 is an equality,

$$\int_0^d (a_0 \circ \gamma - \alpha)|\partial_\ell \gamma||\partial_\theta \gamma|\, d\ell = \int_{(0,d)_+} (\beta - \alpha)(g - \lambda f)\, d\ell + \lambda \int_0^d (a_0 \circ \gamma - \alpha)|\partial_\ell \gamma|\, d\ell\,.$$

Analogously, the construction of a_0 ensures that the estimate of the last term via the a_0-length of the curve $\gamma(\theta, \cdot)$ is also an equality (precisely, $\int_0^d a_0 \circ \gamma|\partial_\ell \gamma| d\ell = d$). We conclude that

$$\int_0^d (a \circ \gamma - \alpha)|\partial_\ell \gamma||\partial_\theta \gamma|\, d\ell \geq \int_0^d (a_0 \circ \gamma - \alpha)|\partial_\ell \gamma||\partial_\theta \gamma|\, d\ell$$

for all θ. Integrating over θ shows the optimality inequality

$$\int_{\mathbb{R}^2} (a - \alpha)_+\, dx \geq \int_{\mathbb{R}^2} (a^0 - \alpha)_+\, dx. \tag{5.14}$$

The value of the right hand side appears in (5.2) and is determined with a straightforward computation, which essentially amounts to integrating (5.6) with the given parameters.

Step 3. Optimality of a_0. It remains to prove that a_0 obeys (5.1) for all curves γ (we have checked it in Step 1 only for our special curves). We define $u \in \mathrm{Lip}(\mathbb{R}^2)$ by $u(re_\theta) = \beta r$ if $r < c \cosh(\eta\theta)$, $u(c \cosh(\eta\theta) e_\theta + s e_\theta^\perp) = \beta c \cosh(\eta\theta)$ for all $s > 0$, u affine on $(c \cosh \eta\pi, \infty) \times \mathbb{R}$. This corresponds to $u(\gamma(\theta, \ell)) = \ell$ on ω. Then $|\nabla u| = a_0$, and for any admissible curve $\hat\gamma$

$$\int_0^L a_0(\hat\gamma(t))|\hat\gamma'(t)| dt \geq \int_0^L \left| \frac{d}{dt} u(\hat\gamma(t)) \right| dt = d\,.$$

This concludes the proof. □

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^n$, $f, g : \Omega \to (0, \infty)$, $0 < \alpha < \beta$, $a : \Omega \to [\alpha, \beta]$, all measurable, $\lambda > 0$. Then*

$$\int_\Omega (a - \alpha) g \geq \int_{\Omega_+} (\beta - \alpha)(g - \lambda f) + \lambda \int_\Omega (a - \alpha) f \,,$$

where $\Omega_+ = \{\lambda f > g\}$. Equality is achieved whenever $a = \beta$ on Ω_+ and $a = \alpha$ on $\Omega_- = \{\lambda f < g\}$.

Proof. We calculate

$$\begin{aligned}\int_\Omega (a - \alpha) g - \lambda \int_\Omega (a - \alpha) f &= \int_\Omega (a - \alpha)(g - \lambda f) \\ &= (\beta - \alpha) \int_{\Omega_+} (g - \lambda f) + \int_{\Omega_+} (\beta - a)(\lambda f - g) + \int_{\Omega_-} (a - \alpha)(g - \lambda f) \,,\end{aligned}$$

where each integral is nonnegative. □

Acknowledgment. This work was initiated while the second author was visiting the University of Pisa. We express our gratitude to Giuseppe Buttazzo for posing this interesting problem and sharing his knowledge, and for encouraging discussions. This work was partially supported by the DFG through SPP1253.

References

[1] Emilio Acerbi and Giuseppe Buttazzo. On the limits of periodic Riemannian metrics. *J. Analyse Math.*, 43:183–201, 1983/84.

[2] Luigi Ambrosio, Camillo De Lellis, and Carlo Mantegazza. Line energies for gradient vector fields in the plane. *Calc. Var. Partial Diff. Eqs.*, 9:327–355, 1999.

[3] Guy Bouchitté and Giuseppe Buttazzo. Characterization of optimal shapes and masses through Monge-Kantorovich equation. *J. Eur. Math. Soc. (JEMS)*, 3(2):139–168, 2001.

[4] Guy Bouchitté, Giuseppe Buttazzo, and Pierre Seppecher. Shape optimization solutions via Monge-Kantorovich equation. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(10):1185–1191, 1997.

[5] Andrea Braides, Giuseppe Buttazzo, and Ilaria Fragalà. Riemannian approximation of Finsler metrics. *Asymptot. Anal.*, 31(2):177–187, 2002.

[6] Yann Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.

[7] Giuseppe Buttazzo, Andrea Davini, Ilaria Fragalà, and Fabrićio Macià. Optimal Riemannian distances preventing mass transfer. *J. Reine Angew. Math.*, 575:157–171, 2004.

[8] Giuseppe Buttazzo and Ben Schweizer. Γ convergence of Hausdorff measures. *J. Convex Anal.*, 12(1):239–253, 2005.

[9] Luis A. Caffarelli, Mikhail Feldman, and Robert J. McCann. Constructing optimal maps for Monge's transport problem as a limit of strictly convex costs. *J. Amer. Math. Soc.*, 15(1):1–26 (electronic), 2002.

[10] Sergio Conti and Ben Schweizer. Rigidity and gamma convergence for solid-solid phase transitions with SO(2) invariance. *Comm. Pure Appl. Math.*, 59(6):830–868, 2006.

[11] Sergio Conti and Ben Schweizer. A sharp-interface limit for a two-well problem in geometrically linear elasticity. *Arch. Ration. Mech. Anal.*, 179(3):413–452, 2006.

[12] Antonio DeSimone, Robert V. Kohn, Stefan Müller, and Felix Otto. A compactness result in the gradient theory of phase transitions. *Proc. Roy. Soc. Edin. A*, 131:833–844, 2001.

[13] Lawrence C. Evans and Wilfrid Gangbo. Differential equations methods for the Monge-Kantorovich mass transfer problem. *Mem. Amer. Math. Soc.*, 137(653):viii+66, 1999.

[14] Pierre-Emmanuel Jabin, Felix Otto, and Benoît Perthame. Line-energy Ginzburg-Landau models: zero-energy states. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 1:187–202, 2002.

[15] W. Jin and Robert V. Kohn. Singular perturbation and the energy of folds. *J. Nonlinear Sci.*, 10:355–390, 2000.

[16] Leonid V. Kantorovich. On the transfer of masses. *Dokl. Akad. Nauk. SSSR*, (37):227–229, 1942.

[17] Cédric Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

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