The Poisson-Sigma Model

Quantum Theory on topological Surfaces

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The Wheel of Time turns, and Ages come and pass, leaving memories that become legend. Legend fades to myth, and even myth is long forgotten when the Age that gave it birth comes again. In one Age, called the Third Age by some, an Age yet to come, an Age long past, a wind rose in the Mountains of Mist. The wind was not the beginning. There are neither beginnings nor endings to the turning of the Wheel of Time. But it was a beginning.

Robert Jordan (1990)

Contents

1 Introduction

Over the last several years the POISSON-sigma model has received more and more interest. It was invented on one hand by P. SCHALLER and T. STROBL [43] as a generalization of two-dimensional gravity-YANG-MILLS systems and on the other hand by N. IKEDA [27] as a non-linear extension of gauge theories.

The POISSON-sigma model is in fact a sigma model with a POISSON manifold as its target space. The great interest is traced back to the possibility of choosing different POISSON structures on the target manifold. Actually, the POISSON-sigma model associates to various POISSON structures on finite dimensional manifolds different two-dimensional field theories [1, 25, 34, 43, 45]. This includes the following models: the topological sigma-model, non-ABELian gauge theories, in particular the two-dimensional YANG-MILLS theory, two-dimensional gravity models and the WESS-ZUMINO-WITTEN model as well.

In the language of gauge theories the POISSON-sigma model involves an open gauge algebra. In such cases the FADDEEV-POPOV method [18] of the path integral quantization fails. Even the much more powerful quantization procedure of the BRST theory [22] is not applicable in these cases, except for some special POISSON structures. The reason is that both procedures need a well-defined cohomology to construct physical variables. This is based on the nilpotency of the corresponding BRST operator, but for theories with an open gauge algebra this operator is only nilpotent modulo the equations of motion. The proper method that works in these cases and even more general situations is the BATALIN-VILKOVISKY formalism [6]. Like the BRST method it is based on an extension of the phase space, the difference is that for each field one introduces a so-called antifield obtaining a much more general structure on the phase space. The fields and antifields are in some sense canonical conjugate to each other and give rise to an odd symplectic structure on the extended phase space, for an overview see [21]. A detailed description of the application to the POISSON-sigma model can be found in [24]. The structure has a beautiful geometric interpretation, first discovered by E. WITTEN [51] and then described by A.Y. ALEXANDROV, M. KONTSEVICH, A. SCHWARZ and O. ZABORONSKY [2]. This construction enables one to obtain the extended action, which is used in the path integral quantization, from fundamental geometric ingredients, which are a nilpotent vectorfield and a symplectic structure on appropriate supermanifolds. This formalism was extended to the case of manifolds with boundaries and applied to the POISSON-sigma model by A. CATTANEO and G. FELDER [12].

An usual gauge theory is based on a LIE algebra structure for the gauge transformations. The extension of the LIE algebra to a polynomial algebra, like for instance a W-algebra and also a POISSON algebra, defines the class of non-linear gauge theories. In the case of the POISSON-sigma model the non-linearity becomes more transparent if one creates the interacting term in the classical action by the procedure invented by G. BARNICH and M. HENNEAUX [5] . That formalism, based on the antifield formalism, consists of a deformation procedure to generate consistent interactions. It is an important question in gauge theories, how one can generate, starting with a free theory, interactions for the gauge fields in such a way that the gauge invariances are taken into consideration, i.e. preserving the number of gauge symmetries. In other words, one looks for a prescription for the deformation of the free action and simultaneously for the gauge transformations. The procedure by G. BARNICH and M. HENNEAUX deforms the classical master equation which contains the whole information of the underlying gauge structure of the theory, such that the consistency of the action and the gauge transformations is ensured. Further, the formulation in the language of cohomology permits one to use the powerful tools of these structures. This procedure was applied to the YANG-MILLS [3] and the CHERN-SIMONS [5] theory. In [28] K.I. IZAWA has shown that the deformation of the two-dimensional ABELian BF-theory leads to the topological POISSON-sigma model. In [7] the formalism was extended to generate couplings between ABELian gauge fields and matter fields. For a review of the the local BRST cohomology in gauge theories see the work of G. BARNICH, F. BRANDT and M. HENNEAUX [4].

A more mathematical source of interest is the connection of the POISSON-sigma model to the problem of quantization of spaces, especially the quantization of POISSON manifolds. It was already shown by P. SCHALLER and T. STROBL [43, 44] that the DIRAC quantization of the model leads to the quantization of certain submanifolds of the POISSON manifold, the symplectic leaves, i.e. they have shown that the leaves must be integral. There the implications for more general situations were also mentioned. In the meantime A. CATTENEO and G. FELDER [11] have shown that the perturbation expansion of the path integral in the covariant gauge reproduces the KONTSEVICH formula for the deformation quantization of the algebra of functions on a POISSON manifold [35].

The connection to gravity models was used by W. KUMMER, H. LIEBL and D.V. VASSILEVICH to investigate the special case of 2d dilaton gravity in the temporal gauge, and they have calculated the generating functional using BRST methods [36]. In further work they have studied the coupling to matter fields [37].

A.C. Hirshfeld and the author have investigated a complete and general derivation of the partition function for the Poisson-sigma model for an arbitrary gauge on closed manifolds [24]. In this calculation we have reproduced the quantization condition for the symplectic leaves to be integral, now for arbitrary closed world sheets. We have shown that for a linear Poisson structure the partition function is completely computable and it is in some sense dual to the one of the Yang-Mills theory, i.e. the partition function for the Yang-Mills theory may be recovered from that of the linear Poisson-sigma model. C. Klimcik [33] has introduced a model where the target spaces are given by so-called Drinfeld doubles [16], such that the Poisson-sigma model with a Poisson-Lie group as the target space is included. In that paper he calculated the partition function for his model, which turned out to be a q-deformation of the ordinary Yang-Mills theory. In a special case his expression coincides with the Verlinde formula of conformal field theory.

The generalization to manifolds with boundary, which was already initiated by the work of A. Catteneo and G. Felder for the case where the world sheet has the topology of a two-dimensional disc, is still under progress. Recently, F. Falceto and K. Gawedzki have clarified the relation of the bordered version of the gauged WZW model with a Poisson-Lie group as the target to the topological Poisson-sigma model with the dual Poisson-Lie group as the target space [20]. In a recent paper [26] A.C. Hirshfeld and the author have generalized their original calculation of the partition function of the Poisson-sigma model to the world sheet with the topology of the disc. By introducing a glueing prescription we were able to calculate the partition function in the linear case on arbitrary oriented two-dimensional manifolds.

The thesis is structured as follows. In chapter two some required mathematical preliminaries are presented. First of all the geometry of Poisson manifolds is reviewed following the book by J.E. Marsden and T.S. Ratiu [40] and the book by I. Vaisman [46]. The third chapter is concerned with the classical theory of the Poisson-sigma model. The introduction of the model will follow the ideas of P. Schaller and T. Strobl [43] and mention the more mathematical description by A. Cattaneo and G. Felder [12]. Further on a review of the Batalin-Vilkovisky approach is presented [21]. Then the application of the formalism to the Poisson-sigma model is performed [24]. The deformation of the two-dimensional Abelian BF theory leading to the Poisson-sigma model [28] is presented. In chapter four the path integral quantization is performed. First the partition function is calculated for base manifolds without boundaries [24], then this calculation is extended to the world sheet with the topology of the disc [26]. Finally, a glueing prescription is formulated for glueing manifolds together by identifying certain boundary components such that the calculation of the partition function of the linear Poisson-sigma model on arbitrary base manifolds is possible [26]. The thesis ends with some concluding remarks and an outlook for further research.

2 Mathematical Preliminaries

In this chapter mathematical facts, relevant for the the following considerations, will be presented. First of all the main geometric properties of POISSON manifolds are reviewed. This is essential because the POISSON-sigma model is based on these structures. A special kind of POISSON structure which is linear in the coordinates of the POISSON manifold, also called POISSON-LIE structure, leads to the world of representation theory of LIE groups. This connection manifests itself in the fact that the symplectic leaves can be identified via the coadjoint orbits with the unitary irreducible representations of a LIE group, for the proofs see [40, 46, 48].

2.1 Geometry of Poisson manifolds

In [39] S. LIE showed that there exists a POISSON bracket on the dual space $\mathfrak{g}^*$ of a LIE algebra $\mathfrak{g}$ defined by

$$\{F, G\}(\mu) = \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta G}{\delta \mu}\right]\right\rangle \ , \tag{2.1}$$

with $\mu \in \mathfrak{g}^*$ and $F, G \in C^\infty(\mathfrak{g}^*)$. This bracket is not associated to a symplectic structure on $\mathfrak{g}^*$; it is an example for the more general concept of POISSON manifolds. It is closely connected to the symplectic structure on coadjoint orbits, see next section. Due to the fact that the notion of POISSON manifolds is used in many different ways in the literature, its development was rather complicated. The notion of POISSON manifolds was initiated by A. LICHNEROWICZ [38].

Definition 2.1
A POISSON bracket, respectively a POISSON structure, on a smooth manifold N is a bilinear map $\{\cdot, \cdot\}$ on the space of functions $C^\infty(N)$ on the manifold N with the following properties

(i) it yields the structure of a LIE algebra on the space of functions

$$\{F, G\} = -\{G, F\} \ \text{(skew-symmetric)}\,, \tag{2.2}$$

$$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\}=0 \ \ (\text{JACOBI identity})\ , \tag{2.3}$$

(ii) it has a natural compatibility with the usual associative product of functions, which is

$$\{H,FG\}=\{H,F\}G+F\{H,G\} \ \ (\text{LEIBNIZ rule of derivation}) \tag{2.4}$$

with $F,G,H\in C^\infty(N)$. A smooth manifold N such that $C^\infty(N)$ is equipped with a POISSON structure is called a POISSON manifold.

Proposition 2.2
Let N be a POISSON manifold. Then $\forall F,G\in C^\infty(N)$ there exists a well defined vector field X_F such that

$$\{F,G\}=X_F[G]=-X_G[F]=\mathrm{d}G(X_F)=-\mathrm{d}F(X_G)\ . \tag{2.5}$$

X_F is called a HAMILTONian vector field.

Remark: The proposition follows from the fact that the POISSON bracket is a derivation.

From equation (2.5) it follows that the bracket is determined by a skew-symmetric bilinear form on T^*N.

Proposition 2.3
There exists naturally a C^∞-tensor field $P\in\wedge^2 TN$ such that

$$\{F,G\}=P(\mathrm{d}F,\mathrm{d}G)=P^{ij}\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial x^j}\ , \tag{2.6}$$

where the (x^i) are local coordinates on N.

Definition 2.4
P is called the POISSON bivector of the POISSON manifold $(N,\{,\})$.

The skew symmetry and the LEIBNIZ rule are fulfilled by the definition of the POISSON tensor. The remaining condition, which P still has to satisfy, is the JACOBI identity. In terms of the bivector this is

$$\sum_{\text{cyclic}(ijk)}(\partial_l P^{ij})P^{kl}=0\ . \tag{2.7}$$

In an invariant notation this condition is the vanishing of the SCHOUTEN-NIJENHUIS bracket [42] of P with itself, $[P,P]_{SN}=0$.

The bivector P has an associated homomorphism

$$\sharp : T^*N \to TM \ , \tag{2.8}$$

where $\sharp\alpha \equiv \alpha^\sharp$ is defined by

$$\beta(\alpha^\sharp) = P(\alpha,\beta) \tag{2.9}$$

for $\alpha,\beta \in T^*N$. P is in general degenerate in which case it does not induce a symplectic structure on N, and the map $\sharp : T^*N \to TN$ induced by P, which maps a 1-form $\alpha_i \mathrm{d}X^i$ on N to the vector field $\alpha_i P^{ij}\partial_j$, is not surjective. For degenerate P there are functions f on N whose HAMILTONian vector fields $X_f = f_{,i}P^{ij}\partial_j$ vanish, where $f_{,i} := \partial f/\partial x^i$ denotes the derivation with respect to the local coordinates of the POISSON manifold. These functions are called CASIMIR functions.

Symplectic foliation The main geometric property of POISSON manifolds is that they are disjoint unions of symplectic manifolds which are POISSON submanifolds.

Definition 2.5
A mapping $\phi : (N_1,P_1) \to (N_2,P_2)$ between two POISSON manifolds is called a POISSON mapping or POISSON homomorphism if $\forall F,G \in C^\infty(N_1)$ one has

$$\{F\circ\phi, G\circ\phi\}_{N_2} = \{F,G\}_{N_1}\circ\phi \ . \tag{2.10}$$

Definition 2.6
Let $S \subset N$ be a submanifold of the POISSON manifold N. If the inclusion $i : S \to N$ is a POISSON mapping then S is called a POISSON submanifold.

Definition 2.7
A set of linear subspaces $S(N) = \{S_{x_0}(N)\}$ of the tangent spaces $T_{x_0}N$ is called a (general) distribution. The distribution defined by

$$S_{x_0}(N) = \{v \in T_{x_0}N \mid \exists F \in C^\infty(N), X_F(x_0) = v\} \ \ (x_0 \in N) \tag{2.11}$$

is called the characteristic distribution.

Definition 2.8
Let $E \subset TM$ be a sub-vectorbundle of the tangentbundle of a differentiable manifold M. Then E is called (completely) integrable, if $\forall m \in M$ exists a local submanifold M' of M, such that $TM' = TM_{|_{M'}}$.

The local submanifolds can be continued to connected maximal integral manifolds which are uniquely determined and regular immersed submanifolds of M. For the case of POISSON manifolds this yields the following theorem.

Theorem 2.9
The characteristic distribution $S(N)$ of the POISSON manifold (N, P) is completely integrable, and the POISSON structure induces symplectic structures on the leaves of $S(N)$.

Definition 2.10
The leaves of $S(N)$ are called symplectic leaves L of the POISSON manifold N, and $S(N)$ is said to be the symplectic foliation.

Theorem 2.11
Let N^n be a differential manifold and $S(N)$ a general foliation such that

(i) every leaf S of $S(N)$ is endowed with a symplectic structure ω_S,

(ii) if $F \in C^\infty(N)$, the vector field X_F defined by $X_F(x)$= the Hamiltonian vector field of $F_{|_{S(x)}}$ on $(S(x), \omega_{S(x)})$ at x is a differentiable vector field on N.

Then N has a unique POISSON structure whose symplectic foliation is $S(N)$.

In other words, a POISSON structure can be defined by its symplectic foliation. A POISSON manifold is a disjoint union of symplectic manifolds, which are POISSON submanifolds, the symplectic leaves.

Definition 2.12
If the symplectic foliation of (N, P) is regular, i.e. the rank of P is constant, the POISSON manifold is called regular.

Proposition 2.13
Let (N, P) be a POISSON manifold, $L \subset N$ a symplectic leaf and C a CASIMIR function of the POISSON structure P. Then C is constant on L.

On a POISSON manifold coordinates can be chosen in such a way that the POISSON bivector is canonical, this is a generalization of the theorem of G. DARBOUX for symplectic manifolds. The local structure of a POISSON manifold N at a point x_0 is described by the *splitting theorem* due to A.WEINSTEIN [48].

Theorem 2.14
Let x_0 be a point of the POISSON manifold (N^n, P) and $\operatorname{rank}(x_0) = 2h$. Then there exists a neighborhood $U(x_0)$ in N and an isomorphism $\phi = \phi_S \times \phi_M : U \to S \times M$ with S being a $2h$-dimensional symplectic and N a POISSON manifold. The rank of N in $\phi_M(x_0)$ vanishes. S and N are unique modulo local isomorphisms. Further, there exist coordinates (p_a, q^a, y^σ) on (N^n, P) such that the following commutation relations are satisfied

$$\{q^a, q^b\} = \{p_a, p_b\} = \{q^a, y^\sigma\} = \{p_a, y^\sigma\} = 0 \ , \tag{2.12}$$

$$\{q^a, p_b\} = \delta^a_b \ . \tag{2.13}$$

These coordinates are called canonical coordinates for the POISSON manifold N at x_0.

The theorem states that for a regular POISSON manifold there exists canonical coordinates, also called CASIMIR-DARBOUX coordinates, on the POISSON manifold N. Let $\{C^I\}$ be a maximal set of independent CASIMIR functions. Then $C^I(X) = \text{const.} = C^I(X_0)$ defines a level surface through X_0 whose connected components may be identified with the symplectic leaves. According to DARBOUX's theorem there are local coordinates X^α on S such that the symplectic form Ω_S is given by

$$\Omega_S = \mathrm{d}X^1 \wedge \mathrm{d}X^2 + \mathrm{d}X^3 \wedge \mathrm{d}X^4 + \dots \ . \tag{2.14}$$

Together with the CASIMIR functions one then has a coordinate system $\{X^I, X^\alpha\}$ on N with $P^{IJ} = P^{I\alpha} = 0$ and $P^{\alpha\beta} = \text{const.}$

Structures on $\mathbb{R}^n$ The most general POISSON bivector on $\mathbb{R}^n = \{(x_i)\}$ is of the form

$$P = \frac{1}{2} P^{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \ , \quad [P, P]_{SN} = 0 \, . \tag{2.15}$$

In particular, any constant coefficient P_{ij} defines a POISSON structure on $\mathbb{R}^n$ and they are called constant POISSON structures.

The linear Poisson structure These structures are of great interest because of their connection to the representation theory of LIE groups. More precisely, the symplectic leaves of a linear POISSON structure can be identified with the coadjoint orbits of a connected LIE group G, see next section.

Definition 2.15
A linear POISSON structure on a manifold N is defined by

$$P^{ij} = c_k^{ij} x_k \quad (c_k^{ij} = -c_k^{ji}) \,. \tag{2.16}$$

These structures are also called LIE-POISSON structures.

The JACOBI identity for the bivector then becomes

$$c_h^{ij} c_l^{hk} + c_h^{ki} c_l^{hj} + c_h^{jk} c_l^{hi} = 0 \,. \tag{2.17}$$

It follows that the coefficients c_k^{ij} define the structure of an n-dimensional LIE algebra $\mathfrak{g}$ on the dual space of the *linear* POISSON manifold. This structure coincides with the structure found by S. LIE (2.1).

2.2 The orbit method for compact groups

The orbit method was introduced by A. KIRILLOV in order to solve some open problems in representation theory of LIE groups [30, 31, 32]. First one seeks a description of the unitary dual of the LIE group, which is defined by $G^\wedge := \{\text{unitary irreducible representations/equivalence}\}$. It is the set of isomorphism classes of unitary irreducible representations which is a topological space with rich structure. The answer provided by the orbit method is then as follows. Denote by $\mathfrak{g}^*$ the dual space to the LIE algebra $\mathfrak{g}$ of the LIE group G. Let $\mathfrak{g}^*/G$ denote the quotient of $\mathfrak{g}^*$ by the action of G, the so-called coadjoint action. Thus it is the space of orbits of the action of G on $\mathfrak{g}^*$, $G^\wedge = \mathfrak{g}^*/G$. The second question is then how to decompose a given representation T of G into irreducible representations. The third problem considered by A. KIRILLOV is the computation of the character of a given unitary irreducible representation $\mathcal{T}$. In fact there are two kinds of characters for a LIE group. One is a distribution (generalized function) on G and the other is the so-called infinitesimal character. The orbit method gives a prescription for the calculation of the characters for different types of LIE groups, in particular for compact groups. The character is a modified FOURIER transform of the symplectic form on a certain orbit corresponding to an unitary irreducible representation. The concern of this section is to present some results of the orbit method which will be needed in the following chapters.

2.2.1 Geometry of coadjoint orbits

The coadjoint representation Let G be a compact connected LIE group and $\mathfrak{g} = \mathrm{Lie}(G)$ be the tangent space $T_e(G)$ to G at the unit point e. The group G acts on itself by *inner automorphism* $g \mapsto hgh^{-1}$. The point e is a fixed point, and the derived map is $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$. The mapping $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ is called the *adjoint action* and the homomorphism $g \mapsto \mathrm{Ad}(g)$ is the *adjoint representation* of G.

Consider the *dual* space of $\mathfrak{g}$ denoted by $\mathfrak{g}^*$. In the matrix case there always exists a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ which is invariant under the inner automorphisms.

Definition 2.16
The coadjoint representation of a LIE group is the homomorphism

$$K : g \mapsto (\mathrm{Ad}_{g^{-1}})^* \,. \tag{2.18}$$

The infinitesimal version of the coadjoint action is

$$\langle K_*(X)[F], Y \rangle = \langle F, -\mathrm{ad}(X)[Y] \rangle = \langle F, [X, Y] \rangle \,, \tag{2.19}$$

where K_* denotes the differential of the coadjoint action K and $\mathrm{ad}(Y)[X] = \mathrm{Ad}_*(Y)[X] = [Y, X]$.

Definition 2.17
The stabilisator (isotropy group) for an arbitrary action Φ_g for a point $x \in M$ is

$$\mathrm{Stab}(x) := \{ g \in G \mid \Phi_g(x) = x \} \subset G \,. \tag{2.20}$$

The corresponding algebra, the stabilisator algebra, is

$$\mathrm{stab}(x) := \{ X \in \mathfrak{g} \mid X_M(x) = 0 \} \,, \tag{2.21}$$

where X_M is a vector field on M.

Note that $\mathrm{Stab}(x)$ is a LIE subgroup of G.

Definition 2.18
An orbit $\mathcal{O}$ of an action Φ_g on the manifold M is given by

$$\mathcal{O}(x) := \{ \Phi_g(x) \mid g \in G \} \subset M \,. \tag{2.22}$$

Definition 2.19
The orbit associated to the coadjoint action is called coadjoint orbit Ω.

An action Φ_g defines an equivalence relation for the orbits given by $x \sim y$; if $\exists g \in G$ such that $g \cdot x = y$, then $x \in \mathcal{O}(y)$ and $y \in \mathcal{O}(x)$. Then M/G is the set of equivalence classes, also called the set of orbits or the orbit space. For compact LIE groups all orbits are closed, embedded submanifolds.

The symplectic form on the coadjoint orbits All coadjoint orbits Ω possess a G-invariant symplectic structure, i.e. on each orbit $\Omega \subset \mathfrak{g}^*$ there exists canonically a closed non-degenerate G-invariant differential 2-form ω. An invariant differential form on a homogenous manifold is uniquely determined by its value at a single point. This value should be invariant with respect to the action of the stabilizer group. So it is sufficient to calculate the value of ω at an arbitrary point $F \in \Omega$.

It exists naturally a skew symmetric bilinear form on $\mathfrak{g}$ with the property that its kernel is exactly the stabilizer algebra, more precisely the kernel of the corresponding mapping of the differential K_* corresponds to the stabilizer algebra of the coadjoint action:

$$B_F(X,Y) = \langle F, [X,Y] \rangle \,, \tag{2.23}$$

$$\begin{aligned} \mathrm{Ker}[B_F] &= \{X \in \mathfrak{g} \mid B_F(X,Y) = 0 \;\; \forall\, Y \in \mathfrak{g}\} \\ &= \{X \in \mathfrak{g} \mid \langle K_*(X)[F], Y \rangle = 0 \;\; \forall\, Y \in \mathfrak{g}\} = \mathrm{stab}(F) \,. \end{aligned} \tag{2.24}$$

Definition 2.20
The value of the form ω_F at the point F is given by

$$\omega_F(K_*(X)[F], K_*(Y)[F]) := B_F(X,Y) \,. \tag{2.25}$$

Proposition 2.21
The 2-form ω defined by equation (2.25) defines a symplectic structure on the coadjoint orbit, i.e. ω is a symplectic form, that is a G-invariant, non-degenerate closed 2-form.

Remark: The non-degenerency and the G-invariance of ω follow directly from the definition. The fact that ω is a closed form follows from the ensuing calculation.

For $F \in \Omega$ let p_F be the submersion $G \to \Omega : g \mapsto K(g)[F]$. Then $p_F^*(\omega)$ is a left invariant 2-form on G. Define a left invariant 1-form to be $\theta_F := \langle F, g^{-1}\mathrm{d}g\rangle$. Then $p_F^*(\omega)$ is the exterior derivative of θ_F. Since p_F is a submersion p_F^* is injective.

$$p_F^*(\mathrm{d}\omega) = \mathrm{d}(p_F^*(\omega)) = \mathrm{d}^2\theta_F = 0 \tag{2.26}$$

Hence, ω is closed and defines a symplectic structure on the orbit Ω.

Definition 2.22
A coadjoint orbit is called integral, iff for any 2-cycle C in Ω and the symplectic structure ω

$$\int_C \omega \quad \in \mathbb{Z}\ . \tag{2.27}$$

Another approach to see that the coadjoint orbits are symplectic manifolds is based on the theory of Poisson manifolds. The symplectic structure of the coadjoint orbits coincides with that on the symplectic leaves of the Lie-Poisson structure.

Proposition 2.23
The Lie-Poisson structure and the symplectic structure on a coadjoint orbit are compatible in the following way. For $F, G : \mathfrak{g}^* \to \mathbb{R}$ and an orbit $\Omega \subset \mathfrak{g}^*$ one has

$$\{F, G\}_{|\Omega} = \{F_{|\Omega}, G_{|\Omega}\}\ , \tag{2.28}$$

where $\{\cdot,\cdot\}$ is the Lie-Poisson bracket and $\{\cdot,\cdot\}_{|\Omega}$ is the bracket associated to the symplectic structure on the orbit Ω.

In other words, the symplectic leaves of the Lie-Poisson structure of $\mathfrak{g}^*$ are the orbits of the coadjoint action of any connected Lie group G whose Lie algebra is $\mathfrak{g}$. Since the coadjoint orbits are symplectic manifolds it follows that

Corollary 2.24
The coadjoint orbits of a finite dimensional Lie group are manifolds of even dimension.

Corollary 2.25
(i) A function $C \in C^\infty(\mathfrak{g}^*)$ is a Casimir function iff $\frac{\delta C}{\delta \mu} \in \mathfrak{g}^*$ for all $\mu \in \mathfrak{g}^*$.
(ii) If $C \in C^\infty(\mathfrak{g}^*)$ is K-invariant, i.e. is constant on the orbits, then C is a Casimir function. Additionally, if all coadjoint orbits are connected then the inverse is also true.

Compact groups If the Lie group is restricted to be compact, the corresponding Lie algebra $\mathfrak{g}$ admits a non-degenerate G-invariant bilinear form. It is possible to identify $\mathfrak{g}^*$ and $\mathfrak{g}$ and hence the coadjoint and the adjoint representation as well. There exist only a finite number of different types of coadjoint orbits as homogeneous manifolds for a given G, i.e. there are finitely many subgroups G_i such that any orbit is isomorphic to $X_i = G/G_i$.

Definition 2.26
Let T denote the maximal Abelian subgroup, the torus of the group. Then

$X := G/T$ is called a full flag manifold and

$X_i := G/G_i$ is called a degenerate flag manifold.

The flag manifolds X_i have a rich geometric structure. As homogenous spaces of a compact Lie group, they admit a G-invariant Riemannian metric. Being coadjoint orbits they have a canonical G-invariant symplectic structure. And they can be endowed with a complex structure.

Let $\Omega \subset \mathfrak{g}^*$ be a coadjoint orbit. The canonical symplectic form defines a cohomology class $[\omega] \in H^2(X;\mathbb{R})$. An orbit is called integral if $[\omega] \in H^2(X;\mathbb{Z}) \subset H^2(X;\mathbb{R})$, which is the same definition as 2.22. It follows that the number of *integrality conditions* is equal to the second Betti number of the orbit. For compact groups the integral orbits form a discrete set.

Identify $\mathfrak{g}^*$ and $\mathfrak{g}$ and also the complexified algebras, such that $h^* \cong h$, h being the Cartan subalgebra. The weight lattice $W \subset h^*$ corresponds to a lattice in $it^* \subset i\mathfrak{g}^* \cong i\mathfrak{g}$ where t is the complement in the Cartan decomposition. The intersection of Ω with t^* is a finite set. Denote the orbit passing through the point $i\lambda \in t^*$ by Ω_λ.

Proposition 2.27
i) The orbit Ω_λ is integral iff $\lambda \in W$ and
ii) The dimension of Ω_λ is equal to the number of roots non-orthogonal to λ.

In particular, all orbits of maximal dimension are isomorphic to the full flag manifold.

The important theorem which connects the theory of the coadjoint orbits with the theory of representations of a Lie group is

Theorem 2.28
All irreducible representations of a compact, connected and simply connected Lie group G correspond to integral coadjoint orbits of maximal dimension.

The other way round, i.e. starting with a coadjoint orbit and determining the corresponding representations, is a bit more complicated. For a given coadjoint orbit Ω there might be several irreducible unitary representations or also possible there might be no representation corresponding to it at all. The topology of the orbit, more precisely the first and second BETTI numbers determine which is the case. As already mentioned above there are $b_2(\Omega) = \dim H^2(\Omega, \mathbb{Z})$ integrality conditions. The first BETTI number $b_1(\Omega) = \dim H^1(\Omega, \mathbb{Z})$ gives rise to additional cyclic parameters describing the irreducible representations.

2.2.2 The universal formula for characters

One of the main problems of the theory of group representations is to obtain explicit formulas for generalized characters of irreducible representations. The method of orbits indicates an approach to solving this problem for all LIE groups. It was extremely successful for the case of nilpotent LIE groups. With a little modification it also provides the answer for compact groups. The idea consists of considering generalized functions on a LIE group which are defined on a coadjoint orbit in $\mathfrak{g}^*$.

Theorem 2.29

The character of a representation $\mathcal{T}$ associated to the integral coadjoint orbit Ω of maximal dimension is given by

$$\chi_\Omega(\exp X) = \frac{1}{j(X)} \int\limits_\Omega \exp[2\pi i \langle F, X\rangle] \frac{\omega^r}{r!} \tag{2.29}$$

with $r = \frac{\dim(G) - \mathrm{rank}(G)}{2} = \frac{\dim(\Omega)}{2}$ and $j(X) = [\det \frac{\sinh(\mathrm{ad}X/2)}{\mathrm{ad}X/2}]^{1/2}$ the JACOBI determinant of the exponential map. In particular, the dimension of the representation is equal to the symplectic volume of the corresponding coadjoint orbit

$$\dim(\lambda) = \chi_\Omega(1) = \int\limits_\Omega \frac{\omega^r}{r!} = \mathrm{Vol}(\Omega)\ . \tag{2.30}$$

Remark: The generalized character of a representation $\mathcal{T}$ of G is the distribution $\chi_\mathcal{T}$ on G. Let $C^\infty(G)$ denote the space of smooth compactly supported functions on G. For $\phi \in C^\infty(G)$ one defines $T_\phi := \int_G \phi(g) T(g) \mathrm{d}g$. Then T_ϕ has a trace and the character is the distribution

$$\chi : \phi \to \mathrm{Tr}\,[T_\phi]\ . \tag{2.31}$$

For nilpotent groups the spaces of functions in G, $\mathfrak{g}$ and $\mathfrak{g}^*$ are in natural correspondence. While the group G and the algebra $\mathfrak{g}$ are connected by the exponential map, $\mathfrak{g}$ and $\mathfrak{g}^*$ are related by the FOURIER transform. The orbit $\Omega \subset \mathfrak{g}^*$ canonically determines a volume form. If one transfers the measure to G by the FOURIER transform and the exponential map, one obtains precisely the distribution $\chi = \chi_\Omega$

$$\chi_\Omega(\exp X) = \int_\Omega \exp[2\pi i \langle F, X\rangle] \frac{\omega^r}{r!} . \tag{2.32}$$

For more general groups one has to modify the formula due to the fact that the exponential map has now a non-trivial JACOBI determinant what leads to (2.29). For a rigorous proof see [30].

The symmetrization map Let $P(\mathfrak{g}^*)$ be the naturally graded algebra of polynomial functions on $\mathfrak{g}^*$ which can be identified with the symmetric algebra $S(\mathfrak{g})$ via the map $\mathfrak{g} \ni X \mapsto 2\pi i \langle \cdot, X\rangle \in P(\mathfrak{g}^*)$.

Definition 2.30
The map $\mathrm{sym} : S(\mathfrak{g}) \to U(\mathfrak{g})$, with $U(\mathfrak{g})$ being the universal enveloping algebra of $\mathfrak{g}$, defined by

$$\mathrm{sym}(X_1 X_2 \cdots X_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) X_{\sigma(1)} X_{\sigma(2)} \cdots X_{\sigma(k)} \tag{2.33}$$

for $X_i \ (0 < i < k)$ is called the symmetrization map.

In particular, for any $X \in \mathfrak{g}$ one has $\mathrm{sym}(X^n) = X^n \in U(\mathfrak{g})$. This map is G-covariant and bijective. By the identification of $S(\mathfrak{g})$ and $P(\mathfrak{g}^*)$ one gets a G-covariant linear homomorphism $U(\mathfrak{g}) \to P(\mathfrak{g}^*) : A \mapsto p_A$ which maps the center $Z(\mathfrak{g})$ of the universal enveloping algebra onto $Y(\mathfrak{g}) = P(\mathfrak{g}^*)^G$, the algebra of G-invariant polynomials on $\mathfrak{g}^*$. This is an algebra homomorphism for nilpotent LIE algebras. Including the case of compact groups it requires a modification which leads to the following definition.

Definition 2.31
For $A \in U(\mathfrak{g})$ define $p_A \in P(\mathfrak{g}^*)$ by

$$A = \mathrm{sym}(j(X) p_A) , \tag{2.34}$$

where $j(X) = [\det \frac{\sinh(\mathrm{ad}X/2)}{\mathrm{ad}X/2}]^{1/2}$.

The symmetrization map for $SU(2)$ As an example for the symmetrization map the case in which the LIE group is $SU(2)$ is presented. Let $G = SU(2)$ and let $\mathfrak{g} = su(2)$ have the standard basis X_1, X_2, X_3 with the commutation relations

$$[X_1, X_2] = X_3 \ , \quad [X_2, X_3] = X_1 \ , \quad [X_3, X_1] = X_2 \ . \tag{2.35}$$

The same elements considered on $\mathfrak{g}^*$ are denoted by (x_1, x_2, x_3), and may further be considered to be the dual coordinates (α, β, γ) on $\mathfrak{g}$. Then define

$$r := \sqrt{x_1^2 + x_2^2 + x_3^2} \quad , \quad \rho := \sqrt{\alpha^2 + \beta^2 + \gamma^2} \ . \tag{2.36}$$

The symmetrization map for r^2 is then

$$C := X_1^2 + X_2^2 + X_3^2 = \mathrm{sym}(r^2) \ . \tag{2.37}$$

The general expression for the symmetrization is more complicated.

Proposition 2.32

The symmetrization map is explicitly given by

$$\mathrm{sym}(r^{2n}) = \frac{(-1)^{n-1}}{4^n} \sum_{k=0}^{n} \binom{2n+1}{2k} B_{2k}(4^k - 2)(1 - 4C)^{n-k} \ , \tag{2.38}$$

where B_{2k} are the BERNOULLI numbers.

Convolution on coadjoint orbits An interesting aspect of the theory of coadjoint orbits is the convolution, which is based on the following theorem [49, 15].

Theorem 2.33

For Ad-G-invariant distributions the convolution operations on G and $\mathfrak{g}$ are related by the map $\Phi : C^\infty(\mathfrak{g}) \to C^\infty(G)$ defined by

$$\langle \Phi(\nu), f \rangle := \langle \nu, j \cdot (f \circ \exp) \rangle \ , \tag{2.39}$$

$$\Phi(\mu) *_G \Phi(\nu) = \Phi(\mu *_{\mathfrak{g}} \nu) \ . \tag{2.40}$$

Definition 2.34

The map $\Phi : C^\infty(\mathfrak{g}) \to C^\infty(G)$ is called the wrapping map.

Remark: The essential tool for the proof of theorem 2.33 is KIROLLOV's character formula. The usual classification of $G^\wedge$ in terms of integral coadjoint orbits can be recovered from theorem 2.33, along the way KIRILLOV's character formula also falls out [50]. The wrapping map *straightens* the group convolution, i.e. the convolution in $\mathfrak{g}$ is now ABELian. Another consequence of theorem 2.33 is the following.

Theorem 2.35
Let G be a compact LIE group , $X_1, X_2 \in \mathfrak{g}$; $\mathcal{O}_i = G \cdot X_i$ the adjoint orbit through X_i and $C_i = G \cdot \exp X_i$ the conjugacy class through $\exp X_i$, $i = 1, 2$. Then

$$C_1 \cdot C_2 \subset \exp(\Omega_1 + \Omega_2) \ . \tag{2.41}$$

3 The Classical Theory

This chapter is concerned with the classical aspects of the Poisson-sigma model. In the first section the model is introduced following the work of P. Schaller and T. Strobl [43]. A remarkable feature of the model is the inclusion of certain two-dimensional field theories, the two cases of the topological sigma model and the Yang-Mills theory are considered. The second section deals with the gauge structure of the model in view of quantization. The antifield formalism by I.A. Batalin and G.A. Vilkoviski [6] is used to obtain the so-called extended and gauge-fixed action for the Poisson-sigma model [24]. As already mentioned in the introduction the Poisson-sigma model can be interpreted as a non-linear gauge theory [27]. The non-linearity becomes clear if one performs the deformation theory for generating consistent interactions by G. Barnich and M. Henneaux [5], which is based on the antifield formalism. In section three it will be shown that the deformation of the the two-dimensional Abelian BF theory yields the Poisson-sigma model [28]. For further aspects of the Hamiltonian formalism see [43]. A. Cattaneo and G. Felder have shown in [13] that the classical phase space is the space of Hamiltonian foliation and has a natural groupoid structure.

3.1 The Poisson-sigma model

The Poisson-sigma model is a semi-topological field theory on a two-dimensional world sheet Σ with local coordinates u^μ ($\mu = 1, 2$). The theory involves a set X^i of bosonic scalar fields which can be interpreted as a set of maps $X^i : \Sigma \to N$ such that for $u^\mu \in \Sigma$ the $X^i(u^\mu)$ can be seen as coordinates of a Poisson manifold which is the target space of the theory. This space is equipped with a Poisson bivector P^{ij} instead of a metric. Hence, the fields $X^i(u)$ are not sufficient to define a sigma model due the fact that the Poisson bivector has only contravariant indices which cannot be contracted with $\mathrm{d}X^i$. In addition one has a 1-form A_i on the world sheet taking values in T^*N, so $A_i = A_{i\mu}\mathrm{d}u^\mu$ is a 1-form on the world sheet Σ

and i is a covariant index in the target space N. The Poisson-sigma model is characterized by the action [43]

$$\mathcal{S}_0[X,A] = \int_\Sigma \big[A_i \wedge \mathrm{d}X^i + \frac{1}{2}\, P^{ij}(X) A_i \wedge A_j + \mathcal{C}(X)\big], \tag{3.1}$$

with $\mathcal{C}(X) = \mu C(X)$, where μ is the volume form on Σ and $C(X)$ is a Casimir function of the Poisson structure given by the bivector $P = P^{ij}(X)\partial_i \wedge \partial_j$.

The model possesses a gauge invariance given by

$$\delta X^i = P^{ij}(X)\varepsilon_j\ , \qquad\qquad \delta A_i = D_i^j \varepsilon_j, \tag{3.2}$$

where $D_i^j = \delta_i^j \mathrm{d} + P^{kj}{}_{,i} A_k$. The important ingredient for the invariance of the model with respect to these transformations is of course the Jacobi identity for the Poisson bivector. For $C \equiv 0$ the model can be seen as a topological field theory, i.e. it is covariant with respect to diffeomorphisms of the target manifold, the Poisson manifold.

In [12] the topological model was considered from a more mathematical point of view, namely as an action functional on the space of vector bundle morphisms $\hat{X} : T\Sigma \to T^*N$ from the tangent bundle of the two-dimensional world sheet Σ to the cotangent bundle of the Poisson manifold. Such a map is given by a base map $X : \Sigma \to N$ and a section A of $Hom(T\Sigma, X^*(T\Sigma))$, for $U \in \Sigma,\ v \in T_u\Sigma,\ \hat{X} = (X(u), A(u)v)$. Denote the pairing between the cotangent and tangent space at a point of Σ by $\langle \cdot, \cdot \rangle$. If X is a map from Σ to N then this pairing induces a pairing between the differential forms on Σ with values in $X^*T\Sigma$. It is defined as the pairing of the values and the exterior product of differential forms, and takes values in the differential forms on Σ. The action functional is

$$\mathcal{S}[X,A] = \int_\Sigma \langle A, \mathrm{d}X\rangle + \frac{1}{2}\langle A, (P \circ X)A\rangle\ , \tag{3.3}$$

where A and $\mathrm{d}X$ are viewed as 1-forms on Σ with values in the pull-back of the (co)tangent bundle and $P(X)$ is viewed as a linear map $T_u^*\Sigma \to T_u\Sigma$. In this work the more physical notation (3.1) is preferred, because it is sufficient for the treatment of the path integral quantization.

Varying the action with respect to the fields yields the equations of motion

$$D_i^j A_j + \frac{\partial \mathcal{C}(X)}{\partial X^i} = 0\ , \qquad\qquad \mathrm{d}X^i + P^{ij}A_j = DX^i = 0. \tag{3.4}$$

The gauge algebra is given by

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]X^i = P^{ji}(P^{mn}{}_{,j}\ \varepsilon_{1n}\varepsilon_{2m})\ , \tag{3.5}$$

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]A_i = D_i^j(P^{mn}{}_{,j}\varepsilon_{1n}\varepsilon_{2m}) - DX^j P^{mn}{}_{,ji}\,\varepsilon_{1n}\varepsilon_{2m}\;. \tag{3.6}$$

Note that in contrast to usual gauge theories the gauge algebra is just closed modulo the equations of motion. This shows the non-linearity of the model.

3.1.1 Including models

An interesting aspect of this model is that it associates to certain Poisson structures on a finite-dimensional manifold two-dimensional field theories. First consider the case in which the Poisson structure P gives rise to a symplectic 2-form on N, i.e. it has an inverse Ω. Then it is possible to eliminate the gauge fields A_i by means of the equations of motion and the resulting action is

$$S_{top} = \int_\Sigma \Omega_{ij} \mathrm{d}X^i \wedge \mathrm{d}X^j\;, \tag{3.7}$$

which evidently is the action of E. Witten's topological sigma model in the approach of L. Baulieu and I.M. Singer [52].

Secondly, choose a linear Poisson structure $P^{ij} = c_k^{ij} X^k$ on $N = \mathbb{R}^3$. The coefficients c_k^{ij} then define a Lie algebra on the dual space which in turn can be identified with N. Now there are two different Casimir functions. The trivial one, $C = 0$, leads to the two-dimensional BF theory. On the other hand if one chooses the quadratic Casimir function $C = X^i X^i$ one has

$$S_{YM} = \int_\Sigma F^i \wedge *F_i\;, \tag{3.8}$$

the two-dimensional Yang-Mills theory. The Poisson-sigma model also covers the gauged Wess-Zumino-Witten model [1]. The main motivation for introducing this model was the interest in including the theory of gravity into a Yang-Mills theory at least in two dimensions. Actually, if one chooses a non-linear Poisson structure the resulting model is a gravity theory with a dilaton field. This line of the model was investigated in detail by T. Klösch and T. Strobl [34].

3.2 The extended action for the Poisson-sigma model

The main idea of gauge theories is the appearance of gauge invariances in the sense that a solution of the equations of motion, which can be achieved by a gauge transformation from another solution, are physically the same. This concept has to be taken into account in the quantization process of the gauge theory. The first approach, invented by L.D. FADDEEV and V.N. POPOV, was the introduction of so-called ghosts and antighosts to take care of the gauge freedom due to the transformations [18]. Further investigations, especially the theory of graded differential algebras, lead to a systematic explanation of the gauge structure, the BRST theory [22]. The main point was the possibility of defining a nilpotent BRST operator leading to a well-defined cohomology of the physical variables. It turned out that the zeroth cohomology is isomorphic to the space of observables of the original theory. If the gauge algebra is not closed this operator is not nilpotent anymore and the BRST theory is not applicable. The next extension was the approach of I.A. BATALIN and G.A. VILKOVISKY [6], allowing also the treatment of gauge theories with an open gauge algebra. The idea is that one extends the action of the theory in such a way that the gauge freedom is under control and this extended action can be used in the path integral quantization.

In this section a review of the BATALIN-VILKOVISKY theory and the application of the formalism to the POISSON-sigma model [24] are presented. In the last part of the section a short review of the deformation approach is given and the application to the two-dimensional BF theory is shown, leading to the POISSON-sigma model [28].

3.2.1 The Batalin-Vilkovisky theory

The structure equations of gauge theories The BATALIN-VILKOVISKY formalism has a beautiful geometric interpretation, first discovered by E. WITTEN [51], and recently described in the paper of M. ALEXANDROV, M. KONTSEVICH, A. SCHWARZ and O. ZABORONSKY [2]. The review below follows the article by J. GOMIS, J. PARIS and S. STUART [21].

Consider a system whose dynamics are governed by a classical action $\mathcal{S}_0[\phi^i]$ which depends on the fields $\phi^i(x),\ \ i = 1, \ldots, n$. In the following the compact notation is used in which the multi-index i may denote the various fields involved, the discrete indices on which they may depend, and the dependence on the spacetime variables as well. The generalized summation convention then means that a repeated index may denote not only a sum over discrete variables, but also integration over the spacetime variables. $\epsilon_i = \epsilon(\phi^i)$ will denote the GRASSMAN parity of the fields ϕ^i. Fields with $\epsilon_i = 0$ are called bosonic, fields with $\epsilon_i = 1$ fermionic. The graded commutation rule is

$$\phi^i(x)\phi^j(y) = (-1)^{\epsilon_i\epsilon_j}\phi^j(y)\phi^i(x). \tag{3.9}$$

For a gauge theory the action is invariant under a set of m gauge transformations with infinitesimal form

$$\delta\phi^i = R^i_\alpha\,\varepsilon^\alpha, \quad \alpha = 1 \text{ or } 2 \text{ or} \ldots m. \tag{3.10}$$

This is a compact notation for

$$\begin{aligned}\delta\phi^i(x) &= (R^i_\alpha(\phi)\varepsilon^\alpha)(x)\\ &= \sum_\alpha \int dy\, R^i_\alpha(x,y)\,\varepsilon^\alpha(y).\end{aligned}$$

The $\varepsilon^\alpha(x)$ are the infinitesimal gauge parameters and the $R^i_\alpha(\phi)$ the generators of the gauge transformations. When $\epsilon_\alpha = \epsilon(\varepsilon^\alpha) = 0$ one has an ordinary symmetry, when $\epsilon_\alpha = 1$ a supersymmetry. The GRASSMAN parity of R^i_α is $\epsilon(R^i_\alpha) = \epsilon_i + \epsilon_\alpha \pmod 2$. When the gauge generators are independent the theory is said to be *irreducible*, otherwise it is reducible. As will be shown the POISSON-sigma model is an irreducible theory, so it will be sufficient to consider the irreducible case only.

A subscript index after a comma denotes the right derivative with respect to the corresponding field, and in general when a derivative is indicated it is to be understood as a right derivative unless specifically noted to be otherwise. The field equations may then be written as

$$\mathcal{S}_{0,i} = 0. \tag{3.11}$$

The classical solutions ϕ_0 are determined by $\mathcal{S}_{0,i}|_{\phi_0} = 0$. The NOETHER identities are

$$\mathcal{S}_{0,i}\,R^i_\alpha = 0. \tag{3.12}$$

The general solution to the NOETHER identity $\mathcal{S}_{0,i}\,\lambda^i = 0$ is

$$\lambda^i = R^i_\alpha\,T^\alpha + \mathcal{S}_{0,j}\,E^{ji}. \tag{3.13}$$

The commutator of two gauge transformations is

$$[\delta_1,\delta_2]\phi^i = (R^i_{\alpha,j}R^j_\beta - (-1)^{\epsilon_\alpha\epsilon_\beta}R^i_{\beta,j}R^j_\alpha)\varepsilon^\beta_1\varepsilon^\alpha_2. \tag{3.14}$$

Since this commutator is a symmetry of the action it satisfies the NOETHER identity

$$\mathcal{S}_{0,i}(R^i_{\alpha,j}R^j_\beta - (-1)^{\epsilon_\alpha\epsilon_\beta}R^i_{\beta,j}R^j_\alpha) = 0, \tag{3.15}$$

which by equation (3.13) implies that

$$R^i_{\alpha,j}R^j_\beta - (-1)^{\epsilon_\alpha\epsilon_\beta}R^i_{\beta,j}R^j_\alpha = R^i_\gamma T^\gamma_{\alpha\beta} - \mathcal{S}_{0,j}E^{ji}_{\alpha\beta}. \tag{3.16}$$

Equations (3.14) and (3.16) lead to the following conditions

$$[\delta_1,\delta_2]\phi^i = (R^i_\gamma T^\gamma_{\alpha\beta} - \mathcal{S}_{0,j}E^{ji}_{\alpha\beta})\varepsilon^\beta_1\varepsilon^\alpha_2. \tag{3.17}$$

The tensor coefficients $T^\gamma_{\alpha\beta}$ are called the structure constants of the gauge algebra, although they depend in general on the fields of the theory. When $E^{ij}_{\alpha\beta} = 0$ the gauge algebra is said to be *closed*, otherwise it is *open*. Equation (3.17) defines a Lie algebra if the algebra is closed and the $T^\gamma_{\alpha\beta}$ are independent of the fields.

The gauge tensors have the following graded symmetry properties:

$$\begin{aligned} E^{ij}_{\alpha\beta} &= -(-1)^{\epsilon_i\epsilon_j}E^{ji}_{\alpha\beta} = -(-1)^{\epsilon_\alpha\epsilon_\beta}E^{ij}_{\beta\alpha}, \\ T^\gamma_{\alpha\beta} &= -(-1)^{\epsilon_\alpha\epsilon_\beta}T^\gamma_{\beta\alpha}. \end{aligned} \tag{3.18}$$

The Grassman parities are

$$\epsilon(T^\gamma_{\alpha\beta}) = \epsilon_\alpha + \epsilon_\beta + \epsilon_\gamma \pmod 2 \tag{3.19}$$

and

$$\epsilon(E^{ij}_{\alpha\beta}) = \epsilon_i + \epsilon_j + \epsilon_\alpha + \epsilon_\beta \pmod 2. \tag{3.20}$$

Various restrictions are imposed by the Jacobi identity

$$\sum_{\text{cyclic}(123)} [\delta_1,[\delta_2,\delta_3]] = 0. \tag{3.21}$$

These restrictions are

$$\sum_{\text{cyclic}(123)} (R^i_\delta A^\delta_{\alpha\beta\gamma} - \mathcal{S}_{0,j}B^{ji}_{\alpha\beta\gamma})\varepsilon^\gamma_1\varepsilon^\beta_2\varepsilon^\alpha_3 = 0, \tag{3.22}$$

where

$$3A^\delta_{\alpha\beta\gamma} \equiv (T^\delta_{\alpha\beta,k}R^k_\gamma - T^\delta_{\alpha\eta}T^\eta_{\beta\gamma})$$

$$+(-1)^{\epsilon_\alpha(\epsilon_\beta+\epsilon_\gamma)}(T^\delta_{\beta\gamma,k}R^k_\alpha - T^\delta_{\beta\eta}T^\eta_{\gamma\alpha}) + (-1)^{\epsilon_\gamma(\epsilon_\alpha+\epsilon_\beta)}(T^\delta_{\gamma\alpha,k}R^k_\beta - T^\delta_{\gamma\eta}T^\eta_{\alpha\beta}) \tag{3.23}$$

and

$$\begin{aligned} 3B^{ji}_{\alpha\beta\gamma} \equiv &(E^{ji}_{\alpha\beta,k}R^k_\gamma - E^{ji}_{\alpha\delta}T^\delta_{\beta\gamma} - (-1)^{\epsilon_i\epsilon_\alpha}R^j_{\alpha,k}E^{ki}_{\beta\gamma} + (-1)^{\epsilon_j(\epsilon_i+\epsilon_\alpha)}R^i_{\alpha,k}E^{kj}_{\beta\gamma}) \\ &+ (-1)^{\epsilon_\alpha(\epsilon_\beta+\epsilon_\gamma)}(\alpha\to\beta,\ \beta\to\gamma,\ \gamma\to\alpha) \\ &+ (-1)^{\epsilon_\gamma(\epsilon_\alpha+\epsilon_\beta)}(\alpha\to\gamma,\ \beta\to\alpha,\ \gamma\to\beta). \end{aligned} \tag{3.24}$$

As in the familiar FADDEEV-POPOV procedure it is useful to introduce ghost fields C^α with opposite GRASSMAN parities to the gauge parameters ε^α;

$$\epsilon(C^\alpha) = \epsilon_\alpha + 1, \tag{3.25}$$

and to replace the gauge parameters by ghost fields. One must then modify the graded symmetry properties of the gauge structure tensors according to

$$T_{\alpha_1\alpha_2\alpha_3\alpha_4\dots} \rightarrow (-1)^{\epsilon_{\alpha_2}+\epsilon_{\alpha_4}+\dots}\, T_{\alpha_1\alpha_2\alpha_3\alpha_4\dots}\ . \tag{3.26}$$

The NOETHER identities then take the form

$$\mathcal{S}_{0,i} R^i_\alpha C^\alpha = 0, \tag{3.27}$$

and the structure relations (3.16) simplifies

$$(2R^i_{\alpha,j} R^j_\beta - R^i_\gamma T^\gamma_{\alpha\beta} + \mathcal{S}_{0,j} E^{ji}_{\alpha\beta}) C^\beta C^\alpha = 0. \tag{3.28}$$

Introducing the antifields One incorporates the ghost fields into the field set $\Phi^A = \{\phi^i, C^\alpha\}$, where $i = 1,\dots,n$ and $\alpha = 1,\dots,m$. Clearly $A = 1,\dots,N$, where $N = n + m$. One then further increases the set by introducing an antifield Φ^*_A for each field Φ^A. The GRASSMAN parity of the antifields is

$$\epsilon(\Phi^*_A) = \epsilon(\Phi^A) + 1 \pmod 2. \tag{3.29}$$

We also assign to each field a *ghost number*, with

$$\mathrm{gh}[\phi^i] = 0, \tag{3.30}$$

$$\mathrm{gh}[C^\alpha] = 1, \tag{3.31}$$

$$\mathrm{gh}[\Phi^*_A] = -\mathrm{gh}[\Phi^A] - 1\,. \tag{3.32}$$

In the space of fields and antifields the *antibracket* is defined by

$$(X, Y) = \frac{\partial_r X}{\partial \Phi^A}\frac{\partial_l Y}{\partial \Phi^*_A} - \frac{\partial_r X}{\partial \Phi^*_A}\frac{\partial_l Y}{\partial \Phi^A}, \tag{3.33}$$

where ∂_r denotes the right, ∂_l the left derivative. The antibracket is graded antisymmetric;

$$(X, Y) = -(-1)^{(\epsilon_X+1)(\epsilon_Y+1)}(Y, X). \tag{3.34}$$

It satisfies a graded JACOBI identity

$$((X,Y),Z) + (-1)^{(\epsilon_X+1)(\epsilon_Y+\epsilon_Z)}((Y,Z),X) + (-1)^{(\epsilon_Z+1)(\epsilon_X+\epsilon_Y)}((Z,X),Y) = 0. \tag{3.35}$$

It is a graded derivation

$$
\begin{aligned}
(X, YZ) &= (X, Y)Z + (-1)^{\epsilon_Y \epsilon_X}(X, Z)Y, \\
(XY, Z) &= X(Y, Z) + (-1)^{\epsilon_X \epsilon_Y} Y(X, Z).
\end{aligned}
\tag{3.36}
$$

It has ghost number

$$
\mathrm{gh}[(X, Y)] = \mathrm{gh}[X] + \mathrm{gh}[Y] + 1 \tag{3.37}
$$

and GRASSMAN parity

$$
\epsilon((X, Y)) = \epsilon(X) + \epsilon(Y) + 1 \pmod 2. \tag{3.38}
$$

For bosonic fields the antibracket simplifies to

$$
(B, B) = 2\,\frac{\partial B}{\partial \Phi^A}\frac{\partial B}{\partial \Phi^*_A}, \tag{3.39}
$$

while for fermionic fields it is

$$
(F, F) = 0, \tag{3.40}
$$

and for any X

$$
((X, X), X) = 0. \tag{3.41}
$$

If one groups the fields and antifields together into the set

$$
z^a = \{\Phi^A, \Phi^*_A\}, \quad a = 1, ..., 2N, \tag{3.42}
$$

then the antibracket is seen to define a symplectic structure on the space of fields and antifields

$$
(X, Y) = \frac{\partial X}{\partial z^a}\zeta^{ab}\frac{\partial Y}{\partial z^b} \tag{3.43}
$$

with

$$
\zeta^{ab} = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^A_B & 0 \end{pmatrix}. \tag{3.44}
$$

The antifields can be thought of as conjugate variables to the fields, since

$$
(\Phi^A, \Phi^*_B) = \delta^A_B. \tag{3.45}
$$

The classical master equation Let $\mathcal{S}[\Phi^A, \Phi^*_A]$ be a functional of the fields and antifields with the dimensions of an action, vanishing ghost number and even GRASSMAN parity. The equation

$$(\mathcal{S}, \mathcal{S}) = 2\,\frac{\partial \mathcal{S}}{\partial \Phi^A}\frac{\partial \mathcal{S}}{\partial \Phi^*_A} = 0 \tag{3.46}$$

is the *classical master equation.* Solutions of the classical master equation with suitable boundary conditions turn out to be generating functionals for the gauge structures of the gauge theory. $\mathcal{S}$ is also the starting point for the quantization of the theory.

One denotes by Σ the subspace of stationary points of the action in the space of fields and antifields

$$\Sigma = \left\{ z^a \,\middle|\, \frac{\partial \mathcal{S}}{\partial z^a} = 0 \right\}. \tag{3.47}$$

Given a classical solution ϕ_0 of $\mathcal{S}_0$ one possible stationary point is

$$\phi^i = \phi^i_0, \quad C^\alpha = 0, \quad \Phi^*_A = 0. \tag{3.48}$$

An action $\mathcal{S}$ which satisfies the classical master equation has its own set of invariances

$$\frac{\partial \mathcal{S}}{\partial z^a}\, R^a_b = 0, \tag{3.49}$$

with

$$R^a_b = \zeta^{ac}\, \frac{\partial_l \partial_r \mathcal{S}}{\partial z^c \partial z^b}\,. \tag{3.50}$$

This equation implies

$$R^c_a\, R^a_b\, |_\Sigma = 0. \tag{3.51}$$

We see that R^a_b is nilpotent on-shell. The rank of a nilpotent $2N \times 2N$ matrix is less than or equal to N. Let r be the rank of the Hessian of $\mathcal{S}$ at the stationary point

$$r = \operatorname{rank} \left.\frac{\partial_l \partial_r \mathcal{S}}{\partial z^a \partial z^b}\right|_\Sigma. \tag{3.52}$$

Then one has $r \leq N$. The relevant solutions of the classical master equation are those for which $r = N$ holds. In this case the number of independent gauge invariances of the type in equation (3.49) equals the number of antifields. When at a later stage the gauge is fixed the non-physical antifields are eliminated.

To ensure the correct classical limit the proper solution must contain the classical action $\mathcal{S}_0$ in the sense that

$$\mathcal{S}\left[\Phi^A, \Phi^*_A\right]\big|_{\Phi^*_A=0} = \mathcal{S}_0[\phi^i]. \tag{3.53}$$

The action $\mathcal{S}[\Phi^A, \Phi^*_A]$ can be expanded in a series in the antifields, while maintaining vanishing ghost number and even GRASSMAN parity

$$\mathcal{S}[\Phi, \Phi^*] = \mathcal{S}_0 + \phi^*_i R^i_\alpha C^\alpha + C^*_\alpha \frac{1}{2} T^\alpha_{\beta\gamma} (-1)^{\epsilon_\beta} C^\gamma C^\beta + \phi^*_i \phi^*_j (-1)^{\epsilon_i} \frac{1}{4} E^{ji}_{\alpha\beta} (-1)^{\epsilon_\alpha} C^\beta C^\alpha + \dots . \tag{3.54}$$

When this is inserted into the classical master equation one finds that this equation implies the gauge structure of the classical theory (see e.g. equation (3.80) below).

Gauge-fixing and quantization Equation (3.49) shows that the action $\mathcal{S}$ still possesses gauge invariances, and hence is not yet suitable for quantization via the path integral approach, a gauge-fixing procedure is necessary. In the BATALIN-VILKOVISKY approach the gauge is fixed, and the antifields eliminated, by use of a gauge-fixing fermion Ψ which has GRASSMAN parity $\epsilon(\Psi) = 1$ and $\mathrm{gh}[\Psi] = -1$. It is a functional of the fields Φ^A only; its relation to the antifields is

$$\Phi^*_A = \frac{\partial \Psi}{\partial \Phi^A}. \tag{3.55}$$

Then one defines a surface in functional space

$$\Sigma_\Psi = \left\{ \left(\Phi^A, \Phi^*_A\right) \middle| \Phi^*_A = \frac{\partial \Psi}{\partial \Phi^A} \right\}, \tag{3.56}$$

so that for any functional $X[\Phi, \Phi^*]$ holds

$$X_{|\Sigma_\Psi} = X\left[\Phi, \frac{\partial \Psi}{\partial \Phi}\right]. \tag{3.57}$$

This kind of surface is also called a Lagrangian submanifold due to the fact that the symplectic structure vanishes on this subspace.

To construct a gauge-fixing fermion Ψ of ghost number -1 one must again introduce additional fields. The simplest choice utilizes a trivial pair $\bar{C}_\alpha, \bar{\pi}_\alpha$ with

$$\epsilon(\bar{C}_\alpha) = \epsilon_\alpha + 1, \qquad \epsilon(\bar{\pi}_\alpha) = \epsilon_\alpha, \tag{3.58}$$

$$\mathrm{gh}[\bar{C}_\alpha] = -1, \qquad \mathrm{gh}[\bar{\pi}_\alpha] = 0. \tag{3.59}$$

The fields $\bar{C}_\alpha$ are the FADDEEV-POPOV antighost. Along with these fields we include the corresponding antifields $\bar{C}^{*\alpha}$, $\bar{\pi}^{*\alpha}$. Adding the term $\bar{\pi}_\alpha \bar{C}^{*\alpha}$ to the action $\mathcal{S}$ does not spoil its properties as a proper solution to the classical master equation, and one gets the non-minimal action

$$\mathcal{S}^{non} = \mathcal{S} + \bar{\pi}_\alpha \bar{C}^{*\alpha}. \tag{3.60}$$

The simplest possibility for Ψ is

$$\Psi = \bar{C}_\alpha \chi^\alpha(\phi), \tag{3.61}$$

where χ^α are the gauge-fixing conditions for the fields ϕ. The gauge-fixed action is denoted by

$$\mathcal{S}_\Psi = \mathcal{S}^{non}\,|_{\Sigma_\Psi}\ . \tag{3.62}$$

Quantization is performed using the path integral to calculate a correlation function X, with the constraint (3.55) implemented by a δ-function:

$$I_\Psi(X) = \int [D\Phi][D\Phi^*]\delta\left(\Phi^*_A - \frac{\partial\Psi}{\partial\Phi^A}\right)\exp\left(\frac{i}{\hbar}W[\Phi,\Phi^*]\right)\ X[\Phi,\Phi^*]. \tag{3.63}$$

Here W is the quantum action, which reduces to $\mathcal{S}$ in the limit $\hbar \to 0$. An admissible Ψ leads to well-defined propagators when the path integral is expressed as a perturbation series expansion.

The results of a calculation should be independent of the gauge-fixing. Consider the integrand in equation (3.63),

$$\mathcal{I}[\Phi,\Phi^*] = \exp\left(\frac{i}{\hbar}W[\Phi,\Phi^*]\right)\ X[\Phi,\Phi^*]\ . \tag{3.64}$$

Under an infinitesimal change in Ψ one has

$$I_{\Psi+\delta\Psi}(X) - I_\Psi(X) \approx \int [D\Phi]\ \triangle\mathcal{I}\ \delta\Psi\ , \tag{3.65}$$

where the Laplacian $\triangle$ is

$$\triangle = (-1)^{\epsilon_A+1}\frac{\partial}{\partial\Phi^A}\frac{\partial}{\partial\Phi^*_A}\ . \tag{3.66}$$

Obviously the integral $I_\Psi(X)$ is independent of Ψ if $\triangle\mathcal{I} = 0$. For $X = 1$ one gets the requirement

$$\triangle\exp\left(\frac{i}{\hbar}W\right) = \exp\left(\frac{i}{\hbar}W\right)\left(\frac{i}{\hbar}\triangle W - \frac{1}{2\hbar^2}(W,W)\right) = 0. \tag{3.67}$$

The formula

$$\frac{1}{2}(W,W) = i\hbar \triangle W \tag{3.68}$$

is the *quantum master equation.* A gauge-invariant correlation function satisfies

$$(X,W) = i\hbar \triangle X. \tag{3.69}$$

The terms of higher order in $\hbar$ by which the quantum action W may differ from the solution of the classical master equation $\mathcal{S}$ correspond to the counter terms of the renormalizable gauge theory if

$$\triangle \mathcal{S} = 0. \tag{3.70}$$

One must of course use a regularization scheme which respects the symmetries of the theory. For $W = \mathcal{S} + O(\hbar)$ the quantum master equation (3.68) reduces in this case to the classical master equation $(\mathcal{S},\mathcal{S}) = 0$. Hence, up to possible counter terms, one may simply choose $W = \mathcal{S}$.

To implement the gauge-fixing we use for the action $W = \mathcal{S}^{non}$. For the path integral $Z = I_\Psi(X=1)$ we perform the integration over the antifields in equation (3.63) by using the δ-function. The result is

$$Z = \int [D\Phi] \exp\left(\frac{i}{\hbar}\mathcal{S}_\Psi\right). \tag{3.71}$$

3.2.2 The antifield formalism of the Poisson-sigma model

In this section the antifield formalism is applied to the POISSON-sigma model. Due to the fact that the gauge algebra is just closed modulo the equations of motion it is a non-trivial application of the formalism.

In the notation of Section 3.2.1 the generators R of the gauge transformations (3.2) for the POISSON-sigma model are P^{ij} and D_i^j. The gauge tensors T and E are $P^{ij}{}_{,k}$ and $P^{mn}{}_{,ji}$. The higher order gauge tensors A and B vanish. Then the NOETHER identities are

$$\int_\Sigma ((D_i^j A_j + \frac{\partial \mathcal{C}(X)}{X^i})P^{ki} + (DX^i)D_i^k)C_k = 0. \tag{3.72}$$

Considering the commutator of two gauge transformations leads to (see equations (3.14-3.16))

$$\int_\Sigma \mu(2P^{mi}{}_{,j}P^{nj} - P^{ji}P^{mn}{}_{,j})C_m C_n = 0 , \tag{3.73}$$

$$\int_{\Sigma} \mu\,(2(P^{jk}{}_{,i}D^l_j + P^{mk}{}_{,ij}A_m P^{jl}) - D^m_i P^{kl}{}_{,m} + (DX^i)P^{kl}{}_{,ji})C_l C_k = 0. \tag{3.74}$$

The JACOBI identity is

$$P^{ij}{}_{,m}P^{mk}C_iC_jC_k = 0. \tag{3.75}$$

Later on the first and second derivatives of the JACOBI identity will be needed

$$(P^{ij}{}_{,mn}P^{mk} + P^{ij}{}_{,m}P^{mk}{}_{,n})C_iC_jC_k = 0, \tag{3.76}$$

$$(P^{ij}{}_{,mnp}P^{mk} + P^{ij}{}_{,mn}P^{mk}{}_{,p} + P^{ij}{}_{,mp}P^{mk}{}_{,n} + P^{ij}{}_{,m}P^{mk}{}_{,np})C_iC_jC_k = 0\ . \tag{3.77}$$

The fields and antifields of the model are

$$\Phi^A = \{A_i, X^i, C_i\} \text{ and } \Phi^*_A = \{A^{i*}, X^*_i, C^{i*}\}\ . \tag{3.78}$$

The extended action is

$$\begin{aligned}\mathcal{S} = \int_{\Sigma} \big[A_i \wedge \mathrm{d}X^i + P^{ij}(X)A_i \wedge A_j + \mathcal{C}(X) + A^{i*}D^j_iC_j + X^*_iP^{ji}(X)C_j \\ + \frac{1}{2}C^{i*}P^{jk}{}_{,i}(X)C_jC_k + \frac{1}{4}A^{i*}\wedge A^{j*}P^{kl}{}_{,ij}(X)C_kC_l\big].\end{aligned} \tag{3.79}$$

The classical master equation is

$$\begin{aligned}(\mathcal{S},\mathcal{S}) = \int_{\Sigma} \Bigg[\Bigg((DX^m)\wedge D^j_m + (D^i_mA_i + \frac{\partial\mathcal{C}(X)}{X^i})P^{jm}\Bigg)C_j \\ - (X^*_iP^{ij}{}_{,m}P^{km} - X^*_iP^{im}\frac{1}{2}P^{jk}{}_{,m})C_jC_k \\ + (DX^m)\wedge A^{j*}\frac{1}{2}P^{kl}{}_{,mj}C_kC_l - A^{i*}\wedge P^{mk}{}_{,i}C_kD^j_mC_j \\ - A^{i*}\wedge P^{jk}{}_{,im}A_jC_kP^{mn}C_n - (D^m_iA^{i*})\frac{1}{2}P^{jk}{}_{,m}C_jC_k \\ + \frac{1}{2}C^{i*}P^{jk}{}_{,im}C_jC_kP^{lm}C_l + C^{i*}P^{mk}{}_{,i}C_k\frac{1}{2}P^{jl}{}_{,m}C_jC_l \\ + A^{i*}\wedge A^{j*}\Bigg(\frac{1}{4}P^{kl}{}_{,ijm}C_kC_lP^{mn}C_n + \frac{1}{4}P^{ml}{}_{,ij}C_lP^{kl}{}_{,m}C_kC_l - \frac{1}{2}P^{mn}{}_{,i}C_nP^{kl}{}_{,mj}C_kC_l\Bigg)\Bigg] = 0\ .\end{aligned} \tag{3.80}$$

Equations (3.72)-(3.77) ensure that the extended action (3.79) is a solution of the classical master equation (3.80).

Gauge-fixing We shall use gauge-fixing conditions of the form $\chi_i(A,X)$, so that the gauge fermion (3.61) becomes $\Psi = \bar{C}^i \chi_i(A,X)$. The antifields are then fixed to be

$$\begin{aligned} A^{i*} &= \bar{C}_j \frac{\partial \chi_j(A,X)}{\partial A_i}, \\ X_i^* &= \bar{C}_j \frac{\partial \chi_j(A,X)}{\partial X^i}, \\ C_i^* &= 0, \\ \bar{C}_i^* &= \chi_i(A,X). \end{aligned} \tag{3.81}$$

The gauge-fixed action is

$$\mathcal{S}_\Psi = \int\limits_\Sigma \Big[(A_i \wedge \mathrm{d}X^i + \frac{1}{2} P^{ij}(X) A_i \wedge A_j) + \mathcal{C}(X) + \bar{C}^k \frac{\partial \chi_k(A,X)}{\partial A_i} \wedge D_i^j C_j \\ + \bar{C}^k \frac{\partial \chi_k(A,X)}{\partial X^i} P^{ij} C_j + \frac{1}{4} \bar{C}^m \frac{\partial \chi_m(A,X)}{\partial A_i} \bar{C}^n \frac{\partial \chi_n(A,X)}{\partial A_j} P^{kl}{}_{,ij}(X) C_k C_l \} + \bar{\pi}^i \chi_i(A,X) \Big]. \tag{3.82}$$

Consider now different gauge conditions:

(i) First, the LANDAU gauge for the gauge potential $\chi_i = \mathrm{d}A_i$, so that the gauge fermion becomes $\Psi = \bar{C}^i *_H \mathrm{d}A_i$ where $*_H$ is the HODGE operator depending on an arbitrary metric. The antifields are fixed to be

$$A^{i*} = *_H \mathrm{d}\bar{C}^i, \tag{3.83}$$

$$X_i^* = C^{*i} = 0, \tag{3.84}$$

$$\bar{C}_i^* = \mathrm{d} *_H A_i. \tag{3.85}$$

For this gauge choice the gauge-fixed action is

$$\mathcal{S}_\Psi = \int\limits_\Sigma \Big[(A_i \wedge \mathrm{d}X^i + \frac{1}{2} P^{ij}(X) A_i A_j) + \mathcal{C}(X) + *_H \mathrm{d}\bar{C}^i \wedge D_i^j C_j \\ + \frac{1}{4} (*_H \mathrm{d}\bar{C}^i) \wedge (*_H \mathrm{d}\bar{C}^j) P^{kl}{}_{,ij}(X) C_k C_l - \bar{\pi}^i (\mathrm{d} *_H A_i) \Big]. \tag{3.86}$$

(ii) Now consider the temporal gauge $\chi_i = A_{0i}$. In this case the gauge fermion is given by $\Psi = \bar{C}^i A_{0i}$. The antifields are fixed to

$$A^{*0i} = \bar{C}^i, \tag{3.87}$$

$$A^{*1i} = 0, \tag{3.88}$$

$$X_i^* = C^{*i} = 0, \tag{3.89}$$

$$\bar{C}_i^* = A_{0i}. \tag{3.90}$$

The gauged-fixed action is

$$\mathcal{S}_\Psi = \int_\Sigma \mu \left[\epsilon^{\mu\nu}(A_{\mu i}\partial_\nu X^i + \frac{1}{2}P^{ij}(X)A_{\mu i}A_{\nu j}) + \mathcal{C}(X) + \bar{C}^i D^j_{0i} C_j - \bar{\pi}^i(A_{0i})\right]. \tag{3.91}$$

(iii) Finally, the Schwinger-Fock gauge $\chi_i = u^\mu A_{\mu i}$ is considered. Then the antifields are fixed to be

$$A^{*\mu i} = u^\mu \bar{C}^i, \tag{3.92}$$

$$X^*_i = C^{*i} = 0, \tag{3.93}$$

$$\bar{C}^*_i = u^\mu A_{\mu i}. \tag{3.94}$$

For this gauge choice the gauge-fixed action is

$$\mathcal{S}_\Psi = \int_\Sigma \mu \left[\epsilon^{\mu\nu}(A_{\mu i}\partial_\nu X^i + \frac{1}{2}P^{ij}(X)A_{\mu i}A_{\nu j}) + \mathcal{C}(X) + \bar{C}^i u^\mu D^j_{\mu i} C_j - \bar{\pi}^i(u^\mu A_{\mu i})\right]. \tag{3.95}$$

Notice that in the non-covariant gauges (ii) and (iii) the action simplifies, in the sense that the term which arose because of the non-closed gauge algebra vanishes.

Gauge fixing in Casimir-Darboux coordinates Important simplifications occur when we write the action in Casimir-Darboux coordinates $X^i \to \{X^I, X^\alpha\}$, so we go through the gauge-fixing procedure again for these coordinates. The extended action is

$$\mathcal{S} = \int_\Sigma \Big[A_I \wedge \mathrm{d}X^I + A_\alpha \wedge \mathrm{d}X^\alpha + \frac{1}{2}P^{\alpha\beta}(X^I)A_\alpha \wedge A_\beta + \mathcal{C}(X^I) + A^{I*} \wedge \mathrm{d}C_I + A^{\alpha *} \wedge \mathrm{d}C_\alpha + X^*_\alpha P^{\beta\alpha}(X^I)C_\beta\Big]. \tag{3.96}$$

This extended action still possesses gauge invariances, so one has to introduce a non-minimal sector. The non-minimal action is

$$\mathcal{S}^{non} = \int_\Sigma \Big[A_I \wedge \mathrm{d}X^I + A_\alpha \wedge \mathrm{d}X^\alpha + \frac{1}{2}P^{\alpha\beta}(X^I)A_\alpha \wedge A_\beta + \mathcal{C}(X) + A^{I*} \wedge \mathrm{d}C_I + A^{\alpha *} \wedge \mathrm{d}C_\alpha + X^*_\alpha P^{\beta\alpha}(X^I)C_\beta - \bar{\pi}^I \bar{C}^*_I - \bar{\pi}^\alpha \bar{C}^*_\alpha\Big]. \tag{3.97}$$

In these coordinates the gauge freedom of the maps $X^i : \Sigma \to N$ is reduced to the freedom of the maps $X^\alpha : \Sigma \to S$, where S is a symplectic leaf of the Poisson manifold N. The gauge transformations $\delta_\varepsilon X^i = P^{ij}\varepsilon_j$ reduce to

$$\delta_\varepsilon X^\alpha = P^{\alpha\beta}\varepsilon_\beta, \quad \delta_\varepsilon X^I = 0. \tag{3.98}$$

After gauge fixing we need to consider only the homotopy classes $[X^\alpha]$.

It is now possible to decompose the gauge condition into two parts depending only on A_I and on X^α respectively, so that the gauge-fixing of the gauge fields is implemented by gauge conditions of the form $\chi_I(A_I)$ and $\chi_\alpha(X^\alpha)$. The gauge fermion may be written as

$$\Psi = \int\limits_\Sigma \left[\bar{C}^I \chi_I(A_I) + \bar{C}^\alpha \chi_\alpha(X^\alpha)\right] . \tag{3.99}$$

The gauge conditions as expressed using the gauge fermion are

$$\begin{aligned} A^{I*} &= \bar{C}^J \frac{\partial \chi_J(A_I)}{\partial A_I}, \\ A^{\alpha *} &= 0, \\ X^*_\alpha &= \bar{C}^\beta \frac{\partial \chi_\beta(X^\alpha)}{\partial X^\alpha}, \\ C^*_i &= 0, \\ \bar{C}^*_I &= \chi_I(A_I), \\ \bar{C}^*_\alpha &= \chi_\alpha(X^\alpha) . \end{aligned} \tag{3.100}$$

The gauge-fixed action in Casimir-Darboux coordinates takes the form

$$\begin{aligned} \mathcal{S}_\psi = \int\limits_\Sigma \Bigg[& A_I \wedge \mathrm{d}X^I + A_\alpha \wedge \mathrm{d}X^\alpha + \frac{1}{2} P^{\alpha\beta} A_\alpha \wedge A_\beta + \mathcal{C}(X^I) \\ & + \bar{C}^J \frac{\partial \chi_J(A_J)}{\partial A_I} \wedge \mathrm{d}C_I + \bar{C}^\alpha \frac{\partial \chi_\alpha(X^\alpha)}{\partial X^\beta} P^{\beta\gamma} C_\gamma - \bar{\pi}^I \chi_I(A_I) - \bar{\pi}^\alpha \chi_\alpha(X^\alpha) \Bigg] . \end{aligned} \tag{3.101}$$

3.3 Cohomological derivation of the couplings

A non-linear gauge theory is based on a non-linear extension of the underlying Lie algebra as a generalization of the usual non-Abelian gauge theory with internal gauge symmetry [27]. The considered extension in the present case is a Poisson algebra which lead to the Poisson-sigma model.

The Poisson-sigma model can be seen as a two-dimensional BF theory with a quadratic interaction in the gauge fields. There exists a cohomological approach to the problem of generating consistent interactions [5]. It essentially is based on the antifield formalism in the sense that a deformation of the solution of the master equation leads to an action functional containing a consistent interaction term.

The algebra of the fields and antifields together with the BRST-differential generated by S through the antibracket

$$s(\cdot) = (\cdot, S) \tag{3.102}$$

yields a complex. The corresponding BRST cohomology is denoted by $H^* = \sum H^p(s)$. One can define a map in the cohomology induced by the antibracket, the *antibracket map*

$$(\cdot,\cdot) : H^p(s) \times H^q(s) \longrightarrow H^{(p+q+1)}(s) \,, \tag{3.103}$$

$$([A],[B]) \mapsto [(A,B)] \,. \tag{3.104}$$

The important result found by G.Barnich and M.Henneaux is the fact that the antibracket map is trivial in the sense that the antibracket of 2 BRST-closed functionals is BRST-exact, for a proof consult [5]. Due to the triviality of the antibracket map one can define higher order maps in the cohomology, however it turns out they are trivial in a similar way.

Now consider a *free* gauge theory with a *free* symmetry given by

$$\text{Free Theory} = \begin{cases} \text{Free action : } \overset{(0)}{S}_0 [\phi^i] \\ \text{Gauge Symmetry: } \delta_\epsilon \phi^i = \overset{(0)}{\mathcal{R}^i}_\alpha \, \epsilon^\alpha \\ \text{Noether Theorem: } \frac{\delta \overset{(0)}{S}_0[\phi^i]}{\delta \phi^i} \overset{(0)}{\mathcal{R}^i}_\alpha = 0 \,. \end{cases}$$

The aim is now to introduce couplings between the fields ϕ^i fulfilling the crucial physical requirement of preserving the number of gauge symmetries, those couplings will be called consistent. This means one has to perturb the action and the symmetries

$$\overset{(0)}{S}_0 \longrightarrow S_0 = \overset{(0)}{S}_0 + g \overset{(1)}{S}_0 + g^2 \overset{(2)}{S}_0 + \ldots \;, \tag{3.105}$$

$$\overset{(0)}{\mathcal{R}^i}_\alpha \longrightarrow \mathcal{R}^i_\alpha = \overset{(0)}{\mathcal{R}^i}_\alpha + g \overset{(1)}{\mathcal{R}^i}_\alpha + g^2 \overset{(2)}{\mathcal{R}^i}_\alpha + \ldots \;, \tag{3.106}$$

such that $\delta_\epsilon \phi^i = \mathcal{R}^i_\alpha \epsilon^\alpha$ is a symmetry of S_0

$$\frac{\delta(\overset{(0)}{S}_0 + g \overset{(1)}{S}_0 + g^2 \overset{(2)}{S}_0 + \ldots)}{\delta \phi^i} (\overset{(0)}{\mathcal{R}^i}_\alpha + g \overset{(1)}{\mathcal{R}^i}_\alpha + g^2 \overset{(2)}{\mathcal{R}^i}_\alpha + \ldots) = 0 \,, \tag{3.107}$$

which expresses the consistency. It is not an easy task to deform simultaneously the action and the symmetry to get a consistent interaction.

This problem can be reformulated as a deformation problem of the solution of the master equation. Basically this procedure is based on the fact that the master equation contains all the information about the gauge structure

$$(\overset{(0)}{S}_{BV}, \overset{(0)}{S}_{BV}) = 0 \longrightarrow (S_{BV}, S_{BV}) = 0 \; , \tag{3.108}$$

$$\overset{(0)}{S}_{BV} \longrightarrow S_{BV} = \overset{(0)}{S}_{BV} + g \overset{(1)}{S}_{BV} + g^2 \overset{(2)}{S}_{BV} + \dots \; . \tag{3.109}$$

The master equation guarantees now the consistency of S_0 and $\mathcal{R}^i_\alpha$ and further, that the original and the deformed gauge theory have the same spectrum of ghosts and antifields.

The advantage of this formulation is that one now can use the cohomological techniques of deformation theory. The deformed master equation can be analyzed order by order in the deformation parameter, the coupling constant. This expansion yields the following relations

$$(\overset{(0)}{S}_{BV}, \overset{(0)}{S}_{BV}) = 0 \; , \tag{3.110}$$

$$2(\overset{(0)}{S}_{BV}, \overset{(1)}{S}_{BV}) = 0 \; , \tag{3.111}$$

$$2(\overset{(0)}{S}_{BV}, \overset{(2)}{S}_{BV}) + (\overset{(1)}{S}_{BV}, \overset{(1)}{S}_{BV}) = 0 \; , \tag{3.112}$$

(+higher orders) .

The first equation (3.110) is fulfilled by assumption, it is exactly the master equation for the free gauge theory. Equation (3.111) shows that $\overset{(1)}{S}_{BV}$ is forced to be a cocycle of the free BRST differential $\overset{(0)}{s}$. Assume now that $\overset{(1)}{S}_{BV}$ is a coboundary, then the corresponding interaction induces a field redefinition, which need not be considered, and the deformation will be called trivial. Therefore $\overset{(1)}{S}_{BV}$ is an element of the zeroth cohomological space $H^0(\overset{(0)}{s})$ which is isomorphic to the space of physical observables of the free theory. Because of the triviality of the antibracket map $(\overset{(1)}{S}_{BV}, \overset{(1)}{S}_{BV})$ is BRST exact and one gets no obstructions from (3.112) for constructing the interaction, and $\overset{(2)}{S}_{BV}$ exists. This is also true for higher orders, so there are no obstructions for the interacting action at all.

Usually the original action is a local functional of a corresponding Lagrange function, so also the deformations must be local functionals. Taking locality into account the analysis gets more involved because the antibracket map is not trivial anymore, e.g. the antibracket of two local BRST cocycles need not necessarily to be the BRST variation of a local functional. Consider $\overset{(k)}{S} = \int \overset{(k)}{\mathcal{L}}$, where $\mathcal{L}$ is the Lagrangian, an n-form. The corresponding (local) antibracket is defined modulo an d-exact term, d being the exterior derivative. This yields for the deformation expansion for the Lagrangian

$$2 \overset{(0)}{s}\overset{(1)}{\mathcal{L}} = \mathrm{d} \overset{(1)}{j} \ , \tag{3.113}$$

$$\overset{(0)}{s}\overset{(2)}{\mathcal{L}} + \{\overset{(1)}{\mathcal{L}}, \overset{(1)}{\mathcal{L}}\} = \mathrm{d} \overset{(2)}{j} \ , \tag{3.114}$$

$$(\text{+higher orders}) \ ,$$

where $\overset{(k)}{j}$ is the symbol for the d-exact term. $\overset{(1)}{\mathcal{L}}$ is BRST closed modulo d, this means that the non-trivial deformations of the master equation belong to $H^0(\overset{(0)}{s} | \ \mathrm{d})$. Because the corresponding local antibracket is no longer trivial, it possesses a lot of structure, one gets obstructions for the construction of the interaction term, the so-called consistency conditions. The construction of local, consistent interaction is strongly constrained.

Deformation of the Abelian BF theory The POISSON-sigma model can be seen as a BF theory with an interaction quadratic in the gauge fields. This interaction can be obtained by the deformation procedure. In [28] it was shown that the deformation of the ABELian BF theory in two-dimensions corresponds with the topological POISSON-sigma model. In this approach to the interacting action the origin of the non-linearity becomes more transparent.

The action of the ABELian BF theory is given by

$$\overset{(0)}{S}_0 = \int\limits_{\Sigma_g} A_i \wedge \mathrm{d}\phi^i \ , \tag{3.115}$$

where the A_i is an ABELian gauge field and ϕ^i is bosonic scalar field. The gauge symmetries are

$$\delta\phi^i = 0, \qquad \delta A_i = \mathrm{d}\varepsilon_i \ . \tag{3.116}$$

Due to the fact that the theory is so simple the minimal solution of the classical master equation is quite plain. It just consists of the first term involving the gauge generators, all higher gauge structure functions are equal to zero, so that

$$\overset{(0)}{\mathcal{S}} = \int\limits_{\Sigma_g} \left[A_i \wedge \mathrm{d}\phi^i + A^{*i}\mathrm{d}C_i\right] \ . \tag{3.117}$$

The first deformation $\overset{(1)}{\mathcal{L}}$ of the Lagrangian associated to the minimal solution $\overset{(0)}{\mathcal{S}}$ should obey the following condition

$$\overset{(0)}{s}\overset{(1)}{\mathcal{L}} + \mathrm{d}a_{[1]} = 0 \ . \tag{3.118}$$

It defines an element of $H^0(\overset{(0)}{s}|\,\mathrm{d})$, so that one gets a set of *descent equations*

$$\overset{(0)}{s}\; a_{[1]} + \mathrm{d}a_{[0]}\;, = 0 \qquad\qquad \overset{(0)}{s}\; a_{[0]} = 0\;. \tag{3.119}$$

It is a simple calculation to get the solution for $\overset{(1)}{\mathcal{L}}$

$$\overset{(1)}{\mathcal{L}} = -\frac{1}{4}\frac{\delta^2 f^{ij}[\phi]}{\delta\phi^k\delta\phi^l}A^{*k}\wedge A^{*l}C_iC_j + \frac{\delta f^{ij}[\phi]}{\delta\phi^k}C^{*k}C_iC_j$$
$$- \frac{\delta f^{ij}[\phi]}{\delta\phi^k}A^{*k}\wedge A_iC_j - f^{ij}[\phi]\phi_i^*C_j + \frac{1}{2}f^{ij}[\phi]A_i\wedge A_j\,. \tag{3.120}$$

The $f^{ij}[\phi]$ are antisymmetric and to yield a consistent interaction they have to satisfy

$$\sum_{\mathrm{cycl}(ijk)}\frac{\delta f^{ij}[\phi]}{\delta\phi^l}f^{kl}[\phi] = 0\;, \tag{3.121}$$

which is a generalized JACOBI identity. Since this condition is fulfilled there are no obstructions in the construction and the second order deformation can be chosen to be zero. This yields for the deformed solution of the master equation

$$S_{BV} = \overset{(0)}{S} + \overset{(1)}{S} = \int\limits_{\Sigma_g}\big[A_i\wedge\mathrm{d}\phi^i - \frac{\delta f^{ij}[\phi]}{\delta\phi^k}A^{*k}\wedge A_iC_j + A^{*i}\wedge\mathrm{d}C_i - f^{ij}[\phi]\phi_i^*C_j + \frac{1}{2}f^{ij}[\phi]A_i\wedge A_j$$
$$-\frac{\delta f^{ij}[\phi]}{\delta\phi^k}A^{*k}\wedge A_iC_j + \frac{\delta f^{ij}[\phi]}{\delta\phi^k}C^{*k}C_iC_j - \frac{1}{4}\frac{\delta^2 f^{ij}[\phi]}{\delta\phi^k\delta\phi^l}A^{*k}\wedge A^{*l}C_iC_j\big]\;. \tag{3.122}$$

By setting the antifields to zero one can read off the classical action including an interaction term quadratic in the gauge fields A_i

$$S_0 = \overset{(0)}{S_0} + \overset{(1)}{S_0} = \int\limits_{\Sigma_g}\big[A_i\wedge\mathrm{d}\phi^i + \frac{1}{2}f^{ij}[\phi]A_i\wedge A_j\big] \tag{3.123}$$

and the deformed gauge symmetries are

$$\delta_\epsilon\phi^i = f^{ji}[\phi]\varepsilon_j\;, \tag{3.124}$$

$$\delta_\epsilon A_i = \mathrm{d}\varepsilon_i - f^{kl}{}_{,i}[\phi]A_k\wedge A_l\,. \tag{3.125}$$

Note that the gauge algebra is only closed on-shell, which reflects the non-linearity of the gauge algebra. One can see that if one chooses for the structure functions $f^{ij}(\phi)$ the POISSON structure P^{ij} the consistency condition is just the vanishing of the SCHOUTEN-NIJENHUIS bracket. Hence, the deformed BF theory is equal to the topological part of the POISSON-sigma model. The origin of the non-linearity stems from the quadratic interaction in the gauge fields. To ensure the gauge invariances of the deformed, interacting gauge theory, the structure *constants* must be field-dependent, such that the gauge algebra is just closed on-shell.

4 Quantum Theory: The Partition Function

In this chapter the quantum theory of the POISSON-sigma model is considered. There exist several ways to obtain a (full) quantum theory of a field theory. First, there is the DIRAC quantization scheme, i.e. one goes from the LAGRANGE to the HAMILTON formalism, determines the constraints and translates them into operator language. Then one needs to solve these equations to obtain the wave functions. This was performed for the POISSON-sigma model by P. SCHALLER and T. STROBL in [43]. Secondly one can use the perturbation expansion of FEYNMAN to calculate correlation functions. In [11] A. CATTANEO and G. FELDER have calculated the correlation function of two functions with support on the boundary of the two-dimensional disc which represents the world sheet. After renormalization it coincides with the KONTSEVICH formula for deformation quantization of the algebra of POISSON manifolds.

This chapter is concerned with the calculation of the partition function for the POISSON-sigma model. In the first section it will be calculated for world sheets without boundary [24]. In the second section this calculation will be generalized to the two-dimensional disc. The restriction to a linear POISSON structure on the target space enables one to complete the calculation in both cases, as will be shown in the third section. In the last section a glueing prescription will be presented to obtain the partition function for the linear case on arbitrary (oriented) two-dimensional manifolds [26].

4.1 The partition function on closed manifolds

In contrast to the DIRAC quantization which is restricted to the world sheet with the topology of a cylinder, the path integral offers the opportunity to perform the quantization for arbitrary world sheet topologies. In this section the world sheet Σ is a closed manifold, i.e. without a boundary, with genus g. The partition function is then

$$Z = \int_{\Sigma_\Psi} [DX^I][DX^\alpha][DA_I][DA_\alpha][DC_I][D\bar{C}_I][DC_\alpha][D\bar{C}_\alpha][D\bar{\pi}_I][D\bar{\pi}_\alpha] \exp\left(-\frac{i}{\hbar}\mathcal{S}_\Psi\right) \qquad (4.1)$$

with $\mathcal{S}_\Psi$ given by equation (3.101). Integrating over the ghost and antighost fields yields the FADDEEV-POPOV determinants

$$\det\left[\frac{\partial\chi_I(A_I)}{\partial A_I}\wedge \mathrm{d}\right]_{\Omega^0(\Sigma)} \qquad \text{and} \qquad \det\left[\frac{\partial\chi_\alpha(X^\alpha)}{\partial X^\gamma}P^{\gamma\beta}(X^I)\right]_{\Omega^0(\Sigma)}\ , \tag{4.2}$$

where the subscripts $\Omega^k(\Sigma)$ indicate that the determinant results from an integration over k-forms on Σ. The resulting expression of the path integral is then

$$\begin{aligned} Z = \int\limits_{\Sigma_\Psi} &[DX^I][DX^\alpha][DA_I][DA_\alpha][D\bar\pi_I][D\bar\pi_\alpha] \\ &\times \det\left[\frac{\partial\chi_I(A_I)}{\partial A_I}\wedge \mathrm{d}\right]_{\Omega^0(\Sigma)} \det\left[\frac{\partial\chi_\alpha(X^\alpha)}{\partial X^\gamma}P^{\gamma\beta}(X^I)\right]_{\Omega^0(\Sigma)} \\ &\times \exp\Bigg(-\frac{i}{\hbar}\int\limits_\Sigma [A_I\wedge \mathrm{d}X^I + A_\alpha\wedge \mathrm{d}X^\alpha + \frac{1}{2}P^{\alpha\beta}A_\alpha\wedge A_\beta + \mathcal{C}(X^I) \\ &\qquad - \bar\pi^I\chi_I(A_I) - \bar\pi^\alpha\chi_\alpha(X^\alpha)]\Bigg)\ , \end{aligned} \tag{4.3}$$

The integrations over $\bar\pi_I$ and $\bar\pi_\alpha$ yield δ-functions which implement the gauge conditions.

$$\begin{aligned} Z = \int\limits_{\Sigma_\Psi} &[DX^I][DX^\alpha][DA_I][DA_\alpha]\det\left[\frac{\partial\chi_I(A_I)}{\partial A_I}\wedge \mathrm{d}\right]_{\Omega^0(\Sigma)} \det\left[\frac{\partial\chi_\alpha(X^\alpha)}{\partial X^\gamma}P^{\gamma\beta}(X^I)\right]_{\Omega^0(\Sigma)} \\ &\times \exp\left(-\frac{i}{\hbar}\int\limits_\Sigma \left[A_I\wedge \mathrm{d}X^I + A_\alpha\wedge \mathrm{d}X^\alpha + \frac{1}{2}P^{\alpha\beta}A_\alpha\wedge A_\beta + \mathcal{C}(X^I)\right]\right)\ , \end{aligned} \tag{4.4}$$

where from now on the integrations extend only over the degrees of freedom which respect the gauge-fixing conditions, e.g. the δ-functions which ensure the gauge condition are implemented. The integration over A_α is GAUSSian, it yields

$$\begin{aligned} Z = \int\limits_{\Sigma_\Psi} &[DX^I][DX^\alpha][DA_I]\det\left[\frac{\partial\chi_I(A^I)}{\partial A_I}\wedge \mathrm{d}\right]_{\Omega^0(\Sigma)} \det\left[\frac{\partial\chi_\alpha(X^\alpha)}{\partial X^\gamma}P^{\gamma\beta}(X^I)\right]_{\Omega^0(\Sigma)} \\ &\times \det{}^{-1/2}\left[P^{\alpha\beta}(X^I)\right]_{\Omega^1(\Sigma)} \\ &\times \exp\left(-\frac{i}{\hbar}\int\limits_\Sigma A_I\wedge \mathrm{d}X^I - \frac{i\hbar}{2}\int\limits_\Sigma \Omega_{\alpha\beta}(X^I)\mathrm{d}X^\alpha\wedge \mathrm{d}X^\beta - \frac{i}{\hbar}\int\limits_\Sigma \mathcal{C}(X^I)\right). \end{aligned} \tag{4.5}$$

Besides the term in the exponent the only dependence on the fields A_I is in the relevant FADDEEV-POPOV determinant. If one chooses a gauge condition linear in A_I this determinant becomes independent of the fields, and can be absorbed into a normalization factor. The integration over A_I then yields a δ-function for $\mathrm{d}X^I$. When this δ-function is implemented the fields X^I become independent of the coordinates $\{u^\mu\}$ on Σ. Hence the CASIMIR functions are constants. The constant modes of the CASIMIR coordinates X_0^I count the symplectic leaves. The path integral is now

$$Z = \int\limits_{\Sigma_\Psi} \mathrm{d}X_0^I [DX^\alpha] \det \left[\frac{\partial \chi_\alpha(X^\alpha)}{\partial X^\gamma} P^{\gamma\beta}(X_0^I) \right]_{\Omega^0(\Sigma)} \det{}^{-1/2} \left[P^{\alpha\beta}(X_0^I) \right]_{\Omega^1(\Sigma)} \exp\left(-\frac{i\hbar}{2} \int\limits_\Sigma \Omega_{\alpha\beta} \mathrm{d}X^\alpha \wedge \mathrm{d}X^\beta \right) \exp\left(-\frac{i}{\hbar} \int\limits_\Sigma \mathcal{C}(X_0^I) \right). \quad (4.6)$$

The gauge-fixing of the fields X^α reduces the integral $[DX^\alpha]$ to a sum over the homotopy classes

$$Z = \int\limits_{\Sigma_\Psi} \mathrm{d}X_0^I \sum_{[\Sigma \to S(X_0^I)]} \det \left[\frac{\partial \chi_\alpha(X)}{\partial X^\gamma} P^{\gamma\beta}(X_0^I) \right]_{\Omega^0(\Sigma)} \det{}^{-1/2} \left[P^{\alpha\beta}(X_0^I) \right]_{\Omega^1(\Sigma)} \exp\left(-\frac{i\hbar}{2} \int\limits_\Sigma \Omega_{\alpha\beta} \mathrm{d}X^\alpha \wedge \mathrm{d}X^\beta \right) \exp\left(-\frac{i}{\hbar} \int\limits_\Sigma \mathcal{C}(X_0^I) \right). \quad (4.7)$$

Since the CASIMIR functions (X_0^I) are independent of the coordinates on Σ the last exponent simplifies to

$$\exp\left(-\frac{i}{\hbar} \int\limits_\Sigma \mathcal{C}(X_0^I) \right) = \exp\left(-\frac{i}{\hbar} \int\limits_\Sigma \mu C(X_0^I) \right) = \exp\left(-\frac{i}{\hbar} A_\Sigma C(X_0^I) \right), \quad (4.8)$$

where A_Σ is the surface area of Σ. The final form of the path integral then becomes

$$Z = \int\limits_{\Sigma_\Psi} \mathrm{d}X_0^I \sum_{[\Sigma \to S(X_0^I)]} \det \left[\frac{\partial \chi_\alpha(X)}{\partial X^\gamma} P^{\gamma\beta}(X_0^I) \right]_{\Omega^0(\Sigma)} \det{}^{-1/2} \left[P^{\alpha\beta}(X_0^I) \right]_{\Omega^1(\Sigma)} \times \exp\left(-\frac{i\hbar}{2} \int\limits_\Sigma \Omega_{\alpha\beta} \mathrm{d}X^\alpha \wedge \mathrm{d}X^\beta \right) \exp\left(-\frac{i}{\hbar} A_\Sigma C(X_0^I) \right). \quad (4.9)$$

Note that equation (4.9) is an almost closed expression for the partition function for the POISSON-sigma model, i.e. all the functional integrations have been performed.

4.2 The partition function on the disc

In order to find the partition function on two-dimensional world sheets with arbitrary topology the calculation will be extended in this section to the disc $\mathbb{D}^2$. The main difference to the previous calculation is that now one has to specify the boundary conditions for the fields A_i

and X^i. Denoting by u the coordinates of the disc, for $u \in \partial\mathbb{D}^2$ the fields A_i are restricted to obey $A_i(u)\cdot v = 0$ where v is a vector tangent to the boundary. In [11] it was pointed out that the HODGE dual antifields have the same boundary condition as the fields. Then it follows for $u \in \partial\mathbb{D}^2$ that $C_i(u) = 0$, $C^*(u) = 0$ and $A^*(u)\cdot w = 0$ for w normal to the boundary. The boundary condition for the maps X^i is as follows: one is to include in the path integral only such maps which map the boundary to a single point in the target manifold.

The partition function for the POISSON-sigma model is then

$$Z(\mathbb{D}^2, \phi(x)) = \int\limits_{\Sigma_\Psi} [DX][DA][D\ldots] \exp(-\frac{i}{\hbar}S_\Psi)\langle\delta_x, \phi(X(u^\partial))\rangle \; , \tag{4.10}$$

where Σ_Ψ denotes the chosen Lagrangian submanifold associated to the gauge fermion Ψ. $\langle\delta_x, \phi(X)\rangle$ is the DIRAC measure, a distribution of order zero. $\phi(X(u^\partial))$ is an arbitrary function with support only on the boundary of the disc, u^∂ denotes an arbitrary point on the boundary. In general functions of the form $X(u^\partial)$ are observables for the POISSON-sigma model [11], because of the boundary condition $C^i(u^\partial) = 0$, $(S, X)_{|\partial D} = P^{ij}C_{i|\partial D} = 0$. This distribution ensures the boundary condition for the fields X, it reflects the freedom of the fields on the boundary of the disc.

If one is interested in submanifolds S of $\mathbb{R}^n$ one has to reduce the DIRAC measure to these submanifolds

$$\langle\delta_x, \phi(x)\rangle\Big|_S = \int\limits_S \omega\phi =: \langle\delta_S, \phi(x)_{|S}\rangle \; , \tag{4.11}$$

where ω is the LERAY form which can be chosen to be proportional to the volume form induced on the submanifold by the EUCLIDian measure on $\mathbb{R}^n$ [14]. Note that the function is restricted to the submanifold S and the dependence of the point x passes over to the choice of the specific submanifold. If one applies this restriction to the symplectic foliation of the POISSON manifold such that the symplectic leaves L are the considered submanifolds, the DIRAC measure then picks a symplectic leaf L given by $C(X^I)$ =constant. The form of the partition function in CASIMIR-DARBOUX coordinates is then

$$Z(\mathbb{D}^2, \phi_L(X^\alpha)) = \int\limits_{\Sigma_\Psi} [D\ldots]\langle\delta_L, \phi_L(X^\alpha)\rangle \exp\Big(-\frac{i}{\hbar}\int\limits_{\mathbb{D}^2}\Big[\, A_I\wedge \mathrm{d}X^I + A_\alpha\wedge \mathrm{d}X^\alpha + \frac{1}{2}P^{\alpha\beta}A_\alpha\wedge A_\beta$$
$$+\,\mathcal{C}(X^I) + \bar{C}^J\frac{\partial\chi_J(A_J)}{\partial A_I}\wedge \mathrm{d}C_I + \bar{C}^\alpha\frac{\partial\chi_\alpha(X_\alpha)}{\partial X_\beta}P^{\gamma\beta}C_\gamma - \bar{\pi}^I\chi_I(A_I) - \bar{\pi}^\alpha\chi_\alpha(X^\alpha)\Big]\Big) \; . \tag{4.12}$$

It is possible to perform all the integrations of the fields. It is the same calculation as in the case of closed manifolds, see the previous section. Integrating over the ghost and antighost fields yields the FADDEEV-POPOV determinants. The integrations over the multipliers yields δ-functions which implement the gauge conditions, from now on the integration extends only over the degrees of freedom which respect the gauge-fixing conditions. The integration over A_α is GAUSSian. Now choose a gauge condition linear in A_I, then the FADDEEV-POPOV determinant does not depend on A_I anymore and one can integrate over these fields, yielding a δ-function for $\mathrm{d}X^I$. When this δ-function is implemented the fields X^I become independent of the coordinates of $\mathbb{D}^2$. Hence the CASIMIR functions are constant and these constant modes X_0^I count the symplectic leaves. The gauge fixing of the fields X^α reduces the integral over X^α to a sum over the homotopy classes of the maps. This leads to the consequence that the function $\phi_L(X^\alpha)$ does not depend on a specific point of the target anymore but just on the homotopy class of the associated map. The partition function is then

$$Z[\mathbb{D}^2, \phi_\Omega(X^\alpha)] = \int\limits_{\Sigma_\Psi} \mathrm{d}X_0^I \sum_{[X^\alpha]} \det\left[\frac{\partial\chi_\alpha(X^\alpha)}{\partial X^\gamma} P^{\gamma\beta}(X_0^I)\right]_{\Omega^0(\Sigma)} \det{}^{-1/2}\left[P^{\alpha\beta}(X_0^I)\right]_{\Omega^1(\Sigma)}$$
$$\times \langle\delta_L, \phi(X^\alpha(u^\partial))\rangle \exp\left(-\frac{i\hbar}{2}\int\limits_{\mathbb{D}^2} \Omega_{\alpha\beta}\mathrm{d}X^\alpha \wedge \mathrm{d}X^\beta\right) \exp\left(-\frac{i}{\hbar}A_{\mathbb{D}^2}C(X_0^I)\right) , \quad (4.13)$$

where the subscript $\Omega^k(\Sigma)$ indicates that the determinant results from an integration over k-forms and $A_{\mathbb{D}^2}$ denotes the surface area of the disc.

All the functional integrations have been performed and one has arrived at an almost closed expression for the partition function. The boundary condition is now restricted to a function on the symplectic leaves which reflects the freedom of the fields X on the boundary. This means that the boundary condition for the fields X is now reduced to each single symplectic leaf characterized by the corresponding constant mode X_0^I. The boundary condition can be interpreted as follows. One maps the boundary of the disc to a point in the target and associates to this point the LERAY form of the leaf, which is of course the same for every point of the leaf, with respect to the chosen function ϕ.

4.3 The linear Poisson structure on the target space $\mathbb{R}^3$

4.3.1 Closed manifolds

In this section the special case where the POISSON manifold $N = \mathbb{R}^3$, and the POISSON structure is linear $P^{ij} = c_k^{ij} X^k$ for closed manifolds is considered. As mentioned in section

2.1 this structure gives rise to a LIE algebra structure on the dual space and there exists a symmetric, non-degenerate bilinear form such that both spaces can be identified. In section 3.1.1 this identification and the choice of the quadratic CASIMIR function $C(X) = \sum_i X^i X^i$ has lead in the classical theory to the action for the two-dimensional YANG-MILLS theory (3.8). The same can be done for the partition function [24]. First the restriction to the linear POISSON structure is used to make further progress in the calculation of the partition function, resulting in a closed expression for the *linear* POISSON-sigma model. Then applying some of the results of the orbit method by KIRILLOV, see section 2.2, one can recalculate the partition function for the YANG-MILLS theory on closed manifolds.

In the case of a three-dimensional target space, i.e. $N = \mathbb{R}^3$, the linear POISSON structure foliates into two-dimensional spheres $\mathbb{S}^2$ characterized by their radius, the zero modes X_0^I, and the point zero, the origin.

For a map $f : X \longrightarrow Y$, where X and Y are k-dimensional oriented manifolds and ω a k-form on Y, the degree of the mapping is given by

$$\int_X f^*\omega = \deg[f] \int_Y \omega \ . \tag{4.14}$$

This formula yields for the remaining part of the action in the partition function (4.9)

$$\frac{1}{2}\int_\Sigma \Omega_{\alpha\beta} \mathrm{d}X^\alpha \wedge \mathrm{d}X^\beta = \frac{n}{2}\int_{\mathbb{S}^2} \Omega(X_0^I) \ , \tag{4.15}$$

where $\Omega(X_0^I)$ is the symplectic form on the corresponding leaf $L(X_0^I)$ induced by the linear POISSON structure $P^{ij} = c_k^{ij} X^k$. Consider the homotopy classes of the maps $X^\alpha : \Sigma \longrightarrow L(X_0^I)(\equiv \mathbb{S}^2)$. The HOPF theorem tells us that the mappings $f, g : \Sigma \longrightarrow \mathbb{S}^2(X_0^I)$ are homotopic if and only if the degree of the mapping f is the same as the degree of g. This means that the sum over the homotopy classes of the maps $[X^\alpha]$ can be expressed as a sum over the degrees $n = \deg[X^\alpha]$, therefore

$$\sum_{[X^\alpha]} \longrightarrow \sum_{n\in\mathbb{Z}} \ . \tag{4.16}$$

This gives for the partition function

$$Z = \int_{\Sigma_\Psi} \mathrm{d}X_0^I \sum_{n\in\mathbb{Z}} \det\left[\frac{\partial\chi_\alpha(X)}{\partial X^\gamma} P^{\gamma\beta}(X_0^I)\right]_{\Omega^0(\Sigma)} \det{}^{-1/2}\left[P^{\alpha\beta}(X_0^I)\right]_{\Omega^1(\Sigma)}$$
$$\times\ \exp\left(-\frac{in\hbar}{2}\int_{\mathbb{S}^2}\Omega(X_0^I)\right)\exp\left(-\frac{1}{\hbar}A_\Sigma C(X_0^I)\right) . \tag{4.17}$$

The sum over n yields a periodic δ-function

$$\sum_{n\in\mathbb{Z}} \exp\left(-\frac{in\hbar}{2}\int\limits_{\mathbb{S}^2}\Omega(X_0^I)\right) = \sum_{n\in\mathbb{Z}} \delta\left(\int\limits_{\mathbb{S}^2}\Omega(X_0^I) - \frac{n\hbar}{2}\right) . \tag{4.18}$$

The δ-function says that the symplectic leaves must be half-integer valued. This fact reduces the number of the symplectic leaves to a countable set, which is labeled by $\mathcal{O}(L)$. In [43] P. Schaller and T. Strobl have found the same result in the Dirac quantization scheme. The form of the partition is now

$$Z = \int\limits_{\Sigma_\Psi} \mathrm{d}X_0^I \sum_{n\in\mathbb{Z}} \det\left[\frac{\partial\chi_\alpha(X^\alpha)}{\partial X^\gamma}P^{\gamma\beta}(X_0^I)\right]_{\Omega^0(\Sigma)} \det{}^{-1/2}\left[P^{\alpha\beta}(X_0^I)\right]_{\Omega^1(\Sigma)}$$
$$\times\delta\left(\int\limits_{\mathbb{S}^2}\Omega(X_0^I) - \frac{n\hbar}{2}\right)\exp\left(-\frac{i}{\hbar}A_\Sigma C(X_0^I)\right) . \tag{4.19}$$

The next step is the calculation of the two determinants in the path integral. Choosing the "unitary gauge" $\chi_\alpha(X^\alpha) = X^\alpha$, such that $\partial\chi_\alpha(X)/\partial X^\gamma = \delta_\gamma^\alpha$, the two determinants have the same form. Due to the Hodge decomposition theorem they are characterized by harmonic forms with form degree zero or one. Now one has to count the linear independent forms, which are characterized by the dimension of the corresponding homology groups, the Betti numbers. These numbers yield for the power of the combined determinant the Euler characteristic $\chi(\Sigma)$. Indeed, this is a similar argument to that used by M. Blau and G. Thompson in [8]. The restriction of the scalar fields to the Casimir-Darboux coordinates X^I corresponds to the restriction of the scalar fields to the invariant Cartan subalgebra considered by M. Blau and G. Thompson. The result is a factor

$$\det\left[P^{\alpha\beta}(X_0^I)\right]^{\chi(\Sigma)} . \tag{4.20}$$

The determinant of a mapping equals the volume of the image of that mapping, hence the determinant $\det(P^{\alpha\beta}(X_0^I))$ corresponds to the symplectic volume of $\mathrm{Vol}(L(X_0^I))$ of the leaf $L(X_0^I)$ specified by the constant mode X_0^I. The path integral then takes the form

$$Z = \int\limits_{\Sigma} \mathrm{d}X_0^I \sum_{n\in\mathbb{Z}} \mathrm{Vol}(L(X_0^I))^{\chi(\Sigma)}\ \delta\left(\int\limits_{\mathbb{S}^2}\Omega(X_0^I) - \frac{n\hbar}{2}\right)\exp\left(-\frac{i}{\hbar}A_\Sigma C(X_0^I)\right) . \tag{4.21}$$

Implementing the δ-function by integrating over X_0^I the sum over the mapping degrees becomes a sum over the set $\mathcal{O}(L)$ of the integral symplectic leaves,

$$Z = \sum_{\Omega\in\mathcal{O}(L)} \mathrm{Vol}(L)^{\chi(\Sigma)} \exp\left(-\frac{1}{\hbar}A_\Sigma C(L)\right) . \tag{4.22}$$

As one can see the restriction to a linear POISSON structure allows one to perform the complete calculation of the partition function.

Now one is in the position to pass over to the partition function of the YANG-MILLS theory. Identifying the symplectic leaves with the orbits of the coadjoint action the orbits are 2-spheres. The integrality of the leaves passes over to the orbits such that the orbits are integral and of maximum dimension so the orbits correspond to unitary irreducible representations of the LIE group G which belongs the LIE algebra $\mathfrak{g}$ dual to the linear POISSON manifold, see section 2. Then the coadjoint orbits are 2-spheres. The BETTI numbers are $b_1(\mathbb{S}^2) = 0$ and $b_2(\mathbb{S}^2) = 1$. So there is just one integrality condition given by equation (4.19) and one parameter, the size of the sphere, for irreducible representations. This corresponds to the fact that any irreducible unitary representation of $SU(2)$ is uniquely determined by its dimension. This can be seen by the special form of the character formula of KIRILLOV, see equation (2.30), which says that the symplectic volume of the coadjoint orbit equals the dimension of the corresponding irreducible unitary representation. Note that for the 2-sphere the symplectic volume and the surface area are the same.

The identification of the integral orbits with the irreducible unitary representations leads then to a sum over the representations λ. Further on one has to take into account the symmetrization map (2.37) which maps the quadratic CASIMIR $C(\Omega)$ into the CASIMIR function of the corresponding representation $C(\lambda)$. So the final form of the partition function is

$$Z = \sum_{\lambda} \dim(\lambda)^{\chi(\Sigma)} \exp\left(-\frac{1}{\hbar} A_\Sigma C(\lambda)\right), \tag{4.23}$$

where $\dim(\lambda)$ is the dimension of the representation λ. This is exactly the partition function for the two-dimensional YANG-MILLS theory, calculated by M. BLAU and G. THOMPSON in [8]. Note that by omitting the CASIMIR term in the action we get just a sum over the dimensions of the representations, which is the correct result for the BF theory.

4.3.2 The disc

In the previous section it was shown for closed world sheet manifolds that a linear POISSON structure on $\mathbb{R}^3$ leads to a closed expression for the partition function and it is in some sense dual to the one of the two-dimensional YANG-MILLS theory. In this section it will be shown that this duality can also be shown in the case of the disc as the world sheet.

One proceeds as in the case for the closed manifolds. Starting with the calculation of the partition function for the linear POISSON structure given by $P^{ij} = c_k^{ij} X^k$ on $\mathbb{R}^3$ one obtains an expression for the partition function and using the results of the orbit method one can recalculate that for the YANG-MILLS theory.

The mappings $X^\alpha : \mathbb{D}^2 \to \mathbb{S}^2$ are again characterized by the their degree,

$$\int_{\mathbb{D}_2} \frac{i\hbar}{2} \Omega_{\alpha\beta} \mathrm{d}X^\alpha \wedge \mathrm{d}X^\beta = \frac{in\hbar}{2} \int_{\mathbb{S}_2} \Omega(X_0^I) \,, \tag{4.24}$$

where $\Omega(X_0^I)$ denotes the symplectic form of the leaf associated to the CASIMIR X_0^I.

As in the case of closed manifolds the sum over the degree defines a periodic δ-function, such that

$$\sum_n \exp\left[-\frac{in\hbar}{2} \int_{\mathbb{S}_2} \Omega(X_0^I) \right] = \sum_n \delta \left(\int_{\mathbb{S}_2} \Omega(X_0^I) - \frac{n\hbar}{2} \right) \,. \tag{4.25}$$

As in the case of closed manifolds this shows that the symplectic leaves, and after the identification the coadjoint orbits, have to be integral, more precisely they are half integer valued. This integrality condition of the leafs reduced them to a countable set $\mathcal{O}(L(X_0^I))$.

Choosing the unitary gauge for the fields X^α both determinants in the partition function have the same form and it is possible to combine them. The number of the linear independent forms on a manifold with boundary with vanishing tangent components like the gauge fields, respectively the ghosts, is equal to the relative BETTI number. Then it follows that the combined determinant has as exponent the sum of the BETTI numbers which is nothing else than the EULER characteristic, now with the boundary components included. For more details see [19]. It follows that the exponent for the disc is just 1. The determinant is the symplectic volume of the symplectic leaf [24].

These considerations lead to the following final form for the partition function of the linear POISSON-sigma model on the disc

$$Z(\mathbb{D}^2, \phi_{L(X_0^I)}) = \sum_{L \in \mathcal{O}(L)} \mathrm{Vol}(L) \chi_L(\phi_L) \exp(-\frac{1}{\hbar} A_{\mathbb{D}^2} C(L)) \,, \tag{4.26}$$

with the notation $\chi_L(\phi_L) = \langle \delta_L, \phi_L(X^\alpha) \rangle = \int_L \Omega_L \phi_L(X^\alpha)$. Further the dependence of the function ϕ of the target space is now still on the coordinates of the leaf, but due to the fact that one integrates over the symplectic leaf with respect to the symplectic form $\Omega(X_0^I)$ the essential differences in choosing this function depends on the leaf.

As in the case of closed manifolds it is possible to identify the partition function of the linear POISSON-sigma model with that of the YANG-MILLS theory. This is essentially based on the duality of a linear Poisson manifold and a Lie algebra. The difference now is of course the appearance of the distribution $\chi_L(\phi_L)$, the remaining part of the boundary condition of the maps X^α. To see the *duality* of the two models one must identify the symplectic leaves with the coadjoint orbits and one has to choose a particular function ϕ on the orbit $\Omega(X_0^I)$

$$\phi_{\Omega(X_0^I)}(X^\alpha) = \exp(2\pi i\langle X^\alpha, \bar{X}\rangle)\ , \tag{4.27}$$

where $\bar{X}$ is a point on the dual space, the Lie algebra and $\langle\cdot,\cdot\rangle$ denotes the *duality* product which is of course non-degenerate. Now the distribution is nothing else than the Fourier transformation of the measure on the orbits, which is the symplectic structure of the orbit. This is turn the starting point of the character formula of Kirillov, equation (2.29)

$$\chi_\Omega(\exp \bar{X}) = \frac{1}{j(\bar{X})}\int\limits_\Omega \exp(2\pi i\langle X^\alpha, \bar{X}\rangle)\frac{\omega^r}{r!}\,. \tag{4.28}$$

To make the calculation a bit more transparent one can think of the formula in the following two steps

1. The map from functions on G to functions on $\mathcal{G} : f \mapsto \phi : \phi(\bar{X}) = j(\bar{X})f(\exp \bar{X})$, where $j(\bar{X}) = \sqrt{\frac{\mathrm{d}(\exp \bar{X})}{\mathrm{d}\bar{X}}}$.

2. The usual Fourier transform which sends functions on $\mathcal{G}$ to function on $\mathcal{G}^*$.

Performing first the Fourier transformation explicitly in the present case one gets

$$Z(\mathbb{D}^2, \bar{X}) = \sum_\Omega \mathrm{Vol}(\Omega)(\frac{\sin(4\pi X_0^I\bar{X})}{\bar{X}})\exp(A_{\mathbb{D}^2}C(\Omega))\,. \tag{4.29}$$

The Fourier transform of the Dirac measure restricted to the 2-spheres is proportional to $\frac{\sin(4\pi X_0^I\bar{X})}{\bar{X}}$ [14], where X_0^I stands now for the quadratic radius such that the argument in the sine is scaled by the volume of the 2-spheres, which is by equation (2.30) the dimension of the corresponding representation. To obtain the partition function for the Yang-Mills theory one has to calculate the determinant $j(\bar{X})$ of the exponential map. For the case of $SU(2)$ it is

$$J(\bar{X}) = \frac{\sin(\bar{X})}{\bar{X}}\,. \tag{4.30}$$

This leads to

$$\frac{\sin(4\pi X_0^I\bar{X})}{\bar{X}} * J^{-1} = \frac{\sin(4\pi X_0^I\bar{X})}{\sin(\bar{X})}\ , \tag{4.31}$$

which is exactly the expression for the character for the Lie group $SU(2)$. The representations are characterized by their dimensions $\dim(\lambda) = \mathrm{Vol}(\Omega) = 4\pi X_0^I$. Taking into account the symmetrization map which maps the quadratic CASIMIR $C(\Omega)$, which are characterizing the coadjoint orbit, into the CASIMIR $C(\lambda)$ of the corresponding representation one gets for the partition function

$$Z(\mathbb{D}^2, \exp \bar{X}) = \sum_{\lambda} \dim(\lambda)\chi_\lambda(\exp \bar{X}) \exp(-\frac{1}{\hbar} A_{\mathbb{D}^2} C(\lambda) \ , \qquad (4.32)$$

where $\chi_\lambda(\exp \bar{X})$ denotes the character of the representation λ of the $SU(2)$ group. The equation (4.32) is the partition of the two-dimensional YANG-MILLS theory on a disc [9]. It can be interpreted as a special case of the linear POISSON-sigma model, with $\exp(2\pi i\langle X^\alpha, \bar{X}\rangle)$ as the specific function on the boundary, which corresponds to $\exp \bar{X}$ by the identification of the POISSON manifold with its dual, the LIE algebra.

4.4 The partition function of the linear Poisson-sigma model on arbitrary surfaces

The two-dimensional oriented manifolds are fully classified. Starting with a few standard manifolds it is possible to obtain an arbitrary manifold with the help of a glueing prescription [41]. This fact was used by M. BLAU and G. THOMPSON to perform the complete quantization for the two-dimensional MAXWELL and YANG-MILLS theories in [9]. The goal of this section is to show that such a glueing prescription for the linear POISSON-sigma model exists, which allows the partition function for the glued manifold to be deduced from the partition functions for the components. The various cases are considered in turn.

4.4.1 $g = 0 \ , \ n \geq 1$

First manifolds with more than one boundary component should be obtained. Geometrically this means that one starts with a boundary component, a circle, and deforms it into a rectangle. After that one identifies two edges of the four which are opposite to each other such that one creates an additional boundary component. There already exists a formula which allows one to perform this calculation, see [49].

For functions $\phi_1, \phi_2 \in C(G)$ define the convolution to be

$$\phi_1 * \phi_2(g) = \int_G \phi_1(g')\phi_2(g'^{-1}g)\mathrm{d}g \ . \qquad (4.33)$$

Then there exists a well-known equation [47] for the generalized character

$$\chi_\lambda * \chi_{\lambda'} = \frac{\delta_{\lambda\lambda'}}{\dim(\lambda)}\chi_\lambda \,, \tag{4.34}$$

where λ denotes an irreducible representation of the group G. From this equation, together with the fact that the characters form an orthogonal basis for the central functions, one gets

$$\chi_\lambda(\phi_1 * \phi_2) = \frac{1}{\dim(\lambda)}\chi_\lambda(\phi_1)\chi_\lambda(\phi_2) \,. \tag{4.35}$$

One can shift the group convolution to the Lie algebra with the so-called wrapping map, see section 2. Let ψ_1, ψ_2 be G-invariant, smooth functions on $\mathfrak{g}$. By $\psi^\wedge$ denote the Fourier transform to the dual space of $\mathcal{G}$. Then since $\int_{L_\lambda} \mathrm{d}\mu = \int_{\Omega_\lambda} \mathrm{d}\mu = \dim(\lambda)$

$$\int\limits_L \psi_1^\wedge \mathrm{d}\mu \int\limits_L \psi_2^\wedge \mathrm{d}\mu = \mathrm{Vol}(L) \int\limits_L \psi_1^\wedge \psi_2^\wedge \mathrm{d}\mu \,, \tag{4.36}$$

where $\mathrm{d}\mu$ stands for the measure corresponding to the symplectic form of the coadjoint orbit L. Translating this into the notation of the previous section, one gets

$$\chi_L(\phi_1)\chi_L(\phi_2) = \mathrm{Vol}(L)\chi_L(\phi_1\phi_2) \,. \tag{4.37}$$

Using formula (4.37) in the partition function on the disc (4.26) yields

$$\begin{aligned} Z(\mathbb{C}, \phi_1\phi_2) &= \sum_{\mathcal{O}(L)} \mathrm{Vol}(L)\chi_L(\phi_1\phi_2)\exp(A_\mathbb{C} C(L)) & (4.38)\\ &= \sum_{\mathcal{O}(L)} \mathrm{Vol}(L)\chi_L(\phi_1)\chi_L(\phi_2)\frac{1}{\mathrm{Vol}(L)}\exp(A_\mathbb{C} C(L)) & (4.39)\\ &= \sum_{\mathcal{O}(L)} \chi_L(\phi_1)\chi_L(\phi_2)\exp(A_\mathbb{C} C(L)) \,. & (4.40) \end{aligned}$$

The result is a partition function containing two functions, one with support on each bound-

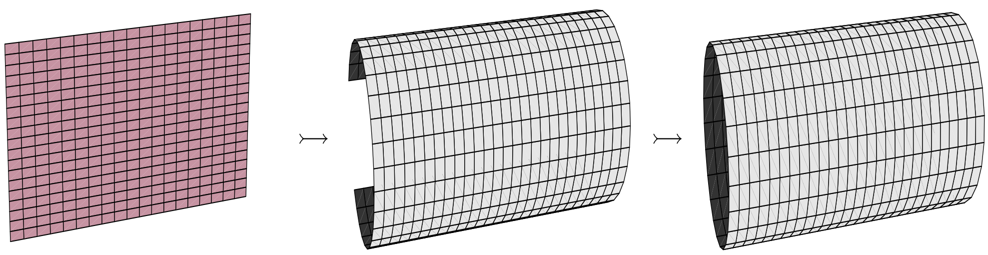

Figure 4.1: $\square \rightarrowtail \mathbb{C}$

ary. Geometrically this process can be interpreted as follows. First one deforms the boundary, the circle, into a rectangle such that each edge of the rectangle has its own degree of freedom, respectively its own function, on the edge. The freedom on the boundary turn into $\chi_L(\phi) = \chi_L(\phi_1, \phi_2, \phi_3, \phi_4)$, where the ϕ_i denote the corresponding parts of the function ϕ with support on the edge i of the rectangle. Then one identifies two opposite edges

$$\chi_L(\phi) \to \chi_L(\phi_1 \phi_a \phi_2 \phi_a^{-1}) = \chi_L(\phi_1 \phi_2) \ . \tag{4.41}$$

The resulting surface is of course a **cylinder**, see figure (4.1). This result can be compared with the results achieved in the Dirac quantization scheme by P. Schaller and T. Strobl in [43]. In that paper they performed the canonical quantization and solved the operator constraint equations for the linear Poisson-sigma model in Casimir-Darboux coordinates. Their result was that the wave functions are restricted to the symplectic leaves, as are the functions ϕ in the present calculation, and hence the distribution $\chi_L(\phi)$. Further, they showed that in the general case each integral orbit corresponds to one quantum state. This fact is given by the integrality condition of the symplectic leaves.

Note that by choosing both functions to be $\exp(2\pi i \langle X^\alpha, \bar{X}\rangle)$ one gets the right result for the partition function of the two-dimensional Yang-Mills theory on the cylinder [9].

The manifold with three boundary components, the next step in the construction, is called the **pants** manifold and the partition function is

$$Z(\mathbb{P}, \phi_1, \phi_2, \phi_3) = \sum_{\mathcal{O}(L)} \chi_L(\phi_1) \chi_L(\phi_a \phi_2 \phi_a^{-1} \phi_3) \exp(A_{\mathbb{P}} C(L))) \tag{4.42}$$

$$= \sum_{\mathcal{O}(L)} \chi_L(\phi_1) \chi_L(\phi_2 \phi_3) \exp(A_{\mathbb{P}} C(L))) \tag{4.43}$$

$$= \sum_{\mathcal{O}(L)} \frac{1}{\mathrm{Vol}(L)} \chi_L(\phi_1) \chi_L(\phi_2) \chi_L(\phi_3) \exp(A_{\mathbb{P}} C(L)) \ . \tag{4.44}$$

In this way one can get any manifold with an arbitrary number $n \geq 1$ of boundaries, for each boundary component there is first an additional factor with a new *boundary function* and an additional factor of $\mathrm{Vol}(L)^{-1}$.

4.4.2 $g = 0\ ,\ n = 0$

Now the partition function for the surface with genus $g = 0$ and no boundary component, which is the 2-sphere, should be calculated. The difference is that now one does not just *deform* the manifold as in the previous section, here one has to *glue* the manifolds together to get the sphere. For this glueing one defines the following product

$$\chi_L(\phi) \circledast \chi_{L'}(\phi^{-1}) := \frac{\chi_L(\phi)}{\mathrm{Vol}(L)} \frac{\chi_{L'}(\phi^{-1})}{\mathrm{Vol}(L')} = \delta_{LL'} \ , \tag{4.45}$$

By choosing $\phi_1 = \phi$ and $\phi_2 = \phi^{-1} = 1/\phi$ in equation (4.37) one gets

$$\chi_L(\phi)\chi_L(\phi^{-1}) = \mathrm{Vol}(L)\chi_L(\phi\phi^{-1}) = \mathrm{Vol}(L)^2. \tag{4.46}$$

The definition of the glueing product (4.45) is now quite natural.

With this product at hand one is in the position to calculate the partition function for the **sphere** by glueing two discs together, see figure (4.2)

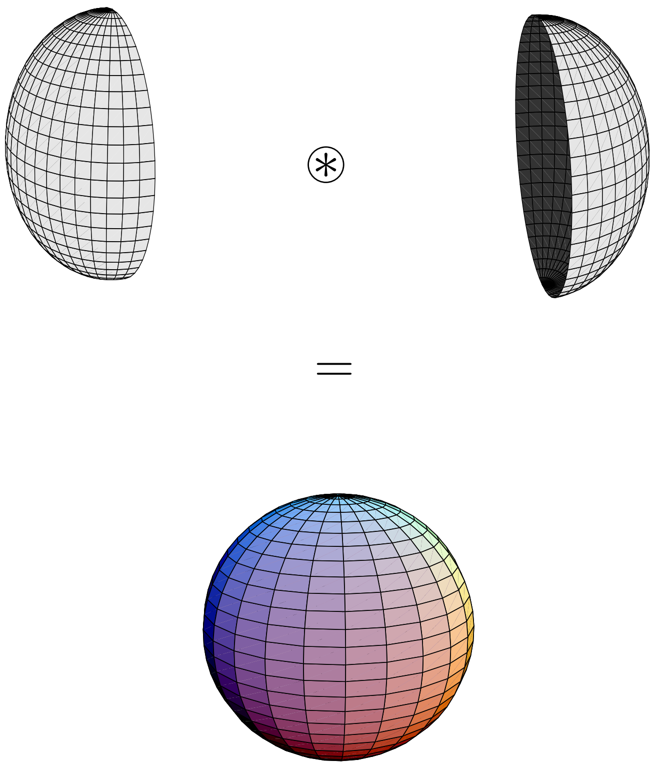

Figure 4.2: $\mathbb{D}^2 \circledast \mathbb{D}'^2 = \mathbb{S}^2$

$$
\begin{aligned}
Z(\mathbb{S}^2) =\ & Z(\mathbb{D}^2,\phi) \circledast Z(\mathbb{D}'^2,\phi^{-1}) \\
=\ & \sum_{\mathcal{O}(L)} \sum_{\mathcal{O}(L')} \mathrm{Vol}(L)\mathrm{Vol}(L') \underbrace{\left[\chi_L(\phi) \circledast \chi_{L'}(\phi^{-1})\right]}_{\delta_{LL'}} \exp(A_{\mathbb{D}^2} C(L)) \exp(A_{\mathbb{D}'^2} C(L')) \\
=\ & \sum_{\mathcal{O}(L)} \mathrm{Vol}(L)^2 \exp\left(A_{\mathbb{S}^2} C(L)\right) \ . \qquad (4.47)
\end{aligned}
$$

This result is exactly the partition function for the linear Poisson-sigma model on the sphere. By specifying equation (4.22) for the sphere, i.e. plugging in the Euler character for the sphere, which is 2, and the surface area of the sphere in the exponent, the above solution is obtained.

Another check for the new product is performed by deforming two discs to rectangles, and then glueing two edges together. The result should again be a rectangle. Hence, the resulting partition function should be the one of the disc. The partition function takes the form

$$
\begin{aligned}
Z(\Box,\phi_1\phi_2\phi_3\phi_4) &= \sum_{\mathcal{O}(L)} \mathrm{Vol}(L)\chi_L(\phi_1\phi_2\phi_3\phi_4) \exp(A_\Box C(L)) \\
&= \sum_{\mathcal{O}(L)} \chi_L(\phi_1\phi_2\phi_3)\chi_L(\phi_4) \exp(A_\Box C(L))
\end{aligned}
$$

$$
\begin{aligned}
\hookrightarrow\ \ & Z(\Box,\phi_1\phi_2\phi_3\phi_a) \circledast Z(\Box',\phi_a^{-1}\phi_4\phi_5\phi_6) \\
&= \sum_{\mathcal{O}(L)} \sum_{\mathcal{O}(L')} \chi_L(\phi_1\phi_2\phi_3)\left[\chi_L(\phi_a) \circledast \chi_{L'}(\phi_a^{-1})\right]\chi_{L'}(\phi_4\phi_5\phi_6) \exp\left(A_\Box\, C(L)\right) \exp\left(A_{\Box'}\, C(L')\right) \\
&= \sum_{\mathcal{O}(L)} \sum_{(L')} \chi_{L'}(\phi_1\phi_2\phi_3)\chi_L(\phi_4\phi_5\phi_6)\delta_{LL'} \exp\left(A_\Box\, C(L)\right) \exp\left(A_{\Box'}\, C(L')\right) \\
&= \sum_{\mathcal{O}(L)} \chi_L(\phi_1\phi_2\phi_3)\chi_L(\phi_4\phi_5\phi_6) \exp\left(A_\Box\, C(L)\right) \\
&= \sum_{\mathcal{O}(L)} \mathrm{Vol}(L)\chi_L(\phi_1\phi_2\phi_3\phi_4\phi_5\phi_6) \exp\left(A_\Box\, C(L)\right) \\
&= \sum_{\mathcal{O}(L)} \mathrm{Vol}(L)\chi_L(\phi) \exp\left(A_\Box\, C(L)\right) = Z(\mathbb{D}^2,\phi) \qquad (4.48)
\end{aligned}
$$

with $\phi = \phi_1\phi_2\phi_3\phi_4\phi_5\phi_6$. This calculation shows that the glueing product (4.45) is self-consistent.

4.4.3 $g = 1\ ,\ n \geq 0$

The next generalization is the possibility of a non-vanishing genus g. If one changes the genus of the surface one has to use the glueing product (4.45). The manifold with genus $g = 1$ and no boundary is the **torus**. One can get it by glueing together two cylinders, see figure (4.3)

$$\begin{aligned}Z(\mathbb{T}) &= Z(\mathbb{C},\phi_a\phi_b)\circledast Z(\mathbb{C}',\phi_a^{-1}\phi_b^{-1})\\ &= \sum_{\mathcal{O}(L)}\sum_{\mathcal{O}(L')}\left[\chi_L(\phi_a)\circledast\chi_{L'}(\phi_a^{-1})\right]\left[\chi_L(\phi_b)\circledast\chi_{L'}(\phi_b^{-1})\right]\exp\left(A_{\mathbb{C}}C(L)\right)\exp\left(A_{\mathbb{C}'}C(L')\right)\\ &= \sum_{\mathcal{O}(L)}\sum_{\mathcal{O}(L')}\delta_{LL'}\exp\left(A_{\mathbb{C}}C(L)+A_{\mathbb{C}'}C(L')\right)\\ &= \sum_{\mathcal{O}(L)}\exp\left(\left(A_{\mathbb{C}}+A_{\mathbb{C}'}\right)C(L)\right)\\ &= \sum_{\mathcal{O}(L)}\exp\left(A_{\mathbb{T}}\,C(L)\right)\ .\end{aligned} \tag{4.49}$$

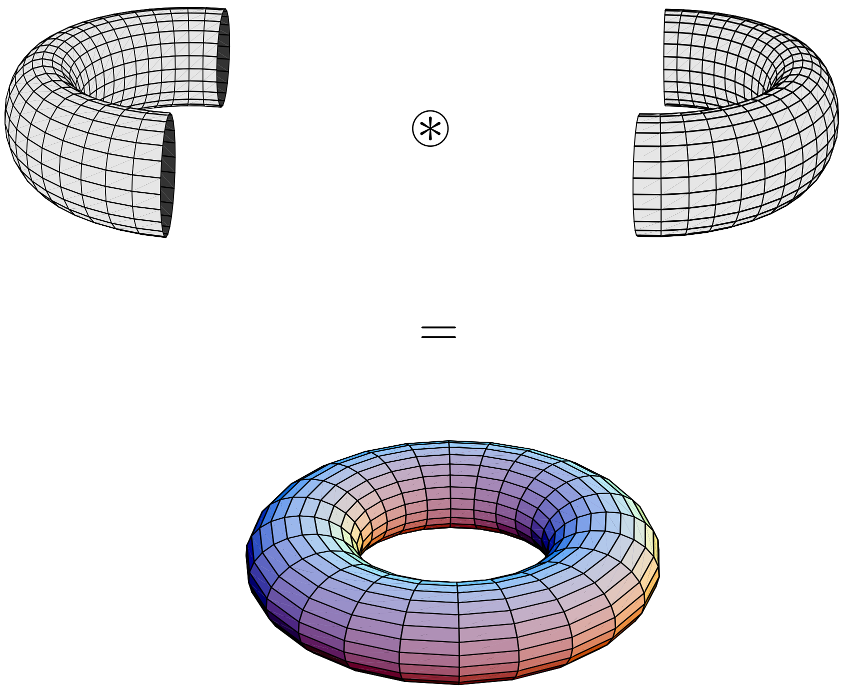

Figure 4.3: $\mathbb{C}\circledast\mathbb{C}'=\mathbb{T}$

The torus is again a manifold without boundary and one can compare it with equation (4.22) of section 4.3. The Euler character for the torus is zero, so in the partition function the symplectic volume of the coadjoint orbit does not appear.

The next manifold under consideration is the **handle** $\mathbb{H}=\mathbf{\Sigma_{1,1}}$, with genus $g=1$ and one boundary component $n=1$. To get this surface one has to take the pants manifold and glue two of the boundary components together, see figure (4.4). Due to the fact that one changes the genus by one one has to use the glueing product (4.45)

$$
\begin{aligned}
Z(\Sigma_{1,1}, \phi) &= Z(\mathbb{P}, \phi, \phi_a, \phi_a^{-1}) \\
&= \sum_{\mathcal{O}(L)} \frac{1}{\mathrm{Vol}(L)} \chi_L(\phi) \big[\chi_L(\phi_a) \circledast \chi_L(\phi_a^{-1})\big] \exp\big(A_{\Sigma_{1,1}}\, C(L)\big) \\
&= \sum_{\mathcal{O}(L)} \frac{1}{\mathrm{Vol}(L)} \chi_L(\phi) \exp\big(A_{\Sigma_{1,1}}\, C(L)\big) \ . \qquad (4.50)
\end{aligned}
$$

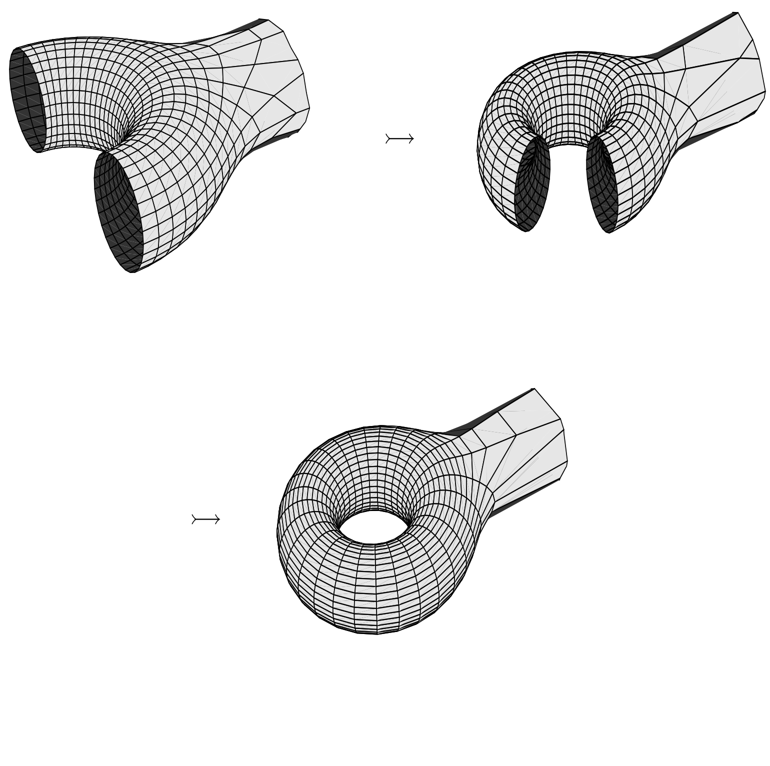

Figure 4.4: $\mathbb{P} \rightarrowtail \mathbb{H}$

This result enables one to calculate the partition function for the torus in yet a third way. Starting from the handle, one glues a disc onto its boundary, see figure (4.5)

$$\begin{aligned}
Z(\mathbb{T}) &= Z(\Sigma_{1,1}, \phi_a) \circledast Z(\mathbb{D}^2, \phi_a^{-1}) \\
&= \sum_{\mathcal{O}(L)} \sum_{\mathcal{O}(L')} \frac{1}{\mathrm{Vol}(L)} \left[\chi_L(\phi_a) \circledast \chi_{L'}(\phi_a^{-1})\right] \mathrm{Vol}(L') \exp\left(A_{\Sigma_{1,1}} C(L)\right) \exp\left(A_{\mathbb{D}^2} C(L')\right) \\
&= \sum_{\mathcal{O}(L)} \frac{\mathrm{Vol}(L')}{\mathrm{Vol}(L)} \delta_{LL'} \exp\left(A_{\Sigma_{1,1}} C(L) + A_{\mathbb{D}^2} C(L')\right) \\
&= \sum_{\mathcal{O}(L)} \exp\left(A_{\mathbb{T}} C(L)\right) \ , \qquad (4.51)
\end{aligned}$$

which is the same result as (4.49).

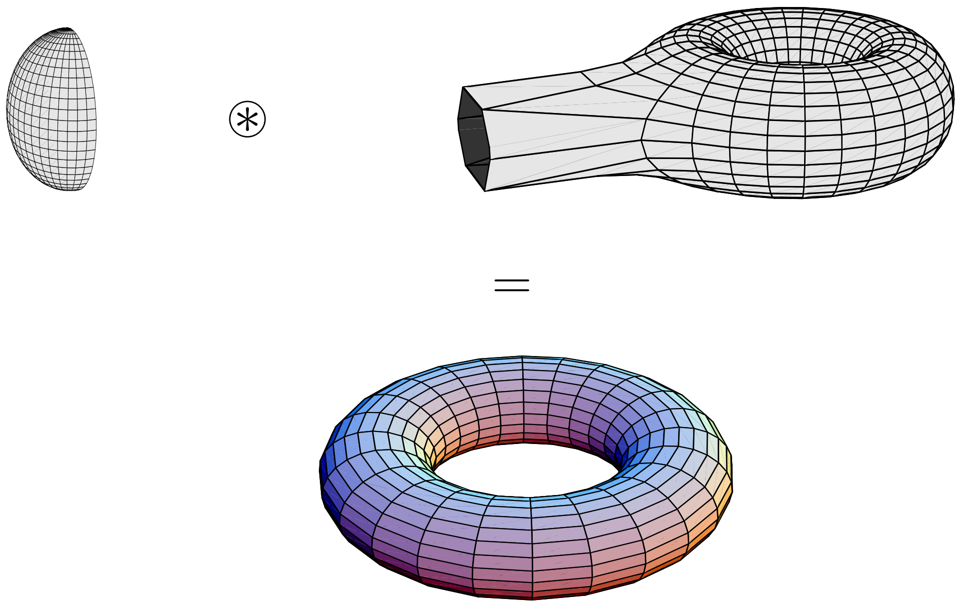

Figure 4.5: $\mathbb{D}^2 \circledast \Sigma_{1,1} = \mathbb{T}$

By glueing two pants together at two boundaries one obtains the manifold $\mathbf{\Sigma_{1,2}}$, see figure (4.6). The resulting partition function is

$$\begin{aligned} Z(\Sigma_{1,2},\phi_1,\phi_2) &= Z(\mathbb{P},\phi_1,\phi_a,\phi_b) \circledast Z(\mathbb{P}',\phi_2,\phi_a^{-1}\phi_b^{-1}) \\ &= \sum_{\mathcal{O}(L)}\sum_{\mathcal{O}(L')} \frac{1}{\mathrm{Vol}(L)}\frac{1}{\mathrm{Vol}(L')}\chi_L(\phi_1)\chi_{L'}(\phi_2)\left[\chi_L(\phi_a)\circledast\chi_{L'}(\phi_a^{-1})\right] \\ &\qquad \times \left[\chi_L(\phi_b)\circledast\chi_{L'}(\phi_b^{-1})\right]\exp\left(A_{\mathbb{P}}\,C(L)\right)\exp\left(A_{\mathbb{P}'}\,C(L')\right) \\ &= \sum_{\mathcal{O}(L)} \frac{1}{\mathrm{Vol}(L)^2}\chi_L(\phi_1)\chi_L(\phi_2)\exp\left(A_{\Sigma_{1,2}}\,C(L)\right) \ . \end{aligned} \tag{4.52}$$

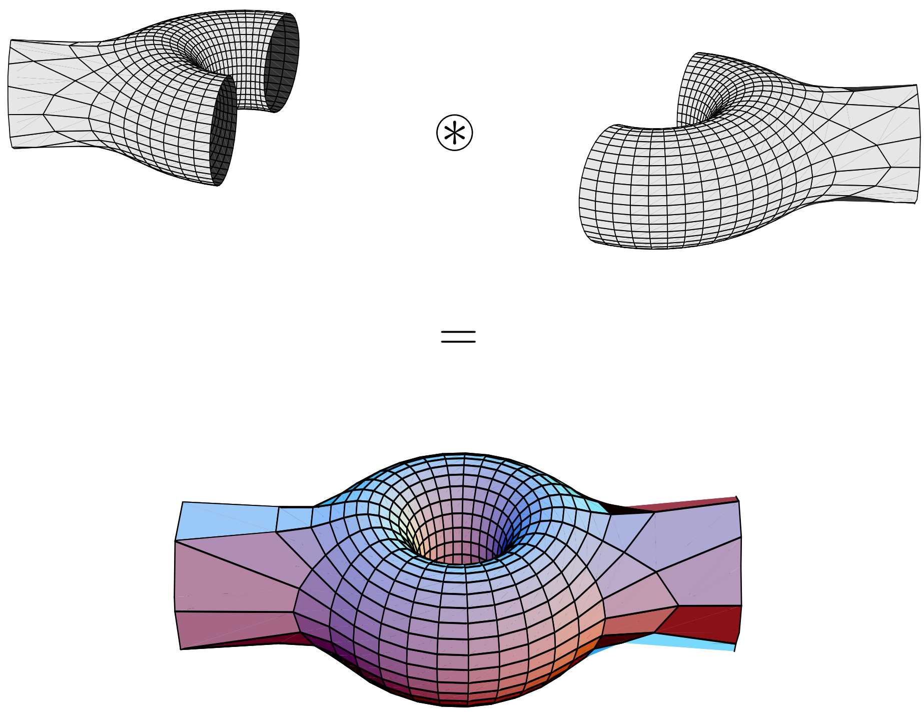

Figure 4.6: $\mathbb{P}\circledast\mathbb{P}'=\Sigma_{1,2}$

Due to the fact that one does not change the genus one can proceed as in the previous section. Starting with the partition function (4.50)

$$Z = \sum_{\mathcal{O}(L)} \frac{1}{\mathrm{Vol}(L)} \chi_L(\phi) \exp\left(A_{\Sigma_{1,1}}\, C(L)\right)$$

and applying (4.37)

$$\begin{aligned} &= \sum_{\mathcal{O}(L)} \frac{1}{\mathrm{Vol}(L)} \chi_L(\phi_1\phi_2) \exp\left(A_{\Sigma_{1,1}}\, C(L)\right) \\ &= \sum_{\mathcal{O}(L)} \frac{1}{\mathrm{Vol}(L)^2} \chi_L(\phi_1)\chi_L(\phi_2) \exp\left(A_{\Sigma_{1,2}} C(L)\right) \ , \end{aligned} \tag{4.53}$$

which is the same result as in equation (4.52). In this way one finds the partition function of any surface with genus $g = 1$ and an arbitrary number of boundary components n, $\mathbf{\Sigma_{1,n}}$.

4.4.4 Arbitrary g and n

With the considerations of the previous sections one is in the position to calculate the partition function for the linear POISSON-sigma model on an arbitrary two-dimensional (oriented) manifold. The fundamental manifold one starts with is the pants manifold $\Sigma_{0,3} = \mathbb{P}$. The question is how one can calculate the partition function on a manifold $\Sigma_{g,n}$ with arbitrary g and n. Starting with the pants manifolds, it should be possible to increase g and n in an arbitrary way. On the other hand, one must have the chance to decrease the number of boundary components to zero. Hence, one has three requirements:

- The adding of a disc, i.e. decreasing the number of boundary components n by one, results in multiplying the partition function by a factor $\mathrm{Vol}(L)$.
- The glueing of the pants manifold, i.e increasing the number of boundary components by one, results in a factor $\mathrm{Vol}(L)^{-1}$. This is similar to the application of (4.37).
- The glueing of $\Sigma_{1,2}$ increases the number of the genus by one, while in the partition function an additional factor of $\mathrm{Vol}(L)^{-2}$ appears.

These considerations lead to the following expression for the partition function of the linear POISSON-sigma model on an arbitrary surface $\Sigma_{g,n}$

$$Z(\Sigma_{g,n}, \phi_1, \ldots, \phi_n) = \sum_L \mathrm{Vol}(L)^{2-2g-n} \prod_{i=1}^{n} \chi_L(\phi_i) \exp[A_{\Sigma_{g,n}} C(L)] \ . \tag{4.54}$$

One sees that the exponent of the volume of the symplectic leaf is exactly the EULER characteristic for a two-dimensional manifold with genus g and n boundary components. This is the result which would be expected by considering the powers of the determinants in the partition function. If one now chooses for each function the specific one, which leads to the FOURIER transformation for the symplectic measure of the orbit, one reproduces the result for the two-dimensional YANG-MILLS theory on arbitrary oriented manifolds [9].

5 Summary and Concluding Remarks

In this thesis various properties of the classical and quantum theory of the semi-topological POISSON-sigma model were presented. By choosing different POISSON structures on the target space several field theories can be recovered. Due to the non-closedness of the gauge algebra, i.e. the algebra of the model is just closed modulo the equations of motion, the analysis of the gauge structure must follow the BATALIN-VILKOVISKY theory. After setting up the antifields for the model, the gauge-fixing was implemented for three different types of gauge conditions. An interesting fact, which has appeared for the resulting *gauge-fixed* action, was that in the non-covariant gauges the action simplified, in the sense that the term which is responsible for the non-closedness of the gauge algebra, vanishes. The appearance of the non-closure of the algebra is based on the non-linearity of the model due to the quadratic interaction in the gauge fields and the associated gauge transformations. This interaction can be generated with the help of the deformation procedure of G. BARNICH and M. HENNEAUX from the ABELian BF theory, such that the non-linearity is established by the deformation procedure.

The partition function was calculated for the case of closed manifolds and then extended to a world sheet with the topology of a two-dimensional disc. In both cases a quantization condition for the symplectic leaves were found. As in the DIRAC quantization procedure considered by P. SCHALLER and T. STROBL [43, 44], the leaves must be integral. Further, in both cases it was possible to show that the partition function of the two-dimensional YANG-MILLS theory can be recalculated by choosing a linear POISSON structure on the target manifold. The orbit method invented by A. KIRILLOV [30, 31, 32] provides the background to show the *duality* of these two models. On the one hand one has the YANG-MILLS theory with its associated gauge group. The partition function of this model involves a sum over the unitary irreducible representations of this group. On the other hand, one may start from the linear POISSON-sigma model, with a LIE algebra which arises as the dual of the linear POISSON structure. The POISSON manifold is however foliated into symplectic leaves, and the periodic delta-function which arises from the evaluation of the partition function limits one to the integral leaves. Each leaf corresponds to a coadjoint orbit of the LIE group, and the integral leaves correspond in turn to the irreducible representations of the group. Hence the sum over the integral leaves in the partition function of the linear POISSON-sigma model becomes after the identification the sum over the irreducible gauge group representations in the YANG-MILLS theory. The reproduction of the boundary condition for the YANG-MILLS theory on the disc, which appeared in the resulting partition function as the character of the corresponding unitary irreducible presentation, is given by the universal character formula by A. KIRILLOV. The corresponding term in the partition function of the linear POISSON-sigma model is a distribution with support on the integral symplectic leaves respectively the coadjoint orbits. Choosing a specific function for the distribution leads to the required FOURIER transform and the identification with the YANG-MILLS partition function. In the last chapter a glueing product was defined, which enables one to glue manifolds together by identifying certain boundary components. The partition function of the glued manifold was inferred by that of the components. This provides an expression for the partition function of the linear POISSON-sigma model on an oriented two-dimensional manifold with an arbitrary number of genus and boundary components. This can be seen as a full quantization for the linear POISSON-sigma model.

An interesting further step towards the general quantization of the POISSON-sigma model would be the calculation of the partition function for more general POISSON structures.

The connection between the POISSON-sigma model and KONTSEVICH 's star product discovered by A. CATTANEO and G. FELDER [11] remains a topic worthy of further research. The use of the star product in the deformation quantization approach to quantum theory provides new insights in quantum mechanics, see [23] and references therein, and quantum field theory [17]. Up to now one has used in these works essentially the MOYAL star product, which is limited to functions defined on symplectic manifolds. To treat gauge theories in this approach it will be necessary to work in the more general context of POISSON manifolds, in which case the KONTSEVICH construction, which yields the star product as a formal series in $\hbar$, is relevant. Here one may hope to gain information on the perturbative expansion from knowledge of the complete partition function. The case, where one deals with a linear POISSON structure like in this thesis, is particularly interesting because of the close connection which here prevails between the KONTSEVICH product and the CAMPBELL-BAKER-HAUSDORFF formula of group theory [29]. Another approach to the quantization of POISSON manifolds is to embed it as a Lagrangian submanifold into a symplectic manifold, which has the structure of a groupoid [10]. The quantization of the latter gives rise to the quantization of the former. The POISSON-sigma model provides in principle a connection between the KONTSEVICH quantization formula and the world of groupoids. A. CATTANEO and G. FELDER [13] have shown that the phase space of the topological POISSON-sigma model in the HAMILTON formalism is given by the space of leaves of the corresponding HAMILTONian foliation. This space comes with a natural structure of a (symplectic) groupoid. The relation is now indicated by the fact that the perturbation expansion of the same model yields, after an appropriate renormalization, the KONTSEVICH deformation formula.

This thesis has demonstrated that the POISSON-sigma model has and will provide interesting insight for the mathematical and physical aspects of the quantization of gauge theories.

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Tell me, I've still a lot to learn
Understand, these fires never stop
Believe me, when this joke is tired of laughing
I will hear the promise of my Orpheus sing
David Sylvian, *Orpheus* taken from *Secrets Of The Beehive*

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