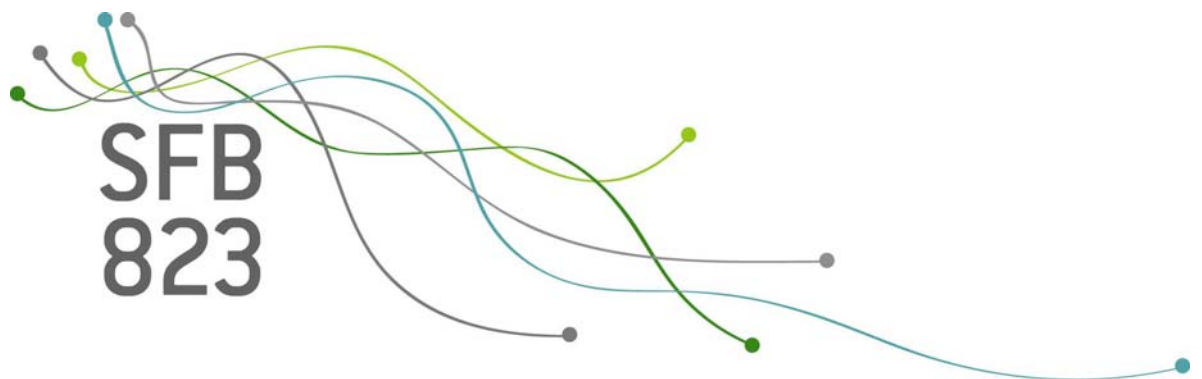


SFB  
823

# Weak convergence of the weighted sequential empirical process of some long-range dependent data

Jannis Buchsteiner

Nr. 29/2014



Discussion Paper



# Weak Convergence of the Weighted Sequential Empirical Process of some Long-Range Dependent Data

Jannis Buchsteiner\*

*Fakultät für Mathematik, Ruhr-Universität Bochum, Germany.*

## Abstract

Let  $(X_k)_{k \geq 1}$  be a Gaussian long-range dependent process with  $EX_1 = 0$ ,  $EX_1^2 = 1$  and covariance function  $r(k) = k^{-D}L(k)$ . For any measurable function  $G$  let  $(Y_k)_{k \geq 1} = (G(X_k))_{k \geq 1}$ . We study the asymptotic behaviour of the associated sequential empirical process  $(R_N(x, t))$  with respect to a weighted sup-norm  $\|\cdot\|_w$ . We show that, after an appropriate normalization,  $(R_N(x, t))$  converges weakly in the space of càdlàg functions with finite weighted norm to a Hermite process.

**Keywords:** Sequential empirical process; long-range dependence; weighted norm; modified functional delta method

## 1 Introduction

Given a stationary stochastic process  $(Y_j)_{j \geq 1}$ , with marginal distribution function  $F(x) = P(Y_1 \leq x)$ , we define the sequential empirical process

$$R_N(x, t) = \sum_{j=1}^{\lfloor Nt \rfloor} \left( 1_{\{Y_j \leq x\}} - F(x) \right), \quad x \in \mathbb{R}, 0 \leq t \leq 1.$$

This process plays an important role in statistics, e.g. in the study of nonparametric change-point tests. The asymptotic distribution of the sequential empirical process was initially determined by Müller (1970), and independently Kiefer (1972), who both studied the case when the underlying data  $(Y_j)_{j \geq 1}$  are independent and identically distributed. In this case,  $N^{-1/2}R_N(x, t)$  converges in distribution towards a mean-zero Gaussian process  $K(x, t)$  with covariance structure  $E(K(x, s)K(y, t)) = (s \wedge t)(F(x \wedge y) - F(x)F(y))$ .

---

\*E-mail: [jannis.buchsteiner@rub.de](mailto:jannis.buchsteiner@rub.de)

Research supported by Collaborative Research Center SFB 823 *Statistical modeling of nonlinear dynamic processes*.

The process  $K(x, t)$  is also called a Kiefer-Müller process. Komlós, Major, and Tusnády (1975) proved an almost sure approximation theorem for the sequential empirical process with sharp rates, again in the case of i.i.d. data.

Sequential empirical processes of dependent data have been studied by a large number of authors, e.g. Berkes and Philipp (1977) and Philipp and Pinzur (1980) for strongly mixing processes, and Berkes, Hörmann, and Schauer (2009) for so called S-mixing processes. For long-range dependent data, the sequential empirical process was first studied by Dehling and Taqqu (1989), in the case of a Gaussian subordinated process. Giraitis and Surgailis (2002) used similiar techniques to establish weak convergence if the underlying data is a long memory moving average process.

Under some technical conditions, Dehling and Taqqu (1989) prove convergence of the normalized sequential empirical process in the space  $D([-\infty, \infty] \times [0, 1])$  towards a process of the type  $J(x)Z(t)$ ,  $x \in \mathbb{R}, 0 \leq t \leq 1$ , where  $J : \mathbb{R} \rightarrow \mathbb{R}$  is a deterministic function and where  $(Z(t))_{0 \leq t \leq 1}$  is a Hermite process.

In the present paper, we consider the above result with regard to the weighted sequential empirical process  $w(x)R_N(x, t)$ , where  $w(x) = (1 + |x|)^\lambda$ , for some  $\lambda > 0$ . Therefore we equip the function space

$$D_w([-\infty, \infty] \times [0, 1]) := \{f \in D([-\infty, \infty] \times [0, 1]) : \sup_{x \in \mathbb{R}, t \in [0, 1]} |w(x)f(x, t)| < \infty\},$$

with the weighted sup-norm  $\|f\|_w := \sup |w(x)f(x, t)|$  and show that the result of Dehling and Taqqu takes place in this normed subspace of  $D([-\infty, \infty] \times [0, 1])$ .

The asymptotic distribution of the weighted one-parameter empirical process  $(R_N(x, 1))$  has been studied for i.i.d. data by Čibisov (1964) and O'Reilly (1974). Shao and Yu (1996) treated the cases when the underlying data are strong mixing,  $\rho$ -mixing and associated. Recently, Beutner, Wu, and Zähle (2012) studied empirical process convergence with respect to weighted norms for linear long-range dependent data.

Weak convergence of the empirical process with respect to weighted supremum norms has been applied by Beutner and Zähle (2010) in their study of the asymptotic behaviour of the distortion risk measure. They developed a modified functional delta method (MFDM) which requires only quasi-Hadamard differentiability on the one hand, but weighted convergence of the empirical process on the other hand. By using the MFDM, Beutner and Zähle (2012) also determined the asymptotic distribution of U- and V-statistics with an unbounded kernel. The weight functions arising in this context are functions of  $x$  only. More generally one could study weight functions  $w(x, t)$ . However, this is beyond the scope of the present paper.

## 2 Definitions and Main Results

We consider a stationary Gaussian process  $(X_j)_{j \geq 1}$  with  $EX_1 = 0$ ,  $EX_1^2 = 1$  and covariance function  $r(k) = EX_1X_{k+1}$ , which satisfies

$$r(k) = k^{-D}L(k), \tag{1}$$

where  $L$  is a slowly varying function at infinity and  $0 < D < 1$ . Such a sequence is called a Gaussian long-range dependent process. For any measurable function  $G : \mathbb{R} \rightarrow \mathbb{R}$  we define the subordinated process  $(Y_j)_{j \geq 1}$  by

$$Y_j := G(X_j).$$

A useful tool to establish weak convergence of  $(R_N(x, t))$  under these circumstances are Hermite polynomials. The Hermite polynomial  $H_n$  of order  $n$  is defined as

$$H_n(x) := (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}.$$

For example  $H_0(x) = 1$ ,  $H_1(x) = x$  and  $H_2(x) = x^2 - 1$ . Since  $(H_n)_{n \geq 0}$  is an orthogonal basis for the space of square integrable functions with respect to the standard normal distribution, we have for any  $x \in \mathbb{R}$  the series expansion

$$1_{\{Y_j \leq x\}} - F(x) = \sum_{q=0}^{\infty} \frac{J_q(x)}{q!} H_q(X_j). \quad (2)$$

As usual, the Hermite coefficients  $J_q(x)$  are given by the inner product, i.e.

$$J_q(x) = E(1_{\{Y_j \leq x\}} - F(x))H_q(X_j) = E1_{\{Y_j \leq x\}}H_q(X_j) = \int_{\{G(s) \leq x\}} H_q(s)\varphi(s)ds,$$

for  $q \geq 1$ , where  $\varphi$  is the standard normal density. With regard to (2) we call the index  $m(x)$  of the first nonzero Hermite coefficient the Hermite rank of  $1_{\{G(\cdot) \leq x\}} - F(x)$ . Since  $E(1_{\{Y_j \leq x\}} - F(x)) = 0$  we have  $m(x) \geq 1$ . If  $0 < D < 1/m(x)$ , then  $(1_{\{Y_j \leq x\}} - F(x))_{j \geq 1}$  exhibits long-range dependence, see Taqqu (1975).

Moreover we set  $m := \min\{m(x) : x \in \mathbb{R}\}$  and call  $m$  the Hermite rank of the class of functions  $\{1_{\{G(\cdot) \leq x\}} - F(x) : x \in \mathbb{R}\}$ .

**Theorem A** (Dehling and Taqqu 1989, Theorem 1.1). *Let  $(X_j)_{j \geq 1}$  be a stationary, mean-zero Gaussian process with covariance (1), let the class of functions  $1_{\{G(X_j) \leq x\}} - F(x)$ ,  $-\infty < x < \infty$ , have Hermite rank  $m$  and let  $0 < D < 1/m$ . Then*

$$\left\{ d_N^{-1} R_N(x, t) : -\infty \leq x \leq \infty, 0 \leq t \leq 1 \right\}$$

converges weakly in  $D([-\infty, \infty] \times [0, 1])$ , equipped with the sup-norm, to

$$\left\{ \frac{J_m(x)}{m!} Z_m(t) : -\infty \leq x \leq \infty, 0 \leq t \leq 1 \right\}.$$

The normalization factor  $d_N$  is asymptotically proportional to  $\sqrt{N^{2-mD} L^m(N)}$ , more precisely

$$d_N^2 = \text{Var} \left( \sum_{j=1}^N H_m(X_j) \right),$$

see Taqqu (1975, Corollary 4.1). The process  $(Z_m(t))_{t \in [0,1]}$  is called an  $m$ th order Hermite process. It can be represented as a multiple Wiener-Itô integral as well as a Wiener-Itô-Dobrushin integral, see Taqqu (1979). For  $m = 1$  it is a fractional Brownian motion and therefore Gaussian, but it is non Gaussian for  $m \geq 2$ .

Heuristically, we have to control  $w(x)F(x)$  and  $w(x)(1 - F(x))$  for  $x \rightarrow -\infty$  resp.  $x \rightarrow \infty$  to get a weighted version of Theorem A. Therefore we require that  $F$  has at least a finite  $\delta$ -th moment, i.e.

$$\int |x|^\delta dF(x) < \infty \quad (3)$$

for some  $\delta > 0$ .

**Theorem 1.** *Let  $(X_j)_{j \geq 1}$  be a stationary, mean-zero Gaussian process with covariance (1), let the class of functions  $1_{\{G(X_j) \leq x\}} - F(x)$ ,  $-\infty < x < \infty$ , have Hermite rank  $m$  and let  $0 < D < 1/m$ . If  $F$  has a finite  $\delta$ -th moment then*

$$\{d_N^{-1} R_N(x, t) : -\infty \leq x \leq \infty, 0 \leq t \leq 1\}$$

*converges weakly in  $D_w([-\infty, \infty] \times [0, 1])$ , equipped with the weighted sup-norm  $\|\cdot\|_w$ , to*

$$\left\{ \frac{J_m(x)}{m!} Z_m(t) : -\infty \leq x \leq \infty, 0 \leq t \leq 1 \right\},$$

*where  $w(x) = (1 + |x|)^\lambda$  and  $\lambda = \delta/3$ .*

If we want to use Theorem 1 to apply the MFDM, we need  $\lambda > 1$ , i.e. the distribution function  $F$  must have a finite  $\delta$ -th moment with  $\delta > 3$ . We conjecture that the choice  $\lambda = \delta/3$  could be improved to  $\delta/2$ , since  $\lambda = \delta/3$  is only necessary to get (7).

To prove Theorem 1 we need a weighted version of Taqqu's weak reduction principle (cf. Taqqu, 1975; Dehling and Taqqu, 1989).

**Theorem 2.** *Under the assumptions of Theorem 1 there exist constants  $C, \kappa > 0$  such that for any  $0 < \varepsilon \leq 1$*

$$\begin{aligned} P \left( \max_{n \leq N} \sup_{-\infty \leq x \leq \infty} d_N^{-1} \left| w(x) \sum_{j=1}^n \left( 1_{\{Y_j \leq x\}} - F(x) - \frac{J_m(x)}{m!} H_m(X_j) \right) \right| > \varepsilon \right) \\ \leq CN^{-\kappa} (1 + \varepsilon^{-3}), \end{aligned} \quad (4)$$

*where  $w(x) = (1 + |x|)^\lambda$  and  $\lambda = \delta/3$ .*

### 3 Proofs

From now on we assume that the conditions of Theorem 1 are satisfied. Especially let  $w(x) = (1 + |x|)^\lambda$  with  $\lambda = \delta/3$ . For consistency reasons we adopt some notations by Dehling and Taqqu, namely

$$\Lambda(x) := F(x) + \int 1_{\{G(s) \leq x\}} \frac{|H_m(s)|}{m!} \varphi(s) ds,$$

$$S_N(n, x) := d_N^{-1} \sum_{j=1}^n \left( 1_{\{Y_j \leq x\}} - F(x) - \frac{J_m(x)}{m!} H_m(X_j) \right).$$

Furthermore for  $x \leq y$  we set

$$\begin{aligned} F(x, y) &:= F(y) - F(x), \\ J_m(x, y) &:= J_m(y) - J_m(x) \\ S_N(n, x, y) &:= S_N(n, y) - S_N(n, x) \\ \Lambda(x, y) &:= \Lambda(y) - \Lambda(x). \end{aligned}$$

Note that  $\Lambda$  is nondecreasing and that  $\Lambda(x, y)$  bounds  $F(x, y)$  as well as  $(1/m!)J_m(x, y)$  if  $x \leq y$ .

Lemma 1 is a modification of Lemma 3.1 by Dehling and Taqqu. The following rearrangement is small but necessary.

**Lemma 1.** *Under the assumptions of Theorem 1 there exist constants  $\gamma > 0$  and  $C$  such that for  $n \leq N$ ,*

$$E |S_N(n, x, y)|^2 \leq C \left( \frac{n}{N} \right) N^{-\gamma} F(x, y) (1 - F(x, y)). \quad (5)$$

We can bound (5) again by  $C(n/N)N^{-\gamma}(1 - F(y))$ , or  $C(n/N)N^{-\gamma}F(x)$ , which is useful for  $y \rightarrow \infty$  resp.  $x \rightarrow -\infty$ . During this paper we will handle  $C$  as a universal constant, possibly growing from line to line and from lemma to lemma, but at the end bounded and independent of  $N, n, x$  and  $\varepsilon$ .

*Proof.* The Hermite expansion

$$\sum_{q=m}^{\infty} \frac{J_q(x, y)}{q!} H_q(X_j) = 1_{\{x \leq Y_j \leq y\}} - F(x, y)$$

yields

$$\sum_{q=m}^{\infty} \frac{J_q^2(x, y)}{q!} = E \left( 1_{\{x \leq Y_j \leq y\}} - F(x, y) \right)^2 = F(x, y) (1 - F(x, y)).$$

Together with  $E H_q(X_j) H_q(X_k) = q! (E X_j X_k)^q = q! (r(j - k))^q$  we get

$$\begin{aligned} & E \left( \sum_{j \leq n} \left( 1_{\{x \leq Y_j \leq y\}} - F(x, y) - \frac{J_m(x, y)}{m!} H_m(X_j) \right) \right)^2 \\ &= E \left( \sum_{j \leq n} \sum_{q=m+1}^{\infty} \frac{J_q(x, y)}{q!} H_q(X_j) \right)^2 \\ &= \sum_{q=m+1}^{\infty} \frac{J_q^2(x, y)}{q!} \frac{1}{q!} \sum_{j, k \leq n} E H_q(X_j) H_q(X_k) \end{aligned}$$

$$\leq F(x, y)(1 - F(x, y)) \sum_{j, k \leq n} |r(j - k)|^{m+1}.$$

Since  $\sum_{j, k \leq n} |r(j - k)|^{m+1} \leq 2n \sum_{k=1}^n k^{-D(m+1)} |L(k)|^{m+1}$ , we have

$$\begin{aligned} \sum_{j, k \leq n} |r(j - k)|^{m+1} &\leq Cn^{2-D(m+1)} |L(n)|^{m+1}, \quad \text{for } D(m+1) < 1, \\ \sum_{j, k \leq n} |r(j - k)|^{m+1} &\leq Cn, \quad \text{for } D(m+1) > 1, \\ \sum_{j, k \leq n} |r(j - k)|^{m+1} &\leq Cn^{1+\alpha} |L(n)|^m, \quad \text{for } D(m+1) = 1 \end{aligned}$$

and  $0 < \alpha < 1 - mD$ . In general we get

$$\sum_{j, k \leq n} |r(j - k)|^{m+1} \leq Cn^{1+\alpha\vee 2-D(m+1)} L'(n),$$

where  $L'$  is some suitable slowly varying function. Therefore

$$\begin{aligned} E|S_N(n, x, y)|^2 &\leq Cd_N^{-2} F(x, y)(1 - F(x, y)) n^{1+\alpha\vee 2-D(m+1)} L'(n) \\ &\leq CF(x, y)(1 - F(x, y)) n^{1+\alpha\vee 2-D(m+1)} N^{mD-2} L'(n) (L(N))^{-m} \\ &= CF(x, y)(1 - F(x, y)) \left(\frac{n}{N}\right)^{1+\alpha\vee 2-D(m+1)} N^{mD+\alpha-1\vee -D} L'(n) (L(N))^{-m} \\ &\leq CF(x, y)(1 - F(x, y)) \left(\frac{n}{N}\right) N^{-\gamma}. \end{aligned}$$

□

**Lemma 2.** *Under the assumptions of Theorem 1 there exist constants  $\rho > 0$  and  $C$  such that for any  $n \leq N$  and  $0 < \varepsilon \leq 1$ ,*

$$P\left(\sup_{x \in \mathbb{R}} |w(x)S_N(n, x)| > \varepsilon\right) \leq CN^{-\rho} \left(\frac{n}{N}\varepsilon^{-3} + \left(\frac{n}{N}\right)^{2-mD}\right),$$

where  $w(x) = (1 + |x|)^\lambda$  and  $\lambda = \delta/3$ .

*Proof.* As Dehling and Taqqu (1989, Lemma 3.2) we will use the classical chaining technique. For simplicity we will bound the probability separately for  $x \in [0, \infty)$  and  $x \in (-\infty, 0]$ , starting with the first case. Since  $\lim_{x \rightarrow \infty} w(x)\Lambda(x) = \infty$ , the refining partitions  $(x_i(k))_{i \in \mathbb{N}}$  of  $[0, \infty)$  should consist of an infinite number of grid points. For  $k \geq 0$  we set

$$x_i(k) := \inf\{x \geq 0 : w(x)\Lambda(x) \geq \Lambda(0) + i2^{-k}\}.$$

By this definition we have

$$\begin{aligned} &w(x_{i+1}(k))\Lambda(x_i(k), x_{i+1}(k)-) \\ &\leq w(x_{i+1}(k))\Lambda(x_{i+1}(k)-) - w(x_i(k))\Lambda(x_i(k)) \end{aligned}$$



$$\leq 2^{-k}. \quad (6)$$

Moreover, using condition (3) together with the assumption  $\delta = 3\lambda$  and  $i + 1 \leq \Lambda(\infty)w(x_{i+1}(0))$  we get

$$\begin{aligned} & \sum_{j=0}^{\infty} w(x_{j+1}(0))^2 (1 - F(x_j(0))) \\ &= \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} w(x_{j+1}(0))^2 (F(x_{i+1}(0)) - F(x_i(0))) \\ &= \sum_{i=0}^{\infty} \sum_{j=0}^i w(x_{j+1}(0))^2 (F(x_{i+1}(0)) - F(x_i(0))) \\ &\leq \sum_{i=0}^{\infty} (i+1) w(x_{i+1}(0))^2 (F(x_{i+1}(0)) - F(x_i(0))) \\ &\leq \Lambda(\infty) \sum_{i=0}^{\infty} w(x_{i+1}(0))^3 (F(x_{i+1}(0)) - F(x_i(0))) \\ &\leq C \sum_{i=0}^{\infty} w(x_i(0))^3 (F(x_{i+1}(0)) - F(x_i(0))) \\ &< \infty. \end{aligned} \quad (7)$$

Notice that for all  $k \in \mathbb{N}$   $(x_j(k+1))_{j \in \mathbb{N}}$  is a refinement of  $(x_i(k))_{i \in \mathbb{N}}$  and so for any index  $i \in \mathbb{N}$  it exists an index  $j \in \mathbb{N}$  with  $x_j(k+1) = x_i(k)$  and  $x_{j-2}(k+1) = x_{i-1}(k)$ . This yields

$$\begin{aligned} & w(x_i(k))^2 (F(x_i(k)) - F(x_{i-1}(k))) \\ &= w(x_j(k+1))^2 (F(x_j(k+1)) - F(x_{j-2}(k+1))) \\ &= w(x_j(k+1))^2 (F(x_j(k+1)) - F(x_{j-1}(k+1))) \\ &\quad + w(x_j(k+1))^2 (F(x_{j-1}(k+1)) - F(x_{j-2}(k+1))) \\ &\geq w(x_j(k+1))^2 (F(x_j(k+1)) - F(x_{j-1}(k+1))) \\ &\quad + w(x_{j-1}(k+1))^2 (F(x_{j-1}(k+1)) - F(x_{j-2}(k+1))). \end{aligned} \quad (8)$$

Since (8) implies

$$\begin{aligned} & \sum_{i=1}^{\infty} w(x_i(k+1))^2 (F(x_i(k+1)) - F(x_{i-1}(k+1))) \\ &\leq \sum_{i=1}^{\infty} w(x_i(k))^2 (F(x_i(k)) - F(x_{i-1}(k))) \end{aligned}$$

and (6) implies

$$w(x_{i+1}(k)) \leq \frac{1}{\Lambda(0)} \Lambda(x_{i+1}(k)-) w(x_{i+1}(k))$$

$$\begin{aligned}
&\leq \frac{1}{\Lambda(0)} \left( 2^{-k} + w(x_i(k))\Lambda(x_i(k)) \right) \\
&\leq \frac{1}{\Lambda(0)} (1 + w(x_i(k))\Lambda(\infty)) \\
&\leq Cw(x_i(k))
\end{aligned}$$

we get

$$\begin{aligned}
&\sum_{i=1}^{\infty} w(x_{i+1}(k+1))^2 (F(x_{i+1}(k+1)) - F(x_{i-1}(k+1))) \\
&= \sum_{i=1}^{\infty} w(x_{i+1}(k+1))^2 (F(x_{i+1}(k+1)) - F(x_i(k+1))) \\
&\quad + \sum_{i=1}^{\infty} w(x_{i+1}(k+1))^2 (F(x_i(k+1)) - F(x_{i-1}(k+1))) \\
&\leq C \sum_{i=1}^{\infty} w(x_i(k+1))^2 (F(x_i(k+1)) - F(x_{i-1}(k+1))) \\
&\leq C \sum_{i=1}^{\infty} w(x_i(k))^2 (F(x_i(k)) - F(x_{i-1}(k))) \\
&\leq C \sum_{i=1}^{\infty} w(x_i(0))^2 (F(x_i(0)) - F(x_{i-1}(0))) \\
&< \infty,
\end{aligned} \tag{9}$$

where (9) is uniform in  $k$ . We will use (6), (7) and (9) as follows. For any  $x \geq 0$  and any  $k \in \{1, \dots, K\}$  there exists an index  $i_k(x)$  such that

$$x_{i_k(x)}(k) \leq x < x_{i_k(x)+1}(k).$$

This nesting yields a stepwise chaining of  $x$ , given by

$$0 \leq x_{i_0(x)}(0) \leq x_{i_1(x)}(1) \leq \dots \leq x_{i_K(x)}(K) \leq x.$$

Using the grid points above, we get

$$\begin{aligned}
|w(x)S_N(n, x)| &\leq |w(x)S_N(n, x_{i_0(x)}(0))| + |w(x)S_N(n, x_{i_0(x)}(0), x_{i_1(x)}(1))| \\
&\quad + \dots + |w(x)S_N(n, x_{i_K(x)}(K), x)| \\
&\leq |w(x_{i_0(x)+1}(0))S_N(n, x_{i_0(x)}(0))| + |w(x_{i_1(x)+1}(1))S_N(n, x_{i_0(x)}(0), x_{i_1(x)}(1))| \\
&\quad + \dots + |w(x)S_N(n, x_{i_K(x)}(K), x)|.
\end{aligned} \tag{10}$$

The last term of the right hand side can be bounded as follows

$$|w(x)S_N(n, x_{i_K(x)}(K), x)| = d_N^{-1} \left| \sum_{j \leq n} \left( w(x) \left( 1_{\{x_{i_K(x)}(K) < Y_j \leq x\}} - F(x_{i_K(x)}(K), x) \right) \right)$$

$$\begin{aligned}
& \left| -w(x) \frac{J_m(x_{i_K(x)}(K), x)}{m!} H_m(X_j) \right| \\
& \leq d_N^{-1} \sum_{j \leq n} \left( w(x_{i_K(x)+1}(K)) 1_{\{x_{i_K(x)}(K) < Y_j < x_{i_K(x)+1}(K)\}} \right. \\
& \quad \left. + w(x_{i_K(x)+1}(K)) F(x_{i_K(x)}(K), x_{i_K(x)+1}(K)-) \right) \\
& \quad + w(x_{i_K(x)+1}(K)) \Lambda(x_{i_K(x)}(K), x_{i_K(x)+1}(K)-) d_N^{-1} \left| \sum_{j \leq n} H_m(X_j) \right| \\
& \leq |w(x_{i_K(x)+1}(K)) S_N(n, x_{i_K(x)}(K), x_{i_K(x)+1}(K)-)| \\
& \quad + 2nd_N^{-1} w(x_{i_K(x)+1}(K)) F(x_{i_K(x)}(K), x_{i_K(x)+1}(K)-) \\
& \quad + 2w(x_{i_K(x)+1}(K)) \Lambda(x_{i_K(x)}(K), x_{i_K(x)+1}(K)-) d_N^{-1} \left| \sum_{j \leq n} H_m(X_j) \right| \\
& \leq |w(x_{i_K(x)+1}(K)) S_N(n, x_{i_K(x)}(K), x_{i_K(x)+1}(K)-)| \\
& \quad + 2nd_N^{-1} 2^{-K} + 2d_N^{-1} 2^{-K} \left| \sum_{j \leq n} H_m(X_j) \right|. \tag{11}
\end{aligned}$$

Because of (10), (11) and  $\sum_{k=0}^{\infty} \varepsilon/(k+3)^2 \leq \varepsilon/2$  the probability  $P(\sup |w(x) S_N(n, x)| > \varepsilon)$  is dominated by

$$\begin{aligned}
& P \left( \max_{x>0} |w(x_{i_0(x)+1}(0)) S_N(n, x_{i_0(x)}(0))| > \varepsilon/9 \right) \\
& + \sum_{k=1}^K P \left( \max_{x>0} |w(x_{i_k(x)+1}(k)) S_N(n, x_{i_{k-1}(x)}(k-1), x_{i_k(x)}(k))| > \varepsilon/(k+3)^2 \right) \\
& + P \left( \max_{x>0} |w(x_{i_K(x)+1}(K)) S_N(n, x_{i_K(x)}(K), x_{i_K(x)+1}(K)-)| > \varepsilon/(K+3)^2 \right) \\
& + P \left( 2d_N^{-1} 2^{-K} \left| \sum_{j \leq n} H_m(X_j) \right| > \varepsilon/2 - 2nd_N^{-1} 2^{-K} \right). \tag{12}
\end{aligned}$$

Using (7) and Lemma 1 we get

$$\begin{aligned}
& P \left( \max_{x \in \mathbb{R}} |w(x_{i_0(x)+1}(0)) S_N(n, x_{i_0(x)}(0))| > \frac{\varepsilon}{9} \right) \\
& \leq \sum_{j=0}^{\infty} P \left( |w(x_{j+1}(0)) S_N(n, x_j(0))| > \frac{\varepsilon}{9} \right) \\
& \leq C \left( \frac{n}{N} \right) N^{-\gamma} 81 \varepsilon^{-2} \sum_{j=0}^{\infty} w(x_{j+1}(0))^2 (1 - F(x_j(0)))
\end{aligned}$$

$$\leq C \left( \frac{n}{N} \right) N^{-\gamma} 81 \varepsilon^{-2}. \quad (13)$$

For  $1 \leq k < K$  we get by (9)

$$\begin{aligned} & P \left( \max_{x>0} |w(x_{i_{k+1}(x)+1}(k+1)) S_N(n, x_{i_k(x)}(k), x_{i_{k+1}(x)}(k+1))| > \frac{\varepsilon}{(k+3)^2} \right) \\ & \leq \sum_{j=0}^{\infty} P \left( |w(x_{j+2}(k+1)) S_N(n, x_j(k+1), x_{j+1}(k+1))| > \frac{\varepsilon}{(k+3)^2} \right) \\ & \leq C \left( \frac{n}{N} \right) N^{-\gamma} (k+3)^4 \varepsilon^{-2} \sum_{j=0}^{\infty} w(x_{j+2}(k+1))^2 (F(x_{j+2}(k+1)) - F(x_j(k+1))) \\ & \leq C \left( \frac{n}{N} \right) N^{-\gamma} (k+3)^4 \varepsilon^{-2} \end{aligned} \quad (14)$$

and similarly

$$\begin{aligned} & P \left( \max_{x>0} |w(x_{i_K(x)+1}(K)) S_N(n, x_{i_K(x)}(K), x_{i_K(x)+1}(K)-)| > \frac{\varepsilon}{(K+3)^2} \right) \\ & \leq C \left( \frac{n}{N} \right) N^{-\gamma} (K+3)^4 \varepsilon^{-2}. \end{aligned} \quad (15)$$

We choose

$$K = \left\lceil \log_2 \left( \frac{8N d_N^{-1}}{\varepsilon} \right) \right\rceil + 1,$$

which implies  $\varepsilon/2 - 2N d_N^{-1} 2^{-K} \geq \varepsilon/4$  and therefore

$$\begin{aligned} & P \left( 2d_N^{-1} 2^{-K} \left| \sum_{j \leq n} H_m(X_j) \right| > \frac{\varepsilon}{2} - 2nd_N^{-1} 2^{-K} \right) \\ & \leq P \left( d_N^{-1} \left| \sum_{j \leq n} H_m(X_j) \right| > \frac{\varepsilon}{4} 2^{K-1} \right) \\ & \leq \left( \frac{d_n}{d_N} \right)^2 \left( \frac{\varepsilon}{4} \right)^{-2} 2^{-2K+2} \\ & \leq \left( \frac{d_n}{d_N} \right)^2 d_N^2 N^{-2} \\ & \leq C \left( \frac{n}{N} \right)^{2-mD} \left( \frac{L(n)}{L(N)} \right)^m N^{-mD+\lambda} \\ & \leq C \left( \frac{n}{N} \right)^{2-mD} N^{-mD+\lambda} \end{aligned} \quad (16)$$

for any  $\lambda > 0$ . Remember that  $P(\sup |w(x) S_N(n, x)| > \varepsilon)$  is dominated by (12). Using (13), (14), (15) and (16), this yields

$$P \left( \sup_{x>0} |w(x) S_N(n, x)| > \varepsilon \right) \leq C \left( \frac{n}{N} \right) N^{-\gamma} \varepsilon^{-2} \sum_{k=0}^K (k+3)^4 + C \left( \frac{n}{N} \right)^{2-mD} N^{-mD+\lambda}$$

$$\begin{aligned}
&\leq C \left(\frac{n}{N}\right) N^{-\gamma} \varepsilon^{-2} (K+3)^5 + C \left(\frac{n}{N}\right)^{2-mD} N^{-mD+\lambda} \\
&\leq CN^{-\rho} \left(\frac{n}{N} \varepsilon^{-3} + \left(\frac{n}{N}\right)^{2-mD}\right)
\end{aligned}$$

for any  $\rho$  with  $0 < \rho < \min(\gamma, mD - \lambda)$ , because of

$$\begin{aligned}
(K+3)^5 &= (\lfloor \log_2(8Nd_N^{-1}\varepsilon^{-1}) \rfloor + 4)^5 \\
&\leq C (\log(\varepsilon^{-1}) + \log(CN))^5 \\
&\leq C\varepsilon^{-1} N^\delta
\end{aligned}$$

for any  $\delta > 0$ .

To prove the second case, i.e.  $x \in (-\infty, 0]$ , we set

$$y_i(k) := \sup\{y \leq 0 : w(y)(\Lambda(0) - \Lambda(y)) \geq i2^{-k}\}.$$

So we get corresponding versions of (6), (7) and (9), namely

$$\begin{aligned}
&w(y_j(k))\Lambda(y_j(k), y_{j-1}(k)-) \\
&= w(y_j(k))(-\Lambda(0) + \Lambda(y_{j-1}(k)-) + \Lambda(0) - \Lambda(y_j(k))) \\
&\leq w(y_j(k))(\Lambda(0) - \Lambda(y_j(k))) - w(y_{j-1}(k)-)(\Lambda(0) - \Lambda(y_{j-1}(k)-)) \\
&\leq 2^{-k},
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\sum_{j=0}^{\infty} w(y_j(0))^2 F(y_j(0)) \\
&= \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} w(y_j(0))^2 (F(y_i(0)) - F(y_{i+1}(0))) \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^i w(y_j(0))^2 (F(y_i(0)) - F(y_{i+1}(0))) \\
&\leq \Lambda(0) \sum_{i=0}^{\infty} w(y_{i+1}(0))^3 (F(y_i(0)) - F(y_{i+1}(0))) \\
&\leq C \sum_{i=0}^{\infty} w(y_i(0))^3 (F(y_i(0)) - F(y_{i+1}(0))) \\
&< \infty,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\sum_{i=0}^{\infty} w(y_{i+1}(k))^2 (F(y_i(k)) - F(y_{i+1}(k))) \\
&\leq \sum_{i=0}^{\infty} w(y_{i+1}(0))^2 (F(y_i(0)) - F(y_{i+1}(0)))
\end{aligned}$$

$$< \infty. \tag{19}$$

Now, for any  $x \leq 0$  and  $K \in \mathbb{N}$  we can find a chain

$$-\infty < y_{i_0(x)}(0) \leq y_{i_1(x)}(1) \leq \dots \leq y_{i_K(x)}(K) \leq x,$$

with  $y_{i_k(x)}(k) \leq x \leq y_{i_{k-1}(x)}(k)$ . Using

$$\begin{aligned} & |w(x)S_N(n, x)| \\ & \leq |w(y_{i_0(x)}(0))S_N(n, y_{i_0(x)}(0))| + |w(y_{i_0(x)}(0))S_N(n, y_{i_0(x)}(0), y_{i_1(x)}(1))| \\ & \quad + |w(y_{i_1(x)}(1))S_N(n, y_{i_1(x)}(1), y_{i_2(x)}(2))| + \dots + |w(x)S_N(n, y_{i_K(x)}(K), x)| \end{aligned}$$

and

$$\begin{aligned} & |w(x)S_N(n, y_{i_K(x)}(K), x)| \\ & \leq |w(y_{i_K(x)}(K))S_N(n, y_{i_K(x)}(K), y_{i_{K-1}(x)}(K))| + 2nd_N^{-1}2^{-K} + 2^{-K}d_N^{-1} \left| \sum_{j \leq n} H_m(X_j) \right| \end{aligned}$$

together with (18) and (19), we can finish the proof in the same way as in the first case.  $\square$

We are now ready to prove the weighted weak reduction principle. Therefore we can use the original proof by Dehling and Taqqu.

*Proof of Theorem 2.* Let  $N = 2^r$  and  $M_N(n) := \sup_{x \in \mathbb{R}} |w(x)S_N(n, x)|$ . Using the stationarity of  $(X_j)_{j \geq 1}$  we get for  $n_1 < n_2 \leq N$

$$\begin{aligned} M_N(n_1, n_2) & := M_N(n_2) - M_N(n_1) \\ & \leq \sup_{x \in \mathbb{R}} |w(x)(S_N(n_2, x) - S_N(n_1, x))| \\ & \stackrel{D}{=} M_N(n_2 - n_1) \end{aligned}$$

Together with Lemma 2 we obtain

$$P \left( \max_{j=1, \dots, 2^{r-k}} |M_N((j-1)2^k, j2^k)| > \varepsilon \right) \leq CN^{-\rho}(\varepsilon^{-3} + 2^{(k-r)(1-mD)}).$$

Since  $n = \sum_{k=0}^r \sigma_k 2^{r-k}$ ,  $\sigma_k \in \{0, 1\}$ , we have

$$M_N(n) = \sum_{k=0}^r \sigma_k M_N((j_k - 1)2^{r-k}, j_k 2^{r-k}),$$

with some suitable  $j_k \in \{1, \dots, 2^k\}$ . This yields

$$P \left( \max_{n \leq N} |M_N(n)| > \varepsilon \right) \leq P \left( \sum_{k=0}^r \max_{j=1, \dots, 2^{r-k}} |M_N((j-1)2^k, j2^k)| > \varepsilon \right)$$

$$\begin{aligned}
&\leq \sum_{k=0}^r P \left( \max_{j=1, \dots, 2^{r-k}} |M_N((j-1)2^k, j2^k)| > \varepsilon(k+2)^{-2} \right) \\
&\leq CN^{-\rho} \left( \varepsilon^{-3} \sum_{k=0}^{\log_2(N)} (k+2)^6 + \sum_{k=0}^{\log_2(N)} 2^{(k-r)(1-mD)} \right) \\
&\leq CN^{-\rho}(\varepsilon^{-3} + 1).
\end{aligned}$$

For  $N \neq 2^r$  we have  $d_N^{-1} < Cd_{2^r}^{-1}$ , with  $r = \min\{r : N \leq 2^r\}$  and  $C$  independent of  $N$  and  $r$ . Hence

$$\begin{aligned}
P \left( \max_{n \leq N} |M_N(n)| > \varepsilon \right) &\leq P \left( \max_{n \leq 2^r} |M_{2^r}(n)| > C^{-1}\varepsilon \right) \\
&\leq C2^{-r\rho}(\varepsilon^{-3} + 1) \\
&\leq CN^{-\rho}(\varepsilon^{-3} + 1).
\end{aligned}$$

□

Before we can prove Theorem 1 we need one last lemma. More precisely we have to show, that the function  $J_m$ , which yields the  $m$ th order Hermite coefficient of  $1_{\{Y_j \leq x\}}$ , is bounded with respect to the weighted norm we use.

**Lemma 3.** *If  $F$  has a finite  $\delta$ -th moment then for all  $q \in \mathbb{N}$  we have*

$$\sup_{x \in \mathbb{R}} |w(x)J_q(x)| < \infty,$$

where  $w(x) = (1 + |x|)^\lambda$  and  $\lambda = \delta/3$ .

*Proof.* Since  $2\lambda < \delta$  and condition (3) we have  $E|w(Y_j)|^2 < \infty$ . The weight function  $w$  is non-increasing on  $(-\infty, 0]$  and increasing on  $[0, \infty)$ . Therefore we get for  $x \leq 0$

$$\begin{aligned}
|w(x)J_q(x)| &\leq E|w(x)1_{\{Y_j \leq x\}}H_q(X_j)| \\
&\leq \sqrt{E|w(x)1_{\{Y_j \leq x\}}|^2 E|H_q(X_j)|^2} \\
&\leq \sqrt{E|w(Y_j)|^2 E|H_q(X_j)|^2} \\
&< \infty
\end{aligned}$$

and for  $x \geq 0$

$$\begin{aligned}
|w(x)J_q(x)| &= |E(w(x)(1 - 1_{\{Y_j \leq x\}})H_q(X_j))| \\
&\leq \sqrt{E|w(x)1_{\{Y_j > x\}}|^2 E|H_q(X_j)|^2} \\
&\leq \sqrt{E|w(Y_j)|^2 E|H_q(X_j)|^2} \\
&< \infty.
\end{aligned}$$

□

*Proof of Theorem 1.* Since  $C[0, 1]$  is separable and  $Z_m \in C[0, 1]$  a.s., we can use the process convergence

$$Z_{m,N}(t) := d_N^{-1} \sum_{j \leq [Nt]} H_m(X_j) \xrightarrow{D} Z_m(t)$$

in  $D[0, 1]$ , investigated by Taqqu (1975, 1979), to apply the a.s. representation theorem (Pollard, 1984, page 71). Therefore it exists processes  $(\tilde{Z}_{m,N}(t))_{t \in [0,1]}$  and  $(\tilde{Z}_m(t))_{t \in [0,1]}$ , with  $(\tilde{Z}_{m,N}(t)) \stackrel{D}{=} (Z_{m,N}(t))$ ,  $(\tilde{Z}_m(t)) \stackrel{D}{=} (Z_m(t))$  and

$$\left\| \tilde{Z}_{m,N}(\cdot) - \tilde{Z}_m(\cdot) \right\|_{\infty} \rightarrow 0 \quad \text{a.s.}$$

Using Lemma 3 we have

$$\left\| J_m(\cdot) \tilde{Z}_{m,N}(\cdot) - J_m(\cdot) \tilde{Z}_m(\cdot) \right\|_w \rightarrow 0 \quad \text{a.s.}$$

and this implies

$$J_m(x) Z_{m,N}(t) \xrightarrow{D} J_m(x) Z_m(t)$$

in  $D_w \subset D([-\infty, \infty] \times [0, 1])$ , equipped with the weighted norm  $\|\cdot\|_w$ . Theorem 1 follows by the weighted weak reduction principle (Theorem 2).  $\square$

## References

- I. Berkes and W. Philipp. An almost sure invariance principle for the empirical distribution function of mixing random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 41:115–137, 1977.
- I. Berkes, S. Hörmann, and J. Schauer. Asymptotic results for the empirical process of stationary sequences. *Stochastic Process. Appl.*, 119:1298–1324, 2009.
- E. Beutner and H. Zähle. A modified functional delta method and its application to the estimation of risk functionals. *J. Multivariate Anal.*, 101:2452–2463, 2010.
- E. Beutner and H. Zähle. Deriving the asymptotic distribution of U- and V-statistics of dependent data using weighted empirical processes. *Bernoulli*, 18:803–822, 2012.
- E. Beutner, W. B. Wu, and H. Zähle. Asymptotics for statistical functionals of long-memory sequences. *Stochastic Process. Appl.*, 122:910–929, 2012.
- D. M. Čibisov. Some theorems on the limiting behaviour of an empirical distribution function. *Trudy Mat. Inst. Steklov.*, 71:104–112, 1964.
- H. Dehling and M. S. Taqqu. The empirical process of some long-range dependent sequences with an application to U-statistics. *Ann. Statist.*, 17:1767–1783, 1989.



- L. Giraitis and D. Surgailis. The reduction principle for the empirical process of a long memory linear process. In *Empirical process techniques for dependent data*, pages 241–255. Birkhäuser Boston, 2002.
- J. Kiefer. Skorohod embedding of multivariate RV's, and the sample DF. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 24:1–35, 1972.
- J. Komlós, P. Major, and G. Tusnády. An approximation of partial sums of independent RV's and the sample DF. I. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 32:111–131, 1975.
- D. W. Müller. On Glivenko-Cantelli convergence. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 16:195–210, 1970.
- N. E. O'Reilly. On the weak convergence of empirical processes in sup-norm metrics. *Ann. Probab.*, 2:642–651, 1974.
- W. Philipp and L. Pinzur. Almost sure approximation theorems for the multivariate empirical process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 54:1–13, 1980.
- D. Pollard. *Convergence of stochastic processes*. Springer-Verlag, New York, 1984.
- Q.-M. Shao and H. Yu. Weak convergence for weighted empirical processes of dependent sequences. *Ann. Probab.*, 24:2098–2127, 1996.
- G. R. Shorack and J. A. Wellner. *Empirical processes with applications to statistics*. John Wiley & Sons Inc., New York, 1986.
- M. S. Taqqu. Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 31:287–302, 1975.
- M. S. Taqqu. Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 50:53–83, 1979.





