Homogenization of compressible fluids in perforated domains

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Homogenization of compressible fluids in perforated domains

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Contents

“Big whorls have little whorls
which feed on their velocity;
And little whorls have lesser whorls,
and so on to viscosity.”

Lewis Fry Richardson, *Weather Prediction by Numerical Process*, 1922

List of symbols

$A : B$	Frobenius scalar product of $A, B \in \mathbb{R}^{d\times d}$; $A : B \coloneqq \sum_{1\le i,j\le d} A_{ij}B_{ij}$
$\mathbf{a}\otimes\mathbf{b}$	tensor product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$; $(\mathbf{a}\otimes\mathbf{b})_{ij} \coloneqq \mathbf{a}_i\mathbf{b}_j$
$\mathcal{B}$	Bogovskiĭ's operator
$B_r(x)$	open ball with radius $r > 0$ around $x \in \mathbb{R}^d$
χ_M	characteristic function of the set M
$C^\infty_{c,0}(D)$	set of smooth functions with compact support and zero mean value over D
$\mathcal{D}(D)$	set of test functions; $\mathcal{D}(D) \coloneqq C^\infty_c(D)$
$\operatorname{diam}(D)$	diameter of a set $D \subset \mathbb{R}^d$; $\operatorname{diam}(D) \coloneqq \sup\{\lvert x-y\rvert : x, y \in D\}$
$\operatorname{dist}_\infty(x, y)$	distance wrt. maximum norm; $\operatorname{dist}_\infty(x, y) \coloneqq \max_{1\le i\le d} \lvert x_i - y_i\rvert$
$\mathbb{E}$	expected value wrt. the probability space $(\Omega, \mathfrak{A}, \mathbb{P})$
$E^{q,p}(D)$	$\{\mathbf{u} \in L^q(D) : \operatorname{div}\mathbf{u} \in L^p(D)\}$ with $\lVert\mathbf{u}\rVert_{E^{q,p}} \coloneqq \lVert\mathbf{u}\rVert_{L^q} + \lVert\operatorname{div}\mathbf{u}\rVert_{L^p}$
$E^{q,p}_0(D)$	$\overline{C^\infty_c(D)}^{\lVert\cdot\rVert_{E^{q,p}}}$
$\tilde{f}$	extension by 0 of a function f defined on a domain $D \subset \mathbb{R}^d$
$\mathbb{I}$	identity matrix/identity mapping
$L^q_0(D)$	set of functions $f \in L^q(D)$ with $\int_D f = 0$
$\mathcal{M}^+(D)$	set of non-negative Radon measures on $D \subset \mathbb{R}^d$
$\mathbb{N}$	natural numbers starting with 0
ϱ	density of the fluid
$\mathbb{S}$	Newtonian viscous stress tensor
$SO(d)$	$\{Q \in \mathbb{R}^{d\times d} : Q^TQ = \mathbb{I},\ \det Q = 1\}$
ϑ	temperature of the fluid
$\mathbf{u}$	velocity of the fluid
$\big[\dot{W}^{1,p'}(D)\big]'$	$\{g \in \big[W^{1,p'}(D)\big]' : \langle g, 1\rangle = 0\}$

The index ε indicates the dependence of functions and sets on the perforation. Further, we use the standard notation for Lebesgue and Sobolev spaces, and denote them even for vector- or matrix-valued functions as in the scalar case, e.g., $L^q(D)$ instead of $L^q(D; \mathbb{R}^d)$. Bold letters indicate vector-valued functions. Finally, we will use the symbol $a \lesssim b$ whenever there is a generic constant $C > 0$ such that $a \le C\,b$. The specific value of C may change from line to line.

Chapter 1

Introduction

The studies of mathematical fluid mechanics have a long history. Here, we will give just a brief overview without claim of completeness. Already in ancient times, Greek mathematicians discovered fundamental principles of hydrodynamics. An example is Archimedes' principle, stating that a body, partially or completely immersed in a fluid, experiences a buoyancy force equal to the weight of the displaced fluid. Later on in the 17th century, R. Boyle and E. Mariotte investigated the behavior of gases under constant temperature, realizing that the product of pressure and volume of a gas is constant. This was later extended to the ideal gas law, stating that the product of pressure and volume of a gas is always a multiple of the temperature, provided the change of volume is slow enough such that the temperature and pressure can adjust. For very quick changes, no heat will be produced, giving rise to adiabatic processes as considered by P. Laplace, S. Poisson, and N. Carnot, among others. I. Newton gave a precise description of the notion of viscosity, adopted by L. Euler and J. d'Alembert to formulate mathematical equations of fluid motions. These equations were modified in the 18th and 19th century by C. Navier and G. Stokes, respectively, yielding the famous Navier-Stokes equations considered in this thesis. If we take additionally into account that fluids may conduct heat, we arrive at the Navier-Stokes-Fourier equations, also called the full system. They describe, in general form, the motion of viscous, heat-conducting, compressible fluids, and are still subject of intensive mathematical research.

Besides the theory of flows of homogeneous fluids, another question asked is what might happen, if we put small inclusions (also called holes or obstacles) into the fluid. This question traces back to a part of Einstein's PhD thesis [Ein06] from 1906, where he derived an effective viscosity for such a suspension, provided the inclusions take up little volume in a certain sense. The process in which such small-scale heterogeneous equations can be well-approximated by homogeneous ones is called *homogenization*. In our setting, given a domain in space which contains many small obstacles, one may ask for an equation that approximates the actual flow, but somewhat "disregards" about the suspension.

For Stokes equations as a simplification of the whole Navier-Stokes equations, the first results in homogenization were obtained by L. Tartar in [Tar80]. He considered an *incompressible* fluid, moving in a domain perforated by periodically arranged holes, the size of which is proportional to their mutual distance. Letting this distance become smaller, he derived in the limit a variant of the well-known porous medium equation, which nowadays is known as *Darcy's* *law*. Empirically, this was already obtained in 1856 by H. Darcy when studying filter beds in fountains (see [Dar56] and [Mus37]). A modification of Darcy's law was given by H. Brinkman in 1949, taking into account "the damping force caused by the porous mass" (see [Bri49]). He proposed an additional friction term in the equations, nowadays known as *Brinkman's law*. A similar but rather surprising result was obtained by Cioranescu and Murat in [CM82], where they considered the Poisson equation $\Delta u = f$ in a spatial domain D, periodically perforated by obstacles the size of which is inverse proportional to their number inside D. In particular, the volume fraction of the inclusions will vanish as their number grows. At first glance, one might therefore expect that the holes do not hinder the flow in the limit. However, it turns out that an additional friction term Mu in the limiting equation $\Delta u + Mu = f$ occurs, which is purely reminiscent from the obstacles. This "strange term coming from nowhere" can be seen as the first rigorous derivation of Brinkman's law.

Based on these results, G. Allaire considered in his PhD thesis [All90] the Stokes equations in a periodically perforated domain $D_\varepsilon \subset \mathbb{R}^d$, $d \geq 3$, where the perforations have mutual distance $\varepsilon > 0$ to each other, and are of size ε^α for some $\alpha \geq 1$. He discovered that there are three regimes of particle sizes, each of them yielding another limiting system. The *subcritical* case is $\alpha > d/(d-2)$, for which the holes are too tiny to significantly hinder the flow, leading within the limit $\varepsilon \to 0$ to the same Stokes equations in D. This is, in three dimensions mostly considered in this thesis, precisely the case when $\alpha > 3$. We will refer to them as *tiny holes*. The *supercritical* case is $\alpha < 3$, which we refer to as *large holes*. Here, the holes are large enough to stop the flow as $\varepsilon \to 0$. Rescaling the velocity by a proper factor, he arrived at a rigorous verification of Darcy's law for all $1 \leq \alpha < 3$. The *critical* case $\alpha = 3$ is precisely when the holes are still too small to stop the flow, but large enough to put friction on it. In accordance with the results obtained by Cioranescu and Murat, Allaire also obtained the additional Brinkman term. Up to this point, the fluid is still assumed to be incompressible, meaning that the density is constant.

The works on *compressible* flows, however, were to this point rather sparse. In his seminal work in 2002 [Mas02], N. Masmoudi gave a homogenization result for compressible fluids inside a perforated domain, assuming that the size of the inclusions is proportional to their mutual distance, or, in the notation above, $\alpha = 1$. The limiting equations are a density dependent analogue to Darcy's law. In the proof he extensively used a right inverse to the divergence operator to bound the density independently of the perforations. This inverse, nowadays known as *Bogovskiĭ's operator*, was known to exist for fixed domains D, acting as an operator $\mathcal{B} : L_0^2(D) \to W_0^{1,2}(D)$, where $L_0^2(D)$ is the space of all mean-free functions $f \in L^2(D)$. However, the crucial point is to explore its dependence on the perforation (that is, on ε) explicitly. In Masmoudi's work, it turns out that the operator norm of $\mathcal{B}$ cannot be bounded uniformly in ε, which corresponds to a kind of bottleneck effect for the flow through the perforated domain. Heuristically, for fixed f, searching a solution $\mathbf{u}$ to the divergence equation $\operatorname{div} \mathbf{u} = f$ with zero boundary data on ∂D is equivalent to ask for a flow with given sources and sinks that "sticks" to the boundary. These sources and sinks will move mass from one point in space to another. Hence, for the case of large holes, the mass has to be transported along these perforations, which become denser and denser in the limit, while the flow still has to be zero on the boundary. Since f is fixed, the velocity gradient $\nabla \mathbf{u}$ of the flow has to become larger and larger in order to pass through the tunnels between obstacles, leading to the unboundedness of its norm. The same effect should be present for any kind of large holes, meaning for any $\alpha < 3$.

For tiny holes with $\alpha > 3$, one should expect that the inverse to the divergence is bounded. E. Feireisl and Y. Lu considered in [FL15] the case of periodically arranged tiny holes, where Bogovskiĭ's operator is indeed bounded. They improved this together with L. Diening in [DFL17] to the case of well-separated obstacles for $\mathcal{B}$ mapping $L_0^q(D)$ to $W_0^{1,q}(D)$ for $1 < q < \infty$, giving an explicit dependence of the bounds on ε for any $\alpha \geq 1$. Using this inverse divergence, they bound the density and, accordingly, the velocity independent of the perforations, obtaining in the limit again the same Navier-Stokes equations. Darcy's law for large holes $\alpha < 3$ was recently rigorously derived by R. Höfer, K. Kowalczyk, and S. Schwarzacher in [HKS21]. As a matter of fact, the critical case $\alpha = 3$ for compressible flows is still mainly open. To shorten the exposition, we refer to more results in the following chapters.

As mentioned before, the available results for compressible fluids required that the perforations are in a certain sense well-separated. In this thesis, we leave this assumption in the direction of *stochastic* perforations, meaning that the holes are distributed according to a stochastic process and may be very close to each other. We will see that we are still be able to construct an inverse operator to the divergence, give explicit bounds on its norm, and apply it to homogenize the Navier-Stokes as well as Navier-Stokes-Fourier equations for compressible fluids in randomly perforated domains.

Organization. This thesis consists of several parts. We start to describe the flow of compressible fluids by mathematical equations in Chapter 2, which we derive from several physical principles. In Chapter 3, we show how to construct an inverse to the divergence in different domains, starting with star-shaped domains and ending in randomly perforated ones. Chapter 4 is devoted to the homogenization of different types of equations. More precisely, we start in Section 4.1 with the homogenization of stationary Navier-Stokes equations in domains that are randomly perforated by *tiny holes*, and assume a certain growth rate for the fluids pressure. In Section 4.2 we will relax the growth condition and explain how to treat time dependent equations. Heat-conducting fluids will be considered in Section 4.3. In all the aforementioned, the limiting equations are the same as in the perforated domain. Section 4.4, however, deals with the *critical* case $\alpha = 3$ in periodically perforated domains. We will show that, under an additional scaling assumption on the pressure, the limiting equations are of Brinkman type, thus providing a first step towards the homogenization of compressible Navier-Stokes equations in the critical regime. Finally, in Chapter 5, we give an outlook on possible future work and open problems.

Chapter 2

Derivation of the Navier-Stokes equations

The Navier-Stokes as well as the Navier-Stokes-Fourier equations are derived from several physical conservation laws such as conservation of momentum and energy, and Newton's laws of motion. Together with the fundamental laws of thermodynamics, in the first section, we give a short derivation of the equations considered in this thesis. The second part of this chapter is devoted to dimensional analysis of the Navier-Stokes-Fourier equations, which will be apparent in further discussion.

2.1 Fundamental assumptions

We assume that we are given a domain $D \subset \mathbb{R}^3$ and a map $S : [0, T] \times D \to \mathbb{R}^3$ for some time $T \in (0, \infty)$, called *motion*, such that

1. $S(t, \cdot)$ is a C^1-diffeomorphism from D to $D_t \coloneqq S(t, D)$.

2. The gradient $\nabla_x S(t, x)$ satisfies $\det \nabla_x S(t, x) > 0$.

3. $S(0, \cdot) = \mathbb{I}$.

A motion transforms a particle p, sitting at time $t = 0$ on the position $x \in D$, to a particle which sits at time $t > 0$ on position $y = S(t, x) \in D_t$ (see [Bar17]). Thus, if $f(t, p)$ is a physical quantity according to the particle p, we can view it in two different frames:

1. The Lagrangian frame defines the observable on D instead of D_t. We thus have $f_L(t, x) = f(t, S(t, x))$, where we use the subscript L to indicate the Lagrangian frame.

2. The Eulerian frame uses D_t as the domain of definition. Thus, using the subscript E to indicate the Eulerian system, $f_E(t, y) = f(t, y)$.

In particular, the time derivatives of f in the different frames are connected through

$$\begin{aligned} \partial_t f_L(t, x) &= \partial_t [f_E(t, S(t, x))] = (\partial_t f_E)(t, S(t, x)) + \nabla_y f_E(t, S(t, x)) \cdot \partial_t S(t, x) \\ &= \partial_t f_E(t, y) + \mathbf{u}(t, y) \cdot \nabla_y f_E(t, y), \end{aligned}$$

where we used the notation

$$\mathbf{u}(t,y) = \partial_t S(t,x), \quad x = S^{-1}(t,y), \quad y \in D_t,$$

for the velocity of the particle. Since this thesis will deal with fluids, which are modeled by the assumption that they are continuous rather than consisting of many discrete particles, we may refer to $\mathbf{u}$ as the *fluid velocity*. We further assume that we can characterize the fluid by the following quantities:

- A non-negative measurable function $\varrho(t,y)$, defined on $\{(t,y) : t \in [0,T], y \in D_t\}$, called the *mass density*;

- A positive measurable function $\vartheta(t,y)$, defined on $\{(t,y) : t \in [0,T], y \in D_t\}$, called the *fluid temperature*;

- The *pressure* $p = p(\varrho,\vartheta)$, the *specific internal energy* $e = e(\varrho,\vartheta)$, and the *specific entropy* $s = s(\varrho,\vartheta)$, also called the *thermodynamic functions*;

- A stress tensor $\mathbb{T} \in \mathbb{R}^{3\times 3}$, measuring the force per unit surface area that one part of the fluid imposes on another part directly opposite of their shared surface element;

- A vector field $\mathbf{q}(\vartheta)$ measuring the *flux of heat*;

- Force terms $\varrho\mathbf{f}$ and $\mathbf{g}$ which represent volumetric (meaning measured per unit volume) forces such as gravity, and other external forces, respectively.

Here, the functions $[\varrho, \mathbf{u}, \vartheta]$ represent the main state variables of the fluid, while all other quantities are recovered from them by constitutive relations. For instance, we assume that the heat flux is governed by Fourier's law

$$\mathbf{q}(\vartheta) = -\kappa(\vartheta)\nabla\vartheta,$$

where the *thermal conductivity* κ is a function of the absolute temperature ϑ. Physically, heat only flows from warm to cold regions, leading to the assumption $\kappa > 0$.

To derive the equations governing the fluid's motion, we need to take into account at least two important physical properties: the fundamental laws of thermodynamics have to be satisfied, and material laws have to be isotropic, that is, invariant under rotations. The laws of thermodynamics will be used in order to derive balance equations for the energy and entropy; we will come back to this later. For now, let us focus a bit more on the hypothesis of isotropy. We call a scalar function $f : \mathbb{R}^{3\times 3} \to \mathbb{R}$ *isotropic* if for any $Q \in SO(3)$ and any $A \in \mathbb{R}^{3\times 3}$, we have

$$f(Q^T A Q) = f(A).$$

Similarly, a tensor-valued function $F : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ is called *isotropic* if

$$F(Q^T A Q) = Q^T F(A) Q.$$

We will show that the stress tensor $\mathbb{T}$ satisfies Stokes' law

$$\mathbb{T} = \mathbb{S} - p\mathbb{I},$$

where $\mathbb{S}$ is the viscous stress tensor. The physical principle behind viscosity is associated to the relative motion of different fluid parts. Thus, $\mathbb{S}$ just depends linearly on $\nabla\mathbf{u}$, hence $\mathbb{S} = \mathcal{A}\nabla\mathbf{u}$ for some fourth order tensor $\mathcal{A} = \{\mathcal{A}_{ijkl}\}_{1\leq i,j,k,l\leq 3}$. Following [Ped14], it turns out that the most general isotropic fourth order tensor is of the form

$$\mathcal{A}_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}, \tag{2.1}$$

where δ_{ij} is the Kronecker delta, and $\alpha, \beta, \gamma \in \mathbb{R}$. Moreover, as a consequence of the conservation of angular momentum, the stress tensor $\mathbb{T}$ and hence $\mathbb{S}$ are symmetric, see [Bar17, Satz 3.4.2] for a proof. Consequently, we have $\mathcal{A}_{ijkl} = \mathcal{A}_{jikl}$, implying that $\beta = \gamma$ and that $\mathbb{S}$ just depends on the symmetric part of the gradient $E = \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u})$. To obtain the particular forms of $\mathbb{T}$ and $\mathbb{S}$, we give the following representation lemma on isotropic scalar and tensor-valued functions.

Lemma 2.1.1. *Let $f : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}$ be an isotropic scalar function, defined on the space of symmetric 3×3-matrices, and $F : \mathbb{R}^{3\times 3}_{\text{sym}} \to \mathbb{R}^{3\times 3}_{\text{sym}}$ be an isotropic tensor-valued and affine linear function. Then, f only depends on $J(A) := \{\operatorname{tr}(A), \det(A), \frac{1}{2}((\operatorname{tr}(A)^2 - \operatorname{tr}(A^2))\}$, and F admits the form*

$$F(A) = 2\mu A + (\kappa + \lambda\operatorname{tr}(A))\mathbb{I}$$

for some $\mu, \kappa, \lambda \in \mathbb{R}$. Here, we denote by $\operatorname{tr}(A)$ the trace of the matrix A.

Proof. We start with f. Obviously, we have $J(Q^TAQ) = J(A)$ for any $Q \in SO(3)$, so we have to show that $f(A) = f(B)$ whenever $J(A) = J(B)$. Since $J(A)$ contains the coefficients of the characteristic polynomial $p_A(x) = \det(A - x\mathbb{I})$, we conclude from $J(A) = J(B)$ that A and B have the same eigenvalues. Thus, there exist $Q \in SO(3)$ with

$$Q^TAQ = B,$$

so $f(A) = f(Q^TBQ) = f(B)$ since f is isotropic.
Let us now turn to F. Since F is affine linear and $F(A)^T = F(A)$ for any $A \in \mathbb{R}^{3\times 3}_{\text{sym}}$, we obtain from (2.1) that there exist $\mu, \alpha \in \mathbb{R}$ and $B \in \mathbb{R}^{3\times 3}$ such that

$$F(A) = 2\mu A + \alpha\operatorname{tr}(A)\mathbb{I} + B.$$

From the isotropy condition, we obtain for any $Q \in SO(3)$

$$2\mu Q^TAQ + \alpha\operatorname{tr}(Q^TAQ)\mathbb{I} + B = F(Q^TAQ) = Q^TF(A)Q = 2\mu Q^TAQ + \alpha\operatorname{tr}(A)\mathbb{I} + Q^TBQ.$$

Since $\operatorname{tr}(Q^TAQ) = \operatorname{tr}(A)$ for any $Q \in SO(3)$, we conclude $Q^TBQ = B$ and thus $B = f\mathbb{I}$ for some isotropic scalar function $f = f(A)$. Since f does just depend on $\operatorname{tr}(A)$, $\det(A)$, and $(\operatorname{tr}(A))^2 - \operatorname{tr}(A^2)$, and F is affine linear in A, in particular f has to be, we obtain

$$f(A) = \kappa + (\lambda - \alpha)\operatorname{tr}(A)$$

for some constants $\kappa, \lambda \in \mathbb{R}$, yielding finally

$$F(A) = 2\mu A + \alpha\,\mathrm{tr}(A)\mathbb{I} + (\kappa + (\lambda - \alpha)\,\mathrm{tr}(A))\mathbb{I} = 2\mu A + (\kappa + \lambda\,\mathrm{tr}(A))\mathbb{I}.$$

□

Since $\mathbb{S}$ is a linear function of E, we obtain *Newton's rheological law*

$$\begin{aligned}\mathbb{S}(\nabla\mathbf{u}) = \mathbb{S}(E) &= 2\mu E - \frac{2\mu}{3}\mathrm{tr}(E)\mathbb{I} + \eta\,\mathrm{tr}(E)\mathbb{I} \\ &= \mu\Big(\nabla\mathbf{u} + \nabla^T\mathbf{u} - \frac{2}{3}\,\mathrm{div}(\mathbf{u})\mathbb{I}\Big) + \eta\,\mathrm{div}(\mathbf{u})\mathbb{I},\end{aligned}$$

where μ and η are scalar functions of the temperature ϑ. The particular splitting of $\mathbb{S}$ in the μ-term and η-term is also physically motivated. Note that

$$\mathrm{tr}\Big(\nabla\mathbf{u} + \nabla^T\mathbf{u} - \frac{2}{3}\,\mathrm{div}(\mathbf{u})\mathbb{I}\Big) = 2\,\mathrm{tr}(\nabla\mathbf{u}) - 2\,\mathrm{div}(\mathbf{u}) = 0,$$

so the first term is the traceless part of the viscosity tensor, thus representing shear stresses only. We may thus refer to μ as the *shear viscosity coefficient*. The second part of $\mathbb{S}$ expresses pure stretching and compression, so we may refer to η as the *bulk viscosity coefficient*. We will later see that we shall assume $\mu > 0$ and $\eta \geq 0$ for physical reasons.

2.2 Balance laws

Balance laws are important to describe basic physical principles. In this thesis, we will always work with the Eulerian frame and assume that the domain D does not change its overall shape in time, meaning $D = D_t$ for all $t \in [0, T]$. We will therefore drop the dependence on t and just write D instead of D_t. In this section, we will further disregard regularity questions and assume that all considered functions are sufficiently smooth.

Mass conservation: We start with the assumption that the fluid's mass is conserved in time for any particular subdomain of D. For $B \subset D$, define the mass inside B by

$$\mathfrak{m}(t, B) := \int_B \varrho(t, x)\,\mathrm{d}x.$$

Together with Reynolds' transport theorem B.3, the conservation of mass leads to

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{m}(t, B) = \int_B \partial_t\varrho + \mathrm{div}(\varrho\mathbf{u})\,\mathrm{d}x.$$

Since we assumed that ϱ and $\mathbf{u}$ are smooth and $B \subset D$ was arbitrary, we derive the *continuity equation*

$$\partial_t\varrho + \mathrm{div}(\varrho\mathbf{u}) = 0 \quad \text{in } D. \tag{2.2}$$

Multiplying this equation with $b'(\varrho)$, where $b : [0, \infty) \to \mathbb{R}$ is a bounded differentiable function, we formally arrive at the *renormalized continuity equation*

$$\begin{aligned} &\partial_t(b(\varrho)) + \operatorname{div}(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho)) \operatorname{div}(\mathbf{u}) \\ &= b'(\varrho)\partial_t\varrho + b'(\varrho)\nabla\varrho \cdot \mathbf{u} + b(\varrho) \operatorname{div}(\mathbf{u}) + \varrho b'(\varrho) \operatorname{div}(\mathbf{u}) - b(\varrho) \operatorname{div}(\mathbf{u}) \\ &= b'(\varrho)(\partial_t\varrho + \nabla\varrho \cdot \mathbf{u} + \varrho \operatorname{div}(\mathbf{u})) \\ &= b'(\varrho)(\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u})) = 0. \end{aligned} \tag{2.3}$$

We remark that this renormalized version of mass conservation "hides" the derivatives $\partial_t\varrho$ and $\nabla\varrho$ in the terms $\partial_t b(\varrho)$ and $\operatorname{div}(b(\varrho)\mathbf{u})$, respectively. Therefore, the form written in the very first line of (2.3) is a good preparation for the notion of a weak formulation of the renormalized equation. It will be crucial to obtain a strong convergence of a sequence of densities $\{\varrho_\varepsilon\}_{\varepsilon>0}$, defined on a perforated domain D_ε, when $\varepsilon \to 0$. We will come back to this in further sections.

Momentum balance: Let us turn to the conservation of linear momentum. Assume that

$$\mathbf{f}, \mathbf{g} : [0, T] \times D \to \mathbb{R}^3,$$

where $\mathbf{f}$ is a volumetric (meaning measured per unit volume) force and $\mathbf{g}$ an additional outer force, then the flux of the momentum $\varrho\mathbf{u}$ is governed by the stress tensor $\mathbb{T} = \mathbb{S} - p\mathbb{I}$, where we abbreviate $\mathbb{S} = \mathbb{S}(\nabla\mathbf{u})$. According to Newton's second law of motion "force equals mass times acceleration", this yields for any $B \subset D$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_B \varrho\mathbf{u} \,\mathrm{d}x = \int_B \varrho\mathbf{f} + \mathbf{g} \,\mathrm{d}x + \int_{\partial B} (\mathbb{S} - p\mathbb{I}) \cdot \mathbf{n} \,\mathrm{d}\sigma,$$

where $\mathbf{n}$ denotes the outward unit normal on ∂B. Using Reynolds' Theorem B.3 and Gauß' divergence theorem, we arrive at

$$\int_B \partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u}) \,\mathrm{d}x = \int_B \varrho\mathbf{f} + \mathbf{g} + \operatorname{div}\mathbb{S} - \nabla p \,\mathrm{d}x,$$

which in differential form gives rise to the *balance of linear momentum*

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\mathbf{u} \otimes \mathbf{u}) = \varrho\mathbf{f} + \mathbf{g} + \operatorname{div}\mathbb{S} - \nabla p \quad \text{in } D. \tag{2.4}$$

Energy balance: A fundamental postulate in physics is that energy can never be created or destroyed, meaning that energy is conserved and can just be transformed into other forms. We thus may define the total energy density as

$$E := \frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta),$$

where $e(\varrho, \vartheta)$ is the specific internal energy density. According to thermodynamic and mechanical principles (see also [NS04, Section 1.2.7]), the rate of change of the total energy is given by the sum of the powers of volume and outer forces, surface forces according to stresses, and heat exchange. We therefore get for any volume $B \subset D$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_B \varrho E \,\mathrm{d}x = \int_B (\varrho\mathbf{f}+\mathbf{g})\cdot\mathbf{u}\,\mathrm{d}x + \int_{\partial B} [(\mathbb{S}-p\mathbb{I})\mathbf{u}]\cdot\mathbf{n}\,\mathrm{d}\sigma + \int_B \varrho r\,\mathrm{d}x - \int_{\partial B}\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}\sigma,$$

where r is the density of internal heat sources or sinks, and $\mathbf{q}$ is the heat flux through the boundary. Note that, according to the physical principle that heat travels from warm to cold only, the sign of $\mathbf{q}$ is chosen such that a "hot" body B shall give off its heat to the "colder" outside, thus losing energy when time goes on. Applying once again Theorem B.3, we obtain

$$\int_B \partial_t(\varrho E) + \mathrm{div}(\varrho E\mathbf{u})\,\mathrm{d}x = \int_B (\varrho\mathbf{f}+\mathbf{g})\cdot\mathbf{u} + \mathrm{div}(\mathbb{S}\mathbf{u}-p\mathbf{u}) + \varrho r - \mathrm{div}(\mathbf{q})\,\mathrm{d}x,$$

finally yielding the *balance of total energy*

$$\partial_t(\varrho E) + \mathrm{div}(\varrho E\mathbf{u}) = \mathrm{div}(\mathbb{S}\mathbf{u}-p\mathbf{u}-\mathbf{q}) + (\varrho\mathbf{f}+\mathbf{g})\cdot\mathbf{u} + \varrho r \quad \text{in } D. \tag{2.5}$$

Moreover, we get from the balance of momentum (2.4) by multiplying with $\mathbf{u}$ and using the continuity equation (2.2) the *balance of kinetic energy*

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) + \mathrm{div}\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}\right) = (\varrho\mathbf{f}+\mathbf{g})\cdot\mathbf{u} + \mathrm{div}(\mathbb{S}\mathbf{u}-p\mathbf{u}) - \mathbb{S}:\nabla\mathbf{u} + p\,\mathrm{div}(\mathbf{u}) \tag{2.6}$$

and thus, subtracting (2.5) and (2.6), the *balance of internal energy*

$$\partial_t(\varrho e) + \mathrm{div}(\varrho e\mathbf{u}) = \mathbb{S}:\nabla\mathbf{u} - p\,\mathrm{div}(\mathbf{u}) + \varrho r - \mathrm{div}(\mathbf{q}) \quad \text{in } D. \tag{2.7}$$

Entropy balance: Finally, we turn to the balance of entropy. Firstly, due to the first law of thermodynamics, the change of the internal energy is given by (infinitesimal) variations of heat plus the work done by the system to its environment, meaning

$$De = \delta Q - \delta W,$$

where we stick to the sign convention commonly used in physical literature. The work done by the system is given by the change of volume due to pressure, meaning $\delta W = pD\big(\frac{1}{\varrho}\big)$. By the second law of thermodynamics, the change of heat is given by the change of the entropy due to temperature, that is, $\delta Q = \vartheta Ds$, leading to *Gibb's equation*

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta)D\left(\frac{1}{\varrho}\right), \tag{2.8}$$

which in turn forces the expression $\frac{1}{\vartheta}\big(De + pD\big(\frac{1}{\varrho}\big)\big)$ to be a perfect gradient, namely $Ds = \frac{1}{\vartheta}\big(De + pD\big(\frac{1}{\varrho}\big)\big)$. This gives rise to

$$\begin{aligned} \partial_t s &= \frac{1}{\vartheta}\left(\partial_t e + p\partial_t\left(\frac{1}{\varrho}\right)\right) = \frac{1}{\vartheta}\left(\partial_t e - \frac{p}{\varrho^2}\partial_t\varrho\right), \\ \nabla s &= \frac{1}{\vartheta}\left(\nabla e - \frac{p}{\varrho^2}\nabla\varrho\right). \end{aligned} \tag{2.9}$$

Dividing (2.7) by ϑ and using the continuity equation (2.2), we obtain

$$\frac{1}{\vartheta}(\varrho\partial_t e + \varrho\mathbf{u}\cdot\nabla e) = \frac{1}{\vartheta}\left(\mathbb{S}:\nabla\mathbf{u} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta}\right) + \frac{\varrho}{\vartheta}r - \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) - \frac{p}{\vartheta}\operatorname{div}(\mathbf{u}).$$

Together with (2.9) and (2.2), we get for the left-hand side

$$\begin{aligned}\frac{1}{\vartheta}(\varrho\partial_t e + \varrho\mathbf{u}\cdot\nabla e) &= \varrho\partial_t s + \varrho\mathbf{u}\cdot\nabla s + \frac{1}{\vartheta}\left(\frac{p}{\varrho}\partial_t\varrho + \frac{p}{\varrho}\mathbf{u}\cdot\nabla\varrho\right)\\ &= \partial_t(\varrho s) + \operatorname{div}(\varrho s\mathbf{u}) + \frac{p}{\vartheta\varrho}(-\varrho\operatorname{div}(\mathbf{u}))\\ &= \partial_t(\varrho s) + \operatorname{div}(\varrho s\mathbf{u}) - \frac{p}{\vartheta}\operatorname{div}(\mathbf{u}),\end{aligned}$$

so we finally obtain the *balance of entropy*

$$\partial_t(\varrho s) + \operatorname{div}(\varrho s\mathbf{u}) = \sigma + \frac{\varrho}{\vartheta}r - \operatorname{div}\left(\frac{\mathbf{q}}{\vartheta}\right) \quad \text{in } D, \tag{2.10}$$

where the *entropy production rate* is given by

$$\sigma := \frac{1}{\vartheta}\left(\mathbb{S}:\nabla\mathbf{u} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta}\right).$$

Physically, entropy can be seen as a measure of disorder inside the system. Additionally, the second law of thermodynamics also states that any physically admissible process can just produce entropy, leading to the assumption that σ is non-negative. Since we will mostly deal with the framework of weak solutions in this thesis, which shall dissipate more kinetic energy than expected from equation (2.4) due to possible concentrations and singularities, we will assume that σ is a non-negative Radon measure satisfying the *entropy inequality*

$$\sigma \geq \frac{1}{\vartheta}\left(\mathbb{S}:\nabla\mathbf{u} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta}\right). \tag{2.11}$$

We will come back to this when dealing with the full Navier-Stokes-Fourier system in Section 4.3.

Ideal fluids and adiabatic pressure law: In subsequent chapters for the case of constant temperature, we will assume that the pressure satisfies the *adiabatic pressure law*

$$p = p(\varrho) = a\varrho^\gamma \tag{2.12}$$

for some constant $a > 0$ and some $\gamma \geq 1$. This dependence is sometimes also known as the *isentropic* or *barotropic pressure law*. It can be derived from Gibb's equation (2.8), together with some assumptions on ideal gases. Let us summarize these assumptions as follows.

1. The ideal gas law $p = R\varrho\vartheta$ holds, where $R > 0$ is the *universal gas constant* (measured here in units of molar mass).

2. The only change of energy is due to heat, that is, $\partial_\varrho e = 0$ and $\partial_\vartheta e = c_v$, where $c_v > 0$ is the *specific heat capacity at constant volume*. This leads to $e = c_v\vartheta$.

3. We are given positive reference values $p_0, \varrho_0, \vartheta_0 > 0$.
4. The *adiabatic exponent* is given by $\gamma \coloneqq c_p/c_v$, where $c_p \coloneqq c_v + R$ is the *specific heat capacity at constant pressure.*

The number γ is also called *isentropic exponent* or, according to its definition, *heat capacity ratio.* From the ideal gas law, we derive

$$\frac{p}{p_0} = \frac{\varrho\vartheta}{\varrho_0\vartheta_0}, \quad \text{or, equivalently,} \quad \frac{\vartheta}{\vartheta_0} = \frac{p/p_0}{\varrho/\varrho_0}.$$

Thus, Gibb's relation leads to

$$\begin{aligned}
Ds &= \frac{1}{\vartheta} De - \frac{p}{\vartheta\varrho^2} D\varrho = \frac{c_v}{\vartheta} D\vartheta - \frac{R}{\varrho} D\varrho \\
&= c_v D \log \frac{\vartheta}{\vartheta_0} - RD \log \frac{\varrho}{\varrho_0} \\
&= c_v D \log \frac{\vartheta}{\vartheta_0} - c_v(\gamma - 1) D \log \frac{\varrho}{\varrho_0} \\
&= c_v D \log \frac{\vartheta/\vartheta_0}{(\varrho/\varrho_0)^{\gamma-1}} \\
&= c_v D \log \frac{p/p_0}{(\varrho/\varrho_0)^{\gamma}},
\end{aligned}$$

yielding the specific entropy for an ideal gas being of the form

$$s(\varrho, \vartheta) = c_v \log\left(\frac{\vartheta}{\varrho^{\gamma-1}}\right) = c_v \log\left(\frac{p}{\varrho^{\gamma}}\right). \tag{2.13}$$

Assuming now that the thermodynamic process is adiabatic, meaning no heat is exchanged, so $\delta Q = 0$ and, in view of $\delta Q = \vartheta Ds$, also $Ds = 0$, we see that (2.13) furnishes (2.12).

Assumptions on coefficients: Together with the entropy inequality and the form of the viscous stress tensor

$$\mathbb{S}(\nabla\mathbf{u}) = \mu\left(\nabla\mathbf{u} + \nabla^T\mathbf{u} - \frac{2}{3}\operatorname{div}(\mathbf{u})\mathbb{I}\right) + \eta \operatorname{div}(\mathbf{u})\mathbb{I},$$

we can give more restrictions on the viscosity coefficients $\mu, \eta \in \mathbb{R}$. Since the entropy production rate must be non-negative, we obtain

$$\mathbb{S} : \nabla\mathbf{u} \geq 0.$$

Seeing that $\nabla\mathbf{u} : \nabla^T\mathbf{u} = |\operatorname{div}(\mathbf{u})|^2$, we may write

$$\begin{aligned}
\mathbb{S} : \nabla\mathbf{u} &= \mu|\nabla\mathbf{u}|^2 + \frac{\mu}{3}|\operatorname{div}(\mathbf{u})|^2 + \eta\,|\operatorname{div}(\mathbf{u})|^2 \\
&= \frac{\mu}{2}\left(|\nabla\mathbf{u}|^2 + 2\nabla\mathbf{u} : \nabla^T\mathbf{u} + |\nabla^T\mathbf{u}|^2 - \frac{4}{3}|\operatorname{div}(\mathbf{u})|^2\right) + \eta\,|\operatorname{div}(\mathbf{u})|^2 \\
&= \frac{\mu}{2}\left(|\nabla\mathbf{u} + \nabla^T\mathbf{u}|^2 - \frac{4}{3}\operatorname{div}(\mathbf{u})\mathbb{I} : (\nabla\mathbf{u} + \nabla^T\mathbf{u}) + \left|\frac{2}{3}\operatorname{div}(\mathbf{u})\mathbb{I}\right|^2\right) + \eta\,|\operatorname{div}(\mathbf{u})|^2
\end{aligned}$$

$$= \frac{\mu}{2}\left|\nabla\mathbf{u} + \nabla^T\mathbf{u} - \frac{2}{3}\operatorname{div}(\mathbf{u})\mathbb{I}\right|^2 + \eta\,|\operatorname{div}(\mathbf{u})|^2$$

to conclude that $\mu, \eta \geq 0$. As we shall work in this thesis with viscous fluids, we require for μ the stronger condition $\mu > 0$.
The same notion gives rise to

$$-\mathbf{q}\cdot\nabla\vartheta \geq 0,$$

which, together with Fourier's law $\mathbf{q}(\vartheta) = -\kappa(\vartheta)\nabla\vartheta$, yields $\kappa \geq 0$ in accordance with the physical principle of heat conduction already mentioned. In Section 4.3, we will assume that the heat flux does not vanish, meaning $\kappa > 0$.

For the physical intuition behind the adiabatic exponent γ, let us note that the thermodynamic principle of equipartition of energy states that the energy of a gas in thermal equilibrium is shared uniformly to all degrees of freedom f, that is,

$$\frac{e}{f} = \frac{1}{2}R\vartheta,$$

where the factor $\frac{1}{2}$ is related to the kinetic energy of the gas; we refer to [Cla57] and [Dem06, Kapitel 10] for the connection between kinetic and heat theory. This and $e = c_v\vartheta$ leads to $c_v = \frac{1}{2}Rf$. Together with $c_p = c_v + R = \frac{1}{2}R(f+2)$ and $\gamma = c_p/c_v$, we obtain

$$\gamma = \frac{\frac{1}{2}R(f+2)}{\frac{1}{2}Rf} = 1 + \frac{2}{f}.$$

For instance, a monoatomic gas has three degrees of freedom, one for each direction in space. Thus, we get $\gamma = \frac{5}{3}$ in this case. Larger molecules have more degrees of freedom, including vibrations and rotations around possible symmetry axes. Thus, the *physical* range for γ is

$$1 \leq \gamma \leq \frac{5}{3}. \tag{2.14}$$

However, due to mathematical reasons, we are not able to hit this range. Indeed, we will always assume that at least $\gamma > 2$, and comment this issue later on in Chapter 4.

2.3 Scaling considerations

In order to obtain reasonable predictions for small as well as large systems of fluids, it is convenient to non-dimensionalize the equations derived in the previous section. To this end, we will take into account the fundamental dimensions length, time, mass, and temperature. In particular, we assume that the system we are interested in has some characteristic values of length L_c, time T_c, density ϱ_c, velocity u_c, and temperature ϑ_c, where the other characteristic parameters $p_c, e_c, \kappa_c, \mu_c, \eta_c$ as well as the sources f_c, g_c, r_c are composed quantities of them. By writing $X' = X/X_c$ for any physical quantity X which may represent, for instance, time or velocity, we obtain for the continuity equation (2.2)

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = \frac{\varrho_c}{T_c} \partial_{t'} \varrho' + \frac{\varrho_c u_c}{L_c} \operatorname{div}_{x'}(\varrho' \mathbf{u}') = 0,$$

which, omitting primes for simplicity, we may write in the form

$$\mathrm{Sr}\, \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0$$

with the dimensionless Strouhal number $\mathrm{Sr} \coloneqq L_c/(T_c u_c)$. Further, due to Gibb's relation (2.8), we have the compatibility condition $p_c = \varrho_c e_c$. Additionally, the viscosity coefficients μ and η share the same physical units, so we may measure them in terms of the same characteristic viscosity, meaning $\eta_c = \mu_c$. The same argument leads to $g_c = \varrho_c f_c$. Thus, similar considerations as for the continuity equation lead for the momentum equation (2.4), the kinetic energy balance (2.6), the internal energy balance (2.7), and the entropy balance (2.10) to

$$\begin{aligned}
\mathrm{Sr}\, \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) &= \frac{1}{\mathrm{Fr}^2}(\varrho \mathbf{f} + \mathbf{g}) + \frac{1}{\mathrm{Re}} \operatorname{div} \mathbb{S} - \frac{1}{\mathrm{Ma}^2} \nabla p, \\
\mathrm{Sr}\, \partial_t(\varrho |\mathbf{u}|^2) + \operatorname{div}(\varrho |\mathbf{u}|^2 \mathbf{u}) &= \frac{1}{\mathrm{Fr}^2}(\varrho \mathbf{f} + \mathbf{g}) + \frac{1}{\mathrm{Re}}(\operatorname{div}(\mathbb{S}\mathbf{u}) - \mathbb{S} : \nabla \mathbf{u}) \\
&\quad + \frac{1}{\mathrm{Ma}^2}(p \operatorname{div}(\mathbf{u}) - \operatorname{div}(p\mathbf{u})), \\
\mathrm{Sr}\, \partial_t(\varrho e) + \operatorname{div}(\varrho e \mathbf{u}) &= \frac{\mathrm{Ma}^2}{\mathrm{Re}} \mathbb{S} : \nabla \mathbf{u} - p \operatorname{div} \mathbf{u} + \mathrm{Hr} \varrho r - \frac{1}{\mathrm{Pe}} \operatorname{div} \mathbf{q}, \\
\mathrm{Sr}\, \partial_t(\varrho s) + \operatorname{div}(\varrho s \mathbf{u}) &= \sigma + \mathrm{Hr} \frac{\varrho}{\vartheta} r - \frac{1}{\mathrm{Pe}} \operatorname{div} \frac{\mathbf{q}}{\vartheta},
\end{aligned}$$

where here

$$\begin{aligned}
\mathbb{S} &= \mu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u} - \frac{2}{3} \operatorname{div}(\mathbf{u}) \mathbb{I} \right) + \eta \operatorname{div}(\mathbf{u}) \mathbb{I}, \\
\mathbf{q} &= -\kappa(\vartheta) \nabla \vartheta, \\
\sigma &= \frac{1}{\vartheta} \left(\frac{\mathrm{Ma}^2}{\mathrm{Re}} \mathbb{S} : \nabla \mathbf{u} - \frac{1}{\mathrm{Pe}} \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta} \right),
\end{aligned}$$

and the occurring characteristic numbers are given in Table 2.1. We remark that these numbers are not unique. However, due to Buckingham's famous Π-theorem (see [CLP82]), one may determine how many of them are *independent*.

Symbol	Name	Definition
Sr	Strouhal number	$L_c/(T_c u_c)$
Fr	Froude number	$u_c/\sqrt{L_c f_c}$
Re	Reynolds number	$\varrho_c u_c L_c/\mu_c$
Ma	Mach number	$u_c/\sqrt{p_c/\varrho_c}$
Hr	Heat release parameter	$\varrho_c r_c L_c/(p_c u_c)$
Pe	Péclet number	$p_c u_c L_c/(\vartheta_c \kappa_c)$

Table 2.1: Characteristic numbers of fluid motions

The meaning of such similarity considerations becomes apparent in physical simulations. For instance, focusing on the Reynolds number, one may describe the behavior of an airplane wing in a wind tunnel by modeling the real wing with characteristic length L_c by a model wing of, say, length $L_c/2$. If the characteristic velocity of air for the real wing is u_c, then the solutions to the equations remain the same if the velocity in the wind tunnel is given by $2u_c$, which is due to the fact that the Reynolds number does not change.
Mathematically, the scaled equations are particularly interesting if one of the parameters tends towards zero or infinity, which gives rise to so-called singular limits. In Section 4.4, we will consider the case of a vanishing Mach number. Note that in the definition of the Mach number, the quantity $\sqrt{p_c/\varrho_c}$ has the physical unit of a velocity, therefore also called (local) speed of sound. A vanishing Mach number thus corresponds to a very slow flow (compared to speed of sound), which is expected to have constant density and thus being incompressible. We will see in Section 4.4 that this is actually the case.

Chapter 3

Bogovskiĭ's operator in different domains

In this chapter, let $D \subset \mathbb{R}^3$ be a bounded domain, $1 < q < \infty$, and $f \in L^q_0(D)$ be given, where we denote $L^q_0(D)$ as the space of all functions $g \in L^q(D)$ with $\int_D g = 0$. We will search for a solution $\mathbf{u} \in W^{1,q}_0(D)$ to the equation

$$\begin{cases} \operatorname{div} \mathbf{u} = f & \text{in } D, \\ \mathbf{u} = 0 & \text{on } \partial D, \end{cases} \tag{3.1}$$

such that $\mathbf{u}$ obeys a bound

$$\|\mathbf{u}\|_{W^{1,q}_0(D)} \leq C \, \|f\|_{L^q(D)},$$

where the constant $C > 0$ is independent of $\mathbf{u}$ and f. To find a solution $\mathbf{u}$ to equation (3.1) which depends linearly on f is equivalent to ask for a bounded linear operator

$$\mathcal{B} : L^q_0(D) \to W^{1,q}_0(D)$$

such that for any $f \in L^q_0(D)$,

$$\operatorname{div} \mathcal{B}(f) = f \quad \text{in } D, \quad \|\mathcal{B}(f)\|_{W^{1,q}_0(D)} \leq C \, \|f\|_{L^q(D)}. \tag{3.2}$$

If such an operator exists, then $\mathbf{u} = \mathcal{B}(f)$ is a solution to (3.1). We will give several results on the existence of such an operator $\mathcal{B}$, depending on the domain D. We start with domains that are star-shaped, where $\mathcal{B}$ can be defined by an explicit formula, following [Gal11] and the original work of M. Bogovskiĭ [Bog80], and continue with domains having a Lipschitz boundary. The next section will focus on the case of so-called John domains, where the existence of an operator $\mathcal{B}$ was shown in [DRS10]. The last two sections are devoted to the case of perforated domains D_ε, which will be defined later. For the existence of an operator $\mathcal{B}_\varepsilon$ in the perforated domain D_ε, the outcomes from [DRS10] will be crucial in order to get a bounded linear operator $\mathcal{B}_\varepsilon$, where we can give an explicit dependence of the constant C on ε. Under suitable assumptions on the perforations defining the domain D_ε, we will show that for some q we can construct a bounded linear map

$$\mathcal{B}_\varepsilon : L^q_0(D_\varepsilon) \to W^{1,q}_0(D_\varepsilon)$$

such that

$$\operatorname{div} \mathcal{B}_\varepsilon(f) = f \quad \text{in } D_\varepsilon, \quad \|\mathcal{B}_\varepsilon(f)\|_{W_0^{1,q}(D_\varepsilon)} \leq C\,\|f\|_{L^q(D_\varepsilon)},$$

where the constant $C > 0$ is now *independent* of ε. This operator will be extensively used in Chapter 4 to obtain uniform bounds on the density.
Since the work to be done is the same in $\mathbb{R}^3$ and $\mathbb{R}^d$ for $d \geq 2$, we will state the results in the sequel for the general case $D \subset \mathbb{R}^d$. We will refer to $\mathcal{B}$ as a right inverse to the divergence or, in dedication to M. Bogovskiĭ, as *Bogovskiĭ's operator*.

3.1 Star-shaped and Lipschitz domains

3.1.1 Bogovskiĭ's operator in star-shaped domains

Let us start with the definition of star-shaped domains. A domain $D \subset \mathbb{R}^d$ is said to be *star-shaped* with respect to some point $x_0 \in D$ (the *star center*) if for every point $x \in D$ the line $\{tx_0 + (1-t)x : t \in [0,1]\} \subset D$. In other words, the point x_0 "sees" all other points $x \in D$. For instance, convex domains are clearly star-shaped with respect to any of their interior points, but also the domain $\mathbb{R}^2 \setminus \{(x_1, 0) \in \mathbb{R}^2 : x_1 \leq 0\}$ is star-shaped with respect to the point $x_0 = (1,0)$. A domain is said to be star-shaped with respect to a ball $\overline{B} \subset D$, if for all $x_0 \in B$ it is star-shaped with respect to x_0. Following [Gal11, Lemma III.3.1] and [FN09, Lemma 10.6], we have the following existence result for the inverse of the divergence.

Lemma 3.1.1. *Let $D \subset \mathbb{R}^d$ be a bounded star-shaped domain with respect to a ball $\overline{B_R(y)} \subset D$, and let $1 < q < \infty$ and $f \in L_0^q(D)$. Then, there exists a solution $\mathbf{u}$ to problem (3.1), which depends linearly on f. The constant appearing in (3.2) admits the bound*

$$C \leq C_0(d,q)\,[\operatorname{diam}(D)/R]^d\,(1 + \operatorname{diam}(D)/R). \tag{3.3}$$

If additionally $f \in C_c^\infty(D)$, then $\mathbf{u} \in C_c^\infty(D)$.

Proof. First, we assume $f \in C_c^\infty(D)$. By the change of variables $x' = (x-y)/R$, we transform f into a function $f'(x') = f(x)$, and D into a domain D' which is star-shaped with respect to the ball $B = B_1(0)$ and satisfies

$$\operatorname{diam}(D') = \operatorname{diam}(D)/R.$$

The system (3.1) transforms into

$$\operatorname{div}_{x'}(\mathbf{u}') = Rf' =: F'.$$

Evidently, we have $F' \in C_c^\infty(D') \cap L_0^q(D')$. Hence, if $\mathbf{u}'$ satisfies $\operatorname{div}_{x'} \mathbf{u}' = F'$, then $\mathbf{u}(x) := \mathbf{u}'(x')$ yields a solution to (3.1). Thus, it is sufficient to show (3.1) for the case that D is star-shaped with respect to the ball $B = B_1(0)$. Note that this also implies $\operatorname{diam}(D) \geq 1$.
Let $\eta \in C_c^\infty(D)$ be such that $\operatorname{supp}\eta \subset B$, $\eta \geq 0$, and $\int_B \eta \,\mathrm{d}x = 1$. We now define $\mathbf{u} = \mathcal{B}(f)$ via the formula

$$\mathbf{u}(x) := \mathcal{B}(f)(x) := \int_D f(y) \left[\frac{x-y}{|x-y|^d} \int_{|x-y|}^\infty \eta\Big(y + \xi \frac{x-y}{|x-y|}\Big) \xi^{d-1} \,\mathrm{d}\xi \right] \mathrm{d}y. \tag{3.4}$$

Using the change of variables $r = \xi/|x-y|$ in (3.4), we recover at once the equivalent formula

$$\mathbf{u}(x) = \int_D f(y)(x-y) \int_1^\infty \eta(y + r(x-y)) r^{d-1} \, \mathrm{d}r \, \mathrm{d}y.$$

This form immediately yields that $\mathbf{u}$ has compact support in D. Indeed, set

$$E \coloneqq \{z \in D : z = tz_1 + (1-t)z_2,\ z_1 \in \operatorname{supp}(f),\ z_2 \in \overline{B},\ t \in [0,1]\}.$$

Since D is star-shaped with respect to every point of B and $\operatorname{supp}(f)$ is compact, we see that E is a compact subset of D. Now, let $x \in D \setminus E$. Then for all $y \in \operatorname{supp}(f)$ and any $r \geq 1$, we have

$$y + r(x-y) \notin \overline{B},$$

hence $\eta(y + r(x-y)) = 0$ and thus, by (3.4), $\mathbf{u}(x) = 0$, which shows that $\operatorname{supp}(\mathbf{u}) \subset E$. On the other hand, using the change of variables $r = \xi - |x-y|$ in (3.4) and the fact that we may extend f outside D to be zero, we obtain

$$\mathbf{u}(x) = \int_{\mathbb{R}^d} f(y) \frac{x-y}{|x-y|^d} \int_0^\infty \eta\left(x + r\frac{x-y}{|x-y|}\right)(r + |x-y|)^{d-1} \, \mathrm{d}r \, \mathrm{d}y.$$

By a further change of variables $z = x - y$, we obtain

$$\mathbf{u}(x) = \int_{\mathbb{R}^d} f(x-z) \frac{z}{|z|^d} \int_0^\infty \eta\left(x + r\frac{z}{|z|}\right)(r + |z|)^{d-1} \, \mathrm{d}r \, \mathrm{d}z.$$

Note that if $L > 0$ is large enough such that $\operatorname{supp}(f) \cup B \subset B_L(0)$, we get $\mathbf{u}(x) = 0$ on $\mathbb{R}^d \setminus B_L(0)$ and for $x \in B_L(0)$, the function $g(r,z) \coloneqq (4L)^{d-1}\chi_{(0,2L)}(r)\chi_{B_{2L}(0)}(z)|z|^{1-d}$ is integrable and dominates $z|z|^{-d}(r+|z|)^{d-1}$. Thus, we are allowed to change integration and differentiation up to any order, showing that $\mathbf{u} \in C_c^\infty(D)$.

Let us now show that $\mathbf{u}$ satisfies (3.1) and the required estimates. First, we rewrite (3.4) in the form

$$\mathbf{u}(x) = \int_D f(y) N(x,y) \, \mathrm{d}y,$$

where the kernel $N(x,y)$ is defined as

$$N(x,y) \coloneqq \frac{x-y}{|x-y|^d} \int_{|x-y|}^\infty \eta\left(y + \xi\frac{x-y}{|x-y|}\right)\xi^{d-1} \, \mathrm{d}\xi.$$

Observe that by definition of N we have

$$\begin{aligned}\frac{\partial N_i}{\partial y_j}(x,y) &= -\frac{\partial N_i}{\partial x_j}(x,y) + \frac{x_i - y_i}{|x-y|^d} \int_{|x-y|}^\infty (\partial_j \eta)\left(y + \xi\frac{x-y}{|x-y|}\right)\xi^{d-1} \, \mathrm{d}\xi \\ &\eqqcolon -\frac{\partial N_i}{\partial x_j}(x,y) + \tilde{N}_{ij}(x,y).\end{aligned}$$

Differentiating the i-th component $\mathbf{u}_i$ with respect to x_j thus yields

$$\partial_j \mathbf{u}_i(x) = \int_D f(y)\partial_{x_j} N_i(x,y)\,\mathrm{d}y = -\int_D f(y)\partial_{y_j} N_i(x,y)\,\mathrm{d}y + \int_D f(y)\tilde{N}_{ij}(x,y)\,\mathrm{d}y.$$

Now, let $\varepsilon > 0$ be sufficiently small such that $B_\varepsilon(x) \subset D$, and split the integral over D into the integral over $B_\varepsilon(x)$ and its remainder $D \setminus B_\varepsilon(x)$. Using integration by parts, we obtain

$$\begin{aligned}
\partial_j \mathbf{u}_i(x) &= \int_{B_\varepsilon(x)} \partial_{y_j} f(y) N_i(x,y)\,\mathrm{d}y - \int_{\partial B_\varepsilon(x)} f(y)N_i(x,y)\mathbf{n}_j\,\mathrm{d}\sigma(y) \\
&\quad - \int_{D\setminus B_\varepsilon(x)} f(y)\partial_{y_j} N_i(x,y)\,\mathrm{d}y + \int_{D\setminus B_\varepsilon(x)} f(y)\tilde{N}_{ij}(x,y)\,\mathrm{d}y + \int_{B_\varepsilon(x)} f(y)\tilde{N}_{ij}(x,y)\,\mathrm{d}y \\
&= \int_{D\setminus B_\varepsilon(x)} f(y)\partial_{x_j} N_i(x,y)\,\mathrm{d}y + \int_{\partial B_\varepsilon(x)} f(y)N_i(x,y)\frac{x_j - y_j}{|x-y|}\,\mathrm{d}\sigma(y) \\
&\quad + \int_{B_\varepsilon(x)} \partial_{y_j} f(y)N_i(x,y) + f(y)\tilde{N}_{ij}(x,y)\,\mathrm{d}y.
\end{aligned}$$

Since $f(y)$ and $\partial_{y_j} f(y)$ are bounded and the singularities of N and $\tilde{N}$ are weak singularities of order $(d-1)$, hence integrable over $B_\varepsilon(x)$, we see that the last term vanishes in the limit $\varepsilon \to 0$. Thus, we get

$$\begin{aligned}
\partial_j \mathbf{u}_i(x) &= \lim_{\varepsilon\to 0}\left(\int_{|x-y|\geq\varepsilon} f(y)\partial_j N_i(x,y)\,\mathrm{d}y + \int_{\partial B_\varepsilon(x)} f(y)\frac{x_j - y_j}{|x-y|}N_i(x,y)\,\mathrm{d}\sigma(y)\right) \\
&= p.v. \int_D f(y)\partial_j N_i(x,y)\,\mathrm{d}y + \lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)} f(y)\frac{x_j - y_j}{|x-y|}N_i(x,y)\,\mathrm{d}\sigma(y), \qquad (3.5)
\end{aligned}$$

where we used the prescript $p.v.$ to indicate that the first integral has to be understood in the Cauchy principal value sense. For the second term, we obtain

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon(x)} f(y)\frac{x_j - y_j}{|x-y|}N_i(x,y)\,\mathrm{d}\sigma(y) = f(x)\int_D \frac{(x_j-y_j)(x_i-y_i)}{|x-y|^2}\eta(y)\,\mathrm{d}y, \qquad (3.6)$$

since, by the below change of variables $z = (x-y)/\varepsilon$ and $r = \xi - \varepsilon|z|$ for the first integral, the fact that $\mathrm{supp}(\eta) \subset B$, the change $z = (x-y)/|x-y|$ and integration over surfaces in the second integral, we have

$$\begin{aligned}
\Delta_\varepsilon(x) &:= \left| \int_{\partial B_\varepsilon(x)} f(y)\frac{x_j - y_j}{|x-y|}N_i(x,y)\,\mathrm{d}\sigma(y) - f(x)\int_D \frac{(x_j-y_j)(x_i-y_i)}{|x-y|^2}\eta(y)\,\mathrm{d}y \right| \\
&= \left| \int_{\partial B_1(0)} z_i z_j f(x-\varepsilon z)\int_0^\infty \eta(x+rz)(r+\varepsilon)^{d-1}\,\mathrm{d}r\,\mathrm{d}\sigma(z) \right. \\
&\qquad \left. - f(x)\int_{\partial B_1(0)} z_i z_j \int_0^\infty \eta(x+rz)r^{d-1}\,\mathrm{d}r\,\mathrm{d}\sigma(z) \right|.
\end{aligned}$$

Thus, if ε is small enough, we obtain

$$\Delta_\varepsilon(x) \leq \int_{\partial B_1(0)} |f(x) - f(x-\varepsilon z)|\,\mathrm{d}\sigma(z) + o(1),$$

which yields $\lim_{\varepsilon\to 0}\Delta_\varepsilon(x) = 0$ and finally (3.6). To handle the remaining integral in (3.5), we apply Calderón-Zygmund theory. First, for fixed y, we find

$$\begin{aligned}
\partial_j N_i(x,y) &= \partial_j \left[(x_i - y_i) \int_1^\infty \eta(y + r(x-y)) r^{d-1} \,\mathrm{d}r \right] \\
&= \delta_{ij} \int_1^\infty \eta(y + r(x-y)) r^{d-1} \,\mathrm{d}r + (x_i - y_i) \int_1^\infty (\partial_j \eta)(y + r(x-y)) r^d \,\mathrm{d}r \\
&= \frac{\delta_{ij}}{|x-y|^d} \int_0^\infty \eta\Big(x + s\frac{x-y}{|x-y|} \Big) (s + |x-y|)^{d-1} \,\mathrm{d}s \\
&\quad + \frac{x_i - y_i}{|x-y|^{d+1}} \int_0^\infty (\partial_j \eta)\Big(x + s\frac{x-y}{|x-y|} \Big) (s + |x-y|)^{d} \,\mathrm{d}s.
\end{aligned}$$

We now expand the powers of d in the last two integrals to write $\partial_j N_i(x,y)$ in the form

$$\partial_j N_i(x,y) = K_{ij}(x, x-y) + G_{ij}(x, x-y),$$

where we define

$$\begin{aligned}
K_{ij}(x, x-y) &:= \frac{\delta_{ij}}{|x-y|^d} \int_0^\infty \eta\Big(x + s\frac{x-y}{|x-y|} \Big) s^{d-1} \,\mathrm{d}s \\
&\quad + \frac{x_i - y_i}{|x-y|^{d+1}} \int_0^\infty (\partial_j \eta)\Big(x + s\frac{x-y}{|x-y|} \Big) s^{d} \,\mathrm{d}s \\
&=: \frac{k_{ij}(x, x-y)}{|x-y|^d},
\end{aligned}$$

and $G_{ij}(x, x-y)$ is given by

$$\begin{aligned}
G_{ij}(x, x-y) &:= \sum_{k=1}^{d-1} \binom{d-1}{k} \frac{\delta_{ij}}{|x-y|^{d-k}} \int_0^\infty \eta\Big(x + s\frac{x-y}{|x-y|} \Big) s^{d-1-k} \,\mathrm{d}s \\
&\quad + \sum_{k=1}^{d} \binom{d}{k} \frac{x_i - y_i}{|x-y|^{d+1-k}} \int_0^\infty (\partial_j \eta)\Big(x + s\frac{x-y}{|x-y|} \Big) s^{d-k} \,\mathrm{d}s \\
&= \sum_{k=1}^{d-1} \binom{d-1}{k} \delta_{ij} \int_0^\infty \eta\big(x + r(x-y)\big) r^{d-1-k} \,\mathrm{d}r \\
&\quad + \sum_{k=1}^{d} \binom{d}{k} (x_i - y_i) \int_0^\infty (\partial_j \eta)\big(x + r(x-y)\big) r^{d-k} \,\mathrm{d}r.
\end{aligned}$$

By $\operatorname{supp}(\eta) \subset B$, we get for the upper bound of the integrals

$$1 \geq |x + r(x-y)| \geq r|x-y| - |x| \Longrightarrow r \leq \frac{1 + |x|}{|x-y|} \leq \frac{2\operatorname{diam}(D)}{|x-y|}$$

since $\operatorname{diam}(D) \geq 1$ and $0 \in D$, thus $|x| \leq \operatorname{diam}(D)$ for any $x \in D$. In turn, we may estimate

$$\begin{aligned}
&\sum_{k=1}^{d-1} \binom{d-1}{k} \int_0^\infty \eta\big(x + r(x-y)\big) r^{d-1-k} \,\mathrm{d}r \leq \binom{d-1}{\lceil \frac{d-1}{2} \rceil} \|\eta\|_{L^\infty(\mathbb{R}^d)} \sum_{k=1}^{d-1} \int_0^{\frac{2\operatorname{diam}(D)}{|x-y|}} r^{d-1-k} \,\mathrm{d}r \\
&= C(\eta, d) \sum_{k=1}^{d-1} \frac{1}{d-k} \left(\frac{2\operatorname{diam}(D)}{|x-y|} \right)^{d-k} \leq C(\eta, d) \frac{\operatorname{diam}(D)^{d-1}}{|x-y|^{d-1}}.
\end{aligned}$$

Similarly, we calculate

$$\begin{aligned}
&\sum_{k=1}^{d}\binom{d}{k}(x_i-y_i)\int_0^\infty(\partial_j\eta)\big(x+r(x-y)\big)r^{d-k}\,\mathrm{d}r\\
&\le\binom{d}{\lceil\frac{d}{2}\rceil}\|\nabla\eta\|_{L^\infty(\mathbb{R}^d)}|x-y|\sum_{k=1}^{d}\int_0^{\frac{2\operatorname{diam}(D)}{|x-y|}}r^{d-k}\,\mathrm{d}r\\
&=C(\eta,d)\sum_{k=1}^{d}\frac{1}{d-k+1}\frac{(2\operatorname{diam}(D))^{d-k+1}}{|x-y|^{d-k}}\\
&\le C(\eta,d)\,\operatorname{diam}(D)\,\frac{\operatorname{diam}(D)^{d-1}}{|x-y|^{d-1}}.
\end{aligned}$$

Thus, we get the estimate

$$|G_{ij}(x,x-y)|\le C(\eta,d)\,(1+\operatorname{diam}(D))\,\frac{\operatorname{diam}(D)^{d-1}}{|x-y|^{d-1}}. \tag{3.7}$$

Concerning the other part of $\partial_j N_i(x,y)$, the kernel $K_{i,j}(x,z)$ respectively its defining function $k_{ij}(x,z)$ satisfies the assumptions of the Calderón-Zygmund Theorem B.9. Indeed, we have from the definition of $k_{ij}(x,z)$ that for any $\lambda>0$, $k_{ij}(x,z)=k_{ij}(x,\lambda z)$, and also for any $|z|=1$

$$\begin{aligned}
|k_{ij}(x,z)|&\le\left|\int_0^\infty\eta(x+rz)r^{d-1}\,\mathrm{d}r\right|+\left|\int_0^\infty(\partial_j\eta)(x+rz)r^{d}\,\mathrm{d}r\right|\\
&\le\|\eta\|_{L^\infty(\mathbb{R}^d)}\frac{\operatorname{diam}(D)^d}{d}+\|\partial_j\eta\|_{L^\infty(\mathbb{R}^d)}\frac{\operatorname{diam}(D)^{d+1}}{d+1},
\end{aligned}$$

therefore

$$\|k_{ij}(x,z)\|_{L^\infty(D\times\{|z|=1\})}\le C(\eta,d)\,\operatorname{diam}(D)^d(1+\operatorname{diam}(D)). \tag{3.8}$$

Third, we find with $\eta\in C_c^\infty(\mathbb{R}^d)$, $\int_{\mathbb{R}^d}\eta\,\mathrm{d}x=1$, and partial integration

$$\begin{aligned}
\int_{|z|=1}k_{ij}(x,z)\,\mathrm{d}z&=\delta_{ij}\int_{|z|=1}\int_0^\infty\eta(x+rz)r^{d-1}\,\mathrm{d}r\,\mathrm{d}z+\int_{|z|=1}z_i\int_0^\infty(\partial_j\eta)(x+rz)r^{d}\,\mathrm{d}r\,\mathrm{d}z\\
&=\int_{\mathbb{R}^d}\delta_{ij}\eta(x+y)+y_i(\partial_j\eta)(x+y)\,\mathrm{d}y=\int_{\mathbb{R}^d}\delta_{ij}\eta(z)+(x_i-z_i)(\partial_j\eta)(z)\,\mathrm{d}z\\
&=\delta_{ij}-\int_{\mathbb{R}^d}\delta_{ij}\eta(z)\,\mathrm{d}z=0.
\end{aligned}$$

Hence, the limits in (3.5) exist and (3.5) can be written as

$$\begin{aligned}
\partial_j\mathbf{u}_i(x)&=p.v.\int_D f(y)K_{ij}(x,x-y)\,\mathrm{d}y+\int_D f(y)G_{ij}(x,x-y)\,\mathrm{d}y\\
&\quad+f(x)\int_D\frac{(x_j-y_j)(x_i-y_i)}{|x-y|^2}\eta(y)\,\mathrm{d}y\\
&=:F_1(x)+F_2(x)+F_3(x).
\end{aligned}$$

From Theorem B.9 and (3.8), we get that

$$\|F_1\|_{L^q(D)} \le C \ \mathrm{diam}(D)^d \,(1 + \mathrm{diam}(D))\, \|f\|_{L^q(D)}.$$

Young's inequality (B.3) and (3.7) furnish

$$\|F_2\|_{L^q(D)} \le C \ \mathrm{diam}(D)^d \,(1 + \mathrm{diam}(D))\, \|f\|_{L^q(D)}.$$

Finally, we have

$$\|F_3\|_{L^q(D)} = \left\| \int_D \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} \eta(y) \,\mathrm{d}y \right\|_{L^\infty_x(D)} \|f\|_{L^q(D)} \le C \,\|f\|_{L^q(D)},$$

where the constants above may depend on η, d, and q, but not on D. Combining the estimates above, together with $\mathrm{diam}(D) \ge 1$, we end up with

$$\|\mathbf{u}\|_{W^{1,q}_0(D)} \le C(d,q) \ \mathrm{diam}(D)^d \,(1 + \mathrm{diam}(D))\, \|f\|_{L^q(D)},$$

which is inequality (3.3) since we assumed at the beginning $R = 1$.
It is left to show that $\mathbf{u}$ also satisfies (3.1). For that, we calculate

$$\begin{aligned}
\mathrm{div}\,\mathbf{u}(x) &= \int_D f(y) \bigg[d \int_1^\infty \eta(y + r(x-y)) r^{d-1} \,\mathrm{d}r \\
&\quad + \sum_{i=1}^d \int_1^\infty (x_i - y_i)(\partial_i \eta)(y + r(x-y)) r^d \,\mathrm{d}r \bigg] \,\mathrm{d}y \\
&\quad + \sum_{i=1}^d f(x) \int_D \frac{|x_i - y_i|^2}{|x-y|^2} \eta(y) \,\mathrm{d}y \\
&= \int_D f(y) \bigg[d \int_1^\infty \eta(y + r(x-y)) r^{d-1} \,\mathrm{d}r \\
&\quad + \int_1^\infty r^d \Big(\frac{\mathrm{d}}{\mathrm{d}r} \eta(y + r(x-y)) \Big) \,\mathrm{d}r \bigg] \,\mathrm{d}y + f(x) \\
&= \big(r^d \eta(y + r(x-y)) \big) \Big|_{r=1}^\infty \int_D f(y) \mathrm{d}y + f(x) \\
&= -\eta(x) \int_D f(y) \,\mathrm{d}y + f(x).
\end{aligned}$$

Since f has zero integral over D, this shows (3.1).
Lastly, if $f \in L^q_0(D)$ is arbitrary, we choose a sequence $\{f_n^*\}_{n\in\mathbb{N}} \subset C_c^\infty(D)$ such that $f_n^* \to f$ in $L^q(D)$, and define

$$f_n := f_n^* - \varphi \int_D f_n^* \,\mathrm{d}x,$$

where $\varphi \in C_c^\infty(D)$ satisfies $\int_D \varphi \,\mathrm{d}x = 1$. Then we still have $f_n \in C_c^\infty(D)$ and $f_n \to f$ in $L^q(D)$, but also $f_n \in L^q_0(D)$. Thus, to any f_n we can find functions $\mathbf{u}_n \in C_c^\infty(D)$ solving (3.1) and obeying the estimate (3.3). This together with the fact that the operator $\mathcal{B}$ in (3.4) is linear in f, the sequence $\{\mathbf{u}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $W_0^{1,q}(D)$ since

$$\|\mathbf{u}_n - \mathbf{u}_k\|_{W_0^{1,q}(D)} = \|\mathcal{B}(f_n) - \mathcal{B}(f_k)\|_{W_0^{1,q}(D)} = \|\mathcal{B}(f_n - f_k)\|_{W_0^{1,q}(D)} \leq C\,\|f_n - f_k\|_{L^q(D)},$$

thus converging strongly in $W_0^{1,q}(D)$ to a function $\mathbf{u} \in W_0^{1,q}(D)$. In particular,

$$f = \lim_{n\to\infty} f_n = \lim_{n\to\infty} \operatorname{div} \mathbf{u}_n = \operatorname{div} \mathbf{u}$$

in the sense of strong limits in $L^q(D)$, showing that $\mathbf{u}$ satisfies (3.1) and (3.3). This finishes the proof of the Lemma. □

Note that the constant appearing in (3.3) is invariant under rotation, scaling, and translation of the domain D. These invariances will thus be true for any upper bound on the constants appearing in future theorems on the existence of an operator $\mathcal{B}$, and will also be used in the proof of Theorem 3.2.9 below.

3.1.2 Bogovskiĭ's operator in Lipschitz domains

We will prove that for bounded domains $D \subset \mathbb{R}^d$ with Lipschitz boundary, there exists a right inverse to the divergence operator. To this end, we first give the definition of a Lipschitz domain, and then state and prove a result on the connection of Lipschitz and star-shaped domains, which can be found in [Gal11, Lemma II.1.3 and Exercise II.1.5] (see also [SBH19, Proposition 10.11]).

Definition 3.1.2. *Let $B_1^{d-1}(0)$ denote the open unit ball in $\mathbb{R}^{d-1}$, and let $D \subset \mathbb{R}^d$ be a bounded domain. Then we say that D has* Lipschitz boundary *(or D is a* Lipschitz domain*), if for any $x_0 \in \partial D$ there exists a ball $B_r(x_0)$ and a Lipschitz function $\zeta : B_1^{d-1}(0) \to \mathbb{R}$ with Lipschitz constant $L > 0$ such that in a proper coordinate system with origin x_0, we have*

$$\begin{aligned} \partial D \cap B_r(x_0) &= \{(x', \zeta(x')) : x' \in B_1^{d-1}(0)\}, \\ D \cap B_r(x_0) &= \{(x', x_d) \in B_1^{d-1}(0) \times \mathbb{R} : x_d < \zeta(x')\}, \end{aligned}$$

where we used the notation $x' = (x_1, \ldots, x_{d-1})$.

Lemma 3.1.3. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists a finite collection of bounded open sets $\{D_i\}_{i=1}^k$ which are star-shaped with respect to a ball such that $D = \bigcup_{i=1}^k D_i$.*

Proof. Let $x_0 \in \partial D$. Since D is a Lipschitz domain, there exists a function $\zeta : B_1^{d-1}(0) \to \mathbb{R}$ with Lipschitz constant $L > 0$ and a ball $B_r(x_0)$ such that for any point $x = (x', x_d) \in \partial D \cap B_r(x_0)$, we have

$$x_d = \zeta(x'), \quad x' \in B_1^{d-1}(0),$$

and for all $x \in D \cap B_r(x_0)$, we have

$$x_d < \zeta(x'), \quad x' \in B_1^{d-1}(0).$$

By a proper rotation and translation, without loss of generality we may assume that $x_0 = 0$. Denote by $y_0 = (0, \ldots, 0, y_d) \in D$ the intersection point of $B_r(x_0)$ and the x_d-axis, and let

$C(y_0, \alpha)$ be the cone with vertex y_0, opening axis equal to the x_d-axis, and semi-aperture $0 < \alpha < \pi/2$. Let R be a ray starting in y_0 and lying inside $C(y_0, \alpha)$. Then, R intersects $\partial D \cap B_r(x_0)$ in exactly one point. Indeed, assume that R cuts $\partial D \cap B_r(x_0)$ in two distinct points $z_1 \neq z_2$, and let $\alpha' < \alpha$ be the angle between R and the x_d-axis. Rotating the coordinate system as the case may be, we can assume that

$$z_1 = (z_1^1 \mathbf{e}_1, \zeta(z_1^1 \mathbf{e}_1)), \quad z_1^1 > 0, \qquad z_2 = (z_2^1 \mathbf{e}_1, \zeta(z_2^1 \mathbf{e}_1)), \quad z_2^1 > 0,$$

where $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^{d-1}$. Since $z_1, z_2 \in R$, we have

$$\tan \alpha' = \frac{z_1^1}{\zeta(z_1^1 \mathbf{e}_1) - y_d} = \frac{z_2^1}{\zeta(z_2^1 \mathbf{e}_1) - y_d} = \frac{|z_1^1 - z_2^1|}{|\zeta(z_1^1 \mathbf{e}_1) - \zeta(z_2^1 \mathbf{e}_1)|} = \frac{|z_1 - z_2|}{|\zeta(z_1^1 \mathbf{e}_1) - \zeta(z_2^1 \mathbf{e}_1)|},$$

hence

$$\frac{|z_1 - z_2|}{|\zeta(z_1^1 \mathbf{e}_1) - \zeta(z_2^1 \mathbf{e}_1)|} = \tan \alpha' \leq \tan \alpha.$$

Choosing α so small that

$$\tan \alpha \leq \frac{1}{2L},$$

this would yield

$$|\zeta(z_1^1 \mathbf{e}_1) - \zeta(z_2^1 \mathbf{e}_1)| \leq L|z_1 - z_2| \leq \frac{1}{2}|\zeta(z_1^1 \mathbf{e}_1) - \zeta(z_2^1 \mathbf{e}_1)|,$$

which is a contradiction. Thus, R intersects $\partial D \cap B_r(x_0)$ in exactly one point. Now, let $z = (z', z_d)$ with $z_d > y_d$, and denote by $S = S(z)$ the intersection of $C(y_0, \alpha/2)$ with the plane $\{(x', x_d) \in \mathbb{R}^d : x_d = z_d\}$. Set further

$$R(z) = \operatorname{dist}(\partial S, z).$$

If z is sufficiently close to y_0 (say, $z = z_0$), the set $S(z_0) \subset D$ and, in addition, any ray originating from a point in $S(z_0)$ that lies completely inside $C(y_0, \alpha/2)$ forms with the x_d-axis an angle less than α, hence it intersects $\partial D \cap B_r(x_0)$ in only one point. Let Z be a cylinder with axis equal to the x_d-axis that fulfils

$$\partial D \cap Z = \partial D \cap C(y_0, \alpha/2),$$

then, by what we have shown till now, the set $Z \cap B_r(x_0) \cap D$ is star-shaped with respect to the ball $B_{R(z_0)}(z_0)$. Since $x_0 \in \partial D$ was arbitrary and ∂D is compact, we can choose a finite number of points $\{x_i\}_{i=1}^{k'} \subset \partial D$, corresponding balls $B_{r_i}(x_i)$, and corresponding cylinders Z_i such that $\partial D \subset \bigcup_{i=1}^{k'} Z_i \cap B_r(x_i)$. Define now $D_i \coloneqq Z_i \cap B_{r_i}(x_i) \cap D$ and consider

$$D_{\text{int}} \coloneqq D \setminus \bigcup_{i=1}^{k'} D_i = \left(\overline{D}^c \cup \left(\bigcup_{i=1}^{k'} D_i \right) \right)^c,$$

then D_{int} is compact and separated from ∂D, so we may choose finitely many balls $\{B_i\}_{i=k'+1}^{k}$

which cover D_{int} and which are strictly contained in D. Since all the sets D_i, $1 \le i \le k'$, are star-shaped with respect to a ball, and the balls B_i, $k'+1 \le i \le k$, are obviously star-shaped with respect to a ball, we may take $\{D_i\}_{i=1}^k := \{D_i\}_{i=1}^{k'} \cup \{B_i\}_{i=k'+1}^k$ to finish the proof. □

Before we present and prove the fact that in any bounded Lipschitz domain there exists a right inverse to the divergence, we show the following decomposition result for functions defined on the union of star-shaped domains (see [Gal11, Lemma III.3.2]).

Lemma 3.1.4. *Let $D = \bigcup_{i=1}^k D_i \subset \mathbb{R}^d$ be a connected set, where any set D_i is a star-shaped domain with respect to a ball, and the sets are numbered in such a way that $|D_i \cap \bigcup_{l=i+1}^k D_l| \neq 0$ for any $1 \le i \le k-1$. Further, let $f \in L_0^q(D)$. Then there exist functions $\{f_i\}_{i=1}^k$ such that*

- $\operatorname{supp}(f_i) \subset \overline{D_i}$,
- $f_i \in L_0^q(D_i)$,
- $f = \sum_{i=1}^k f_i$,
- $\|f_i\|_{L^q(D_i)} \le C_i \|f\|_{L^q(D)}$, *where (with the convention $\prod_{j=1}^0 a_j := 1$)*

$$C_i = \left(1 + \frac{|D_i|^{1-1/q}}{|F_i|^{1-1/q}}\right) \prod_{j=1}^{i-1} \left(1 + \frac{|K_j \setminus D_j|^{1-1/q}}{|F_j|^{1-1/q}}\right), \quad 1 \le i \le k-1,$$
$$C_k = \prod_{j=1}^{k-1} \left(1 + \frac{|K_j \setminus D_j|^{1-1/q}}{|F_j|^{1-1/q}}\right),$$

and the sets F_i and K_i are defined via $F_i := D_i \cap K_i$ and $K_i := \bigcup_{l=i+1}^k D_l$.

Proof. First, note that the constants C_i are well-defined since we may always number the sets D_i in such a way that $|F_i| \neq 0$ for any i. We will give a graph-theoretical argument for this and refer to [Gri21] for the basic concepts of graph theory. Let $\mathcal{G} = (V, E)$, where $V = \{D_i\}_{i=1}^k$ are the vertices of $\mathcal{G}$, and E is the set of edges, where we connect D_i and D_j with an edge if $|D_i \cap D_j| > 0$. Since D is connected, $\mathcal{G}$ is as well, and therefore contains a spanning tree $\mathcal{T}$. We may now relabel the sets D_i such that D_1 is a leaf of $\mathcal{T}$ (that is, a vertex with just one edge), and remove D_1 and its edge from $\mathcal{T}$. Now, let D_2 be a leaf of $\mathcal{T} \setminus \{D_1\}$, D_3 be a leaf of $\mathcal{T} \setminus \{D_1, D_2\}$, and so on. Note that if we remove a leaf from the tree $\mathcal{T}$, the remaining graph is still a tree and in particular connected. Therefore, we may proceed inductively to show the desired. Now, define

$$f_1(x) := \begin{cases} f(x) - \frac{\chi_{F_1}(x)}{|F_1|} \int_{D_1} f \,\mathrm{d}x & \text{if } x \in D_1, \\ 0 & \text{if } x \in K_1 \setminus D_1, \end{cases}$$
$$g_1(x) := \begin{cases} [1 - \chi_{F_1}(x)] f(x) - \frac{\chi_{F_1}(x)}{|F_1|} \int_{K_1 \setminus D_1} f \,\mathrm{d}x & \text{if } x \in K_1, \\ 0 & \text{if } x \in D_1 \setminus K_1. \end{cases}$$

Then, clearly,

$$\int_{D_1} f_1 \,\mathrm{d}x = 0, \quad \operatorname{supp}(f_1) \subset \overline{D_1}, \quad f_1 \in L^q(D_1),$$
$$\operatorname{supp}(g_1) \subset \overline{K_1}, \quad g_1 \in L^q(K_1),$$

and also, since $D = D_1 \cup K_1$ and by dividing into the cases $x \in D_1 \setminus K_1$, $x \in K_1 \setminus D_1$, and $x \in D_1 \cap K_1 = F_1$, we have $f(x) = f_1(x) + g_1(x)$ in D, thus

$$\begin{aligned}\int_{K_1} g_1 \,\mathrm{d}x &= \int_{K_1\setminus D_1} g_1 \,\mathrm{d}x + \int_{K_1\cap D_1} g_1 \,\mathrm{d}x = \int_{K_1\setminus D_1} g_1 \,\mathrm{d}x + \int_{F_1} g_1 \,\mathrm{d}x \\ &= \int_{K_1\setminus D_1} g_1 \,\mathrm{d}x - \int_{K_1\setminus D_1} f \,\mathrm{d}x = -\int_{K_1\setminus D_1} f_1 \,\mathrm{d}x = 0.\end{aligned}$$

Setting $g_0 \coloneqq f$, we define inductively for any $1 \le i \le k-1$

$$f_i(x) \coloneqq \begin{cases} g_{i-1}(x) - \frac{\chi_{F_i}(x)}{|F_i|} \int_{D_i} g_{i-1} \,\mathrm{d}x & \text{if } x \in D_i, \\ 0 & \text{if } x \in K_i \setminus D_i, \end{cases}$$

$$g_i(x) \coloneqq \begin{cases} [1 - \chi_{F_i}(x)] g_{i-1}(x) - \frac{\chi_{F_i}(x)}{|F_i|} \int_{K_i\setminus D_i} g_{i-1} \,\mathrm{d}x & \text{if } x \in K_i, \\ 0 & \text{if } x \in D_i \setminus K_i, \end{cases}$$

and set $f_k(x) \coloneqq g_{k-1}(x)$. Similarly to the case $i = 1$, we have for all $1 \le i \le k$

$$\int_{D_i} f_i \,\mathrm{d}x = 0, \quad \operatorname{supp}(f_i) \subset \overline{D_i}, \quad f_i \in L^q(D_i).$$

Now, by Hölder's inequality (B.2), we get

$$\begin{aligned}\|f_k\|_{L^q(D_k)} = \|g_{k-1}\|_{L^q(D_k)} &\le \|g_{k-2}\|_{L^q(D_k)} + \left| \int_{K_{k-1}\setminus D_{k-1}} g_{k-2} \,\mathrm{d}x \right| |F_{k-1}|^{\frac{1}{q}-1} \\ &\le \|g_{k-2}\|_{L^q(D)} + \|g_{k-2}\|_{L^q(K_{k-1}\setminus D_{k-1})} \frac{|K_{k-1} \setminus D_{k-1}|^{1-1/q}}{|F_{k-1}|^{1-1/q}} \\ &\le \|g_{k-2}\|_{L^q(D)} \left(1 + \frac{|K_{k-1} \setminus D_{k-1}|^{1-1/q}}{|F_{k-1}|^{1-1/q}} \right).\end{aligned}$$

Similarly, for any $1 \le i \le k-1$,

$$\begin{aligned}\|f_i\|_{L^q(D_i)} &\le \|g_{i-1}\|_{L^q(D_i)} + \left| \int_{D_i} g_{i-1} \,\mathrm{d}x \right| |F_i|^{\frac{1}{q}-1} \\ &\le \|g_{i-1}\|_{L^q(D)} + \|g_{i-1}\|_{L^q(D_i)} \frac{|D_i|^{1-1/q}}{|F_i|^{1-1/q}} \\ &\le \|g_{i-1}\|_{L^q(D)} \left(1 + \frac{|D_i|^{1-1/q}}{|F_i|^{1-1/q}} \right),\end{aligned}$$

which, by $\operatorname{supp}(g_{i-1}) \subset \overline{K_{i-1}}$ and iterating $(i-1)$ times, yields for all $1 \le i \le k-1$

$$\begin{aligned}\|f_i\|_{L^q(D_i)} &\le \|g_{i-1}\|_{L^q(D)} \left(1 + \frac{|D_i|^{1-1/q}}{|F_i|^{1-1/q}} \right) \\ &= \|g_{i-1}\|_{L^q(K_{i-1})} \left(1 + \frac{|D_i|^{1-1/q}}{|F_i|^{1-1/q}} \right) \\ &\le \left(\|g_{i-2}\|_{L^q(K_{i-1})} + \left| \int_{K_{i-1}\setminus D_{i-1}} g_{i-2} \,\mathrm{d}x \right| |F_{i-1}|^{\frac{1}{q}-1} \right) \left(1 + \frac{|D_i|^{1-1/q}}{|F_i|^{1-1/q}} \right)\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\|g_{i-2}\|_{L^q(D)} + \|g_{i-2}\|_{L^q(K_{i-1}\setminus D_{i-1})} \frac{|K_{i-1} \setminus D_{i-1}|^{1-1/q}}{|F_{i-1}|^{1-1/q}} \right) \left(1 + \frac{|D_i|^{1-1/q}}{|F_i|^{1-1/q}} \right) \\
&\leq \|g_{i-2}\|_{L^q(D)} \left(1 + \frac{|K_{i-1} \setminus D_{i-1}|^{1-1/q}}{|F_{i-1}|^{1-1/q}} \right) \left(1 + \frac{|D_i|^{1-1/q}}{|F_i|^{1-1/q}} \right) \\
&\leq \|g_0\|_{L^q(D)} \left(1 + \frac{|D_i|^{1-1/q}}{|F_i|^{1-1/q}} \right) \prod_{j=1}^{i-1} \left(1 + \frac{|K_j \setminus D_j|^{1-1/q}}{|F_j|^{1-1/q}} \right) \\
&= \|f\|_{L^q(D)} \left(1 + \frac{|D_i|^{1-1/q}}{|F_i|^{1-1/q}} \right) \prod_{j=1}^{i-1} \left(1 + \frac{|K_j \setminus D_j|^{1-1/q}}{|F_j|^{1-1/q}} \right)
\end{aligned}
$$

as well as

$$
\|f_k\|_{L^q(D_k)} \leq \|f\|_{L^q(D)} \prod_{j=1}^{k-1} \left(1 + \frac{|K_j \setminus D_j|^{1-1/q}}{|F_j|^{1-1/q}} \right).
$$

□

The next lemma states that we can find and bound a Bogovskiĭ operator in any domain that is a union of star-shaped domains. This can be seen as a prototype for the case of Lipschitz domains, see Theorem 3.1.6 below.

Lemma 3.1.5. *Let $D = \bigcup_{i=1}^k D_i \subset \mathbb{R}^d$ be a connected set, where any D_i is star-shaped with respect to a ball B_i with radius $r_i > 0$, and let $f \in L^q_0(D)$. Then there exists a bounded linear map $\mathcal{B} : L^q_0(D) \to W^{1,q}_0(D)$ such that $\mathcal{B}(f)$ is a solution to system (3.1) and $\mathcal{B}$ satisfies the bound (3.2), where the constant C obtains an upper bound*

$$
C \leq c_0\, c_1\, (1 + s)\, s^d.
$$

Here, $c_0 = c_0(d, q)$, c_1 is an upper bound for the constants C_i arising in Lemma 3.1.4, and s is defined as

$$
s := \max_{1 \leq i \leq k} \frac{\operatorname{diam}(D_i)}{r_i}.
$$

Proof. Decompose f as in Lemma 3.1.4, then by Lemma 3.1.1 we can find in every D_i a linear map $\mathcal{B}_i : L^q_0(D_i) \to W^{1,q}_0(D_i)$ and a constant $c_0 = c_0(d, q) > 0$ satisfying

$$
\begin{aligned}
\operatorname{div} \mathcal{B}_i(f_i) &= f_i \text{ in } D_i, \\
\|\mathcal{B}_i(f_i)\|_{W^{1,q}_0(D_i)} &\leq c_0\, \|f_i\|_{L^q(D_i)} \left(1 + \frac{\operatorname{diam}(D_i)}{r_i} \right) \left(\frac{\operatorname{diam}(D_i)}{r_i} \right)^d \\
&\leq c_0\, C_i\, \|f\|_{L^q(D)} \left(1 + \frac{\operatorname{diam}(D_i)}{r_i} \right) \left(\frac{\operatorname{diam}(D_i)}{r_i} \right)^d.
\end{aligned}
$$

Extending any $\mathcal{B}_i$ to be zero outside its domain of definition, we may set

$$
\mathcal{B}(f) := \sum_{i=1}^k \mathcal{B}_i(f_i),
$$

which finishes the proof since

$$\operatorname{div} \mathcal{B}(f) = \sum_{i=1}^{k} \operatorname{div} \mathcal{B}_i(f_i) = \sum_{i=1}^{k} f_i = f \text{ in } D_i,$$

and

$$\begin{aligned}
\|\mathcal{B}(f)\|_{W_0^{1,q}(D)} &\leq \sum_{i=1}^{k} \|\mathcal{B}_i(f_i)\|_{W_0^{1,q}(D_i)} \\
&\leq c_0 \, \|f\|_{L^q(D)} \sum_{i=1}^{k} C_i \left(1 + \frac{\operatorname{diam}(D_i)}{r_i}\right) \left(\frac{\operatorname{diam}(D_i)}{r_i}\right)^d \\
&\leq c_0 \, \|f\|_{L^q(D)} \, k \, (\max_{1 \leq i \leq k} C_i) \left(1 + \max_{1 \leq i \leq k} \frac{\operatorname{diam}(D_i)}{r_i}\right) \left(\max_{1 \leq i \leq k} \frac{\operatorname{diam}(D_i)}{r_i}\right)^d \\
&\leq c_0 \, c_1 \, (1 + s) \, s^d \, \|f\|_{L^q(D)}.
\end{aligned}$$

□

Theorem 3.1.6. *Let $D \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Then there exists a bounded linear map $\mathcal{B} : L_0^q(D) \to W_0^{1,q}(D)$ obeying the bound (3.2) such that for any $f \in L_0^q(D)$, the function $\mathbf{u} = \mathcal{B}(f)$ is a solution to system (3.1).*

Proof. By Lemma 3.1.3, we may write D as $D = \bigcup_{i=1}^{k} D_i$, where any D_i is star-shaped with respect to a ball. The statement is now a direct consequence of Lemma 3.1.5. □

Remark 3.1.7. *Let us remark that due to the previous lemmata, the constant C arising in (3.2) heavily depends on the Lipschitz character L of ∂D. More precisely, we see that the number k from Lemma 3.1.4 of star-shaped domains that cover D depends proportional on L, which shows that also the constant c_1 from Lemma 3.1.5 tends towards infinity as L does. We will come back to this observation in Section 3.3 later on.*

3.2 John domains

John domains are a class of rather general domains that still satisfy some good regularity properties. They were first used by F. John in his work [Joh61] in connections with elasticity problems and are defined as follows.

Definition 3.2.1. *For a constant $c > 0$, a domain $U \subset \mathbb{R}^d$ is said to be a $c-$*John domain *if there exists a point $x_0 \in U$ such that for any point $x \in U$ there is a rectifiable path $\Gamma : [0, \ell] \to U$ which is parametrized by arc length with*

$$\Gamma(0) = x, \quad \Gamma(\ell) = x_0, \quad \forall t \in [0, \ell] : |\Gamma(t) - x| \leq c \operatorname{dist}(\Gamma(t), \partial U). \tag{3.9}$$

John domains may have fractal boundaries or internal cusps, whereas external cusps are forbidden. For instance, the interior of Koch's snowflake as well as any convex domain are John domains, see [Pom13, Theorem 5.9]. In the case of bounded domains, there are several equivalent definitions of John domains, see [Väi88, Section 2.17]. We state the following characterization, which is used in [DRS10, Section 3.1]: a bounded domain U is a $c-$John domain

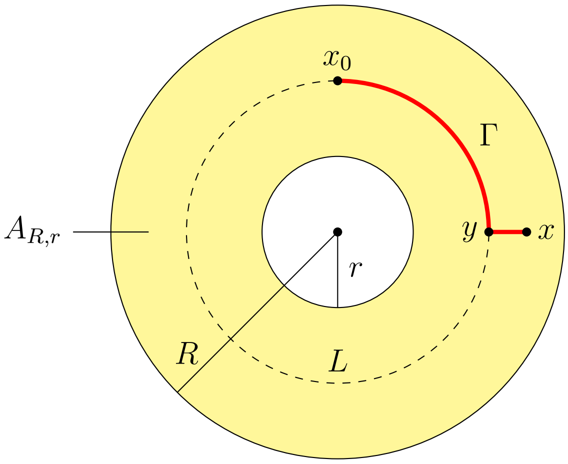


Figure 3.1: The path Γ (red), connecting $x \in A_{R,r}$ first to $y \in L$ and then to x_0 without leaving L (dashed).

in the sense of Definition 3.2.1 if and only if there is a $c_1(c) > 0$ and a point $x_0 \in U$ such that any point $x \in U$ can be connected to x_0 by a rectifiable path $\Gamma : [0, \ell] \to U$ which is parametrized by arc length and

$$\bigcup_{t\in[0,\ell]} B\big(\Gamma(t), t/c_1\big) \subset U. \tag{3.10}$$

Obviously, if U is a $c-$John domain, than it is also a $c'-$John domain for any $c' \geq c$. Further, the condition (3.9) is invariant under rotation, translation, and scaling of the domain U. We will give some examples of John domains, which we will use later on for the existence and boundedness of Bogovskiĭ's operator in perforated domains.

Example 3.2.2. *Let $0 < r < R$ and consider the annulus $A_{R,r} := B_R(0) \setminus \overline{B_r(0)}$. Then $A_{R,r}$ is a John domain with constant $c = \frac{2\pi R}{R-r}$. In particular, if $0 < r_0 < R$ is fixed and $0 < r \leq r_0$, then the John constant of $A_{R,r}$ just depends on r_0 and R but not on r.*

Proof. The ideas given here will show up again later in the proof of Lemma 3.4.4. Let $L := \partial B_{(R+r)/2}(0)$ be the midline in $A_{R,r}$, and fix a point $x_0 \in L$. Then, for $x \in A_{R,r}$, the path Γ which connects x to L and then follows L to x_0 will do the job. More precisely, let $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the shortest line connecting x to L and hitting L in a point y, and $\Gamma_2 \subset L$ is the shortest arc from y to x_0; see Figure 3.1 for an illustration in two dimensions. Since Γ_1 is defined to be the shortest line joining x to L, it is part of the ray $\{tx : t \geq 0\}$, so its length is bounded by $\frac{R-r}{2}$. Similarly, as Γ_2 is the shortest arc in L joining y to x_0, its length is bounded by $\pi\frac{R+r}{2}$. Denoting by ℓ the length of Γ, we then have $\ell \leq \pi\frac{R+r}{2} + \frac{R-r}{2}$, so Γ is rectifiable. Let $t_0 \in [0, \ell]$ be the unique time such that $\Gamma(t_0) = y$. For $t \in [0, t_0]$, we obviously have

$$\begin{aligned}|\Gamma(t) - x| &= |\Gamma_1(t) - x| = \operatorname{dist}(\Gamma_1(t), \partial A_{R,r}) \\ &\leq \frac{2\pi R}{R-r}\operatorname{dist}(\Gamma_1(t), \partial A_{R,r}) = \frac{2\pi R}{R-r}\operatorname{dist}(\Gamma(t), \partial A_{R,r}).\end{aligned}$$

For any $t \in [t_0, \ell]$, we have $\operatorname{dist}(\Gamma_2(t), \partial A_{R,r}) = \frac{R-r}{2}$, thus

$$|\Gamma(t) - x| = |\Gamma_2(t) - x| \leq \pi\frac{R+r}{2} + |y - x| \leq \pi\frac{R+r}{2} + \frac{R-r}{2} \leq \pi R$$

$$= \frac{2\pi R}{R-r}\frac{R-r}{2} = \frac{2\pi R}{R-r}\operatorname{dist}(\Gamma_2(t), \partial A_{R,r}) = \frac{2\pi R}{R-r}\operatorname{dist}(\Gamma(t), \partial A_{R,r}),$$

leading finally to

$$|\Gamma(t) - x| \leq \frac{2\pi R}{R-r}\operatorname{dist}(\Gamma(t), \partial A_{R,r}) \quad \forall t \in [0, \ell].$$

□

Example 3.2.3. *Let $x_0 \in \mathbb{R}^d$ and $S \subset \mathbb{R}^d$ be a bounded star-shaped domain with respect to a ball $B_r(x_0)$. Then S is a John domain with John constant at least $\operatorname{diam}(S)/r$.*

Proof. Let $x \in S$ and choose the path Γ as the straight line from x to x_0. Since S is star-shaped with respect to any point of $B_r(x_0)$ and $|x - x_0| \leq \operatorname{diam}(S)$, we have

$$\bigcup_{t\in[0,|x-x_0|]} B(\Gamma(t), rt/\operatorname{diam}(S)) \subset S.$$

Since for straight paths Γ, equations (3.9) and (3.10) coincide with $c = c_1$, we may choose $c = \operatorname{diam}(S)/r$. □

Example 3.2.4. *Let $x_0, y_0 \in \mathbb{R}^d$ and $S_1, S_2 \subset \mathbb{R}^d$ be bounded star-shaped domains with respect to balls $B_{R_1}(x_0), B_{R_2}(y_0)$, respectively. Assume that there exist $z \in S_1 \cap S_2$ and $0 < r \leq \frac{1}{2}\min\{\operatorname{diam}(B_1(x_0)), \operatorname{diam}(B_2(y_0))\}$ such that $B_r(z) \subset S_1 \cap S_2$ and $S := S_1 \cup S_2 \subset B_{c_0 r}(z)$ for some $c_0 > 0$. Then S is a John domain.*

Proof. We will show that the point x_0 is a proper John center for S. First, since S_1 and S_2 are star-shaped, we see from Example 3.2.3 that they are John domains with constant $c' := \max\left\{\frac{\operatorname{diam}(S_1)}{R_1}, \frac{\operatorname{diam}(S_2)}{R_2}\right\}$. Let $x \in S$. If $x \in S_1$, we are in the situation of Example 3.2.3. If $x \in S_2$, then we connect first x to y_0 with the path $\Gamma_1(t) := x + t\frac{y_0 - x}{|y_0 - x|}$, then connect y_0 to z via $\Gamma_2(t) := y_0 + t\frac{z - y_0}{|z - y_0|}$, and finally connect z to x_0 via $\Gamma_3(t) := z + t\frac{x_0 - z}{|x_0 - z|}$. Since $x \in S_2$ and $z \in S_1$, both Γ_1 and Γ_3 are proper paths with John constant c'. Since S_2 is star-shaped and $z \in B_r(z) \subset S_2$, we have $\operatorname{dist}(\Gamma_2(t), \partial S) \geq r$, leading to

$$|\Gamma_2(t) - z| \leq |y_0 - z| \leq \operatorname{diam}(S_2) \leq 2c_0 r \leq 2c_0 \operatorname{dist}(\Gamma_2(t), \partial S).$$

Finally, we choose $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ to obtain a proper path joining x to x_0 in S, where the constant c occurring in (3.9) can be chosen as $c = c' + 2c_0 + c' = 2(c' + c_0)$. □

Example 3.2.5. *Let $T \subset B_1(0)$ be a simply connected compact set with Lipschitz boundary and $0 \in T$. Then $B_1(0) \setminus T$ is a John domain, where the John constant only depends on the Lipschitz character of ∂T.*

Proof. From Lemma 3.1.3, we can cover $B_1(0) \setminus T$ with finitely many open sets $\{D_i\}_{i=1}^k$ such that each D_i is star-shaped with respect to some ball. If $D_i \cap D_j \neq \emptyset$ for some $i \neq j$, then we may find a a ball $B_r \subset D_i \cap D_j$ such that the conditions of Example 3.2.4 are fulfilled. The statement follows now from iterating the arguments given in Example 3.2.4. □

Finally, we will state the following result for shrinking domains, which will be crucial in the proof of Theorem 3.4.1.

Lemma 3.2.6. *Let $T \subset B_{1/2}(0)$ be a simply connected compact set with Lipschitz boundary and $0 \in T$, and let $0 < r < 1$. Then the domain $B_1(0) \setminus (rT)$ is a John domain, where the John constant does not depend on r.*

Proof. Since $T \subset B_{1/2}(0)$, we have $rT \subset B_{r/2}(0) \subset B_r(0) \subset B_1(0)$. From Example 3.2.5, we know that $B_1(0) \setminus T$ is a John domain, where the John constant just depends on the Lipschitz character of ∂T. Since (3.9) is invariant under scaling, the same holds true for the set $B_r(0) \setminus (rT)$. Further, from Example 3.2.2, the set $B_1(0) \setminus \overline{B_{r/2}(0)}$ is a John domain with constant independent of r. We now choose a star-shaped domain from the covering of $B_r(0) \setminus (rT)$ such that its star center $x_0 \in B_r(0) \setminus \overline{B_{r/2}(0)}$. Note that this choice is always possible: indeed, if all star centers are inside $B_{r/2}(0) \setminus (rT)$, then we choose one of the star-shaped domains that cover $B_r(0) \setminus (rT)$. This one will also cover a part of $B_r(0) \setminus \overline{B_{r/2}(0)}$, which then has Lipschitz boundary, and we may cover this part by finitely many star-shaped domains with star centers in $B_r(0) \setminus \overline{B_{r/2}(0)}$. Now, let $x \in B_1(0) \setminus (rT)$. If $x \in B_1(0) \setminus \overline{B_{r/2}(0)}$, we are in the situation of Example 3.2.2. If $x \in B_r(0) \setminus (rT)$, then first join x to x_0, following the star-shaped sets that connect them as shown in Example 3.2.4, and finally x_0 to the John center of $B_1(0) \setminus \overline{B_{r/2}(0)}$. The path we obtained then fulfills (3.9) with a constant $c > 0$ that may depend on the Lipschitz character of ∂T, but not on r. □

Before we state the existence and boundedness of a right inverse to the divergence in John domains taken from [DRS10], we give the definition of the so-called emanating chain condition. Our arguments how to prove the existence of a Bogovskiĭ operator are then built on this definition.

Definition 3.2.7 ([DRS10, Definition 3.5])**.** *Let $D \subset \mathbb{R}^d$ be a bounded domain and $\sigma_1, \sigma_2 \geq 1$. Then D satisfies the* emanating chain condition *with constants σ_1 and σ_2 if there exists a covering $\mathcal{W} = \{W_i : i \in \mathbb{N}\}$ of D consisting of open cubes or balls such that:*

i) For all $i \in \mathbb{N}$, we have $\sigma_1 W_i \subset D$ and $\sum_{i\in\mathbb{N}} \chi_{\sigma_1 W_i} \leq \sigma_2 \chi_D$ on $\mathbb{R}^d$.

ii) For any $W_i \in \mathcal{W}$ there exists a chain of pairwise different $W_{i,0}, W_{i,1}, \ldots, W_{i,m_i} \in \mathcal{W}$ such that $W_{i,0} = W_i$, $W_{i,m_i} = W_0$ and $W_{i,k} \subset \sigma_2 W_{i,l}$ for all $0 \leq k \leq l \leq m_i$. Further, for any $0 \leq k < m_i$, there exists a ball $B_{i,k} \subset W_{i,k} \cap W_{i,k+1}$ such that $W_{i,k} \cup W_{i,k+1} \subset \sigma_2 B_{i,k}$.

iii) For any compact $K \subset D$, the set $\{i \in \mathbb{N} : W_i \cap K \neq \emptyset\}$ is finite.

As shown in [DRS10], for bounded domains, this condition is equivalent to be a John domain. For our purposes, we will just show that any John domain satisfies the emanating chain condition. The proof of this fact uses a covering result which is known as the Whitney covering lemma, and occurred first in [Whi34, Section 8] (see also [Shv07, Theorem 2.4] and [Guz75, Theorem 2.1 and 2.2]).

Lemma 3.2.8 ([DRS10, Proposition 3.3])**.** *There are constants $1 < \kappa_1 < \kappa_2$ and $N > 0$ which depend only on the dimension d such that for any open proper set $D \subsetneq \mathbb{R}^d$ there is a family $\{Q_i, i \in \mathbb{N}\}$ of open cubes or balls such that*

(W1) $D = \bigcup_{i\in\mathbb{N}} \kappa_1 Q_i$,

(W2) $\frac{1}{2}\kappa_1 \operatorname{diam}(Q_i) \leq \operatorname{dist}(Q_i, \partial D) \leq \kappa_2 \operatorname{diam}(Q_i)$,

(W3) $\sum_{i\in\mathbb{N}} \chi_{2\kappa_1 Q_i} \leq N\chi_D$ *on* $\mathbb{R}^d$.

We are now in the position to state the following result about existence and boundedness of a right inverse to the divergence, which is again taken from [DRS10]. We will use the notation $C^\infty_{c,0}(D)$ for functions $f \in C^\infty_c(D)$ with zero mean value over D.

Theorem 3.2.9 ([DRS10, Theorem 5.2])**.** *Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the emanating chain condition with constants $\sigma_1, \sigma_2 \geq 1$. In particular, by Theorem 3.2.10 below, D may be a John domain. Farther, let $1 < q < \infty$. Then, there is a bounded linear operator*

$$\mathcal{B} : L^q_0(D) \to W^{1,q}_0(D)$$

such that for any $f \in L^q_0(D)$

$$\operatorname{div} \mathcal{B}(f) = f \quad \text{in } D, \quad \|\mathcal{B}(f)\|_{W^{1,q}_0(D)} \leq C \, \|f\|_{L^q(D)},$$

where the constant $C > 0$ just depends on σ_1, σ_2, q, and d. If, in addition, $f \in C^\infty_{c,0}(D)$, then $\mathcal{B}(f) \in C^\infty_c(D)$.

To proof Theorem 3.2.9, we need several decomposition and covering results. The first one states that any John domain satisfies the emanating chain condition.

Theorem 3.2.10 ([DRS10, Theorem 3.8])**.** *Let $D \subset \mathbb{R}^d$ be a bounded $c-$John domain with John center $x_0 \in D$, and let $\{Q_i, i \in \mathbb{N}\}$ be a Whitney covering of D with constants κ_1, κ_2, N, and $x_0 \in Q_0$. Define $\sigma_1 := \frac{4}{3}$, $W_i := \frac{3}{2}\kappa_1 Q_i$, and $\mathcal{W} := \{W_i, i \in \mathbb{N}\}$. Then there exists a constant $\sigma_2 = \sigma_2(\kappa_1, \kappa_2, c, d) \geq 1$ such that D satisfies the emanating chain condition with constants σ_1 and σ_2 and covering $\mathcal{W}$.*

Sketch of the proof. The main idea is to construct for any $i \in \mathbb{N}$ a finite sequence of pairwise different cubes or balls $Q_{i,1}, \ldots, Q_{i,m_i} \in \{Q_i, i \in \mathbb{N}\}$ such that:

- $Q_{i,0} = Q_i$, $Q_{i,m_i} = Q_0$ for $i \in \mathbb{N}$.
- $\overline{\kappa_1 Q_{i,k}} \cap \overline{\kappa_1 Q_{i,k+1}} \neq \emptyset$ for $i \in \mathbb{N}$ and $0 \leq k < m_i$.
- There is a constant $\sigma_2 = \sigma_2(\kappa_1, \kappa_2, c, d) \geq 1$ with $Q_{i,k} \subset \sigma_2 Q_{i,l}$ for any $0 \leq k \leq l \leq m_i$.

The sets $Q_{i,k}$ will be defined inductively, starting with $Q_{i,0} := Q_i$. If we denote by x_i the center of W_i, by the equivalent form (3.10) of the John property (3.9) there is a rectifiable path Γ_i with length ℓ_i joining x_i to x_0 such that

$$\bigcup_{t\in[0,\ell_i]} B(\Gamma_i(t), t/c_1) \subset D.$$

Since the image of Γ_i is a compact subset of D, it only intersects finitely many Q_i, $i \in \mathbb{N}$, from which the chain will be constructed. Assuming that the sets $Q_{i,0}, \ldots, Q_{i,m}$ are already constructed and $Q_{i,m} \neq Q_0$, we set

$$t_{m+1} := \sup\{s : \Gamma_i(s) \in \kappa_1 Q_{i,m}\}.$$

Due to (W1) of Lemma 3.2.8, there exists $Q_{i,m+1} \in \{Q_i, i \in \mathbb{N}\}$ with $\Gamma_i(t_{m+1}) \in \kappa_1 Q_{i,m+1}$ and, by construction, also $\Gamma_i(t_{m+1}) \in \overline{\kappa_1 Q_{i,m}} \cap \overline{\kappa_1 Q_{i,m+1}}$. Furthermore, by definition of the t_k, we have $Q_{i,m+1} \neq Q_{i,k}$ for all $0 \leq k \leq m$. Finally, fixing $0 \leq k \leq m+1$ and denote by $x_{i,k}$ and $x_{i,m+1}$ the centers of $Q_{i,k}$ and $Q_{i,m+1}$, respectively, for fixed $y \in Q_{i,k}$ one may estimate

$$|x_{i,m+1} - y| \leq t_{m+1} + \operatorname{diam}(Q_{i,k}) + \operatorname{diam}(Q_{i,m+1}).$$

Using the John property as well as the properties from the Whitney covering, one can estimate every summand in terms of κ_1, κ_2, c_1, and $\operatorname{diam}(\kappa_1 Q_{i,m+1})$ to obtain

$$|x_{i,m+1} - y| \leq C\,(1 + c_1)(1 + \kappa_2)\operatorname{diam}(\kappa_1 Q_{i,m+1})$$

for some $C = C(d) > 0$, which eventually shows that

$$Q_{i,k} \subset C\,(1 + c_1)(1 + \kappa_2)\,2\,\kappa_1 Q_{i,m+1}.$$

Since the constant on the right is independent of m, one may choose

$$\sigma_2 \coloneqq C(d)\,(1 + c_1)(1 + \kappa_2)\,2\,\kappa_1.$$

Now, it is enough to define for any $i \in \mathbb{N}$ and any $0 \leq k \leq m_i$ the sets $W_{i,k} \coloneqq \frac{3}{2}\kappa_1 Q_{i,k}$ and show that the $W_{i,k}$ fulfill all the conditions of Definition 3.2.7. □

The next step is to show that for a domain satisfying the emanating chain condition with covering $\mathcal{W} = \{W_i, i \in \mathbb{N}\}$, one can decompose a function $f \in L^q_0(D)$ into functions $f_i \in L^q_0(W_i)$. This decomposition result is a generalization of Lemma 3.1.4 to domains satisfying the emanating chain condition rather than domains that are star-shaped.

Theorem 3.2.11 ([DRS10, Theorem 4.2])**.** *Let $D \subset \mathbb{R}^d$ be a bounded domain satisfying the emanating chain condition with constants $\sigma_1, \sigma_2 \geq 1$ and covering $\mathcal{W} = \{W_i, i \in \mathbb{N}\}$. Then there are linear operators $T_i : C^\infty_{c,0}(D) \to C^\infty_{c,0}(W_i)$ such that for all $1 < q < \infty$ we have:*

- *For any $i \in \mathbb{N}$, T_i is continuous from $L^q_0(D)$ to $L^q_0(W_i)$.*
- *For any $i \in \mathbb{N}$ and any $f \in L^q_0(D)$, it holds*

$$|T_i f| \leq C(d)\,\sigma_2 \chi_{W_i} M f \quad \text{almost everywhere,}$$

  where Mf is the Hardy-Littlewood maximal function from Lemma B.6.

- *For any $f \in L^q_0(D)$, it holds $f = \sum_{i\in\mathbb{N}} T_i f$ in $L^q_0(D)$, where the convergence is unconditional.*
- *The map $f \mapsto \|T_i f\|_{L^q_0(W_i)}$ is bounded and obeys the estimate*

$$\frac{1}{C}\,\|f\|_{L^q(D)} \leq \left(\sum_{i\in\mathbb{N}} \|T_i f\|^q_{L^q(W_i)}\right)^{\frac{1}{q}} \leq C\,\|f\|_{L^q(D)},$$

  where $C = C(d, \sigma_1, \sigma_2, q) > 0$.

- *If $f \in C^\infty_{c,0}(D)$, then $\#\{i \in \mathbb{N} : T_i f \neq 0\} < \infty$.*

Sketch of the proof. The proof of the existence of the operators T_i is constructive. First, we assume that $f \in C^\infty_{c,0}(D)$. Since $D \subset \bigcup_{i\in\mathbb{N}} W_i$, one may choose a smooth partition of unity $\{\xi_i\}_{i\in\mathbb{N}}$ with $\operatorname{supp}(\xi_i) \subset W_i$ and define $S_i f := \xi_i f$. It is immediate to see that the operators S_i are linear from $L^1(D)$ to $L^1(W_i)$ and from $C^\infty_c(D)$ to $C^\infty_c(W_i)$, they are bounded since $|S_i f| \leq \chi_{W_i}|f|$, and also $\sum_{i\in\mathbb{N}} S_i f = f$ almost everywhere and in $L^q(D)$ if $f \in L^q(D)$. However, they do not satisfy $\int_D S_i f \,\mathrm{d}x = 0$ in general. To fix this issue, let $W_{i,k}$ and $B_{i,k}$ be as in Definition 3.2.7. One may assume that the balls $B_{i,k}$ stem from a family $\mathfrak{B}$ such that

$$\sum_{B\in\mathfrak{B}} \chi_B \leq \sigma_2 \chi_D \quad \text{on } \mathbb{R}^d .$$

Now, for any $B_{i,k}$ with $0 \leq k < m_i$, let $\eta_{i,k} \in C^\infty_c(B_{i,k})$ be such that $\eta_{i,k} \geq 0$, $\int_{B_{i,k}} \eta_{i,k}\,\mathrm{d}x = 1$, and $\|\eta_{i,k}\|_{L^\infty(\mathbb{R}^d)} \leq C(d)/|B_{i,k}|$. Choose further a function $\eta_{i,m_i} \in C^\infty_c(W_0)$ with $\eta_{i,m_i} \geq 0$, $\int_{W_0} \eta_{i,m_i}\,\mathrm{d}x = 1$, and $\|\eta_{i,m_i}\|_{L^\infty(\mathbb{R}^d)} \leq C(d)/|W_0|$. Finally, one defines the operators T_i as

$$T_i f := S_i f - \eta_{i,0} \int_{W_i} S_i f \,\mathrm{d}x + \sum_{j>0,\, j\neq i} \left(\int_{W_j} S_j f \,\mathrm{d}x \sum_{\substack{k:\, 0<k\leq m_j,\\ W_{j,k}=W_i}} (\eta_{j,k-1} - \eta_{j,k}) \right).$$

Since all cubes (or balls) in a chain are pairwise different, the sum over k consists of at most one summand. However, the sum over j may still be countable. Thus, one has to proof that T_i is well-defined. It turns out that the sum converges almost everywhere absolutely and can be estimated with the help of the maximal function Mf, so $T_i f$ is indeed well-defined for any $f \in C^\infty_{c,0}(D)$ and any $f \in L^q_0(D)$. The fact that $\int_{W_i} T_i f\,\mathrm{d}x = 0$ follows from this convergence and $\int_{\mathbb{R}^d} \eta_{i,k}\,\mathrm{d}x = 1$ for all $i, k \geq 0$ with $0 \leq k \leq m_i$. The estimate for the map $f \mapsto \|T_i f\|_{L^q_0(W_i)}$ finally follows from the fact that the maximal operator M is bounded from $L^q(D)$ to $L^q(D)$ for any $q > 1$ and

$$\begin{aligned}
\left(\sum_{i\in\mathbb{N}} \|T_i f\|^q_{L^q(W_i)} \right)^{\frac{1}{q}} &\leq C \left(\sum_{i\in\mathbb{N}} \|\chi_{W_i} M f\|^q_{L^q(W_i)} \right)^{\frac{1}{q}} \\
&= C \left(\sum_{i\in\mathbb{N}} \int_D \chi_{W_i} |Mf|^q \,\mathrm{d}x \right)^{\frac{1}{q}} \\
&\leq C \,\|Mf\|_{L^q(D)} \\
&\leq C \,\|f\|_{L^q(D)}
\end{aligned}$$

as well as

$$\|f\|_{L^q(D)} = \left\| \sum_{i\in\mathbb{N}} T_i f \right\|_{L^q(D)} \leq C \left(\sum_{i\in\mathbb{N}} \|T_i f\|^q_{L^q(W_i)} \right)^{\frac{1}{q}}.$$

□

With the outcomes from Theorems 3.2.10 and 3.2.11 at hand, we are able to prove the existence of a bounded linear right inverse to the divergence in bounded John domains.

Proof of Theorem 3.2.9. Let $\mathcal{W} = \{W_i, i \in \mathbb{N}\}$ be the covering of D satisfying the properties stated in Definition 3.2.7. Further, let $T_i : L^q_0(D) \to L^q_0(W_i)$ be as in Theorem 3.2.11. By

Lemma 3.1.1, there exists a linear bounded operator $\mathcal{B}_{\text{ref}}$ which is continuous from $C^\infty_{c,0}(B_1(0))$ to $C^\infty_c(B_1(0))$ and also from $L^q_0(B_1(0))$ to $W^{1,q}_0(B_1(0))$. Since we may choose W_i as balls and because of the form of the constant in (3.3), translation and scaling considerations show the existence of linear bounded operators $\mathcal{B}_i : L^q_0(W_i) \to W^{1,q}_0(W_i)$ such that for any $f \in L^q_0(W_i)$,

$$\operatorname{div} \mathcal{B}_i(f) = f \quad \text{in } W_i, \quad \|\mathcal{B}_i(f)\|_{W^{1,q}_0(W_i)} \leq C \, \|f\|_{L^q(W_i)},$$

where the constant $C > 0$ only depends on q and d, but not on i.
Let $f \in L^q_0(D)$ and extend $\mathcal{B}_i T_i f$ outside W_i by zero, which yields $\mathcal{B}_i T_i f \in W^{1,q}_0(D)$. We now define

$$\mathcal{B}(f) := \sum_{i \in \mathbb{N}} \mathcal{B}_i T_i f \text{ almost everywhere in } D.$$

Since $W^{1,q}_0(W_i) \subset W^{1,1}_0(W_i)$, the sum converges in $L^1_{\text{loc}}(D)$ and also in the sense of distributions. The same argument yields $\nabla \mathcal{B}(f) = \sum_{i \in \mathbb{N}} \nabla \mathcal{B}_i T_i f$ in $L^1_{\text{loc}}(D)$.
Together with Theorem 3.2.11 and the estimate on $\mathcal{B}_i$, we obtain

$$\sum_{i \in \mathbb{N}} \|\mathcal{B}_i T_i f\|^q_{L^q(W_i)} \leq C \sum_{i \in \mathbb{N}} \|T_i f\|^q_{L^q(W_i)} \leq C \, \|f\|^q_{L^q(D)}.$$

This and the fact that $\operatorname{supp}(\mathcal{B}_i T_i f) \subset W_i$ yields $\mathcal{B} f = \sum_{i \in \mathbb{N}} \mathcal{B}_i T_i f$ in $W^{1,q}_0(D)$ and also $\nabla \mathcal{B} f = \sum_{i \in \mathbb{N}} \nabla \mathcal{B}_i T_i f$ in $L^q_0(D)$ as well as

$$\|\mathcal{B} f\|^q_{W^{1,q}_0(D)} \leq C \sum_{i \in \mathbb{N}} \|\mathcal{B}_i T_i f\|^q_{L^q(W_i)} \leq C \, \|f\|^q_{L^q(D)}.$$

Since $\nabla \mathcal{B} f = \sum_{i \in \mathbb{N}} \nabla \mathcal{B}_i T_i f$, $\operatorname{div} \mathcal{B}_i(f) = f$ in W_i, and $\sum_{i \in \mathbb{N}} T_i f = f$ in $L^q(D)$, we conclude

$$\operatorname{div} \mathcal{B}(f) = \sum_{i \in \mathbb{N}} \operatorname{div} \mathcal{B}_i T_i f = \sum_{i \in \mathbb{N}} T_i f = f.$$

Finally, let $f \in C^\infty_{c,0}(D)$. Then, by Theorem 3.2.11, we have $T_i f \in C^\infty_c(W_i)$ for all $i \in \mathbb{N}$, and $T_i f \neq 0$ for just finitely many $i \in \mathbb{N}$, which yields by the properties of $\mathcal{B}_i$ that $\mathcal{B}_i T_i f \in C^\infty_c(W_i)$ for any $i \in \mathbb{N}$ and $\mathcal{B}_i T_i f \neq 0$ for only finitely many $i \in \mathbb{N}$. This shows $\mathcal{B} f \in C^\infty_c(D)$ and the proof is complete. □

3.3 Perforated domains: The case of well separated obstacles

We now turn to the case of perforated domains and the existence and boundedness of a right inverse to the divergence in this case. Let us emphasize that the bounds on the Bogovskiĭ operator in the previous sections depend on the domain, whereas in the perforated setting, we want to know the *precise dependence* on the perforation. This uniform boundedness is the main issue in the following two sections, and it will be crucial to obtain the homogenization results in Chapter 4.

Let us start to define the perforated domain for the case of well-separated holes. Since we will apply the results in the sequel for homogenization of the compressible Navier-Stokes equations in $\mathbb{R}^3$, we will focus just on the case $d = 3$ and rely on the results given in [DFL17]. We will not go into a more detailed analysis of boundary regularity, thus assuming that the boundaries of all occurring sets are sufficiently smooth.
Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary. Let $\varepsilon \in (0,1)$, $\alpha \geq 1$, and $\{x_i^\varepsilon\}_{i\in\mathbb{N}} \subset \mathbb{R}^3$ be a collection of points such that $|x_i^\varepsilon - x_k^\varepsilon| \geq 2\varepsilon$ for all $i \neq k$. We then define the perforated domain as

$$D_\varepsilon \coloneqq D \setminus \bigcup_{i\in K_\varepsilon} (\varepsilon^\alpha T + x_i^\varepsilon), \quad K_\varepsilon \coloneqq \{i \in \mathbb{N} : x_i^\varepsilon \in D,\ \operatorname{dist}(x_i^\varepsilon, \partial D) > \varepsilon\},$$

where $T \subset B_{1/2}(0)$ is a simply connected compact set with smooth boundary and $0 \in T$. We call the sets $T_i^\varepsilon \coloneqq (\varepsilon^\alpha T + x_i^\varepsilon)$ *obstacles* or *holes*. We also assume that we just removed those balls from D which are not too close to the boundary in order to avoid boundary issues.
In this section, we will show the following existence theorem:

Theorem 3.3.1 ([DFL17, Theorem 2.3 and Section 3])**.** *Let D_ε be defined as above. Then for all $1 < q < \infty$ there exists a bounded linear operator*

$$\mathcal{B}_\varepsilon : L_0^q(D_\varepsilon) \to W_0^{1,q}(D_\varepsilon)$$

such that

$$\operatorname{div} \mathcal{B}_\varepsilon(f) = f \quad \text{in } D_\varepsilon, \quad \|\mathcal{B}_\varepsilon(f)\|_{W_0^{1,q}(D_\varepsilon)} \leq C \left(1 + \varepsilon^{\frac{(3-q)\alpha - 3}{q}}\right) \|f\|_{L^q(D_\varepsilon)}, \tag{3.11}$$

where the constant $C > 0$ is independent of ε.

Note that for any *fixed* $\varepsilon > 0$, the existence of such an operator is guaranteed by the results obtained in Sections 3.1 and 3.2. However, as mentioned in Remark 3.1.7, the constant in (3.2) depends on the Lipschitz character of D_ε, which becomes unbounded as ε tends towards zero. The crucial point is to derive the explicit ε-dependence on the Bogovskiĭ constant. We will give the full proof here, since it contains many ideas which we will use later in the case of a random perforation.

Proof of Theorem 3.3.1. To start, let $f \in L_0^q(D_\varepsilon)$ and denote by $\tilde{f}$ its zero extension to the whole space, that is,

$$\tilde{f} = f \text{ in } D_\varepsilon, \quad \tilde{f} = 0 \text{ in } \mathbb{R}^3 \setminus D_\varepsilon.$$

Since we assumed D to have a smooth boundary, by Theorem 3.1.6 there is a vector field $\mathbf{u} = \mathcal{B}_D(\tilde{f}) \in W_0^{1,q}(D)$ such that

$$\operatorname{div} \mathbf{u} = \tilde{f} \text{ in } D, \quad \|\mathbf{u}\|_{W_0^{1,q}(D)} \leq C\, \|\tilde{f}\|_{L^q(D)} = C\, \|f\|_{L^q(D_\varepsilon)},$$

where the constant $C > 0$ just depends on D and q. Clearly, $\mathbf{u}$ does in general not satisfy $\mathbf{u} \in W_0^{1,q}(D_\varepsilon)$ since it might not vanish on the holes. We will therefore take a cut-off argument in order to let $\mathbf{u}$ vanish on the holes, however, this will change its divergence. To fix this, we will use local Bogovskiĭ operators around each hole. Estimating the norms of the cut-off

procedure will finally yield the ε-dependence from (3.11).
By the assumption that $|x_i^\varepsilon - x_k^\varepsilon| \geq 2\varepsilon$ for all $i \neq k$, we can enclose each T_i^ε by

$$T_i^\varepsilon \subset B_{2\varepsilon^\alpha}(x_i^\varepsilon) \subset B_{\varepsilon/2}(x_i^\varepsilon) \subset B_\varepsilon(x_i^\varepsilon),$$

where for any $i \neq k$ we have $B_\varepsilon(x_i^\varepsilon) \cap B_\varepsilon(x_k^\varepsilon) = \emptyset$. Now, we consider two cut-off functions, defined by

$$\chi_{\varepsilon,i} \in C_c^\infty\Big(B_\varepsilon(x_i^\varepsilon)\Big), \quad \chi_{\varepsilon,i} \restriction_{B_{\varepsilon/2}(x_i^\varepsilon)} = 1, \quad \|\nabla \chi_{\varepsilon,i}\|_{L^\infty(D)} \leq C\,\varepsilon^{-1}, \tag{3.12}$$

$$\zeta_{\varepsilon,i} \in C_c^\infty\Big(B_{2\varepsilon^\alpha}(x_i)\Big), \quad \zeta_{\varepsilon,i} \restriction_{T_i^\varepsilon} = 1, \quad \|\nabla \zeta_{\varepsilon,i}\|_{L^\infty(B_{2\varepsilon^\alpha}(x_i^\varepsilon))} \leq C\,\varepsilon^{-\alpha}, \tag{3.13}$$

and set

$$D_{\varepsilon,i} \coloneqq B_\varepsilon(x_i^\varepsilon) \setminus \overline{B_{\varepsilon/2}(x_i^\varepsilon)}, \quad E_{\varepsilon,i} \coloneqq B_\varepsilon(x_i^\varepsilon) \setminus T_i^\varepsilon.$$

Defining

$$\begin{aligned} \mathbf{b}_{\varepsilon,i}(\mathbf{u}) &\coloneqq \chi_{\varepsilon,i}(\mathbf{u} - \langle \mathbf{u} \rangle_{D_{\varepsilon,i}}) \in W_0^{1,q}(B_\varepsilon(x_i^\varepsilon)), \\ \beta_{\varepsilon,i}(\mathbf{u}) &\coloneqq \zeta_{\varepsilon,i}\, \langle \mathbf{u} \rangle_{D_{\varepsilon,i}} \in W_0^{1,q}\Big(B_{2\varepsilon^\alpha}(x_i^\varepsilon)\Big), \end{aligned} \tag{3.14}$$

where we denote the mean value of a function $\mathbf{u}$ over a measurable set $S \subset \mathbb{R}^3$ by

$$\langle \mathbf{u} \rangle_S \coloneqq \frac{1}{|S|} \int_S \mathbf{u}\, \mathrm{d}x,$$

Poincaré's inequality (B.6) now implies

$$\|\mathbf{u} - \langle \mathbf{u} \rangle_{D_{\varepsilon,i}}\|_{L^q(D_{\varepsilon,i})} \lesssim \varepsilon\, \|\nabla \mathbf{u}\|_{L^q(D_{\varepsilon,i})},$$

and by (3.12) we get

$$\begin{aligned} \|\nabla \mathbf{b}_{\varepsilon,i}(\mathbf{u})\|_{L^q(D_{\varepsilon,i})} &\leq \|\chi_{\varepsilon,i} \nabla(\mathbf{u} - \langle \mathbf{u} \rangle_{D_{\varepsilon,i}})\|_{L^q(D_{\varepsilon,i})} + \|\nabla \chi_{\varepsilon,i}(\mathbf{u} - \langle \mathbf{u} \rangle_{D_{\varepsilon,i}})\|_{L^q(D_{\varepsilon,i})} \\ &\lesssim \|\nabla(\mathbf{u} - \langle \mathbf{u} \rangle_{D_{\varepsilon,i}})\|_{L^q(D_{\varepsilon,i})} + \varepsilon^{-1}\, \|\mathbf{u} - \langle \mathbf{u} \rangle_{D_{\varepsilon,i}}\|_{L^q(D_{\varepsilon,i})} \\ &\lesssim \|\nabla \mathbf{u}\|_{L^q(D_{\varepsilon,i})}. \end{aligned} \tag{3.15}$$

Similarly, by (3.13) and Hölder's inequality (B.2), we obtain

$$\begin{aligned} \|\nabla \beta_{\varepsilon,i}(\mathbf{u})\|_{L^q(B_{2\varepsilon^\alpha}(x_i^\varepsilon))} &= \|\nabla \zeta_{\varepsilon,i} \cdot \langle \mathbf{u} \rangle_{D_{\varepsilon,i}}\|_{L^q(B_{2\varepsilon^\alpha}(x_i^\varepsilon))} \\ &\lesssim \varepsilon^{\left(\frac{3}{q}-1\right)\alpha}\, |\langle \mathbf{u} \rangle_{D_{\varepsilon,i}}| \lesssim \varepsilon^{\left(\frac{3}{q}-1\right)\alpha}\, |D_{\varepsilon,i}|^{-\frac{1}{q}} \|\mathbf{u}\|_{L^q(D_{\varepsilon,i})} \\ &\lesssim \varepsilon^{\left(\frac{3}{q}-1\right)\alpha - \frac{3}{q}}\, \|\mathbf{u}\|_{L^q(D_{\varepsilon,i})}. \end{aligned} \tag{3.16}$$

Since $\beta_{\varepsilon,i}$ as well as $\mathbf{b}_{\varepsilon,i}$ do not have vanishing divergence, we need to correct them using a local Bogovkiĭ operator on $E_{\varepsilon,i}$, provided $\beta_{\varepsilon,i}$ has been extended by zero outside $B_{2\varepsilon^\alpha}(x_i^\varepsilon)$. By Lemma 3.2.6, $E_{\varepsilon,i}$ is a John domain with a John constant which is independent of ε, so we know from Lemma 3.2.9 that there exists a linear bounded operator $\mathcal{B}_{E_{\varepsilon,i}}$ such that

$$\mathcal{B}_{E_{\varepsilon,i}} : L_0^q(E_{\varepsilon,i}) \to W_0^{1,q}(E_{\varepsilon,i}), \quad \operatorname{div} \mathcal{B}_{E_{\varepsilon,i}}(f) = f, \quad \|\mathcal{B}_{E_{\varepsilon,i}}(f)\|_{W_0^{1,q}(E_{\varepsilon,i})} \le C \|f\|_{L^q(E_{\varepsilon,i})}$$

for all $f \in L_0^q(E_{\varepsilon,i})$, where the constant $C > 0$ is independent of ε and i. We are now ready to define the restriction operator from D to D_ε via

$$R_\varepsilon(\mathbf{u}) := \mathbf{u} - \sum_{i \in K_\varepsilon} \beta_{\varepsilon,i}(\mathbf{u}) + \mathbf{b}_{\varepsilon,i}(\mathbf{u}) - \mathcal{B}_{E_{\varepsilon,i}}\big(\operatorname{div}(\beta_{\varepsilon,i}(\mathbf{u}) + \mathbf{b}_{\varepsilon,i}(\mathbf{u}))\big), \tag{3.17}$$

provided $\mathcal{B}_{E_{\varepsilon,i}}\big(\operatorname{div}(\beta_{\varepsilon,i}(\mathbf{u}) + \mathbf{b}_{\varepsilon,i}(\mathbf{u}))\big)$ is extended to be zero outside its domain of definition. Repeating the arguments shown in [DFL17, Section 3], we check that the operator R_ε is well defined and satisfies the desired norm bounds. First, by the definitions of $\mathbf{b}_{\varepsilon,i}$ and $\beta_{\varepsilon,i}$ in (3.14), we have

$$\begin{aligned}
\int_{B_\varepsilon(x_i^\varepsilon)} \operatorname{div}(\mathbf{b}_{\varepsilon,i}(\mathbf{u}) + \beta_{\varepsilon,i}(\mathbf{u}))\,\mathrm{d}x &= \int_{B_\varepsilon(x_i^\varepsilon)} \operatorname{div} \mathbf{b}_{\varepsilon,i}(\mathbf{u})\,\mathrm{d}x + \int_{B_{2\varepsilon^\alpha}(x_i^\varepsilon)} \operatorname{div} \beta_{\varepsilon,i}(\mathbf{u})\,\mathrm{d}x \\
&= \int_{\partial B_\varepsilon(x_i^\varepsilon)} \mathbf{b}_{\varepsilon,i}(\mathbf{u}) \cdot \mathbf{n}\,\mathrm{d}\sigma(x) + \int_{\partial B_{2\varepsilon^\alpha}(x_i^\varepsilon)} \beta_{\varepsilon,i}(\mathbf{u}) \cdot \mathbf{n}\,\mathrm{d}\sigma(x) = 0.
\end{aligned}$$

On the other hand, $\chi_{\varepsilon,i} = \zeta_{\varepsilon,i} = 1$ and $\operatorname{div} \mathbf{u} = \tilde{f} = 0$ inside T_i^ε, thus

$$\begin{aligned}
&\operatorname{div} \mathbf{b}_{\varepsilon,i}(\mathbf{u}) = \chi_{\varepsilon,i} \operatorname{div} \mathbf{u} + \nabla \chi_{\varepsilon,i} \cdot (\mathbf{u} - \langle \mathbf{u} \rangle_{D_{\varepsilon,i}}) = 0 && \text{in } T_i^\varepsilon, \\
&\operatorname{div} \beta_{\varepsilon,i}(\mathbf{u}) = \nabla \zeta_{\varepsilon,i} \cdot (\mathbf{u} - \langle \mathbf{u} \rangle_{D_{\varepsilon,i}}) = 0 && \text{in } T_i^\varepsilon,
\end{aligned}$$

leading to

$$\int_{E_{\varepsilon,i}} \operatorname{div}(\mathbf{b}_{\varepsilon,i}(\mathbf{u}) + \beta_{\varepsilon,i}(\mathbf{u}))\,\mathrm{d}x = 0$$

as required. Next, for any $x \in T_i^\varepsilon$, we get

$$\begin{aligned}
R_\varepsilon(\mathbf{u})(x) &= \mathbf{u}(x) - \big(\mathbf{b}_{\varepsilon,i}(\mathbf{u}) + \beta_{\varepsilon,i}(\mathbf{u})\big)(x) - \mathcal{B}_{E_{\varepsilon,i}}\big(\operatorname{div}(\mathbf{b}_{\varepsilon,i}(\mathbf{u}) + \beta_{\varepsilon,i}(\mathbf{u}))\big)(x) \\
&= \mathbf{u}(x) - \beta_{\varepsilon,i}(\mathbf{u})(x) - \mathbf{b}_{\varepsilon,i}(\mathbf{u})(x) \\
&= \mathbf{u}(x) - \zeta_{\varepsilon,i}(x)\langle \mathbf{u} \rangle_{D_{\varepsilon,i}} - \chi_{\varepsilon,i}(x)(\mathbf{u}(x) - \langle \mathbf{u} \rangle_{D_{\varepsilon,i}}) \\
&= 0,
\end{aligned}$$

where in the last equality we used that $\chi_{\varepsilon,i}(x) = \zeta_{\varepsilon,i}(x) = 1$ in T_i^ε. This yields that the operator R_ε is well defined and satisfies

$$R_\varepsilon(\mathbf{u}) \in W_0^{1,q}(D_\varepsilon), \quad \operatorname{div} R_\varepsilon(\mathbf{u}) = \operatorname{div} \mathbf{u} = f \text{ in } D_\varepsilon.$$

Finally, by (3.15), (3.16), and the fact that the balls $B_\varepsilon(x_i^\varepsilon)$ are disjoint, we see that

$$\|R_\varepsilon(\mathbf{u})\|_{W_0^{1,q}(D_\varepsilon)} \le C \left(\varepsilon^{(\frac{3}{q}-1)\alpha - \frac{3}{q}} + 1 \right) \|\mathbf{u}\|_{W_0^{1,q}(D)},$$

where the constant $C > 0$ is independent of $\varepsilon > 0$. For $f \in L_0^q(D_\varepsilon)$ we define

$$\mathcal{B}_\varepsilon(f) := (R_\varepsilon \circ \mathcal{B}_D)(\tilde{f})$$

and observe that we get the desired operator, namely $\mathcal{B}_\varepsilon(f) \in W_0^{1,q}(D_\varepsilon)$,

$$\operatorname{div} \mathcal{B}_\varepsilon(f) = f \text{ in } D_\varepsilon, \quad \text{and} \quad \|\mathcal{B}_\varepsilon(f)\|_{W_0^{1,q}(D_\varepsilon)} \leq C\left(1 + \varepsilon^{\frac{(3-q)\alpha-3}{q}}\right)\|f\|_{L^q(D_\varepsilon)}.$$

This finishes the proof. □

3.4 Perforated domains: The case of a random perforation

In this section, we will generalize Theorem 3.3.1 to the case of randomly placed holes which additionally have random radii. Note that we assumed in Section 3.3 the centers x_i^ε fulfill $|x_i^\varepsilon - x_k^\varepsilon| \geq 2\varepsilon$. We drop this assumption and show, under suitable conditions on the radii of the holes, that the centers for a random perforation rather satisfy $|x_i^\varepsilon - x_k^\varepsilon| \geq C\,\varepsilon^{2^+}$, where we indicate with 2^+ any number that is greater than 2. Before defining properly the perforated domain, let us first state an important observation taken from the estimate (3.11):
The operator $\mathcal{B}_\varepsilon$ is uniformly bounded as long as $(3-q)\alpha - 3 \geq 0$, in particular, we need $1 < q < 3$ and $\alpha \geq 3/(3-q)$.
These conditions will occur in a slightly different form later in Theorem 3.4.1, and are optimal in the sense of capacity, see also Remark 3.4.8 below. Since we will use the Bogovskiĭ operator in the homogenization of compressible Navier-Stokes equations to bound the density of the fluid independent of ε, we need that it is uniformly bounded. The condition $\alpha \geq 3/(3-q)$ will play a crucial role there.

Let us begin with the description of the random distributed holes. Let $\Phi = \{z_i\}_{i\in\mathbb{N}}$ be a random collection of points in $\mathbb{R}^3$, and denote for any bounded measurable set $S \subset \mathbb{R}^3$ the number of points inside S by $N(S)$. We assume that the points are distributed according to a Poisson process with intensity rate $\lambda > 0$, which is characterized by the following two properties:

1. For any $k \in \mathbb{N}$, it holds $\mathbb{P}(N(S) = k) = \frac{(\lambda|S|)^k}{k!}e^{-\lambda|S|}$.

2. For any two measurable and *disjoint* sets S_1 and S_2, the random sets $\Phi \cap S_1$ and $\Phi \cap S_2$ are independent.

Furthermore, we associate to every point $z_i \in \Phi$ a random radius. For that, let $\mathcal{R} = \{r_i\}_{z_i\in\Phi}$ be another random process of independent identically distributed random variables $r_i \in (0,\infty)$ with finite m-th moment, i.e.,

$$\mathbb{E}(r_i^m) < \infty \text{ for some } m > 0,$$

and which are independent of Φ. The couple $(\Phi, \mathcal{R}) \subset \mathbb{R}^3 \times \mathbb{R}_+$ is called a *marked Poisson point process* and can be seen as a random variable $\Omega \ni \omega \mapsto (\Phi(\omega), \mathcal{R}(\omega))$, defined on an

abstract probability space $(\Omega, \mathfrak{A}, \mathbb{P})$. We will omit the dependence on ω in the sequel if no ambiguity occurs. The exact range of m we can work with will be specified later.
Let $D \subset \mathbb{R}^3$ be a bounded domain with smooth boundary which is star-shaped with respect to the origin, that is, for any $x \in D$, the segment $\{tx : t \in [0,1]\} \subset D$. We define for $\alpha > 2$ and $\varepsilon > 0$ the perforated domain D_ε as

$$\Phi^\varepsilon(D) := \{z \in \Phi \cap \frac{1}{\varepsilon}D : \operatorname{dist}(\varepsilon z, \partial D) > \varepsilon\}, \quad D_\varepsilon := D \setminus \bigcup_{z_i \in \Phi^\varepsilon(D)} \overline{B_{\varepsilon^\alpha r_i}(\varepsilon z_i)}. \tag{3.18}$$

The assumption of star-shapedness of D can be dropped, however, it ensures that the sets $\Phi^\varepsilon(D)$ are monotonically increasing as $\varepsilon \to 0$, thus simplifying some arguments of the proofs. Moreover, the constraint $\operatorname{dist}(\varepsilon z, \partial D) > \varepsilon$ in the definition of $\Phi^\varepsilon(D)$ prevents us from considering boundary issues in the homogenization process later on. It will also ensure that boxes around holes are well inside D, see Theorem 3.4.2 below.

Our main result in this section reads as follows:

Theorem 3.4.1. *Let $\alpha > 2$, $D \subset \mathbb{R}^3$ be a bounded star-shaped domain with respect to the origin with smooth boundary, and $(\Phi, \mathcal{R}) = (\{z_j\}, \{r_j\})$ be a marked Poisson point process with intensity $\lambda > 0$. We assume the radii $r_j > 0$ fulfill $\mathbb{E}(r_j^m) < \infty$ for some $m > 3/(\alpha - 2)$. Then for all $1 < q < 3$ which satisfy*

$$\alpha - \frac{3}{m} > \frac{3}{3 - q} \tag{3.19}$$

there exists a random almost surely strictly positive $\varepsilon_0 = \varepsilon_0(\omega)$ such that for $0 < \varepsilon \leq \varepsilon_0$ there exists a bounded linear operator

$$\mathcal{B}_\varepsilon : L_0^q(D_\varepsilon) \to W_0^{1,q}(D_\varepsilon; \mathbb{R}^3)$$

with D_ε defined in (3.18), such that for all $f \in L_0^q(D_\varepsilon)$

$$\operatorname{div} \mathcal{B}_\varepsilon(f) = f \ \text{in}\ D_\varepsilon, \quad \|\mathcal{B}_\varepsilon(f)\|_{W_0^{1,q}(D_\varepsilon)} \leq C \, \|f\|_{L^q(D_\varepsilon)},$$

where the deterministic constant $C > 0$ is independent of ε.

To show this result, we will need some preliminaries. The first one states that for small but positive ε, the holes are well separated and that we may group them to clusters which cannot have too many elements.

Theorem 3.4.2. *Let $\alpha > 2$ and $\lambda > 0$ be the intensity of a marked Poisson point process $(\Phi, \mathcal{R}) = (\{z_j\}, \{r_j\})$ with $r_j > 0$ and $\mathbb{E}(r_j^m) < \infty$, where $m > 0$ satisfies*

$$m > \frac{3}{\alpha - 2}.$$

Let $0 < \delta < \alpha - 1 - \frac{3}{m}$, $\kappa \in (\max(1, \delta), \alpha - 1 - \frac{3}{m})$, and $\tau \geq 1$. Then there exists a random variable $\varepsilon_0(\omega)$, which is almost surely strictly positive, satisfying:

1. *For every $0 < \varepsilon \leq \varepsilon_0$ holds*

$$\max_{z_i \in \Phi^\varepsilon(D)} \tau \varepsilon^\alpha r_i \leq \varepsilon^{1+\kappa},$$

and for every $z_i, z_j \in \Phi^\varepsilon(D)$ with $z_i \neq z_j$

$$B_{\tau\varepsilon^{1+\kappa}}(\varepsilon z_i) \cap B_{\tau\varepsilon^{1+\kappa}}(\varepsilon z_j) = \emptyset.$$

2. *Let*

$$N := N(\delta) := 8\left(1 + \left\lceil \frac{1}{\delta} \right\rceil\right). \tag{3.20}$$

Then for each $0 < \varepsilon \leq \varepsilon_0$ there are finitely many open rectangular cuboids $\{I_i^\varepsilon\} \subset D$, having edges parallel to the coordinates axes and which we simply call boxes, satisfying:

(a) *The boxes I_i^ε cover the balls, i.e., for any $z \in \Phi^\varepsilon(D)$ we have $B_{\varepsilon^{1+\kappa}}(\varepsilon z) \subset \bigcup_i I_i^\varepsilon$.*

(b) *Any box I_i^ε contains at most N points from $\varepsilon\Phi^\varepsilon(D)$.*

(c) *Balls are well inside the box: for $\varepsilon z \in I_i^\varepsilon \cap \varepsilon\Phi^\varepsilon(D)$ holds $\operatorname{dist}(B_{\varepsilon^{1+\kappa}}(\varepsilon z), \partial I_i^\varepsilon) \geq \frac{1}{16N}\varepsilon^{1+\delta}$.*

(d) *Any two distinct boxes I_i^ε and I_j^ε are well separated: $\operatorname{dist}_\infty(I_i^\varepsilon, I_j^\varepsilon) \geq \frac{1}{4N}\varepsilon^{1+\delta}$.*

(e) *The shortest side of I_i^ε is at least $\frac{1}{2N}\varepsilon^{1+\delta}$ while the longest side is at most $\varepsilon^{1+\delta}$.*

The condition $\alpha > 2$ ensures that the interval for κ is not empty. The proof of the second part of Theorem 3.4.2 uses that for $0 < \varepsilon \leq \varepsilon_0$ any cube with side length $\varepsilon^{1+\delta}$ contains at most N points from the Poisson point process. This can hold only if $\delta > 0$, since the number of points in a cube of size ε^{1+0} is Poisson distributed, i.e., any number of points appears there with small but positive probability.

Proposition 3.4.3. *Let $d \geq 1$, $\delta > 0$ be fixed, and let $\{z_j\} \subset \mathbb{R}^d$ be points generated by a Poisson point process of intensity $\lambda > 0$. In addition, let $D \subset \mathbb{R}^d$ be a bounded star-shaped domain. Then there exists a deterministic constant $N(\delta, d) \in \mathbb{N}$ and a random variable $\varepsilon_0(\omega, \lambda, D)$, which is almost surely positive, such that for all $0 < \varepsilon \leq \varepsilon_0$ and any $x \in \mathbb{R}^d$ the cube $x + [0, \varepsilon^{1+\delta}]^d$ contains at most N points from $D \cap \varepsilon\Phi$.*

To construct the Bogovskiĭ operator $\mathcal{B}_\varepsilon$ in D_ε from Theorem 3.4.1 we use local Bogovskiĭ operators for each box I_i^ε to modify the Bogovskiĭ operator in D. Instead of making explicit construction in each box I_i^ε, we invoke the result on the existence of Bogovskiĭ's operator for John domains (see Theorem 3.2.9) and show that each box I_i^ε minus the balls is a John domain – for this the outcomes of Theorem 3.4.2 will be crucial. In particular, we need that one box contains at most N balls, the balls are not close to each other, and they are tiny, compared to the size of the box. The following lemma states that any perforated I_i^ε is a $c-$John domain.

Lemma 3.4.4. *Under the assumptions of Theorem 3.4.2 for fixed*

$$0 < \delta < \frac{\alpha - 2 - \frac{3}{m}}{2}, \tag{3.21}$$

let $0 < \varepsilon \leq \varepsilon_0$. Then for every box I_i^ε constructed in Theorem 3.4.2, the domain

$$U := I_i^\varepsilon \setminus \bigcup_{z_j \in \varepsilon^{-1} I_i^\varepsilon \cap \Phi^\varepsilon(D)} \overline{B_{\varepsilon^\alpha r_j}(\varepsilon z_j)} \tag{3.22}$$

is a $c-$John domain, where $c = c(N)$ is independent of ε, and N is defined in (3.20).

In particular, for any $1 < q < \infty$ there exists a uniformly bounded Bogovkiĭ operator $\mathcal{B}_U : L_0^q(U) \to W_0^{1,q}(U)$, i.e., there exists a constant $C > 0$, independent of ε, such that for any $f \in L_0^q(U)$

$$\operatorname{div} \mathcal{B}_U(f) = f, \quad \|\mathcal{B}_U(f)\|_{W_0^{1,q}(U)} \le C \, \|f\|_{L_0^q(U)}.$$

3.4.1 Proofs of Theorem 3.4.2 and Proposition 3.4.3

The goal of this section is to prove Theorem 3.4.2, the second part of which is based on Proposition 3.4.3 about the distribution of the random points, modeled by the Poisson point process. Fixing $\delta > 0$, this proposition states that for ε small enough, for any cube of side length $\varepsilon^{1+\delta}$ inside a fixed domain D there are at most $N = N(\delta, d)$ of the rescaled points εz in the cube. The heuristic explanation of this is as follows: assuming we only need to consider a disjoint set of cubes and fixing $\varepsilon > 0$, the number of cubes in D which we have to consider scales like $\varepsilon^{-(1+\delta)d}$. At the same time, the probability of one cube of side length $\varepsilon^{1+\delta}$ having more than N points scales in the case of the Poisson point process like $(\frac{\varepsilon^{(1+\delta)d}}{\varepsilon^d})^N = \varepsilon^{\delta N d}$. Hence, choosing N large enough so that $\varepsilon^{-(1+\delta)d} \, \varepsilon^{\delta N d} \ll 1$ should lead to the result.

Proof of Proposition 3.4.3. We start with a special case, which will be later used to prove the general case:

Claim: There exists $N_1 \in \mathbb{N}$ and an a.s. positive random variable $\varepsilon_0(\omega)$ such that for any dyadic $\varepsilon = 2^{-l}$ smaller than ε_0, any half-closed cube $Q_{\varepsilon,z} = \varepsilon^{1+\delta} z + [0, \varepsilon^{1+\delta})^d$, $z \in \mathbb{Z}^d$, contains at most N_1 points from $\frac{\varepsilon}{2}\Phi(\omega) \cap D$.

If rescaled by a factor 2 the claim says that in a cube with side length $(2\varepsilon)^{1+\delta}$ there are at most N_1 points, and we are considering points (more precisely cubes) inside $2D$ instead of D only. The reason for this choice will be clear later in the proof.

For $l \in \mathbb{N}$ and $\varepsilon = 2^{-l}$, we define

$$\begin{aligned} B_l := \{\omega \in \Omega : & \text{ one of the dyadic cubes } Q_{2^{-l},z} \text{ contains} \\ & \text{at least } N_1 \text{ points from } 2^{-l-1}\Phi \cap D\}. \end{aligned}$$

In order to prove the claim, it is enough to show $\sum_{l \ge 0} \mathbb{P}(B_l) < \infty$ and apply the Borel-Cantelli Lemma C.1. Recall that for any measurable bounded set $S \subset \mathbb{R}^d$, we denote by $N(S) = \#(S \cap \Phi)$ the number of random points in S. First, by rescaling, we see that

$$B_l = \{\omega \in \Omega : \text{there exists a dyadic cube } Q_{2^{-l},z} \text{ such that } N(2^{l+1}(Q_{2^{-l},z} \cap D)) \ge N_1\}.$$

Since we can cover D with at most $C\,|D|\,(2^l)^d$ cubes $Q_{2^{-l},z}$ and due to the stationarity of the process Φ, we estimate

$$\begin{aligned} \mathbb{P}(B_l) &\le C(D)\, 2^{ld}\, \mathbb{P}\big(N(2^{l+1} Q_{2^{-l},0}) \ge N_1\big) = C(D)\, 2^{ld}\, \mathbb{P}\big(N(2^{l+1} \cdot 2^{-l(1+\delta)}[0,1)^d) \ge N_1\big) \\ &\le C(d,\lambda,D)\, 2^{ld} \cdot 2^{(1-l\delta)N_1 d} = C(d,\lambda,D)\, 2^{ld(1-N_1\delta)}, \end{aligned} \tag{3.23}$$

where in the last inequality we used that the points in Φ are Poisson-distributed, i.e.,

$$\mathbb{P}(N(S) = n) = e^{-\lambda|S|} \frac{(\lambda|S|)^n}{n!}$$

for any $n \in \mathbb{N}$, that for any $x > 0$ we have

$$e^{-x} \sum_{k \geq n} \frac{x^k}{k!} = \frac{x^n}{n!} e^{-x} \sum_{k \geq 0} \frac{x^k n!}{(n+k)!} \leq \frac{x^n}{n!} e^{-x} \sum_{k \geq 0} \frac{x^k}{k!} = \frac{x^n}{n!} \tag{3.24}$$

since $1 \leq \binom{n+k}{k} = \frac{(n+k)!}{k!\,n!}$ for all $n, k \in \mathbb{N}$, and that $\frac{\lambda^n}{n!} \leq C(\lambda)$ for any $\lambda > 0$ and any $n \geq 0$. Choosing now $N_1 = 1 + \lceil \frac{1}{\delta} \rceil$ in (3.23), we have $\mathbb{P}(B_l) \leq C\, 2^{-ql}$ for some $q > 0$, meaning that $\sum_{l \geq 0} \mathbb{P}(B_l) \leq C \sum_{l \geq 0} 2^{-ql} < \infty$. The Borel-Cantelli Lemma C.1 now implies

$$\mathbb{P}\big(\limsup_{l \to \infty} B_l\big) = 0,$$

meaning that almost surely there is an $\varepsilon_0(\omega) > 0$ such that for all $0 < \varepsilon = 2^{-l} \leq \varepsilon_0$, any cube $Q_{2^{-l},z}$ contains not more than N_1 points from $\frac{\varepsilon}{2}\Phi \cap D$, thus proving the claim.

To show the general case, for $\omega \in \Omega$ we consider $\varepsilon_0(\omega)$ coming from the claim. Without loss of generality we assume $\varepsilon_0 = 2^{-l_0}$ for some $l_0 \in \mathbb{N}$ (otherwise replace ε_0 with the largest smaller power of 2). To finish the proof we need to show that for any $0 < \varepsilon \leq \varepsilon_0$, any cube $Q_\varepsilon = x + [0, \varepsilon^{1+\delta}]^d$, $x \in \mathbb{R}^d$, contains at most N points from $D \cap \varepsilon\Phi(\omega)$. Let $0 < \varepsilon \leq \varepsilon_0$ and $Q_\varepsilon = x + [0, \varepsilon^{1+\delta}]^d$ be chosen arbitrary, and let $N := 2^d N_1$. Let $l \geq l_0$ be the unique l such that $2^{-(l+1)} < \varepsilon \leq 2^{-l}$.

Observe that for $\varpi > 0$ we have $\#(Q_\varepsilon \cap \varepsilon\Phi) = \#(\varpi Q_\varepsilon \cap \varpi\varepsilon\Phi)$, where $\varpi Q_\varepsilon = \{x + \varpi(y - x) : y \in Q_\varepsilon\}$, which together with star-shapedness of D yields for $\varpi = \frac{2^{-(l+1)}}{\varepsilon} \in (0, 1)$

$$\#(Q_\varepsilon \cap \varepsilon\Phi \cap D) = \#(\varpi Q_\varepsilon \cap \varpi\varepsilon\Phi \cap \varpi D) \leq \#(\varpi Q_\varepsilon \cap 2^{-(l+1)}\Phi \cap D).$$

We now cover ϖQ_ε with (at most) 2^d cubes $Q_{2^{-l},z}$. Observe that even if ϖQ_ε is closed and $Q_{2^{-l},z}$ are only half-closed, the covering is possible since $\varpi\varepsilon = 2^{-(l+1)} < 2^{-l}$. In particular, the claim implies that any $Q_{2^{-l},z}$ contains at most N_1 points from $\frac{2^{-l}}{2}\Phi \cap D$, thus implying that ϖQ_ε, being covered by at most 2^d cubes $Q_{2^{-l},z}$, contains at most $2^d N_1$ points from $2^{-(l+1)}\Phi \cap D$. This together with the last display implies $\#(Q_\varepsilon \cap \varepsilon\Phi \cap D) \leq 2^d N_1 = N$, thus concluding the proof of the proposition. □

We now turn to the proof of Theorem 3.4.2, the first part of which is based on the following Strong Law of Large Numbers (see Theorem C.2).

Lemma 3.4.5. *Let $d \geq 1$ and $(\Phi, \mathcal{R}) = (\{z_j\}, \{r_j\})$ be a marked Poisson point process with intensity $\lambda > 0$. Assume that the marks $\{r_j\}$ are positive i.i.d. random variables independent of Φ such that $\mathbb{E}(r_j^m) < \infty$ for some $m > 0$. Then, for every bounded measurable set $S \subset \mathbb{R}^d$ which is star-shaped with respect to the origin, we have almost surely*

$$\lim_{\varepsilon \to 0} \varepsilon^d N(\varepsilon^{-1}S) = \lambda|S|, \qquad \lim_{\varepsilon \to 0} \varepsilon^d \sum_{z_j \in \varepsilon^{-1}S} r_j^m = \lambda\mathbb{E}(r^m)|S|.$$

Remark 3.4.6. *Assuming the boundary of the set S from the previous lemma is not too large, the same argument also shows*

$$\lim_{\varepsilon \to 0} \varepsilon^d \sum_{z_j \in \Phi^\varepsilon(S)} r_j^m = \lambda\mathbb{E}(r^m)|S|. \tag{3.25}$$

In particular, it is enough that S has a C^2-boundary.

With Lemma 3.4.5, we obtain for the domain D_ε and for $\varepsilon > 0$ small enough

$$\begin{aligned} |\partial D_\varepsilon| &= |\partial D| + \Big| \bigcup_{z_i \in \Phi^\varepsilon(D)} \partial B_{\varepsilon^\alpha r_i}(\varepsilon z_i) \Big| \le C + C\,\varepsilon^{2\alpha-3}\varepsilon^3 \sum_{z_i\in\Phi^\varepsilon(D)} r_i^2 \le C, \\ |D \setminus D_\varepsilon| &= \Big| \bigcup_{z_i \in \Phi^\varepsilon(D)} B_{\varepsilon^\alpha r_i}(\varepsilon z_i) \Big| \le C\,\varepsilon^{3(\alpha-1)}\varepsilon^3 \sum_{z_i\in\Phi^\varepsilon(D)} r_i^3 \le C\,\varepsilon^{3(\alpha-1)}, \end{aligned} \tag{3.26}$$

which implies $|D_\varepsilon| \to |D|$ as $\varepsilon \to 0$. Thus, for ε possibly even smaller, we can control the measure of D_ε by $\frac{1}{2}|D| \le |D_\varepsilon| \le 2\,|D|$.

Using Remark 3.4.6 as well as Proposition 3.4.3 we can prove Theorem 3.4.2.

Proof of Theorem 3.4.2. **Part (1):** We start with the first part of the theorem, which actually holds for any dimension $d \ge 1$, $\alpha > 2$, $m > \frac{d}{\alpha-2}$, and $\kappa \in (1, \alpha - 1 - \frac{d}{m})$. We follow the lines of [GH19, Proof of Lemma 6.1].

Using (3.25) and the choice of κ, we have for almost all ω

$$\limsup_{\varepsilon\to 0} \varepsilon^{\frac{d}{m}} \max_{z_i\in\Phi^\varepsilon(D)} r_i \le \limsup_{\varepsilon\to 0} \varepsilon^{\frac{d}{m}} \Big(\sum_{z_i\in\Phi^\varepsilon(D)} r_i^m \Big)^{\frac{1}{m}} \le \big[\lambda \mathbb{E}(r^m)|D|\big]^{\frac{1}{m}}.$$

This implies for $\varepsilon > 0$ small enough

$$\max_{z_i\in\Phi^\varepsilon(D)} \tau\varepsilon^\alpha r_i \le 2\tau\varepsilon^{\alpha-\frac{d}{m}} \big[\lambda \mathbb{E}(r^m)|D|\big]^{\frac{1}{m}} \le \varepsilon^{1+\kappa}, \tag{3.27}$$

the last inequality coming from $\alpha - \frac{d}{m} > \kappa + 1$, and therefore being true for ε being possibly even smaller.

To show that two balls do not intersect we consider an event

$$A_\tau^\varepsilon := \{\omega \in \Omega :\ \text{there are 2 intersecting balls in } \{B_{\tau\varepsilon^{1+\kappa}}(\varepsilon z)\}_{z\in\Phi^\varepsilon(D)}\},$$

and it is enough to show

$$\mathbb{P}\Big(\bigcap_{\varepsilon_0 > 0} \bigcup_{\varepsilon \le \varepsilon_0} A_\tau^\varepsilon \Big) = 0. \tag{3.28}$$

We reduce this to the case of dyadic ε, by showing

$$\mathbb{P}\Big(\bigcap_{l_0 \ge 1} \bigcup_{l \ge l_0} A_{\bar\tau}^{\varepsilon_l} \Big) = 0, \tag{3.29}$$

where $\varepsilon_l = 2^{-l}$ and $\bar\tau = 2^{1+\kappa}\tau$.

Indeed, let $l \in \mathbb{N}$ be such that $\varepsilon_{l+1} \le \varepsilon < \varepsilon_l$. Now suppose $z_i, z_j \in \Phi^\varepsilon(D)$, $z_i \ne z_j$ such that

$$B_{\tau\varepsilon^{1+\kappa}}(\varepsilon z_i) \cap B_{\tau\varepsilon^{1+\kappa}}(\varepsilon z_j) \ne \emptyset.$$

Then

$$\varepsilon_{l+1}|z_i - z_j| \le \varepsilon|z_i - z_j| \le 2\tau\varepsilon^{1+\kappa} \le 2\tau\varepsilon_l^{1+\kappa} = 2\tau(2\varepsilon_{l+1})^{1+\kappa} = 2\cdot 2^{1+\kappa}\tau\varepsilon_{l+1}^{1+\kappa},$$

which means that

$$B_{2^{1+\kappa}\tau\varepsilon_{l+1}^{1+\kappa}}(\varepsilon_{l+1}z_i) \cap B_{2^{1+\kappa}\tau\varepsilon_{l+1}^{1+\kappa}}(\varepsilon_{l+1}z_j) \neq \emptyset.$$

The domain D being star-shaped implies monotonicity of $\Phi^\varepsilon(D)$ in ε, in particular $\Phi^\varepsilon(D) \subset \Phi^{\varepsilon_{l+1}}(D)$, which combined with the previous display yields

$$A_\tau^\varepsilon \subset A_{\bar{\tau}}^{\varepsilon_{l+1}},$$

thus showing that (3.29) implies (3.28).

It remains to show (3.29). Let $\varepsilon > 0$ and $\tau \ge 1$ be fixed. Observe that if for $z_i, z_j \in \Phi^\varepsilon(D)$ we have $B_{\tau\varepsilon^{1+\kappa}}(\varepsilon z_i) \cap B_{\tau\varepsilon^{1+\kappa}}(\varepsilon z_j) \neq \emptyset$, then $\varepsilon|z_i - z_j| \le 2\tau\varepsilon^{1+\kappa}$ and after simplifying $|z_i - z_j| \le 2\tau\varepsilon^\kappa$, in other words

$$A_\tau^\varepsilon \subset \{\omega \in \Omega : \exists x \in \frac{1}{\varepsilon}D : \#(\Phi^\varepsilon(D) \cap B_{2\tau\varepsilon^\kappa}(x)) \ge 2\}. \tag{3.30}$$

Recall that for $S \subset \mathbb{R}^d$, we denote by $N(S) = \#(S \cap \Phi)$ the random variable providing the number of points of the process which lie inside S. Let us also note that the points are distributed according to a Poisson distribution with intensity $\lambda > 0$. We now recall a basic estimate from [GH19, Proof of Lemma 6.1]: for $0 < \eta < 1$, define the set of cubes with side length η centered at the grid $\eta\mathbb{Z}^d$ by

$$\mathcal{Q}_\eta := \{y + [-\eta/2, \eta/2]^d : y \in \eta\mathbb{Z}^d\}.$$

Since it is not true that any ball of radius $\frac{\eta}{4}$ is contained in one of these cubes, we need to add (finitely many) shifted copies of $\mathcal{Q}_\eta$. For that let S_η be the vertices of the cube $[0, \eta/2]^d$, i.e.,

$$S_\eta = \{z = (z_1, \dots, z_d) \in \mathbb{R}^d : z_k \in \{0, \eta/2\} \text{ for } k = 1, \dots, d\}.$$

Observe that for any $x \in \mathbb{R}^d$, there exist $z \in S_\eta$ and a cube $Q \in \mathcal{Q}_\eta$ such that $B_{\frac{\eta}{4}}(x) \subset z + Q$, which immediately implies

$$\begin{aligned}\mathbb{P}(\exists x \in \frac{1}{\varepsilon}D : N(B_{\frac{\eta}{4}}(x)) \ge 2)\\ \le \mathbb{P}(\exists Q \in \mathcal{Q}_\eta, z \in S_\eta : (z + Q) \cap \frac{1}{\varepsilon}D \neq \emptyset, N(z + Q) \ge 2).\end{aligned}$$

Since S_η has 2^d elements and the number of cubes $Q \in \mathcal{Q}_\eta$ that intersect $\varepsilon^{-1}D$ is bounded by $C(D)(\varepsilon\eta)^{-d}$, we use the distribution of the Poisson point process to conclude

$$\begin{aligned}\mathbb{P}(\exists x \in \frac{1}{\varepsilon}D : N(B_{\frac{\eta}{4}}(x)) \ge 2) &\le \sum_{z \in S_\eta}\sum_Q \mathbb{P}(N(z + Q) \ge 2)\\ &\le 2^d C(D)(\varepsilon\eta)^{-d} e^{-\lambda\eta^d}\sum_{k=2}^\infty \frac{(\lambda\eta^d)^k}{k!} \le C(d, D)\,(\varepsilon\eta)^{-d}(\lambda\eta^d)^2,\end{aligned}$$

where the last inequality follows from (3.24). Letting $\eta_\varepsilon := 8\tau\varepsilon^\kappa$, this together with (3.30) and the fact that $\#(\Phi^\varepsilon(D) \cap S) \le N(S)$ for any $S \subset \mathbb{R}^d$, yields

$$\mathbb{P}(A^\varepsilon_\tau) \le C\,(\varepsilon^{1+\kappa})^{-d}\varepsilon^{2d\kappa} = C\,\varepsilon^{d(\kappa-1)}.$$

To show (3.29) we take a sum over l with $\varepsilon = \varepsilon_l = 2^{-l}$, which using $\kappa > 1$ can be estimated as

$$\sum_{l=0}^{\infty} \mathbb{P}(A^{\varepsilon_l}_{\tau}) \le C \sum_{l=0}^{\infty} 2^{-ld(\kappa-1)} < \infty,$$

and (3.29) follows from direct application of the Borel-Cantelli Lemma C.1.

Part (2): We now turn to the second part of the theorem, i.e., the construction of boxes I^ε_i. Fixing ε, the first step is to construct a finite collection $\mathcal{I} = \{\tilde{I}_i\}$ of auxiliary boxes such that:

- these boxes cover the points, i.e., $\bigcup_i \tilde{I}_i \supset \varepsilon\Phi^\varepsilon(D)$,
- $\operatorname{dist}_\infty(\tilde{I}_i, \tilde{I}_j) \ge \frac{1}{2N}\varepsilon^{1+\delta}$,
- $s(\tilde{I}_i) \le \varepsilon^{1+\delta}$, where $s(I)$ of a box I denotes the size of its longest side,
- each box $\tilde{I}_i$ satisfies $|\tilde{I}_i \cap \varepsilon\Phi^\varepsilon(D)| \le N$.

Here the crucial condition is the second one, i.e., that the boxes are well-separated.

Let $l := \frac{1}{2N}\varepsilon^{1+\delta}$. We will grow the boxes $\tilde{I}$ from the collection $\mathcal{I}$ step by step, starting with cubes of side length l. At every moment of this growth process, every box $\tilde{I} \in \mathcal{I}$ will satisfy the following conditions:

i. $\tilde{I} = [a_x l, b_x l) \times [a_y l, b_y l) \times [a_z l, b_z l)$ for some $a_x, b_x, a_y, b_y, a_z, b_z \in \mathbb{Z}$, i.e., each box is a union of many small cubes;

ii. for each $a \in [a_x, b_x) \cap \mathbb{Z}$ holds $[al, (a+1)l) \times [a_y l, b_y l) \times [a_z l, b_z l) \cap \varepsilon\Phi^\varepsilon(D) \ne \emptyset$, and similarly for y and z, i.e., in every slice there is some point from $\varepsilon\Phi^\varepsilon(D)$;

iii. $\#(\tilde{I} \cap \varepsilon\Phi^\varepsilon(D)) \le N$.

At the beginning, let $\mathcal{I}$ consist of all cubes $[a_x l, (a_x+1)l) \times [a_y l, (a_y+1)l) \times [a_z l, (a_z+1)l)$ which have a point from $\varepsilon\Phi^\varepsilon(D)$ in it. Since D is bounded, $\mathcal{I}$ consists of finitely many boxes (cubes). We then repeat the following procedure:

If there exist two different boxes $\tilde{I}, \tilde{J} \in \mathcal{I}$ such that $\operatorname{dist}(\tilde{I}, \tilde{J}) = 0$, we fix them and merge them together. That means, we remove $\tilde{I} = [a_x l, b_x l) \times [a_y l, b_y l) \times [a_z l, b_z l)$ and $\tilde{J} = [a'_x l, b'_x l) \times [a'_y l, b'_y l) \times [a'_z l, b'_z l)$ from $\mathcal{I}$ and add

$$\begin{aligned}\tilde{K} &= [A_x l, B_x l) \times [A_y l, B_y l) \times [A_z l, B_z l)\\ &:= [(a_x \wedge a'_x)l, (b_x \vee b'_x)l) \times [(a_y \wedge a'_y)l, (b_y \vee b'_y)l) \times [(a_z \wedge a'_z)l, (b_z \vee b'_z)l)\end{aligned}$$

to $\mathcal{I}$ instead. Here $\wedge$ and $\vee$ stand as usual for minimum and maximum, respectively.

First, observe that (i) trivially follows from the definition of $\tilde{K}$. Next, to verify that $\tilde{K}$ satisfies (ii), let us fix $i \in \{x, y, z\}$, and observe that $\operatorname{dist}(\tilde{I}, \tilde{J}) = 0$ implies $[a_i, b_i] \cap [a'_i, b'_i] \ne \emptyset$.

Hence, for any $a \in [a_i \wedge a_i', b_i \vee b_i')$ either $a \in [a_i, b_i)$, in which case (ii) for $\tilde{I}$ implies (ii) for $\tilde{K}$, or $a \in [a_i', b_i')$, in which case ii for $\tilde{J}$ implies (ii) for $\tilde{K}$.

It remains to argue that $\tilde{K}$ satisfies also (iii). Since $\tilde{I}$ satisfies both (ii) and (iii), in particular to each $a \in [a_i, b_i) \cap \mathbb{Z}$ there is assigned at least one point from $\varepsilon\Phi^\varepsilon(D)$ and there are at most N such points, it follows that $[a_il, b_il)$ has a length of at most Nl. The same argument applies verbatim to $\tilde{J}$, and so the union of $[a_il, b_il)$ and $[a_i'l, b_i'l)$ has a length of at most $2Nl$. Hence, each side of $\tilde{K}$ has a length of at most $s(\tilde{K}) \leq 2Nl = 2N\frac{1}{2N}\varepsilon^{1+\delta} = \varepsilon^{1+\delta}$. In addition $\tilde{K}$ satisfies (i), and so there exists a (closed) cube $Q_{\tilde{K}}$ with a side length of $\varepsilon^{1+\delta}$ such that $\tilde{K} \subset Q_{\tilde{K}}$. By Proposition 3.4.3, the number of points in $Q_{\tilde{K}}$ is at most N, which implies the same for $\tilde{K}$, i.e.,

$$\#(\tilde{K} \cap \varepsilon\Phi^\varepsilon(D)) \leq \#(Q_{\tilde{K}} \cap \varepsilon\Phi^\varepsilon(D)) \leq N,$$

which shows (iii) for $\tilde{K}$; moreover, since $\tilde{K}$ also fulfills (ii), this shows that $\tilde{K}$ has a length of at most $s(\tilde{K}) \leq Nl$.

Since the collection $\mathcal{I}$ was finite at the beginning, and in each iteration we decrease the number of boxes in $\mathcal{I}$ by one (we remove $\tilde{I}$ and $\tilde{J}$ and add $\tilde{K}$), this process has to terminate. In particular, at the end $\mathcal{I}$ consists of boxes which have positive distance from each other, since otherwise the process would not terminate at this point. Since all boxes in $\mathcal{I}$ satisfy (i), this in particular implies that this positive distance has to be at least $l = \frac{1}{2N}\varepsilon^{1+\delta}$. Moreover, since each box has a side length of at most $Nl = \frac{1}{2}\varepsilon^{1+\delta}$, and each point in $\varepsilon\Phi^\varepsilon(D)$ has at least distance ε to ∂D, we see that each box (and actually also its small neighborhood) lies inside D.

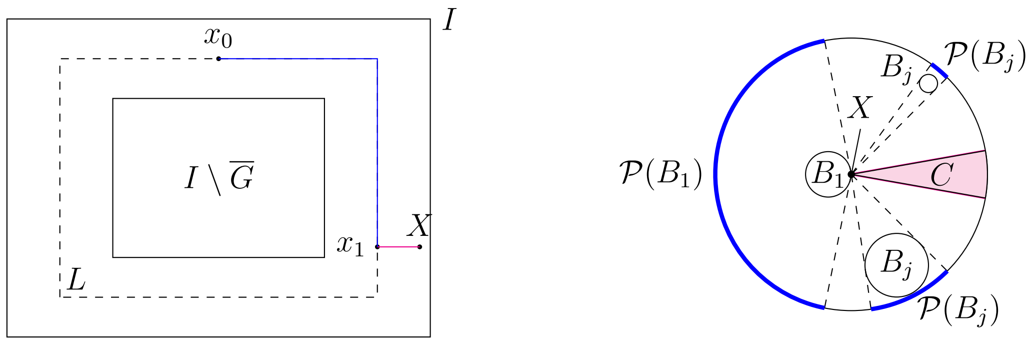
Using boxes from $\mathcal{I}$ we define boxes I_i^ε: for each auxiliary box $\tilde{I}_i \in \mathcal{I}$ set $I_i^\varepsilon := \{x \in \mathbb{R}^3 : \mathrm{dist}_\infty(x, \tilde{I}_i) \leq \frac{1}{8N}\varepsilon^{1+\delta}\}$, and it remains to show that $\{I_i^\varepsilon\}$ satisfies (2a)-(2e). First, by the assumption $\kappa > \delta$, and so for small enough ε we have $\varepsilon^{1+\kappa} \leq \frac{1}{16N}\varepsilon^{1+\delta}$. Therefore, by the triangle inequality we have for any $\varepsilon z \in \tilde{I}_i$ that $\mathrm{dist}_\infty(B_{\varepsilon^{1+\kappa}}(\varepsilon z), \partial I_i^\varepsilon) \geq \frac{1}{8N}\varepsilon^{1+\delta} - \varepsilon^{1+\kappa} \geq \frac{1}{16N}\varepsilon^{1+\delta}$, thus (2a) and (2c) hold. Since by the construction the auxiliary boxes satisfy $\mathrm{dist}_\infty(\tilde{I}_i, \tilde{I}_j) \geq \frac{1}{2N}\varepsilon^{1+\delta}$, and all the points from $\varepsilon\Phi^\varepsilon(D)$ are inside these boxes, we see that $I_i^\varepsilon \setminus \tilde{I}_i$ contains no point from $\varepsilon\Phi^\varepsilon(D)$. Therefore (iii) for $\tilde{I}_i \in \mathcal{I}$ implies (2b) for I_i^ε. Finally, (2d) trivially follows from the definition of I_i^ε and the separation of elements in $\mathcal{I}$ in form of $\mathrm{dist}_\infty(\tilde{I}_i, \tilde{I}_j) \geq \frac{1}{2N}\varepsilon^{1+\delta}$, and (2e) uses that I_i^ε consist in each direction of at least one cube and of at most N of them. □

3.4.2 Proofs of Lemma 3.4.4 and Theorem 3.4.1

Before proving that a box from which we remove finitely many small well-separated balls is a John domain, let us recall Definition 3.2.1 for John domains. For a constant $c > 0$, a domain $U \subset \mathbb{R}^d$ is said to be a $c-$John domain if there exists a point $x_0 \in U$ such that for any point $x \in U$ there is a rectifiable path $\Gamma : [0, \ell] \to U$ which is parametrized by arc length with

$$\Gamma(0) = x, \quad \Gamma(\ell) = x_0, \quad \forall t \in [0, \ell] : |\Gamma(t) - x| \leq c\ \mathrm{dist}(\Gamma(t), \partial U). \tag{3.31}$$

We will use the characterization (3.10), which we also recall here: a bounded domain $U \subset \mathbb{R}^d$ is a $c-$John domain in the sense of Definition 3.2.1 if and only if there is a $c_1(c) > 0$ and a point $x_0 \in U$ such that any point $x \in U$ can be connected to x_0 by a rectifiable path $\Gamma : [0, \ell] \to U$



(a) The case $X \in G$. (b) The case $X \in U \setminus G$.

Figure 3.2: (a) The point $X \in G$, first connected to $x_1 \in L$ (red) and then to x_0 while not leaving L (blue). (b) The projections (blue) of the balls B_1 and B_j onto the sphere S with midpoint X. The cone C illustrated by the red area hits none of the balls and serves as the "outgoing" sector from X to L.

which is parametrized by arc length and

$$\bigcup_{t\in[0,\ell]} B\big(\Gamma(t), t/c_1\big) \subset U.$$

Note that for straight lines Γ, the two definitions coincide with $c_1 = c$. One way how to prove Lemma 3.4.4 is inductively by showing, that under some assumption on a ball one can remove it from a John domain while changing the John constant of the domain by a fixed factor at most. In order to do so, we would need to modify arcs which run close to (or through) this removed ball while estimating how much does this change the situation. For a similar argument with small balls replaced with points, see [HPW08, Theorem 1.4]. Assuming this, since we have to remove at most N balls and at the beginning the domain is rectangle with proportional sides, in particular a John domain, this would lead to the conclusion.

Instead of this, we provide a direct constructive argument.

Proof of Lemma 3.4.4. To start, we use Theorem 3.4.2, part (1), twice: once with $\kappa = \kappa_1 := 1+\delta$ and second time with $\kappa = \kappa_2 := \alpha - 1 - \frac{3}{m} - \delta$. Observe that both values of κ are within the admissible range $(\max(1,\delta), \alpha - 1 - \frac{3}{m})$, and therefore the theorem yields the following: there exists an a.s. positive random variable $\varepsilon_0(\omega)$, obtained as the smaller of the two ε_0, such that for $0 < \varepsilon \le \varepsilon_0$ it holds

$$\max_{z_j \in \Phi^\varepsilon(D)} \varepsilon^\alpha r_j \le \varepsilon^{1+\kappa_2} \quad \text{and} \quad |\varepsilon z_j - \varepsilon z_k| \ge 2\varepsilon^{1+\kappa_1} \quad \text{for any } z_j, z_k \in \Phi^\varepsilon(D). \tag{3.32}$$

Assume now we have $0 < \varepsilon \le \varepsilon_0$ small enough and recall that we want to show that

$$U := I_i^\varepsilon \setminus \bigcup_j \overline{B_j}$$

is a $c(N)-$John domain in the sense of Definition 3.2.1, where

$$\{B_j\}_j := \{B_{\varepsilon^\alpha r_j}(\varepsilon z_j) : \varepsilon z_j \in I_i^\varepsilon\}.$$

For brevity, we set $I := I_i^\varepsilon = p + (-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-l_3/2, l_3/2) \subset \mathbb{R}^3$, where p

is the center and l_i are the side lengths of I. Since (3.31) is scale-invariant, we can assume $l_1 \geq l_2 \geq l_3$. The set

$$L \coloneqq \{x \in I : \mathrm{dist}_\infty(x, \partial I) = \frac{1}{32N}\varepsilon^{1+\delta}\}$$

will serve as a "highway" in the set U, and for the specific point x_0 from Definition 3.2.1 we choose $x_0 \coloneqq p + (0, 0, l_3/2 - \frac{1}{32N}\varepsilon^{1+\delta})$. We also denote the ring around L by $G \coloneqq \{x \in I : \mathrm{dist}_\infty(x, L) < \frac{1}{32N}\varepsilon^{1+\delta}\}$. Let us note that a brief sketch of the ideas developed here for G can be found in Example 3.2.2.

To show that U is a John domain, for each $X \in U$ we need to construct a path from X to x_0 along which $|\cdot - X| \leq c\,\mathrm{dist}(\cdot, \partial U)$. The idea is first to go from X to L, and then run along L to x_0. Observe that for points $x \in L$ the condition is easy to satisfy: for each $x \in L$ we have $\mathrm{dist}(x, \partial U) = \mathrm{dist}(x, \partial G) = \frac{1}{32N}\varepsilon^{1+\delta}$ and $|x - X| \leq \mathrm{diam}(U) \leq \sqrt{3}\,l_1$, and so using (2e) to see $l_1 \leq c(N)\,\varepsilon^{1+\delta}$ we get that $|x - X| \leq c(N)\,\mathrm{dist}(x, \partial U)$ as required.

It remains to describe the path from X to L. For points $X \in G$ this is straightforward (see Figure 3.2a): we just choose the shortest path from X to L and observe that any point x on that path satisfies $\mathrm{dist}(x, \partial U) \geq \mathrm{dist}(x, \partial G) \geq 3^{-1/2}|x - X|$. The $\sqrt{3}$ is optimal as can be seen from points in corners.

For the remaining part of the proof, we deal with the points from the "interior" $U \setminus G$. For $X \in U \setminus G$ we need to construct a path from X to L, while not going too close to the balls $\{B_j\}_j$. We will use two important properties of these balls: the size of the balls is much smaller than their mutual distance (see (3.32)), and there are at most N of them. We fix $X \in U \setminus G$ and show that we can actually use a straight line to connect X with L. Along this line we should be able to move a growing ball without hitting $\{B_j\}_j$, which is equivalent to the existence of a cone with an opening $c(N)$ that avoids all the balls. For this, let S be a unit sphere centered at X, and let $\mathcal{P}$ denote the orthogonal projection on S. We further let $P \coloneqq \mathcal{P}(\bigcup_j B_j)$ denote the projection of balls on S. Observe that if we find a disc on S of fixed radius (depending on N) which does not overlap with P, then we are done since such disc corresponds to a cone at X avoiding the balls $\{B_j\}_j$ (see Figure 3.2b).

Hence, we reduced our task to a problem of finding a not too small disc in $S \setminus P$, with P being a union of at most N discs with some additional properties. First, it can happen that X lies very close to one of the balls, so that the projection of this particular ball on S covers (almost) half of the sphere S. For this reason and without loss of generality, let B_1 denote the ball whose center is closest to X, which we treat separately: let $S' \subset S$ be a half-sphere with the pole being exactly opposite to the center of $\mathcal{P}(B_1)$, in particular $\mathcal{P}(B_1)$ and S' are disjoint. Since B_1 was the closest ball to X, it follows from the second estimate in (3.32) that X is at least $\varepsilon^{1+\kappa_1}$ away from the centers of the remaining balls $\{B_j\}_{j\geq 2}$. On the other hand, the first relation in (3.32) bounds the radii of these balls with $\varepsilon^{1+\kappa_2}$. Therefore, the projections of these remaining balls are discs of radius at most $C\,\frac{\varepsilon^{1+\kappa_2}}{\varepsilon^{1+\kappa_1}} = C\,\varepsilon^{\kappa_2-\kappa_1}$. Since $\kappa_2 - \kappa_1 = \alpha - 2 - \frac{3}{m} - 2\delta > 0$ by the choice of δ in (3.21), we see that for ε small enough these (at most $N-1$) projections are tiny discs (almost points). We can now find a radius $r = r(N)$ with the following property: there are N discs $D_1, \ldots, D_N$ of radius r in S' such that the distance between any two discs is at least r as well. One option is to arrange them along the boundary of S' with necessary spacing between them, thus achieving $r \sim N^{-1}$. Provided now ε is small enough such that the radii of $\mathcal{P}(B_j)$, which are bounded by $C\,\varepsilon^{\kappa_2-\kappa_1}$, are smaller than r, we are done: there are N discs $D_1, \ldots, D_N$ and at most $N-1$ projections $\mathcal{P}(B_j)$, where each projection can touch at most one D_j, so that one disc will not overlap with any of the projections $\mathcal{P}(B_j)$, thus defining the cone we are searching for.

This solution to the last question is naturally far from optimal (in r): consider a well-studied question of finding an optimal cover of a sphere (more precisely half of it) with N identical discs of smallest radius (see [Tót49]). If ϱ denotes the smallest such radius, then for any configuration of $N-1$ points in S' there exists a disc in S' of radius ϱ which avoids them, thus also providing a solution to our problem. □

Since the perforated boxes U from Lemma 3.4.4 are uniform John domains, in particular we have a Bogovkiĭ operator on each U, Theorem 3.4.1 can be proven along the lines of the proof in [DFL17]. First, using a Bogovskiĭ operator on the whole of D we obtain a function $\mathbf{u}$ with the correct divergence but that naturally does not vanish on the holes. To achieve that, we modify $\mathbf{u}$ in each box I_i^ε. More precisely, near ∂I_i^ε in a boundary layer of size $\frac{1}{16N}\varepsilon^{1+\delta}$ we change $\mathbf{u}$ to its average value over this layer, and then inside the box (where also the balls are removed) cut off this constant function near each hole over a scale ε^α. Since we also change the divergence of the function with this modification, we employ Bogovskiĭ's operator both on each box as well as near each hole to fix the divergence.

Proof of Theorem 3.4.1. Let us recall the definition of $D_\varepsilon = D \setminus \bigcup_{z_j \in \Phi^\varepsilon(D)} \overline{B_{\varepsilon^\alpha r_j}(\varepsilon z_j)}$. To prove the theorem we construct a linear operator of Bogovskiĭ type, bounded independently of ε, that is,

$$\mathcal{B}_\varepsilon : L_0^q(D_\varepsilon) \to W_0^{1,q}(D_\varepsilon)$$

satisfying

$$\operatorname{div} \mathcal{B}_\varepsilon(f) = f \text{ in } D_\varepsilon, \quad \|\mathcal{B}_\varepsilon(f)\|_{W_0^{1,q}(D_\varepsilon)} \le C\, \|f\|_{L_0^q(D_\varepsilon)}. \tag{3.33}$$

To this end, we will first use a Bogovskiĭ operator on the whole domain D and then correct this function first to its mean value over a large scale and then to zero near each hole without changing the divergence. The proof is essentially the same as in Section 3.3, however, we will repeat the proof here in order to see the differences due to the random perforation. We will also give some remarks on this procedure after the proof.

For $1 < q < \infty$ and $f \in L_0^q(D_\varepsilon)$ we denote by $\tilde{f} \in L_0^q(D)$ its zero extension in the holes. Using the classical Bogovskiĭ operator in the Lipschitz domain D (see Theorem 3.2.9), the norm of which depends on the Lipschitz character of D, we can find a function $\mathbf{u} = \mathcal{B}_D(\tilde{f}) \in W_0^{1,q}(D)$ satisfying

$$\operatorname{div} \mathbf{u} = \tilde{f} \text{ in } D, \quad \|\mathbf{u}\|_{W_0^{1,q}(D)} \le C\, \|\tilde{f}\|_{L_0^q(D)} = C\, \|f\|_{L_0^q(D_\varepsilon)}$$

with $C = C(D, q)$.

Since $\alpha - 3/m > 2$, by applying Theorem 3.4.2 we find for every $\varepsilon > 0$ small enough a finite collection of boxes I_i^ε such that for any point $z_j \in \Phi^\varepsilon(D)$ there is i such that

$$B_{\varepsilon^\alpha r_j}(\varepsilon z_j) \subset B_{2\varepsilon^\alpha r_j}(\varepsilon z_j) \subset B_{\varepsilon^{1+\kappa}}(\varepsilon z_j) \subset I_i^{\varepsilon,\mathrm{in}},$$

where

$$I_i^{\varepsilon,\text{in}} := \{x \in I_i^\varepsilon : \text{dist}_\infty(x, \partial I_i^\varepsilon) \geq \frac{1}{16N}\varepsilon^{1+\delta}\}.$$

For any box I_i^ε and any ball $B_{\varepsilon^\alpha r_j}(\varepsilon z_j)$ consider the corresponding cut-off functions

$$\chi_{\varepsilon,i} \in C_c^\infty(I_i^\varepsilon), \quad \chi_{\varepsilon,i} \restriction_{I_i^{\varepsilon,\text{in}}} = 1, \quad \|\nabla \chi_{\varepsilon,i}\|_{L^\infty(D)} \lesssim \varepsilon^{-(1+\delta)}, \tag{3.34}$$

$$\zeta_{\varepsilon,j} \in C_c^\infty\Big(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j)\Big), \quad \zeta_{\varepsilon,j} \restriction_{B_{\varepsilon^\alpha r_j}(\varepsilon z_j)} = 1, \quad \|\nabla \zeta_{\varepsilon,j}\|_{L^\infty(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))} \lesssim \frac{1}{r_j}\varepsilon^{-\alpha}, \tag{3.35}$$

and define

$$\begin{aligned} A_i^\varepsilon &:= I_i^\varepsilon \setminus I_i^{\varepsilon,\text{in}} = \{x \in I_i^\varepsilon : \text{dist}_\infty(x, \partial I_i^\varepsilon) < \frac{1}{16N}\varepsilon^{1+\delta}\}, \\ \mathbf{b}_{\varepsilon,i}(\mathbf{u}) &:= \chi_{\varepsilon,i}(\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon}) \in W_0^{1,q}(I_i^\varepsilon), \\ \beta_{\varepsilon,j}(\mathbf{u}) &:= \zeta_{\varepsilon,j}\, \langle \mathbf{u} \rangle_{A_i^\varepsilon} \in W_0^{1,q}\Big(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j)\Big), \end{aligned} \tag{3.36}$$

where as before i and j are related through $\varepsilon z_j \in I_i^\varepsilon$, and $\langle \mathbf{u} \rangle_S$ is the mean value of $\mathbf{u}$ over the set S.

Since all the lengths in the set A_i^ε are proportional to $\varepsilon^{1+\delta}$ (with the proportionality depending on N), Poincaré's inequality (B.6) implies

$$\|\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon}\|_{L^q(A_i^\varepsilon)} \lesssim \varepsilon^{1+\delta}\, \|\nabla \mathbf{u}\|_{L^q(A_i^\varepsilon)},$$

and by (3.34) we get

$$\begin{aligned} \|\nabla \mathbf{b}_{\varepsilon,i}(\mathbf{u})\|_{L^q(A_i^\varepsilon)} &\leq \|\chi_{\varepsilon,i} \nabla(\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon})\|_{L^q(A_i^\varepsilon)} + \|\nabla \chi_\varepsilon(\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon})\|_{L^q(A_i^\varepsilon)} \\ &\lesssim \|\nabla(\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon})\|_{L^q(A_i^\varepsilon)} + \varepsilon^{-(1+\delta)}\, \|\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon}\|_{L^q(A_i^\varepsilon)} \\ &\lesssim \|\nabla \mathbf{u}\|_{L^q(A_i^\varepsilon)}. \end{aligned} \tag{3.37}$$

From (3.35) and Hölder's inequality (B.2), we obtain

$$\begin{aligned} \|\nabla \beta_{\varepsilon,j}(\mathbf{u})\|_{L^q(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))} &= \|\nabla \zeta_{\varepsilon,j} \cdot \langle \mathbf{u} \rangle_{A_i^\varepsilon}\|_{L^q(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))} \\ &\lesssim r_j^{\frac{3}{q}-1} \varepsilon^{\left(\frac{3}{q}-1\right)\alpha}\, |\langle \mathbf{u} \rangle_{A_i^\varepsilon}| \lesssim r_j^{\frac{3}{q}-1} \varepsilon^{\left(\frac{3}{q}-1\right)\alpha}\, |A_i^\varepsilon|^{-\frac{1}{q}} \|\mathbf{u}\|_{L^q(A_i^\varepsilon)} \\ &\lesssim r_j^{\frac{3}{q}-1} \varepsilon^{\left(\frac{3}{q}-1\right)\alpha - \frac{3(1+\delta)}{q}}\, \|\mathbf{u}\|_{L^q(A_i^\varepsilon)}. \end{aligned} \tag{3.38}$$

Since $B_{\varepsilon^\alpha r_j}(\varepsilon z_j) \subset D$, we have $r_j \leq \varepsilon^{1+\kappa_2-\alpha} = \varepsilon^{-(\frac{3}{m}+\delta)}$ by (3.32) and the choice of $\kappa_2 = \alpha - 1 - \frac{3}{m} - \delta$. This yields

$$r_j^{\frac{3}{q}-1} \varepsilon^{\left(\frac{3}{q}-1\right)\alpha - \frac{3(1+\delta)}{q}} \leq \varepsilon^{(\frac{3}{q}-1)(\alpha - \frac{3}{m} - \delta) - \frac{3}{q}(1+\delta)}.$$

Thus, for all $1 < q < 3$ which satisfy (3.19), we can choose δ such that

$$0 < \delta \leq \frac{(3-q)(\alpha - \frac{3}{m}) - 3}{6-q} \tag{3.39}$$

to get uniform bounds on $\|\beta_{\varepsilon,j}(\mathbf{u})\|_{L^q(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))}$. Similar to Section 3.3, the functions $\beta_{\varepsilon,j}$

and $\mathbf{b}_{\varepsilon,i}$ do not have vanishing divergence in general, so we need to correct them using a Bogovkiĭ operator on perforations of I_i^ε. Since $I_i^\varepsilon \setminus \bigcup_{\varepsilon z_j \in I_i^\varepsilon} \overline{B_{\varepsilon^{\alpha} r_j}(\varepsilon z_j)}$ is a uniform John domain, the existence of the Bogovskiĭ operator $\mathcal{B}_{\varepsilon,i}$ for the set $I_i^\varepsilon \setminus \bigcup_{\varepsilon z_j \in I_i^\varepsilon} \overline{B_{\varepsilon^{\alpha} r_j}(\varepsilon z_j)}$ is content of Lemma 3.4.4, provided we choose δ from (3.39) possibly even smaller to satisfy also (3.21). Moreover, note that by construction, any box I_i^ε contains at least one point from $\Phi^\varepsilon(D)$ and at most N of them. We are now ready to define the restriction operator from D to D_ε via

$$R_\varepsilon(\mathbf{u}) \coloneqq \mathbf{u} - \sum_{I_i^\varepsilon \subset D} \sum_{\varepsilon z_j \in I_i^\varepsilon} \left(\frac{\mathbf{b}_{\varepsilon,i}(\mathbf{u})}{\#\{\varepsilon z_j \in I_i^\varepsilon\}} + \beta_{\varepsilon,j}(\mathbf{u}) \right) - \mathcal{B}_{\varepsilon,i} \operatorname{div} \left(\frac{\mathbf{b}_{\varepsilon,i}(\mathbf{u})}{\#\{\varepsilon z_j \in I_i^\varepsilon\}} + \beta_{\varepsilon,j}(\mathbf{u}) \right), \quad (3.40)$$

where all functions were extended by 0 outside their domain of definition. This definition is a generalization of the one in (3.17). Indeed, if any box I_i^ε contains just one point εz_j, the definitions are equivalent. Repeating the arguments shown in Section 3.3, we check that the operator R_ε is well defined and satisfies the desired norm bounds. First, by the definitions of $\mathbf{b}_{\varepsilon,i}$ and $\beta_{\varepsilon,j}$ in (3.36), we have

$$\int_{I_i^\varepsilon} \operatorname{div} \mathbf{b}_{\varepsilon,i}(\mathbf{u}) \, \mathrm{d}x = 0, \quad \int_{I_i^\varepsilon} \operatorname{div} \beta_{\varepsilon,j}(\mathbf{u}) \, \mathrm{d}x = \int_{B_{2\varepsilon^{\alpha} r_j}(\varepsilon z_j)} \operatorname{div} \beta_{\varepsilon,j}(\mathbf{u}) \, \mathrm{d}x = 0.$$

On the other hand, $\chi_{\varepsilon,i} = \zeta_{\varepsilon,j} = 1$ and $\operatorname{div} \mathbf{u} = \tilde{f} = 0$ inside $B_{\varepsilon^{\alpha} r_j}(\varepsilon z_j)$, thus

$$\begin{aligned} \operatorname{div} \mathbf{b}_{\varepsilon,i}(\mathbf{u}) &= \chi_{\varepsilon,i} \operatorname{div} \mathbf{u} + \nabla \chi_{\varepsilon,i} \cdot (\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon}) = 0 && \text{in } B_{\varepsilon^{\alpha} r_j}(\varepsilon z_j), \\ \operatorname{div} \beta_{\varepsilon,j}(\mathbf{u}) &= \nabla \zeta_{\varepsilon,j} \cdot (\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon}) = 0 && \text{in } B_{\varepsilon^{\alpha} r_j}(\varepsilon z_j), \end{aligned}$$

leading to

$$\int_{I_i^\varepsilon \setminus \bigcup\limits_{z_j \in \varepsilon^{-1} I_i^\varepsilon \cap \Phi^\varepsilon(D)} B_{\varepsilon^{\alpha} r_j}(\varepsilon z_j)} \operatorname{div} \left(\frac{\mathbf{b}_{\varepsilon,i}(\mathbf{u})}{\#\{\varepsilon z_j \in I_i^\varepsilon\}} + \beta_{\varepsilon,j}(\mathbf{u}) \right) \mathrm{d}x = 0$$

as required. Next, for any hole $B_{\varepsilon^{\alpha} r_j}(\varepsilon z_j) \subset I_i^\varepsilon$ and any $x \in B_{\varepsilon^{\alpha} r_j}(\varepsilon z_j)$, we get

$$\begin{aligned} R_\varepsilon(\mathbf{u})(x) = \mathbf{u}(x) &- \sum_{\varepsilon z_j \in I_i^\varepsilon} \left(\frac{\mathbf{b}_{\varepsilon,i}(\mathbf{u})}{\#\{\varepsilon z_j \in I_i^\varepsilon\}} + \beta_{\varepsilon,j}(\mathbf{u}) \right)(x) \\ &- \sum_{\varepsilon z_j \in I_i^\varepsilon} \mathcal{B}_{\varepsilon,i} \operatorname{div} \left(\frac{\mathbf{b}_{\varepsilon,i}(\mathbf{u})}{\#\{\varepsilon z_j \in I_i^\varepsilon\}} + \beta_{\varepsilon,j}(\mathbf{u}) \right)(x) \\ &= \mathbf{u}(x) - (\mathbf{b}_{\varepsilon,i}(\mathbf{u}) + \beta_{\varepsilon,j}(\mathbf{u}))(x) - \mathcal{B}_{\varepsilon,i} \operatorname{div} \big(\mathbf{b}_{\varepsilon,i}(\mathbf{u}) + \beta_{\varepsilon,j}(\mathbf{u}) \big)(x) \\ &= \mathbf{u}(x) - \chi_{\varepsilon,i}(x)(\mathbf{u}(x) - \langle \mathbf{u} \rangle_{A_i^\varepsilon}) - \zeta_{\varepsilon,j}(x) \langle \mathbf{u} \rangle_{A_i^\varepsilon} \\ &= 0, \end{aligned}$$

where in the last equality we used that $\chi_{\varepsilon,i}(x) = \zeta_{\varepsilon,j}(x) = 1$ in $B_{\varepsilon^{\alpha} r_j}(\varepsilon z_j)$. This yields that the operator R_ε is well defined and satisfies

$$R_\varepsilon(\mathbf{u}) \in W_0^{1,q}(D_\varepsilon), \quad \operatorname{div} R_\varepsilon(\mathbf{u}) = f \text{ in } D_\varepsilon.$$

Finally, by (3.37), (3.38) and the fact that the boxes I_i^ε are disjoint, we see that

$$\|R_\varepsilon(\mathbf{u})\|_{W_0^{1,q}(D_\varepsilon)} \le C\left(\varepsilon^{(\frac{3}{q}-1)(\alpha-\frac{3}{m}-\delta)-\frac{3}{q}(1+\delta)}+1\right)\|\mathbf{u}\|_{W_0^{1,q}(D)}, \tag{3.41}$$

where the constant $C > 0$ is independent of $\varepsilon > 0$. Note that due to the choice of δ, the exponent of ε on the right-hand side is non-negative, so we may bound R_ε uniformly with respect to ε. For $f \in L_0^q(D_\varepsilon)$ we define

$$\mathcal{B}_\varepsilon(f) := (R_\varepsilon \circ \mathcal{B}_D)(\tilde{f})$$

and observe that we get the desired operator, namely $\mathcal{B}_\varepsilon(f) \in W_0^{1,q}(D_\varepsilon)$,

$$\operatorname{div}\mathcal{B}_\varepsilon(f) = f \text{ in } D_\varepsilon, \quad \text{and} \quad \|\mathcal{B}_\varepsilon(f)\|_{W_0^{1,q}(D_\varepsilon)} \le C\,\|f\|_{L^q(D_\varepsilon)}.$$

This finishes the proof of Theorem 3.4.1. □

Remark 3.4.7. *As holes are well separated, one might think that the construction of the Bogovskiĭ operator is possible in just two steps: first in the whole domain and second with a cut-off argument near each hole. This construction would follow the one from [All90] and its L^q-generalization by Lu in [Lu21, Section 5]. However, following Lu's proof, one recognizes that we would get a worse exponent of ε: the term $\frac{3}{q}(1+\delta)$ would change to $\frac{3}{q}(2+\delta)$. This is due to the fact that in our case, the holes do not have mutual distance of order ε, but rather (more than) $\varepsilon^{2+\delta}$ due to the random distribution of centers.*

Remark 3.4.8. *We note that the ε-dependence in (3.41) seems not to be optimal but "close to optimal" in the sense of capacity (see also [Lu21, Remark 2.4]): Recall that for $1 < q < \infty$ and $S \subset \mathbb{R}^d$, the q-capacity is defined as*

$$\operatorname{Cap}_q(S) := \inf\{\|\nabla f\|^q_{L^q(\mathbb{R}^d)} : f \in W^{1,q}(\mathbb{R}^d), S \subset \{f \ge 1\}\}.$$

We will here focus on the case $d = 3$. For a ball of radius $r > 0$, it is known that for any $1 < q < 3$ there exists a constant $C = C(q) > 0$ such that

$$\operatorname{Cap}_q(B_r(0)) = C\,r^{3-q},$$

see, e.g., [EG15, Theorem 4.15]. Since the capacity is an outer measure, the fact that (for $\varepsilon > 0$ small enough) the balls are well separated and the expected number of holes inside D is of order ε^{-3}, together with (3.27) and the choice of $\kappa_2 = \alpha - 1 - \frac{3}{m} - \delta$, we have

$$\begin{aligned}\operatorname{Cap}_q\Big(\bigcup_{z_i\in\Phi^\varepsilon(D)} B_{r_i\varepsilon^\alpha}(\varepsilon z_i)\Big) &\le \sum_{z\in\Phi^\varepsilon(D)} \operatorname{Cap}_q(B_{r_i\varepsilon^\alpha}(\varepsilon z_i)) \\ &\le C\,\varepsilon^{-3}\big(\max_{z_i\in\Phi^\varepsilon(D)} r_i\varepsilon^\alpha\big)^{3-q} \le C\,\varepsilon^{(1+\kappa_2)(3-q)-3} = C\,\varepsilon^{(3-q)(\alpha-\frac{3}{m}-\delta)-3}.\end{aligned} \tag{3.42}$$

We see that the essential quantity $(3-q)(\alpha - \frac{3}{m} - \delta) - 3$ arising here is almost the same as in (3.41). A possible explanation for the connection between the capacity estimates and the estimate for the Bogovskiĭ operator is as follows. Let $u \in W_0^{1,q}(D_\varepsilon)$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ with $\varphi = 1$ in D. Then $\varphi(1-u)$ is an admissible function in the definition of the q-capacity for the *union of all holes, that is,*

$$\varphi(1-u) \in W^{1,q}(\mathbb{R}^d), \quad \varphi(1-u) = 1 \text{ on } \bigcup_{z_i \in \Phi^\varepsilon(D)} B_{r_i \varepsilon^\alpha}(\varepsilon z_i).$$

Direct calculations yield

$$\|\varphi(1-u)\|_{W^{1,q}(\mathbb{R}^d)} \leq C(\varphi)\left(1 + \|u\|_{W^{1,q}(D_\varepsilon)}\right)$$

as well as

$$C(\varphi)\left(1 + \|u\|^q_{W^{1,q}(D_\varepsilon)}\right) \geq \|\varphi(1-u)\|^q_{W^{1,q}(\mathbb{R}^d)} \geq \mathrm{Cap}_q\Big(\bigcup_{z_i \in \Phi^\varepsilon(D)} B_{r_i \varepsilon^\alpha}(\varepsilon z_i)\Big).$$

If α is large and the radii r_i are almost constant, meaning that the holes inside D should be very well separated, one might expect that the inequality (3.42) is close to an equality, yielding

$$\|u\|^q_{W^{1,q}(D_\varepsilon)} \geq C\left(\varepsilon^{(3-q)(\alpha - \frac{3}{m} - \delta) - 3} - 1\right).$$

For the Bogovskiĭ operator obtained in Theorem 3.4.1, we have $\mathcal{B}_\varepsilon(f) \in W_0^{1,q}(D_\varepsilon)$, so the optimal general estimate on $\|u\|_{W^{1,q}(D_\varepsilon)}$ may be of size $\varepsilon^{(\frac{3}{q}-1)(\alpha-\frac{3}{m}-\delta)-\frac{3}{q}}$. The suboptimal factor $\varepsilon^{-\frac{3}{q}(1+\delta)}$ in (3.41) is due to the fact our construction does not enable us to have a better estimate on $\nabla \beta_{\varepsilon,j}(\mathbf{u})$ in (3.38).

3.5 Extension to negative Sobolev spaces

Here, we will give an extension result for the Bogovskiĭ operator $\mathcal{B}_\varepsilon$ constructed in Section 3.4, for the later use in the homogenization of time-dependent Navier-Stokes equations. We will need to control the Bogovskiĭ operator in some negative Sobolev space to handle terms of the form $\mathcal{B}_\varepsilon(\operatorname{div}(\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon))$ that arise from the renormalized continuity equation (2.3) and the time derivative in the weak (meaning integral) formulation of the momentum equation (2.4), that is, from $\partial_t \mathcal{B}_\varepsilon(\varrho_\varepsilon) = \mathcal{B}_\varepsilon(\partial_t \varrho_\varepsilon)$. We will get rigorous on this in Section 4.2. The statement and proof of such an extension to negative Sobolev spaces requires some additional structure on L^p-functions, precisely, we have to control their divergence in a suitable way. We therefore introduce some new function spaces according to the definitions made in [FN09, Section 10.3].

Definition 3.5.1. *Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $1 < p \leq q < \infty$, and denote the Hölder conjugate of p by $p' := p/(p-1)$. Then we define the following function spaces:*

1. *$E^{q,p}(D) := \{\mathbf{u} \in L^q(D) : \operatorname{div} \mathbf{u} \in L^p(D)\}$, endowed with the norm $\|\mathbf{u}\|_{E^{q,p}(D)} := \|\mathbf{u}\|_{L^q(D)} + \|\operatorname{div} \mathbf{u}\|_{L^p(D)}$;*
2. *$E_0^{q,p}(D) := \overline{C_c^\infty(D)}^{\|\cdot\|_{E^{q,p}}}$;*
3. *$[\dot{W}^{1,p'}(D)]' := \{g \in [W^{1,p'}(D)]' : \langle g, 1\rangle = 0\}$.*

These spaces have some important properties and connections, which we state in the next lemma. Recall that we defined $C_{c,0}^\infty(D)$ as the set of smooth functions with zero mean value over D.

Lemma 3.5.2. *Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $1 < p \le q < \infty$. Then it holds:*

1. *$E^{q,p}(D) \subset E^{p,p}(D)$ and $E_0^{q,p}(D) \subset E_0^{p,p}(D)$.*

2. *The set $C^\infty(\overline{D})$ is dense in $E^{q,p}(D)$.*

3. *$C_{c,0}^\infty(D)$ is dense in $\big[\dot{W}^{1,p'}(D)\big]'$.*

4. *We have $\{\operatorname{div} \mathbf{f} : \mathbf{f} \in E_0^{q,p}(D)\} \subset \big[\dot{W}^{1,p'}(D)\big]'$ via*

$$\langle \operatorname{div} \mathbf{f}, \varphi \rangle_{[\dot{W}^{1,p'}(D)]', \dot{W}^{1,p'}(D)} := -\int_D \mathbf{f} \cdot \nabla\varphi \, \mathrm{d}x \quad \text{for all } \mathbf{f} \in E_0^{q,p}(D),\ \varphi \in W^{1,p'}(D),$$

and we may estimate the norms as

$$\|\operatorname{div} \mathbf{f}\|_{[\dot{W}^{1,p'}(D)]'} \le C\, \|\mathbf{f}\|_{L^q(D)}$$

for some constant $C = C(p, q, |D|) > 0$.

Proof. The first statement is trivial due to $q \ge p$ and $|D| < \infty$. The second one can be found in [FN09, Lemma 10.2]. Since any bounded Lipschitz domain can be decomposed into finitely many star-shaped domains (see Lemma 3.1.3), we may assume that D is star-shaped with respect to a ball centered at the origin. For $\mathbf{u} \in E^{q,p}(D)$ and $\tau \in (0,1)$ write $\mathbf{u}_\tau(x) = \mathbf{u}(x/\tau)$. Then, $\overline{\tau D} \subset D$, $\mathbf{u}_\tau \in E^{q,p}(\tau D)$ with $\operatorname{div}(\mathbf{u}_\tau) = \frac{1}{\tau}(\operatorname{div}\mathbf{u})_\tau$, and also

$$\|\operatorname{div}(\mathbf{u} - \mathbf{u}_\tau)\|_{L^p(D)} \le \big(1 - \tau^{-1}\big)\|\operatorname{div}\mathbf{u}\|_{L^p(D)} + \tau^{-1}\|\operatorname{div}\mathbf{u} - (\operatorname{div}\mathbf{u})_\tau\|_{L^p(D)}.$$

Since translations $h \mapsto \mathbf{v}(\cdot + h)$ are continuous for any $\mathbf{v} \in L^r(\mathbb{R}^3)$, $1 \le r < \infty$, we may write $\mathbf{v}_\tau(x) = \mathbf{v}(x - (1 - \frac{1}{\tau})x)$ to see that also scalings are continuous as $\tau \to 1$. Thus, we see that $\operatorname{div}(\mathbf{u} - \mathbf{u}_\tau) \to 0$ in $L^p(D)$ since $(\operatorname{div}\mathbf{u})_\tau \to \operatorname{div}\mathbf{u}$ in $L^p(D)$, and additionally $\mathbf{u}_\tau \to \mathbf{u}$ in $L^q(D)$, provided all functions have been extended by zero to the whole space. Hence, it suffices to approximate $\mathbf{u}_\tau$ in $E^{q,p}(D)$ by smooth functions. To this end, let $0 < \varepsilon < \frac{1}{2}\operatorname{dist}(\tau D, \partial D)$, and let η_ε be a mollifying kernel. Then $\operatorname{supp}\,(\mathbf{u}_\tau * \eta_\varepsilon) \subset D$ and $\mathbf{u}_\tau * \eta_\varepsilon \in C^\infty(\overline{D}) \cap E^{q,p}(D)$, and $\mathbf{u}_\tau * \eta_\varepsilon \to \mathbf{u}_\tau$ in $E^{q,p}(D)$ as $\varepsilon \to 0$ by the properties of mollifiers (see Proposition B.7).
The third statement is again proven in [FN09, Lemmata 10.4 and 10.5]. More precisely, for fixed $f \in \big[\dot{W}^{1,p'}(D)\big]'$ there is $\mathbf{w} \in L^p(D)$ such that

$$\langle f, \varphi \rangle = \int_D \mathbf{w} \cdot \nabla\varphi \, \mathrm{d}x, \quad \|\mathbf{w}\|_{L^p(D)} = \|f\|_{[\dot{W}^{1,p'}(D)]'},$$

which in turn is a consequence of the Hahn-Banach theorem and the Riesz representation theorem. Approximating $\mathbf{w}$ with smooth functions $\mathbf{w}_n$ such that $\mathbf{w}_n \to \mathbf{w}$ in $L^p(D)$ and $\|\mathbf{w}_n\|_{L^p(D)} \le \|\mathbf{w}_n - \mathbf{w}\|_{L^p(D)} + \|\mathbf{w}\|_{L^p(D)} \le 2\,\|f\|_{[\dot{W}^{1,p'}(D)]'}$, and defining functionals f_n via

$$\langle f_n, \varphi \rangle := \int_D \mathbf{w}_n \cdot \nabla\varphi \, \mathrm{d}x$$

yields the desired by sending $n \to \infty$. Assertion four follows from the observation that the integral on the right is finite for any $\mathbf{f} \in E_0^{q,p}(D)$ since $\|\mathbf{f}\|_{L^p(D)} \le C\,\|\mathbf{f}\|_{L^q(D)}$ by $q \ge p$, so the dual product is well defined. Further, this immediately yields

$$\langle \operatorname{div} \mathbf{f}, 1\rangle_{[\dot W^{1,p'}(D)]', \dot W^{1,p'}(D)} = -\int_D \mathbf{f}\cdot \nabla 1 \, \mathrm{d}x = 0$$

as required. Finally, we have

$$\begin{aligned}
&\|\operatorname{div} \mathbf{f}\|_{[\dot W^{1,p'}(D)]'} = \sup_{\|\varphi\|_{W^{1,p'}}=1} |\langle \operatorname{div} \mathbf{f}, \varphi\rangle_{[\dot W^{1,p'}(D)]', \dot W^{1,p'}(D)}| \\
&\le \sup_{\|\varphi\|_{W^{1,p'}}=1} \int_D |\mathbf{f}|\,|\nabla\varphi| \, \mathrm{d}x \le \sup_{\|\varphi\|_{W^{1,p'}}=1} \|\mathbf{f}\|_{L^q(D)} \|\nabla\varphi\|_{L^{q'}(D)} \\
&\le C \sup_{\|\varphi\|_{W^{1,p'}}=1} \|\mathbf{f}\|_{L^q(D)} \|\nabla\varphi\|_{L^{p'}(D)} \le C \sup_{\|\varphi\|_{W^{1,p'}}=1} \|\mathbf{f}\|_{L^q(D)} \|\varphi\|_{W^{1,p'}(D)} \\
&= C\,\|\mathbf{f}\|_{L^q(D)}
\end{aligned}$$

for some constant $C > 0$ just dependent on p, q and $|D|$, where we used that $q' \le p'$ by $q \ge p$, so $\|\nabla\varphi\|_{L^{q'}(D)} \le C\,\|\nabla\varphi\|_{L^{p'}(D)}$. □

Additionally, due to the fact $E^{q,p}(D) \subset E^{p,p}(D)$, we are able to define a generalized normal trace, which we take from [FN09, Theorem 10.8].

Theorem 3.5.3. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $1 < p < \infty$. There exists a unique bounded linear operator $\mathrm{Tr}_{\mathbf{n}}$ such that*

1. $\mathrm{Tr}_{\mathbf{n}} : E^{p,p}(D) \to \left[W^{1-\frac{1}{p'},p'}(\partial D)\right]'$ *and* $\mathrm{Tr}_{\mathbf{n}}(\mathbf{u}) = \mathrm{Tr}(\mathbf{u})\cdot\mathbf{n}$ *almost everywhere on* ∂D *if* $\mathbf{u} \in C^\infty(\overline{D})$, *where* Tr *is the usual trace operator on* $W^{1,p}(D)$*;*

2. *For any* $\mathbf{u} \in E^{p,p}(D)$ *and any* $v \in W^{1,p'}(D)$, *it holds*

$$\int_D v \operatorname{div} \mathbf{u} \, \mathrm{d}x + \int_D \nabla v \cdot \mathbf{u} \, \mathrm{d}x = \langle \mathrm{Tr}_{\mathbf{n}}(\mathbf{u}), \mathrm{Tr}(v)\rangle,$$

   where $\langle\cdot,\cdot\rangle$ *denotes the duality product between* $\left[W^{1-\frac{1}{p'},p'}(\partial D)\right]'$ *and* $W^{1-\frac{1}{p'},p'}(\partial D)$.

3. *If* $\mathbf{u} \in W^{1,p}(D)$, *then* $\mathrm{Tr}_{\mathbf{n}}(\mathbf{u}) \in L^p(\partial D)$ *and* $\mathrm{Tr}_{\mathbf{n}}(\mathbf{u}) = \mathrm{Tr}(\mathbf{u})\cdot\mathbf{n}$ *almost everywhere on* ∂D.

4. $\{\mathbf{u} \in E^{p,p}(D) : \mathrm{Tr}_{\mathbf{n}}(\mathbf{u}) = 0\} = E_0^{p,p}(D)$.

We are now in the position to state and prove the following extension result from [LS18, Theorem 1.1 and Proposition 2.2] for the case of a random perforation (see also [FN09, Theorem 10.11]). As mentioned earlier, the regularity assumption on D in the second part of the statement can be relaxed. It ensures that we can apply all the results obtained in Section 3.4, however, we will not discuss its optimality here.

Theorem 3.5.4. *Let $D \subset \mathbb{R}^3$ be a bounded Lipschitz domain, $\mathcal{B}$ be the corresponding Bogovskiĭ operator from Theorem 3.2.9, and $1 < p < 3$. Then we can extend $\mathcal{B}$ to an operator*

$$\mathcal{B} : \left[\dot W^{1,p'}(D)\right]' \to L^p(D),$$

and, for $p \le q < 3$, *to an operator mapping* $\{\operatorname{div}\mathbf{f} : \mathbf{f} \in E_0^{q,p}(D)\}$ *to* $L^q(D)$. *More precisely, if* $f \in \left[\dot W^{1,p'}(D)\right]'$, *then*

$$\langle \mathcal{B}(f), \nabla\varphi\rangle = -\langle f, \varphi\rangle \text{ for any } \varphi \in W^{1,p'}(D), \quad \|\mathcal{B}(f)\|_{L^p(D)} \le C\,\|f\|_{[W^{1,p'}(D)]'}$$

for some constant $C > 0$ *independent of* f, *and similarly for* $\mathbf{f} \in E_0^{q,p}(D)$

$$\langle \mathcal{B}\operatorname{div}\mathbf{f}, \nabla\varphi\rangle = \langle \mathbf{f}, \nabla\varphi\rangle \text{ for any } \varphi \in W^{1,p'}(D), \quad \|\mathcal{B}\operatorname{div}\mathbf{f}\|_{L^q(D)} \le C\,\|\mathbf{f}\|_{L^q(D)}.$$

Further, if ∂D *is smooth, let* D_ε *be defined as in* (3.18), *and* $\mathcal{B}_\varepsilon$ *be the operator constructed in Theorem 3.4.1. If* $q > 3/2$ *and* $\mathbf{f} \in E_0^{q,p}(D_\varepsilon)$, *then there is a constant* $C > 0$ *independent of* ε *and* $\mathbf{f}$ *such that*

$$\langle \mathcal{B}_\varepsilon\operatorname{div}\mathbf{f}, \nabla\varphi\rangle = \langle \mathbf{f}, \nabla\varphi\rangle \text{ for any } \varphi \in W^{1,p'}(D_\varepsilon), \quad \|\mathcal{B}_\varepsilon\operatorname{div}\mathbf{f}\|_{L^q(D_\varepsilon)} \le C\,\|\mathbf{f}\|_{L^q(D_\varepsilon)}.$$

Proof. By Lemma 3.5.2, it is enough to prove the first assertion for $\left[\dot W^{1,p'}(D)\right]'$. We will just summarize the ideas, details can be found in [FN09, Theorem 10.11]. As before, for fixed $f \in \left[\dot W^{1,p'}(D)\right]'$ there is $\mathbf{w} \in L^p(D)$ such that

$$\langle f, \varphi\rangle = \int_D \mathbf{w}\cdot\nabla\varphi\,\mathrm{d}x, \quad \|\mathbf{w}\|_{L^p(D)} = \|f\|_{[W^{1,p'}(D)]'}.$$

Taking $\mathbf{w}_n \in C_c^\infty(D)$ such that $\mathbf{w}_n \to \mathbf{w}$ in $L^p(D)$ and $\|\mathbf{w}_n\|_{L^p(D)} \le 2\,\|f\|_{[W^{1,p'}(D)]'}$, we apply the decomposition Theorem 3.2.11 to define similarly to the proof of Theorem 3.2.9

$$\mathcal{B}\operatorname{div}\mathbf{w}_n \coloneqq \sum_{i\in\mathbb{N}} \mathcal{B}_i T_i \operatorname{div}\mathbf{w}_n,$$

where $\mathcal{B}_i$ is the standard Bogovskiĭ operator from Theorem 3.1.6 on the cube W_i. Since $\mathcal{B}$ is linear, we now conclude by

$$\|\mathcal{B}\operatorname{div}\mathbf{w}_n\|_{L^p(D)} \le C\sum_{i\in\mathbb{N}} \|\mathcal{B}_i T_i\operatorname{div}\mathbf{w}_n\|_{L^p(D)} \le C\,\|\mathbf{w}_n\|_{L^p(D)} \le C\,\|f\|_{[W^{1,p'}(D)]'}$$

and sending $n \to \infty$.
The second statement for the perforated domain D_ε requires more care. Let $\mathbf{f} \in E_0^{q,p}(D_\varepsilon)$, then clearly $\tilde{\mathbf{f}} \in E_0^{q,p}(D)$. Setting $\mathbf{u} = \mathcal{B}\operatorname{div}\tilde{\mathbf{f}} \in L^q(D)$, then

$$\|\mathbf{u}\|_{L^q(D)} \le C\,\|\mathbf{f}\|_{L^q(D)} \tag{3.43}$$

for some constant $C = C(q, D) > 0$. We now want to modify $\mathbf{u}$ such that it vanishes on the holes without changing its divergence. Recall the definitions of the cut-off functions $\chi_{\varepsilon,i}$ and $\zeta_{\varepsilon,j}$ as well as $\mathbf{b}_{\varepsilon,i}$ and $\beta_{\varepsilon,j}$ from (3.34)-(3.36) as

$$\begin{aligned}
&\chi_{\varepsilon,i} \in C_c^\infty(I_i^\varepsilon), \quad \chi_{\varepsilon,i}\restriction_{I_i^{\varepsilon,\mathrm{in}}} = 1, \quad \|\nabla\chi_{\varepsilon,i}\|_{L^\infty(D)} \lesssim \varepsilon^{-(1+\delta)},\\
&\zeta_{\varepsilon,j} \in C_c^\infty\Big(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j)\Big), \quad \zeta_{\varepsilon,j}\restriction_{B_{\varepsilon^\alpha r_j}(\varepsilon z_j)} = 1, \quad \|\nabla\zeta_{\varepsilon,j}\|_{L^\infty(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))} \lesssim \frac{1}{r_j}\varepsilon^{-\alpha},\\
&\mathbf{b}_{\varepsilon,i}(\mathbf{u}) = \chi_{\varepsilon,i}(\mathbf{u} - \langle\mathbf{u}\rangle_{A_i^\varepsilon}) \in L^q(I_i^\varepsilon),
\end{aligned}$$

$$\beta_{\varepsilon,j}(\mathbf{u}) = \zeta_{\varepsilon,j} \, \langle \mathbf{u} \rangle_{A_i^\varepsilon} \in L^q\Big(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j)\Big).$$

Since $\operatorname{div} \tilde{\mathbf{f}} \in L^p(D)$ with $\int_D \operatorname{div} \tilde{\mathbf{f}} = \langle \operatorname{div} \tilde{\mathbf{f}}, 1 \rangle = 0$, we have $\mathbf{u} \in W_0^{1,p}(D)$ and $R_\varepsilon(\mathbf{u}) \in W_0^{1,p}(D_\varepsilon)$ is well-defined, where R_ε is the restriction operator from (3.40). The goal is now to get a uniform estimate

$$\|R_\varepsilon(\mathbf{u})\|_{L^q(D_\varepsilon)} \leq C \, \|\mathbf{f}\|_{L^q(D_\varepsilon)}$$

for some constant $C > 0$ independent of ε and $\|\operatorname{div} \mathbf{f}\|_{L^p(D_\varepsilon)}$. First, we have

$$\begin{aligned} \|\mathbf{b}_{\varepsilon,i}(\mathbf{u})\|_{L^q(I_i^\varepsilon)} &\leq \|\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon}\|_{L^q(I_i^\varepsilon)} \leq \|\mathbf{u}\|_{L^q(I_i^\varepsilon)} + |\langle \mathbf{u} \rangle_{A_i^\varepsilon}| \, |I_i^\varepsilon|^{\frac{1}{q}} \\ &\leq \|\mathbf{u}\|_{L^q(I_i^\varepsilon)} + \frac{|I_i^\varepsilon|^{\frac{1}{q}}}{|A_i^\varepsilon|^{\frac{1}{q}}} \|\mathbf{u}\|_{L^q(A_i^\varepsilon)} \leq C \, \|\mathbf{u}\|_{L^q(I_i^\varepsilon)}, \end{aligned}$$

where we used that $|I_i^\varepsilon|$ and $|A_i^\varepsilon|$ are both of order $\varepsilon^{1+\delta}$ and $A_i^\varepsilon \subset I_i^\varepsilon$, so we may choose C independent of ε. Farther, for ε small enough such that $r_j \leq \varepsilon^{-(\frac{3}{m}+\delta)}$,

$$\begin{aligned} \|\beta_{\varepsilon,j}(\mathbf{u})\|_{L^q(I_i^\varepsilon)} &\leq |\langle \mathbf{u} \rangle_{A_i^\varepsilon}| \, |B_{2\varepsilon^\alpha r_j}(\varepsilon z_j)|^{\frac{1}{q}} \leq C \, (\varepsilon^\alpha r_j)^{\frac{3}{q}} \varepsilon^{-\frac{3}{q}(1+\delta)} \|\mathbf{u}\|_{L^q(A_i^\varepsilon)} \\ &\leq C \, \varepsilon^{\frac{3}{q}(\alpha - 1 - \frac{3}{m} - 2\delta)} \|\mathbf{u}\|_{L^q(I_i^\varepsilon)} \leq C \, \|\mathbf{u}\|_{L^q(I_i^\varepsilon)} \end{aligned}$$

since the exponent of ε is non-negative due to the definition of δ in (3.21). Using $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{f}$ and abbreviating $n_i := \#\{\varepsilon z_j \in I_i^\varepsilon\} \in [1, N] \cap \mathbb{N}$, we write

$$\begin{aligned} \operatorname{div}\left(\frac{\mathbf{b}_{\varepsilon,i}(\mathbf{u})}{n_i} + \beta_{\varepsilon,j}(\mathbf{u})\right) &= \frac{1}{n_i}\big(\chi_{\varepsilon,i} \operatorname{div} \mathbf{u} + \nabla \chi_{\varepsilon,i} \cdot (\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon})\big) + \nabla \zeta_{\varepsilon,j} \cdot \langle \mathbf{u} \rangle_{A_i^\varepsilon} \\ &= \frac{1}{n_i}\big(\chi_{\varepsilon,i} \operatorname{div} \mathbf{f} + \nabla \chi_{\varepsilon,i} \cdot (\mathbf{u} - \langle \mathbf{u} \rangle_{A_i^\varepsilon})\big) + \nabla \zeta_{\varepsilon,j} \cdot \langle \mathbf{u} \rangle_{A_i^\varepsilon} \\ &= \frac{1}{n_i} \operatorname{div}(\chi_{\varepsilon,i} \mathbf{f}) + \frac{1}{n_i} \nabla \chi_{\varepsilon,i} \cdot (\mathbf{u} - \mathbf{f}) + \nabla\left(\zeta_{\varepsilon,j} - \frac{\chi_{\varepsilon,i}}{n_i}\right) \cdot \langle \mathbf{u} \rangle_{A_i^\varepsilon}. \end{aligned} \tag{3.44}$$

We will estimate each term separately. Setting $U := I_i^\varepsilon \setminus \bigcup_{z_j \in \varepsilon^{-1} I_i^\varepsilon \cap \Phi^\varepsilon(D)} \overline{B_{\varepsilon^\alpha r_j}(\varepsilon z_j)}$, we have by $\chi_{\varepsilon,i} = 0$ on ∂I_i^ε, $\mathbf{f} \in E_0^{q,p}(D_\varepsilon) \subset E_0^{p,p}(D_\varepsilon)$, and Theorem 3.5.3, that

$$\begin{aligned} 0 &= \langle \operatorname{Tr}_{\mathbf{n}}(\mathbf{f}), \operatorname{Tr}(\chi_{\varepsilon,i}) \rangle_{[W^{1-\frac{1}{p'},p'}(\partial U)]', W^{1-\frac{1}{p'},p'}(\partial U)} \\ &= \int_U \chi_{\varepsilon,i} \operatorname{div} \mathbf{f} \, \mathrm{d}x + \int_U \nabla \chi_{\varepsilon,i} \cdot \mathbf{f} \, \mathrm{d}x = \int_U \operatorname{div}(\chi_{\varepsilon,i} \mathbf{f}) \, \mathrm{d}x, \end{aligned}$$

so we may use the fact that U is a uniform John domain by Lemma 3.4.4 and the first part of the proof to obtain

$$\|\mathcal{B}_{\varepsilon,i}(\operatorname{div}(\chi_{\varepsilon,i} \mathbf{f}))\|_{L^q(U)} \leq C \, \|\chi_{\varepsilon,i} \mathbf{f}\|_{L^q(U)} \leq C \, \|\mathbf{f}\|_{L^q(I_i^\varepsilon)}.$$

Since $\mathbf{u} \in W_0^{1,p}(D)$ has a well defined trace, $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{f}$, $\chi_{\varepsilon,i} = 0$ on ∂I_i^ε, and $\chi_{\varepsilon,i} = 1$ on every hole,

$$0 = \langle \operatorname{Tr}_{\mathbf{n}}(\mathbf{u} - \mathbf{f}), \operatorname{Tr}(\chi_{\varepsilon,i}) \rangle_{[W^{1-\frac{1}{p'},p'}(\partial I_i^\varepsilon)]', W^{1-\frac{1}{p'},p'}(\partial I_i^\varepsilon)}$$

$$= \int_{I_i^\varepsilon} \nabla\chi_{\varepsilon,i} \cdot (\mathbf{u} - \mathbf{f})\,\mathrm{d}x = \int_U \nabla\chi_{\varepsilon,i} \cdot (\mathbf{u} - \mathbf{f})\,\mathrm{d}x.$$

Similarly, since $\zeta_{\varepsilon,j}$ and $\chi_{\varepsilon,i}$ are smooth functions, $\zeta_{\varepsilon,j} = 0$ on $\partial B_{2\varepsilon^\alpha r_j}(\varepsilon z_j)$, in particular on ∂I_i^ε, and $\zeta_{\varepsilon,j} = 1$ on every hole, we have

$$\begin{aligned}
&\int_U \nabla\Big(\zeta_{\varepsilon,j} - \frac{\chi_{\varepsilon,i}}{n_i}\Big) \cdot \langle\mathbf{u}\rangle_{A_i^\varepsilon}\,\mathrm{d}x = \int_{I_i^\varepsilon} \nabla\Big(\zeta_{\varepsilon,j} - \frac{\chi_{\varepsilon,i}}{n_i}\Big) \cdot \langle\mathbf{u}\rangle_{A_i^\varepsilon}\,\mathrm{d}x \\
&\quad = \int_{I_i^\varepsilon} \operatorname{div}\Big(\Big(\zeta_{\varepsilon,j} - \frac{\chi_{\varepsilon,i}}{n_i}\Big)\langle\mathbf{u}\rangle_{A_i^\varepsilon}\Big)\,\mathrm{d}x = \int_{\partial I_i^\varepsilon} \Big(\Big(\zeta_{\varepsilon,j} - \frac{\chi_{\varepsilon,i}}{n_i}\Big)\langle\mathbf{u}\rangle_{A_i^\varepsilon}\Big) \cdot \mathbf{n}\,\mathrm{d}\sigma(x) = 0.
\end{aligned}$$

Let us show why we assumed $q > 3/2$. Let $\tilde{q} \in (1,3)$ such that $1/\tilde{q} = 1/q + 1/3$. Since U is a uniform John domain by Lemma 3.4.4, we have from the first part of the proof and by Sobolev embedding (B.8)

$$\begin{aligned}
\|\mathcal{B}_{\varepsilon,i}(\nabla\chi_{\varepsilon,i} \cdot (\mathbf{u} - \mathbf{f}))\|_{L^q(U)} &\le C\,\|\mathcal{B}_{\varepsilon,i}(\nabla\chi_{\varepsilon,i} \cdot (\mathbf{u} - \mathbf{f}))\|_{W_0^{1,\tilde{q}}(U)} \\
&\le C\,\|\nabla\chi_{\varepsilon,i} \cdot (\mathbf{u} - \mathbf{f})\|_{L^{\tilde{q}}(U)} \\
&\le C\,\|\nabla\chi_{\varepsilon,i}\|_{L^3(U)}(\|\mathbf{f}\|_{L^q(U)} + \|\mathbf{u}\|_{L^q(U)}) \\
&\le C\,\|\nabla\chi_{\varepsilon,i}\|_{L^3(I_i^\varepsilon)}(\|\mathbf{f}\|_{L^q(I_i^\varepsilon)} + \|\mathbf{u}\|_{L^q(I_i^\varepsilon)}).
\end{aligned}$$

Similarly, since $|I_i^\varepsilon|/|A_i^\varepsilon| \le C$, we get with

$$|U|^{\frac{1}{q}}\,|\langle\mathbf{u}\rangle_{A_i^\varepsilon}| \le C\,|I_i^\varepsilon|^{\frac{1}{q}}\,|A_i^\varepsilon|^{-\frac{1}{q}}\|\mathbf{u}\|_{L^q(A_i^\varepsilon)} \le C\,\|\mathbf{u}\|_{L^q(U)}$$

and $1 \le n_i \le N$ for any i the estimate

$$\begin{aligned}
\Big\|\mathcal{B}_{\varepsilon,i}\Big(\nabla\Big(\zeta_{\varepsilon,j} - \frac{\chi_{\varepsilon,i}}{n_i}\Big) \cdot \langle\mathbf{u}\rangle_{A_i^\varepsilon}\Big)\Big\|_{L^q(U)} &\le C\,\|\mathcal{B}_{\varepsilon,i}(\nabla(\zeta_{\varepsilon,j} - \chi_{\varepsilon,i}) \cdot \langle\mathbf{u}\rangle_{A_i^\varepsilon})\|_{W_0^{1,\tilde{q}}(U)} \\
&\le C\,\|\nabla(\zeta_{\varepsilon,j} - \chi_{\varepsilon,i}) \cdot \langle\mathbf{u}\rangle_{A_i^\varepsilon}\|_{L^{\tilde{q}}(U)} \\
&\le C\,(\|\nabla\zeta_{\varepsilon,j}\|_{L^3(U)} + \|\nabla\chi_{\varepsilon,i}\|_{L^3(U)})|\langle\mathbf{u}\rangle_{A_i^\varepsilon}|\,|U|^{\frac{1}{q}} \\
&\le C\,(\|\nabla\zeta_{\varepsilon,j}\|_{L^3(U)} + \|\nabla\chi_{\varepsilon,i}\|_{L^3(U)})\|\mathbf{u}\|_{L^q(U)} \\
&\le C\,(\|\nabla\zeta_{\varepsilon,j}\|_{L^3(I_i^\varepsilon)} + \|\nabla\chi_{\varepsilon,i}\|_{L^3(I_i^\varepsilon)})\|\mathbf{u}\|_{L^q(I_i^\varepsilon)}.
\end{aligned}$$

We further have from $\|\nabla\chi_{\varepsilon,i}\|_{L^\infty} \le C\,\varepsilon^{-(1+\delta)}$ and $\|\nabla\zeta_{\varepsilon,j}\|_{L^\infty} \le C\,(r_j\varepsilon^\alpha)^{-1}$

$$\begin{aligned}
&\|\nabla\chi_{\varepsilon,i}\|_{L^3(I_i^\varepsilon)} \le \|\nabla\chi_{\varepsilon,i}\|_{L^\infty(D)}\,|I_i^\varepsilon|^{\frac{1}{3}} \le C\,\varepsilon^{-(1+\delta)}(\varepsilon^{3(1+\delta)})^{\frac{1}{3}} = C, \\
&\|\nabla\zeta_{\varepsilon,j}\|_{L^3(I_i^\varepsilon)} = \|\nabla\zeta_{\varepsilon,j}\|_{L^3(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))} \le \|\nabla\zeta_{\varepsilon,j}\|_{L^\infty(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))}\,|B_{2\varepsilon^\alpha r_j}(\varepsilon z_j)|^{\frac{1}{3}} \le C
\end{aligned}$$

for some constant $C > 0$ independent of ε, i, j, and $\|\operatorname{div}\mathbf{f}\|_{L^p(D_\varepsilon)}$. Finally, from (3.43) and the fact that all boxes I_i^ε are disjoint, we establish

$$\|R_\varepsilon(\mathbf{u})\|_{L^q(D_\varepsilon)}^q \le C\,(\|\mathbf{u}\|_{L^q(D_\varepsilon)}^q + \|\mathbf{f}\|_{L^q(D_\varepsilon)}^q) \le C\,\|\mathbf{f}\|_{L^q(D_\varepsilon)}^q.$$

Recalling the definition of $\mathcal{B}_\varepsilon$ as $\mathcal{B}_\varepsilon(f) = (R_\varepsilon \circ \mathcal{B}_D)(\tilde{f})$, this completes the proof. □

Chapter 4

Homogenization results for perforated domains

In this chapter, we will give the homogenization results for domains D_ε which are perforated by small holes having radius of order ε^α. In order to obtain bounds on the density that are uniform in $\varepsilon > 0$, the outcomes from Chapter 3 and specifically from Sections 3.3, 3.4, and 3.5 are crucial. We split the chapter in several sections. The first section is devoted to the homogenization of the stationary Navier-Stokes equations for a viscous compressible fluid in a randomly perforated domain as introduced in Section 3.4, where we assume a pressure growth of $p = a\varrho^\gamma$ for some $\gamma > 3$. We will relax this to $\gamma > 2$ in Section 4.2, as well as give arguments how to proceed for the evolutionary system. Let us emphasize that the range of γ we can work with, is far away from the physical range $1 \leq \gamma \leq \frac{5}{3}$ stated in (2.14). This is due to the fact that we need a good control on the density in certain Lebesgue spaces, see Section 4.1.2. In Section 4.3, we focus on the homogenization for the stationary Navier-Stokes-Fourier equations, meaning, that we additionally take into account that the fluid is heat-conducting. In all the aforementioned sections, we further assume that the size of the holes is subcritical, meaning $\alpha > 3$, such that the limiting systems will have the same structure as the ones in the perforated domain. The last section however differs from the ones before: we focus on the case of critically sized holes $\alpha = 3$ for a periodically perforated domain, and scale the pressure by an ε-depending factor, which corresponds to the so-called Low Mach number limit. We will show that in this case, we get in the limiting equations an additional friction term being reminiscent from the holes.

4.1 The case of constant temperature and $\gamma > 3$

4.1.1 Setting and main result

In this section, we assume the holes to be small, in the sense that we assume $\alpha > 3$ in the definition of D_ε in (3.18). Furthermore, we assume the radii r_i to satisfy

$$\mathbb{E}(r_i^M) < \infty \text{ for } M = \max\{3, m\} \text{ and some } m > \frac{3}{\alpha - 3}.$$

The conditions on the moment bound M are needed in order to control the measure of D_ε independently of ε and get a uniformly bounded Bogovskiĭ operator

$$\mathcal{B}_\varepsilon : L^2_0(D_\varepsilon) \to W^{1,2}_0(D_\varepsilon),$$

see also (3.19) in Theorem 3.4.2. For an explanation on the restriction $m > 3/(\alpha-3)$, see Remark 4.1.5 below. Now, in the domain D_ε, we consider the stationary Navier-Stokes equations

$$\begin{cases} \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 & \text{in } D_\varepsilon, \\ \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla p(\varrho_\varepsilon) = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) + \varrho_\varepsilon \mathbf{f} + \mathbf{g} & \text{in } D_\varepsilon, \\ \mathbf{u}_\varepsilon = 0 & \text{on } \partial D_\varepsilon, \end{cases} \tag{4.1}$$

where the Newtonian viscous stress tensor $\mathbb{S}(\nabla \mathbf{u}_\varepsilon)$ as derived in Section 2.1 is of the form

$$\mathbb{S}(\nabla \mathbf{u}) = \mu\left(\nabla \mathbf{u} + \nabla^T \mathbf{u} - \frac{2}{3}\operatorname{div}(\mathbf{u})\mathbb{I}\right) + \eta \operatorname{div}(\mathbf{u})\mathbb{I}, \quad \mu > 0,\, \eta \geq 0, \tag{4.2}$$

and $p(\varrho) = a\varrho^\gamma$ for some $a > 0$ and $\gamma > 3$. We further assume $\mathbf{f}, \mathbf{g} \in L^\infty(D)$.
Before stating our main result, we introduce the notion of finite energy weak solutions.

Definition 4.1.1. *Let $\mathfrak{m} > 0$ be fixed. We call a couple $[\varrho, \mathbf{u}]$ a* renormalized finite energy weak solution *to equations (4.1) if:*

$$\varrho \geq 0 \text{ a.e. in } D_\varepsilon, \quad \int_{D_\varepsilon} \varrho \,\mathrm{d}x = \mathfrak{m},$$
$$\varrho \in L^{\beta(\gamma)}(D_\varepsilon) \text{ for some } \gamma \leq \beta(\gamma) \leq \infty, \quad \mathbf{u} \in W^{1,2}_0(D_\varepsilon),$$
$$\int_{D_\varepsilon} p(\varrho)\operatorname{div}\varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla\varphi - \mathbb{S}(\nabla \mathbf{u}) : \nabla\varphi + (\varrho \mathbf{f} + \mathbf{g}) \cdot \varphi \,\mathrm{d}x = 0$$

for all all test functions $\varphi \in C^\infty_c(D_\varepsilon; \mathbb{R}^3)$, the energy inequality

$$\int_{D_\varepsilon} \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} \,\mathrm{d}x \leq \int_{D_\varepsilon} (\varrho \mathbf{f} + \mathbf{g}) \cdot \mathbf{u} \,\mathrm{d}x \tag{4.3}$$

holds, and the zero extension $[\tilde{\varrho}, \tilde{\mathbf{u}}]$ satisfies in $\mathcal{D}'(\mathbb{R}^3)$

$$\operatorname{div}(\tilde{\varrho}\tilde{\mathbf{u}}) = 0, \quad \operatorname{div}(b(\tilde{\varrho})\tilde{\mathbf{u}}) + (\tilde{\varrho} b'(\tilde{\varrho}) - b(\tilde{\varrho}))\operatorname{div}\tilde{\mathbf{u}} = 0 \tag{4.4}$$

for any $b \in C([0,\infty)) \cap C^1((0,\infty))$ such that there are constants

$$c > 0, \quad \lambda_0 < 1, \quad -1 < \lambda_1 \leq \gamma - 1$$

with

$$b'(s) \leq cs^{-\lambda_0} \text{ for } s \in (0,1], \quad b'(s) \leq cs^{\lambda_1} \text{ for } s \in [1,\infty).$$

Remark 4.1.2. *Due to the DiPerna-Lions transport theory (see [DL89]), for any smooth* *domain $D \subset \mathbb{R}^3$, any $r \in L^\beta(D)$ with $\beta \geq 2$, and any $\mathbf{v} \in W_0^{1,2}(D)$ such that*

$$\operatorname{div}(r\mathbf{v}) = 0 \text{ in } \mathcal{D}'(D),$$

the couple $[r, \mathbf{v}]$, extended by zero outside D, satisfies the renormalized equation

$$\operatorname{div}((b(r)\mathbf{v}) + (rb'(r) - b(r)) \operatorname{div} \mathbf{v} = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3),$$

where $b \in C([0,\infty)) \cap C^1((0,\infty))$ is as in (4.4). We remark that if $\beta = \beta(\gamma)$ is as in Definition 4.1.1, the available existence theory requires $\gamma \geq \frac{5}{3}$ for $\beta(\gamma) \geq 2$ to hold, see Theorem 4.1.3 below.

Formally, the energy inequality (4.3) can be derived from the kinetic energy balance (2.6), which in our stationary case reads

$$\operatorname{div}\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}\right) = (\varrho\mathbf{f} + \mathbf{g}) \cdot \mathbf{u} + \operatorname{div}(\mathbb{S}\mathbf{u} - p\mathbf{u}) - \mathbb{S} : \nabla\mathbf{u} + p \operatorname{div} \mathbf{u}. \tag{4.5}$$

We multiply the continuity equation (2.2) by $a\varrho^{\gamma-1}$ to obtain

$$0 = a\varrho^\gamma \operatorname{div} \mathbf{u} + \frac{a}{\gamma}\mathbf{u} \cdot \nabla\varrho^\gamma = p \operatorname{div} \mathbf{u} + \frac{1}{\gamma}\mathbf{u} \cdot \nabla p. \tag{4.6}$$

Therefore, we get by partial integration and the homogeneous Dirichlet boundary conditions for $\mathbf{u}$, together with $\gamma > 1$,

$$\int_{D_\varepsilon} \mathbf{u} \cdot \nabla p \, \mathrm{d}x = -\int_{D_\varepsilon} p \operatorname{div} \mathbf{u} \, \mathrm{d}x = \frac{1}{\gamma}\int_{D_\varepsilon} \mathbf{u} \cdot \nabla p \, \mathrm{d}x \Longrightarrow \int_{D_\varepsilon} \mathbf{u} \cdot \nabla p \, \mathrm{d}x = 0.$$

Substituting (4.6) into (4.5) and integrating over D_ε yields

$$\int_{D_\varepsilon} \mathbb{S} : \nabla\mathbf{u} \, \mathrm{d}x + \int_{D_\varepsilon} \operatorname{div}\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u} - \mathbb{S}\mathbf{u} + p\mathbf{u}\right) \mathrm{d}x = \int_{D_\varepsilon} (\varrho\mathbf{f} + \mathbf{g}) \cdot \mathbf{u} \, \mathrm{d}x.$$

Since $\mathbf{u} = 0$ on ∂D_ε, the second integral is zero, thus we get the energy *equality*

$$\int_{D_\varepsilon} \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u} \, \mathrm{d}x = \int_{D_\varepsilon} (\varrho\mathbf{f} + \mathbf{g}) \cdot \mathbf{u} \, \mathrm{d}x.$$

We remark that this equality was obtained for *smooth* functions ϱ and $\mathbf{u}$. Since we deal with mere *weak* solutions, which are expected to dissipate more energy than indicated from the momentum equation, we get inequality rather than equality, which precisely yields (4.3).

The existence of renormalized finite energy weak solutions to system (4.1) for *fixed* $\varepsilon > 0$ is guaranteed by Theorem 4.3 in [NS04], which we cite here for further use.

Theorem 4.1.3. *Let $D \subset \mathbb{R}^3$ be a bounded domain of class C^2, $\mathbf{f}, \mathbf{g} \in L^\infty(D)$, and $\mathfrak{m} > 0$. Then there exists a renormalized finite energy weak solution $[\varrho, \mathbf{u}] \in L^{\beta(\gamma)}(D) \times W_0^{1,2}(D)$ in the* *sense of Definition 4.1.1, where*

$$\beta(\gamma) = \begin{cases} 2\gamma & \text{if } \gamma > 3, \\ 3(\gamma - 1) & \text{if } 3/2 < \gamma \leq 3. \end{cases}$$

Note that the restriction $\gamma > \frac{3}{2}$ is necessary to ensure that the convective term $\varrho \mathbf{u} \otimes \mathbf{u}$ is integrable over D, which is needed in order to interpret this term meaningfully. Besides this interpretation, it is worth mentioning that P. Plotnikov and W. Weigant obtained a similar existence result for weak solutions for all $\gamma > 1$, see [PW15].

Back to the homogenization, our main result in this section reads as follows.

Theorem 4.1.4. *Assume $\alpha > 3$. Let $D \subset \mathbb{R}^3$ be a bounded star-shaped domain with respect to the origin with smooth boundary and let $(\Phi, \mathcal{R}) = (\{z_j\}, \{r_j\})$ be a marked Poisson point process with intensity $\lambda > 0$, and $r_j > 0$ with $\mathbb{E}(r_j^M) < \infty$, $M = \max\{3, m\}$, $m > 3/(\alpha - 3)$. Farther let*

$$\mathfrak{m} > 0, \quad \gamma > 3.$$

Then for almost every $\omega \in \Omega$ there exists $\varepsilon_0 = \varepsilon_0(\omega) > 0$, such that the following holds: For $0 < \varepsilon < 1$ let D_ε be as in (3.18) and let $\{[\varrho_\varepsilon, \mathbf{u}_\varepsilon]\}_\varepsilon$ be a family of renormalized finite energy weak solutions to (4.1). Then there is a constant $C > 0$, which is independent of ε, such that

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \|\tilde{\varrho}_\varepsilon\|_{L^{2\gamma}(D)} + \|\tilde{\mathbf{u}}_\varepsilon\|_{W_0^{1,2}(D)} \leq C$$

and, up to a subsequence,

$$\tilde{\varrho}_\varepsilon \rightharpoonup \varrho \text{ weakly in } L^{2\gamma}(D), \quad \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u} \text{ weakly in } W_0^{1,2}(D),$$

where the limit $[\varrho, \mathbf{u}]$ is a renormalized finite energy weak solution to the problem (4.1) in the limit domain D.

Remark 4.1.5. *We note that the condition $m > 3/(\alpha - 3)$ on the size of radii of the perforations is not just needed for technical purposes, but it is in a sense an optimal assumption. Let us give a heuristic explanation on this. Fix $\varepsilon > 0$, then in D_ε we have an expected number of $n \approx \varepsilon^{-3}$ holes with n radii $r_i \in (0, \infty)$. We ask for the probability of having at least one "large" hole inside D, that is, for the distribution of $\max_{1 \leq i \leq n} r_i$. Since the radii $\{r_i\}$ are i.i.d. random variables, we have*

$$\mathbb{P}\Big(\max_{1 \leq i \leq n} r_i \leq R\Big) = \mathbb{P}\Big(\bigcap_{1 \leq i \leq n} \{r_i \leq R\}\Big) = \mathbb{P}(r_i \leq R)^n$$

for any $R > 0$. By the assumption on the moment bound for the radii, we have

$$\int_0^\infty t^m \, \mathrm{d}\mathbb{P}(t) = \mathbb{E}(r^m) < \infty.$$

This yields

$$\mathbb{P}(r_i > R) = \int_R^\infty 1 \,\mathrm{d}\mathbb{P}(t) \leq \int_R^\infty \frac{t^m}{R^m} \,\mathrm{d}\mathbb{P}(t) \leq R^{-m}\mathbb{E}(r^m),$$

so

$$\mathbb{P}\Big(\max_{1\leq i\leq n} r_i \leq R\Big) = \mathbb{P}(r_i \leq R)^n = \big(1 - \mathbb{P}(r_i > R)\big)^n \gtrsim (1 - R^{-m})^n$$

and hence

$$\mathbb{P}\Big(\max_{1\leq i\leq n} r_i > R\Big) = 1 - \mathbb{P}\Big(\max_{1\leq i\leq n} r_i \leq R\Big) \lesssim 1 - (1 - R^{-m})^n \lesssim 1 - \exp(-nR^{-m}),$$

meaning that the probability of having at least one hole with a radius of size R is of order $1 - \exp(-nR^{-m})$. To obtain a non-vanishing probability on this as $n \to \infty$, which is equivalent to $\varepsilon \to 0$, we must have $nR^{-m} = \varepsilon^{-3}R^{-m} \gtrsim 1$, thus $R \lesssim \varepsilon^{-3/m}$. Assuming that the largest radius satisfies $r_{\max} = \max_{1\leq i\leq n} r_i = \varepsilon^{-3/m}$, we end up with $\varepsilon^\alpha r_{\max} = \varepsilon^{\alpha - \frac{3}{m}}$. If now $m = 3/(\alpha - 3)$, we have $\varepsilon^\alpha r_{\max} = \varepsilon^3$, which is precisely the critical scaling of radii in three dimensions. While one *large ball of size ε^3 does not necessarily mean that the system should behave as in the critical case (which is expected to lead to a law of Brinkman type), nevertheless the presence of such a large ball might change some of the properties of the system. Moreover, in the case $m < 3/(\alpha - 3)$, the size of the largest ball would scale like $\varepsilon^{\alpha - \frac{3}{m}}$ with $\alpha - \frac{3}{m} < 3$, and there might be many balls of the size at least ε^3. Thus, our assumption $m > 3/(\alpha - 3)$ seems to be necessary to obtain in the limit $\varepsilon \to 0$ the same Navier-Stokes equations in D as in the perforated domain D_ε.*

The proof of Theorem 4.1.4 in the case of periodically arranged holes with fixed radii $r_i = 1$ for all i was developed in a series of works [DFL17, FL15, LS18], and can be split into two parts. First, using Bogovskiĭ's operator, we construct a good test function for the momentum equation, which leads to uniform in ε estimates on the density as well as the velocity, subsequently providing the (weak) compactness of $\{[\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon]\}_\varepsilon$. To identify the limiting "effective" equation, we need to construct a suitable cut-off function in order to compare the limiting equation with the equation in D_ε. The rest of the proof does not refer in any way to the location or structure of the holes, in particular it applies verbatim in our context, so for the remaining part we follow [DFL17, FL15].

Before we show the homogenization result, we prove a modification of [LS18, Lemma 2.1] in the random setting as the last ingredient in the proof of Theorem 4.1.4, which makes a reference to the randomness in the structure of the holes.

Lemma 4.1.6. *Let $\alpha > 2$, $D \subset \mathbb{R}^3$ be a bounded star-shaped domain with smooth boundary and $0 \in D$, and $(\Phi, \mathcal{R}) = (\{z_i\}, \{r_i\})$ be a marked Poisson point process with intensity $\lambda > 0$ and $r_i > 0$ with $\mathbb{E}(r_i^M) < \infty$ for $M = \max\{3, m\}$, where $m > 3/(\alpha - 2)$. Then for any $1 < q < 3$ such that $(3 - q)\alpha - 3 > 0$ and for almost every ω there exist a positive $\varepsilon_0(\omega)$ and a family of functions $\{g_\varepsilon\}_{\varepsilon>0} \subset C^\infty(D)$ such that for $0 < \varepsilon \leq \varepsilon_0$,*

$$g_\varepsilon = 0 \quad \text{in} \bigcup_{z_j \in \Phi^\varepsilon(D)} B_{\varepsilon^\alpha r_j}(\varepsilon z_j), \qquad g_\varepsilon \to 1 \quad \text{in } W^{1,q}(D) \text{ as } \varepsilon \to 0, \tag{4.7}$$

and there is a constant $C > 0$ such that

$$\|1 - g_\varepsilon\|_{L^q(D)} \le C\,\varepsilon^{\frac{3(\alpha-1)}{q}}, \quad \|\nabla g_\varepsilon\|_{L^q(D)} \le C\,\varepsilon^{\frac{(3-q)\alpha-3}{q}}. \tag{4.8}$$

Proof. By $M > 3/(\alpha - 2)$ and Theorem 3.4.2, there exists an a.s. positive random variable $\varepsilon_0(\omega)$ such that for $0 < \varepsilon \le \varepsilon_0$, all the balls $\{B_{2\varepsilon^\alpha r_j}(\varepsilon z_j)\}_{z_j \in \Phi^\varepsilon(D)}$ are disjoint. Thus, there exist functions $g_\varepsilon \in C^\infty(D)$ such that

$$0 \le g_\varepsilon \le 1, \quad g_\varepsilon = 0 \text{ in } \bigcup_{z_j \in \Phi^\varepsilon(D)} B_{\varepsilon^\alpha r_j}(\varepsilon z_j), \quad g_\varepsilon = 1 \text{ in } D \setminus \bigcup_{z_j \in \Phi^\varepsilon(D)} B_{2\varepsilon^\alpha r_j}(\varepsilon z_j),$$
$$\|\nabla g_\varepsilon\|_{L^\infty(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))} \le C\,(\varepsilon^\alpha r_j)^{-1} \text{ for all } z_j \in \Phi^\varepsilon(D),$$

where the constant $C > 0$ is independent of ε and r_j. Moreover, since $M \ge 3$, (3.25) yields $\lim_{\varepsilon \to 0} \varepsilon^3 \sum_{z_j \in \Phi^\varepsilon(D)} r_j^3 = C$, thus implying

$$\Big| \bigcup_{z_j \in \Phi^\varepsilon(D)} B_{2\varepsilon^\alpha r_j}(\varepsilon z_j) \Big| = |B_2|\,\varepsilon^{3\alpha} \sum_{z_j \in \Phi^\varepsilon(D)} r_j^3 \le C\,\varepsilon^{3(\alpha-1)}$$

for $\varepsilon > 0$ small enough. This yields for any $1 < q < 3$

$$\begin{aligned}\|1 - g_\varepsilon\|_{L^q(D)}^q &= \|1 - g_\varepsilon\|^q_{L^q(\bigcup_{z_j \in \Phi^\varepsilon(D)} B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))} = \sum_{z_j \in \Phi^\varepsilon(D)} \|1 - g_\varepsilon\|^q_{L^q(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))} \\ &\le \sum_{z_j \in \Phi^\varepsilon(D)} |B_{2\varepsilon^\alpha r_j}(\varepsilon z_j)| \le C\,\varepsilon^{3(\alpha-1)}\end{aligned}$$

as well as

$$\begin{aligned}\|\nabla g_\varepsilon\|^q_{L^q(D)} &= \|\nabla g_\varepsilon\|^q_{L^q(\bigcup_{z_j \in \Phi^\varepsilon(D)} B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))} = \sum_{z_j \in \Phi^\varepsilon(D)} \|\nabla g_\varepsilon\|^q_{L^q(B_{2\varepsilon^\alpha r_j}(\varepsilon z_j))} \\ &\le C \sum_{z_j \in \Phi^\varepsilon(D)} (\varepsilon^\alpha r_j)^{-q} |B_{2\varepsilon^\alpha r_j}(\varepsilon z_j)| \le C\,\varepsilon^{(3-q)\alpha-3}\,\varepsilon^3 \sum_{z_j \in \Phi^\varepsilon(D)} r_j^{3-q} \le C\,\varepsilon^{(3-q)\alpha-3},\end{aligned}$$

which completes the proof of the lemma. □

4.1.2 Proof of Theorem 4.1.4: Uniform bounds

We want to give bounds independent of ε for the velocity $\mathbf{u}_\varepsilon$ and the density ϱ_ε arising in the Navier-Stokes equations (4.1). First, we calculate

$$\begin{aligned}\mu \int_{D_\varepsilon} |\nabla \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x &\le \int_{D_\varepsilon} \mu |\nabla \mathbf{u}_\varepsilon|^2 + \Big(\frac{\mu}{3} + \eta\Big) |\operatorname{div} \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x \\ &= \int_{D_\varepsilon} \mu \Big(|\nabla \mathbf{u}_\varepsilon|^2 + |\operatorname{div} \mathbf{u}_\varepsilon|^2 - \frac{2}{3} |\operatorname{div} \mathbf{u}_\varepsilon|^2 \Big) + \eta |\operatorname{div} \mathbf{u}_\varepsilon|^2 \,\mathrm{d}x \\ &= \int_{D_\varepsilon} \mu \Big(\nabla \mathbf{u}_\varepsilon + \nabla^T \mathbf{u}_\varepsilon - \frac{2}{3} \operatorname{div}(\mathbf{u}_\varepsilon)\mathbb{I} \Big) : \nabla \mathbf{u}_\varepsilon + \eta \operatorname{div}(\mathbf{u}_\varepsilon)\mathbb{I} : \nabla \mathbf{u}_\varepsilon \,\mathrm{d}x \\ &= \int_{D_\varepsilon} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : \nabla \mathbf{u}_\varepsilon \,\mathrm{d}x,\end{aligned} \tag{4.9}$$

where in the two last equalities we used the definition of $\mathbb{S}(\nabla \mathbf{u}_\varepsilon)$ and the fact that

$$\int_{D_\varepsilon} |\operatorname{div} \mathbf{u}_\varepsilon|^2 \, \mathrm{d}x = -\langle \nabla \operatorname{div} \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon \rangle_{W^{-1,2}(D_\varepsilon), W_0^{1,2}(D_\varepsilon)}$$
$$= -\langle \operatorname{div} \nabla^T \mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon \rangle_{W^{-1,2}(D_\varepsilon), W_0^{1,2}(D_\varepsilon)} = \int_{D_\varepsilon} \nabla^T \mathbf{u}_\varepsilon : \nabla \mathbf{u}_\varepsilon \, \mathrm{d}x.$$

By the energy inequality (4.3) and Hölder's inequality (B.2), (4.9) yields

$$\|\nabla \mathbf{u}_\varepsilon\|^2_{L^2(D_\varepsilon)} \leq C \left(\|\mathbf{f}\|_{L^\infty(D_\varepsilon)} \|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)} + \|\mathbf{g}\|_{L^\infty(D_\varepsilon)} \right) \|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)}.$$

Since $\mathbf{u}_\varepsilon \in W_0^{1,2}(D_\varepsilon)$, we can use Poincaré's inequality (B.6) and Sobolev embedding (B.8) to obtain $\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)} \leq C \|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)}$, which combined with the previous display yields

$$\begin{aligned} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)} + \|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)} &\leq C \left(\|\mathbf{f}\|_{L^\infty(D_\varepsilon)} \|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)} + \|\mathbf{g}\|_{L^\infty(D_\varepsilon)} \right) \\ &\leq C \left(\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)} + 1 \right). \end{aligned} \tag{4.10}$$

Hence, we have uniform bounds on $\mathbf{u}_\varepsilon$ once we establish bounds on the density ϱ_ε. To this end, we define a test function

$$\varphi := \mathcal{B}_\varepsilon \left(\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon^\gamma \rangle_{D_\varepsilon} \right), \tag{4.11}$$

where $\langle \varrho_\varepsilon^\gamma \rangle_{D_\varepsilon} := |D_\varepsilon|^{-1} \int_{D_\varepsilon} \varrho_\varepsilon^\gamma \, \mathrm{d}x$ is the mean value of $\varrho_\varepsilon^\gamma$ over the domain D_ε, and $\mathcal{B}_\varepsilon$ is the Bogovskiĭ operator constructed in Theorem 3.4.1. We remark that φ is well-defined due to the fact $\varrho_\varepsilon^\gamma \in L^2(D_\varepsilon)$. By the properties of $\mathcal{B}_\varepsilon$, we obtain $\operatorname{div} \varphi = \varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon^\gamma \rangle_{D_\varepsilon}$ in D_ε and

$$\|\varphi\|_{W_0^{1,2}(D_\varepsilon)} \leq C \left(\|\varrho_\varepsilon^\gamma\|_{L^2(D_\varepsilon)} + \|\varrho_\varepsilon^\gamma\|_{L^1(D_\varepsilon)} \right) \leq C \|\varrho_\varepsilon\|^\gamma_{L^{2\gamma}(D_\varepsilon)}.$$

Integrating the second equation of (4.1) against φ yields

$$\int_{D_\varepsilon} p(\varrho_\varepsilon) \varrho_\varepsilon^\gamma \, \mathrm{d}x = \sum_{j=1}^4 I_j,$$

where the integrals I_j are defined as

$$I_1 := \int_{D_\varepsilon} p(\varrho_\varepsilon) \langle \varrho_\varepsilon^\gamma \rangle_{D_\varepsilon} \, \mathrm{d}x, \qquad I_2 := \int_{D_\varepsilon} \mu \nabla \mathbf{u} : \nabla \varphi + \left(\frac{\mu}{3} + \eta \right) \operatorname{div} \mathbf{u}_\varepsilon \operatorname{div} \varphi \, \mathrm{d}x,$$
$$I_3 := -\int_{D_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla \varphi \, \mathrm{d}x, \qquad I_4 := -\int_{D_\varepsilon} (\varrho_\varepsilon \mathbf{f} + \mathbf{g}) \cdot \varphi \, \mathrm{d}x.$$

By the definition of the pressure as $p(\varrho_\varepsilon) = a \varrho_\varepsilon^\gamma$, interpolation between Lebesgue spaces (B.4), and the fact that the total mass of the fluid is fixed and given by $\mathfrak{m} = \|\varrho_\varepsilon\|_{L^1(D_\varepsilon)} > 0$, we estimate I_1 by

$$|I_1| \leq \frac{a}{|D_\varepsilon|} \|\varrho_\varepsilon\|^{2\gamma}_{L^\gamma(D_\varepsilon)} \leq \frac{a}{|D_\varepsilon|} \left(\|\varrho_\varepsilon\|^{\theta_1}_{L^1(D_\varepsilon)} \|\varrho_\varepsilon\|^{1-\theta_1}_{L^{2\gamma}(D_\varepsilon)} \right)^{2\gamma} \leq C \|\varrho_\varepsilon\|^{2\gamma(1-\theta_1)}_{L^{2\gamma}(D_\varepsilon)},$$

where we used (3.26) to control $|D_\varepsilon|$, and $\theta_1 \in (0,1)$ is determined by

$$\frac{1}{\gamma} = \frac{\theta_1}{1} + \frac{1-\theta_1}{2\gamma}.$$

For I_2, we get with (4.10)

$$\begin{aligned}|I_2| &\le C\,\|\nabla \mathbf{u}\|_{L^2(D_\varepsilon)}\|\nabla\varphi\|_{L^2(D_\varepsilon)} \le C\,(\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)}+1)\|\varrho_\varepsilon\|^{\gamma}_{L^{2\gamma}(D_\varepsilon)}\\ &\le C\,\|\varrho_\varepsilon\|^{\gamma}_{L^{2\gamma}(D_\varepsilon)}(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)}+1).\end{aligned}$$

If $\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} \ge 1$, then we obtain

$$\|\varrho_\varepsilon\|^{\gamma}_{L^{2\gamma}(D_\varepsilon)}(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)}+1) \le 2\,\|\varrho_\varepsilon\|^{\gamma+1}_{L^{2\gamma}(D_\varepsilon)} \le 2\left(1+\|\varrho_\varepsilon\|^{\gamma+1}_{L^{2\gamma}(D_\varepsilon)}\right).$$

On the other hand, if $\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} \le 1$, we get similar

$$\|\varrho_\varepsilon\|^{\gamma}_{L^{2\gamma}(D_\varepsilon)}(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)}+1) = \|\varrho_\varepsilon\|^{\gamma+1}_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|^{\gamma}_{L^{2\gamma}(D_\varepsilon)} \le \|\varrho_\varepsilon\|^{\gamma+1}_{L^{2\gamma}(D_\varepsilon)}+1,$$

so finally,

$$|I_2| \le C\left(1+\|\varrho_\varepsilon\|^{\gamma+1}_{L^{2\gamma}(D_\varepsilon)}\right).$$

For I_3, we get analogously

$$\begin{aligned}|I_3| &\le \|\varrho_\varepsilon\|_{L^6(D_\varepsilon)}\|\mathbf{u}_\varepsilon\|^2_{L^6(D_\varepsilon)}\|\nabla\varphi\|_{L^2(D_\varepsilon)} \le C\,\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)}(\|\varrho_\varepsilon\|^2_{L^{\frac{6}{5}}(D_\varepsilon)}+1)\|\varrho_\varepsilon\|^{\gamma}_{L^{2\gamma}(D_\varepsilon)}\\ &\le C\,\|\varrho\|^{\gamma+1}_{L^{2\gamma}(D_\varepsilon)}(\|\varrho_\varepsilon\|^{2\theta_2}_{L^1(D_\varepsilon)}\|\varrho_\varepsilon\|^{2(1-\theta_2)}_{L^{2\gamma}(D_\varepsilon)}+1) \le C\,(\|\varrho\|^{\gamma+3-2\theta_2}_{L^{2\gamma}(D_\varepsilon)}+\|\varrho_\varepsilon\|^{\gamma+1}_{L^{2\gamma}(D_\varepsilon)})\\ &\le C\left(1+\|\varrho_\varepsilon\|^{\gamma+3-2\theta_2}_{L^{2\gamma}(D_\varepsilon)}\right),\end{aligned}$$

where we used that $\gamma > 3$, and $\theta_2 \in (0,1)$ is obtained by

$$\frac{5}{6} = \frac{\theta_2}{1} + \frac{1-\theta_2}{2\gamma}.$$

For I_4, we get as for I_2

$$\begin{aligned}|I_4| &\le C\,(\|\varrho_\varepsilon\|_{L^2(D_\varepsilon)}+1)\|\varphi\|_{L^2(D_\varepsilon)} \le C\,(\|\varrho_\varepsilon\|^{\gamma+1}_{L^{2\gamma}(D_\varepsilon)}+\|\varrho_\varepsilon\|^{\gamma}_{L^{2\gamma}(D_\varepsilon)})\\ &\le C\left(1+\|\varrho_\varepsilon\|^{\gamma+1}_{L^{2\gamma}(D_\varepsilon)}\right).\end{aligned}$$

Finally, we obtain

$$a\|\varrho_\varepsilon\|^{2\gamma}_{L^{2\gamma}(D_\varepsilon)} = \int_{D_\varepsilon} p(\varrho_\varepsilon)\varrho_\varepsilon^\gamma\,\mathrm{d}x \le C\left(\|\varrho_\varepsilon\|^{2\gamma-\beta}_{L^{2\gamma}(D_\varepsilon)}+1\right),$$

where due to our assumption $\gamma > 3$ we can choose

$$\beta = \min\{2\gamma\theta_1, \gamma-1, \gamma-3+2\theta_2\} > 0,$$

which yields $\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} \le C$. In view of (4.10), we also have $\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(D_\varepsilon)} \le C$, where the constant $C > 0$ does not depend on ε. This completes the proof for the uniform bounds.

4.1.3 Proof of Theorem 4.1.4: The limiting system

In the following we want to identify the limiting equations governing the fluid's motion as $\varepsilon \to 0$. First, using the fact that $[\varrho_\varepsilon, \mathbf{u}_\varepsilon]$ is a renormalized weak solution in D_ε, we get that the zero extensions of ϱ_ε and $\mathbf{u}_\varepsilon$ solve

$$\operatorname{div}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon) = 0, \quad \operatorname{div}(b(\tilde{\varrho}_\varepsilon)\tilde{\mathbf{u}}_\varepsilon) + (\tilde{\varrho}_\varepsilon b'(\tilde{\varrho}_\varepsilon) - b(\tilde{\varrho}_\varepsilon)) \operatorname{div} \tilde{\mathbf{u}}_\varepsilon = 0 \text{ in } \mathcal{D}'(D),$$

where $b \in C([0,\infty)) \cap C^1((0,\infty))$ is as in Definition 4.1.1. Letting $\varepsilon \to 0$, we obtain

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \text{ in } \mathcal{D}'(D).$$

Due to the DiPerna-Lions theory (see Remark 4.1.2), this shows that $[\varrho, \mathbf{u}]$ also satisfy the renormalized continuity equations.

Considering the momentum equation in the whole domain, we get an error F_ε on the right-hand side of the equation. Since the balls are tiny ($\alpha > 3$), this friction term is in the limit negligible. More precisely, the zero prolongations of the density and velocity satisfy

$$\nabla p(\tilde{\varrho}_\varepsilon) + \operatorname{div}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) - \operatorname{div} \mathbb{S}(\nabla \tilde{\mathbf{u}}_\varepsilon) = \tilde{\varrho}_\varepsilon \mathbf{f} + \mathbf{g} + F_\varepsilon, \tag{4.12}$$

where F_ε is a distribution satisfying for all $\varphi \in C_c^\infty(D)$

$$|\langle F_\varepsilon, \varphi \rangle_{\mathcal{D}'(D),\mathcal{D}(D)}| \leq C \left(\varepsilon^\sigma \|\varphi\|_{L^{r_2}(D)} + \varepsilon^{\frac{3(\alpha-1)\sigma_0}{2(2+\sigma_0)}} \|\nabla \varphi\|_{L^{2+\sigma_0}(D)} \right) \tag{4.13}$$

for some constants σ, r_2, σ_0 defined in (4.14) and (4.15) below. To show this, we will use the cut-off functions g_ε from Lemma 4.1.6. For any test function $\varphi \in C_c^\infty(D)$, we test the momentum equation in D_ε with $g_\varepsilon \varphi$ to get

$$\begin{aligned}
&\int_D \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon : \nabla \varphi + p(\tilde{\varrho}_\varepsilon) \operatorname{div} \varphi - \mathbb{S}(\nabla \tilde{\mathbf{u}}_\varepsilon) : \nabla \varphi + (\tilde{\varrho}_\varepsilon \mathbf{f} + \mathbf{g}) \cdot \varphi \, \mathrm{d}x \\
&= I_\varepsilon + \int_D \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon : \nabla(g_\varepsilon \varphi) + p(\tilde{\varrho}_\varepsilon) \operatorname{div}(g_\varepsilon \varphi) - \mathbb{S}(\nabla \tilde{\mathbf{u}}_\varepsilon) : \nabla(g_\varepsilon \varphi) \\
&\qquad + (\tilde{\varrho}_\varepsilon \mathbf{f} + \mathbf{g}) \cdot (g_\varepsilon \varphi) \, \mathrm{d}x \\
&= I_\varepsilon,
\end{aligned}$$

where we used that $g_\varepsilon \varphi \in C_c^\infty(D)$ with $g_\varepsilon \varphi = 0$ on $D \setminus D_\varepsilon$ is an appropriate test function, and the term I_ε is given by

$$\begin{aligned}
I_\varepsilon := \sum_{j=1}^4 I_{\varepsilon,j} := &\int_D \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon : (1-g_\varepsilon)\nabla \varphi - \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon : (\nabla g_\varepsilon \otimes \varphi) \, \mathrm{d}x \\
&+ \int_D p(\tilde{\varrho}_\varepsilon)(1-g_\varepsilon) \operatorname{div} \varphi - p(\tilde{\varrho}_\varepsilon) \nabla g_\varepsilon \cdot \varphi \, \mathrm{d}x \\
&+ \int_D -\mathbb{S}(\nabla \tilde{\mathbf{u}}_\varepsilon) : (1-g_\varepsilon)\nabla \varphi + \mathbb{S}(\nabla \tilde{\mathbf{u}}_\varepsilon) : (\nabla g_\varepsilon \otimes \varphi) \, \mathrm{d}x \\
&+ \int_D (\tilde{\varrho}_\varepsilon \mathbf{f} + \mathbf{g}) \cdot (1-g_\varepsilon) \varphi \, \mathrm{d}x.
\end{aligned}$$

We will estimate each $I_{\varepsilon,j}$ separately. For $I_{\varepsilon,1}$, we have

$$|I_{\varepsilon,1}| \leq C\,\|\tilde{\varrho}_\varepsilon\|_{L^{2\gamma}(D)}\|\tilde{\mathbf{u}}_\varepsilon\|^2_{L^6(D)}(\|(1-g_\varepsilon)\nabla\varphi\|_{L^2(D)} + \|\nabla g_\varepsilon \otimes \varphi\|_{L^2(D)}).$$

By the uniform bounds on ϱ_ε and $\mathbf{u}_\varepsilon$ established in Section 4.1.2, we get by Hölder's inequality (B.2)

$$|I_{\varepsilon,1}| \leq C\,(\|1-g_\varepsilon\|_{L^{\frac{2(2+\sigma_0)}{\sigma_0}}(D)}\|\nabla\varphi\|_{L^{2+\sigma_0}(D)} + \|\nabla g_\varepsilon\|_{L^{r_1}(D)}\|\varphi\|_{L^{r_2}(D)}),$$

where

$$\sigma_0 \in (0,\infty), \quad r_1, r_2 \in (2,\infty), \quad \frac{1}{2} = \frac{1}{r_1} + \frac{1}{r_2}. \tag{4.14}$$

This together with Lemma 4.1.6 yields

$$|I_{\varepsilon,1}| \leq C\,\varepsilon^{\frac{3(\alpha-1)\sigma_0}{2(2+\sigma_0)}}\|\nabla\varphi\|_{L^{2+\sigma_0}(D)} + \varepsilon^{\frac{(3-r_1)\alpha-3}{r_1}}\|\varphi\|_{L^{r_2}(D)},$$

where the number

$$\sigma := \frac{(3-r_1)\alpha-3}{r_1} = \frac{3(\alpha-1)}{r_1} - \alpha = \frac{3(\alpha-1)}{2} - \alpha - \frac{3(\alpha-1)}{r_2} = \frac{\alpha-3}{2} - \frac{3(\alpha-1)}{r_2}$$

is strictly positive if we choose

$$r_2 = \frac{12(\alpha-1)}{\alpha-3}, \quad \text{which yields} \quad \sigma = \frac{\alpha-3}{4}. \tag{4.15}$$

By the uniform estimates on $\tilde{\varrho}_\varepsilon$ and $\tilde{\mathbf{u}}_\varepsilon$, we get

$$\|p(\tilde{\varrho}_\varepsilon)\|_{L^2(D)} + \|\mathbb{S}(\nabla\tilde{\mathbf{u}}_\varepsilon)\|_{L^2(D)} \leq C,$$

such that the estimates for the integrals $I_{\varepsilon,2}$ and $I_{\varepsilon,3}$ are exactly the same as for $I_{\varepsilon,1}$. For $I_{\varepsilon,4}$, we obtain

$$\begin{aligned}|I_{\varepsilon,4}| &\leq C\,(\|\tilde{\varrho}_\varepsilon\|_{L^2(D)} + 1)\|(1-g_\varepsilon)\varphi\|_{L^2(D)} \leq C\,\|1-g_\varepsilon\|_{L^{r_1}(D)}\|\varphi\|_{L^{r_2}(D)}\\ &\leq C\,\varepsilon^{\frac{3(\alpha-1)}{r_1}}\|\varphi\|_{L^{r_2}(D)} \leq C\,\varepsilon^{\sigma}\|\varphi\|_{L^{r_2}(D)}.\end{aligned}$$

Combining the estimates above finally yields (4.12).

By the uniform estimates on ϱ_ε and $\mathbf{u}_\varepsilon$, we can extract a subsequence (not relabeled) such that

$$\tilde{\varrho}_\varepsilon \rightharpoonup \varrho \text{ weakly in } L^{2\gamma}(D), \quad \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u} \text{ weakly in } W_0^{1,2}(D).$$

By the Rellich-Kondrachev theorem in Proposition B.5, this yields

$$\begin{aligned}&\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \text{ strongly in } L^q(D) \text{ for any } 1 \leq q < 6,\\ &\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \varrho\mathbf{u} \text{ weakly in } L^q(D) \text{ for any } 1 \leq q < \frac{6\gamma}{\gamma+3},\end{aligned}$$

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \varrho \mathbf{u} \otimes \mathbf{u} \text{ weakly in } L^q(D) \text{ for any } 1 \le q < \frac{6\gamma}{2\gamma+3}.$$

Letting $\varepsilon \to 0$ in the first and second equation of (4.1), we get the following equations in the sense of distributions:

$$\begin{cases} \operatorname{div}(\varrho \mathbf{u}) = 0 & \text{in } D, \\ \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \overline{p(\varrho)} = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) + \varrho \mathbf{f} + \mathbf{g} & \text{in } D, \\ \mathbf{u} = 0 & \text{on } \partial D, \end{cases} \tag{4.16}$$

where $\overline{p(\varrho)}$ is the weak limit of $p(\tilde{\varrho}_\varepsilon)$ in $L^2(D)$. To finish the proof of Theorem 4.1.4, we have to prove $\overline{p(\varrho)} = p(\varrho)$, arguing as in [FL15, Section 2.4.2]. This will be shown in Appendix A.

4.2 Lower γ and time-dependent equations

In this section, we give some arguments how one can improve the adiabatic exponent in the direction of physical relevance, as well as how one can treat also time-dependent equations. In both cases, for mathematical reasons, we are still not able to achieve $\gamma = \frac{5}{3}$, which would be the first "meaningful" exponent, see (2.14). In fact, for the steady setting we may improve γ to be strictly larger than 2, while for time-dependent equations, we need the even worse bound $\gamma > 6$. We will comment this issue later on in this section.

4.2.1 The case $\gamma > 2$

To start, let us state the homogenization result for steady compressible Navier-Stokes equations (4.1) in D_ε for the case $2 < \gamma \le 3$.

Theorem 4.2.1. *Assume $\alpha > 3$ and $2 < \gamma \le 3$. Let $D \subset \mathbb{R}^3$ be a bounded star-shaped domain with respect to the origin with smooth boundary, and let $(\Phi, \mathcal{R}) = (\{z_j\}, \{r_j\})$ be a marked Poisson point process with intensity $\lambda > 0$, and $r_j > 0$ with $\mathbb{E}(r_j^M) < \infty$, $M = \max\{3, m\}$, $m > 3/(\alpha - 3)$, and assume*

$$\alpha - \frac{3}{m} > \frac{2\gamma - 3}{\gamma - 2}. \tag{4.17}$$

Additionally, let $\mathfrak{m} > 0$ be given. Then for almost every $\omega \in \Omega$ there exists $\varepsilon_0 = \varepsilon_0(\omega) > 0$ such that the following holds: For $0 < \varepsilon < 1$ let D_ε be as in (3.18) and let $\{[\varrho_\varepsilon, \mathbf{u}_\varepsilon]\}_\varepsilon$ be a family of renormalized finite energy weak solutions to (4.1) in the sense of Definition 4.1.1. Then there is a constant $C > 0$, which is independent of ε, such that

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \|\tilde{\varrho}_\varepsilon\|_{L^{3(\gamma-1)}(D)} + \|\tilde{\mathbf{u}}_\varepsilon\|_{W_0^{1,2}(D)} \le C$$

and, up to a subsequence,

$$\tilde{\varrho}_\varepsilon \rightharpoonup \varrho \text{ weakly in } L^{3(\gamma-1)}(D), \quad \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u} \text{ weakly in } W_0^{1,2}(D),$$

where the limit $[\varrho, \mathbf{u}]$ is a renormalized finite energy weak solution to the problem (4.1) in the limit domain D in the sense of Definition 4.1.1.

The restriction (4.17) will appear again in Theorem 4.3.3, showing that $\gamma > 2$ is necessary according to the available mathematical theory. The proof of Theorem 4.2.1 is based on the similar result obtained in [DFL17] for the case of well separated holes as well as the bounds for the Bogovskiĭ operator $\mathcal{B}_\varepsilon$ obtained in Section 3.4. Recall that the bound for $\mathcal{B}_\varepsilon$ is uniform with respect to ε as long as $1 < q < 3$ fulfills condition (3.19).

As it does not seem to cause any trouble or new difficulties at first glance, let us recall from Theorem 4.1.3 that for $\gamma > 3$, the density is known to satisfy $\varrho_\varepsilon \in L^{2\gamma}(D_\varepsilon)$, whereas for $\gamma \leq 3$ we only have the weaker control $\varrho_\varepsilon \in L^{3(\gamma-1)}(D_\varepsilon)$. Thus, we are not allowed to use $\varrho_\varepsilon^\gamma$ in the definition of the test function φ in (4.11). Instead, one may use

$$\varphi := \mathcal{B}_\varepsilon(\varrho_\varepsilon^{2\gamma-3} - \langle \varrho_\varepsilon^{2\gamma-3} \rangle_{D_\varepsilon})$$

as test function in the second equation of (4.1). Note that φ is well defined due to the fact that by $2 < \gamma \leq 3$, we have

$$\varrho_\varepsilon^{2\gamma-3} \in L^{\frac{3(\gamma-1)}{2\gamma-3}}(D_\varepsilon), \quad 2 \leq \frac{3(\gamma-1)}{2\gamma-3} < 3,$$

and also condition (3.19) is satisfied as long as

$$\alpha - \frac{3}{m} > \frac{3}{3 - \frac{3\gamma-3}{2\gamma-3}} = \frac{2\gamma-3}{\gamma-2},$$

which is precisely condition (4.17). This leads to

$$\begin{aligned} &\operatorname{div} \varphi = \varrho_\varepsilon^{2\gamma-3} - \langle \varrho^{2\gamma-3} \rangle_{D_\varepsilon} \quad \text{in } D_\varepsilon, \\ &\|\varphi\|_{W_0^{1,\frac{3(\gamma-1)}{2\gamma-3}}(D_\varepsilon)} \leq C \, \|\varrho_\varepsilon^{2\gamma-3}\|_{L^{\frac{3(\gamma-1)}{2\gamma-3}}(D_\varepsilon)} = C \, \|\varrho_\varepsilon\|_{L^{3(\gamma-1)}(D_\varepsilon)}^{2\gamma-3}, \end{aligned}$$

where the constant $C > 0$ is independent of ε. Using the same techniques as shown in the last section, we will finally arrive at

$$\|\varrho_\varepsilon\|_{L^{3(\gamma-1)}(D_\varepsilon)} \leq C, \quad \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(D_\varepsilon)} \leq C,$$

where the constant $C > 0$ is independent of ε. Choosing weakly convergent subsequences $\tilde{\varrho}_\varepsilon \rightharpoonup \varrho$ in $L^{3(\gamma-1)}(D)$, $\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u}$ in $W_0^{1,2}(D)$, we may extend the momentum equation to the whole of D to obtain

$$\nabla p(\tilde{\varrho}_\varepsilon) + \operatorname{div}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) - \operatorname{div} \mathbb{S}(\nabla \tilde{\mathbf{u}}_\varepsilon) = \tilde{\varrho}_\varepsilon \mathbf{f} + \mathbf{g} + F_\varepsilon, \tag{4.18}$$

where F_ε is a distribution now satisfying for all $\varphi \in C_c^\infty(D)$

$$|\langle F_\varepsilon, \varphi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)}| \leq C \, \varepsilon^\nu \bigg(\|\varphi\|_{L^r(D)} + \|\nabla \varphi\|_{L^{\frac{3(\gamma-1)}{2\gamma-3}+\xi}(D)} \bigg),$$

and ν, ξ, and r are chosen such that

$$0 < \xi < 1, \quad 0 < h(\xi) := 3\,(\alpha - 1) \bigg(\frac{3(\gamma-1)}{2\gamma-3} + \xi \bigg)^{-1} - \alpha,$$

$$1 < r < \infty, \quad \frac{1}{r} + \left(\frac{3(\gamma-1)}{2\gamma-3} + \xi \right)^{-1} = \frac{2\gamma-3}{3(\gamma-1)},$$
$$0 < \nu < \infty, \quad \nu := \min \left\{ \frac{3(\alpha-1)}{r}, h(\xi) \right\}.$$

These choices are appropriate since by (4.17),

$$3\,(\alpha-1)\frac{2\gamma-3}{3(\gamma-1)} - \alpha = \frac{\alpha(\gamma-2)-(2\gamma-3)}{\gamma-1} > 0,$$

and they occur due to various use of Hölder's inequality (B.2). Let us remark that similar numbers will occur in Lemma 4.3.10, where we will see how exactly the numbers ν, ξ, and r show up. Finally, we may pass to the limit $\varepsilon \to 0$ to obtain the desired result.

4.2.2 Evolutionary system

Let us show now how to deal with time-dependent Navier-Stokes equations. We will rely on the results given in [LS18], where they considered well separated obstacles. First, for $T > 0$, the system now reads

$$\begin{cases} \partial_t \varrho_\varepsilon + \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 & \text{in } (0,T) \times D_\varepsilon, \\ \partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla p(\varrho_\varepsilon) = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) + \varrho_\varepsilon \mathbf{f} + \mathbf{g} & \text{in } (0,T) \times D_\varepsilon, \\ \mathbf{u}_\varepsilon = 0 & \text{on } (0,T) \times \partial D_\varepsilon, \end{cases} \tag{4.19}$$

where as before $\mathbf{f}, \mathbf{g} \in L^\infty((0,T) \times D)$,

$$\mathbb{S}(\nabla \mathbf{u}) = \mu \left(\nabla \mathbf{u} + \nabla^T \mathbf{u} - \frac{2}{3} \operatorname{div}(\mathbf{u}) \mathbb{I} \right) + \eta \operatorname{div}(\mathbf{u}) \mathbb{I}, \quad \mu > 0,\, \eta \geq 0,$$

and $p(\varrho) = a\varrho^\gamma$ for some $a > 0$ and $\gamma > 6$. The main difficulty in the evolutionary case is that, for $\gamma > \frac{3}{2}$, the pressure is not known to be in $L^2(D_\varepsilon)$, but only in $L^{\frac{5}{3}-\frac{1}{\gamma}}(D_\varepsilon)$ with $\frac{5}{3} - \frac{1}{\gamma} \in (1, \frac{5}{3})$, which is much worse than for the stationary case. The condition $\gamma > 6$ is therefore made to make sense of the term

$$\int_0^T \int_D p(\tilde{\varrho}_\varepsilon)\, \varphi \cdot \nabla g_\varepsilon \, \mathrm{d}x \, \mathrm{d}t,$$

which will arise by testing the momentum equation by $\varphi \in \mathcal{D}((0,T) \times D)$ and split $\varphi = g_\varepsilon \varphi + (1-g_\varepsilon)\varphi$, where g_ε are the functions from Lemma 4.1.6. It also ensures that there exists $q \in (\frac{5}{2}, 3)$ such that

$$(3-q)\alpha - 3 > 0, \quad \left(\frac{5}{3} - \frac{1}{\gamma} \right)^{-1} + \frac{1}{q} < 1, \quad \frac{1}{\gamma} + \frac{1}{3} + \frac{1}{q} < 1, \tag{4.20}$$

so the additional distribution F_ε in the extended momentum equation, which arises similarly as in the stationary case (4.18), will vanish in the limit $\varepsilon \to 0$. In addition, we will need to control the Bogovskiĭ operator in some negative Sobolev space to handle terms of the form $\mathcal{B}_\varepsilon(\operatorname{div}(\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon))$ that arise from the renormalized continuity equation (4.23) and the time derivative in the weak formulation (4.21) below, that is, from $\partial_t \mathcal{B}_\varepsilon(\varrho_\varepsilon) = \mathcal{B}_\varepsilon(\partial_t \varrho_\varepsilon)$. To this end, we will use the outcomes and notations from Section 3.5.

Let us turn to the homogenization of the evolutionary Navier-Stokes equations. To this end, we will transfer the notion of finite energy weak solutions to the time-dependent case.

Definition 4.2.2. *Let $T > 0$ be fixed, and assume for the initial data*

$$\varrho(0,\,\cdot) = \varrho_0, \quad (\varrho\mathbf{u})(0,\,\cdot) = \mathbf{p}_0,$$

together with the compatibility conditions

$$\varrho_0 \geq 0 \text{ a.e. in } D_\varepsilon, \quad \varrho_0 \in L^\gamma(D_\varepsilon), \quad \mathbf{p}_0 = 0 \text{ whenever } \varrho_0 = 0, \quad \frac{|\mathbf{p}_0|^2}{\varrho_0} \in L^1(D_\varepsilon).$$

We call a couple $[\varrho, \mathbf{u}]$ a renormalized finite energy weak solution *to equations (4.1) in the space-time cylinder $(0,T) \times D_\varepsilon$ if:*

- *It holds*

$$\begin{aligned} &\varrho \geq 0 \text{ a.e. in } (0,T) \times D_\varepsilon, \quad \varrho \in C(0,T;\, L^\gamma_{\text{weak}}(D_\varepsilon)),\\ &\mathbf{u} \in L^2(0,T;\, W^{1,2}_0(D_\varepsilon)), \quad \varrho\mathbf{u} \in C(0,T;\, L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(D_\varepsilon)), \end{aligned}$$

  where $C(0,T;\, L^q_{\text{weak}}(D_\varepsilon))$ is defined as the set of all functions f, defined on $(0,T) \times D_\varepsilon$, such that $f(t,\,\cdot) \in L^q(D_\varepsilon)$ for all $t \in [0,T]$, and the map

$$t \mapsto \int_{D_\varepsilon} f(t,x) g(x) \,\mathrm{d}x$$

  is continuous for all $g \in L^{\frac{q}{q-1}}(D_\varepsilon)$;

- *We have for any $0 \leq \tau \leq T$ and any $\varphi \in C_c^\infty([0,T) \times D_\varepsilon)$*

$$\int_0^\tau \int_{D_\varepsilon} \varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_{D_\varepsilon} \varrho(\tau,\,\cdot)\varphi(\tau,\,\cdot) - \varrho_0\varphi(0,\,\cdot)\,\mathrm{d}x;$$

- *We have for any $0 \leq \tau \leq T$ and any $\psi \in C_c^\infty([0,T) \times D_\varepsilon; \mathbb{R}^3)$*

$$\begin{aligned} &\int_0^\tau \int_{D_\varepsilon} \varrho\mathbf{u}\cdot\partial_t\psi + p(\varrho)\operatorname{div}\psi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\psi - \mathbb{S}(\nabla\mathbf{u}) : \nabla\psi + (\varrho\mathbf{f} + \mathbf{g})\cdot\psi \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{D_\varepsilon} (\varrho\mathbf{u})(\tau,\,\cdot)\psi(\tau,\,\cdot) - \varrho_0\mathbf{u}_0\psi(0,\,\cdot)\,\mathrm{d}x; \end{aligned} \tag{4.21}$$

- *The energy inequality*

$$\begin{aligned} &\int_{D_\varepsilon} \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a\varrho^\gamma}{\gamma-1}\right)(\tau,\,\cdot)\,\mathrm{d}x + \int_0^\tau \int_{D_\varepsilon} \mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leq \int_{D_\varepsilon} \left(\frac{|\mathbf{p}_0|^2}{2\,\varrho_0} + \frac{a\varrho_0^\gamma}{\gamma-1}\right)\mathrm{d}x + \int_0^\tau \int_{D_\varepsilon} (\varrho\mathbf{f} + \mathbf{g})\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.22}$$

  holds for almost every $0 \leq \tau \leq T$;

- *The zero extension $[\tilde\varrho, \tilde{\mathbf{u}}]$ satisfies in $\mathcal{D}'((0,T)\times\mathbb{R}^3)$*

$$\partial_t\tilde\varrho + \operatorname{div}(\tilde\varrho\tilde{\mathbf{u}}) = 0, \quad \partial_t b(\tilde\varrho) + \operatorname{div}(b(\tilde\varrho)\tilde{\mathbf{u}}) + (\tilde\varrho b'(\tilde\varrho) - b(\tilde\varrho))\operatorname{div}\tilde{\mathbf{u}} = 0 \tag{4.23}$$

for any $b \in C([0,\infty)) \cap C^1((0,\infty))$ such that there are constants

$$c > 0, \quad \lambda_0 < 1, \quad -1 < \lambda_1 \le \gamma - 1$$

with

$$b'(s) \le cs^{-\lambda_0} \text{ for } s \in (0,1], \quad b'(s) \le cs^{\lambda_1} \text{ for } s \in [1,\infty).$$

Here, we chose the integrability of the initial data such that the right-hand side of the energy inequality is finite. Similar to the stationary case in Section 4.1, the energy inequality (4.22) is formally obtained as an equality from (2.6). Recalling

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) + \operatorname{div}\left(\frac{1}{2}\varrho|\mathbf{u}|^2\mathbf{u}\right) = (\varrho\mathbf{f} + \mathbf{g})\cdot\mathbf{u} + \operatorname{div}(\mathbb{S}\mathbf{u} - p\mathbf{u}) - \mathbb{S} : \nabla\mathbf{u} + p\operatorname{div}\mathbf{u}, \tag{4.24}$$

we again multiply the continuity equation (2.2) by $a\varrho^{\gamma-1}$ to obtain

$$0 = \frac{a}{\gamma}\partial_t\varrho^\gamma + a\varrho^\gamma\operatorname{div}\mathbf{u} + \frac{a}{\gamma}\mathbf{u}\cdot\nabla\varrho^\gamma = \frac{1}{\gamma}\partial_t p + p\operatorname{div}\mathbf{u} + \frac{1}{\gamma}\mathbf{u}\cdot\nabla p.$$

Hence, we integrate by parts and use the zero boundary data on $\mathbf{u}$ to get

$$\begin{aligned} &\int_{D_\varepsilon}\mathbf{u}\cdot\nabla p\,\mathrm{d}x = -\int_{D_\varepsilon} p\operatorname{div}\mathbf{u}\,\mathrm{d}x = \frac{1}{\gamma}\int_{D_\varepsilon}\partial_t p + \mathbf{u}\cdot\nabla p\,\mathrm{d}x \\ &\Longrightarrow (\gamma-1)\int_{D_\varepsilon}\mathbf{u}\cdot\nabla p\,\mathrm{d}x = \int_{D_\varepsilon}\partial_t p\,\mathrm{d}x \\ &\Longrightarrow -\int_{D_\varepsilon} p\operatorname{div}\mathbf{u}\,\mathrm{d}x = \frac{1}{\gamma-1}\partial_t\int_{D_\varepsilon} p\,\mathrm{d}x = \partial_t\int_{D_\varepsilon}\frac{a}{\gamma-1}\varrho^\gamma\,\mathrm{d}x. \end{aligned} \tag{4.25}$$

Substituting (4.25) into (4.24), integrating over $(0,T)\times D_\varepsilon$, and noting that the space integral over divergence parts vanish, we obtain for any $\tau \in [0,T]$

$$\begin{aligned} &\int_{D_\varepsilon}\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a\varrho^\gamma}{\gamma-1}\right)(\tau,\cdot)\,\mathrm{d}x + \int_0^\tau\int_{D_\varepsilon}\mathbb{S}(\nabla\mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{D_\varepsilon}\left(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a\varrho_0^\gamma}{\gamma-1}\right)\mathrm{d}x + \int_0^\tau\int_{D_\varepsilon}(\varrho\mathbf{f} + \mathbf{g})\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_{D_\varepsilon}\left(\frac{|\mathbf{p}_0|^2}{2\varrho_0} + \frac{a\varrho_0^\gamma}{\gamma-1}\right)\mathrm{d}x + \int_0^\tau\int_{D_\varepsilon}(\varrho\mathbf{f} + \mathbf{g})\cdot\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

As before, this equality was obtained for *smooth* functions ϱ and $\mathbf{u}$, but we expect inequality rather than equality for mere *weak* solutions, thus yielding (4.3). We remark that the composed quantity $\mathbf{p}_0$ is the "right" quantity rather than working with ϱ_0 and $\mathbf{u}_0$ as separate variables. Indeed, identifying $\varrho\mathbf{u}$ with the momentum of the fluid, it enjoys some additional time continuity, which is not (known to be) true for the velocity $\mathbf{u}$ itself.

For any fixed $\varepsilon > 0$ and any $\gamma > \frac{3}{2}$, the existence of a renormalized finite energy weak solution to (4.19) is guaranteed by the results of Lions [Lio98] and Feireisl-Novotný-Petzeltová [FNP01]. Our main theorem concerning the homogenization of time-dependent Navier-Stokes equations now reads as follows.

Theorem 4.2.3. *Assume $\alpha > 3$. Let $D \subset \mathbb{R}^3$ be a bounded star-shaped domain with respect to the origin with smooth boundary, let $(\Phi, \mathcal{R}) = (\{z_j\}, \{r_j\})$ be a marked Poisson point process with intensity $\lambda > 0$, and $r_j > 0$ with $\mathbb{E}(r_j^M) < \infty$, $M = \max\{3, m\}$, where $m > 3/(\alpha - 3)$. Furthermore, let D_ε be defined as in (3.18) and*

$$\mathfrak{m} > 0, \ \gamma > 6.$$

For $0 < \varepsilon < 1$ let $\{[\varrho_\varepsilon, \mathbf{u}_\varepsilon]\}_\varepsilon$ be a family of finite energy weak solutions for the no-slip compressible Navier-Stokes equations (4.19) in $(0, T) \times D_\varepsilon$. Assume that the initial conditions

$$\varrho_\varepsilon(0, \cdot) = \varrho_{0,\varepsilon} \quad \text{and} \quad (\varrho_\varepsilon \mathbf{u}_\varepsilon)(0, \cdot) = \mathbf{p}_{\varepsilon,0}$$

satisfy

$$\varrho_{0,\varepsilon} \in L^\gamma(D_\varepsilon), \quad \mathbf{p}_{\varepsilon,0} = 0 \text{ whenever } \varrho_{\varepsilon,0} = 0, \quad \left\| \frac{|\mathbf{p}_{\varepsilon,0}|^2}{\varrho_{\varepsilon,0}} \right\|_{L^1(D_\varepsilon)} \leq C,$$
$$\tilde{\varrho}_{\varepsilon,0} \rightharpoonup \varrho_0 \text{ weakly in } L^\gamma(D), \quad \tilde{\mathbf{p}}_{\varepsilon,0} \rightharpoonup \mathbf{p}_0 \text{ weakly in } L^{\frac{2\gamma}{\gamma+1}}(D),$$

where $C > 0$ is independent of ε. Then for almost every $\omega \in \Omega$ there exists $\varepsilon_0 = \varepsilon_0(\omega) > 0$ such that

$$\sup_{\varepsilon \in (0, \varepsilon_0)} \left(\|\varrho_\varepsilon\|_{L^\infty(0,T;L^\gamma(D_\varepsilon))} + \|\varrho_\varepsilon\|_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)} + \|\mathbf{u}_\varepsilon\|_{L^2(0,T;W_0^{1,2}(D_\varepsilon))} \right) \leq C$$

and, up to a subsequence, the zero extensions satisfy

$$\tilde{\varrho}_\varepsilon \overset{*}{\rightharpoonup} \varrho \text{ weakly-* in } L^\infty(0, T; L^\gamma(D)), \quad \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u} \text{ weakly in } L^2(0, T; W_0^{1,2}(D)),$$

where the limit $[\varrho, \mathbf{u}]$ is a renormalized finite energy weak solution to the problem (4.19) in the limit domain D with initial data $\varrho(0, \cdot) = \varrho_0$ and $(\varrho \mathbf{u})(0, \cdot) = \mathbf{p}_0$, provided

$$\frac{\gamma - 6}{2\gamma - 3} \left(\alpha - \frac{3}{m} \right) > 3. \tag{4.26}$$

The uniform bounds on ϱ_ε and $\mathbf{u}_\varepsilon$ are obtained from the energy inequality (4.22). For the force term on the right-hand side, we use Hölder's inequality (B.2), Poincaré's inequality (B.7),

$$ab^{\frac{1}{p}} \leq b + a^{p'} \quad \forall a, b \geq 0, \ \frac{1}{p} + \frac{1}{p'} = 1 \tag{4.27}$$

as a consequence of Young's inequality (B.1), and (4.9) to estimate

$$\begin{aligned}\int_{D_\varepsilon} (\varrho_\varepsilon \mathbf{f} + \mathbf{g}) \cdot \mathbf{u}_\varepsilon \, \mathrm{d}x &\leq C \left(1 + \|\varrho_\varepsilon\|_{L^\gamma(D_\varepsilon)}\right) \|\mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)} \\ &\leq C \left(1 + \|\varrho_\varepsilon\|_{L^\gamma(D_\varepsilon)}^2\right) + \frac{\mu}{2} \|\mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)}^2\end{aligned}$$

$$
\begin{aligned}
&\le C\,(1+\|\varrho_\varepsilon^\gamma\|_{L^1(D_\varepsilon)}^{\frac{2}{\gamma}}) + \frac{\mu}{2}\|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)}^2 \\
&\le C\,(1+\|\varrho_\varepsilon^\gamma\|_{L^1(D_\varepsilon)}) + \frac{1}{2}\int_{D_\varepsilon} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : \nabla \mathbf{u}_\varepsilon \\
&\le C + C\int_{D_\varepsilon} \frac{a\varrho_\varepsilon^\gamma}{\gamma-1}\,\mathrm{d}x + \frac{1}{2}\int_{D_\varepsilon} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : \nabla \mathbf{u}_\varepsilon \,\mathrm{d}x,
\end{aligned}
$$

yielding for the whole energy inequality

$$
\begin{aligned}
&\int_{D_\varepsilon} \left(\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{a\varrho_\varepsilon^\gamma}{\gamma-1}\right)(T,\,\cdot)\,\mathrm{d}x + \frac{1}{2}\int_0^T\int_{D_\varepsilon} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : \nabla \mathbf{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\
&\le \int_{D_\varepsilon} \left(\frac{|\mathbf{p}_{\varepsilon,0}|^2}{2\varrho_{\varepsilon,0}} + \frac{a\varrho_{\varepsilon,0}^\gamma}{\gamma-1}\right)\mathrm{d}x + C + C\int_0^T\int_{D_\varepsilon} \frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{a\varrho_\varepsilon^\gamma}{\gamma-1}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

By application of Grönwall's lemma (B.5), we deduce

$$
\begin{aligned}
\{\varrho_\varepsilon\}_{\varepsilon>0} &\quad \text{uniformly bounded in} \quad L^\infty(0,T;L^\gamma(D_\varepsilon)),\\
\{\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2\}_{\varepsilon>0} &\quad \text{uniformly bounded in} \quad L^\infty(0,T;L^1(D_\varepsilon)),\\
\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} &\quad \text{uniformly bounded in} \quad L^2(0,T;W_0^{1,2}(D_\varepsilon)).
\end{aligned}
$$

Moreover, due to Proposition B.5, we get for any $1 \le q \le 6$

$$
\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \quad \text{uniformly bounded in} \quad L^2(0,T;L^q(D_\varepsilon)).
$$

Further, for fixed ε, the results from [NS04, Theorem 7.7] show

$$
\varrho_\varepsilon \in L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon).
$$

However, the bound on ϱ_ε is not uniform in ε. As in the stationary case, using the Bogovskiĭ operator from Theorem 3.5.4, we test the second equation in (4.19) with

$$
\varphi(t,x) = \psi(t)\mathcal{B}_\varepsilon\Big(\varrho_\varepsilon^{\frac{2\gamma}{3}-1} - \langle \varrho_\varepsilon^{\frac{2\gamma}{3}-1}\rangle_{D_\varepsilon}\Big)
$$

for some function $\psi \in C_c^\infty(0,T)$, where $\langle f\rangle_{D_\varepsilon}$ is the mean value of a function f over D_ε. Abbreviating $\theta := \frac{2\gamma}{3} - 1$, this yields

$$
\varphi \in W_0^{1,\frac{5\gamma-3}{2\gamma-3}}((0,T)\times D_\varepsilon), \quad \operatorname{div}\varphi = \psi(t)(\varrho_\varepsilon^\theta - \langle \varrho_\varepsilon^\theta\rangle_{D_\varepsilon}).
$$

Recall from Theorem 3.4.1, the bound on the Bogovskiĭ operator for $q = \frac{5\gamma-3}{2\gamma-3}$ is uniform as long as

$$
1 < q < 3, \quad \alpha - \frac{3}{m} > \frac{3}{3-q}.
$$

Indeed, we obviously have $q > 5/2 > 1$, and from $\gamma > 6$ we obtain

$$
q < 3 \iff 5\gamma - 3 < 6\gamma - 9 \iff \gamma > 6.
$$

Moreover, we calculate

$$\alpha - \frac{3}{m} > \frac{3}{3 - \frac{5\gamma-3}{2\gamma-3}} = 3\,\frac{2\gamma-3}{\gamma-6},$$

which is precisely condition (4.26). Thus, we get

$$\|\varphi\|_{W_0^{1,\frac{5\gamma-3}{2\gamma-3}}((0,T)\times D_\varepsilon)} \le C\,\|\varrho_\varepsilon^\theta - \langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon}\|_{L^{\frac{5\gamma-3}{2\gamma-3}}((0,T)\times D_\varepsilon)} \le C\,\|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)}$$

for some constant $C > 0$ independent of ε. Moreover,

$$\begin{aligned}\|\varphi\|_{L^\infty(0,T;W_0^{1,\frac{\gamma}{\theta}}(D_\varepsilon))} &\le C\,\|\mathcal{B}_\varepsilon(\varrho^\theta - \langle\varrho^\theta\rangle_{D_\varepsilon})\|_{L^\infty(0,T;W_0^{1,\frac{\gamma}{\theta}}(D_\varepsilon))}\\ &\le C\,\|\varrho_\varepsilon^\theta - \langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon}\|_{L^\infty(0,T;L^{\frac{\gamma}{\theta}}(D_\varepsilon))} \le C\,\|\varrho_\varepsilon\|^\theta_{L^\infty(0,T;L^\gamma(D_\varepsilon))} \le C\end{aligned}$$

since $\gamma/\theta = 3\gamma/(2\gamma-3)$ is strictly less than 3 for all $\gamma > 3$, so we may apply Theorem 3.5.4 with $q = \gamma/\theta$. Using the function φ as test function in the momentum equation, we get

$$\int_0^T \int_{D_\varepsilon} \psi p(\varrho_\varepsilon)\varrho_\varepsilon^\theta \,\mathrm{d}x\,\mathrm{d}t = \sum_{j=1}^6 I_j,$$

where

$$\begin{aligned}
I_1 &:= \int_0^T \int_{D_\varepsilon} \psi p(\varrho_\varepsilon)\langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon}\,\mathrm{d}x\,\mathrm{d}t, \quad I_2 := -\int_0^T \int_{D_\varepsilon} \psi'(t)\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta - \langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon})\,\mathrm{d}x\,\mathrm{d}t,\\
I_3 &:= -\int_0^T \int_{D_\varepsilon} \psi\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon : \nabla\mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta - \langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon})\,\mathrm{d}x\,\mathrm{d}t,\\
I_4 &:= \int_0^T \int_{D_\varepsilon} \psi\mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \nabla\mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta - \langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon})\,\mathrm{d}x\,\mathrm{d}t,\\
I_5 &:= -\int_0^T \int_{D_\varepsilon} \psi\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathcal{B}_\varepsilon(\partial_t\varrho_\varepsilon^\theta - \partial_t\langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon})\,\mathrm{d}x\,\mathrm{d}t,\\
I_6 &:= -\int_0^T \int_{D_\varepsilon} \psi(\varrho_\varepsilon\mathbf{f} + \mathbf{g})\cdot\mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta - \langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon})\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

We estimate each integral separately, following [LS18, Section 3.2]. However, due to our uniform bounds on $\mathcal{B}_\varepsilon$, we do not have to bootstrap the integrability from $\theta = \frac{\gamma}{2}$ to $\theta = \frac{2\gamma}{3} - 1$ but rather start with the desired value $\theta = \frac{2\gamma}{3} - 1$. For I_1, we get with $\theta \le \gamma$ and $\theta + \gamma = \frac{5\gamma}{3} - 1$

$$\begin{aligned}|I_1| &\le C \sup_{t\in[0,T]} |\langle\varrho_\varepsilon^\theta(t)\rangle_{D_\varepsilon}| \int_{D_\varepsilon} |\varrho_\varepsilon^\gamma(t)|\,\mathrm{d}x \le C\,\|\varrho_\varepsilon\|^\theta_{L^\infty(0,T;L^\theta(D_\varepsilon))}\|\varrho_\varepsilon\|^\gamma_{L^\infty(0,T;L^\gamma(D_\varepsilon))}\\ &\le C\,\|\varrho_\varepsilon\|^{\frac{5\gamma}{3}-1}_{L^\infty(0,T;L^\gamma(D_\varepsilon))} \le C.\end{aligned}$$

For I_2, we obtain by $1 - \frac{1}{\gamma} - \frac{1}{6} = \frac{5\gamma-6}{6\gamma}$ and $\frac{\gamma}{\theta} = \frac{3\gamma}{2\gamma-3} \ge \frac{6\gamma}{5\gamma-6}$ for any $\gamma \ge 0$

$$\begin{aligned}|I_2| &\le C\,\|\varrho_\varepsilon\|_{L^\infty(0,T;L^\gamma(D_\varepsilon))}\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^6(D_\varepsilon))}\|\mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta - \langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon})\|_{L^\infty(0,T;L^{\frac{6\gamma}{5\gamma-6}}(D_\varepsilon))}\\ &\le C\,\|\mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta - \langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon})\|_{L^\infty(0,T;L^{\frac{\gamma}{\theta}}(D_\varepsilon))} \le C\,\|\varrho_\varepsilon^\theta - \langle\varrho_\varepsilon^\theta\rangle_{D_\varepsilon}\|_{L^\infty(0,T;L^{\frac{\gamma}{\theta}}(D_\varepsilon))} \le C.\end{aligned}$$

For I_3 with $\frac{1}{\gamma} + \frac{2}{6} + \frac{\theta}{\gamma} = 1$,

$$|I_3| \leq C \, \|\varrho_\varepsilon\|_{L^\infty(0,T;L^\gamma(D_\varepsilon))} \|\mathbf{u}_\varepsilon\|^2_{L^2(0,T;L^6(D_\varepsilon))} \|\nabla \mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta - \langle \varrho_\varepsilon^\theta \rangle_{D_\varepsilon})\|_{L^\infty(0,T;L^{\frac{\gamma}{\theta}}(D_\varepsilon))}$$
$$\leq C \, \|\varrho_\varepsilon^\theta - \langle \varrho_\varepsilon^\theta \rangle_{D_\varepsilon}\|_{L^\infty(0,T;L^{\frac{\gamma}{\theta}}(D_\varepsilon))} \leq C.$$

For I_4 with $2\theta = \frac{4\gamma}{3} - 2 \leq \frac{5\gamma}{3} - 1$ for all $\gamma \geq 0$,

$$|I_4| \leq C \, \|\nabla \mathbf{u}_\varepsilon\|_{L^2(0,T;L^2(D_\varepsilon))} \|\nabla \mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta - \langle \varrho_\varepsilon^\theta \rangle_{D_\varepsilon})\|_{L^2(0,T;L^2(D_\varepsilon))} \leq C \, \|\varrho_\varepsilon^\theta\|_{L^2(0,T;L^2(D_\varepsilon))}$$
$$\leq C \, \|\varrho_\varepsilon\|^\theta_{L^{2\theta}(0,T;L^{2\theta}(D_\varepsilon))} \leq C \, \|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)}.$$

For I_6,

$$|I_6| \leq C \, (\|\varrho_\varepsilon\|_{L^\infty(0,T;L^2(D_\varepsilon))} + 1) \|\varrho_\varepsilon^\theta\|_{L^2(0,T;L^2(D_\varepsilon))}$$
$$\leq C \, \|\varrho_\varepsilon\|^\theta_{L^{2\theta}(0,T;L^{2\theta}(D_\varepsilon))} \leq C \, \|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)}.$$

Let us turn to I_5, which is the most challenging term. We will first assume that $\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon \in E_0^{\frac{6(5\gamma-3)}{17\gamma-21}, \frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)$, and later give arguments how to eliminate this assumption using time regularization. Let us further note that

$$\frac{10\gamma - 6}{9\gamma - 9} = \frac{2(5\gamma - 3)}{9\gamma - 9} \leq \frac{6(5\gamma - 3)}{17\gamma - 21} \iff 17\gamma - 21 \leq 27\gamma - 27 \iff \gamma \geq \frac{5}{3}$$

as well as, by $\gamma > 6$,

$$\frac{6(5\gamma - 3)}{17\gamma - 21} > \frac{3}{2}, \quad \frac{6(5\gamma - 3)}{17\gamma - 21} < 3 \iff 10\gamma - 6 < 17\gamma - 21 \iff \gamma > \frac{15}{7}.$$

To handle the time derivative of $\varrho_\varepsilon^\theta$, we use the extended Bogovskiĭ operator from Theorem 3.5.4. The renormalized continuity equation (4.23) for $b(s) = s^\theta$ yields

$$\partial_t(\varrho_\varepsilon^\theta) + (\theta - 1)\varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon + \operatorname{div}(\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon) = 0.$$

Since $\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon \in E_0^{\frac{6(5\gamma-3)}{17\gamma-21}, \frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)$, we have with Lemma 3.5.2

$$\langle \operatorname{div}(\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon), 1 \rangle_{[\dot{W}^{1,(\frac{10\gamma-6}{9\gamma-9})'}(D_\varepsilon)]', \dot{W}^{1,(\frac{10\gamma-6}{9\gamma-9})'}(D_\varepsilon)} = 0,$$

so we may write

$$I_5 = \int_0^T \int_{D_\varepsilon} \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}_\varepsilon(\operatorname{div}(\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon)) \, \mathrm{d}x \, \mathrm{d}t$$
$$+ (\theta - 1) \int_0^T \int_{D_\varepsilon} \psi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon - \langle \varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon \rangle_{D_\varepsilon}) \, \mathrm{d}x \, \mathrm{d}t$$
$$=: I_7 + I_8.$$

Let us start to estimate $\varrho_\varepsilon \mathbf{u}_\varepsilon$, which we do in the same way as in [LS18]. Since

$$\left(2\left(\frac{5\gamma}{3}-1\right)\right)^{-1}+\frac{1}{2}=\frac{3}{10\gamma-6}+\frac{1}{2}=\frac{5\gamma}{10\gamma-6},$$

we get by the uniform estimate on $\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2$ in $L^\infty(0,T;L^1(D_\varepsilon))$

$$\begin{aligned}
&\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{3}}(0,T;L^{\frac{10\gamma-6}{5\gamma}}(D_\varepsilon))} = \|\sqrt{\varrho_\varepsilon}\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{3}}(0,T;L^{\frac{10\gamma-6}{5\gamma}}(D_\varepsilon))}\\
&\le \|\sqrt{\varrho_\varepsilon}\|_{L^{\frac{10\gamma-6}{3}}(0,T;L^{\frac{10\gamma-6}{3}}(D_\varepsilon))}\|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^\infty(0,T;L^2(D_\varepsilon))} \le C\,\|\varrho_\varepsilon\|^{\frac{1}{2}}_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon))}.
\end{aligned}$$

Similarly,

$$\left(\frac{5\gamma}{3}-1\right)^{-1}+\frac{1}{2}=\frac{5\gamma+3}{10\gamma-6},\quad \left(\frac{5\gamma}{3}-1\right)^{-1}+\frac{1}{6}=\frac{5\gamma+15}{6(5\gamma-3)},$$

so we estimate

$$\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{5\gamma+3}}(0,T;L^{\frac{6(5\gamma-3)}{5\gamma+15}}(D_\varepsilon))} \le \|\varrho_\varepsilon\|_{L^{\frac{5\gamma}{3}-1}((0,T\times D_\varepsilon))}\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^6(D_\varepsilon))} \le C\,\|\varrho_\varepsilon\|_{L^{\frac{5\gamma}{3}-1}((0,T\times D_\varepsilon))}.$$

Using interpolation between Lebesgue spaces (B.4), we have

$$\frac{\gamma+3}{10\gamma-6}=\frac{4}{5}\frac{3}{10\gamma-6}+\frac{1}{5}\frac{5\gamma+3}{10\gamma-6},\quad \frac{13\gamma+3}{6(5\gamma-3)}=\frac{4}{5}\frac{5\gamma}{10\gamma-6}+\frac{1}{5}\frac{5\gamma+15}{6(5\gamma-3)},$$

which yields

$$\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{\gamma+3}}(0,T;L^{\frac{6(5\gamma-3)}{13\gamma+3}}(D_\varepsilon))} \le C\,\|\varrho_\varepsilon\|^{\frac{3}{5}}_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)}.$$

We further have

$$\frac{\gamma+3}{10\gamma-6}+\frac{9\gamma-9}{10\gamma-6}=1,\quad \frac{13\gamma+3}{6(5\gamma-3)}+\frac{17\gamma-21}{6(5\gamma-3)}=1, \tag{4.28}$$

and also

$$\frac{1}{2}+\frac{2\gamma-3}{5\gamma-3}=\frac{9\gamma-9}{10\gamma-6},\quad \frac{1}{6}+\frac{2\gamma-3}{5\gamma-3}=\frac{17\gamma-21}{6(5\gamma-3)}<\frac{2}{3}, \tag{4.29}$$

so we recall $\theta=\frac{2\gamma}{3}-1$ to get for I_7 the estimate

$$\begin{aligned}
|I_7| &\le C\,\|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{\gamma+3}}(0,T;L^{\frac{6(5\gamma-3)}{13\gamma+3}}(D_\varepsilon))}\|\mathcal{B}_\varepsilon \operatorname{div}(\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon)\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(0,T;L^{\frac{6(5\gamma-3)}{17\gamma-21}}(D_\varepsilon))}\\
&\le C\,\|\varrho_\varepsilon\|^{\frac{3}{5}}_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)}\|\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(0,T;L^{\frac{6(5\gamma-3)}{17\gamma-21}}(D_\varepsilon))}\\
&\le C\,\|\varrho_\varepsilon\|^{\frac{3}{5}}_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon))}\|\varrho_\varepsilon^\theta\|_{L^{\frac{5\gamma-3}{2\gamma-3}}((0,T)\times D_\varepsilon))}\|\mathbf{u}_\varepsilon\|_{L^2(0,T;L^6(D_\varepsilon))}\\
&\le C\,\|\varrho_\varepsilon\|^{\frac{3}{5}+\theta}_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon))}\\
&= C\,\|\varrho_\varepsilon\|^{\frac{2\gamma}{3}-\frac{2}{5}}_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon))}.
\end{aligned}$$

For I_8, Hölder's inequality (B.2) and (4.29) imply

$$\|\varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{9\gamma-9}}((0,T)\times D_\varepsilon)} \leq \|\varrho_\varepsilon^\theta\|_{L^{\frac{5\gamma-3}{2\gamma-3}}((0,T)\times D_\varepsilon)} \|\operatorname{div} \mathbf{u}_\varepsilon\|_{L^2((0,T)\times D_\varepsilon)} \leq C\, \|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)}.$$

Thus, we obtain by Sobolev embedding (B.8),

$$\left(\frac{10\gamma-6}{9\gamma-9}\right)^* = \frac{3\,\frac{10\gamma-6}{9\gamma-9}}{3-\frac{10\gamma-6}{9\gamma-9}} = \frac{(10\gamma-6)/(3\gamma-3)}{(17\gamma-21)/(9\gamma-9)} = \frac{6(5\gamma-3)}{17\gamma-21},$$

and the fact $\frac{10\gamma-6}{9\gamma-9} < 3$ for any $\gamma > 21/17$ the estimate

$$\begin{aligned}
&\|\mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon - \langle \varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon\rangle_{D_\varepsilon})\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(0,T;L^{\frac{6(5\gamma-3)}{17\gamma-21}}(D_\varepsilon))} \\
&\leq C\, \|\mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon - \langle \varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon\rangle_{D_\varepsilon})\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(0,T;W_0^{1,\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon))} \\
&\leq C\, \|\varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{9\gamma-9}}((0,T)\times D_\varepsilon)} \leq C\, \|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)}.
\end{aligned}$$

Together with (4.28), we estimate I_8 as

$$\begin{aligned}
|I_8| &\leq C\, \|\varrho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{\gamma+3}}(0,T;L^{\frac{6(5\gamma-3)}{13\gamma+3}}(D_\varepsilon))} \|\mathcal{B}_\varepsilon(\varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon - \langle \varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon\rangle_{D_\varepsilon})\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(0,T;L^{\frac{6(5\gamma-3)}{17\gamma-21}}(D_\varepsilon))} \\
&\leq C\, \|\varrho_\varepsilon\|^{\frac{3}{5}}_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)} \|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)} \\
&= C\, \|\varrho_\varepsilon\|^{\frac{2\gamma}{3}-\frac{2}{5}}_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)},
\end{aligned}$$

which eventually yields for I_5

$$|I_5| \leq |I_7| + |I_8| \leq C\, \|\varrho_\varepsilon\|^{\frac{2\gamma}{3}-\frac{2}{5}}_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)}.$$

Finally, since $\theta = \frac{2\gamma}{3} - 1 < \frac{2\gamma}{3} - \frac{2}{5}$, we arrive at

$$\int_0^T \int_{D_\varepsilon} \psi\, a \varrho_\varepsilon^{\frac{5\gamma}{3}-1} \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_{D_\varepsilon} \psi p(\varrho_\varepsilon) \varrho_\varepsilon^\theta \,\mathrm{d}x \,\mathrm{d}t \leq C \left(1 + \|\varrho_\varepsilon\|^{\frac{2\gamma}{3}-\frac{2}{5}}_{L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)}\right)$$

for arbitrary $\psi \in C_c^\infty(0,T)$, so we may choose a sequence $\psi_n \to 1$ strongly in $L^\infty(0,T)$ to obtain a uniform bound on ϱ_ε in $L^{\frac{5\gamma}{3}-1}((0,T) \times D_\varepsilon)$, provided $\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon \in E_0^{\frac{6(5\gamma-3)}{17\gamma-21}, \frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)$.

To overcome this additional assumption on $\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon$, we briefly sketch the arguments from [FN09, Section 2.2.5]. First, note that the exponents are optimal in the sense that, by (4.29),

$$\begin{aligned}
\|\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon\|_{L^{\frac{6(5\gamma-3)}{17\gamma-21}}(D_\varepsilon)} &\leq \|\varrho_\varepsilon^\theta\|_{L^{\frac{5\gamma-3}{2\gamma-3}}(D_\varepsilon)} \|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)} \leq C\, \|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}(D_\varepsilon)} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(D_\varepsilon)}, \\
\|\varrho_\varepsilon^\theta \operatorname{div} \mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)} &\leq \|\varrho_\varepsilon^\theta\|_{L^{\frac{5\gamma-3}{2\gamma-3}}(D_\varepsilon)} \|\operatorname{div} \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)} \leq \|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}(D_\varepsilon)} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)}.
\end{aligned}$$

The main observation now is that all estimates above remain valid if we choose as test function

$$\varphi_\delta(t,x) = \psi(t) \mathcal{B}_\varepsilon[\varrho_\varepsilon^\theta - \langle \varrho_\varepsilon^\theta \rangle_{D_\varepsilon}]_\delta,$$

where $[f]_\delta$ denotes the mollification in the time variable t (see Proposition B.7 for the definition and properties of mollifiers). Since ϱ_ε and $\mathbf{u}_\varepsilon$ fulfill the renormalized continuity equation (4.23), we have

$$\partial_t[\varrho_\varepsilon^\theta]_\delta + (\theta-1)[\varrho_\varepsilon^\theta \operatorname{div}\mathbf{u}_\varepsilon]_\delta + \operatorname{div}[\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon]_\delta = 0,$$

or, equivalently,

$$\operatorname{div}[\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon]_\delta = (1-\theta)[\varrho_\varepsilon^\theta \operatorname{div}\mathbf{u}_\varepsilon]_\delta - \partial_t[\varrho_\varepsilon^\theta]_\delta.$$

Thus, we estimate by Young's inequality (B.3)

$$\|[\varrho_\varepsilon^\theta \operatorname{div}\mathbf{u}_\varepsilon]_\delta\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)} \le \|\varrho_\varepsilon^\theta \operatorname{div}\mathbf{u}_\varepsilon\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)} \le \|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}(D_\varepsilon)} \|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)}$$

and

$$\|\partial_t[\varrho_\varepsilon^\theta]_\delta\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)} \le C\,\|\varrho_\varepsilon^\theta\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)} = C\,\|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}(D_\varepsilon)},$$

so indeed $\operatorname{div}[\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon]_\delta \in L^{\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)$ and $[\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon]_\delta \in E^{\frac{6(5\gamma-3)}{17\gamma-21},\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)$, hence $[\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon]_\delta$ has a well-defined normal trace on ∂D_ε. Since $\mathbf{u}_\varepsilon = 0$ on ∂D_ε, Theorem 3.5.3 now yields

$$\int_{D_\varepsilon} \operatorname{div}[\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon]_\delta \,\mathrm{d}x = \langle \operatorname{Tr}_{\mathbf{n}}(\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon), 1\rangle = 0,$$

so $[\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon]_\delta \in E_0^{\frac{6(5\gamma-3)}{17\gamma-21},\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)$ and further

$$\partial_t \mathcal{B}_\varepsilon[\varrho_\varepsilon^\theta - \langle \varrho_\varepsilon^\theta\rangle_{D_\varepsilon}]_\delta = -\mathcal{B}_\varepsilon \operatorname{div}[\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon]_\delta - \mathcal{B}_\varepsilon[\varrho_\varepsilon^\theta \operatorname{div}\mathbf{u}_\varepsilon - \langle \varrho_\varepsilon^\theta \operatorname{div}\mathbf{u}_\varepsilon\rangle_{D_\varepsilon}]_\delta.$$

Seeing finally that, by Theorem 3.5.4,

$$\begin{aligned}
\|\mathcal{B}_\varepsilon[\varrho_\varepsilon^\theta - \langle \varrho_\varepsilon^\theta\rangle_{D_\varepsilon}]_\delta\|_{W_0^{1,\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)} &\le C\,\|[\varrho_\varepsilon^\theta]_\delta\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)} \le C\,\|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}(D_\varepsilon)},\\
\|\partial_t \mathcal{B}_\varepsilon[\varrho_\varepsilon^\theta - \langle \varrho_\varepsilon^\theta\rangle_{D_\varepsilon}]_\delta\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)} &\le C\left(\|[\varrho_\varepsilon^\theta \mathbf{u}_\varepsilon]_\delta\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)} + \|[\varrho_\varepsilon^\theta \operatorname{div}\mathbf{u}_\varepsilon]_\delta\|_{L^{\frac{10\gamma-6}{9\gamma-9}}(D_\varepsilon)}\right)\\
&\le C\,\|\varrho_\varepsilon\|^\theta_{L^{\frac{5\gamma}{3}-1}(D_\varepsilon)} \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(D_\varepsilon)}
\end{aligned}$$

for some constant $C>0$ independent of ε and δ, we follow the lines above to get a uniform bound on $[\varrho_\varepsilon]_\delta$ in $L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)$. Letting $\delta \to 0$, this yields uniform bounds on ϱ_ε in $L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)$ since $[\varrho_\varepsilon]_\delta \to \varrho_\varepsilon$ strongly in $L^{\frac{5\gamma}{3}-1}((0,T)\times D_\varepsilon)$.

Once established the uniform bounds on ϱ_ε and $\mathbf{u}_\varepsilon$, the remaining homogenization proof follows essentially the same lines as done in previous sections. Let us briefly sketch the occurring differences here, following [LS18]. For the extended momentum equation, we obtain

$$\partial_t(\tilde\varrho_\varepsilon \tilde{\mathbf{u}}_\varepsilon) + \operatorname{div}(\tilde\varrho_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) + \nabla p(\tilde\varrho_\varepsilon) = \operatorname{div}\mathbb{S}(\nabla \tilde{\mathbf{u}}_\varepsilon) + \tilde\varrho_\varepsilon \mathbf{f} + \mathbf{g} + F_\varepsilon \text{ in } \mathcal{D}'((0,T)\times D),$$

where the additional distribution F_ε satisfies

$$|\langle F_\varepsilon, \varphi\rangle| \le C\,\varepsilon^{(3-q)\alpha-3}\big(\|\partial_t\varphi\|_{L^2(0,T;L^2(D))} + \|\nabla\varphi\|_{L^r(0,T;L^3(D))} + \|\varphi\|_{L^r(0,T;L^r(D))}\big)$$

for any $\varphi \in C_c^\infty((0,T)\times D)$. Here, $1 < r < \infty$ occurs out of various interpolations between Lebesgue spaces (B.4), C and r are independent of ε, and $5/2 < q < 3$ satisfies (4.20). The extended continuity equation

$$\partial_t\tilde\varrho_\varepsilon + \operatorname{div}(\tilde\varrho_\varepsilon\tilde{\mathbf{u}}_\varepsilon) = 0 \quad \text{in } (0,T)\times\mathbb{R}^3 \tag{4.30}$$

is obtained as follows. Let $\psi \in C_c^\infty((0,T)\times\mathbb{R}^3)$, and $\{\varphi_n\}_{n\ge1} \subset C_c^\infty(D_\varepsilon)$ be a sequence of smooth functions with $0 \le \varphi_n \le 1$, $|\nabla\varphi_n| \le 4n$, and

$$\varphi_n = 1 \text{ on } \{\operatorname{dist}(x,\partial D_\varepsilon) \ge 1/n\}, \quad \varphi_n = 0 \text{ on } \{\operatorname{dist}(x,\partial D_\varepsilon) \le 1/(2n)\}.$$

Note that this implies for any $1 \le q \le \infty$

$$\|1-\varphi_n\|_{L^q(D_\varepsilon)} = \|1-\varphi_n\|_{L^q(\{\operatorname{dist}(x,\partial D_\varepsilon)\}<1/n\})} \le |\{\operatorname{dist}(x,\partial D_\varepsilon) < 1/n\}|^{\frac1q} \le C\,n^{-\frac1q} \tag{4.31}$$

as well as

$$\begin{aligned}\|\operatorname{dist}(x,\partial D_\varepsilon)\nabla\varphi_n\|_{L^q(D_\varepsilon)} &\le \|\nabla\varphi_n\|_{L^\infty(D_\varepsilon)}\|\operatorname{dist}(x,\partial D_\varepsilon)\|_{L^q(\{\operatorname{dist}(x,\partial D_\varepsilon)\}<1/n\})}\\ &\le 4n\cdot\frac{1}{n}\cdot|\{\operatorname{dist}(x,\partial D_\varepsilon)\} < 1/n\}|^{\frac1q} \le C\,n^{-\frac1q}.\end{aligned} \tag{4.32}$$

Then we may split

$$\begin{aligned}\int_0^T\int_D \tilde\varrho_\varepsilon\partial_t\psi + \tilde\varrho_\varepsilon\tilde{\mathbf{u}}_\varepsilon\cdot\nabla\psi\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{D_\varepsilon}\varrho_\varepsilon\partial_t(\psi\varphi_n) + \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla(\psi\varphi_n)\,\mathrm{d}x\,\mathrm{d}t\\ &+ \int_0^T\int_{D_\varepsilon}\varrho_\varepsilon(1-\varphi_n)\partial_t\psi + \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot(1-\varphi_n)\nabla\psi - \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\psi\nabla\varphi_n\,\mathrm{d}x\,\mathrm{d}t\\ &= \int_0^T\int_{D_\varepsilon}\varrho_\varepsilon(1-\varphi_n)\partial_t\psi + \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot(1-\varphi_n)\nabla\psi - \varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\psi\nabla\varphi_n\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where we used that $\psi\varphi_n$ is a good test function for the continuity equation $\partial_t\varrho_\varepsilon + \operatorname{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon) = 0$. By the uniform estimates on ϱ_ε and $\varrho_\varepsilon\mathbf{u}_\varepsilon$, the first two terms vanish as $n\to\infty$ by (4.31). The third term is handled by Hardy's inequality (B.10) and (4.32).

The convergence of the non-linear terms $\tilde\varrho_\varepsilon\tilde{\mathbf{u}}_\varepsilon$ and $\tilde\varrho_\varepsilon\tilde{\mathbf{u}}_\varepsilon\otimes\tilde{\mathbf{u}}_\varepsilon$ to $\varrho\mathbf{u}$ and $\varrho\mathbf{u}\otimes\mathbf{u}$, respectively, are obtained by getting the uniform bounds

$$\|\partial_t\tilde\varrho_\varepsilon\|_{L^2(0,T;W^{-1,\frac{6\gamma}{6+\gamma}}(D_\varepsilon))} + \|\tilde\varrho_\varepsilon\tilde{\mathbf{u}}_\varepsilon\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(D_\varepsilon))} \le C$$

from equation (4.30) and the uniform bounds on $\varrho_\varepsilon\mathbf{u}_\varepsilon$ already mentioned. Together with Lemma 5.1 in [Lio98], this yields

$$\tilde\varrho_\varepsilon\tilde{\mathbf{u}}_\varepsilon \to \varrho\mathbf{u} \text{ in } \mathcal{D}'((0,T)\times D),$$

and also

$$\tilde{\varrho}_\varepsilon \to \varrho \text{ in } C(0,T; L^\gamma_{\text{weak}}(D)), \quad \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \to \varrho \mathbf{u} \text{ in } C(0,T; L^{\frac{2\gamma}{\gamma+1}}_{\text{weak}}(D)).$$

A similar argument can be used to show

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon \to \varrho \mathbf{u} \otimes \mathbf{u} \text{ in } \mathcal{D}'((0,T) \times D),$$

leading for the continuity equation, momentum equation, and renormalized continuity equation to

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \quad & \text{in } \mathcal{D}'((0,T) \times D), \\ \partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \overline{p(\varrho)} = \operatorname{div} \mathbb{S}(\nabla \mathbf{u}) + \varrho \mathbf{f} + \mathbf{g} \quad & \text{in } \mathcal{D}'((0,T) \times D), \\ \partial_t b(\varrho) + \operatorname{div}(b(\varrho)\mathbf{u}) + (\varrho b'(\varrho) - b(\varrho)) \operatorname{div} \mathbf{u} = 0 \quad & \text{in } \mathcal{D}'((0,T) \times D), \end{aligned}$$

where $\overline{p(\varrho)}$ is the weak limit of $p(\tilde{\varrho}_\varepsilon)$ in $L^{\frac{5}{3}-\frac{1}{\gamma}}((0,T) \times D)$. The proof of $\overline{p(\varrho)} = p(\varrho)$ requires as before the strong convergence of the density, which can be handled similar to Appendix A.

4.3 Heat-conducting fluids

4.3.1 The model

In this section, we consider the stationary compressible Navier-Stokes-Fourier equations in perforated domains D_ε, which describe the steady motion of a compressible and heat-conducting Newtonian fluid. In contrast to the previous sections, we additionally have to take into account equations that cover the behavior of energy as well as entropy. However, we will assume that there are no internal sources or sinks of heat inside the fluid, meaning $r = 0$ in (2.5) and (2.10). As derived in Section 2.2, for $\varepsilon > 0$, the unknown density $\varrho_\varepsilon : D_\varepsilon \to [0, \infty)$, velocity $\mathbf{u}_\varepsilon : D_\varepsilon \to \mathbb{R}^3$, and temperature $\vartheta_\varepsilon : D_\varepsilon \to (0, \infty)$ of a viscous compressible fluid are described by

$$\begin{cases} \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 & \text{in } D_\varepsilon, \\ \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla p(\varrho_\varepsilon, \vartheta_\varepsilon) = \operatorname{div} \mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) + \varrho_\varepsilon \mathbf{f} + \mathbf{g} & \text{in } D_\varepsilon, \\ \operatorname{div}(\varrho_\varepsilon E_\varepsilon \mathbf{u}_\varepsilon + p(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon - \mathbb{S}(\vartheta_\varepsilon, \mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon + \mathbf{q}_\varepsilon) = (\varrho_\varepsilon \mathbf{f} + \mathbf{g}) \cdot \mathbf{u}_\varepsilon & \text{in } D_\varepsilon, \\ \operatorname{div}\left(\varrho_\varepsilon s_\varepsilon \mathbf{u}_\varepsilon + \frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon}\right) = \sigma_\varepsilon & \text{in } D_\varepsilon, \end{cases} \tag{4.33}$$

where $\mathbb{S}$ denotes the Newtonian viscous stress tensor, which is now given by

$$\mathbb{S}(\vartheta, \nabla \mathbf{u}) = \mu(\vartheta)\left(\nabla \mathbf{u} + \nabla^T \mathbf{u} - \frac{2}{3} \operatorname{div}(\mathbf{u})\mathbb{I}\right) + \eta(\vartheta) \operatorname{div}(\mathbf{u})\mathbb{I}, \tag{4.34}$$

and the entropy production rate $\sigma \in \mathcal{M}^+(\overline{D_\varepsilon})$ is a non-negative Radon-measure satisfying

$$\sigma \geq \frac{\mathbb{S}(\vartheta, \nabla \mathbf{u}) : \nabla \mathbf{u}}{\vartheta} - \frac{\mathbf{q} \cdot \nabla \vartheta}{\vartheta^2}. \tag{4.35}$$

We further assume the viscosity coefficients $\mu(\cdot), \eta(\cdot)$ being continuous functions on $(0,\infty)$, $\mu(\cdot)$ is moreover Lipschitz continuous, and

$$C_1(1+\vartheta) \leq \mu(\vartheta) \leq C_2(1+\vartheta), \quad 0 \leq \eta(\vartheta) \leq C_2(1+\vartheta). \tag{4.36}$$

We also impose boundary conditions on ∂D_ε as

$$\begin{aligned} \mathbf{u}_\varepsilon &= 0, \\ \mathbf{q}_\varepsilon \cdot \mathbf{n} &= L(\vartheta_\varepsilon - \vartheta_0), \end{aligned} \tag{4.37}$$

where $\vartheta_0 \geq T_0 > 0$ is a prescribed temperature distribution in D and $L > 0$ a given constant, and fix the total mass by

$$\int_{D_\varepsilon} \varrho_\varepsilon \,\mathrm{d}x = \mathfrak{m} > 0, \tag{4.38}$$

where $\mathfrak{m} > 0$ is independent of ε.

For the constitutive law of the pressure, we assume that it can be written as the sum of adiabatic pressure and the pressure of an ideal gas, meaning

$$p(\varrho,\vartheta) = a\varrho^\gamma + c_v(\gamma-1)\varrho\vartheta, \tag{4.39}$$

where $a > 0$, $\gamma > 2$ is the adiabatic exponent, and $c_v > 0$ is the specific heat capacity at constant volume, see also Section 2.2 for a derivation of this. The heat flux is governed by Fourier's law

$$\mathbf{q}(\vartheta, \nabla\vartheta) = -\kappa(\vartheta)\nabla\vartheta, \tag{4.40}$$

where we assume the heat conductivity κ to satisfy

$$C_3(1+\vartheta^{m_\vartheta}) \leq \kappa(\vartheta) \leq C_4(1+\vartheta^{m_\vartheta}) \tag{4.41}$$

for some $m_\vartheta > 2$. The total energy density is given by

$$E = \frac{1}{2}|\mathbf{u}|^2 + e, \tag{4.42}$$

where the specific energy e satisfies Gibb's relation

$$\frac{1}{\vartheta}\left(De + p(\varrho,\vartheta)D\left(\frac{1}{\varrho}\right)\right) = Ds(\varrho,\vartheta). \tag{4.43}$$

Assuming the specific entropy for an ideal fluid as $s(\varrho,\vartheta) = c_v \log\left(\frac{\vartheta}{\varrho^{\gamma-1}}\right)$ (see (2.13)), this leads to

$$\vartheta Ds(\varrho,\vartheta) = \vartheta\partial_\varrho s D\varrho + \vartheta\partial_\vartheta s D\vartheta = c_v D\vartheta - c_v(\gamma-1)\frac{\vartheta}{\varrho}D\varrho.$$

By Gibb's relation, this should be equal to

$$De + (a\varrho^\gamma + c_v(\gamma-1)\varrho\vartheta)D\left(\frac{1}{\varrho}\right) = \partial_\varrho e D\varrho + \partial_\vartheta e D\vartheta - a\varrho^{\gamma-2}D\varrho - c_v(\gamma-1)\frac{\vartheta}{\varrho}D\varrho.$$

Comparing the differentials of ϱ and ϑ, we obtain

$$\partial_\vartheta e = c_v, \quad \partial_\varrho e = a\varrho^{\gamma-2},$$

yielding

$$e(\varrho,\vartheta) = c_v\vartheta + \frac{a\varrho^{\gamma-1}}{\gamma-1}. \tag{4.44}$$

Further, the entropy s fulfills *formally* the balance of entropy

$$\operatorname{div}\left(\varrho s\mathbf{u} + \frac{\mathbf{q}}{\vartheta}\right) = \sigma = \frac{\mathbb{S}:\nabla\mathbf{u}}{\vartheta} - \frac{\mathbf{q}\cdot\nabla\vartheta}{\vartheta^2},$$

see also (2.10) for its derivation. Since weak solutions are expected to dissipate more kinetic energy than indicated from the second equation in (4.33), we should for the entropy production rate σ expect inequality rather than equality, which is precisely the notion of (4.35); see [FN09, Chapter 2] for details. Finally, we assume the external forces $\mathbf{f},\mathbf{g} \in L^\infty(\mathbb{R}^3)$.

The existence of classical solutions to (4.33) is known only if the data are in a certain sense "small" (see, e.g., [DV87, PP14] and the references therein). Therefore, we will work with weak solutions, which are known to exist under even weaker assumptions of m_ϑ and γ as made above.

4.3.2 Weak formulation, weak solutions, and main result

Here, we state the weak formulation of the problem in D_ε. To simplify notation, we will identify a function with D_ε as its domain of definition with its zero extension to the whole of $\mathbb{R}^3$.

First, the weak formulation of the continuity equation reads

$$\int_{\mathbb{R}^3} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\psi \,\mathrm{d}x = 0 \tag{4.45}$$

for all $\psi \in C_c^1(\mathbb{R}^3)$. We will moreover work with a renormalized version of this, that is,

$$\int_{\mathbb{R}^3} b(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla\psi + (b(\varrho_\varepsilon) - \varrho_\varepsilon b'(\varrho_\varepsilon))\operatorname{div}(\mathbf{u}_\varepsilon)\psi \,\mathrm{d}x = 0 \tag{4.46}$$

for any $\psi \in C_c^1(\mathbb{R}^3)$, where $b \in C([0,\infty)) \cap C^1((0,\infty))$ is as in (4.4).

The weak formulation of the momentum equation reads

$$\int_{D_\varepsilon} p(\varrho_\varepsilon,\vartheta_\varepsilon)\operatorname{div}\varphi + (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon):\nabla\varphi - \mathbb{S}(\vartheta_\varepsilon,\nabla\mathbf{u}_\varepsilon):\nabla\varphi + (\varrho_\varepsilon\mathbf{f}+\mathbf{g})\cdot\varphi \,\mathrm{d}x = 0 \tag{4.47}$$

for any $\varphi \in C_c^1(D_\varepsilon;\mathbb{R}^3)$.

The weak formulation of the energy balance reads

$$
\begin{aligned}
- \int_{D_\varepsilon} \Big(\varrho_\varepsilon E_\varepsilon \mathbf{u}_\varepsilon + p(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon - \mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon)\mathbf{u}_\varepsilon + \mathbf{q}_\varepsilon \Big) \cdot \nabla \psi \,\mathrm{d}x + \int_{\partial D_\varepsilon} L(\vartheta_\varepsilon - \vartheta_0)\psi \,\mathrm{d}\sigma(x) \\
= \int_{D_\varepsilon} (\varrho_\varepsilon \mathbf{f} + \mathbf{g}) \cdot \mathbf{u}_\varepsilon \psi \,\mathrm{d}x
\end{aligned}
\tag{4.48}
$$

for all $\psi \in C^1(\overline{D_\varepsilon})$. Farther, we also have the balance of entropy

$$
\langle \sigma_\varepsilon, \psi \rangle_{\mathcal{M}^+} + \int_{\partial D_\varepsilon} \frac{L\vartheta_0}{\vartheta_\varepsilon} \psi \,\mathrm{d}\sigma(x) = - \int_{D_\varepsilon} \Big(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)\mathbf{u}_\varepsilon + \frac{\mathbf{q}_\varepsilon}{\vartheta_\varepsilon} \Big) \cdot \nabla \psi \,\mathrm{d}x + L \int_{\partial D_\varepsilon} \psi \,\mathrm{d}\sigma(x)
\tag{4.49}
$$

for all $\psi \in C^1(\overline{D}_\varepsilon)$ with $\psi \geq 0$, where we used the notation $\langle \sigma_\varepsilon, \psi \rangle_{\mathcal{M}^+} = \int_{D_\varepsilon} \psi \,\mathrm{d}\sigma_\varepsilon(x)$.

Definition 4.3.1. *The triple $[\varrho, \mathbf{u}, \vartheta]$ is said to be a* renormalized weak entropy solution *to problem (4.33)–(4.44) if $\varrho \geq 0, \vartheta > 0$ a.e. in D_ε, $\varrho \in L^\gamma(D_\varepsilon)$, $\mathbf{u} \in W_0^{1,2}(D_\varepsilon; \mathbb{R}^3)$, $\vartheta^{m_\vartheta/2}$ and $\log \vartheta \in W^{1,2}(D_\varepsilon)$ such that $\varrho|\mathbf{u}|^3$, $|\mathbb{S}(\vartheta, \nabla \mathbf{u})\mathbf{u}|$ and $p(\varrho, \vartheta)|\mathbf{u}| \in L^1(D_\varepsilon)$, and the relations (4.45)–(4.49) are fulfilled.*

For $\varepsilon > 0$ fixed, the existence of weak solutions is guaranteed by the following result, see [NP11] for details.

Theorem 4.3.2. *Let $\mathbf{f}, \mathbf{g} \in L^\infty(\mathbb{R}^3)$, $\vartheta_0 \in L^1(\partial D_\varepsilon)$, $\vartheta_0 \geq T_0 > 0$ a.e. on ∂D_ε, $L > 0$ and $\mathfrak{m} > 0$. Let $\gamma > \frac{5}{3}$ and $m_\vartheta > 1$. Then there exists a renormalized weak entropy solution $[\varrho, \mathbf{u}, \vartheta]$ to problem (4.33)–(4.44) in the sense of Definition 4.3.1.*

We are now in the position to state our main result, which generalizes [LP21, Theorem 2.2] to the case of a random perforation.

Theorem 4.3.3. *Let $(\Phi, \mathcal{R}) = (\{z_i\}, \{r_i\})$ and D_ε be defined as in Section 3.4. Let $\mathbf{f}, \mathbf{g} \in L^\infty(\mathbb{R}^3)$, $\mathfrak{m} > 0$, $L > 0$, and $\vartheta_0 \geq T_0 > 0$ in D be defined such that it possesses a uniform finite upper bound on its L^q-norm over all smooth two-dimensional surfaces with finite surface area contained in D for some $q > 1$. Let $\{[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon]\}_{\varepsilon > 0}$ be a sequence of renormalized weak entropy solutions to problem (4.33)–(4.44), extended in a suitable way to the whole domain D as shown in Section 4.3.4 below. Let $\alpha > 3$, $\gamma > 2$, $m_\vartheta > 2$, and $m > \max\{3/(\alpha - 3), 3\}$ satisfy the relation*

$$
\alpha - \frac{3}{m} > \max \left\{ \frac{2\gamma - 3}{\gamma - 2}, \frac{3m_\vartheta - 2}{m_\vartheta - 2} \right\}.
\tag{4.50}
$$

Then, there exists an almost surely positive random variable $\varepsilon_0(\omega)$ such that for all $0 < \varepsilon \leq \varepsilon_0$ there hold the uniform bounds

$$
\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D_\varepsilon)} + \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(D_\varepsilon)} + \|\vartheta_\varepsilon\|_{W^{1,2}(D_\varepsilon) \cap L^{3m_\vartheta}(D_\varepsilon)} \leq C,
$$

where $\Theta := \min\{2\gamma - 3, \gamma \frac{3m_\vartheta - 2}{3m_\vartheta + 2}\}$. Moreover, the corresponding weak limit as $\varepsilon \to 0$ is a renormalized weak solution to problem (4.33)–(4.44) in the limit domain D, i.e., $\varrho \geq 0$ and $\vartheta > 0$ a.e. in D, and the equations (4.45)-(4.48) are fulfilled.

Note that we are not able to prove the balance of entropy (4.49), which is due to the mere weak control on $1/\vartheta_\varepsilon$; see Remark 4.3.12 for a more detailed explanation on this issue.

4.3.3 Uniform bounds

In this section, we give uniform bounds on the velocity and the density. Note that the assumption $m \geq 3$ is made to control the measure of the boundary ∂D_ε and the measure of D_ε itself.

The entropy balance (4.49) together with (4.35) enables us to get several bounds on the sequence $\{[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon]\}_{\varepsilon>0}$ in D_ε. With the help of (3.26), we obtain for the entropy balance (4.49) with $\psi \equiv 1$

$$\sigma_\varepsilon(\overline{D}_\varepsilon) + \int_{\partial D_\varepsilon} \frac{L\vartheta_0}{\vartheta_\varepsilon}\, \mathrm{d}\sigma(x) \leq L\,|\partial D_\varepsilon| \leq C,$$

and in view of (4.35), (4.40) and (4.41) also

$$\int_{D_\varepsilon} \frac{\mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) : \mathbf{u}_\varepsilon}{\vartheta_\varepsilon} + \frac{(1+\vartheta_\varepsilon^{m_\vartheta})|\nabla \vartheta_\varepsilon|^2}{\vartheta_\varepsilon^2}\, \mathrm{d}x \leq C\, \sigma_\varepsilon(\overline{D}_\varepsilon) \leq C. \tag{4.51}$$

If we take also $\psi \equiv 1$ in the weak formulation of the energy balance (4.48), we obtain

$$L\int_{\partial D_\varepsilon} \vartheta_\varepsilon\, \mathrm{d}\sigma(x) \leq C\left(1 + \int_{D_\varepsilon} (\varrho_\varepsilon + 1)|\mathbf{u}_\varepsilon|\, \mathrm{d}x\right) \leq C\left(1 + (\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)} + 1)\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)}\right).$$

Hence, due to the form of the stress tensor in (4.34) and similar to the calculation made in (4.9), we have

$$\begin{aligned} &\sigma_\varepsilon(\overline{D}_\varepsilon) + \|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(D_\varepsilon)} + \|\nabla \log \vartheta_\varepsilon\|_{L^2(D_\varepsilon)} + \|\nabla |\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\|_{L^2(D_\varepsilon)} + \|\vartheta_\varepsilon^{-1}\|_{L^1(\partial D_\varepsilon)} \leq C, \\ &\|\vartheta_\varepsilon\|_{L^1(\partial D_\varepsilon)} \leq C\,(1 + \|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)}). \end{aligned} \tag{4.52}$$

Note that the bounds in (4.52) imply, by Sobolev inequality (B.8) and Poincaré's inequality (B.7), that we can control the norm $\|\vartheta_\varepsilon\|_{L^{3m_\vartheta}(D_\varepsilon)}$ by

$$\begin{aligned} \|\vartheta_\varepsilon\|^{\frac{m_\vartheta}{2}}_{L^{3m_\vartheta}(D_\varepsilon)} &= \|\vartheta_\varepsilon^{\frac{m_\vartheta}{2}}\|_{L^6(D_\varepsilon)} \leq C\,\|\vartheta_\varepsilon^{\frac{m_\vartheta}{2}}\|_{W^{1,2}(D_\varepsilon)} \\ &\leq C\left(\|\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\|_{L^2(D_\varepsilon)} + \|\vartheta_\varepsilon\|_{L^1(\partial D_\varepsilon)}\right) \leq C\left(1 + \|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)}\right). \end{aligned}$$

However, we do not know whether ϑ_ε is *uniformly* bounded. To prove this, we need some additional tools. We will do this in the next subsection independent of the following results. For now, we will assume that ϑ_ε is uniformly bounded in $L^{3m_\vartheta}(D_\varepsilon)$ and prove this fact later on.

To get uniform bounds on the density, we will use Lemma 3.4.1 and proceed similar to [BO21, DFL17, LP21].

Lemma 4.3.4 (see [LP21, Lemma 3.3])**.** *Under the assumptions of Lemma 3.4.1, assume additionally that $\|\vartheta_\varepsilon\|_{L^{3m_\vartheta}(D_\varepsilon)}$ is uniformly bounded. Then, for $\varepsilon > 0$ small enough, we have*

$$\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D_\varepsilon)} \leq C,$$

where $C > 0$ is independent of ε, and

$$\Theta := \min\left\{2\gamma - 3, \gamma\frac{3m_\vartheta - 2}{3m_\vartheta + 2}\right\}. \tag{4.53}$$

Proof. In the weak formulation of the momentum balance (4.47), we will use the test function

$$\varphi := \mathcal{B}_\varepsilon\big(\varrho_\varepsilon^\Theta - \langle\varrho_\varepsilon^\Theta\rangle_{D_\varepsilon}\big), \quad \langle\varrho_\varepsilon^\Theta\rangle_{D_\varepsilon} := \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon} \varrho_\varepsilon^\Theta \,\mathrm{d}x,$$

where $\mathcal{B}_\varepsilon$ is the operator from Theorem 3.4.1, and Θ to be determined. We then have for any $1 < q < 3$ satisfying (3.19)

$$\|\nabla\varphi\|_{L^q(D_\varepsilon)} \le C(q)\,\|\varrho_\varepsilon^\Theta\|_{L^q(D_\varepsilon)}.$$

Using φ as test function in (4.47) and recalling the pressure as $p(\varrho,\vartheta) = a\varrho^\gamma + c_v(\gamma-1)\varrho\vartheta$, we get

$$\begin{aligned}\int_{D_\varepsilon} a\varrho_\varepsilon^{\gamma+\Theta}\,\mathrm{d}x = \int_{D_\varepsilon} p(\varrho_\varepsilon,\vartheta_\varepsilon)\langle\varrho_\varepsilon^\Theta\rangle_{D_\varepsilon} - c_v(\gamma-1)\varrho_\varepsilon^{\Theta+1}\vartheta_\varepsilon + \mathbb{S}(\vartheta_\varepsilon,\nabla\mathbf{u}_\varepsilon) : \nabla\varphi\,\mathrm{d}x \\ - \int_{D_\varepsilon} (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \nabla\varphi - (\varrho_\varepsilon\mathbf{f} + \mathbf{g})\cdot\varphi\,\mathrm{d}x.\end{aligned} \tag{4.54}$$

We will estimate the right-hand side term by term and start with the most restrictive ones, which will give bounds on Θ. First, we take the convective term to estimate

$$\begin{aligned}\int_{D_\varepsilon} \big|(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \nabla\varphi\big|\,\mathrm{d}x &\le \|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)}^2\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D_\varepsilon)}\|\nabla\varphi\|_{L^{q_1}(D_\varepsilon)} \\ &\le C(q_1)\,\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)}^2\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D_\varepsilon)}\|\varrho_\varepsilon^\Theta\|_{L^{q_1}(D_\varepsilon)} \\ &= C(q_1)\,\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)}^2\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D_\varepsilon)}\|\varrho_\varepsilon\|_{L^{q_1\Theta}(D_\varepsilon)}^\Theta,\end{aligned}$$

where q_1 is determined by

$$\frac{1}{q_1} = 1 - \frac{2}{6} - \frac{1}{\gamma+\Theta}.$$

In order to get as high integrability of ϱ_ε as possible, we choose Θ such that $q_1\Theta = \gamma + \Theta$. This together with $\gamma > 2$ leads to

$$\Theta = \Theta_1 := 2\gamma - 3 > 1, \quad q_1 = \frac{3(\gamma-1)}{2\gamma-3} \in (\frac{3}{2}, 3), \quad \frac{3}{3-q_1} = \frac{2\gamma-3}{\gamma-2}.$$

Note that the exponents $\Theta_1 = 2\gamma - 3$ and $\frac{3(\gamma-1)}{2\gamma-3}$ showed up earlier in the proof of Theorem 4.2.1 due to the same reasons. Using Sobolev embedding (B.8) and the uniform bound on $\mathbf{u}_\varepsilon$ from (4.52) to obtain $\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)} \le C\,\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(D_\varepsilon)} \le C$, we deduce

$$\int_{D_\varepsilon} \big|(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \nabla\varphi\big|\,\mathrm{d}x \le C\,\|\varrho_\varepsilon\|_{L^{\gamma+\Theta_1}(D_\varepsilon)}^{1+\Theta_1},$$

where $C > 0$ is independent of ε, and $1 + \Theta_1 < \gamma + \Theta_1$.

Second, we consider the diffusive term to obtain

$$\begin{aligned}\int_{D_\varepsilon} \big|\mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) : \nabla\varphi\big| \,\mathrm{d}x &\le C\,(1 + \|\vartheta_\varepsilon\|_{L^{3m_\vartheta}(D_\varepsilon)})\|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)}\|\nabla\varphi\|_{L^{q_2}(D_\varepsilon)}\\ &\le C(q_2)\,\|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)}\|\varrho_\varepsilon^\Theta\|_{L^{q_2}(D_\varepsilon)}\\ &= C(q_2)\,\|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)}\|\varrho_\varepsilon\|^\Theta_{L^{q_2\Theta}(D_\varepsilon)},\end{aligned}$$

where we set (recall $m_\vartheta > 2$)

$$q_2 := \frac{6m_\vartheta}{3m_\vartheta - 2} \in (2,3), \qquad \frac{3}{3-q_2} = \frac{3m_\vartheta - 2}{m_\vartheta - 2}.$$

As before, we choose Θ such that $q_2\Theta = \gamma + \Theta$, which leads to

$$\Theta = \Theta_2 := \gamma\frac{3m_\vartheta - 2}{3m_\vartheta + 2} > 1.$$

This yields

$$\int_{D_\varepsilon} \big|\mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) : \nabla\varphi\big| \,\mathrm{d}x \le C\,\|\varrho_\varepsilon\|^\Theta_{L^{\gamma+\Theta}(D_\varepsilon)}.$$

In particular, if we set

$$\Theta := \min\{\Theta_1, \Theta_2\} > 1, \quad \alpha - \frac{3}{m} > \max\left\{\frac{2\gamma-3}{\gamma-2}, \frac{3m_\vartheta - 2}{m_\vartheta - 2}\right\} > 3,$$

then q_1 and q_2 satisfy (3.19) and we infer

$$\int_{D_\varepsilon} \big|(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla\varphi\big| \,\mathrm{d}x + \int_{D_\varepsilon} \big|\mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) : \nabla\varphi\big| \,\mathrm{d}x \le C\,\big(1 + \|\varrho_\varepsilon\|^{1+\Theta}_{L^{\gamma+\Theta}(D_\varepsilon)}\big).$$

Since $m_\vartheta > 2$, we have $\Theta \le \gamma\frac{3m_\vartheta-2}{3m_\vartheta+2} < \gamma$, yielding $2\Theta < \gamma + \Theta$. Thus we deduce

$$\begin{aligned}\int_{D_\varepsilon} \big|(\varrho_\varepsilon \mathbf{f} + \mathbf{g})\cdot\varphi\big| \,\mathrm{d}x &\le C\,(\|\varrho_\varepsilon\|_{L^2(D_\varepsilon)} + 1)\|\varphi\|_{L^2(D_\varepsilon)}\\ &\le C(2)\,(\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D_\varepsilon)} + 1)\|\varrho_\varepsilon\|^\Theta_{L^{2\Theta}(D_\varepsilon)}\\ &\le C(2)\,(\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D_\varepsilon)} + 1)\|\varrho_\varepsilon\|^\Theta_{L^{\gamma+\Theta}(D_\varepsilon)}\\ &\le C(2)\,(\|\varrho_\varepsilon\|^{1+\Theta}_{L^{\gamma+\Theta}(D_\varepsilon)} + 1),\end{aligned}$$

where in the last inequality we used (4.27) for $a = 1$, $b = \|\varrho_\varepsilon\|^{1+\Theta}_{L^{\gamma+\Theta}(D_\varepsilon)}$, and $p = (1+\Theta)/\Theta$.

Furthermore, the estimate for the pressure reads

$$\begin{aligned}\int_{D_\varepsilon} \big|p(\varrho_\varepsilon, \vartheta_\varepsilon)\langle\varrho_\varepsilon^\Theta\rangle_{D_\varepsilon}\big| \,\mathrm{d}x &\le C\int_{D_\varepsilon} (\varrho_\varepsilon^\gamma + \varrho_\varepsilon\vartheta_\varepsilon)\langle\varrho_\varepsilon^\Theta\rangle_{D_\varepsilon} \,\mathrm{d}x\\ &\le C\bigg(\|\varrho_\varepsilon\|^\gamma_{L^\gamma(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)}\|\vartheta_\varepsilon\|_{L^6(D_\varepsilon)}\bigg)\|\varrho_\varepsilon\|^\Theta_{L^\Theta(D_\varepsilon)}\\ &\le C\bigg(\|\varrho_\varepsilon\|^\gamma_{L^\gamma(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^\gamma(D_\varepsilon)}\bigg)\|\varrho_\varepsilon\|^\Theta_{L^\Theta(D_\varepsilon)}.\end{aligned}$$

Here we assumed that ϑ_ε is bounded in $L^{3m_\vartheta}(D_\varepsilon) \subset L^6(D_\varepsilon)$. Using (4.27) for $b = \|\varrho_\varepsilon\|^{\gamma}_{L^\gamma(D_\varepsilon)}$, $p = \gamma$, and $a = 1$, we get

$$\|\varrho_\varepsilon\|_{L^\gamma(D_\varepsilon)} \le \|\varrho_\varepsilon\|^{\gamma}_{L^\gamma(D_\varepsilon)} + 1.$$

Together with $\Theta < \gamma$, which implies $\|\varrho_\varepsilon\|^{\Theta}_{L^\Theta(D_\varepsilon)} \le 1 + \|\varrho_\varepsilon\|^{\gamma}_{L^\gamma(D_\varepsilon)}\|\varrho_\varepsilon\|^{\Theta}_{L^\Theta(D_\varepsilon)}$, and interpolation between the norms of $L^1(D_\varepsilon)$ and $L^{\gamma+\Theta}(D_\varepsilon)$ (see (B.4)), this yields

$$\begin{aligned}
\int_{D_\varepsilon} \left|p(\varrho_\varepsilon, \vartheta_\varepsilon)\langle \varrho_\varepsilon^{\Theta}\rangle_{D_\varepsilon}\right| \mathrm{d}x &\le C\left(\|\varrho_\varepsilon\|^{\gamma}_{L^\gamma(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^\gamma(D_\varepsilon)}\right)\|\varrho_\varepsilon\|^{\Theta}_{L^\Theta(D_\varepsilon)} \\
&\le C\left(\|\varrho_\varepsilon\|^{\gamma}_{L^\gamma(D_\varepsilon)}\|\varrho_\varepsilon\|^{\Theta}_{L^\Theta(D_\varepsilon)} + \|\varrho_\varepsilon\|^{\Theta}_{L^\Theta(D_\varepsilon)}\right) \\
&\le C\left(\|\varrho_\varepsilon\|^{\gamma}_{L^\gamma(D_\varepsilon)}\|\varrho_\varepsilon\|^{\Theta}_{L^\Theta(D_\varepsilon)} + 1\right) \\
&\le C\left(\|\varrho_\varepsilon\|^{\gamma+\Theta}_{L^\gamma(D_\varepsilon)} + 1\right) \\
&\le C\left(\|\varrho_\varepsilon\|^{(1-\varpi)(\gamma+\Theta)}_{L^1(D_\varepsilon)}\|\varrho_\varepsilon\|^{\varpi(\gamma+\Theta)}_{L^{\gamma+\Theta}(D_\varepsilon)} + 1\right) \\
&\le C\left(\|\varrho_\varepsilon\|^{\varpi(\gamma+\Theta)}_{L^{\gamma+\Theta}(D_\varepsilon)} + 1\right),
\end{aligned}$$

where we used that we control the total mass $\mathfrak{m} = \|\varrho_\varepsilon\|_{L^1(D_\varepsilon)}$, and $\varpi \in (0,1)$ is determined by

$$\frac{1}{\gamma} = \frac{1-\varpi}{1} + \frac{\varpi}{\gamma+\Theta}.$$

Lastly, we estimate

$$\int_{D_\varepsilon} |\varrho_\varepsilon^{\Theta+1}\vartheta_\varepsilon| \,\mathrm{d}x \le \|\vartheta_\varepsilon\|_{L^q(D_\varepsilon)}\|\varrho_\varepsilon^{\Theta+1}\|_{L^{\frac{\gamma+\Theta}{\Theta+1}}(D_\varepsilon)} \le C\,\|\varrho_\varepsilon\|^{\Theta+1}_{L^{\gamma+\Theta}(D_\varepsilon)},$$

where we set $q := (\gamma+\Theta)/(\gamma-1)$. Recalling that $\Theta < \gamma$ and $\gamma > 2$, this yields $q \in (1,4)$, which entails in $\|\vartheta_\varepsilon\|_{L^q(D_\varepsilon)} \le C$ since we assumed $\|\vartheta_\varepsilon\|_{L^{3m_\vartheta}(D_\varepsilon)} \le C$ and $m_\vartheta > 2$.
Finally, we obtain from (4.54)

$$\|\varrho_\varepsilon\|^{\gamma+\Theta}_{L^{\gamma+\Theta}(D_\varepsilon)} \le C\left(1 + \|\varrho_\varepsilon\|^{\beta}_{L^{\gamma+\Theta}(D_\varepsilon)}\right) \qquad \text{for some } 1 < \beta < \gamma+\Theta,$$

which yields the uniform bound on ϱ_ε in $L^{\gamma+\Theta}(D_\varepsilon)$, provided ϑ_ε is uniformly bounded in $L^{3m_\vartheta}(D_\varepsilon)$. □

Combining the uniform estimates on ϱ_ε with those from (4.52), we obtain

$$\begin{gathered}
\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D_\varepsilon)} + \|\nabla \log\vartheta_\varepsilon\|_{L^2(D_\varepsilon)} + \|\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\|_{L^2(D_\varepsilon)} \le C, \\
\|\vartheta_\varepsilon\|_{L^1(\partial D_\varepsilon)} + \|\vartheta_\varepsilon^{-1}\|_{L^1(\partial D_\varepsilon)} \le C.
\end{gathered}$$

Note that these bounds are obtained by using the assumption that ϑ_ε is uniformly bounded in $L^{3m_\vartheta}(D_\varepsilon)$. This assumption will be proven in the next section.

4.3.4 Extension of functions

In order to work in the fixed domain D instead of the variable domain D_ε, we can extend the functions $\mathbf{u}_\varepsilon$ and ϱ_ε as well as the measure σ_ε simply by zero, which will preserve their regularity and their norms. In particular, the extended functions are still uniformly bounded. In the sequel we will denote this zero extension of a function f by $\tilde{f}$.

However, the extension of the temperature is more delicate since an extension by zero will in general not preserve its regularity. This extension was previously done in [LP21, Section 3], so we rely on their proofs. First recall that, by Theorem 3.4.2 and for $\varepsilon > 0$ small enough, the balls $\{B_{2\varepsilon^\alpha r_i}(\varepsilon z_i)\}_{z_i \in \Phi^\varepsilon(D)}$ are disjoint. The first lemma we need thus follows the same lines of the proof of [LP21, Lemma 3.1].

Lemma 4.3.5. *Let D_ε be defined as in (3.18) and let the assumptions of Theorem 3.4.2 hold. Then there is an almost surely positive random variable $\varepsilon_0(\omega)$ such that for all $0 < \varepsilon \le \varepsilon_0$ there exists an extension operator $\tilde{E}_\varepsilon : W^{1,2}(D_\varepsilon) \to W^{1,2}(D)$ such that for any $\varphi \in W^{1,2}(D_\varepsilon)$ and any $z_i \in \Phi^\varepsilon(D)$,*

$$\begin{aligned}
&\tilde{E}_\varepsilon \varphi = \varphi \text{ in } D_\varepsilon,\\
&\|\nabla \tilde{E}_\varepsilon \varphi\|_{L^2(B_{\varepsilon^\alpha r_i}(\varepsilon z_i))} \le C\, \|\nabla \varphi\|_{L^2(B_{2\varepsilon^\alpha r_i}(\varepsilon z_i) \setminus \overline{B_{\varepsilon^\alpha r_i}(\varepsilon z_i)})}
\end{aligned}$$

and hence $\|\nabla \tilde{E}_\varepsilon \varphi\|_{L^2(D)} \le C\, \|\nabla \varphi\|_{L^2(D_\varepsilon)}$. Farther, for any $1 \le q \le \infty$,

$$\|\tilde{E}_\varepsilon \varphi\|_{L^q(B_{\varepsilon^\alpha r_i}(\varepsilon z_i))} \le C\, \|\varphi\|_{L^q(B_{2\varepsilon^\alpha r_i}(\varepsilon z_i) \setminus \overline{B_{\varepsilon^\alpha r_i}(\varepsilon z_i)})}$$

and therefore $\|\tilde{E}_\varepsilon \varphi\|_{L^q(D)} \le C\, \|\varphi\|_{L^q(D_\varepsilon)}$, where the constant $C > 0$ is independent of ε and i. Furthermore, there exists an operator $E_\varepsilon : W^{1,2}_{\ge 0}(D_\varepsilon) \to W^{1,2}_{\ge 0}(D)$ with the same properties as above. Here $W^{1,2}_{\ge 0}$ denotes the Sobolev space of all non-negative functions in $W^{1,2}$. In particular, one may choose $E_\varepsilon \varphi := \max\{0, \tilde{E}_\varepsilon \varphi\}$.

Proof. To begin, let $\varphi \in W^{1,2}(B_2(0) \setminus \overline{B_1(0)})$, and write it in the form

$$\varphi = M\varphi + \psi, \quad M\varphi := \frac{1}{|B_2(0) \setminus \overline{B_1(0)}|} \int_{B_2(0) \setminus \overline{B_1(0)}} \varphi \, \mathrm{d}x, \quad M\psi = 0.$$

Since $B_2(0) \setminus \overline{B_1(0)}$ is a bounded Lipschitz domain, from [Ste70, Chapter VI, Theorem 5] we infer that there exists an extension operator $\tilde{S} : W^{1,2}(B_2(0) \setminus \overline{B_1(0)}) \to W^{1,2}(B_2(0))$ such that

$$\begin{aligned}
&\tilde{S}\psi = \psi \quad \text{in } B_2(0) \setminus \overline{B_1(0)},\\
&\|\tilde{S}\psi\|_{W^{1,2}(B_2(0))} \le C\, \|\psi\|_{W^{1,2}(B_2(0))},\\
&\|\tilde{S}\psi\|_{L^q(B_2(0))} \le C\, \|\psi\|_{L^q(B_2(0) \setminus \overline{B_1(0)})} \quad \forall 1 \le q \le \infty,
\end{aligned}$$

where the constant $C > 0$ is independent of q. Since $M\psi = 0$, we further deduce with Poincaré's inequality (B.6)

$$\begin{aligned}
\|\tilde{S}\psi\|_{W^{1,2}(B_2(0))} &\le C\, \|\psi\|_{W^{1,2}(B_2(0) \setminus \overline{B_1(0)})}\\
&\le C\, \|\nabla \psi\|_{L^2(B_2(0) \setminus \overline{B_1(0)})} = C\, \|\nabla \varphi\|_{L^2(B_2(0) \setminus \overline{B_1(0)})}.
\end{aligned}$$

Setting

$$S\varphi := M\varphi + \tilde{S}\psi \quad \text{in } B_2(0),$$

then still

$$\begin{aligned} &S\varphi = \varphi \quad \text{in } B_2(0) \setminus \overline{B_1(0)}, \\ &\|S\varphi\|_{W^{1,2}(B_2(0))} \le C\, \|\varphi\|_{W^{1,2}(B_2(0)\setminus\overline{B_1(0)})}, \\ &\|S\varphi\|_{L^q(B_2(0)\setminus\overline{B_1(0)})} \le C\, \|\varphi\|_{L^q(B_2(0)\setminus\overline{B_1(0)})} \quad \forall 1 \le q \le \infty. \end{aligned}$$

Now, for $\varphi \in W^{1,2}(B_{2\varepsilon^\alpha r_i}(\varepsilon z_i) \setminus \overline{B_{\varepsilon^\alpha r_i}(\varepsilon z_i)})$, set

$$\tilde{\varphi}(y) := \varphi(\varepsilon z_i + r_i\varepsilon^\alpha y),$$

then $\tilde{\varphi} \in W^{1,2}(B_2(0) \setminus \overline{B_1(0)})$. We can now define the extension operator in each hole $\tilde{E}^i_\varepsilon$ by

$$\tilde{E}^i_\varepsilon\varphi(x) := (S\tilde{\varphi})\left(\frac{x - \varepsilon z_i}{r_i\varepsilon^\alpha}\right).$$

By the properties of the operator S, we clearly have $\tilde{E}^i_\varepsilon\varphi \in W^{1,2}(B_{2\varepsilon^\alpha r_i}(\varepsilon z_i))$ and $\tilde{E}^i_\varepsilon\varphi = \varphi$ in $B_{2\varepsilon^\alpha r_i}(\varepsilon z_i) \setminus \overline{B_{\varepsilon^\alpha r_i}(\varepsilon z_i)}$. Farther,

$$\begin{aligned} \int_{B_{2\varepsilon^\alpha r_i}(\varepsilon z_i)} |\nabla_x \tilde{E}^i_\varepsilon\varphi|^2 \,\mathrm{d}x &= \int_{B_{2\varepsilon^\alpha r_i}(\varepsilon z_i)} \left|\nabla_x S\tilde{\varphi}\left(\frac{x - \varepsilon z_i}{r_i\varepsilon^\alpha}\right)\right|^2 \mathrm{d}x \\ &= \int_{B_{2\varepsilon^\alpha r_i}(\varepsilon z_i)} (r_i\varepsilon^\alpha)^{-2} \left|(\nabla_y S\tilde{\varphi})\left(\frac{x - \varepsilon z_i}{r_i\varepsilon^\alpha}\right)\right|^2 \mathrm{d}x \\ &= r_i\varepsilon^\alpha \int_{B_2(0)} |(\nabla_y S\tilde{\varphi})(y)|^2 \,\mathrm{d}y \\ &\le C\, r_i\varepsilon^\alpha \int_{B_2(0)\setminus\overline{B_1(0)}} |\nabla_y \tilde{\varphi}(y)|^2 \,\mathrm{d}y \\ &= C \int_{B_{2\varepsilon^\alpha r_i}(\varepsilon z_i)\setminus\overline{B_{\varepsilon^\alpha r_i}(\varepsilon z_i)}} |\nabla_x \varphi(x)|^2 \,\mathrm{d}x. \end{aligned}$$

Similarly,

$$\|\tilde{E}^i_\varepsilon\varphi\|_{L^q(B_{2\varepsilon^\alpha r_i}(\varepsilon z_i))} \le C\, \|\varphi\|_{L^q(B_{2\varepsilon^\alpha r_i}(\varepsilon z_i)\setminus\overline{B_{\varepsilon^\alpha r_i}(\varepsilon z_i)})} \quad \forall 1 \le q \le \infty.$$

Finally, the extension operator $\tilde{E}_\varepsilon$ is defined by

$$\tilde{E}_\varepsilon\varphi := \varphi\chi + \sum_{z_i \in \Phi^\varepsilon(D)} \tilde{E}^i_\varepsilon\varphi \quad \forall \varphi \in W^{1,2}(D_\varepsilon),$$

provided each $\tilde{E}^i_\varepsilon\varphi$ is extended to be zero outside its domain of definition, and χ is the characteristic function of $D \setminus \bigcup_{z_i \in \Phi^\varepsilon(D)} \overline{B_{2r_i\varepsilon^\alpha}(\varepsilon z_i)}$. This operator clearly obeys all the required properties of the lemma. The last assertion for $E_\varepsilon\varphi = \max\{0, \tilde{E}_\varepsilon\varphi\}$ is a simple consequence of

$$\|\nabla E_\varepsilon\varphi\|_{L^2(D)} \le \|\nabla\tilde{E}_\varepsilon\varphi\|_{L^2(D)}$$

and

$$\|E_\varepsilon\varphi\|_{L^q(D)} \leq \|\tilde{E}_\varepsilon\varphi\|_{L^q(D)} \quad \forall 1 \leq q \leq \infty.$$

□

With the help of the extension operator E_ε, we can bound the temperature uniformly with respect to ε.

Lemma 4.3.6. *For $\varepsilon > 0$ small enough, we have $\|E_\varepsilon\vartheta_\varepsilon\|_{W^{1,2}(D)} + \|E_\varepsilon\vartheta_\varepsilon\|_{L^{3m_\vartheta}(D)} \leq C$ for some $C > 0$ independent of ε. In particular, we have $\|\vartheta_\varepsilon\|_{W^{1,2}(D_\varepsilon)} + \|\vartheta_\varepsilon\|_{L^{3m_\vartheta}(D_\varepsilon)} \leq C$ uniformly in ε.*

Proof. First, as $\vartheta_\varepsilon \in W^{1,2}(D_\varepsilon)$ and $\vartheta_\varepsilon > 0$ almost everywhere in D_ε, we have $E_\varepsilon\vartheta_\varepsilon \in W^{1,2}(D)$ and $E_\varepsilon\vartheta_\varepsilon \geq 0$ almost everywhere in D. By $m_\vartheta > 2$, the fact that $\vartheta_\varepsilon^2 \leq 1 + \vartheta_\varepsilon^{m_\vartheta}$, and (4.51), we get

$$\int_{D_\varepsilon} |\nabla\vartheta_\varepsilon|^2 \,\mathrm{d}x \leq \int_{D_\varepsilon} \frac{(1+\vartheta_\varepsilon^{m_\vartheta})|\nabla\vartheta_\varepsilon|^2}{\vartheta_\varepsilon^2} \,\mathrm{d}x \leq C$$

uniformly in ε. By Lemma 4.3.5, the same holds true for $E_\varepsilon\vartheta_\varepsilon \in W^{1,2}(D)$. As we also have uniform control on the L^1-norm of ϑ_ε over ∂D (see (4.52)) and this value does not change by applying the extension, we have a uniform L^6-control of $E_\varepsilon\vartheta_\varepsilon$ over the whole of D. As $E_\varepsilon\vartheta_\varepsilon$ coincides with ϑ_ε in D_ε, we have also a uniform control on the L^6-norm of ϑ_ε in D_ε.
Assume for now $m_\vartheta \leq 12$, and recall that $\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}$ is uniformly bounded in $L^2(D_\varepsilon)$ by (4.52). By the arguments given above, we already know that $\vartheta_\varepsilon^{\frac{m_\vartheta}{2}}$ has uniform controlled L^1-norm over D_ε, so we may estimate with the help of Poincaré's inequality (B.6)

$$\begin{aligned}\|\vartheta_\varepsilon^{\frac{m_\vartheta}{2}}\|_{L^2(D_\varepsilon)} &\leq \|\vartheta_\varepsilon^{\frac{m_\vartheta}{2}} - \langle\vartheta_\varepsilon^{\frac{m_\vartheta}{2}}\rangle_{D_\varepsilon}\|_{L^2(D_\varepsilon)} + \langle\vartheta_\varepsilon^{\frac{m_\vartheta}{2}}\rangle_{D_\varepsilon}|D_\varepsilon|^{\frac12}\\ &\leq C\,\|\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\|_{L^2(D_\varepsilon)} + |D_\varepsilon|^{-\frac12}\|\vartheta_\varepsilon^{\frac{m_\vartheta}{2}}\|_{L^1(D_\varepsilon)} \leq C,\end{aligned}$$

where we used the notation $\langle\vartheta_\varepsilon^{\frac{m_\vartheta}{2}}\rangle_{D_\varepsilon}$ for the mean value of $\vartheta_\varepsilon^{\frac{m_\vartheta}{2}}$ over D_ε, and the fact that, for ε small enough, the measure of D_ε is controlled by (3.26). Thus, $\vartheta_\varepsilon^{\frac{m_\vartheta}{2}}$ is uniformly bounded in $W^{1,2}(D_\varepsilon)$, and by Lemma 4.3.5, the same holds true for $(E_\varepsilon\vartheta_\varepsilon)^{\frac{m_\vartheta}{2}}$ in $W^{1,2}(D)$. By the Rellich-Kondrachev theorem from Proposition B.5, $(E_\varepsilon\vartheta_\varepsilon)^{\frac{m_\vartheta}{2}}$ is bounded uniformly in $L^6(D)$, that is, $E_\varepsilon\vartheta_\varepsilon$ is uniformly bounded in $L^{3m_\vartheta}(D_\varepsilon)$. Again by Lemma 4.3.5, this yields a uniform bound on ϑ_ε in $L^{3m_\vartheta}(D_\varepsilon)$.
Let now $m_\vartheta > 12$. From the steps done above, we have a uniform control on ϑ_ε in $L^{36}(D_\varepsilon)$. By repeating the arguments for $2 < m_\vartheta \leq 12$, we may cover all $2 < m_\vartheta \leq 36$. Going further yields the desired for any $m_\vartheta > 2$. □

We further need to estimate the trace of ϑ_ε in ∂D_ε. Indeed, for fixed $\varepsilon > 0$, the trace of ϑ_ε belongs to $L^{2m_\vartheta}(\partial D_\varepsilon)$, as can be seen from the standard trace theorem, see also the proof of the next lemma. This lemma enables us to control the norm of the trace of ϑ_ε in a quantitative way.

Lemma 4.3.7. *Under the assumptions of Theorem 4.3.3, we have for any $z_i \in \Phi^\varepsilon(D)$ and for $\varepsilon > 0$ small enough*

$$\|\vartheta_\varepsilon\|^{2m_\vartheta}_{L^{2m_\vartheta}(\partial B_i)} \le C\left(\|\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\|^2_{L^2(2B_i\setminus\overline{B_i})} + \|\vartheta_\varepsilon\|^{3m_\vartheta}_{L^{3m_\vartheta}(2B_i\setminus\overline{B_i})} + \|\vartheta_\varepsilon\|^{2m_\vartheta}_{L^{3m_\vartheta}(2B_i\setminus\overline{B_i})}\right),$$

where we set $B_i := B_{\varepsilon^\alpha r_i}(\varepsilon z_i)$ and $2B_i := B_{2\varepsilon^\alpha r_i}(\varepsilon z_i)$.

Proof. Following the standard proof of the trace theorem for Sobolev functions (see, for instance, [Eva10, Chapter 5.5, Theorem 1]), we may arrive at

$$\int_{\partial B_i} |\vartheta_\varepsilon|^{2m_\vartheta}\,\mathrm{d}\sigma(x) \le C\int_{2B_i\setminus\overline{B_i}} |\nabla(\varphi_\varepsilon|\vartheta_\varepsilon|^{2m_\vartheta})|\,\mathrm{d}x,$$

where $\varphi_\varepsilon \in C_c^\infty(2B_i)$ is a non-negative smooth cut-off function satisfying $\varphi_\varepsilon|_{\partial B_i} = 1$ and $\|\nabla\varphi_\varepsilon\|_{L^\infty(2B_i)} \le C\,(r_i\varepsilon^\alpha)^{-1}$. Since $\nabla|\vartheta_\varepsilon|^{2m_\vartheta} = 4\,\vartheta_\varepsilon^{\frac{3m_\vartheta}{2}}\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}$ and $|2B_i\setminus\overline{B_i}| \lesssim (r_i\varepsilon^\alpha)^3$, we see at once that $\vartheta_\varepsilon \in L^{2m_\vartheta}(\partial D_\varepsilon)$. Using further Hölder's inequality (B.2) and Young's inequality (B.1), we calculate

$$\begin{aligned}
\|\vartheta_\varepsilon\|^{2m_\vartheta}_{L^{2m_\vartheta}(\partial B_i)} &= \int_{\partial B_i} |\vartheta_\varepsilon|^{2m_\vartheta}\,\mathrm{d}\sigma(x) \lesssim \int_{2B_i\setminus\overline{B_i}} |\nabla(\varphi_\varepsilon|\vartheta_\varepsilon|^{2m_\vartheta})|\,\mathrm{d}x \\
&\lesssim \int_{2B_i\setminus\overline{B_i}} |\nabla\varphi_\varepsilon|\,|\vartheta_\varepsilon|^{2m_\vartheta} + \varphi_\varepsilon\big|\nabla|\vartheta_\varepsilon|^{2m_\vartheta}\big|\,\mathrm{d}x \\
&\lesssim (r_i\varepsilon^\alpha)^{-1}\int_{2B_i\setminus\overline{B_i}} |\vartheta_\varepsilon|^{2m_\vartheta} + \big|\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\big|\,|\vartheta_\varepsilon|^{\frac{3m_\vartheta}{2}}\,\mathrm{d}x \\
&\lesssim \left(\int_{2B_i\setminus\overline{B_i}} |\vartheta_\varepsilon|^{3m_\vartheta}\,\mathrm{d}x\right)^{\frac{2}{3}} + \left(\int_{2B_i\setminus\overline{B_i}} \big|\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\big|^2\,\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_{2B_i\setminus\overline{B_i}} |\vartheta_\varepsilon|^{3m_\vartheta}\,\mathrm{d}x\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{2B_i\setminus\overline{B_i}} |\vartheta_\varepsilon|^{3m_\vartheta}\,\mathrm{d}x\right)^{\frac{2}{3}} + \int_{2B_i\setminus\overline{B_i}} \big|\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\big|^2\,\mathrm{d}x + \int_{2B_i\setminus\overline{B_i}} |\vartheta_\varepsilon|^{3m_\vartheta}\,\mathrm{d}x \\
&= \|\vartheta_\varepsilon\|^{2m_\vartheta}_{L^{3m_\vartheta}(2B_i\setminus\overline{B_i})} + \|\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\|^2_{L^2(2B_i\setminus\overline{B_i})} + \|\vartheta_\varepsilon\|^{3m_\vartheta}_{L^{3m_\vartheta}(2B_i\setminus\overline{B_i})}.
\end{aligned}$$

□

The last ingredient we need is a trace estimate for the whole boundary of the holes.

Corollary 4.3.8. *Under the assumptions of Theorem 3.4.2 and Theorem 4.3.3, we have for any $z_i \in \Phi^\varepsilon(D)$ and for $\varepsilon > 0$ small enough*

$$\|\vartheta_\varepsilon\|_{L^{2m_\vartheta}(\cup_{z_i\in\Phi^\varepsilon(D)}\partial B_{\varepsilon^\alpha r_i}(\varepsilon z_i))} \le C\,\varepsilon^{-\frac{1}{2m_\vartheta}}.$$

Proof. For $z_i \in \Phi^\varepsilon(D)$, we set again $B_i := B_{\varepsilon^\alpha r_i}(\varepsilon z_i)$ and $2B_i := B_{2\varepsilon^\alpha r_i}(\varepsilon z_i)$. Then, using Hölder's inequality (B.2) and Lemma 4.3.7, we get

$$\begin{aligned}
\int_{\cup_{z_i\in\Phi^\varepsilon(D)}\partial B_i} |\vartheta_\varepsilon|^{2m_\vartheta}\,\mathrm{d}\sigma(x) &= \sum_{z_i\in\Phi^\varepsilon(D)} \int_{\partial B_i} |\vartheta_\varepsilon|^{2m_\vartheta}\,\mathrm{d}\sigma(x) \\
&\le C\sum_{z_i\in\Phi^\varepsilon(D)} \left(\int_{2B_i\setminus\overline{B_i}} |\vartheta_\varepsilon|^{3m_\vartheta}\,\mathrm{d}x\right)^{\frac{2}{3}}
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{z_i\in\Phi^\varepsilon(D)} \int_{2B_i\setminus\overline{B_i}} \left|\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\right|^2 \mathrm{d}x + \sum_{z_i\in\Phi^\varepsilon(D)} \int_{2B_i\setminus\overline{B_i}} |\vartheta|^{3m_\vartheta} \,\mathrm{d}x \\
&\leq C\left(\sum_{z_i\in\Phi^\varepsilon(D)} \int_{2B_i\setminus\overline{B_i}} |\vartheta_\varepsilon|^{3m_\vartheta} \,\mathrm{d}x\right)^{\frac{2}{3}} \left(\sum_{z_i\in\Phi^\varepsilon(D)} 1\right)^{\frac{1}{3}} \\
&\quad + \int_{D_\varepsilon} \left|\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}\right|^2 \mathrm{d}x + \int_{D_\varepsilon} |\vartheta|^{3m_\vartheta} \,\mathrm{d}x \\
&\leq C\left[\left(\#\{z_i\in\Phi^\varepsilon(D)\}\right)^{\frac{1}{3}} + 1\right],
\end{aligned}
$$

where in the last inequality we used the uniform bounds on ϑ_ε and $\nabla|\vartheta_\varepsilon|^{\frac{m_\vartheta}{2}}$. From Remark 3.4.6, for $\varepsilon>0$ small enough, the number of points $z_i\in\Phi^\varepsilon(D)$ is bounded by $C\,\varepsilon^{-3}$, which immediately implies our desired assertion. □

Summarizing all the above results, we know the existence of an almost surely positive random variable $\varepsilon_0(\omega)$ such that for all $0<\varepsilon\leq\varepsilon_0$ the solution $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon]$ to (4.33)-(4.44) and the measure σ_ε, suitably extended to the whole of D, satisfy

$$
\tilde{\sigma}_\varepsilon(\overline{D}) + \|\tilde{\mathbf{u}}_\varepsilon\|_{W_0^{1,2}(D)} + \|\tilde{\varrho}_\varepsilon\|_{L^{\gamma+\Theta}(D)} + \|E_\varepsilon\vartheta_\varepsilon\|_{W^{1,2}(D)\cap L^{3m_\vartheta}(D)} + \|E_\varepsilon \log(\vartheta_\varepsilon)\|_{W^{1,2}(D)} \leq C, \tag{4.55}
$$

where Θ is defined in (4.53). Furthermore, ϑ_ε has a well-defined trace on each $\partial B_{\varepsilon^\alpha r_i}(\varepsilon z_i)$, the norm of which is controlled by Corollary 4.3.8.

4.3.5 Equations in fixed domain

This section is devoted to show the the homogenization result for the Navier-Stokes-Fourier equations in a randomly perforated domain in the subcritical case $\alpha>3$. The proof of such a result in the case of well separated holes is given in [LP21, Section 4]. Their methods apply almost verbatim to our situation, so we will mainly focus on the differences due to the random setting. Again, we will always assume that the moment bound $m\geq 3$ in (3.19) to bound the measures of D_ε and ∂D_ε.

First, the bounds in (4.52) and (4.55) enable us to extract subsequences (not relabeled) such that

$$
\begin{aligned}
&\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u} \text{ weakly in } W_0^{1,2}(D), \quad \tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \text{ strongly in } L^q(D) \text{ for all } 1\leq q<6, \\
&\tilde{\varrho}_\varepsilon \rightharpoonup \varrho \text{ weakly in } L^{\gamma+\Theta}(D), \\
&E_\varepsilon\vartheta_\varepsilon \rightharpoonup \vartheta \text{ weakly in } W^{1,2}(D), \quad E_\varepsilon\vartheta_\varepsilon \to \vartheta \text{ strongly in } L^q(D) \text{ for all } 1\leq q<3m_\vartheta, \\
&E_\varepsilon \log(\vartheta_\varepsilon) \rightharpoonup \overline{\log(\vartheta)} \text{ weakly in } W^{1,2}(D),
\end{aligned}
$$

where we denote by $\overline{\log(\vartheta)}$ the weak limit of $E_\varepsilon \log(\vartheta_\varepsilon)$ in $W^{1,2}(D)$.

Let us start with the limit passage for the energy, continuity, and momentum equations. To pass to the limit in the energy balance in (4.33), we use its weak formulation (4.48) and the fact $\tilde{\mathbf{u}}_\varepsilon = 0$ in $D \setminus D_\varepsilon$ to write

$$
\begin{aligned}
&-\int_D \Big(\tilde{\varrho}_\varepsilon E(\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, \tilde{\vartheta}_\varepsilon)\tilde{\mathbf{u}}_\varepsilon + p(\tilde{\varrho}_\varepsilon, \tilde{\vartheta}_\varepsilon)\tilde{\mathbf{u}}_\varepsilon - \mathbb{S}(\tilde{\vartheta}_\varepsilon, \nabla \tilde{\mathbf{u}}_\varepsilon)\tilde{\mathbf{u}}_\varepsilon - \kappa(\tilde{\vartheta}_\varepsilon)\nabla E_\varepsilon \vartheta_\varepsilon \Big) \cdot \nabla \psi \,\mathrm{d}x \\
&\quad + L \int_{\partial D} (\vartheta_\varepsilon - \vartheta_0)\psi \,\mathrm{d}\sigma(x) - \int_D (\tilde{\varrho}_\varepsilon \mathbf{f} + \mathbf{g}) \cdot \tilde{\mathbf{u}}_\varepsilon \psi \,\mathrm{d}x \\
&= \int_{D\setminus D_\varepsilon} \kappa(\vartheta_\varepsilon)\nabla\vartheta_\varepsilon \cdot \nabla \psi \,\mathrm{d}x - L \int_{\cup_{z_i\in\Phi^\varepsilon(D)} \partial B_{\varepsilon^\alpha r_i}(\varepsilon z_i)} (\vartheta_\varepsilon - \vartheta_0)\psi \,\mathrm{d}\sigma(x) \\
&=: I_1 + I_2
\end{aligned}
\tag{4.56}
$$

for any $\psi \in C^1(\overline{D})$, where $E(\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon)$ is the total energy from (4.42). We want to show that both integrals on the right-hand side vanish as $\varepsilon \to 0$. For I_1, by Hölder's inequality (B.2), we get

$$
|I_1| \leq C \, \|\nabla \psi\|_{L^\infty(D)} (1 + \|\vartheta_\varepsilon\|^{m_\vartheta}_{L^{3m_\vartheta}(D\setminus D_\varepsilon)}) \, \|\nabla \vartheta_\varepsilon\|_{L^2(D\setminus D_\varepsilon)} \, |D \setminus D_\varepsilon|^{\frac{1}{6}} \to 0,
$$

where we used that $|D \setminus D_\varepsilon| \to 0$ by (3.26). For I_2, let us set $B_i := B_{\varepsilon^\alpha r_i}(\varepsilon z_i)$. Using Corollary 4.3.8 and that $\|\vartheta_0\|_{L^q(\partial D_\varepsilon)}$ is uniformly bounded for some $q > 1$ with respect to ε, together with $\alpha > 3$, $m > 2$, and Lemma 3.4.5, we obtain

$$
\begin{aligned}
|I_2| &\leq C \bigg[\|\vartheta_\varepsilon\|_{L^{2m_\vartheta}(\cup_{z_i\in\Phi^\varepsilon(D)}\partial B_i)} \Big| \bigcup_{z_i \in \Phi^\varepsilon(D)} \partial B_i \Big|^{\frac{2m_\vartheta - 1}{2m_\vartheta}} + \|\vartheta_0\|_{L^q(\cup_{z_i\in\Phi^\varepsilon(D)}\partial B_i)} \Big| \bigcup_{z_i \in \Phi^\varepsilon(D)} \partial B_i \Big|^{\frac{q-1}{q}} \bigg] \\
&\leq C \bigg[\varepsilon^{-\frac{1}{2m_\vartheta}} \Big(\sum_{z_i \in \Phi^\varepsilon(D)} \varepsilon^{2\alpha} r_i^2 \Big)^{\frac{2m_\vartheta - 1}{2m_\vartheta}} + \Big(\sum_{z_i \in \Phi^\varepsilon(D)} \varepsilon^{2\alpha} r_i^2 \Big)^{\frac{q-1}{q}} \bigg] \\
&\leq C \bigg[\varepsilon^{\frac{(2\alpha-3)(2m_\vartheta - 1) - 1}{2m_\vartheta}} + \varepsilon^{\frac{(2\alpha - 3)(q-1)}{q}} \bigg] \to 0,
\end{aligned}
$$

where we used that $(2\alpha - 3)(2m_\vartheta - 1) > 1$ due to our assumptions $\alpha > \frac{3m_\vartheta - 2}{m_\vartheta - 2}$ and $m_\vartheta > 2$. Hence, letting $\varepsilon \to 0$ on the left-hand side of (4.56), we get by the strong convergences of $\mathbf{u}_\varepsilon$ and ϑ_ε

$$
\begin{aligned}
&-\int_D \Big(\big(\overline{\varrho e(\varrho, \vartheta)} + \frac{1}{2}\varrho |\mathbf{u}|^2 + \overline{p(\varrho, \vartheta)} - \mathbb{S}(\vartheta, \nabla \mathbf{u})\big)\mathbf{u} - \kappa(\vartheta)\nabla\vartheta \Big) \cdot \nabla \psi \,\mathrm{d}x \\
&\quad + L \int_{\partial D} (\vartheta - \vartheta_0)\psi \,\mathrm{d}\sigma(x) = \int_D (\varrho \mathbf{f} + \mathbf{g}) \cdot \mathbf{u} \psi \,\mathrm{d}x.
\end{aligned}
$$

Here, $\overline{f(\varrho, \vartheta)}$ denotes the weak limit of a function $f(\varrho_\varepsilon, \vartheta_\varepsilon)$ in some suitable L^q-space. Also, the temperature $\vartheta > 0$ almost everywhere in D and $\overline{\log(\vartheta)} = \log(\vartheta)$, which can be proven as shown in [LP21, Lemma 4.1]. For convenience, we repeat the proof here.

Lemma 4.3.9. *The limiting temperature $\vartheta > 0$ a.e. in D, and further $\overline{\log(\vartheta)} = \log(\vartheta)$.*

Proof. First, since $E_\varepsilon \vartheta_\varepsilon \to \vartheta$ strongly in, say, $L^2(D)$, we can extract a subsequence (not relabeled) such that $E_\varepsilon \vartheta_\varepsilon \to \vartheta$ a.e. in D, which yields that the limit temperature cannot be negative. It thus suffices to prove that it can be zero just on a set of measure zero. To this end we assume the contrary, that is, there exists $\delta > 0$ such that $|\{\vartheta = 0\}| = \delta$. Take a sequence $\{\varepsilon_l\}_{l\in\mathbb{N}} \subset (0, \varepsilon_0(\omega))$ with $\varepsilon_l \leq l^{-1}$, where $\varepsilon_0(\omega) > 0$ is as in Theorem 3.4.2, and consider the sets

$$D_{l_0} := \bigcup_{l=l_0}^{\infty} \bigcup_{z_i \in \Phi^{\varepsilon_l}(D)} \overline{B_{\varepsilon_l^\alpha r_i}(\varepsilon_l z_i)}.$$

Since for $\varepsilon_l > 0$ small enough we have

$$\Big| \bigcup_{z_i \in \Phi^{\varepsilon_l}(D)} B_{\varepsilon_l^\alpha r_i}(\varepsilon_l z_i) \Big| \leq \frac{C}{l^{3(\alpha-1)}}$$

by (3.26) and $\alpha > 3$, we can find $l_0 \in \mathbb{N}$ such that $|D_{l_0}| \leq \frac{\delta}{2}$.

We have $E_{\varepsilon_l} \log(\vartheta_{\varepsilon_l}) \rightharpoonup \overline{\log(\vartheta)}$ weakly in $L^q(D)$ for all $1 \leq q \leq 2$, in particular $\overline{\log(\vartheta)} > -\infty$ a.e. in D. Since we have also $\tilde{\vartheta}_{\varepsilon_l} \to \vartheta$ a.e. in D and thus a.e. in $D \setminus D_{l_0}$, we infer by Vitali's convergence theorem (see Proposition B.8) $\log(\tilde{\vartheta}_{\varepsilon_l}) \to \log(\vartheta)$ in $L^q(D \setminus D_{l_0})$ for some $q > 1$. Since by definition of E_ε we have $\log(\tilde{\vartheta}_{\varepsilon_l}) = E_{\varepsilon_l} \log(\vartheta_{\varepsilon_l})$ in $D \setminus D_{l_0}$, we have $\overline{\log(\vartheta)} = \log(\vartheta)$ a.e. in $D \setminus D_{l_0}$, which yields $\log(\vartheta) > -\infty$ a.e. in $D \setminus D_{l_0}$. This means that ϑ can be zero at most on the set D_{l_0} which has a measure less than $\delta/2$, which is a contradiction. Thus $\vartheta > 0$ and $\overline{\log(\vartheta)} = \log(\vartheta)$ a.e. in D. □

It remains to show the energy balance for the limit functions, which is in fact a consequence of the strong convergence of the density ϱ_ε to ϱ at least in $L^1(D)$. More precisely, the strong convergence holds in $L^q(D)$ for any $1 \leq q < \gamma + \Theta$. The proof of this fact follows the same lines as done in Appendix A.

We now turn to the continuity and momentum equation. Recall that the continuity equation holds in the weak and renormalized sense (4.45) and (4.46), so we obtain by the strong convergence of $\tilde{\mathbf{u}}_\varepsilon$ to $\mathbf{u}$

$$\operatorname{div}(\varrho \mathbf{u}) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3). \tag{4.57}$$

Moreover, by Remark 4.1.2, (4.57) implies that the couple $[\varrho, \mathbf{u}]$ fulfills the renormalized continuity equation (4.46) for any $b \in C([0,\infty)) \cap C^1((0,\infty))$ satisfying the conditions of Remark 4.1.2.

To pass to the limit in the momentum equation, we need to construct suitable test functions. To this end, we recall Lemma 4.1.6, which guarantees for any $1 < r < 3$ with $(3-r)\alpha - 3 > 0$ the existence of a family of functions $\{g_\varepsilon\}_{\varepsilon > 0} \subset W^{1,r}(D)$ such that for $0 < \varepsilon \leq \varepsilon_0$,

$$g_\varepsilon = 0 \quad \text{in} \bigcup_{z_j \in \Phi^\varepsilon(D)} B_{\varepsilon^\alpha r_j}(\varepsilon z_j), \qquad g_\varepsilon \to 1 \quad \text{in } W^{1,r}(D) \text{ as } \varepsilon \to 0, \tag{4.58}$$

and there is a constant $C > 0$ such that

$$\|1 - g_\varepsilon\|_{L^r(D)} \leq C \varepsilon^{\frac{3(\alpha-1)}{r}} \quad \|\nabla g_\varepsilon\|_{L^r(D)} \leq C\, \varepsilon^{\frac{(3-r)\alpha - 3}{r}} = C\, \varepsilon^{\frac{3(\alpha-1)}{r} - \alpha}. \tag{4.59}$$

Using these cut-off functions, we obtain a similar statement as given in Lemma 4.2 in [LP21].

Lemma 4.3.10. *Under the assumptions of Theorem 4.3.3, there holds*

$$\operatorname{div}(\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) + \nabla p(\tilde{\varrho}_\varepsilon, E_\varepsilon \vartheta_\varepsilon) - \operatorname{div} \mathbb{S}(E_\varepsilon \vartheta_\varepsilon, \nabla \tilde{\mathbf{u}}_\varepsilon) = \tilde{\varrho}_\varepsilon \mathbf{f} + \mathbf{g} + F_\varepsilon \text{ in } \mathcal{D}'(D),$$

where F_ε is a distribution satisfying

$$|\langle F_\varepsilon, \varphi \rangle_{\mathcal{D}'(D), \mathcal{D}(D)}| \leq C \varepsilon^\nu \big(\|\nabla \varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}+\xi}(D)} + \|\varphi\|_{L^r(D)} \big)$$

for all $\varphi \in \mathcal{D}(D)$, where Θ is defined in (4.53), and ν, ξ, r are defined such that the following conditions are fulfilled:

$$\begin{aligned}
&0 < \xi < 1, \quad 0 < h(\xi) := 3\,(\alpha - 1) \left(\frac{\gamma + \Theta}{\Theta} + \xi \right)^{-1} - \alpha, \\
&1 < r < \infty, \quad \frac{1}{r} + \left(\frac{\gamma + \Theta}{\Theta} + \xi \right)^{-1} = \frac{\Theta}{\gamma + \Theta}, \\
&0 < \nu < \infty, \quad \nu := \min \left\{ \frac{3(\alpha - 1)}{r}, h(\xi) \right\}.
\end{aligned}$$

Let us remark that the conditions on ξ, r, and ν occurred earlier in Section 4.2 for the case of constant temperature, $\gamma > 2$, and $\Theta = 2\gamma - 3$, where we have $\frac{\gamma+\Theta}{\Theta} = \frac{3(\gamma-1)}{2\gamma-3}$.

Proof of Lemma 4.3.10. The proof is similar to the one given earlier in Section 4.1.3. For legibility, we will identify functions $[\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon]$, defined on the domain D_ε, with their extensions $[\tilde{\varrho}_\varepsilon, \tilde{\mathbf{u}}_\varepsilon, E_\varepsilon \vartheta_\varepsilon]$ to the whole of D.

Let $\varphi \in \mathcal{D}(D)$ and decompose $\varphi = g_\varepsilon \varphi + (1 - g_\varepsilon)\varphi$, then $g_\varepsilon \varphi$ is a proper test function in the second equation of (4.33). Hence,

$$\begin{aligned}
&\int_D \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla \varphi + p(\varrho_\varepsilon, \vartheta_\varepsilon) \operatorname{div} \varphi - \mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) : \nabla \varphi + (\varrho_\varepsilon \mathbf{f} + \mathbf{g}) \cdot \varphi \,\mathrm{d}x \\
&\quad = \int_{D_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla(g_\varepsilon \varphi) + p(\varrho_\varepsilon, \vartheta_\varepsilon) \operatorname{div}(g_\varepsilon \varphi) \,\mathrm{d}x \\
&\qquad + \int_D -\mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) : \nabla(g_\varepsilon \varphi) + (\varrho_\varepsilon \mathbf{f} + \mathbf{g}) \cdot (g_\varepsilon \varphi) \,\mathrm{d}x + I_\varepsilon \\
&\quad = I_\varepsilon,
\end{aligned}$$

where the remainder is given by

$$\begin{aligned}
I_\varepsilon := \sum_{j=1}^4 I_{j,\varepsilon} := &\int_D \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : (1 - g_\varepsilon) \nabla \varphi - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : (\nabla g_\varepsilon \otimes \varphi) \,\mathrm{d}x \\
&+ \int_D p(\varrho_\varepsilon, \vartheta_\varepsilon)(1 - g_\varepsilon) \operatorname{div} \varphi - p(\varrho_\varepsilon, \vartheta_\varepsilon) \nabla g_\varepsilon \cdot \varphi \,\mathrm{d}x \\
&+ \int_D -\mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) : (1 - g_\varepsilon) \nabla \varphi + \mathbb{S}(\vartheta_\varepsilon, \nabla \mathbf{u}_\varepsilon) : (\nabla g_\varepsilon \otimes \varphi) \,\mathrm{d}x \\
&+ \int_D (\varrho_\varepsilon \mathbf{f} + \mathbf{g}) \cdot (1 - g_\varepsilon) \varphi \,\mathrm{d}x.
\end{aligned}$$

We start with $I_{\varepsilon,2}$, which is the most restrictive one. We split the integral due to the definition of the pressure as $p = a\varrho^\gamma + c_v(\gamma-1)\varrho\vartheta$ into

$$\begin{aligned} I_{\varepsilon,2} &= \int_D p(\varrho_\varepsilon,\vartheta_\varepsilon)[(1-g_\varepsilon)\operatorname{div}\varphi - \nabla g_\varepsilon\cdot\varphi]\,\mathrm{d}x \\ &= \int_D a\varrho_\varepsilon^\gamma[(1-g_\varepsilon)\operatorname{div}\varphi - \nabla g_\varepsilon\cdot\varphi]\,\mathrm{d}x \\ &\quad + \int_D c_v(\gamma-1)\varrho_\varepsilon\vartheta_\varepsilon[(1-g_\varepsilon)\operatorname{div}\varphi - \nabla g_\varepsilon\cdot\varphi]\,\mathrm{d}x \\ &\quad =: I^1 + I^2. \end{aligned}$$

For I^1, we estimate

$$\begin{aligned} |I^1| &\le C\,\|\varrho_\varepsilon^\gamma\|_{L^{\frac{\gamma+\Theta}{\gamma}}(D)}\big(\|(1-g_\varepsilon)\operatorname{div}\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)} + \|\nabla g_\varepsilon\cdot\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}\big) \\ &= C\,\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D)}^\gamma\big(\|(1-g_\varepsilon)\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)} + \|\nabla g_\varepsilon\cdot\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}\big) \\ &\le C\,\big(\|(1-g_\varepsilon)\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)} + \|\nabla g_\varepsilon\cdot\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}\big) \\ &\le C\,\big(\|1-g_\varepsilon\|_{L^r(D)}\|\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}+\xi}(D)} + \|\nabla g_\varepsilon\|_{L^{\frac{\gamma+\Theta}{\Theta}+\xi}(D)}\|\varphi\|_{L^r(D)}\big), \end{aligned}$$

where we used the uniform bound on ϱ_ε in $L^{\gamma+\Theta}(D)$, and $\xi \in (0,1)$ and $r \in (1,\infty)$ are determined by

$$\frac{1}{r} + \left(\frac{\gamma+\Theta}{\Theta} + \xi\right)^{-1} = \frac{\Theta}{\gamma+\Theta}. \tag{4.60}$$

From (4.59), we obtain

$$\|1-g_\varepsilon\|_{L^r(D)} \le C\,\varepsilon^{\frac{3(\alpha-1)}{r}}, \quad \|\nabla g_\varepsilon\|_{L^{\frac{\gamma+\Theta}{\Theta}+\xi}(D)} \le C\,\varepsilon^{3(\alpha-1)\left(\frac{\gamma+\Theta}{\Theta}+\xi\right)^{-1}-\alpha}$$

as well as

$$\begin{aligned} 3\,(\alpha-1)\left(\frac{\gamma+\Theta}{\Theta}\right)^{-1} - \alpha = \frac{3(\alpha-1)\Theta - \alpha(\gamma+\Theta)}{\gamma+\Theta} = \frac{\alpha(2\Theta-\gamma)-3\Theta}{\gamma+\Theta} > 0 \\ \iff \alpha(2\Theta-\gamma) > 3\Theta. \end{aligned}$$

We distinguish two cases of Θ from its definition in (4.53). First, we assume that

$$\Theta = \min\left\{2\gamma-3, \gamma\frac{3m_\vartheta-2}{3m_\vartheta+2}\right\} = 2\gamma-3,$$

then

$$\alpha(2\Theta-\gamma) = \alpha(3\gamma-6) > 3\Theta = 3(2\gamma-3) \iff \alpha > \frac{2\gamma-3}{\gamma-2},$$

which is true by condition (4.50). Second, if

$$\Theta = \min\left\{2\gamma-3, \gamma\frac{3m_\vartheta-2}{3m_\vartheta+2}\right\} = \gamma\frac{3m_\vartheta-2}{3m_\vartheta+2},$$

then

$$\begin{aligned}\alpha(2\Theta-\gamma)=\alpha\gamma\left(\frac{6m_\vartheta-4}{3m_\vartheta+2}-1\right)>3\Theta=\gamma\frac{9m_\vartheta-6}{3m_\vartheta+2}\\ \iff \alpha\frac{3m_\vartheta-6}{3m_\vartheta+2}>\frac{9m_\vartheta-6}{3m_\vartheta+2}\\ \iff \alpha>\frac{3m_\vartheta-2}{m_\vartheta-2},\end{aligned}$$

which again holds by (4.50). We therefore may choose $\xi\in(0,1)$ small enough such that

$$h(\xi):=3\,(\alpha-1)\left(\frac{\gamma+\Theta}{\Theta}+\xi\right)^{-1}-\alpha>0.$$

For this ξ, let r be defined by (4.60), and

$$\nu:=\min\left\{\frac{3(\alpha-1)}{r},h(\xi)\right\}>0,$$

then we may estimate I^1 by

$$|I^1|\le C\,\varepsilon^\nu\big(\|\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}+\xi}(D)}+\|\varphi\|_{L^r(D)}\big).$$

Let us further note that

$$\frac{3(\gamma+\Theta)}{2(\gamma+\Theta)-3}\le\frac{\gamma+\Theta}{\Theta}\iff 3\Theta\le 2(\gamma+\Theta)-3\iff\Theta\le 2\gamma-3, \tag{4.61}$$

which is always true by the definition of Θ in (4.53). Now, we get for I^2

$$\begin{aligned}|I^2|&\le C\,\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D)}\|\vartheta_\varepsilon\|_{L^3(D)}\big(\|(1-g_\varepsilon)\operatorname{div}\varphi\|_{L^{\frac{3(\gamma+\Theta)}{2(\gamma+\Theta)-3}}(D)}+\|\nabla g_\varepsilon\cdot\varphi\|_{L^{\frac{3(\gamma+\Theta)}{2(\gamma+\Theta)-3}}(D)}\big)\\ &\le C\left(\|(1-g_\varepsilon)\nabla\varphi\|_{L^{\frac{3(\gamma+\Theta)}{2(\gamma+\Theta)-3}}(D)}+\|\nabla g_\varepsilon\cdot\varphi\|_{L^{\frac{3(\gamma+\Theta)}{2(\gamma+\Theta)-3}}(D)}\right)\\ &\le C\left(\|(1-g_\varepsilon)\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}+\|\nabla g_\varepsilon\cdot\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}\right),\end{aligned}$$

where we used the uniform bounds on ϱ_ε in $L^{\gamma+\Theta}(D)$ and on ϑ_ε in any $L^q(D)$ for $1\le q\le 3m_\vartheta$. We may therefore proceed as for I^1 to eventually get for $I_{\varepsilon,2}$ the bound

$$|I_{\varepsilon,2}|\le|I^1|+|I^2|\le C\,\varepsilon^\nu\big(\|\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}+\xi}(D)}+\|\varphi\|_{L^r(D)}\big).$$

For $I_{\varepsilon,1}$, we get

$$\begin{aligned}|I_{\varepsilon,1}|&\le\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D)}\|\mathbf{u}_\varepsilon\|_{L^6(D)}^2\big(\|(1-g_\varepsilon)\nabla\varphi\|_{L^{\frac{3(\gamma+\Theta)}{2(\gamma+\Theta)-3}}(D)}+\|\nabla g_\varepsilon\otimes\varphi\|_{L^{\frac{3(\gamma+\Theta)}{2(\gamma+\Theta)-3}}(D)}\big)\\ &\le C\left(\|(1-g_\varepsilon)\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}+\|\nabla g_\varepsilon\otimes\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}\right),\end{aligned}$$

where we used the uniform bounds on ϱ_ε and $\mathbf{u}_\varepsilon$ as well as (4.61). Arguing similar as for $I_{\varepsilon,2}$, we deduce for $I_{\varepsilon,1}$ the bound

$$\begin{aligned}|I_{\varepsilon,1}| &\le C\left(\|1-g_\varepsilon\|_{L^r(D)}\|\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}+\xi}(D)} + \|\nabla g_\varepsilon\|_{L^{\frac{\gamma+\Theta}{\Theta}+\xi}(D)}\|\varphi\|_{L^r(D)}\right)\\ &\le C\,\varepsilon^\nu\big(\|\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}+\xi}(D)} + \|\varphi\|_{L^r(D)}\big).\end{aligned}$$

For $I_{\varepsilon,3}$, we estimate

$$\begin{aligned}|I_{\varepsilon,3}| &\le C\,(1+\|\vartheta_\varepsilon\|_{L^{\frac{2(\gamma+\Theta)}{\gamma-\Theta}}(D)})\|\nabla\mathbf{u}_\varepsilon\|_{L^2(D)}\big(\|(1-g_\varepsilon)\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)} + \|\nabla g_\varepsilon\otimes\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}\big)\\ &\le C\left(\|(1-g_\varepsilon)\nabla\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)} + \|\nabla g_\varepsilon\otimes\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}\right),\end{aligned}$$

where we used the uniform bound on ϑ_ε in $L^q(D)$ for any $1\le q\le 3m_\vartheta$, and the fact that

$$\frac{2(\gamma+\Theta)}{\gamma-\Theta}\le 3m_\vartheta \iff 2\gamma+2\Theta\le 3\gamma m_\vartheta-3\Theta m_\vartheta \iff \Theta\le\gamma\frac{3m_\vartheta-2}{3m_\vartheta+2},$$

which is true by (4.53).
For $I_{\varepsilon,4}$, we repeat the arguments for $I_{\varepsilon,2}$ since

$$\begin{aligned}|I_{\varepsilon,4}| &\le C\,(1+\|\varrho_\varepsilon\|_{L^{\frac{\gamma+\Theta}{\gamma}}(D)})\|(1-g_\varepsilon)\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}\\ &\le C\,(1+\|\varrho_\varepsilon\|_{L^{\gamma+\Theta}(D)})\|(1-g_\varepsilon)\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)} \le C\,\|(1-g_\varepsilon)\varphi\|_{L^{\frac{\gamma+\Theta}{\Theta}}(D)}.\end{aligned}$$

□

We want now to pass to the limit in the entropy balance (4.49) and show that the limits $[\sigma,\mathbf{u},\vartheta]$ fulfill also (4.35). Since this point is missing in [LP21], we follow the proof of [PS21]. We first show that the entropy balance (4.49) is satisfied for the extended functions "up to a small error".

Lemma 4.3.11. *Under the assumptions of Theorem 4.3.3, we have*

$$\begin{aligned}\langle\tilde\sigma_\varepsilon,\psi\rangle_{\mathcal{M}^+(\overline{D})} + \int_{\partial D_\varepsilon}\frac{L\vartheta_0}{\vartheta_\varepsilon}\psi\,\mathrm{d}\sigma(x) = -&\int_D\big(\tilde\varrho_\varepsilon s(\tilde\varrho_\varepsilon,\tilde\vartheta_\varepsilon)\tilde{\mathbf{u}}_\varepsilon - \kappa(\tilde\vartheta_\varepsilon)\nabla E_\varepsilon\log(\vartheta_\varepsilon)\big)\cdot\nabla\psi\,\mathrm{d}x\\ &+ L\int_{\partial D}\psi\,\mathrm{d}\sigma(x) + \langle R_\varepsilon,\psi\rangle\end{aligned}\tag{4.62}$$

with $\langle R_\varepsilon,\psi\rangle\to 0$ *for any* $\psi\in C^1(\overline{D})$ *with* $\psi\ge 0$. *Here, we denote* $\tilde\varrho_\varepsilon s(\tilde\varrho_\varepsilon,\tilde\vartheta_\varepsilon) = c_v\tilde\varrho_\varepsilon E_\varepsilon\log(\vartheta_\varepsilon) - c_v(\gamma-1)\tilde\varrho_\varepsilon\log(\tilde\varrho_\varepsilon)$ *with the convention* $0\cdot\log(0)=0$.

Proof. Let $\psi\in C^1(\overline{D})$ with $\psi\ge 0$, then $\psi\chi_{\overline{D}_\varepsilon}$ is a proper test function in the entropy balance (4.49) in $\overline{D}_\varepsilon$. We further have $\psi=\psi\chi_{\overline{D}_\varepsilon}+\psi\chi_{\overline{D}\setminus\overline{D}_\varepsilon}$ and hence

$$\begin{aligned}&\langle\tilde\sigma_\varepsilon,\psi\rangle_{\mathcal{M}^+(\overline{D})} + \int_{\partial D_\varepsilon}\frac{L\vartheta_0}{\vartheta_\varepsilon}\psi\,\mathrm{d}\sigma(x) + \int_D\big(\tilde\varrho_\varepsilon s(\tilde\varrho_\varepsilon,\tilde\vartheta_\varepsilon)\tilde{\mathbf{u}}_\varepsilon - \kappa(\tilde\vartheta_\varepsilon)\nabla E_\varepsilon\log(\vartheta_\varepsilon)\big)\cdot\nabla\psi - L\int_{\partial D}\psi\,\mathrm{d}x\\ &= \langle\sigma_\varepsilon,\psi\rangle_{\mathcal{M}^+(\overline{D}_\varepsilon)} + \int_{\partial D_\varepsilon}\frac{L\vartheta_0}{\vartheta_\varepsilon}\psi\,\mathrm{d}\sigma(x)\\ &+ \int_{D_\varepsilon}\big(\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)\mathbf{u}_\varepsilon - \kappa(\vartheta_\varepsilon)\nabla\log(\vartheta_\varepsilon)\big)\cdot\nabla\psi\,\mathrm{d}x - L\int_{\partial D_\varepsilon}\psi\,\mathrm{d}\sigma(x)\end{aligned}$$

$$
\begin{aligned}
&+\langle \tilde\sigma_\varepsilon, \psi\rangle_{\mathcal{M}^+(\overline{D}\setminus\overline{D}_\varepsilon)} + \int_{D\setminus D_\varepsilon} \big(\tilde\varrho_\varepsilon s(\tilde\varrho_\varepsilon, \tilde\vartheta_\varepsilon)\tilde{\mathbf{u}}_\varepsilon - \kappa(\tilde\vartheta_\varepsilon)\nabla E_\varepsilon \log(\vartheta_\varepsilon)\big)\cdot\nabla\psi \,\mathrm{d}x + L\int_{\partial(D\setminus D_\varepsilon)} \psi \,\mathrm{d}\sigma(x)\\
&=: \sum_{i=1}^{7} I_i.
\end{aligned}
$$

Clearly $\sum_{i=1}^{4} I_i = 0$ because of (4.49). Further, $I_5 = 0$ since σ_ε has been extended to zero outside D_ε. For I_7 we obtain $I_7 \to 0$ by (3.26). By $\tilde\varrho_\varepsilon = 0$ outside D_ε, we get

$$
\begin{aligned}
I_6 &= -\int_{D\setminus D_\varepsilon} \kappa(\tilde\vartheta_\varepsilon)\nabla E_\varepsilon \log(\vartheta_\varepsilon)\cdot\nabla\psi\,\mathrm{d}x\\
&\le C\,\|\nabla\psi\|_{L^\infty(D\setminus D_\varepsilon)}\|\nabla\log(\vartheta_\varepsilon)\|_{L^2(D\setminus D_\varepsilon)}\|\kappa(\vartheta_\varepsilon)\|_{L^3(D\setminus D_\varepsilon)}|D\setminus D_\varepsilon|^{\frac{1}{6}} \to 0,
\end{aligned}
$$

where we used that $\kappa(\vartheta) \le C(1+\vartheta^{m_\vartheta})$ for some $m_\vartheta > 2$ and $\|\vartheta_\varepsilon\|_{L^{3m_\vartheta}(D_\varepsilon)} \le C$ as well as $|D\setminus D_\varepsilon| \to 0$ by (3.26). □

Remark 4.3.12. *Note that due to the mere low control $\|\vartheta_\varepsilon^{-1}\|_{L^1(\partial D_\varepsilon)} \le C$, we are not able to prove $\int_{\partial D_\varepsilon\setminus\partial D} L\psi\vartheta_0/\vartheta_\varepsilon \to 0$ as $\varepsilon \to 0$, which would finally yield that the weak-* limit of $\tilde\sigma_\varepsilon$ in $\mathcal{M}^+(\overline{D})$ would satisfy the balance of entropy in the limiting domain D. Due to $\int_{\partial D_\varepsilon\setminus\partial D} L\vartheta_0\psi/\vartheta_\varepsilon \ge 0$ we rather have that $\limsup_{\varepsilon\to0}\tilde\sigma_\varepsilon \le \sigma$ in the sense of measures, where $\sigma \in \mathcal{M}^+(\overline{D})$ is defined as the entropy production rate for the limiting system in D.*

We now turn to the limit $\varepsilon \to 0$ in (4.62). We will again follow the arguments given in [PS21, Section 3.2]. First, by the uniform estimates developed in (4.52) and (4.55) and the strong convergence of the temperature and velocity, we have

$$
\tilde\varrho_\varepsilon s(\tilde\varrho_\varepsilon, \tilde\vartheta_\varepsilon) \rightharpoonup \overline{\varrho s(\varrho,\vartheta)} = c_v\varrho\log(\vartheta) - c_v(\gamma-1)\overline{\varrho\log(\varrho)}
$$

weakly in $L^q(D)$ for some $q > 1$ as well as

$$
\tilde\varrho_\varepsilon s(\tilde\varrho_\varepsilon, \tilde\vartheta_\varepsilon)\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \overline{\varrho s(\varrho,\vartheta)\mathbf{u}} = c_v\varrho\log(\vartheta)\mathbf{u} - c_v(\gamma-1)\overline{\varrho\log(\varrho)\mathbf{u}}
$$

weakly in $L^q(D)$ for some $q > 1$. The term $\kappa(\tilde\vartheta_\varepsilon)\nabla E_\varepsilon\log(\vartheta_\varepsilon)$ can be handled by $\tilde\vartheta_\varepsilon \to \vartheta$ strongly in $L^q(D)$ for any $1 \le q < 3m_\vartheta$ and $\nabla E_\varepsilon\log(\vartheta_\varepsilon) \rightharpoonup \nabla\log(\vartheta)$ weakly in $L^2(D)$. As mentioned in Remark 4.3.12, we infer

$$
\langle\sigma,\psi\rangle_{\mathcal{M}^+(\overline{D})} + \int_{\partial D}\frac{L\vartheta_0}{\vartheta}\psi\,\mathrm{d}\sigma(x) \ge -\int_D \big(\overline{\varrho s(\vartheta,\varrho)\mathbf{u}} - \kappa(\vartheta)\nabla\log(\vartheta)\big)\cdot\nabla\psi\,\mathrm{d}x + L\int_{\partial D}\psi\,\mathrm{d}\sigma(x).
$$

Last, let us prove that σ fulfills inequality (4.35). To this end, we notice that

$$
\frac{\mathbb{S}(\tilde\vartheta_\varepsilon, \nabla\tilde{\mathbf{u}}_\varepsilon):\nabla\tilde{\mathbf{u}}_\varepsilon}{\tilde\vartheta_\varepsilon} = \frac{1}{2}\left|\sqrt{\frac{\mu(\tilde\vartheta_\varepsilon)}{\tilde\vartheta_\varepsilon}}\big(\nabla\tilde{\mathbf{u}}_\varepsilon + \nabla^T\tilde{\mathbf{u}}_\varepsilon - \frac{2}{3}\operatorname{div}\tilde{\mathbf{u}}_\varepsilon\mathbb{I}\big)\right|^2 + \left|\sqrt{\frac{\eta(\tilde\vartheta_\varepsilon)}{\tilde\vartheta_\varepsilon}}\operatorname{div}\tilde{\mathbf{u}}_\varepsilon\right|^2
$$

and use weak lower semi-continuity of the L^2-norm to infer

$$
\frac{\mathbb{S}(\vartheta,\nabla\mathbf{u}):\nabla\mathbf{u}}{\vartheta} \le \liminf_{\varepsilon\to0}\frac{\mathbb{S}(\tilde\vartheta_\varepsilon,\nabla\tilde{\mathbf{u}}_\varepsilon):\nabla\tilde{\mathbf{u}}_\varepsilon}{\tilde\vartheta_\varepsilon}
$$

in the sense of distributions. Let us now focus on the term

$$\kappa(\tilde{\vartheta}_\varepsilon)|\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2.$$

By assumption (4.41), it is enough to consider this term for $\kappa(\vartheta) = 1 + \vartheta^{m_\vartheta}$. In this case, we get

$$\kappa(\tilde{\vartheta}_\varepsilon)|\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 = |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 + \tilde{\vartheta}_\varepsilon^{m_\vartheta - 2}|\nabla \tilde{\vartheta}_\varepsilon|^2,$$

where we used that $E_\varepsilon \varphi = \varphi$ in D_ε and $\tilde{\vartheta}_\varepsilon(x) = 0$ whenever $x \in B_{\varepsilon^\alpha r_j}(\varepsilon z_j)$.
Let us focus on the first term and fix $\delta > 0$. Then

$$\begin{aligned}
&\int_D |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 \,\mathrm{d}x \\
&\geq - \int_{D \setminus D_\varepsilon} |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^{2-\delta} \chi_{\{|\nabla E_\varepsilon \log(\vartheta_\varepsilon)| > 1\}} \,\mathrm{d}x + \int_D |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^{2-\delta} \,\mathrm{d}x \\
&\quad + \int_{D_\varepsilon} \Big(|\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 - |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^{2-\delta} \Big) \chi_{\{|\nabla E_\varepsilon \log(\vartheta_\varepsilon)| \leq 1\}} \,\mathrm{d}x =: \sum_{i=1}^3 I_i.
\end{aligned} \tag{4.63}$$

We now estimate, using Hölder's inequality (B.2),

$$|I_1| = \int_{D \setminus D_\varepsilon} |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^{2-\delta} \chi_{\{|\nabla E_\varepsilon \log(\vartheta_\varepsilon)| > 1\}} \,\mathrm{d}x \leq \|\nabla E_\varepsilon \log(\vartheta_\varepsilon)\|_{L^2(D)}^{2-\delta} |D \setminus D_\varepsilon|^{\frac{2}{\delta}}.$$

Hence, for fixed $\delta > 0$, we have $I_1 \to 0$ as $\varepsilon \to 0$ since $|D_\varepsilon| \to |D|$ by (3.26). Further, we get $|I_3| \leq C(\delta) \to 0$ as $\delta \to 0$ uniformly in ε, since the function $z \mapsto |z^2 - z^{2-\delta}|$ obtains in $(0,1)$ its maximum at $z_0 = \big(1 - \frac{\delta}{2}\big)^{\frac{1}{\delta}}$. Thus, I_3 is bounded independently of ε.
Let us now pass to the limit $\varepsilon \to 0$ in (4.63). Due to the strong convergence of the temperature, the fact that the second term in (4.63) is bounded in $L^q(D)$ for some $q > 1$, and the weak lower semicontinuity of the L^q-norm, we obtain

$$\liminf_{\varepsilon \to 0} \int_D |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 \,\mathrm{d}x \geq \int_D |\nabla \log(\vartheta)|^{2-\delta} \,\mathrm{d}x + C(\delta).$$

Since $|\nabla \log(\vartheta)|^{2-\delta}$ converges for $\delta \to 0$ almost everywhere in D to $|\nabla \log(\vartheta)|^2$ and is bounded by

$$\begin{aligned}
|\nabla \log(\vartheta)|^{2-\delta} &= |\nabla \log(\vartheta)|^{2-\delta} \chi_{\{|\nabla \log(\vartheta)| > 1\}} + |\nabla \log(\vartheta)|^{2-\delta} \chi_{\{|\nabla \log(\vartheta)| \leq 1\}} \\
&\leq |\nabla \log(\vartheta)|^2 + 1 \in L^1(D),
\end{aligned}$$

together with Lebesgue's convergence theorem, we infer in the limit $\delta \to 0$

$$\liminf_{\varepsilon \to 0} \int_D |\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 \,\mathrm{d}x \geq \int_D |\nabla \log(\vartheta)|^2 \,\mathrm{d}x.$$

Seeing that the above inequalities remain valid if the integrands are multiplied by arbitrary $\psi \in C^1(\overline{D})$, $\psi \geq 0$, and that the term $\tilde{\vartheta}_\varepsilon^{m_\vartheta - 2}|\nabla \tilde{\vartheta}_\varepsilon|^2$ can be handled similarly due to the fact that $\nabla|\tilde{\vartheta}_\varepsilon|^{\frac{m_\vartheta}{2}} = \frac{m_\vartheta}{2}\tilde{\vartheta}_\varepsilon^{\frac{m_\vartheta-2}{2}}\nabla\tilde{\vartheta}_\varepsilon$ is bounded in $L^2(D)$, we arrive at

$$\liminf_{\varepsilon\to 0}\int_D \kappa(\tilde{\vartheta}_\varepsilon)|\nabla E_\varepsilon \log(\vartheta_\varepsilon)|^2 \,\mathrm{d}x \geq \int_D \kappa(\vartheta)|\nabla \log(\vartheta)|^2 \,\mathrm{d}x,$$

which eventually yields for any $\psi \in C^1(\overline{D})$ with $\psi \geq 0$

$$\begin{aligned}
&\int_D \left(\frac{\mathbb{S}(\vartheta,\nabla\mathbf{u}) : \nabla\mathbf{u}}{\vartheta} + \kappa(\vartheta)|\nabla\log(\vartheta)|^2\right)\psi \,\mathrm{d}x \\
&\leq \liminf_{\varepsilon\to 0}\int_D \left(\frac{\mathbb{S}(\tilde{\vartheta}_\varepsilon,\nabla\tilde{\mathbf{u}}_\varepsilon) : \nabla\tilde{\mathbf{u}}_\varepsilon}{\tilde{\vartheta}_\varepsilon} + \kappa(\tilde{\vartheta}_\varepsilon)|\nabla E_\varepsilon\log(\vartheta_\varepsilon)|^2\right)\psi \,\mathrm{d}x \\
&\leq \liminf_{\varepsilon\to 0}\langle\tilde{\sigma}_\varepsilon,\psi\rangle \leq \langle\sigma,\psi\rangle.
\end{aligned}$$

To finish the proof of Theorem 4.3.3, we have to show

$$\overline{\varrho e(\varrho,\vartheta)} = \varrho e(\varrho,\vartheta), \quad \overline{\varrho s(\varrho,\vartheta)} = \varrho s(\varrho,\vartheta), \quad \overline{\varrho s(\varrho,\vartheta)\mathbf{u}} = \varrho s(\varrho,\vartheta)\mathbf{u}, \quad \overline{p(\varrho,\vartheta)} = p(\varrho,\vartheta).$$

By the strong convergence of ϑ_ε to ϑ in any $L^q(D)$ for $1 \leq q < 3m_\vartheta$, it is sufficient to show the strong convergence of ϱ_ε to ϱ, which is done in Appendix A. To summarize, we finally proved that the weak limit $[\varrho,\mathbf{u},\vartheta]$ is a solution to problem (4.33)–(4.44) in the limit domain D. This completes the proof of Theorem 4.3.3.

4.4 Brinkman's law in the Low Mach number limit

This section is devoted to the homogenization of compressible Navier-Stokes equations in a perforated domain with *critical* perforations. As mentioned in the introduction to this chapter and in contrast to the previous sections, the radii of the holes will not scale like ε^α for some $\alpha > 3$ but rather like ε^3. For the scalar Laplace equation $\Delta u_\varepsilon = f$, this scaling was first considered by Cioranescu and Murat in [CM82], where they obtained an additional term Mu (which they called "strange term") that occurs just in the limiting equation $\Delta u + Mu = f$ and is reminiscent from the holes. An even more general framework of having randomly placed holes with random radii was considered by Giunti, Höfer, and Velázquez in [GHV18] for "almost minimal assumptions on the size of the holes". In the language of Section 3.4, they assumed $\mathbb{E}(r) < \infty$. This assumption, however, allows clusters of holes, meaning the obstacles may overlap. Nonetheless, the additional coefficient M also occurs there, and is related to the harmonic capacity of the holes. The assumption $\mathbb{E}(r) < \infty$ is minimal in the sense that this capacity is finite in average.

Back to the case of periodically arranged holes, Allaire showed in [All90], that for the Stokes equations for an *incompressible* fluid, the limiting system has an extra friction term $M\mathbf{u}$, called *Brinkman term*, that is not seen in the equations for the perforated domain. This "strange term" is related to the drag force around each particle and represents a kind of "boundary layer energy" of the holes (see [All90, Remark 2.1.5]). As a matter of fact, if the holes are spherical with common radius $r > 0$, the matrix M is equal to $6\pi r\mathbb{I}$, which is Stokes' famous drag law. Generalizations to these results were given by several authors. For instance, Hillairet left in [Hil18] the periodic setting and considered randomly placed obstacles, but he still required some kind of hard sphere condition on the perforations. He considered the incompressible Stokes

equations, putting on each hole a different prescribed velocity instead of the no-slip condition. These velocities will show up in the limit system as an additional forcing term working against the frictional Brinkman term. As a continuation to [GHV18], Giunti and Höfer considered in [GH19] the case of randomly placed holes with $E(r^{1+\beta}) < \infty$ for some $\beta > 0$. This tighter assumption gives some information on the geometry of the holes and rules out clusters that are made of many holes of similar size. In turn, this enables one to enclose these clusters in a tiny bit larger set that can be controlled better than the clusters itself, thus showing that the randomness does not effect the limiting behavior of the equations.
The setting for *compressible* fluids, however, is rather different. In order to be able to pass to the limit $\varepsilon \to 0$ here, we need to control the density near each obstacle in a good way. We will do this by imposing the so-called Low Mach number limit, which scales the pressure by a negative power of ε, hence forcing the density to become constant in the limit. The homogenized system will therefore be the *incompressible* Navier-Stokes equations with the additional Brinkman term, which will be the same as found by Allaire.

4.4.1 Setting and main result

As before, we consider a bounded domain $D \subset \mathbb{R}^3$ with smooth boundary. This time, let $\varepsilon > 0$ and $\{x_i^\varepsilon\}_{i\in\mathbb{Z}} \subset \mathbb{R}^3$ be a collection of points in space with $|x_i^\varepsilon - x_j^\varepsilon| \geq 2\varepsilon$ for any $i \neq j$. For simplicity, we will assume that the points x_i^ε lie on a regular mesh of size 2ε, that is, $x_i^\varepsilon \in (2\varepsilon\,\mathbb{Z})^3$ is the center of the i-th cell $P_i^\varepsilon := x_i^\varepsilon + (-\varepsilon, \varepsilon)^3$. Further, let $T \subset B_1(0)$ be a compact and simply connected set with smooth boundary and $0 \in T$, and set $T_i^\varepsilon := x_i^\varepsilon + \varepsilon^3 T$. We now define the perforated domain as

$$D_\varepsilon := D \setminus \bigcup_{i\in K_\varepsilon} T_i^\varepsilon, \quad K_\varepsilon := \{i : \overline{P_i^\varepsilon} \subset D\}. \tag{4.64}$$

By the periodic distribution of the holes, the number of holes inside D_ε satisfies

$$|K_\varepsilon| \leq C\,\frac{|D|}{\varepsilon^3} \quad \text{for some } C > 0 \text{ independent of } \varepsilon.$$

This in particular yields

$$|D \setminus D_\varepsilon| = \left| \bigcup_{i\in K_\varepsilon} T_i^\varepsilon \right| \leq |K_\varepsilon|\,|T_i^\varepsilon| \leq C\,\varepsilon^6.$$

In D_ε, we consider the steady compressible Navier-Stokes equations

$$\begin{cases} \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 & \text{in } D_\varepsilon, \\ \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) - \operatorname{div}\mathbb{S}(\nabla\mathbf{u}_\varepsilon) + \frac{1}{\varepsilon^\beta}\nabla\varrho_\varepsilon^\gamma = \varrho_\varepsilon\mathbf{f} + \mathbf{g} & \text{in } D_\varepsilon, \\ \mathbf{u}_\varepsilon = 0 & \text{on } \partial D_\varepsilon. \end{cases} \tag{4.65}$$

Note that in contrast to the previous sections, the pressure term $\varrho_\varepsilon^\gamma$ is now scaled by a factor $\varepsilon^{-\beta}$, which represents the vanishing Mach number, and corresponds to $\mathrm{Ma}^2 = \varepsilon^\beta$ and setting all other characteristic numbers in Section 2.3 equal to one. Further, we assume that $\gamma \geq 3$, $\beta > 3(\gamma + 1)$, and $\mathbf{f}, \mathbf{g} \in L^\infty(D)$ are given. Since the equations (4.65) are invariant under adding a constant to the pressure term $\varepsilon^{-\beta}\varrho_\varepsilon^\gamma$, we define

$$p_\varepsilon := \varepsilon^{-\beta}(\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon^\gamma \rangle_{D_\varepsilon}), \tag{4.66}$$

where $\langle \cdot \rangle_{D_\varepsilon}$ denotes the mean value over D_ε, given by

$$\langle f \rangle_{D_\varepsilon} := \frac{1}{|D_\varepsilon|} \int_{D_\varepsilon} f \, \mathrm{d}x.$$

Formally, let us assume that the Low Mach limit and the limit in the perforated domain decouple, meaning the pressure reads $p_{\delta,\varepsilon} = \delta^{-\beta}(\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon^\gamma \rangle_{D_\varepsilon})$ in a perforated domain D_ε. Just focusing on the limit $\delta \to 0$ in the second equation of (4.65) in a *fixed* domain D_ε, we obtain

$$\nabla p_\varepsilon = 0 \Longrightarrow p_\varepsilon = C \Longrightarrow \varrho_\varepsilon = C$$

for some constant C (which may depend on ε). Thus, we expect the limit system to be *incompressible* rather than compressible as the system we started with. Back to the perforated domain, we now have an *incompressible* system in D_ε, which by the results of Allaire in [All90] is expected to converge to a system of Brinkman type as $\varepsilon \to 0$. We will indeed show convergence of the velocity $\mathbf{u}_\varepsilon$ and the pressure p_ε to limiting functions $\mathbf{u}$ and p, respectively, such that the couple $[p, \mathbf{u}]$ solves the incompressible steady Navier-Stokes-Brinkman equations

$$\begin{cases} \operatorname{div}(\mathbf{u}) = 0 & \text{in } D, \\ \operatorname{div}(\varrho_0 \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} + \nabla p + \mu M \mathbf{u} = \varrho_0 \mathbf{f} + \mathbf{g} & \text{in } D, \\ \mathbf{u} = 0 & \text{on } \partial D, \end{cases}$$

where the resistance matrix M is introduced in the next section, and the constant ϱ_0 is the strong limit of ϱ_ε in $L^{2\gamma}(D)$, which is determined by the mass constraint on ϱ_ε as formulated in Definition 4.4.1 below.

Before stating our main result, we recall the concept of finite energy weak solutions as done in Definition 4.1.1.

Definition 4.4.1. *Let D_ε be as in (4.64) and $\gamma \geq 3$, $\mathfrak{m} > 0$ be fixed. We say a couple $[\varrho, \mathbf{u}]$ is a* finite energy weak solution *to system (4.65) if*

$$\varrho \in L^{2\gamma}(D_\varepsilon), \quad \mathbf{u} \in W_0^{1,2}(D_\varepsilon),$$

$$\varrho \geq 0 \ a.e. \ in \ D_\varepsilon, \quad \int_{D_\varepsilon} \varrho \, \mathrm{d}x = \mathfrak{m},$$

$$\int_{D_\varepsilon} \varrho \mathbf{u} \cdot \nabla \psi \, \mathrm{d}x = 0,$$

$$\int_{D_\varepsilon} \varepsilon^{-\beta} \varrho^\gamma \operatorname{div} \varphi + (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla \varphi - \mathbb{S}(\nabla \mathbf{u}) : \nabla \varphi + (\varrho \mathbf{f} + \mathbf{g}) \cdot \varphi \, \mathrm{d}x = 0$$

for all test functions $\psi \in C_c^\infty(D_\varepsilon)$ and all test functions $\varphi \in C_c^\infty(D_\varepsilon; \mathbb{R}^3)$, and the energy *inequality*

$$\int_{D_\varepsilon} \mathbb{S}(\nabla \mathbf{u}) : \nabla \mathbf{u} \, \mathrm{d}x \leq \int_{D_\varepsilon} (\varrho \mathbf{f} + \mathbf{g}) \cdot \mathbf{u} \, \mathrm{d}x \tag{4.67}$$

holds.

Remark 4.4.2. *Existence of finite energy weak solutions to system* (4.65) *is known for all values* $\gamma > 3/2$*, see Theorem 4.1.3. However, we need the assumption* $\gamma \geq 3$ *to bound the convective term* $\operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)$ *in a proper way, see Section 4.4.3.*

Let us as before denote the zero extension of a function f with D_ε as its domain of definition by $\tilde{f}$, that is,

$$\tilde{f} = f \text{ in } D_\varepsilon, \quad \tilde{f} = 0 \text{ in } \mathbb{R}^3 \setminus D_\varepsilon.$$

Our main result for the stationary Navier-Stokes equations now reads as follows.

Theorem 4.4.3. *Let* $D \subset \mathbb{R}^3$ *be a bounded domain with smooth boundary,* $0 < \varepsilon < 1$*,* D_ε *be as in* (4.64)*,* $\gamma \geq 3$*,* $\mathfrak{m} > 0$ *and* $\mathbf{f}, \mathbf{g} \in L^\infty(D)$*. Let* $\{[\varrho_\varepsilon, \mathbf{u}_\varepsilon]\}_{\varepsilon > 0}$ *be a sequence of finite energy weak solutions to problem* (4.65) *and assume*

$$\beta > 3(\gamma + 1). \tag{4.68}$$

Then, with p_ε *defined in* (4.66)*, we can extract subsequences (not relabeled) such that*

$$\begin{aligned} \tilde{\varrho}_\varepsilon &\to \varrho_0 && \text{strongly in } L^{2\gamma}(D), \\ \tilde{p}_\varepsilon &\rightharpoonup p && \text{weakly in } L^2(D), \\ \tilde{\mathbf{u}}_\varepsilon &\rightharpoonup \mathbf{u} && \text{weakly in } W_0^{1,2}(D), \end{aligned}$$

where $\varrho_0 = \mathfrak{m}/|D|$ *is constant and* $[p, \mathbf{u}] \in L^2(D) \times W_0^{1,2}(D)$ *with* $\int_D p = 0$ *is a weak solution to the steady incompressible Navier-Stokes-Brinkman equations*

$$\begin{cases} \operatorname{div}(\mathbf{u}) = 0 & \text{in } D, \\ \operatorname{div}(\varrho_0 \mathbf{u} \otimes \mathbf{u}) + \nabla p - \mu \Delta \mathbf{u} + \mu M \mathbf{u} = \varrho_0 \mathbf{f} + \mathbf{g} & \text{in } D, \\ \mathbf{u} = 0 & \text{on } \partial D, \end{cases} \tag{4.69}$$

where M *is defined in* (4.75) *below.*

Remark 4.4.4. *It is well known that the solution to system* (4.69) *is unique if* $\mathbf{f}$ *and* $\mathbf{g}$ *are "sufficiently small", see, e.g., [Tem77, Chapter II, Theorem 1.3]. This smallness assumption can be dropped in the case of Stokes equations, i.e., without the convective term* $\operatorname{div}(\varrho_0 \mathbf{u} \otimes \mathbf{u})$*.*

4.4.2 The cell problem and oscillating test functions

In this section, we introduce oscillating test functions and define the resistance matrix M, following the original work of Allaire [All90]. Here, we repeat the definition of these functions as well as the estimates given in [HKS21].

Consider for a single particle T the solution $[q_k, \mathbf{w}_k]$ to the cell problem

$$\begin{cases} \operatorname{div}(\mathbf{w}_k) = 0 & \text{in } \mathbb{R}^3 \setminus T, \\ \nabla q_k - \Delta \mathbf{w}_k = 0 & \text{in } \mathbb{R}^3 \setminus T, \\ \mathbf{w}_k = 0 & \text{on } \partial T, \\ \mathbf{w}_k = \mathbf{e}_k & \text{at infinity}, \end{cases} \tag{4.70}$$

where $\mathbf{e}_k$ is the k-th unit basis vector of the canonical basis of $\mathbb{R}^3$. Note that the solution exists and is unique, see, e.g., [Gal11, Chapter V]. Let us further recall the definition of oscillating test functions as made in [All90] (see also [HKS21]):

We set

$$\mathbf{w}_k^\varepsilon = \mathbf{e}_k, \quad q_k^\varepsilon = 0 \text{ in } P_i^\varepsilon \cap D$$

for each P_i^ε with $P_i^\varepsilon \cap \partial D \neq \emptyset$. Now, we denote $B_i^r := B_r(x_i^\varepsilon)$ and split each cell P_i^ε entirely included in D into the following four parts:

$$\overline{P_i^\varepsilon} = T_i^\varepsilon \cup \overline{C_i^\varepsilon} \cup \overline{D_i^\varepsilon} \cup \overline{K_i^\varepsilon},$$

where C_i^ε is the open ball centered at x_i^ε with radius $\varepsilon/2$ and perforated by the hole T_i^ε, $D_i^\varepsilon = B_i^\varepsilon \setminus \overline{B_i^{\varepsilon/2}}$ is the ball with radius ε perforated by the ball with radius $\varepsilon/2$, and $K_i^\varepsilon = P_i^\varepsilon \setminus \overline{B_i^\varepsilon}$ are the remaining corners, see Figure 4.1.

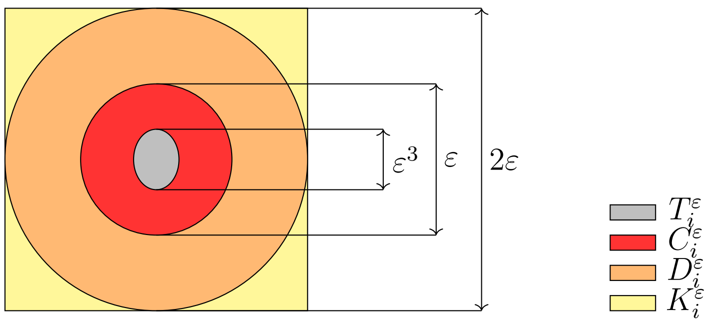


Figure 4.1: Splitting of the cell P_i^ε

In these parts, we define

$$\begin{cases} \mathbf{w}_k^\varepsilon = \mathbf{e}_k \\ q_k^\varepsilon = 0 \end{cases} \text{in } K_i^\varepsilon, \qquad \begin{cases} \nabla q_k^\varepsilon - \Delta \mathbf{w}_k^\varepsilon = 0 \\ \operatorname{div}(\mathbf{w}_k^\varepsilon) = 0 \end{cases} \text{in } D_i^\varepsilon,$$

$$\begin{cases} \mathbf{w}_k^\varepsilon(x) = \mathbf{w}_k\left(\frac{x}{\varepsilon^3}\right) \\ q_k^\varepsilon(x) = \frac{1}{\varepsilon^3} q_k\left(\frac{x}{\varepsilon^3}\right) \end{cases} \text{in } C_i^\varepsilon, \qquad \begin{cases} \mathbf{w}_k^\varepsilon = 0 \\ q_k^\varepsilon = 0 \end{cases} \text{in } T_i^\varepsilon,$$

where we impose matching Dirichlet boundary conditions, and $[q_k, \mathbf{w}_k]$ is the solution to the cell problem (4.70). As shown in [HKS21, Lemma 3.5], we have for the functions $[q_k^\varepsilon, \mathbf{w}_k^\varepsilon]$ the following estimates:

Lemma 4.4.5. *Let $p > \frac{3}{2}$. Then*

$$\|\nabla \mathbf{w}_k^\varepsilon\|_{L^p(D)} + \|q_k^\varepsilon\|_{L^p(D)} \leq C\,\varepsilon^{3\left(\frac{2}{p}-1\right)}, \tag{4.71}$$

$$\|\nabla q_k^\varepsilon\|_{L^p(\cup_i C_i^\varepsilon)} \leq C\,\varepsilon^{6\left(\frac{1}{p}-1\right)}, \tag{4.72}$$

$$\|\nabla \mathbf{w}_k^\varepsilon\|_{L^2\left(\cup_i B_i^\varepsilon \setminus \overline{B_i^{\varepsilon/4}}\right)} + \|q_k^\varepsilon\|_{L^2\left(\cup_i B_i^\varepsilon \setminus \overline{B_i^{\varepsilon/4}}\right)} \leq C\,\varepsilon, \tag{4.73}$$

where the constant $C > 0$ does not depend on ε.

Proof. By the rescaling $x' = (x - x_i^\varepsilon)/\varepsilon^3$ in each C_i^ε, we first find

$$\|\nabla \mathbf{w}_k^\varepsilon\|_{L^p(\cup_i C_i^\varepsilon)} + \|q_k^\varepsilon\|_{L^p(\cup_i C_i^\varepsilon)} \leq C\,\varepsilon^{-\frac{3}{p}}\varepsilon^{-3+\frac{9}{p}}(\|\nabla \mathbf{w}_k\|_{L^p(\mathbb{R}^3\setminus T)} + \|q_k\|_{L^p(\mathbb{R}^3\setminus T)}),$$
$$\|\nabla q_k^\varepsilon\|_{L^p(\cup C_i^\varepsilon)} \leq C\,\varepsilon^{-\frac{3}{p}}\varepsilon^{-6+\frac{9}{p}}\|\nabla q_k\|_{L^p(\mathbb{R}^3\setminus T)},$$

where the factor $\varepsilon^{-\frac{3}{p}}$ occurs due to the fact that the number of holes in D is of order ε^{-3}. By the standard regularity theory for Stokes equations (see [Gal11, Chapter V]), we have for any $l \in \mathbb{N}$

$$|\mathbf{w}_k(x) - \mathbf{e}_k| \leq \frac{C}{|x|}, \quad |\nabla^{l+1}\mathbf{w}_k(x)| + |\nabla^l q_k(x)| \leq \frac{C_l}{|x|^{l+2}} \text{ in } \mathbb{R}^3 \setminus T, \tag{4.74}$$

in particular $\nabla \mathbf{w}_k \in L^p(\mathbb{R}^3 \setminus T)$ and $q_k \in W^{1,p}(\mathbb{R}^3 \setminus T)$ for any $p > \frac{3}{2}$, so (4.72) holds. Thus, since $\varepsilon \leq \varepsilon^{3\left(\frac{2}{p}-1\right)}$ for any $p \geq \frac{3}{2}$, we infer (4.71) if we prove (4.73).

To show (4.73), we split $B_i^\varepsilon \setminus \overline{B_i^{\varepsilon/4}} = D_i^\varepsilon \cup (B_i^{\varepsilon/2} \setminus \overline{B_i^{\varepsilon/4}})$ and estimate with the change of variables $x' = (x - x_i^\varepsilon)/\varepsilon$

$$\|\nabla \mathbf{w}_k^\varepsilon\|_{L^p(\cup_i D_i^\varepsilon)} + \|q_k^\varepsilon\|_{L^p(\cup_i D_i^\varepsilon)} \leq C\,\varepsilon^{-1}(\|\nabla \mathbf{v}_k^\varepsilon\|_{L^p\left(B_1(0)\setminus\overline{B_{\frac{1}{2}}(0)}\right)} + \|p_k^\varepsilon\|_{L^p\left(B_1(0)\setminus\overline{B_{\frac{1}{2}}(0)}\right)}),$$

where $[p_k^\varepsilon, \mathbf{v}_k^\varepsilon]$ is the solution to the homogeneous Stokes problem

$$\begin{cases} \nabla p_k^\varepsilon - \Delta \mathbf{v}_k^\varepsilon = 0 & \text{in } B_1(0) \setminus \overline{B_{\frac{1}{2}}(0)}, \\ \operatorname{div} \mathbf{v}_k^\varepsilon = 0 & \text{in } B_1(0) \setminus \overline{B_{\frac{1}{2}}(0)}, \\ \mathbf{v}_k^\varepsilon = 0 & \text{on } \partial B_1(0), \\ \mathbf{v}_k^\varepsilon = \mathbf{w}_k\left(\frac{\varepsilon\,\cdot}{2\varepsilon^3}\right) - \mathbf{e}_k & \text{on } \partial B_{\frac{1}{2}}(0). \end{cases}$$

Again, by standard theory for the Stokes equations, we have

$$\begin{aligned} \|\nabla \mathbf{v}_k^\varepsilon\|_{L^p\left(B_1(0)\setminus\overline{B_{\frac{1}{2}}(0)}\right)} + \|p_k^\varepsilon\|_{L^p\left(B_1(0)\setminus\overline{B_{\frac{1}{2}}(0)}\right)} &\leq C\|\mathbf{w}_k\left(\frac{\varepsilon\,\cdot}{2\varepsilon^3}\right) - \mathbf{e}_k\|_{W^{1-\frac{1}{p},p}\left(\partial B_{\frac{1}{2}}(0)\right)} \\ &\leq C\|\nabla\varphi\|_{L^p\left(B_1(0)\setminus\overline{B_{\frac{1}{2}}(0)}\right)} \end{aligned}$$

for any $\varphi \in W^{1,p}(B_1(0) \setminus \overline{B_{\frac{1}{2}}(0)})$ that satisfies the same boundary conditions as $\mathbf{v}_k^\varepsilon$. We thus may choose $\varphi = \eta\,(\mathbf{w}_k\left(\frac{\varepsilon\,\cdot}{2\varepsilon^3}\right) - \mathbf{e}_k)$, where $\eta \in C_c^\infty(B_1(0))$ with $\eta = 1$ on $\partial B_{\frac{1}{2}}(0)$. By (4.74), we have $\|\nabla\varphi\|_{L^\infty(B_1(0))} \leq C\,\varepsilon^2$, leading to

$$\|\nabla \mathbf{w}_k^\varepsilon\|_{L^p(\cup_i D_i^\varepsilon)} + \|q_k^\varepsilon\|_{L^p(\cup_i D_i^\varepsilon)} \leq C\,\varepsilon.$$

For $B_i^{\varepsilon/2} \setminus \overline{B_i^{\varepsilon/4}}$, the same scaling and cut-off arguments as made above yield

$$\|\mathbf{w}_k^\varepsilon\|_{L^p\left(\cup_i B_i^{\varepsilon/2}\setminus\overline{B_i^{\varepsilon/4}}\right)} + \|q_k^\varepsilon\|_{L^p\left(\cup_i B_i^{\varepsilon/2}\setminus\overline{B_i^{\varepsilon/4}}\right)} \leq C\,\varepsilon,$$

thus proving (4.73). □

The choice of $B_i^\varepsilon \setminus \overline{B_i^{\varepsilon/4}}$ instead of D_i^ε in the estimate (4.73) will become important when passing to the limit $\varepsilon \to 0$, see Section 4.4.4. Moreover, we have the following theorem due to Allaire.

Theorem 4.4.6 ([All90, page 214, Proposition 1.1.2 and Lemma 2.3.6]). *The functions $[q_k^\varepsilon, \mathbf{w}_k^\varepsilon]$ fulfill:*

(H1) $q_k^\varepsilon \in L^2(D), \quad \mathbf{w}_k^\varepsilon \in W^{1,2}(D)$*;*

(H2) $\operatorname{div} \mathbf{w}_k^\varepsilon = 0$ *in* D *and* $\mathbf{w}_k^\varepsilon = 0$ *on the holes* T_i^ε*;*

(H3) $\mathbf{w}_k^\varepsilon \rightharpoonup \mathbf{e}_k$ *weakly in* $W^{1,2}(D)$*,* $\quad q_k^\varepsilon \rightharpoonup 0$ *weakly in* $L^2(D)/\mathbb{R}$*;*

(H4) For any $\nu_\varepsilon, \nu \in W^{1,2}(D)$ *with* $\nu_\varepsilon = 0$ *on the holes* T_i^ε *and* $\nu_\varepsilon \rightharpoonup \nu$ *weakly in* $W^{1,2}(D)$*, and any* $\varphi \in \mathcal{D}(D)$*, we have*

$$\langle \nabla q_k^\varepsilon - \Delta \mathbf{w}_k^\varepsilon, \varphi\nu_\varepsilon\rangle_{W^{-1,2}(D),W_0^{1,2}(D)} \to \langle M\mathbf{e}_k, \varphi\nu\rangle_{W^{-1,2}(D),W_0^{1,2}(D)},$$

where the resistance matrix $M \in W^{-1,\infty}(D)$ *is defined by its entries* M_{ik} *via*

$$\langle M_{ik}, \varphi\rangle_{\mathcal{D}'(D),\mathcal{D}(D)} = \lim_{\varepsilon\to 0}\int_D \varphi \nabla \mathbf{w}_i^\varepsilon : \nabla \mathbf{w}_k^\varepsilon \,\mathrm{d}x \tag{4.75}$$

for any test function $\varphi \in \mathcal{D}(D)$*.*

Further, for any $p \geq 1$*,*

$$\|\mathbf{w}_k^\varepsilon - \mathbf{e}_k\|_{L^p(D)} \to 0.$$

Remark 4.4.7. *This definition of* M *yields that the matrix is symmetric and positive definite in the sense that for all test functions* $\varphi_i \in \mathcal{D}(D)$ *and* $\Phi = (\varphi_i)_{1\leq i\leq 3}$*,*

$$\langle M\Phi, \Phi\rangle_{\mathcal{D}'(D),\mathcal{D}(D)} = \lim_{\varepsilon\to 0}\int_D \left|\sum_{i=1}^3 \varphi_i \nabla \mathbf{w}_i^\varepsilon\right|^2 \mathrm{d}x \geq 0,$$

thus implying that there exists at least one solution to system (4.69).

4.4.3 Bogovskiĭ's operator and uniform bounds

From Theorem 3.3.1, we obtain the following result for the inverse of the divergence operator.

Theorem 4.4.8. *Let* $1 < q < \infty$ *and* D_ε *be defined as in* (4.64). *There exists a bounded linear operator*

$$\mathcal{B}_\varepsilon : L_0^q(D_\varepsilon) \to W_0^{1,q}(D_\varepsilon)$$

such that for any $f \in L_0^q(D_\varepsilon)$,

$$\operatorname{div} \mathcal{B}_\varepsilon(f) = f \text{ in } D_\varepsilon, \quad \|\mathcal{B}_\varepsilon(f)\|_{W_0^{1,q}(D_\varepsilon)} \leq C \left(1 + \varepsilon^{3\left(\frac{2}{q}-1\right)}\right) \|f\|_{L^q(D_\varepsilon)},$$

where the constant $C > 0$ *does not depend on* ε.

We will use this result to bound the pressure p_ε by the density ϱ_ε. Since the main ideas how to get uniform bounds on $\mathbf{u}_\varepsilon$, ϱ_ε, and p_ε are given in [HKS21], we just sketch the proof in our case. First, by (4.9) and (4.67), we find

$$\mu \|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)}^2 \leq \|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)} \|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)} \|\mathbf{f}\|_{L^\infty(D)} + \|\mathbf{g}\|_{L^\infty(D)} \|\mathbf{u}_\varepsilon\|_{L^1(D_\varepsilon)}.$$

Together with Sobolev embedding (B.8), we obtain

$$\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)} \leq C \, \|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)},$$

which yields

$$\|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)} + \|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)} \leq C \, (\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)} + 1). \tag{4.76}$$

To get uniform bounds on the velocity, we first have to estimate the density. To this end, let $\mathcal{B}_\varepsilon$ be as in Theorem 4.4.8. Testing the second equation in (4.65) with $\mathcal{B}_\varepsilon(p_\varepsilon) \in W_0^{1,2}(D_\varepsilon)$ yields

$$\begin{aligned}
\|p_\varepsilon\|_{L^2(D_\varepsilon)}^2 &= \int_{D_\varepsilon} p_\varepsilon \operatorname{div} \mathcal{B}_\varepsilon(p_\varepsilon) \, \mathrm{d}x \\
&= \int_{D_\varepsilon} \mathbb{S}(\nabla \mathbf{u}_\varepsilon) : \nabla \mathcal{B}_\varepsilon(p_\varepsilon) - (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla \mathcal{B}_\varepsilon(p_\varepsilon) - (\varrho_\varepsilon \mathbf{f} + \mathbf{g}) \cdot \mathcal{B}_\varepsilon(p_\varepsilon) \, \mathrm{d}x.
\end{aligned}$$

Recalling $\varrho_\varepsilon \in L^{2\gamma}(D_\varepsilon)$ and $\gamma \geq 3$, this leads to

$$\begin{aligned}
\|p_\varepsilon\|_{L^2(D_\varepsilon)}^2 &\leq C \, (\|\nabla \mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^6(D_\varepsilon)} \|\mathbf{u}_\varepsilon\|_{L^6(D_\varepsilon)}^2) \|\nabla \mathcal{B}_\varepsilon(p_\varepsilon)\|_{L^2(D_\varepsilon)} \\
&\quad + C \left(\|\mathbf{f}\|_{L^\infty(D_\varepsilon)} \|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\mathbf{g}\|_{L^\infty(D_\varepsilon)} \right) \|\mathcal{B}_\varepsilon(p_\varepsilon)\|_{L^2(D_\varepsilon)} \\
&\overset{(4.76)}{\leq} C \, (\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)} + 1 + \|\varrho_\varepsilon\|_{L^6(D_\varepsilon)} (\|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)}^2 + 1)) \|\nabla \mathcal{B}_\varepsilon(p_\varepsilon)\|_{L^2(D_\varepsilon)} \\
&\quad + C \, (\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + 1) \|\mathcal{B}_\varepsilon(p_\varepsilon)\|_{L^2(D_\varepsilon)} \\
&\leq C \, (\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^6(D_\varepsilon)} \|\varrho_\varepsilon\|_{L^{\frac{6}{5}}(D_\varepsilon)}^2 + 1) \|\mathcal{B}_\varepsilon(p_\varepsilon)\|_{W_0^{1,2}(D_\varepsilon)} \\
&\leq C \, (\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)}^3 + 1) \|\mathcal{B}_\varepsilon(p_\varepsilon)\|_{W_0^{1,2}(D_\varepsilon)}
\end{aligned}$$

$$\leq \; C\,(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|^3_{L^{2\gamma}(D_\varepsilon)} + 1)\|p_\varepsilon\|_{L^2(D_\varepsilon)},$$

that is,

$$\|p_\varepsilon\|_{L^2(D_\varepsilon)} \leq C\,(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|^3_{L^{2\gamma}(D_\varepsilon)} + 1). \tag{4.77}$$

Further, we have

$$\langle \varrho_\varepsilon \rangle_{D_\varepsilon} = \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon} \varrho_\varepsilon \,\mathrm{d}x = \frac{\mathfrak{m}}{|D_\varepsilon|}$$

and, using Lemma B.14,

$$\frac{1}{\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon \rangle^\gamma_{D_\varepsilon}\|_{L^2(D_\varepsilon)} \leq \frac{C}{\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon^\gamma \rangle_{D_\varepsilon}\|_{L^2(D_\varepsilon)} \overset{(4.66)}{=} C\,\|p_\varepsilon\|_{L^2(D_\varepsilon)}.$$

This yields

$$\begin{aligned}
&\frac{1}{\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon \rangle^\gamma_{D_\varepsilon}\|_{L^2(D_\varepsilon)} \leq C\,\|p_\varepsilon\|_{L^2(D_\varepsilon)} \leq C\,(\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} + \|\varrho_\varepsilon\|^3_{L^{2\gamma}(D_\varepsilon)} + 1)\\
&\leq C\left(\|\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon \rangle^\gamma_{D_\varepsilon}\|^{\frac{1}{\gamma}}_{L^2(D_\varepsilon)} + \frac{\mathfrak{m}}{|D_\varepsilon|^{1-1/(2\gamma)}} + \|\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon \rangle^\gamma_{D_\varepsilon}\|^{\frac{3}{\gamma}}_{L^2(D_\varepsilon)} + \frac{\mathfrak{m}^3}{|D_\varepsilon|^{3-3/(2\gamma)}} + 1\right).
\end{aligned}$$

Together with (4.27), we obtain, using $\gamma \geq 3$ and the fact that we may assume $\varepsilon \leq 1$ small enough,

$$\begin{aligned}
\frac{1}{\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon \rangle^\gamma_{D_\varepsilon}\|_{L^2(D_\varepsilon)} &\leq \frac{1}{4\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon \rangle^\gamma_{D_\varepsilon}\|_{L^2(D_\varepsilon)} + C + \frac{1}{4\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon \rangle^\gamma_{D_\varepsilon}\|_{L^2(D_\varepsilon)} + C'\\
&= \frac{1}{2\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon \rangle^\gamma_{D_\varepsilon}\|_{L^2(D_\varepsilon)} + C.
\end{aligned}$$

Using that $|\varrho_\varepsilon - \langle \varrho_\varepsilon \rangle_{D_\varepsilon}|^\gamma \leq |\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon \rangle^\gamma_{D_\varepsilon}|$, which is a consequence of the triangle inequality for the metric $d(a,b) = |a-b|^{\frac{1}{\gamma}}$ for $\gamma \geq 1$, we conclude

$$\frac{1}{\varepsilon^\beta}\|\varrho_\varepsilon - \langle \varrho_\varepsilon \rangle_{D_\varepsilon}\|^\gamma_{L^{2\gamma}(D_\varepsilon)} \leq \frac{1}{\varepsilon^\beta}\|\varrho_\varepsilon^\gamma - \langle \varrho_\varepsilon \rangle^\gamma_{D_\varepsilon}\|_{L^2(D_\varepsilon)} \leq C,$$

which further gives rise to

$$\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} \leq \|\varrho_\varepsilon - \langle \varrho_\varepsilon \rangle_{D_\varepsilon}\|_{L^{2\gamma}(D_\varepsilon)} + C\,\langle \varrho_\varepsilon \rangle_{D_\varepsilon} \leq C.$$

In view of (4.76) and (4.77), we finally establish

$$\begin{aligned}
\|\mathbf{u}_\varepsilon\|_{W_0^{1,2}(D_\varepsilon)} &\leq C,\\
\|\varrho_\varepsilon\|_{L^{2\gamma}(D_\varepsilon)} &\leq C,\\
\|p_\varepsilon\|_{L^2(D_\varepsilon)} &\leq C,\\
\|\varrho_\varepsilon - \langle \varrho_\varepsilon \rangle_{D_\varepsilon}\|_{L^{2\gamma}(D_\varepsilon)} &\leq C\,\varepsilon^{\frac{\beta}{\gamma}}
\end{aligned} \tag{4.78}$$

for some constant $C > 0$ independent of ε.

4.4.4 Convergence proof

The proof of convergence we give here is essentially the same as in [HKS21]. We thus just sketch the steps done there while highlighting the differences.

Proof of Theorem 4.4.3. Step 1: Recall that, for a function f defined on D_ε, we denote by $\tilde{f}$ its zero prolongation to $\mathbb{R}^3$. By the uniform estimates (4.78), we can extract subsequences (not relabeled) such that

$$\begin{aligned}
&\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u} \text{ weakly in } W_0^{1,2}(D),\\
&\tilde{p}_\varepsilon \rightharpoonup p \text{ weakly in } L^2(D),\\
&\tilde{\varrho}_\varepsilon \to \varrho_0 \text{ strongly in } L^{2\gamma}(D),
\end{aligned}$$

where $\varrho_0 = \mathfrak{m}/|D| > 0$ is constant. The strong convergence of the density is obtained by

$$\begin{aligned}
\|\tilde{\varrho}_\varepsilon - \varrho_0\|_{L^{2\gamma}(D)} &\leq \|\varrho_0\|_{L^{2\gamma}(D\setminus D_\varepsilon)} + \|\varrho_\varepsilon - \langle\varrho_\varepsilon\rangle_{D_\varepsilon}\|_{L^{2\gamma}(D_\varepsilon)} + \|\langle\varrho_\varepsilon\rangle_{D_\varepsilon} - \varrho_0\|_{L^{2\gamma}(D_\varepsilon)}\\
&\leq \varrho_0|D\setminus D_\varepsilon|^{\frac{1}{2\gamma}} + C\,\varepsilon^{\frac{\beta}{\gamma}} + \mathfrak{m}|D_\varepsilon|^{\frac{1}{2\gamma}}\left(\frac{1}{|D_\varepsilon|} - \frac{1}{|D|}\right) \to 0,
\end{aligned}$$

since $|D_\varepsilon| \to |D|$. Due to the Rellich-Kondrachev theorem stated in Proposition B.5, we further have

$$\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u} \text{ strongly in } L^q(D) \text{ for all } 1 \leq q < 6.$$

Step 2: We begin by proving that the limiting velocity $\mathbf{u}$ is solenoidal. To this end, let $\varphi \in \mathcal{D}(\mathbb{R}^3)$. By the second equation of (4.65), we have

$$0 = \int_{\mathbb{R}^3} \tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \cdot \nabla\varphi \,\mathrm{d}x \to \varrho_0 \int_D \mathbf{u}\cdot\nabla\varphi\,\mathrm{d}x.$$

This together with the compactness of the trace operator yields

$$\begin{cases} \operatorname{div}\mathbf{u} = 0 & \text{in } D,\\ \mathbf{u} = 0 & \text{on } \partial D. \end{cases} \tag{4.79}$$

Step 3: To prove convergence of the momentum equation, let $\varphi \in \mathcal{D}(D)$ and use $\varphi\mathbf{w}_k^\varepsilon$ as test function in the first equation of (4.1). This yields

$$\begin{aligned}
\int_D \mathbb{S}(\nabla\tilde{\mathbf{u}}_\varepsilon) : \nabla(\varphi\mathbf{w}_k^\varepsilon)\,\mathrm{d}x &= \int_D (\tilde{\varrho}_\varepsilon\tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) : \nabla(\varphi\mathbf{w}_k^\varepsilon)\,\mathrm{d}x + \int_D \tilde{p}_\varepsilon \operatorname{div}(\varphi\mathbf{w}_k^\varepsilon)\,\mathrm{d}x\\
&\quad + \int_D (\tilde{\varrho}_\varepsilon\mathbf{f} + \mathbf{g})\cdot(\varphi\mathbf{w}_k^\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

Using the definition of $\mathbb{S}$ in (4.2) and the fact that $\operatorname{div}(\mathbf{w}_k^\varepsilon) = 0$ by (H2) of Theorem 4.4.6, we rewrite the left-hand side as

$$\int_D \mathbb{S}(\nabla\tilde{\mathbf{u}}_\varepsilon) : \nabla(\varphi\mathbf{w}_k^\varepsilon)\,\mathrm{d}x = \mu\int_D \nabla\tilde{\mathbf{u}}_\varepsilon : \nabla(\varphi\mathbf{w}_k^\varepsilon)\,\mathrm{d}x + \left(\frac{\mu}{3} + \eta\right)\int_D \operatorname{div}(\tilde{\mathbf{u}}_\varepsilon)\operatorname{div}(\varphi\mathbf{w}_k^\varepsilon)\,\mathrm{d}x$$

$$= \mu \int_D \nabla \mathbf{w}_k^\varepsilon : \nabla(\varphi \tilde{\mathbf{u}}_\varepsilon) + \nabla \tilde{\mathbf{u}}_\varepsilon : (\mathbf{w}_k^\varepsilon \otimes \nabla \varphi) - \nabla \mathbf{w}_k^\varepsilon : (\tilde{\mathbf{u}}_\varepsilon \otimes \nabla \varphi) \, \mathrm{d}x + \left(\frac{\mu}{3} + \eta \right) \int_D \operatorname{div}(\tilde{\mathbf{u}}_\varepsilon) \mathbf{w}_k^\varepsilon \cdot \nabla \varphi \, \mathrm{d}x$$

and add the term $-\int_D q_k^\varepsilon \operatorname{div}(\varphi \tilde{\mathbf{u}}_\varepsilon) \, \mathrm{d}x$ to both sides to obtain

$$\begin{aligned} &\underbrace{\mu \int_D \nabla \mathbf{w}_k^\varepsilon : \nabla(\varphi \tilde{\mathbf{u}}_\varepsilon) - q_k^\varepsilon \operatorname{div}(\varphi \tilde{\mathbf{u}}_\varepsilon) \, \mathrm{d}x}_{I_1} \\ &+ \underbrace{\mu \int_D \nabla \tilde{\mathbf{u}}_\varepsilon : (\mathbf{w}_k^\varepsilon \otimes \nabla \varphi) - \nabla \mathbf{w}_k^\varepsilon : (\tilde{\mathbf{u}}_\varepsilon \otimes \nabla \varphi) \, \mathrm{d}x}_{I_2} + \underbrace{\left(\frac{\mu}{3} + \eta \right) \int_D \operatorname{div}(\tilde{\mathbf{u}}_\varepsilon) \mathbf{w}_k^\varepsilon \cdot \nabla \varphi \, \mathrm{d}x}_{I_3} \\ &= \underbrace{\int_D (\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon) : \nabla(\varphi \mathbf{w}_k^\varepsilon) \, \mathrm{d}x}_{I_4} + \underbrace{\int_D \tilde{p}_\varepsilon \mathbf{w}_k^\varepsilon \cdot \nabla \varphi + (\tilde{\varrho}_\varepsilon \mathbf{f} + \mathbf{g}) \cdot (\varphi \mathbf{w}_k^\varepsilon) \, \mathrm{d}x}_{I_5} - \underbrace{\int_D q_k^\varepsilon \operatorname{div}(\varphi \tilde{\mathbf{u}}_\varepsilon) \, \mathrm{d}x}_{I_6}. \end{aligned}$$

Since $\nu_\varepsilon := \tilde{\mathbf{u}}_\varepsilon$ and $\nu := \mathbf{u}$ fulfill hypothesis (H4) of Theorem 4.4.6, we have

$$I_1 \to \mu \langle M \mathbf{e}_k, \varphi \mathbf{u} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual product between $W^{-1,2}(D)$ and $W_0^{1,2}(D)$. Further, by $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ strongly in $L^2(D)$ and $\nabla \mathbf{w}_k^\varepsilon \rightharpoonup 0$ by hypothesis (H3),

$$I_2 \to \mu \int_D \nabla \mathbf{u} : (\mathbf{e}_k \otimes \nabla \varphi) \, \mathrm{d}x.$$

Because of $\mathbf{w}_k^\varepsilon \to \mathbf{e}_k$ strongly in $L^2(D)$ and (4.79), we deduce

$$I_3 \to 0, \quad I_5 \to \int_D p \, \mathbf{e}_k \cdot \nabla \varphi + (\varrho_0 \mathbf{f} + \mathbf{g}) \cdot (\varphi \mathbf{e}_k) \, \mathrm{d}x.$$

Step 4: To show convergence of I_4, we proceed as follows. First, since $\mathbf{u}_\varepsilon = 0$ on ∂D_ε and $\tilde{\mathbf{u}}_\varepsilon \rightharpoonup \mathbf{u}$ weakly in $W^{1,2}(D)$, we have $\widetilde{\nabla \mathbf{u}_\varepsilon} = \nabla \tilde{\mathbf{u}}_\varepsilon \rightharpoonup \nabla \mathbf{u}$ weakly in $L^2(D)$. Second, as shown above for $\gamma \geq 3$, $\tilde{\varrho}_\varepsilon \to \varrho_0$ strongly in $L^{2\gamma}(D)$ and $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ strongly in $L^q(D)$ for any $1 \leq q < 6$, in particular in $L^4(D)$. Together with the strong convergence of $\mathbf{w}_k^\varepsilon$ in any $L^p(D)$ (see Theorem 4.4.6), in particular in $L^{12}(D)$, we get

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \mathbf{w}_k^\varepsilon \to \varrho_0 \mathbf{u} \otimes \mathbf{e}_k \text{ strongly in } L^2(D).$$

This, together with $\operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$, yields

$$\begin{aligned} I_4 &= \int_{D_\varepsilon} (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla(\varphi \mathbf{w}_k^\varepsilon) \, \mathrm{d}x = -\int_{D_\varepsilon} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon \cdot \varphi \mathbf{w}_k^\varepsilon \, \mathrm{d}x = -\int_{D_\varepsilon} \varphi \nabla \mathbf{u}_\varepsilon : (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{w}_k^\varepsilon) \, \mathrm{d}x \\ &= -\int_D \varphi \nabla \tilde{\mathbf{u}}_\varepsilon : (\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \mathbf{w}_k^\varepsilon) \, \mathrm{d}x \to -\int_D \varphi \nabla \mathbf{u} : (\varrho_0 \mathbf{u} \otimes \mathbf{e}_k) \, \mathrm{d}x = \int_D (\varrho_0 \mathbf{u} \otimes \mathbf{u}) : \nabla(\varphi \mathbf{e}_k) \, \mathrm{d}x. \end{aligned}$$

In the case $\gamma > 3$, one can also proceed by seeing that

$$\tilde{\varrho}_\varepsilon \tilde{\mathbf{u}}_\varepsilon \otimes \tilde{\mathbf{u}}_\varepsilon \to \varrho_0 \mathbf{u} \otimes \mathbf{u} \text{ strongly in } L^2(D),$$

where we used that $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ strongly in $L^q(D)$ for $q = 4\gamma/(\gamma-1) < 6$.

Step 5: It remains to show convergence of I_6. Denote again $B_i^r = B_r(x_i^\varepsilon)$. We follow the idea of [HKS21] and introduce a further splitting of the integral:

Let $\psi \in C_c^\infty(B_{1/2}(0))$ be a cut-off function with $\psi = 1$ on $B_{1/4}(0)$, define for $x \in B_i^{\varepsilon/2}$ the function $\psi_\varepsilon^i(x) := \psi((x - x_i^\varepsilon)/\varepsilon)$, and extend ψ_ε^i by zero to the whole of D. Set finally $\psi_\varepsilon(x) := \sum_{i:\overline{P_i^\varepsilon}\subset D} \psi_\varepsilon^i(x)$, where P_i^ε is the cell of size 2ε with center $x_i^\varepsilon \in (2\varepsilon\,\mathbb{Z})^3$. Then we have $\psi_\varepsilon \in C_c^\infty(\bigcup_i B_i^{\varepsilon/2})$ and

$$\psi_\varepsilon = 1 \text{ in } \bigcup_i B_i^{\varepsilon/4}, \quad |\nabla \psi_\varepsilon| \le C\varepsilon^{-1}. \tag{4.80}$$

With this at hand, we write

$$\begin{aligned}
\langle \varrho_\varepsilon \rangle_{D_\varepsilon} \cdot I_6 &= \langle \varrho_\varepsilon \rangle_{D_\varepsilon} \int_{D_\varepsilon} q_k^\varepsilon \psi_\varepsilon \operatorname{div}(\varphi \mathbf{u}_\varepsilon)\,\mathrm{d}x + \langle \varrho_\varepsilon \rangle_{D_\varepsilon} \int_{D_\varepsilon} q_k^\varepsilon (1-\psi_\varepsilon)\varphi \operatorname{div}(\mathbf{u}_\varepsilon)\,\mathrm{d}x \\
&\quad + \langle \varrho_\varepsilon \rangle_{D_\varepsilon} \int_{D_\varepsilon} q_k^\varepsilon (1-\psi_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla\varphi\,\mathrm{d}x \\
&=: I^1 + I^2 + I^3.
\end{aligned}$$

Observe that since $\operatorname{supp} \psi_\varepsilon \subset \cup_i B_i^{\varepsilon/2}$, the term I^1 covers the behavior of q_k^ε "near" the holes, whereas I^2 and I^3 cover the behavior "far away". Since q_k^ε and ψ_ε are (2ε)-periodic functions and $q_k^\varepsilon \psi_\varepsilon \in L^2(D)$, we have $q_k^\varepsilon \psi_\varepsilon \rightharpoonup 0$ weakly in $L^2(D)/\mathbb{R}$. This together with $\tilde{\mathbf{u}}_\varepsilon \to \mathbf{u}$ strongly in $L^2(D)$ yields

$$|I^3| \to 0.$$

For I^2, we use the definition of q_k^ε and (4.73) to find

$$\begin{aligned}
|I^2| &\le C \int_{D\setminus \cup_i \overline{B_i^{\varepsilon/4}}} |q_k^\varepsilon|\,|\operatorname{div}(\mathbf{u}_\varepsilon)|\,\mathrm{d}x \overset{(4.78)}{\le} C\,\|q_k^\varepsilon\|_{L^2\left(D\setminus \cup_i \overline{B_i^{\varepsilon/4}}\right)} \\
&= C\,\|q_k^\varepsilon\|_{L^2\left(\cup_i B_i^\varepsilon \setminus \overline{B_i^{\varepsilon/4}}\right)} \le C\,\varepsilon \to 0.
\end{aligned}$$

To prove $I^1 \to 0$, we write, using $\operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0$,

$$\begin{aligned}
I^1 &= \int_{D_\varepsilon} \nabla(q_k^\varepsilon \psi_\varepsilon \varphi) \cdot (\varrho_\varepsilon \mathbf{u}_\varepsilon)\,\mathrm{d}x - \int_{D_\varepsilon} \nabla(q_k^\varepsilon \psi_\varepsilon \varphi) \cdot (\langle \varrho_\varepsilon \rangle_{D_\varepsilon} \mathbf{u}_\varepsilon)\,\mathrm{d}x + \langle \varrho_\varepsilon \rangle_{D_\varepsilon} \int_{D_\varepsilon} q_k^\varepsilon \psi_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla\varphi\,\mathrm{d}x \\
&= \int_{D_\varepsilon} \nabla(q_k^\varepsilon \psi_\varepsilon \varphi)(\varrho_\varepsilon - \langle \varrho_\varepsilon \rangle_{D_\varepsilon}) \cdot \mathbf{u}_\varepsilon\,\mathrm{d}x + o(1).
\end{aligned}$$

Here, we used again the periodicity of q_k^ε and ψ_ε to conclude $q_k^\varepsilon \psi_\varepsilon \rightharpoonup 0$ weakly in $L^2(D)/\mathbb{R}$. This and the strong convergence of $\tilde{\mathbf{u}}_\varepsilon$ to $\mathbf{u}$ in $L^2(D)$ show that the last term vanishes in the limit $\varepsilon \to 0$. For the remaining integral, we find, recalling $\operatorname{supp} \psi_\varepsilon \subset \cup_i B_i^{\varepsilon/2}$ and $C_i^\varepsilon = B_i^{\varepsilon/2} \setminus T_i^\varepsilon$,

$$|I^1| \le \|\nabla(q_k^\varepsilon \psi_\varepsilon \varphi)\|_{L^{\frac{2\gamma}{\gamma-1}}(\cup_i C_i^\varepsilon)} \|\varrho_\varepsilon - \langle \varrho_\varepsilon \rangle_{D_\varepsilon}\|_{L^{2\gamma}(D_\varepsilon)} \|\mathbf{u}_\varepsilon\|_{L^2(D_\varepsilon)} + o(1)$$

$$\leq C\,\varepsilon^{\frac{\beta}{\gamma}} \|\nabla(q_k^\varepsilon \psi_\varepsilon \varphi)\|_{L^{\frac{2\gamma}{\gamma-1}}(\cup_i C_i^\varepsilon)} + o(1).$$

Since $|\nabla \psi_\varepsilon| \leq C\,\varepsilon^{-1}$, we have

$$|\nabla(q_k^\varepsilon \psi_\varepsilon \varphi)| \leq C\left(|\nabla q_k^\varepsilon| + \frac{1}{\varepsilon}|q_k^\varepsilon|\right),$$

thus

$$|I^1| \leq C\,\varepsilon^{\frac{\beta}{\gamma}}\left(\|\nabla q_k^\varepsilon\|_{L^{\frac{2\gamma}{\gamma-1}}(\cup_i C_i^\varepsilon)} + \frac{1}{\varepsilon}\|q_k^\varepsilon\|_{L^{\frac{2\gamma}{\gamma-1}}(\cup_i C_i^\varepsilon)}\right) + o(1).$$

Together with (4.71), (4.72) for $p = 2\gamma/(\gamma-1) > 3/2$, and the assumption $\beta > 3\,(\gamma+1)$ from (4.68), we establish

$$|I^1| \leq C\,\varepsilon^{\frac{\beta}{\gamma}}\left(\varepsilon^{-3-\frac{3}{\gamma}} + \varepsilon^{-1-\frac{3}{\gamma}}\right) + o(1) \leq C\,\varepsilon^{-3+\frac{\beta-3}{\gamma}} + o(1) \to 0.$$

To summarize, we have in the limit $\varepsilon \to 0$ for all functions $\varphi \in \mathcal{D}(D)$

$$\mu\,\langle M\mathbf{e}_k, \varphi\mathbf{u}\rangle - \mu\,\langle \Delta\mathbf{u}, \varphi\mathbf{e}_k\rangle = -\langle \operatorname{div}(\varrho_0\mathbf{u}\otimes\mathbf{u}), \varphi\mathbf{e}_k\rangle + \langle \varrho_0\mathbf{f} + \mathbf{g} - \nabla p, \varphi\mathbf{e}_k\rangle.$$

Since M is symmetric, this is

$$\nabla p + \varrho_0\mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} + \mu M\mathbf{u} = \varrho_0\mathbf{f} + \mathbf{g} \text{ in } \mathcal{D}'(D),$$

which is the second equation of (4.69). This finishes the proof. □

Chapter 5

Outlook and open problems

In this chapter, we give some remarks on how the results obtained in this thesis could be extended, as well as state still open problems in the field of homogenization of Navier-Stokes equations.

Let us start to give some remarks on the construction of the Bogovskiĭ operator as done in Chapter 3. From Theorem 3.3.1, we see that for the case of well-separated obstacles the construction of a uniformly bounded Bogovskiĭ operator $\mathcal{B}_\varepsilon : L_0^2(D_\varepsilon) \to W_0^{1,2}(D_\varepsilon)$ is possible even for the critical case $\alpha = 3$. However, our construction of $\mathcal{B}_\varepsilon$ in Theorem 3.4.1 for a randomly perforated domain requires $\alpha > 3$ for the L^2-setting. A natural question is how to avoid this stricter assumption on the size of the holes. In view of the applications of the operator $\mathcal{B}_\varepsilon$ for stationary Navier-Stokes equations in Section 4.1, we see that the need to bound the gradient of $\mathcal{B}_\varepsilon$ in L^2 is unavoidable to get proper estimates on the density and the velocity. However, we have some freedom on the space where the operator is defined on. Thus, one possible way is to extend $\mathcal{B}_\varepsilon$ as an operator acting on L_0^q for some $q > 2$ rather than L_0^2, but still maps into $W_0^{1,2}$. This might change the dependence of the Bogovskiĭ constant on ε into the correct direction, thus one may reach the borderline case $\alpha = 3$. On the other hand, according to the functions $\varphi = \mathcal{B}_\varepsilon(\varrho_\varepsilon^\Theta - \langle \varrho_\varepsilon^\Theta \rangle_{D_\varepsilon})$ used as test functions in the momentum equation, this procedure might give a worse restriction on the allowed exponent Θ and, in turn, on the adiabatic exponent γ.

Another necessary condition in Theorem 3.4.1 is $\alpha > 2$. As we have seen in Theorem 3.4.2, it is needed in order to show that the balls $B_{\varepsilon^\alpha r_i}(\varepsilon z_i)$ do not overlap, so we can take some space around each single hole and, loosely speaking, cut off constants without paying too much, as shown in the proof of Theorem 3.4.1. One is willing to believe that this condition is not optimal, in particular, one could be "smarter" to allow also clusters of not too many overlapping holes. One particular problem arising is the possibility that holes might not even overlap but rather touch, producing an external cusp in the perforated box I_i^ε as introduced in Theorem 3.4.2. For domains exhibiting such cusps, it is known that an inverse to the divergence does not exist in general, see, for instance, [ADLG13]. To overcome this issue, one needs a much better understanding of the geometry of holes. A better geometrical understanding would probably also yield the optimal dependence on the Bogovskiĭ constant on ε as mentioned in Remark 3.4.8.

Back to homogenization of compressible Navier-Stokes equations, there are still many open questions. One is how the limiting system would look like, if we assumed the holes to be large, that is, $\alpha < 3$. For periodic obstacles and without assuming that the Mach number vanishes, there are results only for the case $\alpha = 1$, see, e.g., [Mas02], and also [FNT10] for heat-conducting fluids. As mentioned in Section 4.4, the supercritical case $\alpha \in (1, 3)$ for the low Mach number limit was considered in [HKS21], even if the exponent of the additional ε-dependent scaling of the pressure is rather large (see also (4.68)). One might therefore ask whether this can be improved. Furthermore, to the best of the author's knowledge, there is no literature for the homogenization of Navier-Stokes-Fourier equations in the supercritical regime $\alpha \in (1, 3)$, even assuming vanishing Mach number. Another aspect apart of the size of the obstacles is the question whether one can improve the range of the adiabatic exponent γ in the pressure law $p(\varrho) = a\varrho^\gamma$ from the known case $\gamma > 2$ to the direction of physical relevance $\gamma \in [1, \frac{5}{3}]$. The most interesting open question, however, is that of the possible homogenization for the borderline case $\alpha = 3$, even for the seemingly simpler setting of periodically arranged holes, without the assumption of a vanishing Mach number, and even for large enough adiabatic exponents γ.

Appendix A

Strong convergence of the density

In order to complete the proofs of Theorems 4.1.4, 4.2.1, 4.2.3, and 4.3.3, we need to show the strong convergence of $\tilde{\varrho}_\varepsilon \to \varrho$ at least in $L^1(D)$. The proof of this fact is nowadays well understood, see, e.g., [FL15, LP21]. For simplicity and legibility, we will focus on the case of constant temperature and $\gamma > 3$, and assume that the equations are stationary. The proof in the case of variable temperature follows the same lines with slight changes of the exponents of $\tilde{\varrho}_\varepsilon$. For the time-dependent setting, one has to apply an Aubin-Lions type argument. We will comment these issues in Remark A.2 below. To simplify notation, we will identify a function f, defined on D_ε, with its zero prolongation $\tilde{f}$ to the whole of $\mathbb{R}^3$.

First, we start with the compactness of the so-called effective viscous flux:

Lemma A.1. *Under the assumptions of Theorem 4.1.4, there holds for any* $\psi \in C_c^\infty(D)$

$$\lim_{\varepsilon\to 0} \int_D \psi \left(p(\varrho_\varepsilon) - \left(\frac{4\mu}{3} + \eta \right) \operatorname{div} \mathbf{u}_\varepsilon \right) \varrho_\varepsilon \,\mathrm{d}x = \int_D \psi \left(\overline{p(\varrho)} - \left(\frac{4\mu}{3} + \eta \right) \operatorname{div} \mathbf{u} \right) \varrho \,\mathrm{d}x. \tag{A.1}$$

Proof. (See [FN09, Section 3.6.5].) The main idea is to use test functions

$$\psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon), \quad \psi \nabla \Delta^{-1}(\chi_D \varrho),$$

where $\psi \in \mathcal{D}(D)$. By the Sobolev embedding theorem (see (B.8)), we conclude from Lemma B.11

$$\|\nabla \Delta^{-1}(f)\|_{L^{r^*}(\mathbb{R}^3)} \leq C \|f\|_{L^r(\mathbb{R}^3)}, \quad r^* = \frac{3r}{3-r} \text{ if } 1 < r < 3,$$
$$\|\nabla \Delta^{-1}(f)\|_{L^{r^*}(\mathbb{R}^3)} \leq C \|f\|_{L^r(\mathbb{R}^3)} \quad \text{for any } r^* < \infty \text{ if } r \geq 3.$$

By the uniform estimates on ϱ_ε and ϱ as well as the fact $\gamma \geq 3$, we get for any $1 \leq r < \infty$

$$\|\psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon)\|_{L^r(D)} + \|\psi \nabla \Delta^{-1}(\chi_D \varrho)\|_{L^r(D)} \leq C,$$
$$\|\nabla(\psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon))\|_{L^{2\gamma}(D)} + \|\nabla(\psi \nabla \Delta^{-1}(\chi_D \varrho))\|_{L^{2\gamma}(D)} \leq C.$$

Hence, by (4.13), we get with $\varsigma := \min\left(\sigma, \frac{3(\alpha-1)\sigma_0}{2(2+\sigma_0)}\right) > 0$

$$|\langle F_\varepsilon, \psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon) \rangle|$$
$$\leq C\varepsilon^\varsigma \big(\|\psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon)\|_{L^r(D)} + \|\nabla(\psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon))\|_{L^{2+\sigma_0}(D)} \big) \leq C\varepsilon^\varsigma \to 0.$$

Now we choose $\psi\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)$ as test function in (4.12). We obtain

$$\int_D p(\varrho_\varepsilon)\,\mathrm{div}\,\big(\psi\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)\big) + (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \nabla\big(\psi\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)\big)\,\mathrm{d}x$$
$$+ \int_D -\mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \nabla\big(\psi\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)\big) + (\varrho_\varepsilon\mathbf{f}+\mathbf{g})\cdot\psi\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)\,\mathrm{d}x$$
$$= -\langle F_\varepsilon, \psi\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)\rangle.$$

Using product rule for differentiation and the definition of the Riesz operators $\mathcal{R} = (\mathcal{R}_{ij})_{1\le i,j\le 3}$ in Definition B.10, we obtain for the first integral

$$\int_D p(\varrho_\varepsilon)\,\mathrm{div}\,\big(\psi\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)\big) + (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \nabla\big(\psi\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)\big)\,\mathrm{d}x$$
$$= \int_D p(\varrho_\varepsilon)\big(\nabla\psi\cdot\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon) + \psi\varrho_\varepsilon\big)\,\mathrm{d}x$$
$$+ \int_D (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \big(\nabla\psi\otimes\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon) + \psi(\nabla\otimes\nabla)\Delta^{-1}(\chi_D\varrho_\varepsilon)\big)\,\mathrm{d}x$$
$$= \int_D p(\varrho_\varepsilon)\big(\nabla\psi\cdot\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon) + \psi\varrho_\varepsilon\big)\,\mathrm{d}x$$
$$+ \int_D (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \nabla\psi\otimes\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon) + \psi(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \mathcal{R}[\chi_D\varrho_\varepsilon]\,\mathrm{d}x.$$

For the second integral, we similarly have

$$\int_D \mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \nabla\big(\psi\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)\big)\,\mathrm{d}x$$
$$= \int_D \mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \big(\nabla\psi\otimes\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon) + \psi(\nabla\otimes\nabla)\Delta^{-1}(\chi_D\varrho_\varepsilon)\big)\,\mathrm{d}x$$
$$= \int_D \mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \big(\nabla\psi\otimes\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)\big) + \psi\mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \mathcal{R}[\chi_D\varrho_\varepsilon]\,\mathrm{d}x.$$

By the standard theory for elliptic problems, the operator $\nabla\Delta^{-1}$ maps $L^{2\gamma}(D)$ to $W^{1,2\gamma}(D)$. By Morrey's inequality (B.9) and $2\gamma > 3$, we have $W^{1,2\gamma}(D) \subset C(D)$, thus $\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)$ converges strongly to $\nabla\Delta^{-1}(\chi_D\varrho)$ in $L^\infty(D)$. Similarly, choosing $\psi\nabla\Delta^{-1}(\chi_D\varrho)$ as test function in (4.16), we get the same integral relations with ϱ_ε and $\mathbf{u}_\varepsilon$ replaced by ϱ and $\mathbf{u}$, respectively. Subtracting both outcomes, performing the limit $\varepsilon\to 0$, and using the strong convergences of $\mathbf{u}_\varepsilon\to\mathbf{u}$ and $\nabla\Delta^{-1}(\chi_D\varrho_\varepsilon)\to\nabla\Delta^{-1}(\chi_D\varrho)$, we obtain

$$\lim_{\varepsilon\to 0}\int_D \psi\Big(p(\varrho_\varepsilon)\varrho_\varepsilon + (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon) : \mathcal{R}[\chi_D\varrho_\varepsilon] - \mathbb{S}(\nabla\mathbf{u}_\varepsilon) : \mathcal{R}[\chi_D\varrho_\varepsilon]\Big)\,\mathrm{d}x$$
$$= \int_D \psi\Big(\overline{p(\varrho)}\varrho + (\varrho\mathbf{u}\otimes\mathbf{u}) : \mathcal{R}[\chi_D\varrho] - \mathbb{S}(\nabla\mathbf{u}) : \mathcal{R}[\chi_D\varrho]\Big)\,\mathrm{d}x.$$

Further, using $\chi_D\nabla\Delta^{-1}(\psi\varrho_\varepsilon\mathbf{u}_\varepsilon)$ as test function in the weak formulation of the continuity equation $(4.1)_1$, and using $\chi_D\nabla\Delta^{-1}(\psi\varrho\mathbf{u})$ as test function in $(4.16)_1$, we obtain

$$\int_D \chi_D\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathcal{R}[\psi\varrho_\varepsilon\mathbf{u}_\varepsilon]\,\mathrm{d}x = 0, \qquad \int_D \chi_D\varrho\mathbf{u}\cdot\mathcal{R}[\psi\varrho\mathbf{u}]\,\mathrm{d}x = 0,$$

so

$$\lim_{\varepsilon\to0}\int_D \psi\Big(p(\varrho_\varepsilon)\varrho_\varepsilon - \mathbb{S}(\nabla\mathbf{u}_\varepsilon):\mathcal{R}(\chi_D\varrho_\varepsilon)\Big)\,\mathrm{d}x - \int_D \psi\Big(\overline{p(\varrho)}\varrho - \mathbb{S}(\nabla\mathbf{u}):\mathcal{R}(\chi_D\varrho)\Big)\,\mathrm{d}x$$
$$= \int_D (\varrho\mathbf{u}\otimes\mathbf{u}):\mathcal{R}[\chi_D\varrho] - \chi_D\varrho\mathbf{u}\cdot\mathcal{R}[\psi\varrho\mathbf{u}]\,\mathrm{d}x$$
$$- \lim_{\varepsilon\to0}\int_D (\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon):\mathcal{R}[\chi_D\varrho_\varepsilon] - \chi_D\varrho_\varepsilon\mathbf{u}_\varepsilon\cdot\mathcal{R}[\psi\varrho_\varepsilon\mathbf{u}_\varepsilon]\,\mathrm{d}x.$$

By Lemma B.13, the right-hand side vanishes, yielding

$$\lim_{\varepsilon\to0}\int_D \psi\Big(p(\varrho_\varepsilon)\varrho_\varepsilon - \mathbb{S}(\nabla\mathbf{u}_\varepsilon):\mathcal{R}[\chi_D\varrho_\varepsilon]\Big)\,\mathrm{d}x = \int_D \psi\Big(\overline{p(\varrho)}\varrho - \mathbb{S}(\nabla\mathbf{u}):\mathcal{R}[\chi_D\varrho]\Big)\,\mathrm{d}x. \tag{A.2}$$

It remains to show that we might replace the terms $\mathbb{S}(\nabla\mathbf{u}_\varepsilon):\mathcal{R}[\chi_D\varrho_\varepsilon]$ and $\mathbb{S}(\nabla\mathbf{u}):\mathcal{R}[\chi_D\varrho]$ by $\big(\frac{4\mu}{3}+\eta\big)\operatorname{div}\mathbf{u}_\varepsilon$ and $\big(\frac{4\mu}{3}+\eta\big)\operatorname{div}\mathbf{u}$, respectively. By (B.11), we write

$$\int_D \psi\mu\big(\nabla\mathbf{u}_\varepsilon+\nabla^T\mathbf{u}_\varepsilon\big):\mathcal{R}[\chi_D\varrho_\varepsilon]\,\mathrm{d}x = \int_D \varrho_\varepsilon\mathcal{R}:\big[\mu\psi\big(\nabla\mathbf{u}_\varepsilon+\nabla^T\mathbf{u}_\varepsilon\big)\big]\,\mathrm{d}x,$$
$$\int_D \psi\mu\big(\nabla\mathbf{u}+\nabla^T\mathbf{u}\big):\mathcal{R}[\chi_D\varrho]\,\mathrm{d}x = \int_D \varrho\mathcal{R}:\big[\mu\psi\big(\nabla\mathbf{u}+\nabla^T\mathbf{u}\big)\big]\,\mathrm{d}x.$$

Observing that

$$\mathcal{R}:\big[\nabla\mathbf{u}+\nabla^T\mathbf{u}\big] = \sum_{i,j}\partial_i\partial_j\Delta^{-1}(\partial_i\mathbf{u}^j+\partial_j\mathbf{u}^i)$$
$$= \sum_{i,j}\partial_j\partial_i^2\Delta^{-1}\mathbf{u}^j + \partial_i\partial_j^2\Delta^{-1}\mathbf{u}^i = 2\operatorname{div}\mathbf{u},$$

we rewrite

$$\mathcal{R}:\big[\mu\psi\big(\nabla\mathbf{u}+\nabla^T\mathbf{u}\big)\big] = 2\mu\psi\operatorname{div}\mathbf{u} + K(\mathbf{u})$$

with the commutator

$$K(\mathbf{u}) \coloneqq \mathcal{R}:\big[\mu\psi\big(\nabla\mathbf{u}+\nabla^T\mathbf{u}\big)\big] - \mu\psi\mathcal{R}:\big[\nabla\mathbf{u}+\nabla^T\mathbf{u}\big],$$

and similar for ϱ and $\mathbf{u}$ replaced by ϱ_ε and $\mathbf{u}_\varepsilon$, respectively. It now remains to show that $K(\mathbf{u}_\varepsilon)\varrho_\varepsilon \rightharpoonup K(\mathbf{u})\varrho$ weakly in $L^1(D)$ to finish the proof. Since K is linear in $\mathbf{u}$ and $\nabla\mathbf{u}_\varepsilon \rightharpoonup \nabla\mathbf{u}$ weakly in $L^2(D)$, we have $K(\mathbf{u}_\varepsilon) \rightharpoonup K(\mathbf{u})$ weakly in $L^2(D)$. By the continuity equation, we obtain $\operatorname{div}(\varrho_\varepsilon\mathbf{u}_\varepsilon)=0$, in particular, it is bounded in $W^{-1,p}(D)$ for some $p>1$. Furthermore, since $\mathcal{R}=\nabla\otimes\nabla\Delta^{-1}$, we see that $\operatorname{curl}\mathcal{R}=0$. Using the uniform bounds on $\mathbf{u}_\varepsilon$, we thus obtain a uniform bound on $\operatorname{curl}K(\mathbf{u}_\varepsilon)$ in $W^{-1,p}(D)$. Defining the vector fields

$$\mathbf{U}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon \quad\text{and}\quad \mathbf{V}_\varepsilon = K(\mathbf{u}_\varepsilon),$$

we have a uniform bound on $\{\mathbf{U}_\varepsilon\}_{\varepsilon>0}$ in $L^{\frac{6\gamma}{\gamma+3}}(D)$, and also on $\{\mathbf{V}_\varepsilon\}_{\varepsilon>0}$ in $L^2(D)$. Together with $\frac{\gamma+3}{6\gamma}+\frac{1}{2}=\frac{2}{3}+\frac{1}{2\gamma}<1$ for any $\gamma>3/2$, an application of Lemma B.12 yields the desired. □

Remark A.2. *Let us remark that the proof of equation* (A.1) *applies also in the case of temper-*

ature dependent viscosity coefficients $\mu = \mu(\vartheta)$ and $\eta = \eta(\vartheta)$ as well as to the time-dependent case $[\varrho, \mathbf{u}] = [\varrho, \mathbf{u}](t, x)$. For the Navier-Stokes-Fourier equations, the proof is verbatim to the one given above, only replacing the estimate on $\nabla(\psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon))$ and $\nabla(\psi \nabla \Delta^{-1}(\chi_D \varrho))$ by

$$\|\nabla(\psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon))\|_{L^{\gamma+\Theta}(D)} + \|\nabla(\psi \nabla \Delta^{-1}(\chi_D \varrho))\|_{L^{\gamma+\Theta}(D)} \leq C,$$

which holds since $\gamma + \Theta > 3$. By $\Theta > 1$, we may choose $\xi > 0$ in Lemma 4.3.10 such that

$$\frac{\gamma + \Theta}{\Theta} + \xi \leq \gamma + \Theta,$$

yielding

$$|\langle F_\varepsilon, \psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon) \rangle| \leq C \, \varepsilon^\nu \to 0.$$

The same argument holds for the case of constant temperature and $\gamma > 2$ as $\Theta = 2\gamma - 3$ there. The main difference for the evolutionary case is that we have to replace integrals in space by integrals over space and time, and use the bounds on ϱ_ε and $\mathbf{u}_\varepsilon$ in their particular space-time spaces $L^p(0, T; L^q(D))$. Precisely for the density, we get

$$\begin{aligned}
&\|\varphi \psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon)\|_{L^\infty(0,T;L^r(D))} + \|\varphi \psi \nabla \Delta^{-1}(\chi_D \varrho)\|_{L^\infty(0,T;L^r(D))} \leq C, \\
&\|\nabla(\varphi \psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon))\|_{L^\infty(0,T;L^\gamma(D))} + \|\nabla(\varphi \psi \nabla \Delta^{-1}(\chi_D \varrho))\|_{L^\infty(0,T;L^\gamma(D))} \leq C
\end{aligned}$$

for any $\varphi \in C_c^\infty(0, T)$, $\psi \in C_c^\infty(D)$, and any $r \in (1, \infty)$ since $\gamma > 6$. From the continuity equation, we get

$$\partial_t(\nabla \Delta^{-1} \varrho_\varepsilon) = -\nabla \Delta^{-1} \operatorname{div}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = -\operatorname{div} \nabla \Delta^{-1}(\varrho_\varepsilon \mathbf{u}_\varepsilon) = -\varrho_\varepsilon \mathbf{u}_\varepsilon,$$

thus

$$\|\partial_t(\nabla \Delta^{-1}(\chi_D \varrho_\varepsilon))\|_{L^2(0,T;L^{\frac{6\gamma}{6+\gamma}}(D))} \leq C \, \|\varrho_\varepsilon\|_{L^\infty(0,T;L^\gamma(D))} \|\mathbf{u}_\varepsilon\|_{L^2(0,T;W_0^{1,2}(D))} \leq C.$$

The Aubin-Lions-Simon theorem [Sim86, Section 8, Corollary 4] thus states that for any $r \in (1, \infty)$, the sequence $\{\nabla \Delta^{-1}(\chi_D \varrho_\varepsilon)\}_{\varepsilon > 0}$ is relatively compact in $L^\infty(0, T; L^r(D))$. Thus, up to a choice of a subsequence, we obtain

$$\nabla \Delta^{-1}(\chi_D \varrho_\varepsilon) \to \nabla \Delta^{-1}(\chi_D \varrho) \text{ strongly in } C(0, T; L^r_{\text{weak}}(D)),$$

so again $|\langle F_\varepsilon, \varphi \psi \nabla \Delta^{-1}(\chi_D \varrho_\varepsilon) \rangle| \to 0$ as $\varepsilon \to 0$. Finally, we may replace $\mathbb{S}(\nabla \mathbf{u}_\varepsilon) : \mathcal{R}[\chi_D \varrho_\varepsilon]$ by $\left(\frac{4\mu}{3} - \eta\right) \operatorname{div} \mathbf{u}_\varepsilon$ in (A.2) by applying Lemma B.12 to the four-component vector fields

$$\mathbf{U}_\varepsilon = (\varrho_\varepsilon, \varrho_\varepsilon \mathbf{u}_\varepsilon), \quad \mathbf{V}_\varepsilon = (K(\mathbf{u}_\varepsilon), 0, 0, 0).$$

The last ingredient we need to show the strong convergence of the density is a monotonicity lemma, which can be found in [FN09, Theorem 10.19].

Lemma A.3. *Let $P, G \in C(\mathbb{R})$ be non-decreasing functions and let $\{\varrho_n\}_{n \in \mathbb{N}} \subset L^1(D)$ be a* *sequence such that*

$$\left.\begin{array}{r}P(\varrho_n) \rightharpoonup \overline{P(\varrho)},\\ G(\varrho_n) \rightharpoonup \overline{G(\varrho)},\\ P(\varrho_n)G(\varrho_n) \rightharpoonup \overline{P(\varrho)G(\varrho)}\end{array}\right\} \text{ weakly in } L^1(D).$$

Then we have

$$\overline{P(\varrho)}\,\overline{G(\varrho)} \leq \overline{P(\varrho)G(\varrho)} \text{ a.e. in } D.$$

If additionally $G(z) = z$ and (with the notation $\varrho = \overline{G(\varrho)}$)

$$\overline{P(\varrho)\varrho} = \overline{P(\varrho)}\varrho,$$

then

$$\overline{P(\varrho)} = P(\varrho).$$

Lemma A.4. *It holds $\overline{\varrho^{\gamma+1}} = \overline{\varrho^\gamma}\varrho$ a.e. in D and thus $\varrho_\varepsilon \to \varrho$ strongly in $L^1(D)$ and $L^r(D)$, $1 \leq r < 2\gamma$.*

Remark A.5. *We note that the range $1 \leq r < 2\gamma$ should be replaced by $1 \leq r < 3(\gamma - 1)$ if $\gamma > 2$, and by $1 \leq r < \gamma + \Theta$ in the case of Navier-Stokes-Fourier equations. For the evolutionary setting, it is enough to observe that $\overline{\varrho^{\gamma+1}} = \overline{\varrho^\gamma}\varrho$ a.e. in $(0,T) \times D$ is sufficient to conclude $\overline{p(\varrho)} = p(\varrho)$.*

Proof of Lemma A.4. We will follow the proof of [LP21, Lemma 4.6], which applies similarly in the case when $\gamma > 2$, the viscosity coefficients μ and η are temperature dependent, and in time-dependent setting. First, we take the function $b(s) = s\log(s)$ in the renormalized continuity equation (4.4), integrate over D and send $\varepsilon \to 0$ to obtain

$$\int_D \overline{\varrho \operatorname{div} \mathbf{u}} \,\mathrm{d}x = 0. \tag{A.3}$$

Now, we use the same function b in the renormalized equations for the limit $[\varrho, \mathbf{u}]$ to get

$$\int_D \varrho \operatorname{div} \mathbf{u} \,\mathrm{d}x = 0. \tag{A.4}$$

Let us further write (A.1) in the form

$$\int_D \psi\left(\overline{\varrho^{\gamma+1}} - \left(\frac{4\mu}{3} + \eta\right)\overline{\varrho \operatorname{div} \mathbf{u}}\right)\mathrm{d}x = \int_D \psi\left(\overline{\varrho^\gamma}\varrho - \left(\frac{4\mu}{3} + \eta\right)\varrho \operatorname{div} \mathbf{u}\right)\mathrm{d}x,$$

which yields due to the fact $\frac{4\mu}{3} + \eta > 0$ that

$$\frac{\overline{\varrho^{\gamma+1}}}{\frac{4\mu}{3} + \eta} - \overline{\varrho \operatorname{div} \mathbf{u}} = \frac{\overline{\varrho^\gamma}\varrho}{\frac{4\mu}{3} + \eta} - \varrho \operatorname{div} \mathbf{u} \tag{A.5}$$

a.e. in D. Integrating over D, we deduce with (A.3) and (A.4)

$$\int_D \frac{\overline{\varrho^{\gamma+1}}}{\frac{4\mu}{3}+\eta}\,\mathrm{d}x = \int_D \frac{\overline{\varrho^{\gamma}}\varrho}{\frac{4\mu}{3}+\eta}\,\mathrm{d}x.$$

Together with Lemma A.3, we see that

$$\overline{\varrho^{\gamma}}\varrho \leq \overline{\varrho^{\gamma+1}} \text{ a.e. in } D,$$

which finally leads to $\overline{\varrho^{\gamma}}\varrho = \overline{\varrho^{\gamma+1}}$ and thus, applying again Lemma A.3,

$$\overline{\varrho^{\gamma}} = \varrho^{\gamma} \text{ a.e. in } D.$$

Hence, we get (up to the choice of a subsequence) $\tilde{\varrho}_\varepsilon \to \varrho$ strongly in $L^\gamma(D)$, hence also a.e. in D and in $L^r(D)$ for all $1 \leq r < 2\gamma$. □

Remark A.6. *We remark that for time-dependent equations, the renormalized continuity equation yields*

$$\partial_t(\overline{\varrho\log\varrho}) + \operatorname{div}\overline{(\varrho\log\varrho)\mathbf{u}} + \overline{\varrho\operatorname{div}\mathbf{u}} = 0.$$

Thus, we may obtain for any $\tau \in [0,T]$

$$\int_D (\overline{\varrho\log\varrho} - \varrho\log\varrho)(\tau,\cdot)\,\mathrm{d}x + \int_0^\tau \int_D \overline{\varrho\operatorname{div}\mathbf{u}} - \varrho\operatorname{div}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{A.6}$$

Integrating (A.5) *over* $(0,T)\times D$ *and using* (A.6)*, we get*

$$\int_D (\overline{\varrho\log\varrho} - \varrho\log\varrho)(\tau,\cdot)\,\mathrm{d}x + \left(\frac{4\mu}{3}+\eta\right)^{-1}\int_0^\tau \int_D \left(\overline{\varrho^{\gamma+1}} - \overline{\varrho^{\gamma}}\varrho\right)\mathrm{d}x\,\mathrm{d}t = 0.$$

By convexity of $s \mapsto s\log(s)$ *and* $s \mapsto s^\gamma$*, we obtain with Lemma A.3 that* $\overline{\varrho\log\varrho} \geq \varrho\log\varrho$ *and* $\overline{\varrho^{\gamma+1}} \geq \overline{\varrho^{\gamma}}\varrho$*. Hence, by the same token,*

$$\overline{\varrho\log\varrho} = \varrho\log\varrho, \quad \overline{\varrho^{\gamma+1}} = \overline{\varrho^{\gamma}}\varrho,$$

which finally yields the desired convergence $\tilde{\varrho}_\varepsilon \to \varrho$ *a.e. in* D*.*

Appendix B

Some analytic results

Lemma B.1 (Derivative of the determinant)**.** *Let* $A(t) = (a_{ij}(t))_{i,j=1}^d \in C^1([0,T];\mathbb{R}^{d\times d})$ *be an invertible matrix. Then* $\det(A(t))$ *is differentiable with*

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(A(t)) = \det(A(t))\ \mathrm{tr}(\dot{A}(t)A^{-1}(t)),$$

where $\mathrm{tr}(A)$ *is the trace of the matrix* A*, and* $\dot{A} = \frac{\mathrm{d}}{\mathrm{d}t}A$*.*

Proof. Let A_{ij} be the matrix that arises from A by removing the i-th row and j-th column. Then we have by expansion along the i-th row

$$\det(A) = \sum_{j=1}^{d}(-1)^{i+j}a_{ij}\det(A_{ij})$$

and so for any $l \in \{1,\dots,d\}$

$$\frac{\partial \det(A)}{\partial a_{il}} = (-1)^{i+l}\det(A_{il}).$$

Differentiation with respect to t yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(A) = \sum_{i,l=1}^{d}\frac{\partial \det(A)}{\partial a_{il}}\ \frac{\mathrm{d}a_{il}}{\mathrm{d}t} = \sum_{i,l=1}^{d}(-1)^{i+l}\det(A_{il})\,\dot{a}_{il}.$$

Since

$$A^{-1} = \frac{1}{\det(A)}\mathrm{Adj}(A), \quad \text{where} \quad \mathrm{Adj}(A) \coloneqq ((-1)^{i+j}\det(A_{ji}))_{i,j=1}^d,$$

we conclude

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(A) = \sum_{l=1}^{d}(\mathrm{Adj}(A)\dot{A})_{ll} = \det(A)\ \mathrm{tr}(A^{-1}\dot{A}).$$

□

Definition B.2. *A* motion *of a domain $D \subset \mathbb{R}^d$ is a map*

$$S : [0,T] \times D \to \mathbb{R}^d$$

having the following properties:

1. *$S(t,\cdot)$ is a C^1-diffeomorphism from D to $D_t \coloneqq S(t,D)$.*
2. *The gradient $\nabla_y S(t,y)$ satisfies $\det(\nabla_y S(t,y)) > 0$.*
3. *$S(0,\cdot) = \mathbb{I}$.*

Theorem B.3 (Reynolds' transport theorem)**.** *Let $D \subset \mathbb{R}^d$ be a domain, $S : [0,T] \times D \to \mathbb{R}^d$ be a motion, $D_t \coloneqq S(t,D)$, and $f \in C^1(\{(t,x) : t \in [0,T] \text{ and } x \in D_t\}; \mathbb{R})$. Then for any test volume $V(t) \subset D_t$ it holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} f(t,x)\,\mathrm{d}x = \int_{V(t)} [\partial_t f + \operatorname{div}(f\mathbf{u})](t,x)\,\mathrm{d}x$$

with $\mathbf{u}(t,x) \coloneqq \partial_t S(t,y)$, $y = S^{-1}(t,x)$. If further $\mathbf{f} \in C^1\big(\{(t,x) : t \in [0,T] \text{ and } x \in D_t\}; \mathbb{R}^d\big)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \mathbf{f}(t,x)\,\mathrm{d}x = \int_{V(t)} [\partial_t \mathbf{f} + \operatorname{div}(\mathbf{f} \otimes \mathbf{u})](t,x)\,\mathrm{d}x.$$

Proof. With $x = S(t,y)$, we first obtain

$$\mathcal{F}(t) \coloneqq \int_{V(t)} f(t,x)\,\mathrm{d}x = \int_{V_0} f(t,S(t,y))\,|\det(\nabla_y S(t,y))|\,\mathrm{d}y,$$

where $V_0 \coloneqq V(0) \subset D$ and $y \in V_0$. Since $\det(\nabla_y S(t,y)) > 0$, we get

$$\begin{aligned}
\dot{\mathcal{F}} &= \int_{V_0} \frac{\mathrm{d}}{\mathrm{d}t}[f(t,S(t,y))\det(\nabla_y S(t,y))]\,\mathrm{d}y \\
&= \int_{V_0} \big(\partial_t f(t,S(t,y)) + \nabla_x f(t,S(t,y)) \cdot \partial_t S(t,y)\big)\det(\nabla_y S(t,y)) \\
&\qquad + f(t,S(t,y))\,\frac{\mathrm{d}}{\mathrm{d}t}\det(\nabla_y S(t,y))\,\mathrm{d}y.
\end{aligned}$$

Together with Lemma B.1, $x = S(t,y)$, and $(\nabla_y S(t,y))^{-1} = (\nabla_x S^{-1})(t,x)$ we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\det(\nabla_y S(t,y)) &= \det(\nabla_y S(t,y)) \cdot \operatorname{tr}\big((\nabla_y S(t,y))^{-1}\partial_t \nabla_y S(t,y)\big) \\
&= \det(\nabla_y S(t,y)) \cdot \operatorname{tr}\big((\nabla_x S)^{-1}(t,x)\,\partial_t \nabla_y S(t,y)\big) \\
&= \det(\nabla_y S(t,y)) \cdot \operatorname{tr}\big(\nabla_x[(\partial_t S)(t,S^{-1}(t,x))]\big) \\
&= \det(\nabla_y S(t,y)) \cdot \operatorname{tr}(\nabla_x \mathbf{u}(t,x)) \\
&= \det(\nabla_y S(t,y)) \cdot \operatorname{div}_x(\mathbf{u})(t,S(t,y)),
\end{aligned}$$

thus

$$\dot{\mathcal{F}} = \int_{V_0} \big[\partial_t f + \mathbf{u} \cdot \nabla_x f + f\operatorname{div}_x(\mathbf{u})\big](t,S(t,y))\,\det(\nabla_y S(t,y))\,\mathrm{d}y$$

$$= \int_{V(t)} [\partial_t f + \operatorname{div}(f\mathbf{u})](t,x)\,\mathrm{d}x.$$

The second statement follows the same lines, yielding similarly

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \mathbf{f}(t,x)\,\mathrm{d}x &= \int_{V_0} \mathbf{f}(t,S(t,y))\,|\det(\nabla_y S(t,y))|\,\mathrm{d}y \\
&= \int_{V_0} \big[\partial_t \mathbf{f} + \nabla_x \mathbf{f}\cdot\mathbf{u} + \mathbf{f}\operatorname{div}_x(\mathbf{u})\big](t,S(t,y))\,\det(\nabla_y S(t,y))\,\mathrm{d}y \\
&= \int_{V(t)} [\partial_t \mathbf{f} + \operatorname{div}(\mathbf{f}\otimes\mathbf{u})](t,x)\,\mathrm{d}x.
\end{aligned}$$

□

Lemma B.4. *The following inequalities hold:*

1. *(Young's inequality for products.) Let $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for any $a, b \geq 0$,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{B.1}$$

2. *(Hölder's inequality.) Let $(D, \mathfrak{A}, \mu)$ be a measure space, $1 \leq p, q, r \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, and let $f \in L^p(D)$ and $g \in L^q(D)$. Then $fg \in L^r(D)$ with*

$$\|fg\|_{L^r(D)} \leq \|f\|_{L^p(D)}\|g\|_{L^q(D)}. \tag{B.2}$$

3. *(Young's inequality for convolutions.) Let $1 \leq p, q, r \leq \infty$ with*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

and let $f \in L^p(\mathbb{R}^d)$, $g \in L^q(\mathbb{R}^d)$. Then, $f * g \in L^r(\mathbb{R}^d)$ and

$$\|f * g\|_{L^r(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)}\,\|g\|_{L^q(\mathbb{R}^d)}. \tag{B.3}$$

4. *(Interpolation in L^q-spaces.) Let $D \subset \mathbb{R}^d$ be a domain, $1 \leq p \leq q \leq \infty$, $\theta \in (0,1)$, and $p \leq r \leq q$ be defined as*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

If $u \in L^p(D) \cap L^q(D)$, then $u \in L^r(D)$, and

$$\|u\|_{L^r(D)} \leq \|u\|^{\theta}_{L^p(D)}\|u\|^{1-\theta}_{L^q(D)}. \tag{B.4}$$

5. *(Grönwall's inequality.) Let $T > 0$ and $u \in L^1([0,T])$ with*

$$u(t) \leq C_1 \int_0^t u(s)\,\mathrm{d}s + C_2$$

for any $t \in [0,T]$. Then

$$u(t) \le C_2\,(1 + C_1 t e^{C_1 t}). \tag{B.5}$$

Proof. The proof of 3. can be found in [Duo01, Corollary 1.21]. All other proofs are given in [Eva10, Section B.2]. □

Proposition B.5. *Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain, $1 \le p < \infty$, and $u \in W^{1,p}(D)$ be real-, vector-, or matrix-valued. Then, the following assertions hold true:*

1. *(Poincaré's inequality.) Denoting by $\langle u\rangle_D$ the mean value of u over D, we have*

$$\|u - \langle u\rangle_D\|_{L^p(D)} \le C\,\|\nabla u\|_{L^p(D)} \tag{B.6}$$

for some constant $C > 0$ independent of u. If additionally $u \in W_0^{1,p}(D)$, then

$$\|u\|_{L^p(D)} \le C\,\|\nabla u\|_{L^p(D)}. \tag{B.7}$$

2. *(Gagliardo-Nirenberg-Sobolev inequality.) If $1 < p < d$, then $W^{1,p}(D) \subset L^{p^*}(D)$ with $p^* := dp/(d-p)$. Moreover,*

$$\|u\|_{L^{p^*}(D)} \le C\,\|u\|_{W^{1,p}(D)}. \tag{B.8}$$

3. *(Morrey's inequality.) If $d < p \le \infty$, then $W^{1,p}(D) \subset C^{0,1-\frac{d}{p}}(D)$ with*

$$\|u\|_{C^{0,1-\frac{d}{p}}(D)} \le C\,\|u\|_{W^{1,p}(D)}. \tag{B.9}$$

4. *(Rellich-Kondrachev theorem.) The embedding $W^{1,p}(D) \subset L^q(D)$ is compact for any $1 \le q < p^*$.*

5. *(Hardy's inequality.) Let $u \in W_0^{1,2}(D)$. Then $\operatorname{dist}(x,\partial D)^{-1}u \in L^2(D)$ and*

$$\|\operatorname{dist}(x,\partial D)^{-1}u\|_{L^2(D)} \le C\,\|\nabla u\|_{L^2(D)}. \tag{B.10}$$

Proof. For statements 1.–4., see [Eva10]. More precisely, the first assertion can be found in Theorem 3 in Section 5.6 and Theorem 1 in Section 5.8, the second and third are Theorem 6 in Section 5.6, and the fourth one is Theorem 1 in Section 5.7. Statement 5. can be found in [Neč62, Theorem 1.6]. □

Lemma B.6. *For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, we define the Hardy-Littlewood maximal function by*

$$(Mf)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,\mathrm{d}y,$$

where the supremum runs over all cubes $Q \subset \mathbb{R}^d$ that contain x. Then, we have for any $1 < q \le \infty$ and any $f \in L^q(\mathbb{R}^d)$

$$\|Mf\|_{L^q(\mathbb{R}^d)} \le C(q,d)\,\|f\|_{L^q(\mathbb{R}^d)}.$$

Proof. See [Duo01, Theorem 2.5]. □

Proposition B.7 (Properties of mollifiers). *Let $\eta \in C^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}(\eta) \subset B_1(0)$ and $\int_{\mathbb{R}^d} \eta = 1$. Define for $\varepsilon > 0$*

$$\eta_\varepsilon(x) := \varepsilon^{-d}\eta(x/\varepsilon).$$

Let $U \subset \mathbb{R}^d$ be open and set $U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}$. Further, let $u \in L^1_{\mathrm{loc}}(U; \mathbb{R}^n)$, and define for $x \in U_\varepsilon$

$$u_\varepsilon := \eta_\varepsilon * u = \int_U \eta_\varepsilon(\cdot - y)u(y)\,\mathrm{d}y.$$

Then, we have the following properties:

1. *$u_\varepsilon \in C^\infty(U_\varepsilon)$.*
2. *$u_\varepsilon \to u$ almost everywhere for $\varepsilon \to 0$.*
3. *If $1 \le p < \infty$ and $u \in L^p_{\mathrm{loc}}(U)$, then $u_\varepsilon \to u$ strongly in $L^p_{\mathrm{loc}}(U)$. Further, if U is bounded, then $\|u_\varepsilon\|_{L^p(U)} \le \|u\|_{L^p(U)}$ for ε small enough.*

Proof. The case $d = 1$ is proven in [Eva10, Section C.5, Theorem 7] and applies verbatim to any $d \ge 1$. □

Proposition B.8 (Vitali's convergence theorem). *Suppose that $D \subset \mathbb{R}^d$ is a bounded domain, $1 \le p < \infty$, and $\{f_n\}_{n\in\mathbb{N}} \subset L^p(D)$ are measurable. Then $f_n \to f$ strongly in $L^p(D)$ if and only if*

1. *$f_n \to f$ in measure, that is, for any $\varepsilon > 0$ it holds $\lim_{n\to\infty} |\{|f_n - f| \ge \varepsilon\}| = 0$, and*
2. *$\{|f_n|^p\}_{n\in\mathbb{N}}$ is uniformly integrable, that is, for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $U \subset D$ with $|U| < \delta$ and any $n \in \mathbb{N}$, we have $\int_U |f_n|^p < \varepsilon$.*

In particular, if $f_n \to f$ almost everywhere in D, $\|f_n\|_{L^p(D)} \le M$ uniformly in n for some $M > 0$, and $1 \le q < p$, then $f_n \to f$ strongly in $L^q(D)$.

Proof. See [Bog07, Theorem 4.5.4] for the case $p = 1$. Let us show how 1. and 2. imply the convergence $f_n \to f$ if $p > 1$. Set for $\varepsilon > 0$ and $n \in \mathbb{N}$

$$A_n := \{|f_n - f| \ge \varepsilon\}.$$

Then $|A_n| < \eta$ for $\eta > 0$ arbitrary and n large enough, so

$$\int_D |f_n - f|^p\,\mathrm{d}x = \int_{A_n} |f_n - f|^p\,\mathrm{d}x + \int_{D\setminus A_n} |f_n - f|^p\,\mathrm{d}x \le C\,\eta + \varepsilon^p|D|$$

by uniform integrability. The second statement follows from $f_n \to f$ in measure since $f_n \to f$ almost everywhere, and for any $\varepsilon > 0$ we set $\delta > 0$ such that $M\delta^{\frac{pq^2}{p-q}} = \varepsilon$ to get for any $|U| < \delta$

$$\int_U |f_n|^q\,\mathrm{d}x \le \|f_n\|^q_{L^p(U)}|U|^{\frac{pq^2}{p-q}} < \varepsilon.$$

□

Theorem B.9 (Calderón-Zygmund theorem). *Assume $D \subset \mathbb{R}^d$ and $K(x,y) = \frac{k(x,y)}{|y|^d}$, where k is a given regular function and satisfies:*

- *For any $x \neq y$ and any $\lambda > 0$ we have*

$$k(x,y) = k(x, \lambda y).$$

- *For any $x \in D$, $k(x,y) \in L^1(\{|y| = 1\})$ and*

$$\int_{|y|=1} k(x,y)\,\mathrm{d}y = 0.$$

- *There exists a constant $A > 0$ such that*

$$\|k(x,y)\|_{L^\infty(D\times\{|y|=1\})} \leq A.$$

Then for any $1 < q < \infty$ and any $f \in L^q(\mathbb{R}^d)$, the principal value integral

$$\Psi(x) := p.v. \int_{\mathbb{R}^d} K(x, x-y) f(y)\,\mathrm{d}y := \lim_{\varepsilon\to 0} \int_{|x-y|\geq\varepsilon} K(x, x-y) f(y)\,\mathrm{d}y$$

exists for a.e. $x \in D$ and satisfies

$$\Psi \in L^q(\mathbb{R}^d), \quad \|\Psi\|_{L^q(\mathbb{R}^d)} \leq C\,\|f\|_{L^q(\mathbb{R}^d)},$$

where the constant $C > 0$ satisfies $C \leq C(q)A$.

Proof. See [CZ57, Section 5, Theorem 2]. □

Definition B.10. *The* Riesz operators $(\mathcal{R}_{ij})_{1\leq i,j\leq d}$ *are defined as*

$$\mathcal{R}_{ij}[f](x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \frac{\xi_i\xi_j}{|\xi|^2} \mathcal{F}_{x\to\xi}[f] e^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi = \mathcal{F}^{-1}_{\xi\to x}\left[\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}_{x\to\xi}[f]\right],$$

where we denote for a function in the Schwartz space $f \in \mathcal{S}(\mathbb{R}^d)$ its Fourier transform by $\mathcal{F}_{x\to\xi}[f] \in \mathcal{S}(\mathbb{R}^d)$.

Obviously, the Riesz operators can be written as

$$(\mathcal{R}_{ij})_{1\leq i,j\leq d} = (\nabla\otimes\nabla)\Delta^{-1},$$

where Δ^{-1} is the Fourier multiplier with symbol $-|\xi|^2$. For a short survey through the concept of Fourier multipliers and Riesz transforms, see, for instance, [NS04, Section 1.3.7.2] and [Duo01]. We recall also that the Riesz operators satisfy for any $f, g \in \mathcal{S}(\mathbb{R}^d)$ and any $1 \leq i, j \leq d$

$$\int_{\mathbb{R}^d} \mathcal{R}_{ij}[f] g\,\mathrm{d}x = \int_{\mathbb{R}^d} f\mathcal{R}_{ij}[g]\,\mathrm{d}x, \tag{B.11}$$

which is a consequence of Plancherel's theorem and the Fourier multiplier property of $\mathcal{R}_{ij}$, and note that this relation can be extended to any $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$ by density.

Lemma B.11. *For any $1 < p < \infty$, the Riesz operators satisfy*

$$\|(\nabla \otimes \nabla)\Delta^{-1}(f)\|_{L^p(\mathbb{R}^d)} \leq C(p,d)\|f\|_{L^p(\mathbb{R}^d)}.$$

Proof. This is a special case of Theorem 1.56 in [NS04, Section 1.3.7.2]. □

Lemma B.12 (Div-Curl lemma)**.** *Let $1 < p, q < \infty$ with*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Suppose

$$\mathbf{U}_n \rightharpoonup \mathbf{U} \text{ weakly in } L^p(\mathbb{R}^d;\mathbb{R}^d), \quad \mathbf{V}_n \rightharpoonup \mathbf{V} \text{ weakly in } L^q(\mathbb{R}^d;\mathbb{R}^d),$$

and that

$$\begin{aligned}
&\operatorname{div}\mathbf{U}_n \text{ is bounded in } W^{-1,p}(\mathbb{R}^d),\\
&\operatorname{curl}\mathbf{V}_n = (\nabla\mathbf{V}_n - \nabla^T\mathbf{V}_n) \text{ is bounded in } W^{-1,q}(\mathbb{R}^d;\mathbb{R}^{d\times d}).
\end{aligned}$$

Then

$$\mathbf{U}_n \cdot \mathbf{V}_n \to \mathbf{U}\cdot\mathbf{V} \text{ in } \mathcal{D}'(\mathbb{R}^d).$$

Moreover, if

$$\mathbf{U}_n \rightharpoonup \mathbf{U} \text{ weakly in } L^p(\mathbb{R}^d;\mathbb{R}^d), \quad \mathbf{V}_n \rightharpoonup \mathbf{V} \text{ weakly in } L^q(\mathbb{R}^d;\mathbb{R}^d),$$

where

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1,$$

and

$$\operatorname{div}\mathbf{U}_n \text{ is bounded in } W^{-1,s}(\mathbb{R}^d), \quad \operatorname{curl}\mathbf{V}_n \text{ is bounded in } W^{-1,s}(\mathbb{R}^d;\mathbb{R}^{d\times d})$$

for some $s > 1$, then

$$\mathbf{U}_n \cdot \mathbf{V}_n \rightharpoonup \mathbf{U}\cdot\mathbf{V} \text{ weakly in } L^r(\mathbb{R}^d).$$

Proof. See [FN09, Lemma 10.11 and Theorem 10.21]. □

Lemma B.13. *Let $1 < p, q < \infty$ satisfy*

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} < 1.$$

Suppose

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^p(\mathbb{R}^d), \quad v_\varepsilon \rightharpoonup v \text{ weakly in } L^q(\mathbb{R}^d).$$

Then for all $1 \le i, j \le d$ *we have*

$$u_\varepsilon \mathcal{R}_{ij}[v_\varepsilon] - v_\varepsilon \mathcal{R}_{ij}[u_\varepsilon] \rightharpoonup u\mathcal{R}_{ij}[v] - v\mathcal{R}_{ij}[u] \text{ weakly in } L^r(\mathbb{R}^d). \tag{B.12}$$

Proof. We follow [FNP01, Lemma 3.4]. We will prove the more general statement

$$\mathbf{U}_\varepsilon \cdot \mathcal{R}[\mathbf{V}_\varepsilon] - \mathbf{V}_\varepsilon \cdot \mathcal{R}[\mathbf{U}_\varepsilon] \rightharpoonup \mathbf{U} \cdot \mathcal{R}[\mathbf{V}] - \mathbf{V} \cdot \mathcal{R}[\mathbf{U}] \text{ weakly in } L^r(\mathbb{R}^d) \tag{B.13}$$

for vector fields $\mathbf{U}_\varepsilon \rightharpoonup \mathbf{U}$ weakly in $L^p(\mathbb{R}^d)$, $\mathbf{V}_\varepsilon \rightharpoonup \mathbf{V}$ weakly in $L^q(\mathbb{R}^d)$, where we denote $\mathcal{R} = (\mathcal{R}_{ij})_{1\le i,j\le d}$. Indeed, embedding the functions u_ε and v_ε as functions $\mathbf{U}_\varepsilon = u_\varepsilon \mathbf{e}_i \in L^p(\mathbb{R}^d;\mathbb{R}^d)$ and $\mathbf{V}_\varepsilon = v_\varepsilon \mathbf{e}_j \in L^q(\mathbb{R}^d;\mathbb{R}^d)$, where $\mathbf{e}_i$ denotes the i-th vector of the canonical basis of $\mathbb{R}^d$, then (B.12) is equivalent to (B.13). To show (B.13), we rewrite

$$\mathbf{U}_\varepsilon \cdot \mathcal{R}[\mathbf{V}_\varepsilon] - \mathbf{V}_\varepsilon \cdot \mathcal{R}[\mathbf{U}_\varepsilon] = (\mathbf{U}_\varepsilon - \mathcal{R}[\mathbf{U}_\varepsilon]) \cdot \mathcal{R}[\mathbf{V}_\varepsilon] - (\mathbf{V}_\varepsilon - \mathcal{R}[\mathbf{V}_\varepsilon]) \cdot \mathcal{R}[\mathbf{U}_\varepsilon]$$

and consider the k-th component of the vector $\mathcal{R}[\mathbf{U}_\varepsilon]$, that is,

$$(\mathcal{R}[\mathbf{U}_\varepsilon])_k = ((\partial_i\partial_j)_{1\le i,j\le d}\Delta^{-1}[\mathbf{U}_\varepsilon])_k = \sum_{j=1}^d \partial_k\partial_j\Delta^{-1}[\mathbf{U}_\varepsilon]_j. \tag{B.14}$$

Using the divergence operator, we get

$$\operatorname{div}\mathcal{R}[\mathbf{U}_\varepsilon] = \sum_{k=1}^d \partial_k \sum_{j=1}^d \partial_k\partial_j\Delta^{-1}[\mathbf{U}_\varepsilon]_j = \sum_{j=1}^d \partial_j \sum_{k=1}^d \partial_k^2\Delta^{-1}[\mathbf{U}_\varepsilon]_j = \sum_{j=1}^d \partial_j[\mathbf{U}_\varepsilon]_j = \operatorname{div}\mathbf{U}_\varepsilon.$$

The same argument applies to $\mathbf{V}_\varepsilon$, hence

$$\operatorname{div}(\mathbf{U}_\varepsilon - \mathcal{R}[\mathbf{U}_\varepsilon]) = \operatorname{div}(\mathbf{V}_\varepsilon - \mathcal{R}[\mathbf{V}_\varepsilon]) = 0.$$

Similarly, we get from (B.14)

$$\mathcal{R}[\mathbf{U}_\varepsilon] = \nabla(\Delta^{-1}(\operatorname{div}\mathbf{U}_\varepsilon)), \quad \mathcal{R}[\mathbf{V}_\varepsilon] = \nabla(\Delta^{-1}(\operatorname{div}\mathbf{V}_\varepsilon)),$$

which entails in

$$\operatorname{curl}\mathcal{R}[\mathbf{U}_\varepsilon] = \operatorname{curl}\mathcal{R}[\mathbf{V}_\varepsilon] = 0.$$

Thus, Lemma B.12 yields (B.13). □

Lemma B.14. *Let* $D \subset \mathbb{R}^d$ *be a bounded smooth domain and* $q \in (\frac{1}{2}, \infty)$*. Then there exist constants* $C_1, C_2 > 0$ *such that for any* $f \in L^{2q}(D;[0,\infty))$*, we have*

$$\langle |f^q - \langle f\rangle_D^q|^2\rangle_D \le C_1 \langle |f^q - \langle f^q\rangle_D|^2\rangle_D \le C_2 \langle |f^q - \langle f\rangle_D^q|^2\rangle_D,$$

where we set

$$\langle f\rangle_D := \frac{1}{|D|}\int_D f \,\mathrm{d}x.$$

Proof. This lemma occurred earlier in [GS19, Lemma 2.2]. We will use the symbol $a \sim b$ to express that there are constants $C_1, C_2 > 0$ with $a \leq C_1\, b \leq C_2\, a$. First, by the fundamental theorem of calculus and the fact that $\int_D a(f - \langle f\rangle_D) = 0$ for any constant $a \in \mathbb{R}$, we have

$$\begin{aligned}\int_D |f^q - \langle f\rangle_D^q|^2 \,\mathrm{d}x &\sim \int_D |f^{2q} - \langle f\rangle_D^{2q}| \,\mathrm{d}x \sim \int_D (f^{2q-1} - \langle f\rangle_D^{2q-1})(f - \langle f\rangle_D) \,\mathrm{d}x \\ &= \int_D (f^{2q-1} - \langle f^q\rangle_D^{\frac{2q-1}{q}})(f - \langle f\rangle_D) \,\mathrm{d}x.\end{aligned}$$

Noting that for any $a, b \geq 0$ and $q > \frac{1}{2}$

$$\frac{a^{2q-1} - b^{2q-1}}{a - b} \sim (a+b)^{2q-2},$$

we get with $a = f$ and $b = \langle f^q\rangle_D^{\frac{1}{q}}$

$$|f^{2q-1} - \langle f^q\rangle_D^{\frac{2q-1}{q}}| \sim (f + \langle f^q\rangle_D^{\frac{1}{q}})^{2q-2} |f - \langle f^q\rangle_D^{\frac{1}{q}}|.$$

Now, we divide into the cases $|f - \langle f\rangle_D| \leq C\,|f - \langle f^q\rangle_D^{\frac{1}{q}}|$ and $|f - \langle f\rangle_D| \geq C\,|f - \langle f^q\rangle_D^{\frac{1}{q}}|$ to conclude

$$\begin{aligned}\int_D |f^q - \langle f\rangle_D^q|^2 \,\mathrm{d}x &\sim \int_D (f + \langle f^q\rangle_D^{\frac{1}{q}})^{2q-2} |f - \langle f^q\rangle_D^{\frac{1}{q}}| |f - \langle f\rangle_D| \,\mathrm{d}x \\ &\sim \int_D (f + \langle f^q\rangle_D^{\frac{1}{q}})^{2q-2} |f - \langle f^q\rangle_D^{\frac{1}{q}}|^2 \,\mathrm{d}x = \int_D |(f + \langle f^q\rangle_D^{\frac{1}{q}})^{q-1} (f - \langle f^q\rangle_D^{\frac{1}{q}})|^2 \,\mathrm{d}x \\ &\sim \int_D |f^q - \langle f^q\rangle_D|^2 \,\mathrm{d}x.\end{aligned}$$

□

Appendix C

Some probabilistic results

Lemma C.1 (Borel-Cantelli lemma)**.** *Let* $(\Omega, \mathfrak{A}, \mathbb{P})$ *be a probability space and* $\{A_k\}_{k\in\mathbb{N}} \subset \mathfrak{A}$ *be a sequence of measurable sets. Define*

$$\limsup_{k\to\infty} A_k := \bigcap_{k\in\mathbb{N}} \bigcup_{n\geq k} A_n.$$

Assume further

$$\sum_{k=0}^{\infty} \mathbb{P}(A_k) < \infty. \tag{C.1}$$

Then $\mathbb{P}(\limsup_{k\to\infty} A_k) = 0$.

Proof. Let $\varepsilon > 0$ be fixed. Then by (C.1) there exists an $N \in \mathbb{N}$ such that

$$\sum_{k\geq N} \mathbb{P}(A_k) < \varepsilon.$$

Since also for any $k \in \mathbb{N}$ we have $\limsup_{k\to\infty} A_k \subset \bigcup_{n\geq k} A_n$ and the subadditive property of the measure $\mathbb{P}$, we get

$$0 \leq \mathbb{P}(\limsup_{k\to\infty} A_k) \leq \mathbb{P}(\bigcup_{n\geq N} A_n) \leq \sum_{k\geq N} \mathbb{P}(A_k) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this finishes the proof. $\square$

Theorem C.2 (Strong Law of Large Numbers)**.** *Let* $\{X_i\}_{i\in\mathbb{N}}$ *be pairwise independent identically distributed random variables with* $\mathbb{E}(X_i) < \infty$. *Then*

$$\frac{1}{n+1} \sum_{i=0}^{n} X_i \to \mathbb{E}(X_i)$$

almost surely as $n \to \infty$.
In particular, for the Poisson point process $(\Phi, \mathcal{R})$ *defined in Section 3.4, we have*

$$\lim_{\varepsilon\to 0} \varepsilon^d N(\varepsilon^{-1} S) = \lambda |S|, \qquad \lim_{\varepsilon\to 0} \varepsilon^d \sum_{z_j\in\varepsilon^{-1}S} r_j^m = \lambda \mathbb{E}(r^m)|S|$$

almost surely for any bounded measurable set $S \subset \mathbb{R}^d$ which is star-shaped with respect to the origin.

Proof. For the first statement see, e.g., [Dur19, Theorem 2.4.1]. The second statement (in more general settings) can be found, e.g., in [GHV18, Lemma 6.1] and [LP17, Theorem 8.14]. We remark that the proof in [LP17], although showing just convergence in $L^1(\mathbb{P})$, remains valid if one uses Birkhoff's ergodic theorem instead of the Mean ergodic theorem, provided their sequence $\{a_n\}_{n\in\mathbb{N}}$ grows fast enough. □

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