Discount curve estimation by monotonizing McCulloch Splines

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Abstract

In this paper a new and very simple method for monotone estimation of discount curves is proposed. The main idea of this approach is a simple modification of the commonly used (unconstrained) McCulloch Spline. We construct an integrated density estimate from the predicted values of the discount curve. It can be shown that this statistic is an estimate of the inverse of the discount function and the final estimate can easily be obtained by a numerical inversion. The resulting procedure is extremely simple and we have implemented it in Excel and VBA, respectively. The performance is illustrated by three examples, in which the curve was previously estimated with an unconstrained McCulloch Spline.

1 Introduction

Yield curve estimation plays a central role in pricing fixed-income derivatives, risk management and for national central banks. Because the yield curve is not directly observable and there do not exist enough zero coupon bonds, it has to be derived from observed market prices of coupon bearing bonds.
From the mathematical point of view it is equivalent to estimate the spot rates, the forward rates or the discount factors, where the discount factor $\delta(t)$ is the value of 1 unit money, which is payed in time t [see for example Deacon, Derry (1994)]. In practice one has different shape restrictions on

these curves, which are based on economic theory. One very crucial condition is, that forward rates do not become negative. In other words, the discount curve has to be monotone decreasing. A second very crucial constraint is the initial condition $\delta(0) = 1$.
Although there exist a lot of different approaches and literature for estimating the yield curve [see Anderson et.al. (1996) for a review], there are only a few articles concerning the problem of nonnegative estimation of the forward rates[1]. Wets, Bianchi and Yang (2002) used, so called, EpiCurves to obtain a monotone estimate of the discount curve. For this approach, which functional behavior is restricted to a subfamily of smooth curves, linear optimization with many constraints has to be used. Manzano and Blomvall (2004) presented a non-linear dynamic programming algorithm, which implements the forward rate positivity constraint for a one-parametric family of smoothness measures. Hagan and West (2005) introduced a monotone and convex spline [see also Kvasov (2000) for a textbook on shape restricted splines]. Here the curve is guaranteed to be positive if all the inputs are positive.
In this article we present an alternative procedure for the estimation of the discount curve, which can be used to monotonize any previously estimated (discount) curve. The method is extremely simple and uses an integrated density estimate from the predicted values of the unconstrained estimate of the discount function. It can be shown that this statistic is an estimate of the inverse of the discount function and the final (non-increasing) estimate can be obtained by numerical inversion. We will apply the new method to the widely used McCulloch Spline and illustrate its performance by three examples. The resulting procedure is very easy to handle and shows better results than ad hoc methods for monotonizing curves. A program implemented in Excel and VBA is available from the second author.

2 The McCulloch Spline

Splines are very popular for estimating the term structure of interest rates. These functions are highly flexible and so suited for most circumstances. A good review of the statistical aspects of this estimation method can be found in Eubank (1988).
McCulloch (1975) proposed a simple cubic spline model for the estimation of the discount curve, which has a rather reliable performance in most circumstances. He proposed to chose $l-1$ knots, $k_1, \cdots, k_{l-1}$, with $k_1 = 0$ and k_{l-1} corresponding to the longest maturity of a bond. For $j < l$ the j-th spline

[1] The following three articles are the only ones known to the authors.

function is defined as

$$\delta_j(t) = \begin{cases} 0 & \text{if } t < k_{j-1} \\ \frac{(t-k_{j-1})^3}{6(k_j-k_{j-1})} & \text{if } k_{j-1} \leq t < k_j \\ \frac{(k_j-k_{j-1})^2}{6} + \frac{(k_j-k_{j-1})(t-k_j)}{2} + \frac{(t-k_j)^2}{2} - \frac{(t-k_j)^3}{6(k_{j+1}-k_j)} & \text{if } k_j \leq t < k_{j+1} \\ (k_{j+1}-k_{j-1})[\frac{2k_{j+1}-k_j-k_{j-1}}{6} + \frac{t-k_{j+1}}{2}] & \text{if } k_{j+1} \leq t, \end{cases} \tag{1}$$

and for $j = l$ the function is given as:

$$\delta_l(t) = t, \forall t. \tag{2}$$

The resulting spline estimator is finally obtained as the least squares fit of the function $\delta(t) = \sum_{j=1}^{l} \hat{\alpha}_j \delta_j(t)$ to the data.
From a practical point of view it is not easy to handle this method, because splines are sometimes too flexible. Shea (1984) points out that spline functions, like any other numerical approximation technique, cannot yield reasonable estimates without the intelligent use of constraints. First, the user must choose the number and location of polynomial pieces that will serve as the building blocks of the spline model. The polynomial order and the degree of continuity of the spline function are also a matter of choice. If n is the number of observed bonds, McCulloch (1975) suggested to place l knots, where l is the nearest integer to $\sqrt{n}$. Additionally these knots have to be located, such that between two knots there are (nearly) the same number of bonds. There are more sophisticated rules for chosing smoothing parameters in spline estimation [see e.g. Eubank (1988)], but with this heuristic rule the McCulloch Spline is easy to implement and yields, under normal conditions, satisfactory results. Unfortunately the estimates obtained with the McCulloch Spline (and by other spline estimators) do not necessarily satisfy the shape restrictions from economic theory. This is a general drawback of splines and discussed in detail by two papers of Shea (1984, 1985). In particular it is possible that the estimates of the forward rates become negative and we have illustrated this phenomenon in Figure 1 analyzing data from 81 Swiss mortgage bonds issued by the Pfandbriefbank of Swiss Cantonal Banks with settlement date 01.11.2003. This Figure shows the (unconstrained) estimates of the forward rates obtained with the McCulloch Spline, which are negative at some places. Nonnegative estimates of the forward rates can be avoided using parametric estimates for the discount curve. However, such methods are usually not very flexible. To illustrate this phenomenon, Figure 1 also shows the forward rates estimated with the techniques proposed by Nelson and Siegel (1987) and Svensson (1994). The parametric models proposed by these authors are often used by federal banks [see for example

Schich (1996)]. The shapes of the parametric and spline curves are totally different. It is obvious that the parametric estimates are much smoother, but unfortunately the two methods are too inflexible and imprecise for many practical investigations.

-insert Figure 1 about here-

3 Monotone smoothing by inversion

In this section we briefly explain a very simple procedure for monotonizing curve estimates, which was recently proposed by Dette, Neumeyer and Pilz (2006). In the following we will use this method for monotonizing the McCulloch Spline described in Section 2, but it should be noted that this approach of monotonization can be used to monotonize any unconstrained estimate. For this reason we describe it in the general nonparametric regression model

$$Y_i = \delta(X_i) + \sigma(X_i)\epsilon_i, \qquad i = 1, \cdots, n, \tag{3}$$

where $\{(X_i, Y_i)\}_{i=1}^n$ is a bivariate sample of i.i.d. observations [see e.g. Fan and Gijbels (1996)]. This means in our case, that X_i is some time to maturity, Y_i the corresponding discount factor and the function δ represents the discount curve. After an appropriate scaling we may assume without loss of generality that the explanatory variables X_i vary in the interval [0,1]. We further assume that the random variables ϵ_i are i.i.d. with $E[\epsilon_i] = 0$, $E[\epsilon_i^2] = 1$. The regression function $\delta : [0, 1] \to \mathbf{R}$ is assumed to be twice continuously differentiable. For the asymptotic analysis of the following estimate some more assumptions have to be made [see Dette, Neumeyer and Pilz (2006) for details], but these will not be repeated, because the focus of the present paper are applications.

Our procedure starts with an unconstrained curve estimate, say $\hat{\delta}$. There are several proposals to estimate the function δ in model (3) [see e.g. Härdle (1990), Fan and Gijbels (1996) and Eubank (1988)] and in the following any of these estimates could be used as initial (unconstrained) estimate in our method. In the applications discussed in the following section $\hat{\delta}$ will be the McCulloch Spline. If there is evidence that the regression function δ is (strictly) decreasing we define for N$\in \mathbf{N}$

$$\hat{\delta}_A^{-1}(t) := \frac{1}{Nh_d} \sum_{i=1}^{N} \int_t^\infty K_d\left(\frac{\hat{\delta}(\frac{i}{N}) - u}{h_d}\right) du, \tag{4}$$

where K_d denotes a positive and symmetric kernel with compact support, say [-1,1], existing second moment and h_d is the corresponding bandwidth converging to 0 with increasing sample sizes. The statistic in (4) can easily be motivated by replacing the points $1/N, 2/N, \ldots, 1$ in (4) with an i.i.d. sample of uniformly distributed random variables, say $U_1, \ldots, U_N \sim \mathcal{U}([0,1])$. If δ is a strictly decreasing function on the interval $[0,1]$ with negative derivative, then

$$\frac{1}{Nh_d} \sum_{i=1}^{N} K_d\Big(\frac{\delta(U_i) - u}{h_d}\Big) \tag{5}$$

is the classical kernel estimate of the density of the random variable $\delta(U_1)$ [see Silverman (1986)]. From elementary probability it follows that this density is given by

$$-(\delta^{-1})'(u) I_{[\delta(1),\delta(0)]}(u).$$

Consequently, if one integrates (5) appropriately, one obtains an estimate of the function δ^{-1} at the point t. Finally, the unknown regression function δ is replaced by an appropriate estimate $\hat{\delta}$ and the random variabes U_i are substituted by the deterministic points i/N $(i = 1, \ldots, N)$, which yields the statistic defined in (4).

As the Kernel K_d is positive, it follows that the estimate $\hat{\delta}_A^{-1}$ is strictly decreasing. Finally an antitone estimate, say $\hat{\delta}_A$, of the regression function δ is simply obtained by reflection of the function $\hat{\delta}_A^{-1}$ at the line $y = x$. Note that the estimator $\hat{\delta}_A^{-1}(t)$ is equal to 0 and 1 if

$$t > \max_{i=1}^{N} \hat{\delta}(\frac{i}{N}) + h_d \quad \text{and} \quad t < \min_{i=1}^{N} \hat{\delta}(\frac{i}{N}) - h_d,$$

respectively. Because of this, the second crucial condition, $\hat{\delta}_A(0) = 1$, is trivially fulfilled, if h_d is chosen sufficiently small and the preliminary unconstrained estimate satisfies $\hat{\delta}(\frac{i}{N}) \leq 1$ for all i, which is normally the case in applications [see our examples in the following section].

For an increasing sample size and a bandwidth h_d converging sufficiently fast to 0 it is shown in Dette, Neumeyer and Pilz (2006) that in cases where the "true" regression function δ is in fact decreasing the constrained estimate $\hat{\delta}_A$ is consistent and first order asymptotically equivalent to the unconstrained estimate $\hat{\delta}$. Thus from an asymptotic point of view the new estimate $\hat{\delta}_A$ shares the same nice properties as the unconstrained estimate $\hat{\delta}$ and is additionally antitone.

4 Implementation and tests

In the present section we illustrate the monotonization method for the problem of constructing a decreasing estimate of the discount curve, where the McCulloch Spline is used as initial unconstrained estimate. We have implemented the monotonizing procedure in Excel and VBA, respectively.
For the application of the procedure we have used the McCulloch Spline, with knots placed as suggested in McCulloch (1975), for the preliminary estimate $\hat{\delta}$ in formula (4), which also requires the specification of a kernel K_d, a number N of evaluation points and a bandwidths h_d. For sake of transparency we restrict ourselves to the Epanechnikov-kernel, but it is notable that other types of kernel estimators yield very similar results. The Epanechnikov-kernel is defined by

$$K_d(u) := \begin{cases} \frac{3}{4}(1-u^2) & \text{if } |u| < 1 \\ 0 & \text{else,} \end{cases} \tag{6}$$

easy to handle and fulfills the above assumptions. After integrating equation (4) is basically a sum of N terms, where for the i-th summand the integration yields 0, if

$$\hat{\delta}(\frac{i}{N}) + h_d < t,$$

h, if

$$\hat{\delta}(\frac{i}{N}) - h_d > t$$

and

$$\frac{3}{4}\Big[\hat{\delta}(\frac{i}{N}) - t + \frac{2}{3}h_d - \frac{(\hat{\delta}(\frac{i}{N}) - t)^3}{3h_d^2}\Big]$$

else. It was observed by Dette, Neumeyer and Pilz (2006) empirically that the choice of the bandwidth h_d is not too critical as long as it is chosen sufficiently small. We use $h_d = 0,00001$ as bandwidth in estimate (4).
The three charts below show the resulting unconstrained and constrained estimates in three examples. The dotted lines represent the estimates obtained with the classical McCulloch Spline, while the solid lines are the corresponding monotonizations obtained from (4).
Our first example considers the situation investigated in Section 2. We have estimated the term structure of interest rates of 81 Swiss mortgage bonds issued by the Pfandbriefbank of Swiss Cantonal Banks with settlement date 01.11.2003. The classical estimation with the MCulloch spline yields highly fluctuating forward rates, which become even negative [see the upper panel in Figure 2]. The new estimate $\hat{\delta}_A$, which was obtained with $N = 2340$, modifes the initial estimate in regions where it is not decreasing. Note that

there is a maximal difference of 79 base points between the discount factors of the constrained and the unconstrained estimate [see Figure 3].

-insert Figure 2 about here-

-insert Figure 3 about here-

Figure 4 shows an constructed example. We took Finnish government bonds with settlement date 30.06.2005 and manipulated these data. As the sample size was only 12 changing the coupon of 1 bond was enough to get negative forward rates [see the upper panel in Figure 4]. This example is not unrealistic from a practical point of view as some errors in the data often occur. We calculated the monotone estimate with $N = 2627$ and observe a similar picture as for the swiss bonds.

-insert Figure 4 about here-

The last picture shows an estimate of the term structure of a portfolio. This portfolio contains 18 European government and corporate bonds, which are all listed in Iboxx indices. The settlement date is 03.03.2006. Again, the unconstrained estimate yields negative forward rates and is corrected by the constrained estimate, which is obtained with $N = 1860$ [see the upper panel in Figure 5].

-insert Figure 5 about here-

In all three cases the estimated forward rates seem to be more reliable and less fluctuating after monotonization. In addition to that the two estimates of the discount curves do not differ in regions, where the unconstrained estimate is already antitone. Additionally the shapes of the curves are much closer to each other then to curves obtained by parametrical models.

5 Conclusions

We have presented a new procedure to monotonize any prior unconstrained estimate of the discount curve. The method is very easy to implement and a procedure in Excel and VBA has been implemented, which is available from the second author. We have illustrated the new estimation procedure by three examples. These tests show some improvement with respect to the estimation of the forward rates. The new estimate coincides with the initial unconstrained estimate in regions where this is already decreasing and corrects all other parts. The resulting statistic is a decreasing estimate over the full observation region. Because of these advantages and its simplicity we recommend this method for practical investigations.

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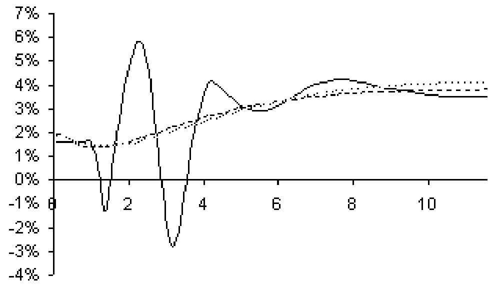

Figure 1: *Unconstrained estimates of the forward rates from* 81 *Swiss mortgage bonds. Solid line: McCulloch Spline; dashed line: Parametric estimate of Nelson and Siegel (1987); dotted line: Parametric estimate of Svensson (1994).*

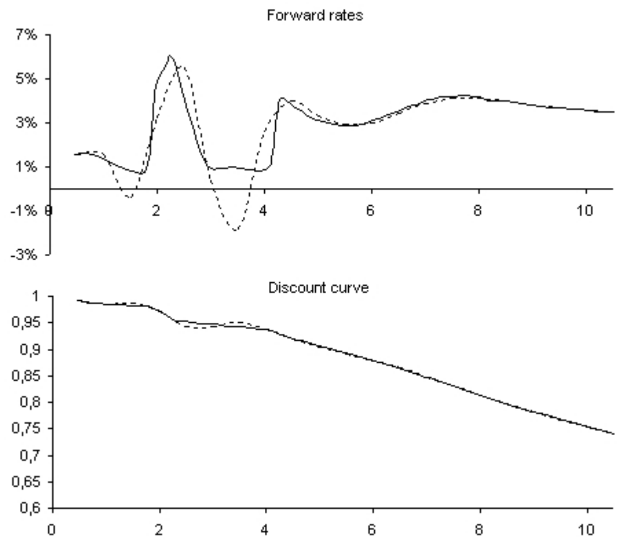

Figure 2: *Estimated discount curves and forward rates of Swiss mortgage bonds. Dotted lines: unconstrained McCulloch Spline; solid lines: monotonized McCulloch Spline.*

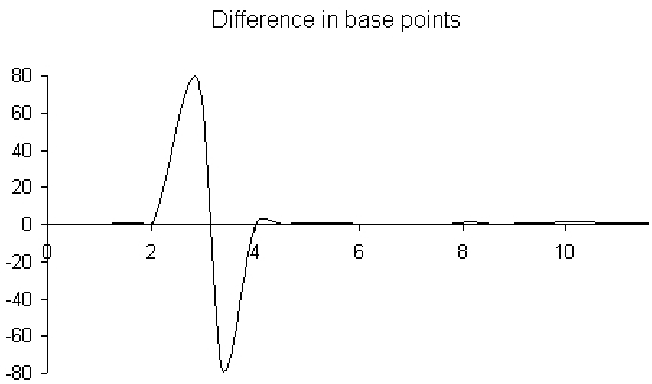

Figure 3: *Difference in base points between discount factors estimated from* 81 *Swiss mortgage bonds with the unconstrained and the constrained McCulloch Spline.*

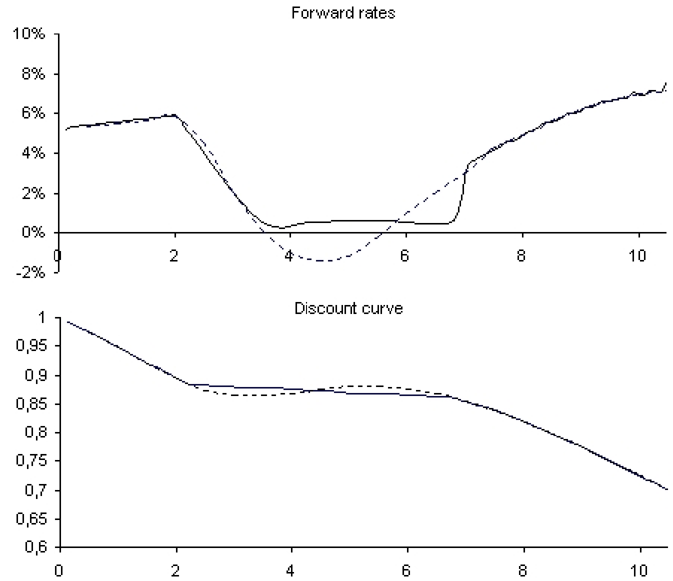

Figure 4: *Estimated discount curves and forward rates of Finnish government bonds. Dotted lines: unconstrained McCulloch Spline; solid lines: monotonized McCulloch Spline.*

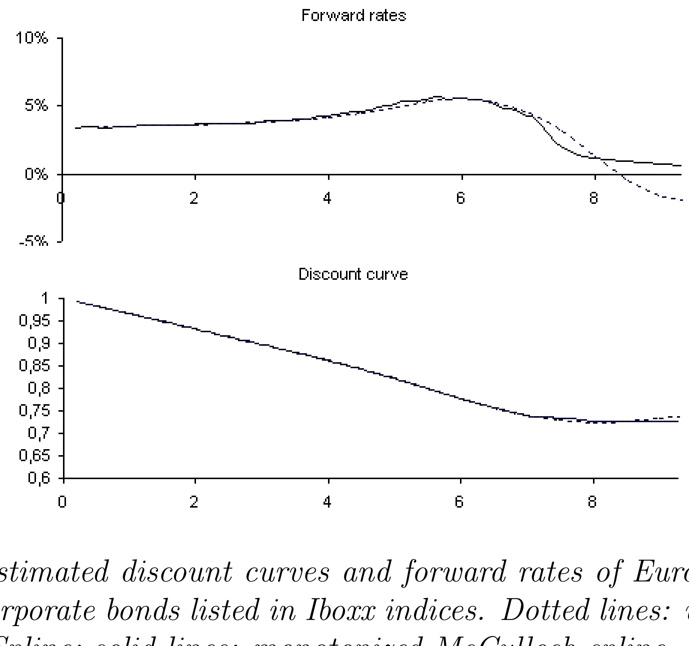

Figure 5: *Estimated discount curves and forward rates of European government and corporate bonds listed in Iboxx indices. Dotted lines: unconstrained McCulloch Spline; solid lines: monotonized McCulloch spline.*