# Freezing Limits for General Random Matrix Ensembles and Applications to Classical $\beta$-Ensembles

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**Dissertation**

Freezing Limits for General Random Matrix Ensembles and Applications to Classical $\beta$-Ensembles

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# Contents

# 1 Introduction

Expressions of type

$$\prod_{1\leq i<j\leq N}(x_j-x_i)^\beta \prod_{i=1}^{N} e^{-V(x_i)} \tag{1.1}$$

appear in several mathematical and physical contexts with $\beta > 0$ and some external potential function $V : \mathbb{R} \to \mathbb{R}$. In the physical context, they appear in Calogero-Mother-Sutherland models, which describe interacting particles on the line. In these models, the term

$$\prod_{1\leq i<j\leq N}(x_j-x_i)^\beta$$

can be interpreted as the repulsion of the particles among themselves, and the term

$$\prod_{i=1}^{N} e^{-V(x_i)}$$

is an external force that acts on all particles. The parameter $\beta$ can be interpreted as the inverse temperature. Therefore, the limit $\beta \to \infty$ is known as the freezing limit or the freezing regime.

In the mathematical context the eigenvalue density of certain random matrix ensembles, such as the Gaussian ensemble, the Wishart ensemble and the MANOVA ensemble, can be described with (1.1)[AGZ10; Dei99; Meh04; Tao12] as well.

For those three classical ensembles, the densities of the eigenvalues appear with the factors $\beta = 1, 2, 4$, depending on whether they are defined over the (screw-)field of real, complex or quaternionic numbers. Furthermore, there exist tridiagonal matrix models for the Gaussian ensemble, the Wishart ensemble and the the MANOVA ensemble [DE02; KN04], where the density of the eigenvalues of these matrices can be expressed by (1.1) for any $\beta > 0$.

The main problem studied in this thesis is the behavior of densities involving the term (1.1) in the limit $\beta \to \infty$ for general convex potentials V. This includes the three classical ensembles, the $\beta$-Hermite ensemble, the $\beta$-Laguerre ensemble, the $\beta$-Jacobi ensemble and some of their edge cases.

Furthermore, for the classical cases, the behavior of these freezing regimes will be studied when the numbers of particles $N$ approaches infinity.

The thesis is organized as follows:

In the next section, some mathematical background is given. Classical random matrix ensembles, such as the Gaussian ensemble, the Wishart ensemble and the MANOVA ensemble, will be introduced. An important characteristic of an random matrix is its set of eigenvalues. For the ensembles mentioned above, the eigenvalues are the $\beta$-Hermite ensemble, the $\beta$-Laguerre ensemble and the $\beta$-Jacobi ensemble. Furthermore, results for tridiagonal matrix models for these three classical cases are summarized.

In the third section, limit theorems for $\beta \to \infty$ are derived. To do so, the multivariate version of Laplace's method is reformulated and extended into a central limit theorem. This result is then applied to a certain class of $\beta$-ensembles. This class contains the three classical ensembles and is associated with orthogonal polynomials [Chi78; Sze75; Ism05]. Afterwards, the case where the maximum of (1.1) is obtained at a boundary point of the domain will be treated. The classical version of Laplace's method no longer works here. Thus, an extension of Laplace's method as introduced in [Bre94; Foc54; FS61; Hsu51; Won01] will be used, extended and rewritten to suit our stochastic point of view. Depending on whether the gradient of (1.1) at its maximum is zero or not, this leads to different types of limit theorems. Those limit theorems are similar and related to central limit theorems. One resulting limit distribution is the normal distribution projected onto a subspace and the other is the direct product of a normal distribution and an exponential distribution.

In the fourth section, the central limit theorems from section three are applied to the three classical ensembles: The $\beta$-Hermite ensemble, the $\beta$-Laguerre ensemble and the $\beta$-Jacobi ensemble. Additionally, some generalizations of the Hermite and the Laguerre ensemble will be studied, where $V(x) = x^2$ and $V(x) = x$ will be replaced with higher-order polynomials.
Moreover, the Laguerre ensemble depends on an additional parameter $\alpha > -1$ besides $\beta$, and the Jacobi ensemble depends on two additional parameters. When these parameters approach the value $-1$, the position of the maximum of (1.1) approaches the boundary of its domain. In this setting, theorems for boundary maxima from section three can be applied. This leads to (central) limit theorems for the freezing regime of the edge cases of classical $\beta$-ensembles.

In the fifth section, which is based on the work [AV19] and [HV21], covariance matrices from the central limit theorems in section four will be analyzed. Explicit formulas for the corresponding eigenvalues and eigenvectors will be derived. The eigenvectors are expressed with the help of some discrete orthogonal polynomials. These polynomials will turn out to be the de Boor-Saff dual polynomials of the classical Hermite, Laguerre and Jacobi polynomials. This duality will be summarized and then used to express the covariance matrices in terms of the Hermite, Laguerre and Jacobi polynomials themselves, instead of their dual polynomials.

In the sixth section, the limit behavior for $N \to \infty$ in the freezing regime will be analyzed. For random variables $X_{\beta,N}$ with the densities (1.1) this is the limit $\beta \to \infty$ and then $N \to \infty$. With the formulas from section five, the limit $N \to \infty$ of $\sigma_{N,N}$, the $(N,N)$-th entry of the covariance matrix, will be derived. For the largest eigenvalue of the Hermite and Laguerre ensemble, methods of ordinary differential equations are used to express the limit of $\sigma_{N,N}$ as an integral over the Airy function. Comparing these findings with the corresponding results in [DE05; GK22] leads to interesting integral identities for the Airy function.
The limit result for $\sigma_{N,N}$ is then restated as a limit theorem for the largest component of (1.1) for $\beta \to \infty$ and then $N \to \infty$. Finally, for the smallest eigenvalue of the Laguerre and Jacobi ensemble, similar results involving Bessel functions and Whittaker functions are derived.

# 2 Preliminaries and background

## 2.1 Random matrices

A random matrix is a real, complex or sometime quaternionic matrix whose entries are all random variables, or equivalently a random variable into the space of real, complex or quaternionic matrices. Of special interest are the space of $n \times n$-dimensional Hermitian matrices

$$\mathcal{H}_n(\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} : \overline{A}^T = A\}$$

and the space of real symmetric matrices

$$\mathcal{H}_n(\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} : A^T = A\}.$$

When equipped with the scalar products

$$< A, B >_{\mathcal{H}_n(\mathbb{C})} := \operatorname{tr}(A\overline{B}^T), \text{ and } < A, B >_{\mathcal{H}_n(\mathbb{R})} := \operatorname{tr}(AB^T),$$

they become Hilbert spaces and together with the associated Borel $\sigma$-algebra they become measurable spaces. Furthermore they are real vector spaces with

$$\dim_{\mathbb{R}}(\mathcal{H}_n(\mathbb{C})) = n^2 \text{ and } \dim_{\mathbb{R}}(\mathcal{H}_n(\mathbb{R})) = \frac{n(n+1)}{2}.$$

Unless otherwise stated, every density is always with respect to the Lebesgue measure. The Lebesgue measure on $\mathcal{H}_n(\mathbb{C})$ and $\mathcal{H}_n(\mathbb{R})$ is the push-forward measure of the standard Lebesgue measure on $\mathbb{R}^{n^2}$ and $\mathbb{R}^{\frac{n(n+1)}{2}}$ under the canonical isomorphism. Integration with respect to the Lebesgue measure will be denoted with $dA$. For details on random matrices, see for example the textbooks [AGZ10; Dei99; Meh04; Tao12] and references within.

### 2.1.1 Gaussian ensembles

**Gaussian orthogonal ensemble (GOE)**

Let $(X_{i,j})_{i,j=1,..,n}$ be independent, identically distributed (i.i.d) random variables with $X_{1,1} \sim \mathcal{N}(0, 2)$. Then, the random matrix $Z$ with

$$Z_{i,j} = \frac{1}{2}(X_{i,j} + X_{j,i})$$

is called *Gaussian orthogonal ensemble (GOE)*. $Z$ is symmetric and has the density

$$Ce^{-\frac{1}{2}\operatorname{tr}(A^2)}.$$

Here, $C > 0$ is the normalization constant such that

$$C \int_{\mathcal{H}_n(\mathbb{R})} e^{-\frac{1}{2}\operatorname{tr}(A^2)} dA = 1.$$

All entries in the upper right triangle of $Z$ are independent. The distributions on the diagonal are given by $\mathcal{N}(0,2)$ and those on the off-diagonal are given by $\mathcal{N}(0,1)$. Now, the ordered eigenvalues of $Z$ form an $n$-dimensional random variable with values on

$$\Lambda := \{\lambda \in \mathbb{R}^n : \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_n\}. \tag{2.1}$$

They have the density function

$$C_2 \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \prod_{i=1}^{n} e^{-\frac{\lambda_i^2}{2}}. \tag{2.2}$$

Here, $C_2 > 0$ is the normalization constant such that

$$C_2 \int_\Lambda \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \prod_{i=1}^{n} e^{-\frac{\lambda_i^2}{2}} = 1.$$

The constants $C, C_2$ can be calculated explicitly, see for example [Meh04] Chapter 17. For details on the orthogonal ensemble and the unitary ensemble and their eigenvalues for example [Dei99] Chapter 5.

**Gaussian unitary ensemble (GUE)**

Let $(X_{i,j}^R)_{i,j=1,..,n}$, $(X_{i,j}^I)_{i,j=1,..,n}$ be i.i.d random variables with $X_{1,1} \sim \mathcal{N}(0,2)$. Consider the complex random variables

$$X_{i,j} := X_{i,j}^R + iX_{i,j}^I.$$

Then, the random matrix $Z$ with

$$Z_{i,j} = \frac{1}{2}\left(X_{i,j} + \overline{X_{j,i}}\right)$$

is called *Gaussian unitary ensemble (GUE)*. $Z$ is hermitian and has the density

$$Ce^{-\frac{1}{2}\mathrm{tr}(A^2)}.$$

Here, $C > 0$ is the normalization constant such that

$$C \int_{\mathcal{H}_n(\mathbb{C})} e^{-\frac{1}{2}\mathrm{tr}(A^2)} dA = 1.$$

Now, the ordered eigenvalues of $Z$ form an $n$-dimensional random variable with values in

$$\Lambda = \{\lambda \in \mathbb{R}^n : \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_n\}.$$

They have the density function

$$C_2 \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)^2 \prod_{i=1}^{n} e^{-\frac{\lambda_i^2}{2}} \tag{2.3}$$

with

$$C_2 \int_\Lambda \prod_{1\le i<j\le n} (\lambda_j - \lambda_i)^2 \prod_{i=1}^n e^{-\frac{\lambda_i^2}{2}} = 1$$

The constants $C, C_2$ can be calculated explicitly, see for example [Meh04] Chapter 17. For details on the Gaussian unitary ensemble and its eigenvalues see for example [AGZ10] Chapter 2, [Meh04] Chapter2/Chapter 3, [Dei99] Chapter 5

**Orthogonal ensembles (OE) and unitary ensembles (UE)**

The GOE and the GUE can be generalized as follows: Let $Q$ be a polynomial of even degree with a positive leading coefficient. A random symmetric matrix with the density

$$Ce^{-\operatorname{tr}(Q(A))}$$

and $C > 0$ such that

$$\int_{\mathcal{H}_n(\mathbb{R})} e^{-\operatorname{tr}(Q(A))} dA = 1$$

is called *orthogonal ensemble (OE)*. Its ordered eigenvalues form a random variable with values in $\Lambda$ (2.1) and have the density

$$C_2 \prod_{1\le i<j\le n} (\lambda_j - \lambda_i) \prod_{i=1}^n e^{-Q(\lambda_i)} \tag{2.4}$$

with

$$C_2 \int_\Lambda \prod_{1\le i<j\le n} (\lambda_j - \lambda_i) \prod_{i=1}^n e^{-Q(\lambda_i)} = 1.$$

A random hermitian matrix with the density

$$C_3 e^{-\operatorname{tr}(Q(A))}$$

and $C_3 > 0$ such that

$$C_3 \int_{\mathcal{H}_n(\mathbb{C})} e^{-\operatorname{tr}(Q(A))} dA = 1$$

is called *unitary ensemble (UE)*. Its ordered eigenvalues form a random variable with values in $\Lambda$ (2.1) and have the density

$$C_4 \prod_{1\le i<j\le n} (\lambda_j - \lambda_i)^2 \prod_{i=1}^n e^{-Q(\lambda_i)} \tag{2.5}$$

with

$$C_4 \int_\Lambda \prod_{1\le i<j\le n} (\lambda_j - \lambda_i)^2 \prod_{i=1}^n e^{-Q(\lambda_i)} = 1.$$

### $\beta$-Hermite ensembles

For $\beta = 1, 2$, the ordered eigenvalues of the Gaussian orthogonal ensemble and the Gaussian uniform ensemble have values in

$$\{\lambda \in \mathbb{R}^n : \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_n\}.$$

They have the densities

$$C_{\beta,n} \prod_{1\leq i<j\leq n} (\lambda_j - \lambda_i)^\beta \prod_{i=1}^{n} e^{-\frac{\lambda_i^2}{2}} \tag{2.6}$$

with $C_{\beta,n}$ such that

$$C_{\beta,n} \int_{R^n} \prod_{1\leq i<j\leq n} (\lambda_j - \lambda_i)^\beta \prod_{i=1}^{n} e^{-\frac{\lambda_i^2}{2}} = 1.$$

For arbitrary $\beta > 0$, random variables with the densities (2.6) are also known as $\beta$-Hermite ensembles. It is known that the Gaussian symplectic ensemble (GSE), a quaternionic analogy of the GOE and GUE, has eigenvalues with densities according to (2.6) with $\beta = 4$ [Meh04]. For general $\beta > 0$, [DE02] introduced a tridiagonal matrix model which has the $\beta$-Hermite ensemble as eigenvalues. It is defined as follows:

### Tridiagonal Gaussian ensembles

For $r > 0$, consider a chi-squared distributed random variable with $r$ degrees of freedom. Then the root of such a random variable has the density

$$\frac{2^{1-\frac{r}{2}}}{\Gamma(\frac{r}{2})} x^{r-1} e^{-\frac{x^2}{2}}.$$

Denote the associated measure with $\chi_r$ and for $\beta > 0$ let $Y_1, ..., Y_n, Z_1, ..., Z_{n-1}$ be independent random variables with

$$\begin{aligned} Y_l &\sim \mathcal{N}(0, 2) && \text{for } l = 1, ..., n \\ \text{and } Z_l &\sim \chi_{\beta l} && \text{for } l = 1, ..., n-1. \end{aligned}$$

Then the ordered eigenvalues of the symmetric random tridiagonal matrix

$$\begin{pmatrix} Y_1 & Z_1 & & & \\ Z_1 & Y_2 & Z_2 & & \\ & \ddots & \ddots & \ddots & \\ & & Z_{n-2} & Y_{n-1} & Z_{n-1} \\ & & & Z_{n-1} & Y_n \end{pmatrix}$$

are distributed according to the $\beta$-Hermite ensemble. For details see [DE05].

### 2.1.2 Wishart ensembles

For $m, n \in \mathbb{N}$ with $m \geq n$ and $i = 1, ..., n,\ j = 1, ..., m$, let $X_{i,j}$ be i.i.d. random variables with $X_{1,1} \sim \mathcal{N}(0, 2)$. Then, the random matrix

$$Z = \begin{pmatrix} X_{1,1} & \dots & X_{1,m} \\ \vdots & & \vdots \\ X_{n,1} & \dots & X_{n,m} \end{pmatrix} \cdot \begin{pmatrix} X_{1,1} & \dots & X_{n,1} \\ \vdots & & \vdots \\ X_{1,m} & \dots & X_{n,m} \end{pmatrix}$$

is called *real Wishart ensemble*. $Z$ is symmetric and has the density

$$C \det(A)^{\frac{m-n-1}{2}} e^{-\frac{1}{2}\operatorname{tr}(A)}.$$

Here, $C > 0$ is the normalisation constant such that

$$C \int_{\mathcal{H}_n(\mathbb{R})} \det(A)^{\frac{m-n-1}{2}} e^{-\frac{1}{2}\operatorname{tr}(A)} dA = 1.$$

The ordered eigenvalues of $Z$ form an $n$-dimensional random variable with values in

$$\Lambda_0 := \{\lambda \in \mathbb{R}^n : 0 \leq \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_n\}.$$

They have the density

$$C_2 \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \prod_{i=1}^{n} \lambda_i^{\frac{1}{2}(m-n-1)} e^{-\frac{\lambda_i}{2}}.$$

with

$$C_2 \int_{\Lambda_0} \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \prod_{i=1}^{n} \lambda_i^{\frac{1}{2}(m-n-1)} e^{-\frac{\lambda_i}{2}} = 1.$$

The constants $C, C_2$ can be calculated explicitly, see for example [Meh04] Chapter 17. For details on the Wishart ensemble and its eigenvalues see for example [LNV18] Chapter 13.

#### $\beta$-Laguerre ensembles

For $m \geq n \in \mathbb{N}$ and $\beta = 1$, the ordered eigenvalues of the real Wishart ensemble have values in

$$\Lambda_0 = \{\lambda \in \mathbb{R}^n : \lambda_1 \leq \lambda_2 \leq .... \leq \lambda_n\}$$

and have the density

$$C_{\beta,m,n} \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)^{\beta} \prod_{i=1}^{n} \lambda_i^{\frac{\beta}{2}(m-n+1)-1} e^{-\frac{\lambda_i}{2}} \tag{2.7}$$

with $C_{\beta,m,n}$ such that

$$C_{\beta,m,n} \int_{\Lambda_0} \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)^\beta \prod_{i=1}^{n} \lambda_i^{\frac{\beta}{2}(m-n+1)-1} e^{-\frac{\lambda_i}{2}} = 1.$$

For arbitrary $\beta > 0$, random variables with the densities (2.6) are also known as $\beta$-Laguerre ensembles. It is known that complex and quaternionic analogies of the real Wishart ensemble, the *complex Wishart ensemble* and the *quaternion Wishart ensembles* have eigenvalues with densities according to (2.7) with $\beta = 2$ and $\beta = 4$. For general $\beta > 0$, [DE02] introduced a tridiagonal matrix model which has the $\beta$-Hermite ensemble as eigenvalues. It is defined as follows:

**Tridiagonal Wishart ensembles**

For $m \ge n$ and $r > 0$, consider a chi-squared distributed random variable with $r$ degrees of freedom. Then, the root of such a random variable has the density

$$\frac{2^{1-\frac{r}{2}}}{\Gamma(\frac{r}{2})} x^{r-1} e^{\frac{-x^2}{2}}.$$

Denote the associated measure with $\chi_r$ and let $Y_1, ..., Y_n, Z_1, ..., Z_{n-1}$ be independent random variables with

$$\begin{aligned} Y_l &\sim \chi_{(m-l+1)\beta} && \text{for } l = 1, ..., n \\ \text{and } Z_l &\sim \chi_{\beta l} && \text{for } l = 1, ..., n-1. \end{aligned}$$

Then the random matrix

$$L_\beta := \begin{pmatrix} Y_n & & & & \\ Z_{n-1} & Y_{n-1} & & & \\ & \ddots & \ddots & & \\ & & Z_2 & Y_2 & \\ & & & Z_1 & Y_1 \end{pmatrix} \cdot \begin{pmatrix} Y_n & Z_{n-1} & & & \\ & Y_{n-1} & \ddots & & \\ & & \ddots & Z_2 & \\ & & & Y_2 & Z_1 \\ & & & & Y_1 \end{pmatrix}$$

is symmetric and tridiagonal, and the ordered eigenvalues of $L_\beta$ are distributed according to the $\beta$-Laguerre ensemble. For details see [DE02].

### 2.1.3 Real MANOVA ensembles

For $m_1, m_2, n \in \mathbb{N}$ with $m_1 \ge n$ and $m_2 \ge n$, consider the random $n \times m_1$ matrix $X$, where all entries are independent and $\mathcal{N}(0,2)$-distributed, and the random $n \times m_2$ matrix $Y$, where all entries are independent and $\mathcal{N}(0,2)$-distributed. Then, the random matrix

$$XX^T + YY^T$$

is symmetric and almost surely positive definite, and therefore the matrix

$$(XX^T + YY^T)^{-\frac{1}{2}}$$

exists. The random matrix

$$Z := (XX^T + YY^T)^{-\frac{1}{2}} XX^T (XX^T + YY^T)^{-\frac{1}{2}}$$

is called the *real MANOVA ensemble*. $Z$ is symmetric and has the density

$$C \det(A)^{\frac{m_1-n-1}{2}} \det(I-A)^{\frac{m_2-n-1}{2}}.$$

Here, $C > 0$ is the normalization constant such that

$$C \int_{\mathcal{H}_n(\mathbb{R})} \det(A)^{\frac{m_1-n-1}{2}} \det(I-A)^{\frac{m_2-n-1}{2}} dA = 1.$$

The ordered eigenvalues of $Z$ form an $n$-dimensional random variable with values in

$$\Lambda_{0,1} := \{\lambda \in \mathbb{R}^n : 0 \leq \lambda_1 \leq ... \leq \lambda_n \leq 1\}.$$

It has the density

$$C_2 \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \prod_{i=1}^{n} \lambda_i^{\frac{m_1-n-1}{2}} (1-\lambda_i)^{\frac{m_2-n-1}{2}}$$

with

$$C_2 \int_{\Lambda_{0,1}} \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \prod_{i=1}^{n} \lambda_i^{\frac{m_1-n-1}{2}} (1-\lambda_i)^{\frac{m_2-n-1}{2}} = 1.$$

The constants $C, C_2$ can be calculated explicitly, see for example [Meh04] Chapter 17. For details on the MANOVA ensemble and its eigenvalues see for example Section 3.3 of [Mui82].

### $\beta$-Jacobi ensemble

For $m_1, m_2, n \in \mathbb{N}$ with $m_1, m_2 \geq n$ and for $\beta = 1$, the ordered eigenvalues of the real MANOVA ensemble take values in

$$\Lambda_{0,1} = \{\lambda \in \mathbb{R}^n : 0 \leq \lambda_1 \leq ... \leq \lambda_n \leq 1\}$$

and have the density

$$C_{\beta,n,m_1,m_2} \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)^{\beta} \prod_{i=1}^{n} \lambda_i^{\frac{\beta}{2}(m_1-n+1)-1} (1-\lambda_i)^{\frac{\beta}{2}(m_1-n+1)-1} \tag{2.8}$$

with $C_{\beta,n,m_1,m_2}$ such that

$$C_{\beta,n,m_1,m_2} \int_{\Lambda_{0,1}} \prod_{1\le i<j\le n} (\lambda_j - \lambda_i)^\beta \prod_{i=1}^n \lambda_i^{\frac{\beta}{2}(m_1-n+1)-1}(1-\lambda_i)^{\frac{\beta}{2}(m_1-n+1)-1} d\lambda = 1.$$

For arbitrary $\beta > 0$, and $m_1, m_2 > 0$ random variables with the densities (2.8) are also known as $\beta$-Jacobi ensembles. It is known that the complex and quaternionic analogies of the real MANOVA ensemble, the *complex MANOVA ensemble* and the *quarternion MANOVA ensemble*, have eigenvalues with densities according to (2.8) with $\beta = 2$ and $\beta = 4$. For $\beta > 0$, [KN04] introduced a tridiagonal matrix model which has the $\beta$-Jacobi ensemble as eigenvalues. It is defined as follows:

**Tridiagonal MANOVA ensemble**

Real-valued random variables are said to be beta-distributed with parameters $s, t > 0$ if they have the densities

$$2^{1-s-t}\frac{\Gamma(s+t)}{\Gamma(s)\Gamma(t)}(1-x)^{s-1}(1+x)^{t-1}\mathbb{1}_{[-1,1]}(x).$$

Denote the associated measure with $B(s,t)$. Let $a = \frac{\beta}{2}(m_1 - n + 1) - 1$, $b = \frac{\beta}{2}(m_1 - n + 1) - 1$, and for $k = 0, ..., 2n-2$ let $X_k$ be distributed as follows:

$$X_k \sim \begin{cases} B\left(\frac{2n-k-2}{4}\beta + a + 1, \frac{2n-k-2}{4}\beta + b + 1\right) & \text{for } k \text{ even },\\ B\left(\frac{2n-k-3}{4}\beta + a + b, \frac{2n-k-1}{4}\beta\right) & \text{for } k \text{ odd }. \end{cases}$$

Let $X_{2n-1} = X_{-1} = 0$ and for $k = 0, .., n-1$ define

$$Y_{k+1} = (1 - X_{2k-1})X_{2k} - (1 + X_{2k-1})X_{2k-2}$$
$$Z_{k+1} = \sqrt{(1 - X_{2k-1})(1 - X_{2k}^2)(1 + X_{2k+1})}.$$

Consider the random matrix

$$J_\beta = \begin{pmatrix} Y_1 & Z_1 & & & \\ Z_1 & Y_2 & Z_2 & & \\ & \ddots & \ddots & \ddots & \\ & & Z_{n-2} & Y_{n-1} & Z_{n-1} \\ & & & Z_{n-1} & Y_n \end{pmatrix}.$$

Then the ordered eigenvalues of the random matrix

$$\frac{1}{4}(J_\beta + 2I_n)$$

are distributed according to the $\beta$-Jacobi ensemble. For details see [KN04]. Note that the eigenvalues of $\frac{1}{4}(J_\beta + 2I_n)$ range from $-2$ to 2 and therefore their distribution is not exactly as in (2.8). This can be adjusted by a simple linear transformation.

# 3 Central limit theorems for $\beta$-matrix ensembles in the freezing regime

In this section, we will derive a central limit theorem for general matrix $\beta$-ensembles in the so-called freezing regime. The name originates from physical contexts, where the parameter $\beta$ can be interpreted as the inverse temperature. To be consistent with other betas occurring later, we will however denote this inverse temperature with the parameter $k$ or $\kappa$, while still calling these objects $\beta$-ensembles. We will postulate central limit theorems for different parametrizations and edge cases of matrix $\beta$-ensembles.

## 3.1 Central limit theorems from Laplace's method

In this subsection Laplace's method will be rewritten and extended to a more stochastic perspective. Therefore recall Laplace's method:

**Theorem 3.1** (Laplace's method)**.** *Let $\Omega \subset \mathbb{R}^N$ be a domain and $\phi \in C^2(\Omega)$, $\psi \in C^0(\Omega)$ with $\psi \geq 0$ such that for a $z \in \Omega$ it holds that*

1. *$\phi$ has a global maximum at $z = (z_1, ..., z_N) \in \Omega$ such that for every neighborhood $V$ of $z$ it holds that*
$$\sup\{\phi(x) : x \in \Omega \setminus V\} < \max\{\phi(x) : x \in \Omega\} = \phi(z),$$
2. *$\phi$ has a negative definite Hessian $H_\phi(z)$ in $z$,*
3. *$\int_\Omega \psi(x)e^{k\phi(x)}dx < \infty$ for all $k > 1$,*
4. *$\psi(z) \neq 0$.*

*Then*

$$\int_\Omega \psi(x)e^{k\phi(x)}dx \sim \left(\frac{2\pi}{k}\right)^{\frac{N}{2}} \frac{\psi(z)e^{k\phi(z)}}{\sqrt{\det(-H_\phi(z))}}.$$

*Proof.* The original proof goes back to Laplace [Sti86]. Other variants can be found in, for example, [Hsu51] or [Foc54]. The proof for the particular version stated above can be found in [Won01] Chapter IX, Section 5, Theorem 3 and Chapter IX, Exercise 5 or in [Bre94] Theorem 41, together with Lemma 38. □

Laplace's method is a theorem about the asymptotic behavior of integrals. It will now be extended to a theorem about the convergence of random variables which have the integrands as densities.

**Theorem 3.2.** *Let $\Omega \subset \mathbb{R}^N$ be a domain, $\phi \in C^3(\Omega)$ and $\psi \in C^0(\Omega)$ with $\psi, \phi > 0$ such that*

*1. $\phi$ has a global maximum at $z = (z_1, ..., z_N) \in \Omega$ such that for every neighborhood $V$ of $z$ it holds that*

$$\sup\{\phi(x) : x \in \Omega \setminus V\} < \max\{\phi(x) : x \in \Omega\} = \phi(z),$$

*2. $\phi$ has a negative definite Hessian matrix $H_\phi(z)$,*

*3. $\int_\Omega \psi(x)\phi(x)^k dx < \infty$ for all $k > 1$,*

*4. $\psi(z) \neq 0$.*

*Then for random variables $X_k$ with densities given by*

$$f_k(x) := c_k \psi(x)\phi(x)^k$$

*with normalization constants*

$$c_k^{-1} := \int_\Omega \psi(x)\phi(x)^k,$$

*it holds that*

$$\sqrt{k}(X_k - z) \to \mathcal{N}(0, \Sigma)$$

*weakly for $k \to \infty$. The covariance matrix $\Sigma$ is given by*

$$\Sigma^{-1} = -H_{\log\phi}(z) = -\frac{H_\phi(z)}{\phi(z)}. \tag{3.1}$$

*Proof.* The Lebesgue densities $\tilde{f}_k(x)$ of $\sqrt{k}(X_k - z)$ satisfy $\tilde{f}_k(x) = k^{-\frac{N}{2}} f_k(\frac{x}{\sqrt{k}} + z)$ and therefore

$$\begin{aligned}\tilde{f}_k(x) :&= c_k k^{-\frac{N}{2}} \psi\left(z + \frac{x}{\sqrt{k}}\right)\left(\phi\left(z + \frac{x}{\sqrt{k}}\right)\right)^k \\ &= c_k k^{-\frac{N}{2}} \psi\left(z + \frac{x}{\sqrt{k}}\right) \exp\left(k \log\left(\phi(z + \frac{x}{\sqrt{k}})\right)\right)\end{aligned} \tag{3.2}$$

on $\sqrt{k}(\Omega - z)$ and zero elsewhere.

We can now use a Taylor approximation of $\log(\phi)$ around $z$:

$$\log\left(\phi\left(z + \frac{x}{\sqrt{k}}\right)\right) = \log(\phi(z)) + \frac{1}{2}\frac{x}{\sqrt{k}}^T H_{\log\phi}(z)\frac{x}{\sqrt{k}} + R\left(\frac{x}{\sqrt{k}}\right), \tag{3.3}$$

where $R$ denotes the remainder term of the Taylor approximation, which will be estimated later. For now, it is sufficient to note that $R\left(\frac{x}{\sqrt{k}}\right) = O(k^{-\frac{3}{2}})$ for fixed values of $x$. Inserting the Taylor approximation (3.3) into (3.2) gives

$$\tilde{f}_k(x) = c_k k^{-\frac{N}{2}} \psi\left(z + \frac{x}{\sqrt{k}}\right) \phi(z)^k \exp\left(\frac{1}{2} x^T H_{\log\phi}(z) x + O(k^{-\frac{1}{2}})\right). \tag{3.4}$$

As $\psi \in C^0(\Omega)$ is continuous, it is obvious that $\psi(z+\frac{x}{\sqrt{k}})$ converges pointwise to $\psi(z)$ for $k \to \infty$. To approximate the limit of $c_k = \left(\int_\Omega \psi(x)\phi(x)^k\right)^{-1}$, we will use Theorem 3.1 for the functions $\psi$ and $\tilde{\phi} = \log(\phi)$. The conditions (1),(2),(3) and (4) carry over from $\phi$ to $\tilde{\phi}$. Therefore, we have

$$\lim_{k\to\infty} c_k k^{-\frac{N}{2}} \phi(z)^k = \frac{\sqrt{\det(-H_{\log\phi}(z))}}{\psi(z)(2\pi)^{\frac{N}{2}}}. \tag{3.5}$$

We conclude from (3.5) and (3.4) that for $k \to \infty$,

$$\tilde{f}_k(x) \to \frac{\det(-H_{\log\phi}(z))^{\frac{1}{2}}}{(2\pi)^{\frac{N}{2}}} \exp\left(-\frac{1}{2}x^T(-H_{\log\phi}(z))x\right)$$

pointwise. This is the density of the normal distribution $\mathcal{N}(0, H_{\log\phi}(z)^{-1})$. Now let $g \in C_c(\mathbb{R}^N)$. It remains to find an integrable upper bound for $\tilde{f}_k(x)g(x)$ to show distributional convergence. As the support of g is compact, there exists an $M > 0$ such that $g(x) = 0$ for $x \in \mathbb{R}^N$ with $|x| > M$. By the Lagrange estimate of the remainder term of the Taylor approximation, there exists a $\theta \in [0,1]$ with

$$R\left(\frac{x}{\sqrt{k}}\right) = \frac{1}{6}\sum_{|\alpha|=3}\left(\frac{x}{\sqrt{k}}\right)^\alpha D^\alpha \log\phi\left(z+\theta\frac{x}{\sqrt{k}}\right).$$

Here the multi-index notation is used. For details, see for example 7.17 in [Ste24] or similar Analysis textbooks. Because $\phi \in C^3(\Omega)$ and $\psi \in C^0(\Omega)$, there exists a $C_1 > 0$ such that

$$|D^\alpha \log\phi(z+y)| \leq C_1 \text{ and } \psi(z+y) \leq C_1$$

for all $y \in [-M,M]^N \cap (\Omega - z)$ and for all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = 3$. Therefore we have

$$\begin{aligned} &\left|g(x)\psi\left(z+\frac{x}{\sqrt{k}}\right)\exp\left(R_2\left[\log\phi; z+\frac{x}{\sqrt{k}}; z\right]\right)\right| \\ &\leq \left(\max_{y\in[-M,M]^N}\{g(y)\}\right) C_1 \exp\left(\frac{1}{6}\sum_{|\alpha|=3} M^\alpha C_1\right) \leq C_2 \end{aligned} \tag{3.6}$$

for some $C_2 > 0$, all $k > 1$ and all $x \in \sqrt{k}(\Omega - z)$. Furthermore, from (3.5) it follows that

$$c_k k^{-\frac{N}{2}} \phi(z)^k \leq \frac{\det(-H_{\log\phi}(z))^{\frac{1}{2}}}{\psi(z)(2\pi)^{\frac{N}{2}}} + 1 =: C_3 \tag{3.7}$$

for $k$ sufficiently large. If we combine (3.3) , (3.6) and (3.7), we obtain the integrable upper bound

$$|g(x)\tilde{f}_k(x)| \leq C_2 C_3 \exp\left(\frac{1}{2}x^T H_{\log\phi}(z)x\right)$$

for $k$ sufficiently large. Now we can conclude vague convergence by

$$\lim_{k\to\infty}\int_{\sqrt{k}(\Omega-z)} g(x)\tilde{f}_k(x)dx = \int_{\mathbb{R}^{\mathbb{N}}}\lim_{k\to\infty}\mathbb{1}_{\sqrt{k}(\Omega-z)}(x)g(x)\tilde{f}_k(x)dx$$
$$= \int_{\mathbb{R}^{\mathbb{N}}} g(x)\frac{\det(-H_{\log\phi}(z))^{\frac{1}{2}}}{(2\pi)^{\frac{N}{2}}}\exp\left(-\frac{1}{2}x^T - H_{\log\phi}(z)x\right)dx.$$

This also results in weak convergence because all measures are probability measures. □

Typically, a limit theorem of the form $\sqrt{k}(X_k - E(X_k)) \to \mathcal{N}(0,\Sigma)$ is called a central limit theorem. In Theorem 3.2, the random variables are centered around a given vector $z$ instead of the expected value $E(X_k)$. The following corollary shows that it is still appropriate to call Theorem 3.2 a central limit theorem.

**Corollary 3.3.** *Let $\phi, \psi \in C^\infty$ obey the assumptions of Theorem 3.2. Then it holds that*

$$\sqrt{k}(X_k - E(X_k)) \to \mathcal{N}(0,\Sigma)$$

*weakly for $k \to \infty$ with a covariance matrix $\Sigma$ as in* (3.1).

*Proof.* Because of the identity

$$\sqrt{k}(X_k - E(X_k)) = \sqrt{k}(X_k - z) - \sqrt{k}(E(X_k) - z),$$

we only have to show that the deterministic sequence $\sqrt{k}(E(X_{k,i}) - z_i)$ converges to 0 for $i = 1, ..., N$. We will therefore use higher-order terms of the asymptotic expansion given in Theorem 3.1. From [Won01] Chapter IX, Section 5, Theorem 3, we obtain

$$\int_\Omega \psi(x)\phi(x)^k dx \sim \frac{\phi(z)^k}{k^{\frac{N}{2}}}\left((2\pi)^{\frac{N}{2}}\frac{\psi(z)}{\sqrt{\det(-H_{\log\phi}(z))}} + O(k^{-1})\right)$$

for $\phi, \psi \in C^\infty$. For $i = 1, ..., N$, the mapping $x = (x_1, ..., x_N) \mapsto x_i$ is a an element of $C^\infty(\Omega)$, so the same holds true for $x \mapsto x_i\psi(x)$. Therefore we have

$$E(X_{k,i}) = \frac{\int_\Omega x_i\psi(x)\phi(x)^k dx}{\int_\Omega \psi(x)\phi(x)^k dx}$$
$$= \frac{\frac{\phi(z)^k}{k^{\frac{N}{2}}}\left((2\pi)^{\frac{N}{2}}\frac{z_i\psi(z)}{\sqrt{\det(-H_{\log\phi}(z))}} + O(k^{-1})\right)}{\frac{\phi(z)^k}{k^{\frac{N}{2}}}\left((2\pi)^{\frac{N}{2}}\frac{\psi(z)}{\sqrt{\det(-H_{\log\phi}(z))}} + O(k^{-1})\right)} = z_i + O(k^{-1}).$$

□

Note that in [Won01], asymptotic expansions for integrals of the above structure were derived. To achieve this, it is canonical and necessary to assume $\psi, \phi \in C^\infty$. However, we only use the first and second term of this asymptotic expansion introduced in [Won01]. It is likely that Corollary 3.3 holds true for weaker assumptions, such as $\phi \in C^3$ and $\psi \in C^1$, as well.

In the following, we will apply the above central limit theorem to a special type of $\beta$-ensembles. Let $I = (a, b) \subseteq \mathbb{R}$ be an interval with $-\infty \leq a < b \leq \infty$. Consider the functions $V \in C^2(I)$ and $\psi \in C_b^0(I)$ with

$$V'' \geq 0 \text{ and } \psi > 0.$$

Furthermore assume that

$$\lim_{x\to a} V(x) = \lim_{x\to b} V(x) = \infty$$

if $a$ and $b$ are finite, and

$$\liminf_{x\to-\infty} \frac{V(x)}{\log(-x)} \geq c > N, \text{ or } \liminf_{x\to\infty} \frac{V(x)}{\log(x)} \geq c > N \tag{3.8}$$

if $a = -\infty$ or $b = \infty$, respectively. Under the above conditions, we can apply Theorem 3.2 to the $N$-dimensional random variables $X_\kappa$ with densities

$$c_\kappa \prod_{1\leq i<j\leq N} (x_j - x_i)^\kappa \prod_{i=1}^N \psi(x_i) \exp\left(-\kappa V(x_i)\right) \tag{3.9}$$

on $\Omega = \{x \in I^N : x_1 < x_2 < ... < x_N\}$ and normalization constants

$$c_\kappa = \left( \int_\Omega \prod_{1\leq i<j\leq N} (x_j - x_i)^\kappa \prod_{i=1}^N \psi(x_i) \exp\left(-\kappa V(x_i)\right) dx \right)^{-1}.$$

The result is as follows.

**Theorem 3.4.** *For $\kappa > 1$, consider the random variables $X_\kappa$ with densities given by* (3.9)*, fulfilling the above conditions. Then, for $\kappa \to \infty$, it holds that*

$$\sqrt{\kappa}(X_\kappa - z) \to \mathcal{N}(0, \Sigma)$$

*weakly. Here, $z = (z_1, ..., z_N)$ is the unique maximum of* (3.9) *and $\Sigma$ is given by*

$$(\Sigma^{-1})_{i,j} = (-H_{\log\phi}(z))_{i,j} = \begin{cases} \sum_{l=1, l\neq i}^N (z_i - z_l)^{-2} + V''(z_i) & \text{for } i = j \\ -(z_i - z_j)^{-2} & \text{for } i \neq j. \end{cases} \tag{3.10}$$

*Proof.* We will use Theorem 3.2 and will therefore show that all necessary conditions hold. Consider the function

$$\phi(x) := \prod_{1\le i<j\le N}(x_j - x_i)\prod_{i=1}^{N}\exp(-V(x_i)).$$

One can see that

$$\log(\phi(x)) = \sum_{i<j}\log(x_j - x_i) - \sum_{i=1}^{N} V(x_i)$$

is a strictly concave function: For $1 \le i < j \le N$, the functions $x \mapsto x_j, x \mapsto -x_i$ are concave functions. Clearly they are both convex and concave, but in particular they are concave. Therefore the function $x \mapsto \log(x_j - x_i)$ is strictly concave, if considered only in two variables. Therefore $x \mapsto \sum_{i<j}\log(x_j - x_i)$ is a strictly concave function, and so is $x \mapsto \log(\phi(x))$. The strict concavity together with the boundary condition (3.8) yields that $\log\phi$ has a unique global maximum, which implies conditions (1) and (2) of Theorem (3.2) for $\log\phi$. This is equivalent to conditions (1) and (2) for $\phi$ because the logarithm is a twice differentiable monotone function. The integrability condition (3) follows from Equation (3.8). We will first show that $\phi$ itself is integrable. Because $\phi$ is positive and measurable ($\phi$ is continuous), we can apply the Tonelli-Fubini theorem and verify via integration by parts, see for example [Kön93] page 293. To integrate $\phi$ with respect to $x_N$, note that

$$\begin{aligned}\phi(x) &= \prod_{1\le i<j\le N}(x_j - x_i)\prod_{i=1}^{N}\exp(-V(x_i))\\ &= \exp(-V(x_N))\prod_{i=1}^{N-1}(x_N - x_i)\prod_{1\le i<j\le N-1}(x_j - x_i)\prod_{i=1}^{N-1}\exp(-V(x_i)).\end{aligned}$$

Hence, it is sufficient to show that $\exp(-V(x_N))\prod_{i=1}^{N-1}(x_N - x_i)$ is integrable with respect to $x_N$. Due to symmetry it is only necessary to consider one edge. If $b < \infty$, the integrability is obvious and if $b = \infty$, (3.8) gives that there exists an $M > 0$ such that

$$V(x) \ge c\log(x)$$

for $x \ge M$. Now an integrable upper bound can be given on the interval $[M, \infty)$ by

$$\exp(-V(x_N))\prod_{i=1}^{N-1}(x_N - x_i) \le x_N^{-c}\prod_{j=1}^{N-1}(x_N - x_1) = x^{N-1-c}\prod_{j=1}^{N-1}\left(1 - \frac{x_i}{x_N}\right).$$

Integration with respect to $x_{N-1}, x_{N-2}, ...$ follows by induction. □

Sometimes, a different parametrization can be useful. Therefore, in addition to the assumptions of Theorem 3.4, consider a monotonically increasing function $d \in C^1(I)$. We will consider the simple multivariate transformation, where every component is transformed with the same function $d$. This means $d(X_k) = (d(X_{\kappa,1}), ..., d(X_{\kappa,N}))$, i.e. $d : \mathbb{R}^N \to \mathbb{R}$ is $d : \mathbb{R} \to \mathbb{R}$ applied component-wise. The density of $d(X_k)$ is then given by

$$c_\kappa \prod_{1 \le i < j \le N} (d^{-1}(x_j) - d^{-1}(x_i))^\kappa \prod_{i=1}^{N} \psi(x_i) \exp\Big( - \kappa V(d^{-1}(x_i)\Big) \tag{3.11}$$

on $\Omega = \{x \in \mathbb{R}^N | d(a) < x_1 < x_2 < ... < x_N < d(b)\}$ with normalization constants

$$c_\kappa^{-1} = \int_\Omega \left( \prod_{1 \le i < j \le N} (d^{-1}(x_j) - d^{-1}(x_i))^\kappa \prod_{i=1}^{N} \psi(x_i) \exp\Big( - \kappa V(d^{-1}(x_i)\Big) \right) dx.$$

Now, by the Delta method of [Vaa98] Theorem 3.1, these transformed random variables obey the following central limit theorem.

**Theorem 3.5.** *For $\kappa > 1$, consider the random variables $X_\kappa$ with densities given by* (3.11)*, fulfilling the above conditions. Then, for $\kappa \to \infty$, there is*

$$\sqrt{\kappa}(X_\kappa - d(z)) \to \mathcal{N}(0, D\Sigma D),$$

*weakly. Here, $D = diag(d'(z_1), ..., d'(z_N))$ and $z = (z_1, ..., z_N)$ is the unique maximum of* (3.9) *from Theorem 3.4. Thus, $d(z) = (d(z_1), ..., d(z_N))$ is the unique maximum of* (3.11)*. Furthermore, $\Sigma^{-1}$ is given as in* (3.10)*, which yields*

$$\big((D\Sigma D)^{-1}\big)_{i,j} = \begin{cases} \sum_{l=1, l \neq i}^{N} (d'(z_i)(z_i - z_l))^{-2} + (d'(z_i))^{-2} V''(z_i) & \text{for } i = j \\ -\big(d'(z_i)d'(z_j)(z_i - z_j)^2\big)^{-1} & \text{for } i \neq j. \end{cases} \tag{3.12}$$

*Proof.* This is a direct application of Theorem 3.1 of [Vaa98] and Theorem 3.4. Note that to get $\psi$ as above, $\tilde{\psi}(\cdot) = \psi(d(\cdot))d'(\cdot) > 0$ has to be used in Theorem 3.4, as then the inverse function rule yields $\psi(x_i) = \tilde{\psi}(d^{-1}(x_i))(d^{-1})'(x_i)$.

□

Central limit theorems in the freezing regimes of the three classical $\beta$-ensembles (the Hermite ensemble, the Laguerre ensemble and the Jacobi ensembles) can be derived with Theorem 3.4. For this,

$$I = \mathbb{R} \text{ and } V(x) = x^2$$

must be selected for the Hermite ensemble,

$$I = (0, \infty) \text{ and } V(x) = (\alpha + 1) \log(x)$$

must be selected for the Laguerre ensemble, and

$$I = (-1, 1) \text{ and } V(x) = (\alpha + 1)\log(1 - x) + (\beta + 1)\log(1 + x)$$

must be selected for the Jacobi ensemble. This is explained in more detail in Section 3.
Each of these three ensembles is associated with a system of orthogonal polynomials. It is therefore interesting to take a closer look at generalized $\beta$-ensembles with a connection to orthogonal polynomials.

## 3.2 Freezing limits for $\beta$-ensembles associated with orthogonal polynomials

In this subsection, we will apply the above theorems to ensembles associated with an orthogonal polynomials system. This will give us more information about the location of the maximum.
Let $I = (a, b)$ be an interval with $-\infty \leq a < b \leq \infty$. Consider a function $v \in C^3((a, b))$ and a probability measure $\mu$ with the density given by $e^{-v(x)}$. Assume that

$$\lim_{x \to a} e^{-v(x)} = \lim_{x \to b} e^{-v(x)} = 0 \tag{3.13}$$

and

$$\int_a^b y^n e^{-v(y)} dy < \infty \tag{3.14}$$

is finite for all $n \in \mathbb{N}$. Let $\{p_n(x)\}_{n \in \mathbb{N}}$ be the associated system of orthonormal polynomials with respect to $\mu$, i.e

$$\int_a^b p_n(x) p_m(x) e^{-v(x)} dx = \delta_{mn}.$$

This system then satisfies the three-term equation

$$\begin{aligned} &p_0(x) = 1, p_1(x) = (x - b_0)/a_1 \\ &x p_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), \quad n \geq 1 \end{aligned} \tag{3.15}$$

for sequences $(a_n)_{n \in \mathbb{N}} \subseteq (0, \infty)$ and $(b_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$. For details see for example [Ism05; Chi78; Sze75]. Furthermore, for $n \in \mathbb{N}$ the functions $A_n, B_n$, which are defined by

$$\begin{aligned} \frac{A_n(x)}{a_n} &= \int_a^b \frac{v'(x) - v'(y)}{x - y} p_n^2(y) e^{-v(y)} dy \qquad (3.16) \\ \frac{B_n(x)}{a_n} &= \int_a^b \frac{v'(x) - v'(y)}{x - y} p_n(y) p_{n-1}(y) e^{-v(y)} dy, \end{aligned}$$

satisfy

$$p_n'(x) = -B_n(x)p_n(x) + A_n(x)p_{n-1}(x). \tag{3.17}$$

For further details and proofs see [Ism05], Chapter 3.2 ff.
Now consider the function

$$R_n(x) := -\left(v'(x) + \frac{A_n'(x)}{A_n(x)}\right)$$

and assume that for all $x \in I$ it holds that

$$v''(x) > 0 \text{ and } \frac{d^2}{dx^2}\Big(v(x) + \log(A_N(x))\Big) = -R_N'(x) > 0. \tag{3.18}$$

Then the function

$$\phi(x) := \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{i=1}^{N} \frac{\exp(-v(x_i))a_N}{A_N(x_i)} \tag{3.19}$$

has a unique maximum in $\Omega := \{x \in (a,b)^N : a < x_1 < x_2 < ... < x_N < b\}$ at $z = (z_1, .., z_N)$. Here, $z_1 < ... < z_N$ are the ordered zeros of $p_N$. The Hessian $H_{\log\phi}$ is negative definite and given by

$$(H_{\log\phi})_{i,j} = \begin{cases} -2\sum_{l=1, l\neq i}^{N} (z_i - z_l)^{-2} + R_N'(z_i) \text{ for } i = j \\ 2\,(z_i - z_j)^{-2} \text{ for } i \neq j. \end{cases} \tag{3.20}$$

For further details and proofs, see Theorem 3.5.1 in [Ism05].
Thus, if we modify $V$ from Theorem 3.4 to $\frac{1}{2}(v + \log(A_N))$, we get a central limit theorem which is centered around the zeros $z_1 < ... < z_N$ of the $N$-th orthogonal polynomial $p_N$. The result is as follows.

**Theorem 3.6.** *Let $\mu$, $(p_n(x))_{n\in\mathbb{N}}$, $v, A_n, B_n, R_n, S_n, \phi, \Omega, z$ be as described above. In particular, $v \in C^3(a,b)$, (3.13), (3.14) and (3.18) are fulfilled. Furthermore, let $\psi : \Omega \to (0,\infty)$ and let $X_\kappa$ be a random variable with the densities*

$$f_k(x) := c_\kappa \psi(x)\phi(x)^\kappa$$

*on $\Omega$ and normalization constants*

$$c_\kappa^{-1} := \int_\Omega \psi(x)\phi(x)^\kappa dx.$$

*Then*

$$\sqrt{\kappa}(X_\kappa - z) \to \mathcal{N}(0, \Sigma)$$

*weakly for $\kappa \to \infty$. Furthermore, the inverse covariance matrix $\Sigma^{-1}$ is given by*

$$\Sigma^{-1} = -H_{\log\phi}(z),$$

*with $H_{\log\phi}(z)$ as in (3.20).*

*Proof.* For this proof, we will apply Theorem 3.4 to the potential function

$$V(x) := \frac{1}{2}\left(v(x) + \log\left(\frac{A_N(x)}{a_N}\right)\right).$$

Therefore, we need to show that all conditions hold for $V$. First, condition (3.18) directly implies $V'' \geq 0$. It remains to show the boundary condition (3.8). Note that there exists a $C > 0$ such that for all $x$ close to $b$, we have

$$A_N(x) \geq C \text{ if } b < \infty \text{ and } A_N(x) \geq \frac{C}{x} \text{ if } b = \infty.$$

This holds because $v'$ is a monotonically increasing function due to (3.18), and for arbitrary $a < w_1 < w_2 < w_3 < x < b$ we can estimate

$$\begin{aligned}
\frac{A_N(x)}{a_N} &= \int_a^b \frac{v'(x) - v'(y)}{x - y} p_n^2(y) e^{-v(y)} dy \\
&\geq \int_{w_1}^{w_2} \frac{v'(x) - v'(y)}{x - y} p_n^2(y) e^{-v(y)} dy \\
&\geq \frac{v'(w_3) - v'(w_2)}{x - w_1} \int_{w_1}^{w_2} p_n^2(y) e^{-v(y)} dy \\
&\geq \begin{cases} \frac{v'(w_3)-v'(w_2)}{b-w_1} \int_{w_1}^{w_2} p_n^2(y) e^{-v(y)} dy \geq C, & \text{for } b < \infty \\ \frac{C}{x-w_1} = \frac{C}{x} \frac{1}{1-\frac{w_1}{x}} \geq \frac{C}{x}, & \text{for } b = \infty. \end{cases}
\end{aligned}$$

If $b < \infty$, (3.13) gives that $0 = \lim\limits_{x \to b} e^{-v(x)} = e^{-v(b)}$ and we can thus use the lower bound for $A_N$ to obtain

$$\lim_{x \to b} V(x) = \lim_{x \to b} \frac{1}{2}\left(v(x) + \log\left(\frac{A_N(x)}{a_N}\right)\right) = \infty.$$

Therefore, $\lim\limits_{x \to b} e^{-V(x)} = 0$. In the case where $b = \infty$, the function $y \mapsto y^n e^{-v(y)}$ is monotonically decreasing for $n \in \mathbb{N}$ and $y$ sufficiently large because

$$\begin{aligned}
&v'' \geq 0 \Rightarrow v' \text{ monotonically increasing}, \\
&v(x) \to \infty (x \to \infty) \Rightarrow v'(y) > 0 \text{ for } y \geq y_0, \\
\Rightarrow &(y^n e^{-v(y)})' = e^{-v(y)}\left(n - y v'(y)\right) < 0 \text{ for } y \geq y_0.
\end{aligned}$$

As $\int_a^b y^n e^{-v(y)} dy$ converges, we have

$$0 = \lim_{x \to b} x^n e^{-v(x)} \Rightarrow -\infty = \lim_{x \to b}\left(n \log(x) - v(x)\right) \Rightarrow \liminf_{x \to b} \frac{v(x)}{\log(x)} \geq n,$$

and thus

$$\begin{aligned}
\liminf_{x \to b} \frac{v(x)}{\log(x)} = \infty \Rightarrow \liminf_{x \to b} \frac{V(x)}{\log(x)} &= \liminf_{x \to b}\left(\frac{v(x)}{2\log(x)} + \frac{\log\left(\frac{A_N(x)}{a_N}\right)}{2\log(x)}\right) \\
&\geq \liminf_{x \to b}\left(\frac{v(x) + \log(C) - \log(x)}{2\log(x)}\right) = \infty.
\end{aligned}$$

The boundary condition for $x$ close to $a$ can be shown similarly.
Hence, as all conditions are fulfilled, we can apply Theorem 3.4 to $X_{\frac{\kappa}{2}} \sim \psi(x)\phi(x)^{\frac{1}{2}\cdot\kappa}$ with according density, with a potential

$$V(x) = \frac{1}{2}\left(v(x) + \log\left(\frac{A_N(x)}{a_n}\right)\right).$$

$\phi^{\frac{1}{2}}$ still obtains its maximum at $z$ and has the negative Hessian

$$-H_{\log(\sqrt{\phi})}(z) = -\frac{1}{2}H_{\log\phi(z)} = \frac{1}{2}\Sigma^{-1}.$$

Therefore, for $\kappa \to \infty$ we have the weak convergence

$$\sqrt{\kappa}(X_{\frac{\kappa}{2}} - z) \to \mathcal{N}(0, 2\Sigma)$$
$$\Rightarrow \sqrt{\frac{\kappa}{2}}\left(X_{\frac{\kappa}{2}} - z\right) \to \mathcal{N}(0, \Sigma).$$

Replacing $\frac{\kappa}{2}$ by $\kappa$ concludes the proof.

□

As already mentioned, Section 3 contains various example applications of Theorem 3.4 and Theorem 3.6. In particular, the three classical cases and generalizations thereof. Another important question is what the behavior is when the maximum is assumed at the boundary. This is what the next subsection deals with.

## 3.3 Limit theorems for the freezing regime at edge cases

In this subsection we will show similar limit theorems for the case that the maximizing point $z$ is located at the boundary of the domain. We differentiate between two cases: the gradient at $z$ is zero and the gradient at $z$ is not zero. We will start with the case where the gradient at $z$ is zero. For this, we will need an extension of Laplace's method. Let $l \in \mathbb{N}$ with $0 \leq l \leq N$.
We will first consider the case where the maximum is located at 0 and the domain is given by

$$\Omega = \{x \in \mathbb{R}^N : x_1 \geq 0, .., x_l \geq 0\}.$$

In this setting, Laplace's method is as follows:

**Theorem 3.7.** *For $\Omega = \{x \in \mathbb{R}^N : x_1 \geq 0, .., x_l \geq 0\}$, let $\phi \in C^3(\mathbb{R}^N)$, $\psi \in C^0(\mathbb{R}^N)$ such that*

1. *$\phi$ has a global maximum at 0 such that for every neighborhood $V$ of 0 it holds that*

$$\sup\{\phi(x) : x \in \Omega \setminus V\} < \sup\{\phi(x) : x \in \Omega\} = \phi(0),$$

*2. $\phi$ has a negative definite Hessian $H_\phi(0)$,*

*3. $\int_\Omega \psi(x)e^{k\phi(x)}dx < \infty$ for all $k > 1$,*

*4. $\psi \geq 0$ and $\psi(0) \neq 0$.*

*Then for $k \to \infty$ it holds that*

$$\int_\Omega \psi(x)e^{k\phi(x)}dx \sim \frac{1}{2^l}\left(\frac{2\pi}{k}\right)^{\frac{N}{2}} \frac{\psi(0)e^{k\phi(0)}}{\sqrt{\det(-H_\phi(0))}}.$$

*Proof.* There are various proofs of the standard version of Laplace's method, see Theorem 3.1. Most of them can be extended to prove this theorem. For $l = 1$, see for example [Won01] (5.15). We will now sketch a modification of the proof given in [Bre94], Theorem 41.
Consider the function

$$h_k(x) := \psi\left(\frac{x}{\sqrt{k}}\right)\exp\left(k\left(\phi\left(\frac{x}{\sqrt{k}}\right) - \phi(0)\right)\right).$$

Then, for $k \to \infty$ we have the pointwise convergence

$$h_k(x) \to \psi(0)\exp\left(\frac{1}{2}x^T H_\phi(0)x\right).$$

Furthermore, an integrable upper bound for $h_k$ can be given, as for example in [Bre94], proof of Theorem 41, Equation (5.32). Note that there, a compactification argument is used and crucial. The same applies here, however we will skip it here for simplicity. Now, the transformation formula yields

$$\frac{k^{\frac{N}{2}}}{e^{k\phi(0)}}\int_\Omega \psi(x)e^{k\phi(x)}dx = \int_\Omega h_k(x)dx \to \psi(0)\int_\Omega \exp\left(\frac{1}{2}x^T H_\phi(0)x\right)dx.$$

Due to symmetry we have

$$\int_{\{x_1\geq 0,\dots,x_l\geq 0\}} \exp\left(\frac{1}{2}x^T H_\phi(0)x\right)dx = \frac{1}{2^l}\frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det(-H_\phi(0))}},$$

which yields the claim. □

Similar to Theorem 3.2, Laplace's method can be used to proof a central limit theorem. This will be the next theorem. Furthermore, the domain of integration is extended to arbitrary intersections of hyperplanes. Therefore consider the following notations and conditions:

For $a_1, \dots, a_m \in \mathbb{R}^N$ and $b_1, \dots, b_m \in \mathbb{R}$, define

$$\Omega := \{x \in \mathbb{R}^N : a_1^T x + b_1 \leq 0, \dots, a_m^T x + b_m \leq 0\},$$

and let $\psi \in C^0(\mathbb{R}^N)$, $\phi \in C^3(\mathbb{R}^N)$ such that

- $\phi$ has a global maximum at $z \in \overline{\Omega}$ such that for every neighborhood $V$ of $z$ it holds that

$$\sup\{\phi(x) : x \in \Omega \setminus V\} < \sup\{\phi(x) : x \in \Omega\} = \phi(z),$$

- There is an $l \in \{1, ..., m\}$ such that

$$a_i^T z + b_i = 0 \text{ for } i = 1, ..., l$$
$$a_i^T z + b_i < 0 \text{ for } i = l+1, ..., m,$$

- $a_1, .., a_l$ are linearly independent,
- $\phi$ has a negative definite Hessian $H_\phi(z)$,
- $\int_\Omega \psi(x) e^{k\phi(x)} dx < \infty$ for all $k > 1$.

Now, let $X_k$ be random variables with densities given by

$$f_k(x) := c_k \psi(x) \exp(k\phi(x)) \mathbb{1}_\Omega(x)$$

on $\Omega$ and normalization constants

$$c_k^{-1} := \int_\Omega \psi(x) \exp(k\phi(x))\, dx.$$

**Theorem 3.8.** *Under the above notations and conditions*

$$\sqrt{k}(X_k - z)$$

*converges weakly for $k \to \infty$ to a random variable with a density given by*

$$\prod_{i=1}^{l} ||a_i||_2^2 \frac{2^l \det(-H_\phi(z))}{(2\pi)^{\frac{N}{2}} \det G} e^{\frac{1}{2} x^T H_\phi(z) x} \mathbb{1}_{\Omega^*}(x). \tag{3.21}$$

*Here, $\Omega^* = \{x \in \mathbb{R}^{\mathbb{N}} : a_1^T x \le 0, ..., a_l^T x \le 0\}$ and $G = (a_i^T a_j)_{i,j=1,...,l}$.*

Note that

$$H_{\tilde{\phi}}(0) := (G_I)^{-1} M^T H_\phi(z) M G_I^{-1}$$

is negative definite and symmetric, and

$$2^l \frac{\det(-H_{\tilde{\phi}}(0))}{(2\pi)^{\frac{N}{2}}} e^{\frac{1}{2} x^T H_{\tilde{\phi}}(0) x} \mathbb{1}_{\{x \in \mathbb{R}^N : x_1 \ge 0, .., x_l \ge 0\}}(x)$$

is the density of a normal distribution on the halfspace, quaterspace, etc., and (3.21) is the density of the push-forward measure of this density under the mapping $x \mapsto M(G_I)^{-1} x$.

*Proof.* Without loss of generality, it can be assumed that

$$||a_i||_2 = 1 \text{ for } i = 1, ..., m.$$

Because $a_1, .., a_l$ are linearly independent, one can find vectors $c_1, ..., c_{N-l}$ which form an orthonormal basis of

$$\text{span}(a_1, a_2, ..., a_l)^\perp.$$

The idea is to now transform the problem into the setting of Theorem 3.7. Therefore consider the matrices $M, G_I, G_0 \in \mathbb{R}^{N\times N}$ with

$$M = (a_1, a_2, .., a_l, c_1, c_2, ..., c_{N-l}), G_I = \begin{pmatrix} G & 0 \\ 0 & I_{N-l} \end{pmatrix} \text{ and } G_0 = \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned}
(\Omega - z) &= \left\{ x \in \mathbb{R}^N : \begin{array}{ll} a_i^T x \leq 0, & i = 1, ..., l \\ a_i^T x \leq \underbrace{-(a_i z + b_i)}_{=:\rho_i > 0}, & i = l+1, ..., m \end{array} \right\}, \\
M^{-1}(\Omega - z) &= \left\{ x \in \mathbb{R}^N : \begin{array}{ll} a_i^T M x \leq 0, & i = 1, ..., l \\ a_i^T M x \leq \rho_i, & i = l+1, ..., m \end{array} \right\} \\
&= \left\{ x \in \mathbb{R}^N : \begin{array}{ll} (G_0 x)_i \leq 0, & i = 1, ..., l \\ a_i^T M x \leq \rho_i, & i = l+1, ..., m \end{array} \right\}, \\
G_I M^{-1}(\Omega - z) &= \left\{ x \in \mathbb{R}^N : \begin{array}{ll} (G_0 (G_I)^{-1} x)_i \leq 0, & i = 1, ..., l \\ a_i^T M (G_I)^{-1} x \leq \rho_i, & i = l+1, ..., m \end{array} \right\} \\
&= \left\{ x \in \mathbb{R}^N : \begin{array}{ll} x_i \leq 0, & i = 1, ..., l \\ a_i^T M (G_I)^{-1} x \leq \rho_i, & i = l+1, ..., m \end{array} \right\}, \\
\sqrt{k} G_I M^{-1}(\Omega - z) &= \left\{ x \in \mathbb{R}^N : \begin{array}{ll} x_i \leq 0, & i = 1, ..., l \\ a_i^T M (G_I)^{-1} x \leq \rho_i \sqrt{k}, & i = l+1, ..., m \end{array} \right\},
\end{aligned}$$

and

$$\lim_{k\to\infty} \mathbb{1}_{\sqrt{k} G_I M^{-1}(\Omega - z)}(x) = \mathbb{1}_{\{x \in \mathbb{R}^{\mathbb{N}} : x_i \leq 0, i=1,...,l\}}(x).$$

Now, for $\tilde{\phi}(x) := \phi(MG_I^{-1}x + z)$ and $\tilde{\psi}(x) := \psi(MG_I^{-1}x + z)$, it follows that

$$\begin{aligned}
c_k^{-1} &= \int_\Omega \psi(x) \exp\left(k\phi(x)\right) dx \\
&= \frac{\det M}{\det G} \int_{\sqrt{k}G_I M^{-1}(\Omega - z)} \tilde{\psi}(x) \exp(k\tilde{\phi}(x)) dx \\
&= \frac{\det M}{\det G} \int_{\{x \in \mathbb{R}^N : x_i \leq 0,\, i=0,\ldots,l\}} \tilde{\psi}(x) \exp(k\tilde{\phi}(x)) dx \\
&- \frac{\det M}{\det G} \int_{\{x \in \mathbb{R}^N : x_i \leq 0,\, i=0,\ldots,l\} \setminus G_I M^{-1}(\Omega - z)} \tilde{\psi}(x) \exp(k\tilde{\phi}(x)) dx.
\end{aligned}$$

From Theorem 3.7, there is

$$\int_{\{x \in \mathbb{R}^N : x_i \leq 0,\, i=0,\ldots,l\}} \tilde{\psi}(x) \exp(k\tilde{\phi}(x)) dx \sim \frac{\det M}{\det G} \frac{1}{2^l} \left(\frac{2\pi}{k}\right)^{\frac{N}{2}} \frac{\tilde{\psi}(0) e^{k\tilde{\phi}(0)}}{\sqrt{\det(-H_{\tilde{\phi}}(0))}}$$

and

$$\frac{k^{\frac{N}{2}}}{e^{k\tilde{\phi}(0)}} \int_{\{x \in \mathbb{R}^N : x_i \leq 0,\, i=0,\ldots,l\} \setminus G_I M^{-1}(\Omega - z)} \tilde{\psi}(x) \exp(k\tilde{\phi}(x)) dx$$

converges pointwise to 0, because

$$\sup\{\phi(x) : x \in \Omega \setminus G_I M^{-1}(\Omega - z)\} < \tilde{\phi}(0)$$

and an integrable upper bound is given by Corollary 3.7. Thus there is

$$c_k^{-1} \sim \frac{\det M}{\det G} \frac{1}{2^l} \left(\frac{2\pi}{k}\right)^{\frac{N}{2}} \frac{\tilde{\psi}(0) e^{k\tilde{\phi}(0)}}{\sqrt{\det(-H_{\tilde{\phi}}(0))}}.$$

The densities of the random variables

$$\tilde{X_k} := \sqrt{k} G_I M^{-1}(X_k - z)$$

are given by

$$c_k k^{-\frac{N}{2}} \det(G_I^{-1}) \det(M) \tilde{\psi}\left(\frac{x}{\sqrt{k}}\right) \exp\left(k\tilde{\phi}\left(\frac{x}{\sqrt{k}}\right)\right) \mathbb{1}_{\sqrt{k}G_I M^{-1}(\Omega - z)}(x). \quad (3.22)$$

Like in the proof of Theorem 3.4, the Taylor expansion gives

$$\tilde{\phi}\left(\frac{x}{\sqrt{k}}\right) = \tilde{\phi}(0) + \frac{1}{2} \frac{x}{\sqrt{k}}^T H_{\tilde{\phi}}(z) \frac{x}{\sqrt{k}} + O(k).$$

Together with the above calculations, (3.22) converges pointwise to

$$2^l \left(\frac{1}{2\pi}\right)^{\frac{N}{2}} \sqrt{\det(-H_{\tilde{\phi}}(0))} \exp\left(\frac{1}{2} x^T H_{\tilde{\phi}}(0)) x\right) \mathbb{1}_{\{x \in \mathbb{R}^{\mathbb{N}} : x_i \leq 0, i=1,\dots,l\}}(x). \quad (3.23)$$

This is the density of a normal distribution with mean 0 and covariance matrix $(-H_{\tilde{\phi}}(0))^{-1}$ restricted to $\{x \in \mathbb{R}^{\mathbb{N}} : x_i \leq 0, i = 1, ..., l\}$. Now, as in the proof of Theorem 3.4, we can estimate the remainder part of the Taylor expansion to conclude that $\sqrt{k} G_I M^{-1}(X_k - z)$ converges vaguely for $k \to \infty$ to a random variable $Z$ whose density is given by (3.23). Again, as all measure involved are probability measures, weak convergence follows. Therefore

$$M G_I^{-1} \sqrt{k} G_I M^{-1}(X_k - z) = \sqrt{k}(X_k - z) \to M G_I^{-1} Z$$

weakly for $k \to \infty$. Now, because $H_{\tilde{\phi}}(0) = (M G_I^{-1})^T H_\phi(z) M G_I^{-1}$, the density of $M G_I^{-1} Z$ is given by (3.21), as was to be shown. □

Next, we will consider the edge cases where the gradient at the maximum is not zero. We again start with an extension of Laplace's method, which can be found in [Bre94] (Theorem 48).

**Theorem 3.9.** *For $m \in \mathbb{N}$ let $\phi, g_1, ..., g_m \in C^2(\mathbb{R}^N)$ and $\psi \in C^0(\mathbb{R}^N)$. The functions $g_1, ..., g_m$ define a compact set $\Omega = \cap_{i=1}^m \{x \in \mathbb{R}^{\mathbb{N}} : g_i(x) \leq 0\}$. Assume that the following conditions are fulfilled:*

*a) The function $\phi$ achieves its global maximum with respect to $\Omega$ only at the point $z$.*

*b) There is an $l \in \{1, ..., m\}$ such that*

$$g_i(z) = 0 \text{ for } i = 1, ..., l \text{ and}$$
$$g_i(z) < 0 \text{ for } i = l + 1, .., m.$$

*c) The gradients $\nabla g_i(z)$ with $i = 1, .., l$ are linearly independent.*

*d) The gradient $\nabla \phi(z)$ has a unique representation in the form*

$$\nabla \phi(z) = \sum_{i=1}^{l} \gamma_i \nabla g_i(z)$$

*with $\gamma_i > 0$ for $i = 1, ..., k$.*

*e) The matrix $H^*(z)$ is regular. Here*

$$H^*(z) = C^T(z) H(z) C(z)$$

*with*

$$H(z) = H_\phi(z) - \sum_{i=1}^{l} \gamma_i H_{g_i}(z)$$

*and*

$$C(z) = (c_1(z), ..., c_{N-l}(z)).$$

*$c_1(z), ..., c_{N-l}(z)$ form an orthonormal basis of the $N-l$-dimensional subspace of $\mathbb{R}^{\mathbb{N}}$, which is orthogonal to the subspace span$[\nabla g_1(z), ..., \nabla g_l(z)]$.*

*Then, the following asymptotic relation holds:*

$$\int_\Omega \psi(x) e^{k\phi(x)} \sim (2\pi)^{\frac{N-l}{2}} \frac{\psi(z)}{\sqrt{\det G \cdot |\det H^*(z)|} \prod_{i=1}^l \gamma_i} e^{k\phi(z)} k^{-\frac{N+l}{2}} dx.$$

*Here $G = ((\nabla g_i(z))^T \cdot \nabla g_j(z))_{i,j=1,...,l}$.*

We can now formulate a limit theorem for this edge case. The idea is to transform the random variable into directions where the gradient is zero and directions where the gradient is not zero. The scaling is then dependent on the direction. Unlike in Theorem 3.8, the transformation cannot be inverted after scaling. The resulting limit theorem looks somewhat similar to a central limit theorem, but the limit distribution is not a normal distribution. Consider the following notations and conditions:

For $a_1, ..., a_m \in \mathbb{R}^N$ and $b_1, ..., b_m \in \mathbb{R}$, define

$$\Omega := \{x \in \mathbb{R}^N : a_1^T x + b_1 \leq 0, ..., a_m^T x + b_m \leq 0\},$$

and let $\psi \in C^0(\mathbb{R}^N)$,$\phi \in C^3(\mathbb{R}^N)$ such that

- $\phi_{|\Omega}, \psi_{|\Omega} > 0$ and $\phi$ achieves its global maximum with respect to $\Omega$ only at the point $z \in \partial\Omega$.
- There is an $l \in \{1, ..., m\}$ such that

$$a_i^T z + b_i = 0 \text{ for } i = 1, ..., l$$
$$a_i^T z + b_i < 0 \text{ for } i = l+1, ..., m.$$

- $a_1, .., a_l$ are linearly independent. The gradient $\nabla\phi(z)$ has a unique representation of the form

$$\nabla\phi(z) = \sum_{i=1}^l \gamma_i a_i$$

  with $\gamma_i > 0$ for $i = 1, ..., l$.
- The matrix $H^*(z)$ is negative definite. Here

$$H^*(z) = C^T H_\phi(z) C$$

with

$$C = (c_1, ..., c_{N-l}).$$

$c_1, ..., c_{N-l}$ form an orthonormal basis of the subspace of $\mathbb{R}^{\mathbb{N}}$, which is orthogonal to the subspace $\text{span}(a_1, ..., a_l)$.

- $\int_\Omega \psi(x) e^{k\phi(x)} dx < \infty$ for all $k > 1$.

Furthermore, let $K = \text{diag}(k, .., k, \sqrt{k}, .., \sqrt{k})$, where $k$ appears $l$ times and $\sqrt{k}$ appears $N - l$ times. Consider the random variables $X_k$ with the densities given by

$$f_k(x) := c_k \psi(x) \exp(k\phi(x)) \mathbb{1}_\Omega(x),$$

normalization constants

$$c_k^{-1} := \int_\Omega \psi(x) \exp(k\phi(x)) \, dx$$

and

$$M = (A, C) = (a_1, a_2, .., a_l, c_1, c_2, ..., c_{N-l}),$$
$$G = (a_i^T a_j)_{i,j=1,...,l}, G_I = \begin{pmatrix} G & 0 \\ 0 & I_{N-l} \end{pmatrix}.$$

**Theorem 3.10.** *With the notations and conditions above, the random variable* $KG_I M^{-1}(X_k - z)$ *converges weakly for* $k \to \infty$ *to a random variable with a distribution given by the density*

$$\left(\prod_{i=1}^{l} ||a_i||_2^2 \gamma_i\right) \exp\left(((\gamma_1, ..., \gamma_l) \cdot x_l^*)^\top\right)$$
$$\cdot \frac{\sqrt{|\det H^*(z)|}}{(2\pi)^{\frac{N-l}{2}}} \exp\left(\frac{1}{2} \bar{x}_{N-l}^\top H^*(z) \bar{x}_{N-l}\right) \mathbb{1}_{\Omega^*}(x)$$

*where* $\Omega^* = \{x \in \mathbb{R}^{\mathbb{N}} : x_1 \leq 0, ..., x_l \leq 0\}$ *and* $x = (x_l^*, \bar{x}_{N-l})^T \in \mathbb{R}^l \times \mathbb{R}^{N-l} = \mathbb{R}^N$.

*Proof.* Without loss of generality, it can be assumed that

$$||a_i||_2 = 1 \text{ for } i = 1, ..., m.$$

Like in the proof of Theorem 3.8, we can see that

$$KG_I M^{-1}(\Omega - z) = \left\{ x \in \mathbb{R}^N : \begin{matrix} x_i \leq 0, & i = 1, ..., l \\ a_i^T M (G_I)^{-1} x \leq \rho_i \sqrt{k}, & i = l+1, ..., m \end{matrix} \right\},$$

where $\rho_i := -(a_i z + b_i) > 0$ for $i = l+1, ..., m$.
Now write $x = (x_l^*, \bar{x}_{N-l})^T \in \mathbb{R}^l \times \mathbb{R}^{N-l} = \mathbb{R}^N$ and define

$$\tilde{\phi}(x) := \phi(MG_I^{-1}x + z) \text{ and } \tilde{\psi}(x) := \psi(MG_I^{-1}x + z).$$

Then

$$G_I M^{-1}(X_k - z) \sim c_k \frac{\det M}{\det G} \tilde{\psi}(x) \exp(k\tilde{\phi}(x)) \mathbb{1}_{G_I M^{-1}(\Omega - z)}(x)$$

and the Taylor expansion

$$\begin{aligned}
\tilde{\phi}(x) &= \tilde{\phi}(0) + x^T \nabla \tilde{\phi}(0) + \frac{1}{2} x^T H_{\tilde{\phi}}(0) x + O(1) \\
&= \phi(z) + x^\top (MG_I^{-1})^\top \nabla \phi(z) + \frac{1}{2} x^\top (MG_I^{-1})^\top H_\phi(z) MG_I^{-1} x + O(1) \\
&= \phi(z) + x^\top \sum_{i=1}^{l} \gamma_i G_I^{-1} M^\top a_i + \frac{1}{2} x^\top (MG_I^{-1})^\top H_\phi(z) MG_I^{-1} x + O(1) \\
&= \phi(z) + O(1) \\
&+ x^\top \sum_{i=1}^{l} \gamma_i e_i + \frac{1}{2} (AG^{-1}x_l^* + C\bar{x}_{N-l})^\top H_\phi(z) (AG^{-1}x_l^* + C\bar{x}_{N-l}) \\
&= \phi(z) + O(1) \\
&+ ((\gamma_1, ..., \gamma_l) \cdot x_l^*)^\top + \frac{1}{2} (AG^{-1}x_l^* + C\bar{x}_{N-l})^\top H_\phi(z) (AG^{-1}x_l^* + C\bar{x}_{N-l})
\end{aligned}$$

holds. Here, $O(1)$ is locally uniform in $x$ for $k \to \infty$. Thus, it follows that the density of $KG_I M^{-1}(X_k - z)$ is given by

$$\begin{aligned}
&c_k \frac{\det M}{\det G} k^{-\frac{N+l}{2}} \tilde{\psi}(K^{-1}x) \exp\left(k\phi(z) + ((\gamma_1, ..., \gamma_l) \cdot x_l^*)^\top\right) \\
&\cdot \exp\left(\frac{1}{2} \bar{x}_{N-l}^\top C^\top H_\phi(z) C \bar{x}_{N-l} + O(\sqrt{k})\right)
\end{aligned}$$

on $KG_I M^{-1}(\Omega - z)$. From Theorem 3.9 we know that

$$c_k \sim \frac{1}{(2\pi)^{\frac{N-l}{2}}} \frac{\sqrt{\det G \cdot |\det H^*(z)|} \prod_{i=1}^l \gamma_i}{\psi(z)} e^{-k\phi(z)} k^{\frac{N+l}{2}}.$$

Therefore, as $k \to \infty$, the density of $KG_I M^{-1}(X_k - z)$ converges locally uniformly to

$$\begin{aligned}
&\frac{\prod_{i=1}^l \gamma_i}{(2\pi)^{\frac{N-l}{2}}} \sqrt{|\det H^*(z)|} \exp\left(((\gamma_1, ..., \gamma_l) \cdot x_l^*)^\top + \frac{1}{2} \bar{x}_{N-l}^\top H^*(z) \bar{x}_{N-l}\right) \\
&= \prod_{i=1}^l \gamma_i \exp\left(((\gamma_1, ..., \gamma_l) \cdot x_l^*)^\top\right) \frac{\sqrt{|\det H^*(z)|}}{(2\pi)^{\frac{N-l}{2}}} \exp\left(\frac{1}{2} \bar{x}_{N-l}^\top H^*(z) \bar{x}_{N-l}\right).
\end{aligned}$$

As before, local uniform convergence and the fact that all measures involved are probability measures together yield weak convergence. □

# 4 The freezing regimes of classical matrix models and other examples

In this section, we will apply the limit theorems from the previous section to three classical matrix ensembles: The Hermite, the Laguerre and the Jacobi ensemble. Furthermore, we will investigate related ensembles and their so-called edge cases.

## 4.1 Freezing regime of $\beta$-Hermite ensembles

Let $(\tilde{H}_n)_{n\in\mathbb{N}}$ be the Hermite polynomials orthonormalized with respect to the measure with the density $\pi^{-\frac{1}{2}}e^{-x^2}$, and let $z_N = (z_{1,N}, ..., z_{N,N})$ be the vector of the ordered zeros of $\tilde{H}_N$ with $z_{1,N} < ... < z_{N,N}$.

**Theorem 4.1** (Dumitriu, Edelmann(2005),Voit(2019)).
*Let $X_\kappa$ be $N$-dimensional random variables with the densities*

$$c_\kappa \prod_{1\leq i<j\leq N}(x_j - x_i)^{2\kappa}\prod_{i=1}^{N} e^{-\frac{\kappa}{2}x_i^2}$$

*on $\Omega := \{x \in \mathbb{R} : x_1 \leq x_2 \leq ... \leq x_N\}$ with*

$$c_\kappa^{-1} = \int_\Omega \prod_{1\leq i<j\leq N}(x_j - x_i)^{2\kappa}\prod_{i=1}^{N} e^{-\frac{\kappa}{2}x_i^2}.$$

*Then*

$$\sqrt{\kappa}(X_\kappa - \sqrt{2}z_N) \to \mathcal{N}(0, \Sigma)$$

*weakly for $\kappa \to \infty$.*
*Here, the entries of the covariance matrix $\Sigma = \left(\Sigma_{i,j}\right)_{i,j=1,\ldots,N}$ are given by*

$$\frac{\sum_{l=0}^{N-1}\tilde{H}_l^2(z_{i,N})\tilde{H}_l^2(z_{j,N}) + \sum_{l=0}^{N-2}\tilde{H}_{l+1}(z_{i,N})\tilde{H}_l(z_{i,N})\tilde{H}_{l+1}(z_{j,N})\tilde{H}_l(z_{j,N})}{\sum_{l=0}^{N-1}\tilde{H}_l^2(z_{i,N})\cdot\sum_{l=0}^{N-1}\tilde{H}_l^2(z_{j,N})}. \tag{4.1}$$

*Moreover, the entries of the inverse covariance matrix $\Sigma^{-1}$ are given by*

$$(\Sigma^{-1})_{i,j} := \begin{cases} 1 + \sum_{l\neq i}(z_{i,N} - z_{l,N})^{-2} & \text{for} \quad i = j \\ -(z_{i,N} - z_{j,N})^{-2} & \text{for} \quad i \neq j. \end{cases} \tag{4.2}$$

Using their tridiagonal $\beta$-matrix model in [DE02], Dumitriu and Edelman derived this limit theorem together with formula (4.2) for the covariance matrix. For details, see Theorem 3.1 in [DE05] Furthermore, in [Voi19], this limit theorem was proven in Theorem 2.2. A different method, which can be generalized to Theorem 3.2 and Theorem 3.6, was used. Furthermore, it is a special case of

Theorem 3.6 with $v(x) = \frac{x^2}{2}$. In this case, the orthonormal polynomials are the orthonormal Hermite polynomials and, up to scaling, given by $2^{\frac{1}{4}} \tilde{H}_m(x 2^{-\frac{1}{2}})$ for $m \in \mathbb{N}$, $x \in \mathbb{R}$. This means that the vector of ordered zeros is $\sqrt{2} z_N$. Here $A_N'(x) = 0$, which means that $R_N'(z_{i,N}) = 1$ so that the inverse covariance matrix (4.2) matches the one given in (3.20) for $v(x) = \frac{x^2}{2}$. The fact that $A_N'(x) = 0$ can easily be seen in Definition (3.16) or in [Ism05] (3.4.13)-(3.4.14).

## 4.2 Freezing regime of $\beta$-ensembles associated with Freud weights

In this subsection we will take a look at the so-called Freud weights studied in [Fre76]. Here, for $m > 0$, we have the weight function

$$w(x) = Ce^{-|x|^m}$$

with $C > 0$ fitting such that $\int_R w(x)dx = 1$. In the following, we will consider the case where $\frac{m}{2} \in \mathbb{N}$. For $m = 2$ we have $w(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ and the situation is, up to scaling, as in Theorem 4.1. Before we turn to the case for arbitrary $m$, we take a look at another case, namely $m = 4$. Here we have

$$w(x) = \frac{e^{-x^4}}{2\Gamma(5/4)}.$$

The aim is to apply Theorem 3.6 here. Therefore, we will look at the functions and conditions occurring there. The polynomials $(p_n)_{n \in \mathbb{N}}$, orthonormal with respect to $w$, obey the three-term recurrence relation

$$xp_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x).$$

We have $v(x) = x^4$ and $v'(x) = 4x^3$. To get $A_N(x)$, we follow [Ism05] p. 57. We have

$$\begin{aligned}
\frac{A_N(x)}{a_N} &= \int_{\mathbb{R}} \frac{v'(x) - v'(y)}{x - y} (p_N(y))^2 w(y) dy \\
&= 4 \int_{\mathbb{R}} \frac{x^3 - y^3}{x - y} (p_N(y))^2 w(y) dy \\
&= 4 \int_{\mathbb{R}} (x^2 + xy + y^2)(p_N(y))^2 w(y) dy \\
&= x^2 + \int_{\mathbb{R}} (y p_N(y))^2 w(y) dy \\
&= x^2 + a_{N+1}^2 + a_N^2.
\end{aligned}$$

Therefore, in this case

$$\left( \prod_{1 \le i < j \le N} (x_i - x_j)^2 \prod_{i=1}^{N} \frac{\exp(-v(x_i)) a_N}{A_N(x_i)} \right)^{\kappa}$$

becomes

$$\prod_{1\le i<j\le N}(x_j-x_i)^{2\kappa}\prod_{i=1}^{N}\frac{e^{-\kappa x_i^4}}{(x_i^2+a_N^2+a_{N+1}^2)^{\kappa}}. \tag{4.3}$$

Again, denote the zeros of $p_N$ by $z_N = (z_{1,N}, ..., z_{N,N})$. Then we can apply Theorem 3.6 to get the following central limit theorem:

**Theorem 4.2.** *Fix a sufficiently large $N \in \mathbb{N}$ and let $X_\kappa$ be the $N$-dimensional random variables with densities*

$$c_\kappa \prod_{1\le i<j\le N}(x_j-x_i)^{2\kappa}\prod_{i=1}^{N}\frac{e^{-\kappa x_i^4}}{(x_i^2+a_N^2+a_{N+1}^2)^{\kappa}}$$

*on $\Omega := \{x \in \mathbb{R} : x_1 \le x_2 \le ... \le x_N\}$ with*

$$c_\kappa^{-1} = \int_\Omega \prod_{1\le i<j\le N}(x_j-x_i)^{2\kappa}\prod_{i=1}^{N}\frac{e^{-\kappa x_i^4}}{(x_i^2+a_N^2+a_{N+1}^2)^{\kappa}}dx.$$

*Then*

$$\sqrt{\kappa}(X_\kappa - z_N) \to \mathcal{N}(0,\Sigma)$$

*weakly for $\kappa \to \infty$. Here the covariance matrix $\Sigma$ is given by*

$$(\Sigma^{-1})_{i,j} := \begin{cases} 2\sum_{l\neq i}(z_{i,N}-z_{l,N})^{-2} + 12z_{i,N}^2 + 2\frac{a_N^2+a_{N+1}^2-z_{i,N}^2}{(a_N^2+a_{N+1}^2+z_{i,N}^2)^2} & \textit{for} \quad i=j \\ -2(z_{i,N}-z_{j,N})^{-2} & \textit{for} \quad i\neq j. \end{cases} \tag{4.4}$$

*Proof.* The goal is to apply Theorem 3.6 for $v(x) = x^4$ and hence

$$\phi(x) = \prod_{1\le i<j\le N}(x_j-x_i)^{2}\prod_{i=1}^{N}\frac{e^{-x_i^4}}{(x_i^2+a_N^2+a_{N+1}^2)}$$

and $\psi(x) \equiv 1$. Thus we have to show that all conditions are fulfilled. The only non-trivial condition is that $\frac{d^2}{dx^2}\left(v(x) + \log(A_N(x))\right)$ is positive. We have

$$-R_N'(x) = \frac{d^2}{dx^2}\left(v(x)+\log(A_N(x))\right) = 12x^2 + 2\frac{a_N^2+a_{N+1}^2-x^2}{(a_N^2+a_{N+1}^2+x^2)^2}.$$

For $x \in [-\sqrt{a_N^2+a_{N+1}^2}, \sqrt{a_N^2+a_{N+1}^2}]$, we obviously have $-R_N'(x) \ge 0$ and for

$x \notin [-\sqrt{a_N^2 + a_{N+1}^2}, \sqrt{a_N^2 + a_{N+1}^2}]$, we can estimate

$$\begin{aligned}
-R_N'(x) &= 12x^2 - \frac{x^2}{(x^2 + a_N^2 + a_{N+1}^2)^2} + \frac{a_N^2 + a_{N+1}^2}{(x^2 + a_N^2 + a_{N+1}^2)^2} \\
&> 12x^2 - \frac{x^2}{4(a_N^2 + a_{N+1}^2)^2} \overset{!}{\geq} 0 \\
&\Leftrightarrow 12 \geq \frac{1}{4(a_N^2 + a_{N+1}^2)^2} \Leftrightarrow a_N^2 + a_{N+1}^2 \geq \sqrt{\frac{1}{48}}.
\end{aligned}$$

The latter can be assumed to be true because for large $N$, [Fre76] gives that

$$\lim_{N \to \infty} \frac{a_N}{N^{\frac{1}{4}}} = \sqrt{\frac{\Gamma(2)\Gamma(3)}{\Gamma(4)}}.$$

Hence it follows that $a_N = O(N^{\frac{1}{4}})$ and thus $a_N^2 + a_{N+1}^2 \geq \sqrt{\frac{1}{48}}$ for $N$ sufficiently large. □

**Remark 4.3.**

- Note that Theorem 4.2 may be correct for all $N \in \mathbb{N}$, meaning that the condition that $N$ has to be large may be obsolete.
- In [Voi19; HV21] Selberg's integral formula and related formulas were used to calculate

$$\lim_{\kappa \to \infty} \left(\frac{\kappa}{2\pi}\right)^{\frac{N}{2}} e^{-\kappa\phi(z)} \int_\Omega \prod_{1 \leq i < j \leq N} (x_i - x_j)^{2\kappa} \prod_{i=1}^N \left(\frac{\exp(-v(x_i))a_N}{A_N(x_i)}\right)^\kappa$$
$$= \sqrt{\det(\Sigma)}$$

  and derive a formula for the determinant of $\Sigma$. It is an interesting question whether this is also possible for (4.4).

Next, we will consider a polynomial $r_m(x) = \sum_{k=0}^m c_k x^k$ of degree $m$ with nonnegative coefficients $c_k \geq 0$ and a weight function

$$w(x) = Ce^{-r_m(x^2)}.$$

Here $C > 0$ is chosen such that $\int_{\mathbb{R}} w(x)dx = 1$. The aim is to apply Theorem 3.6. Therefore, we will look at the functions and conditions occurring there. For $r_m(x) = x^2$ and $r_m(x) = x^4$, the situation is similar to that in Theorem 4.1 and Theorem 4.2. Therefore, the resulting theorem can be seen as a generalization of these two. Furthermore, up to the factor $\frac{A_N(z_i)}{a_N}$, this density is similar to the density of the eigenvalues of the orthogonal ensemble and the unitary ensemble 2.1.1.

Because $w$ is symmetric, the polynomials $(p_n)_{n\in\mathbb{N}}$, orthonormal with respect to $w$, obey the three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + a_n p_{n-1}(x).$$

It holds that

$$v(x) = r_m(x^2) = \sum_{k=0}^{m} c_k x^{2k} \text{ and } v'(x) = \sum_{k=0}^{m-1} 2(k+1)c_{k+1}x^{2k+1}.$$

To get $A_N(x)$, we proceed as in the proof of Theorem 4.2. Note that Exercise 3.1 in [Ism05] also addresses this idea. It holds that

$$\begin{aligned}
\frac{A_N(x)}{a_N} &= \int_{\mathbb{R}} \frac{v'(x) - v'(y)}{x-y}(p_N(y))^2 w(y)dy \\
&= \sum_{k=0}^{m-1} 2(k+1)c_{k+1} \int_{\mathbb{R}} \frac{x^{2k+1} - y^{2k+1}}{x-y}(p_N(y))^2 w(y)dy \\
&= \sum_{k=0}^{m-1} 2(k+1)c_{k+1} \int_{\mathbb{R}} \sum_{l=0}^{k} x^{2l} y^{2(k-l)}(p_N(y))^2 w(y)dy \\
&= \sum_{k=0}^{m-1} \left( 2(k+1)c_{k+1} \sum_{l=0}^{k} \left( x^{2l} \int_{\mathbb{R}} y^{2(k-l)}(p_N(y))^2 w(y)dy \right) \right) \\
&=: s_{m-1}(x^2).
\end{aligned}$$

Therefore,

$$\left( \prod_{1\le i<j\le N} (x_i - x_j)^2 \prod_{i=1}^{N} \frac{\exp(-v(x_i))a_N}{A_N(x_i)} \right)^{\kappa}$$

becomes

$$\prod_{1\le i<j\le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} \frac{e^{-\kappa r_m(x_i^2)}}{(s_{m-1}(x_i{}^2))^{\kappa}}.$$

It is not easy to show that $x \mapsto r_m(x^2) + \log s_{m-1}(x^2)$ is a convex function. There may even be some counterexamples. Therefore we will assume convexity, i.e. for all $x \in \mathbb{R}$ assume that

$$r_m(x) + \log s_{m-1}(x) > 0. \tag{4.5}$$

Again, denote the zeros of $p_N$ by $z_N = (z_{1,N}, ..., z_{N,N})$. Now we can apply Theorem 3.6 for $v(x) = r_m(x^2)$ and $\psi(x) \equiv 1$ and get the following result:

**Theorem 4.4.** *Under the conditions mentioned above, let $X_\kappa$ be the $N$-dimensional random variables with densities*

$$c_\kappa \prod_{1\le i<j\le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} \frac{e^{-\kappa r_m(x_i^2)}}{(s_{m-1}(x_i{}^2))^\kappa}$$

*on $\Omega := \{x \in \mathbb{R} : x_1 \le x_2 \le ... \le x_N\}$ with*

$$c_\kappa^{-1} = \int_\Omega \prod_{1\le i<j\le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} \frac{e^{-\kappa r_m(x_i^2)}}{(s_{m-1}(x_i{}^2))^\kappa} dx.$$

*Then*

$$\sqrt{\kappa}(X_\kappa - z_N) \to \mathcal{N}(0, \Sigma)$$

*weakly for $\kappa \to \infty$. Here, the inverse covariance matrix $\left(\Sigma_{i,j}^{-1}\right)_{i,j=1,\dots,N}$ is given by*

$$\begin{aligned} &2\sum_{l\ne i}(z_{i,N} - z_{l,N})^{-2} + \frac{d^2}{dy^2}\left(r_m(x^2) + \log(s_{m-1}(x^2))\right)\Big|_{x=z_{i,N}} && \textit{for} \quad i = j, \\ &-2(z_{i,N} - z_{j,N})^{-2} && \textit{for} \quad i \ne j. \end{aligned}$$

## 4.3 Freezing regime of $\beta$-Laguerre ensembles and their edge case

In this section, let $(\tilde{L}_n^{(\alpha)})_{n\in\mathbb{N}}$ be the Laguerre Polynomials for $\alpha > -1$, orthonormalized with respect to $\frac{1}{\Gamma(\alpha+1)} e^{-x} x^\alpha \mathbb{1}_{[0,\infty)}(x)$ and with $z_N^{(\alpha)} = (z_{1,N}^{(\alpha)}, ..., z_{N,N}^{(\alpha)})$ the vector of ordered zeros $z_{1,N}^{(\alpha)} < ... < z_{N,N}^{(\alpha)}$. For a detailed description of the Laguerre polynomials, see for example [Chi78; Ism05; Sze75]. The vector $z_N^{(\alpha)}$ is the unique global maximum of the function

$$x \mapsto 2 \sum_{1\le i<j\le N} \log(x_j - x_i) + \sum_{i=1}^{N} \left((\alpha+1)\log(x_i) - x_i\right) \mathbb{1}_{\{x\in\mathbb{R}^N : 0\le x_1\le \dots \le x_N\}}(x).$$

Therefore, the gradient of this function is zero at $z_N^{(\alpha)}$, which leads to the identity

$$2\sum_{\substack{l=1\\ l\ne i}}^{n} \frac{1}{z_{i,N}^\alpha - z_{l,N}^\alpha} = 1 - \frac{\alpha+1}{z_{i,N}^{(\alpha)}}. \tag{4.6}$$

**Theorem 4.5** (Dumitriu, Edelmann(2005),Voit(2019))**.** *For $\alpha > -1$, let $X_\kappa$ be $N$-dimensional random variables with a the densities*

$$c_{\kappa,\alpha} \prod_{1\le i<j\le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} x_i^{(\alpha+1)\kappa - 1} e^{-x_i}$$

*on* $\Omega = \{0 \leq x_1 \leq ... \leq x_N\}$ *with* $c_{\kappa,\alpha}$ *given by*

$$c_{\kappa,\alpha}^{-1} := \int_\Omega \prod_{1\leq i<j\leq N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} x_i^{(\alpha+1)\kappa-1} e^{-x_i}.$$

*Then*

$$\sqrt{\kappa}\left(\frac{X_\kappa}{\kappa} - z_N^{(\alpha)}\right) \to \mathcal{N}(0,\Sigma)$$

*weakly for* $\kappa \to \infty$*. The inverse covariance matrix* $\Sigma_N^{-1}$ *is given by*

$$(\Sigma_N^{-1})_{i,j} := \begin{cases} 2\sum_{l\neq i}(z_{i,N}^{(\alpha)} - z_{l,N}^{(\alpha)})^{-2} + \frac{\alpha+1}{(z_{i,N}^{(\alpha)})^2} & \text{for} \quad i = j \\ -2(z_{i,N}^{(\alpha)} - z_{j,N}^{(\alpha)})^{-2} & \text{for} \quad i \neq j. \end{cases} \tag{4.7}$$

*Proof.* Using their tridiagonal $\beta$-matrix model in [DE02], Dumitriu and Edelman derived this limit theorem together with a rather complicated formula for the covariance matrix, see Theorem 4.1 in [DE05]. Furthermore, it is a special case of Theorem 3.2 and Theorem 3.6. If one chooses the weight function

$$w(x) = \frac{1}{\Gamma(\alpha+1)} e^{-x} x^\alpha$$

of the Laguerre polynomials, then, following [Ism05] (3.3.2), we obtain the following equations:

$$\begin{aligned} v(x) &= x - \alpha\log(x), \\ \frac{A_N(x)}{a_N} &= \frac{1}{x}, \\ \prod_{i=1}^{N} \frac{e^{-v(x_i)}a_N}{A_N(x_i)} &= \prod_{i=1}^{N} x_i^{(\alpha+1)} e^{-x_i}, \\ R_N'(z_{i,N}) &= \frac{d^2}{dx^2}\left(-v(x) - \log\left(\frac{A_N(x)}{a_N}\right)\right)\Big|_{x=z_{i,N}} = \frac{\alpha+1}{(z_{i,N})^2}. \end{aligned}$$

□

Theorem 4.5 can be transformed using the Delta method of [Vaa98] Theorem 3.1, compare Theorem 3.5 above with $d(x) = \sqrt{x}$. The result is as follows:

**Theorem 4.6** (Voit(2019),Dumitriu, Edelmann(2005))**.** *For* $\alpha > -1$*, let* $X_\kappa$ *be* $N$*-dimensional random variables with the densities*

$$\tilde{c}_{\kappa,\alpha} \prod_{1\leq i<j\leq N} (x_j^2 - x_i^2)^{2\kappa} \prod_{i=1}^{N} x_i^{2(\alpha+1)\kappa} e^{-x_i^2}$$

*on* $\Omega = \{0 \le x_1 \le ... \le x_N\}$ *with* $c_{\kappa,\alpha}$ *given by*

$$\tilde{c}_{\kappa,\alpha}^{-1} := \int_\Omega \prod_{1\le i<j\le N} (x_j^2 - x_i^2)^{2\kappa} \prod_{i=1}^N x_i^{2(\alpha+1)\kappa} e^{-x_i^2},$$

*and define* $r_N^\alpha := (r_1^\alpha, r_2^\alpha, ..., r_N^\alpha) := (\sqrt{z_{1,N}^{(\alpha)}}, ..., \sqrt{z_{N,N}^{(\alpha)}})$. *Then*

$$\sqrt{\kappa}\left(\frac{X_\kappa}{\sqrt{\kappa}} - r_N^\alpha\right) \to \mathcal{N}(0, \Sigma_N)$$

*in distribution for* $\kappa \to \infty$. *The inverse covariance matrix* $\Sigma_N^{-1}$ *is given by*

$$(\Sigma_N^{-1})_{i,j} := \begin{cases} 8z_{i,N}^{(\alpha)} \sum_{l\neq i} \frac{1}{(z_{i,N}^{(\alpha)} - z_{l,N}^{(\alpha)})^2} + 4\frac{\alpha+1}{z_{i,N}^{(\alpha)}} & \textit{for} \quad i = j \\ -8\frac{\sqrt{z_{i,N}^{(\alpha)} z_{j,N}^{(\alpha)}}}{(z_{i,N}^{(\alpha)} - z_{j,N}^{(\alpha)})^2} & \textit{for} \quad i \neq j. \end{cases} \tag{4.8}$$

$$= \begin{cases} 4\sum_{l\neq i} \frac{z_{i,N}^{(\alpha)} + z_{l,N}^{(\alpha)}}{(z_{i,N}^{(\alpha)} - z_{l,N}^{(\alpha)})^2} + 2\frac{\alpha+1}{z_{i,N}^{(\alpha)}} + 2 & \textit{for} \quad i = j \\ -8\frac{\sqrt{z_{i,N}^{(\alpha)} z_{j,N}^{(\alpha)}}}{(z_{i,N}^{(\alpha)} - z_{j,N}^{(\alpha)})^2} & \textit{for} \quad i \neq j. \end{cases} \tag{4.9}$$

A direct proof can be found in Theorem 3.3 of [Voi19]. Alternatively, it can be obtained using the Delta method from [Vaa98] Theorem 3.1. Therefore, we apply Theorem 3.5 for

$$v(x) = x - \alpha\log(x),$$
$$V(x) = \frac{1}{2}\left(v(x) + \log\left(\frac{A_N(x)}{a_N}\right)\right) = \frac{1}{2}\left(x - (\alpha+1)\log(x)\right),$$

and $d(x) = \sqrt{x}$.

Note that if Theorem 3.5 is used as described above, the inverse covariance matrix from (4.7) transforms into (4.8), and the matrix in (4.9) is in accordance with the corresponding matrix in Theorem 3.3 of [Voi19]. The central limit theorem there is scaled by a factor of $\frac{1}{\sqrt{2}}$ and therefore the inverse covariance matrix is scaled by a factor of 2. The two matrices (4.8) and (4.9) look different, but are identical. To see this identity, Equation (4.6) gives

$$2\sum_{\substack{l=1\\l\neq i}}^n \frac{1}{z_{i,N}^{(\alpha)} - z_{l,N}^{(\alpha)}} = 2V'(z_{i,N}^{(\alpha)}) = 1 - \frac{\alpha+1}{z_{i,N}^\alpha},$$

which has to be used to simplify

$$\sum_{\substack{l=1\\l\neq i}}^n \frac{z_{l,N}^{(\alpha)}}{(z_{i,N}^{(\alpha)} - z_{l,N}^{(\alpha)})^2} = z_{i,N}^{(\alpha)} \sum_{\substack{l=1\\l\neq i}}^n \frac{1}{(z_{i,N}^{(\alpha)} - z_{l,N}^{(\alpha)})^2} - \sum_{\substack{l=1\\l\neq i}}^n \frac{1}{z_{i,N}^{(\alpha)} - z_{l,N}^{(\alpha)}}.$$

Afterwards, the equality of both matrices (4.8) and (4.9) is obvious.

If one considers the limit case $\alpha = -1$, the situation for the Laguerre ensemble given by

$$c_{\kappa,\alpha} \prod_{1\le i<j\le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} x_i^{(\alpha+1)\kappa-1} e^{-x_i}$$

as in Theorem 4.5 is different from that for the "squared" Laguerre ensemble given by

$$\tilde{c}_{\kappa,\alpha} \prod_{1\le i<j\le N} (x_j^2 - x_i^2)^{2\kappa} \prod_{i=1}^{N} x_i^{2(\alpha+1)\kappa} e^{-x_i^2}$$

as in Theorem 4.6. For the latter, compare [Voi19] Section 5, or Corollary 5.5 in [Voi19]. The situation here is as follows:

**Theorem 4.7** (Voit(2019)). *Let $X_\kappa$ be $N$-dimensional random variables with the densities*

$$\tilde{c}_{\kappa,-1} \prod_{1\le i<j\le N} (x_j^2 - x_i^2)^{2\kappa} \prod_{i=1}^{N} e^{-x_i^2}$$

*on $\Omega = \{0 \le x_1 \le ... \le x_N\}$ and let*

$$r := (r_1, .., r_N) := (0, \sqrt{z_{1,N-1}^{(1)}}, \sqrt{z_{2,N-1}^{(1)}}, ..., \sqrt{z_{N-1,N-1}^{(1)}}).$$

*Then*

$$\sqrt{\kappa}\left(\frac{X_\kappa}{\sqrt{\kappa}} - r\right)$$

*converges in distribution for $\kappa \to \infty$ to the "one-sided normal distribution", which is given by the density*

$$\frac{2}{(2\pi)^{\frac{N}{2}} \det((-H(r)^{-1})} e^{\frac{1}{2}x^T H(r) x} \mathbb{1}_{\{x\in\mathbb{R}^N : x_1 \ge 0\}}(x),$$

*where $H(r)$ is the Hessian of $x \mapsto 2\sum_{1\le i<j\le N} \log(x_j^2 - x_i^2) - \sum_{i=1}^{N} x_i^2$ at the*

*point* $x = r$, *i.e*

$$(-H(r))_{i,j} := \begin{cases} 4\sum_{l\neq i} \frac{r_i^2+r_l^2}{(r_i^2-r_l^2)^2} + 2 & for \quad i=j \\ -8\frac{r_ir_j}{(r_i^2-r_j^2)^2} & for \quad i\neq j. \end{cases}$$

$$= \begin{cases} 4\displaystyle\sum_{l=2,l\neq i}^{N} \frac{z_{i-1,N-1}^{(1)} + z_{l-1,N-1}^{(1)}}{(z_{i-1,N-1}^{(1)} - z_{l-1,N-1}^{(1)})^2} & for \quad 2\leq i=j \\ +2+\dfrac{4}{z_{i-1,N-1}^{(1)}} & \\ -8\frac{\sqrt{z_{i-1,N-1}^{(1)}z_{j-1,N-1}^{(1)}}}{(z_{i-1,N-1}^{(1)}-z_{j-1,N-1}^{(1)})^2} & for \quad i\neq j \text{ and } i,j\geq 2, \\ 2+4\sum_{l=1}^{N-1}\frac{1}{z_{l,N-1}^{(1)}} & for \quad i=j=1 \\ 0 & for \quad i\neq j \text{ and } i=1 \text{ or } j=1. \end{cases}$$

*Proof.* This theorem can be found (up to scaling) in Corollary 5.3 of [Voi19]. Note that the central limit theorem there is scaled by a factor of $\sqrt{2}$. Furthermore, after the rescaling $\frac{X_\kappa}{\sqrt{\kappa}}$, we have a situation similar to that in Theorem 3.8. Here we have $l=1$, $m=N$ and

$$\begin{aligned} \phi(x) = 2\sum_{1\leq i<j\leq N} \log(x_j^2-x_i^2) - \sum_{i=1}^N x_i^2, \psi(x)\equiv 1, \\ a_1 = 0-e_1, a_2 = e_1-e_2, a_3 = e_1-e_3, ..., a_N = e_{N-1}-e_N, \\ b_1 = b_2 = ... = b_N = 0, \\ \Omega^* = \{x\in\mathbb{R}^N : x_1\geq 0\}. \end{aligned}$$

Furthermore, $r$ is the unique global maximum of $\phi$, because $\phi$ is monotonically decreasing in $x_1$ and the maximum of

$$(x_2,...,x_N)\mapsto \phi(0,x_2,...,x_N) = 2\sum_{2\leq i<j\leq N}\log(x_j^2-x_i^2) - \sum_{i=2}^N (x_i^2 - 4\log(x_i))$$

is given by $\left(\sqrt{z_{1,N-1}^{(1)}}, \sqrt{z_{2,N-1}^{(1)}}, ..., \sqrt{z_{N-1,N-1}^{(1)}}\right)$. $\phi$ is a concave function because the Hessian is negative definite at every point. Thus, the first condition of Theorem 3.8 is also fulfilled. Then, Theorem 3.8 yields the claim. □

Now we turn to the edge case $\alpha=-1$ for Laguerre ensembles with densities given by

$$c_{\kappa,\alpha}\prod_{1\leq i<j\leq N}(x_j-x_i)^{2\kappa}\prod_{i=1}^N x_i^{(\alpha+1)\kappa-1}e^{-x_i}, \tag{4.10}$$

as in Theorem 4.5. Here, the gradient at the maximum is not zero and we will apply Theorem 3.10. Note that the term $\prod_{i=1}^N \frac{1}{x_i}$ causes some technical problems. For $\alpha > -1$ in Theorem 4.5, this was irrelevant because for $\psi(x) = \prod_{i=1}^{N} \frac{1}{x_i}$, it held that $\psi(z) \neq 0$. This is no longer the case here. Therefore we will investigate

$$c_{\kappa,\alpha} \prod_{1 \le i < j \le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} x_i^{(\alpha+1),\kappa} e^{-x_i} \tag{4.11}$$

with $c_{\kappa,\alpha} > 0$ accordingly instead. Note that it is very likely that the behavior for (4.10) and (4.11) is the same. There are several results which extend the classical version of Laplace's method to the case where $\psi(z) = 0$, see for example [FS61; Bre94]. Similar results are available for boundary maxima. Those can be used to extend Theorem 3.10 and apply it to (4.10). An example with similar behavior are the Beta-Cauchy ensembles. For details see [Voi22]. For (4.11), however, the situation is as follows:

**Theorem 4.8.** *Let $X_\kappa = (X_\kappa^1, .., X_\kappa^N)$ be $N$-dimensional random variables with the densities*

$$c_{\kappa,-1} \prod_{1 \le i < j \le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} e^{-x_i}.$$

*on $\Omega = \{0 \le x_1 \le ... \le x_N\}$ and let $z_N^{(-1)} = (0, z_{1,N-1}^{(1)}, z_{2,N-1}^{(1)}, ..., z_{N-1,N-1}^{(1)})$. Then*

$$\sqrt{\kappa}\left(\left(\frac{X_\kappa^1}{\sqrt{\kappa}}, \frac{X_\kappa^2}{\kappa}, ..., \frac{X_\kappa^N}{\kappa}\right) - z_N^{(-1)}\right) = \left(X_\kappa^1, \frac{X_\kappa^2}{\sqrt{\kappa}}, ..., \frac{X_\kappa^N}{\sqrt{\kappa}}\right) - \sqrt{\kappa} z_N^{(-1)}$$

*converges in distribution for $\kappa \to \infty$ to a measure whose density is given by*

$$\exp(-\gamma_1 x_1) \frac{\sqrt{|\det H^*(z)|}}{(2\pi)^{\frac{N-1}{2}}} \exp\left(\frac{1}{2}(x_2, ..., x_N) H^*(z)(x_2, ..., x_N)^T\right) \mathbb{1}_{\{x_1 \ge 0\}}(x).$$

*Here, $\gamma_1 := \left(2\sum_{j=2}^{N} \frac{1}{z_{j,N-1}^1} + 1\right)$ and $H^*(z) = \left(H_\phi(z)_{i,j}\right)_{i,j=2,\dots,N}$ is given by the submatrix of the Hessian of $\phi(x) = 2\sum_{1 \le i < j \le N} \log(x_j - x_i) - \sum_{i=1}^{N} x_i$, i.e*

$$(-H^*(z))_{i,j} = \begin{cases} 2\sum_{l=1, l \neq i}^{N-1} (z_{i,N-1}^{(1)} - z_{l,N-1}^{(1)})^{-2} + \frac{2}{(z_{i,N-1}^{(1)})^2} & \text{for} \quad i = j \\ -2(z_{i,N-1}^{(1)} - z_{j,N-1}^{(1)})^{-2} & \text{for} \quad i \neq j. \end{cases}$$

*Proof.* This theorem is an application of Theorem 3.10 for the rescaled variables $\frac{X_\kappa}{\kappa}$ with the densities

$$C \prod_{1 \le i < j \le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} e^{-\kappa x_i} \mathbb{1}_\Omega(x)$$

and $C > 0$ fitting. Here, we have $l = 1$, $m = N$ and

$$\begin{aligned}
\phi(x) &= 2 \sum_{1 \le i < j \le N} \log(x_j - x_i) - \sum_{i=1}^{N} x_i, \psi(x) \equiv 1, \\
a_1 &= 0 - e_1, a_2 = e_1 - e_2, a_3 = e_1 - e_3, ..., a_N = e_{N-1} - e_N, \\
b_1 &= b_2 = ... = b_N = 0, \\
M &= (-e_1, e_2, ..., e_N), G_I = I_N, K = \operatorname{diag}(k, \sqrt{k}, \sqrt{k}, ..., \sqrt{k}), \\
z &= (z_N^{(-1)})^T, \\
\nabla\phi(z) &= -\left(1 + 2\sum_{i=2}^{N} \frac{1}{z_{i,N-1}^{(1)}}\right) \cdot (1, 0, 0..., 0)^T = \gamma_1 a_1, \\
\Omega^* &= \{x \in \mathbb{R}^N : x_1 \ge 0\}.
\end{aligned}$$

Therefore it follows that $KG_I M^{-1} z = Kz = \sqrt{z}$ and $KGM^{-1}\left(\frac{X_\kappa}{\kappa}\right)^T = \left(-X_\kappa^1, \frac{X_\kappa^2}{\sqrt{\kappa}}, ..., \frac{X_\kappa^N}{\sqrt{\kappa}}\right)^T$. Furthermore, $z \in \delta\Omega$, and $\phi$ attains its global maximum on $\Omega$ in $z$. This holds true because $\phi$ is monotonically decreasing in $x_1$ and the only maximum of $\tilde{\phi} : \mathbb{R}^{N-1} \to \mathbb{R}$,

$$\tilde{\phi}(x_2, ..., x_N) := \phi(0, x_2, ..., x_N) = 2 \sum_{2 \le i < j \le N} \log(x_j - x_i) + \sum_{i=2}^{N} (2\log(x_i) - x_i)$$

is given by $\left(z_{1,N-1}^{(1)}, z_{2,N-1}^{(1)}, ..., z_{N-1,N-1}^{(1)}\right)$. $\tilde{\phi}$ is a strictly concave function and hence $H^*(z)$ is negative definite. Thus, all conditions of Theorem 3.10 are fulfilled. This, together with applying the function $(x_1, ..., x_N) \mapsto (-x_1, x_2..., x_N)$ on both sides, yields the claim. □

## 4.4 Freezing regime of $\beta$-ensembles that generalize Laguerre ensembles

Similar to Theorem 4.4, we can extend the Laguerre ensembles by considering higher polynomial degrees in the weight function. For the Interval $I = (0, \infty)$, consider a polynomial $r_m$ of degree $m \in \mathbb{N}$, some $\alpha > 0$, and $C > 0$ fitting such that $\int_I w(x)dx = 1$ for the weight function

$$w(x) = Ce^{-r_m(x)} x^\alpha. \tag{4.12}$$

We will first take a look at the case $r_m(x) = x^2$. The aim is to apply Theorem 3.6 here. Therefore, we will look at the functions and conditions occurring there. The weight function becomes

$$w(x) = \frac{2}{\Gamma\left(\frac{\alpha+1}{2}\right)} e^{-x^2} x^\alpha$$

and the associated orthonormal polynomials obey the three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x).$$

Thus, we have $v(x) = x^2 - \alpha \log(x)$ and $v'(x) = 2x - \frac{\alpha}{x}$. To get $A_N(x)$, we can proceed as in the Laguerre case, see [Ism05], Subsection 3.3. We have

$$\begin{aligned}\frac{A_N(x)}{a_N} &= \int_0^\infty \frac{v'(x) - v'(y)}{x - y}(p_N(y))^2 w(y)dy \\ &= \int_0^\infty \frac{2x - \frac{\alpha}{x} - 2y + \frac{\alpha}{y}}{x - y}(p_N(y))^2 w(y)dy \\ &= 2 + \frac{\alpha}{x}\frac{2}{\Gamma(\frac{\alpha+1}{2})}\int_0^\infty (p_N(y))^2 e^{-y^2} y^{\alpha-1}dy.\end{aligned}$$

Furthermore, using integration by parts and orthogonality, we get

$$\begin{aligned}&\int_0^\infty (p_N(y))^2 e^{-y^2} y^{\alpha-1}dy \\ =& \frac{y^\alpha}{\alpha}(p_N(y))^2 e^{-y^2}\Big|_0^\infty - \int_0^\infty \frac{y^\alpha}{\alpha}((p_N(y))^2 e^{-y^2})'dy \\ =&\frac{1}{\alpha}\int_0^\infty 2y(p_N(y))^2 e^{-y^2} y^\alpha dy - \frac{1}{\alpha}\int_0^\infty 2p_N'(y)p_N(y)e^{-y^2} y^\alpha dy \\ =&\frac{2}{\alpha}\int_0^\infty (a_{N+1}p_{N+1}(y) + b_N p_N(y) + a_N p_{N-1}(y))p_N(y)e^{-y^2} y^\alpha dy \\ =&\frac{2}{\alpha}\frac{\Gamma(\frac{\alpha+1}{2})}{2}b_N.\end{aligned}$$

Overall, we have

$$\frac{A_N(x)}{a_N} = 2 + \frac{2}{x}b_N = \frac{2x + 2b_N}{x}. \tag{4.13}$$

Therefore,

$$\left(\prod_{1\le i<j\le N}(x_i - x_j)^2 \prod_{i=1}^N \frac{\exp(-v(x_i))a_N}{A_N(x_i)}\right)^\kappa$$

becomes

$$\prod_{1\le i<j\le N}(x_j - x_i)^{2\kappa}\prod_{i=1}^N \frac{e^{-\kappa x_i^2} x_i^{(\alpha+1)\kappa}}{(2x_i + 2b_N)^\kappa}. \tag{4.14}$$

Again, denote the zeros of $p_N$ by $z_N = (z_{1,N}, ..., z_{N,N})$. Then we can apply Theorem 3.6 to get the following central limit theorem:

**Theorem 4.9.** *Let $X_\kappa$ be $N$-dimensional random variables with the densities*

$$c_\kappa \prod_{1\le i<j\le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} \frac{e^{-\kappa x_i^2} x_i^{(\alpha+1)\kappa}}{(2x_i + 2b_N)^\kappa}$$

*on* $\Omega := \{x \in \mathbb{R} : 0 < x_1 \le x_2 \le ... \le x_N\}$ *with*

$$c_\kappa^{-1} = \int_\Omega \prod_{1\le i<j\le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} \frac{e^{-\kappa x_i^2} x_i^{(\alpha+1)\kappa}}{(2x_i + 2b_N)^\kappa}.$$

*Then*

$$\sqrt{\kappa}(X_\kappa - z_N) \to \mathcal{N}(0, \Sigma)$$

*weakly for* $\kappa \to \infty$*, where the covariance matrix* $\Sigma$ *is given by*

$$(\Sigma^{-1})_{i,j} := \begin{cases} 2\sum_{l\ne i}(z_{i,N} - z_{l,N})^{-2} + 2 + \frac{\alpha+1}{z_{i,N}^2} - \frac{1}{(z_{i,N}+b_N)^2} & \text{for} \quad i = j \\ -2(z_{i,N} - z_{j,N})^{-2} & \text{for} \quad i \ne j. \end{cases}$$

*Proof.* The goal is to apply Theorem 3.6 for $\psi \equiv 1$, $v(x) = x^2 - \alpha \log(x)$ and

$$\phi(x) = \prod_{1\le i<j\le N} (x_j - x_i)^2 \prod_{i=1}^{N} \frac{e^{-x_i^2} x_i^{(\alpha+1)}}{(2x_i + 2b_N)}.$$

We therefore verify all conditions. The only non-trivial condition is that $\frac{d^2}{dx^2}(v(x) + \log(A_N(x)))$ is positive. Because $b_N = \int_0^\infty y(p_N(y))^2 w(y)dy > 0$ and $\alpha \ge 0$, we get that

$$\frac{d^2}{dx^2}(v(x) + \log(A_N(x))) = 2 + \frac{\alpha+1}{x^2} - \frac{1}{(x+b_N)^2} > 0.$$

□

Similar to Theorem 4.4 and Theorem 4.2, we can generalize Theorem 4.9. We now turn to weight functions

$$w(x) = Ce^{-r_m(x)}x^\alpha$$

as given in 4.12. The aim is to apply Theorem 3.6 here. Therefore we will look at the functions and conditions occurring there.
Consider the polynomial $r_m(x) = \sum_{k=0}^m c_k x^k$ with only nonnegative coefficients $c_k \ge 0$. The system of polynomials $(p_n)_{n\in\mathbb{N}}$, orthonormal with respect to $w$, obey the three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x).$$

We have

$$v(x) = \alpha \log(x) + \sum_{k=0}^{m} c_k x^k \text{ and } v'(x) = \frac{\alpha}{x} + \sum_{k=0}^{m-1} (k+1) c_{k+1} x^k.$$

To get $A_N(x)$, we proceed as in Theorem 4.9 and Theorem 4.4. Note that Exercise 3.1 in [Ism05] addresses a similar issue. We have

$$\begin{aligned} \frac{A_N(x)}{a_N} &= \int_{\mathbb{R}} \frac{v'(x) - v'(y)}{x - y} (p_N(y))^2 w(y) dy \\ &= \sum_{k=0}^{m-1} (k+1) c_{k+1} \int_{\mathbb{R}} \frac{x^k - y^k}{x - y} (p_N(y))^2 w(y) dy \\ &+ \int_{\mathbb{R}} \frac{\frac{\alpha}{x} - \frac{\alpha}{y}}{x - y} (p_N(y))^2 w(y) dy. \end{aligned}$$

Furthermore we have

$$\begin{aligned} &\sum_{k=0}^{m-1} (k+1) c_{k+1} \int_{\mathbb{R}} \frac{x^k - y^k}{x - y} (p_N(y))^2 w(y) dy \\ =& \sum_{k=0}^{m-1} (k+1) c_{k+1} \int_{\mathbb{R}} \sum_{l=0}^{k-1} x^l y^{(k-l-1)} (p_N(y))^2 w(y) dy \\ =& \sum_{k=0}^{m-1} \left( (k+1) c_{k+1} \sum_{l=0}^{k-1} \left( x^l \int_{\mathbb{R}} y^{(k-l-1)} (p_N(y))^2 w(y) dy \right) \right) \end{aligned}$$

and using integration by parts and orthogonality, we get

$$\begin{aligned} &\int_{\mathbb{R}} \frac{\frac{\alpha}{x} - \frac{\alpha}{y}}{x - y} (p_N(y))^2 w(y) dy \\ =& - C \frac{\alpha}{x} \int_0^\infty (p_N(y))^2 e^{-r_m(y)} y^{\alpha - 1} dy \\ =& - C \frac{1}{x} y^\alpha (p_N(y))^2 e^{-r_m(y)} \Big|_0^\infty + C \frac{\alpha}{x} \int_0^\infty \frac{y^\alpha}{\alpha} ((p_N(y))^2 e^{-r_m(y)})' dy \\ =& \frac{C}{x} \int_0^\infty r_m'(y) (p_N(y))^2 e^{-r_m(y)} y^\alpha dy - \frac{C}{x} \int_0^\infty 2 p_N'(y) p_N(y) e^{-r_m(y)} y^\alpha dy \\ =& \frac{1}{x} \int_0^\infty r_m'(y) (p_N(y))^2 w(y) dy \\ =& \frac{1}{x} \sum_{k=0}^{m-1} \left( (k+1) c_{k+1} \int_0^\infty y^k (p_N(y))^2 w(y) dy \right) \\ =:& \frac{M}{x}. \end{aligned}$$

Hence, if we use the abbreviations

$$M := \sum_{k=0}^{m-1} \left( (k+1)c_{k+1} \int_0^\infty y^k (p_N(y))^2 w(y) dy \right),$$

$$s_{m-2}(x) := \sum_{k=0}^{m-1} \left( (k+1)c_{k+1} \left( \sum_{l=0}^{k-1} \left( x^l \int_{\mathbb{R}} y^{(k-l-1)} (p_N(y))^2 w(y) dy \right) \right) \right),$$

we get

$$\frac{A_N(x)}{a_N}(x) = s_{m-2}(x) + \frac{M}{x}.$$

Therefore,

$$\left( \prod_{1 \le i < j \le N} (x_i - x_j)^2 \prod_{i=1}^N \frac{\exp(-v(x_i)) a_N}{A_N(x_i)} \right)^\kappa$$

becomes

$$\prod_{1 \le i < j \le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^N \frac{e^{-\kappa r_m(x)} x_i^{\alpha\kappa}}{(s_{m-2}(x) + \frac{M}{x})^\kappa}. \tag{4.15}$$

It is not easy to show that $x \mapsto v(x) + \log A_N(x)$ is a strictly convex function. There may even be some counterexamples. Therefore we will assume convexity, i.e. for all $x \in \mathbb{R}$ assume that

$$\frac{d^2}{dx^2} \left( r_m(x) + \log \left( s_{m-2}(x) + \frac{M}{x} \right) \right) > 0. \tag{4.16}$$

Again, denote the zeros of $p_N$ by $z_N = (z_{1,N}, ..., z_{N,N})$. Then we can apply Theorem 3.6 to get the following central limit theorem:

**Theorem 4.10.** *Let the above conditions be fulfilled. In particular, let $r_m$ have nonnegative coefficients and assume that 4.16 holds. Now, let $x_\kappa$ be $N$-dimensional random variables with densities*

$$c_\kappa \prod_{1 \le i < j \le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^N \frac{e^{-\kappa r_m(x)} x_i^{\alpha\kappa}}{(s_{m-2}(x) + \frac{M}{x})^\kappa}$$

*on $\Omega := \{x \in \mathbb{R} : x_1 \le x_2 \le ... \le x_N\}$ with*

$$c_\kappa^{-1} = \int_\Omega \prod_{1 \le i < j \le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^N \frac{e^{-\kappa r_m(x)} x_i^{\alpha\kappa}}{(s_{m-2}(x) + \frac{M}{x})^\kappa}.$$

*Then*

$$\sqrt{\kappa}(X_\kappa - z_N) \to \mathcal{N}(0, \Sigma)$$

*weakly for $\kappa \to \infty$. Here, the inverse covariance matrix $\left(\Sigma_{i,j}^{-1}\right)_{i,j=1,\ldots,N}$ is given by*

$$\begin{cases} 2\sum_{l\neq i}(z_{i,N}-z_{l,N})^{-2} + \frac{d^2}{dy^2}\left(r_m(x) + \log(s_{m-2}(x) + \frac{M}{x})\right)\Big|_{x=z_{i,N}} & \text{for} \quad i=j, \\ -2(z_{i,N}-z_{j,N})^{-2} & \text{for} \quad i\neq j. \end{cases}$$

## 4.5 Freezing regime of $\beta$-Jacobi ensembles and their edge cases

In this section, let $(\tilde{P}_n^{(\alpha,\beta)})_{n\in\mathbb{N}}$ be the Jacobi polynomials for $\alpha, \beta > -1$, orthonormalized with respect to

$$w(x) = \frac{\Gamma(\alpha-\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^{\alpha}(1+x)^{\beta}\mathbb{1}_{[-1,1]}(x). \tag{4.17}$$

Further, denote with $z_N^{(\alpha,\beta)} = (z_{1,N}^{(\alpha,\beta)}, \ldots, z_{N,N}^{(\alpha,\beta)})$ the vector of ordered zeros $z_{1,N}^{(\alpha,\beta)} < \ldots < z_{N,N}^{(\alpha,\beta)}$. For a detailed description of the Jacobi polynomials, see for example [Chi78; Ism05; Sze75]. The vector $z_N^{(\alpha,\beta)}$ is the unique global maximum of the function

$$x \mapsto 2\sum_{1\leq i<j\leq N}\log(x_j - x_i) + \sum_{i=1}^{N}\big((\alpha+1)\log(1-x_i) + (\beta+1)\log(1+x_i)\big) \tag{4.18}$$

on $\Omega = \{x \in \mathbb{R}^N : -1 \leq x_1 \leq \ldots \leq x_N \leq 1\}$. Therefore, the gradient of the above function is zero at $z_N^{(\alpha,\beta)}$, which leads to the identity

$$2\sum_{\substack{l=1\\ l\neq i}}^{n}\frac{1}{z_{i,N}^{(\alpha,\beta)} - z_{l,N}^{(\alpha,\beta)}} = \frac{\alpha+1}{1-z_{i,N}^{(\alpha,\beta)}} - \frac{\beta+1}{1+z_{i,N}^{(\alpha,\beta)}}. \tag{4.19}$$

Here, the limit behavior depends on multiple parameters $\alpha, \beta$ and, in some cases, also on the parametrization of the Jacobi ensemble. There are two different parametrizations frequently considered. First, there is the algebraic Jacobi ensemble. Here, a central limit theorem for the so-called freezing regime is given as follows.

**Theorem 4.11** ([HV21])**.** *For $\alpha, \beta > -1$ let $X_\kappa$ be $N$-dimensional random variables with densities*

$$c_{\kappa,\alpha,\beta}\prod_{1\leq i<j\leq N}(x_j - x_i)^{2\kappa}\prod_{i=1}^{N}(1-x_i)^{(\alpha+1)\kappa-\frac{1}{2}}(1+x_i)^{(\beta+1)\kappa-\frac{1}{2}}$$

*on* $\Omega = \{-1 \le x_1 \le ... \le x_N \le 1\}$ *with*

$$c_{\kappa,\alpha,\beta}^{-1} := \int_\Omega \prod_{1\le i<j\le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^N (1-x_i)^{(\alpha+1)\kappa-\frac{1}{2}}(1+x_i)^{(\beta+1)\kappa-\frac{1}{2}}.$$

*Then*

$$\sqrt{\kappa}\left(X_\kappa - z_N^{(\alpha,\beta)}\right) \to \mathcal{N}(0,\Sigma)$$

*weakly for* $\kappa \to \infty$. *The inverse covariance matrix* $\Sigma_N^{-1} = ((\Sigma_N^{-1})_{i,j})_{i,j=1,\dots,N}$ *is given by*

$$\begin{cases} 2\sum_{l\neq i}\left(z_{i,N}^{(\alpha,\beta)} - z_{l,N}^{(\alpha,\beta)}\right)^{-2} + \frac{\alpha+1}{(1-z_{i,N}^{(\alpha,\beta)})^2} + \frac{\beta+1}{(1+z_{i,N}^{(\alpha,\beta)})^2} & \text{for} \quad i=j \\ -2\left(z_{i,N}^{(\alpha,\beta)} - z_{j,N}^{(\alpha,\beta)}\right)^{-2} & \text{for} \quad i \neq j. \end{cases} \tag{4.20}$$

*Proof.* This theorem is, up to scaling, Theorem 2.6 in [HV21]. The theorem there was proven directly, which leads to a nice formula for the determinant of the covariance matrix. However, it can also be obtained as an example of Theorem 3.6. If one chooses the weight $w(x)$ according to (4.17) of the Jacobi polynomials, then, as seen in [Ism05] (3.3.13), it holds that

$$\begin{aligned} v(x) &= -\alpha\log(1-x) - \beta\log(1+x) \\ \frac{A_N(x)}{a_N} &= \frac{(\alpha+\beta+1+2N)}{1-x^2} \\ \prod_{i=1}^N \frac{e^{-v(x_i)}a_N}{A_N(x_i)} &= (\alpha+\beta+1+2N)^N \prod_{i=1}^N \left((1-x_i)^{(\alpha+1)}(1+x_i)^{\beta+1}\right) \\ R_N'(z_{i,N}) &= -\frac{d^2}{dx^2}\left(v(x) + \log\left(\frac{A_N(x)}{a_N}\right)\right)\Big|_{x=z_{i,N}} \\ &= \frac{\alpha+1}{(1-z_{i,N}^{(\alpha,\beta)})^2} + \frac{\beta+1}{(1+z_{i,N}^{(\alpha,\beta)})^2}. \end{aligned}$$

This completes the proof. □

We will now state a central limit theorem for the so-called trigonometric Jacobi ensemble. The algebraic Jacobi ensemble from Theorem 4.11 can be transformed into the trigonometric ensemble with the mapping $t \mapsto \cos(2t)$. Applying the Delta method from [Vaa98] yields:

**Theorem 4.12** (H.,Voit(2021))**.** *For* $\alpha, \beta > -1$ *let* $X_\kappa$ *be* $N$*-dimensional random variables with the densities*

$$\tilde{c}_{\kappa,\alpha,\beta} \prod_{1\le i<j\le N} (\cos(2t_j) - \cos(2t_i))^{2\kappa} \prod_{i=1}^N \sin(t_i)^{2(\alpha+1)\kappa}\cos(t_i)^{2(\beta+1)\kappa} \tag{4.21}$$

*on* $\Omega = \{0 \le t_N \le ... \le t_1 \le \frac{\pi}{2}\}$ *with*

$$\tilde{c}_{\kappa,\alpha,\beta}^{-1} := \int_\Omega \prod_{1\le i<j\le N} (\cos(2t_j) - \cos(2t_i))^{2\kappa} \prod_{i=1}^N \sin(t_i)^{2(\alpha+1)\kappa} \cos(t_i)^{2(\beta+1)\kappa}.$$

*Furthermore define*

$$\frac{1}{2}\arccos(z_N^{(\alpha,\beta)}) := \frac{1}{2}\left(\arccos z_{1,N}^{(\alpha,\beta)}, ..., \arccos z_{N,N}^{(\alpha,\beta)}\right)$$

*and denote the matrix*

$$\begin{cases} 8\sum_{l\neq i} \frac{1-(z_{i,N}^{(\alpha,\beta)})^2}{\left(z_{i,N}^{(\alpha,\beta)}-z_{l,N}^{(\alpha,\beta)}\right)^2} + 4(\alpha+1)\frac{1+z_{i,N}^{(\alpha,\beta)}}{1-z_{i,N}^{(\alpha,\beta)}} + 4(\beta+1)\frac{1-z_{i,N}^{(\alpha,\beta)}}{1+z_{i,N}^{(\alpha,\beta)}} & \text{for } i=j \\ -8\frac{\sqrt{1-(z_{i,N}^{(\alpha,\beta)})^2}\sqrt{1-(z_{j,N}^{(\alpha,\beta)})^2}}{(z_{i,N}^{\alpha}-z_{j,N}^{\alpha})^2} & \text{for } i\neq j \end{cases} \tag{4.22}$$

$$= \begin{cases} 8\sum_{l\neq i} \frac{1-z_{i,N}^{(\alpha,\beta)}z_{l,N}^{(\alpha,\beta)}}{(z_{i,N}^{(\alpha,\beta)}-z_{l,N}^{(\alpha,\beta)})^2} + \frac{4(\alpha+1)}{1-z_{i,N}^{(\alpha,\beta)}} + \frac{4(\beta+1)}{1+z_{i,N}^{(\alpha,\beta)}} & \text{for } i=j \\ -8\frac{\sqrt{1-(z_{i,N}^{(\alpha,\beta)})^2}\sqrt{1-(z_{j,N}^{(\alpha,\beta)})^2}}{(z_{i,N}^{(\alpha,\beta)}-z_{j,N}^{(\alpha,\beta)})^2} & \text{for } i\neq j. \end{cases} \tag{4.23}$$

*by* $\Sigma_N^{-1}$. *Then*

$$\sqrt{\kappa}\left(X_\kappa - \frac{1}{2}\arccos(z_N^{(\alpha,\beta)})\right) \to \mathcal{N}(0, \Sigma_N)$$

*weakly for* $\kappa \to \infty$.

*Proof.* Theorem 4.12 and Theorem 4.11 can be transformed into each other using Theorem 3.3 of [Vaa98]. This means that this theorem can be obtained applying Theorem 3.5 with

$$\begin{aligned} v(x) &= -\alpha\log(1-x) - \beta\log(1+x), \\ V(x) &= \frac{1}{2}\left(v(x) + \log\left(\frac{A_N(x)}{a_N}\right)\right) \\ &= \frac{1}{2}\left(-(\alpha+1)\log(1-x) - (\beta+1)\log(1+x)\right), \\ d(x) &= \frac{1}{2}\arccos(x), \\ d^{-1}(t) &= \cos(2t), \\ \text{and } d'(x) &= \frac{-1}{2\sqrt{1-x^2}}. \end{aligned}$$

Note that here, $d$ is monotonically decreasing instead of increasing and therefore the inequalities in $\Omega$ are inverted. The rest of Theorem 3.5 remains the same and we have

$$\begin{aligned}(d^{-1})'(t) &= -2\sin(2t) = -2\sqrt{1-x^2},\\ 1-d^{-1}(t) &= 1-\cos(2t) = 2\sin^2(t),\\ 1+d^{-1}(t) &= 1+\cos(2t) = 2\cos^2(t).\end{aligned}$$

Now, inserting the above into (3.11) and (3.12) from Theorem 3.5 yields the formulas for the densities (4.21) and for the inverse covariance matrix (4.22), respectively. We thus get the central limit theorem as introduced.

On the other hand, if one applies Theorem 3.2 directly to

$$\phi(t) = \prod_{1\le i<j\le N} (\cos(2t_j)-\cos(2t_i))^2 \prod_{i=1}^{N} \sin(t_i)^{2(\alpha+1)}\cos(t_i)^{2(\beta+1)},$$

we have

$$\begin{aligned}\log(\phi(t)) =& 2\sum_{1\le i<j\le N} \log\left(\cos(2t_j)-\cos(2t_i)\right) \qquad (4.24)\\ &+ 2(\alpha+1)\sum_{i=1}^{N}\log(\sin(t_i)) + 2(\beta+1)\sum_{i=1}^{N}\log(\cos(t_i)),\\ \frac{d}{dt_i}\log(\phi(t_i)) =& 4\sum_{l\neq i}\frac{\sin(2t_i)}{\cos(2t_j)-\cos(2t_i)} \qquad (4.25)\\ &+ 2(\alpha+1)\frac{\cos(t_i)}{\sin(t_i)} - 2(\beta+1)\frac{\sin(t_i)}{\cos(t_i)},\end{aligned}$$

$$(-H_{\log\phi}(t))_{i,j} = \begin{cases} 8\sum_{l\neq i}\frac{1-\cos(2t_l)\cos(2t_i)}{(\cos(2t_l)-\cos(2t_i))^2} + \frac{2(\alpha+1)}{\sin^2(t_i)} + \frac{2(\beta+1)}{\cos^2(t_i)} & \text{for } i=j\\ -8\frac{\sin(2t_i)\sin(2t_j)}{(\cos(2t_j)-\cos(2t_i))^2} & \text{for } i\neq j.\end{cases} \qquad (4.26)$$

Inserting $t = \frac{1}{2}\arccos(z_N^{(\alpha,\beta)})$ here yields the formula for the inverse covariance matrix in the form of (4.23). Note that similar to Theorem 4.6, the equality of the matrices can also be shown directly. To do so we use the identity (4.19):

$$2\sum_{\substack{l=1\\ l\neq i}}^{N} \frac{1}{z_{i,N}^{(\alpha,\beta)} - z_{l,N}^{(\alpha,\beta)}} = 2V'(z_{i,N}^{(\alpha,\beta)}) = \frac{\alpha+1}{1-z_{i,N}^{(\alpha,\beta)}} - \frac{\beta+1}{1+z_{i,N}^{(\alpha,\beta)}}$$

and

$$2\sum_{\substack{l=1\\ l\neq i}}^{N} \frac{1-z_{i,N}^{(\alpha,\beta)} z_{l,N}^{(\alpha,\beta)}}{(z_{i,N}^{(\alpha,\beta)} - z_{l,N}^{(\alpha,\beta)})^2} = 2\sum_{\substack{l=1\\ l\neq i}}^{N} \frac{1-(z_{i,N}^{(\alpha,\beta)})^2}{(z_{i,N}^{(\alpha,\beta)} - z_{l,N}^{(\alpha,\beta)})^2} + 2z_{i,N}^{(\alpha,\beta)}\sum_{\substack{l=1\\ l\neq i}}^{N} \frac{1}{z_{i,N}^{(\alpha,\beta)} - z_{l,N}^{(\alpha,\beta)}},$$

which leads to

$$
\begin{aligned}
&2\sum_{\substack{l=1\\l\neq i}}^{N}\frac{1-z_{i,N}^{(\alpha,\beta)}z_{l,N}^{(\alpha,\beta)}}{(z_{i,N}^{(\alpha,\beta)}-z_{l,N}^{(\alpha,\beta)})^2}\\
=&2\sum_{\substack{l=1\\l\neq i}}^{N}\frac{1-(z_{i,N}^{(\alpha,\beta)})^2}{(z_{i,N}^{(\alpha,\beta)}-z_{l,N}^{(\alpha,\beta)})^2}+z_{i,N}^{(\alpha,\beta)}\left(\frac{\alpha+1}{1-z_{i,N}^{(\alpha,\beta)}}-\frac{\beta+1}{1+z_{i,N}^{(\alpha,\beta)}}\right).
\end{aligned}
$$

□

Note that in Equation (2.6) of [HV21], a factor of $\sin(2t_i)$ instead of $\cos^2(t_i)$ appears. This is the case because there, $\beta \geq \alpha$ is assumed and the formula $2\sin(t)\cos(t) = \sin(2t)$ is used. Both notations have their benefits and describe the "repulsion" on the edges. The difference is that in one case one term, namely $\sin(2t)$, describes the repulsion both edges have in common and the other term describes how much stronger the repulsion is on one edge, whereas in the other case each term describes the repulsion of one edge. For the edge cases $\alpha, \beta \to -1$, the chosen notation seems most suitable.
As we have two parameters $\alpha$ and $\beta$, there are several edge cases to be considered. For the trigonometric Jacobi ensemble, the gradient at the maximum is zero in the edge cases. The one-sided edge case in the trigonometric setting, if $-1 = \alpha < \beta$, is as follows:

**Theorem 4.13.** *For $\beta > -1$ let $X_\kappa$ be $N$-dimensional random variables with densities*

$$\tilde{c}_{\kappa,-1,\beta}\prod_{1\leq i<j\leq N}(\cos(2t_j)-\cos(2t_i))^{2\kappa}\prod_{i=1}^{N}\cos(t_i)^{2(\beta+1)\kappa}$$

*on $\Omega = \{0 \leq t_N \leq ... \leq t_1 \leq \frac{\pi}{2}\}$ with*

$$\tilde{c}_{\kappa,-1,\beta}^{-1} := \int_\Omega \prod_{1\leq i<j\leq N}(\cos(2t_j)-\cos(2t_i))^{2\kappa}\prod_{i=1}^{N}\cos(t_i)^{2(\beta+1)\kappa}.$$

*Furthermore define*

$$
\begin{aligned}
z_N^{(-1,\beta)} &:= (z_{1,N}^{(-1,\beta)}, z_{2,N}^{(-1,\beta)}, ..., z_{N,N}^{(-1,\beta)}) = (z_{1,N-1}^{(1,\beta)}, ..., z_{N-1,N-1}^{(1,\beta)}, 1) \text{ and}\\
\tilde{z}_N^{(-1,\beta)} &:= (\tilde{z}_{1,N}^{(-1,\beta)}, \tilde{z}_{2,N}^{(-1,\beta)}, ..., \tilde{z}_{N,N}^{(-1,\beta)})\\
&:= \frac{1}{2}\left(\arccos z_{1,N-1}^{(1,\beta)}, ..., \arccos z_{N-1,N-1}^{(1,\beta)}, 0\right).
\end{aligned}
$$

*Then*

$$\sqrt{\kappa}\left(X_\kappa - \tilde{z}_N^{(-1,\beta)}\right)$$

*converges in distribution for $\kappa \to \infty$ to the "one-sided normal distribution", which is given by the density*

$$\frac{2\det(-H(\tilde{z}_N^{(-1,\beta)}))}{(2\pi)^{\frac{N}{2}}} e^{\frac{1}{2}x^T H(\tilde{z}_N^{(-1,\beta)})x} \mathbb{1}_{\{x\in\mathbb{R}^N : x_N \geq 0\}}(x).$$

*Here, $H(\tilde{z}_N^{(-1,\beta)})$ is the Hessian matrix of*

$$t \mapsto 2 \sum_{1\leq i<j\leq N} \log\left(\cos(2t_j) - \cos(2t_i)\right) + 2(\beta+1)\sum_{i=1}^{N} \log(\cos(t_i))$$

*at the point $x = \tilde{z}_N^{(-1,\beta)}$, i.e. $(-H(\tilde{z}_N^{(-1,\beta)}))_{i,j}$ is given by*

$$\begin{cases} 8\displaystyle\sum_{l=1, l\neq i}^{N-1} \frac{1 - z_{i,N-1}^{(1,\beta)} z_{l,N-1}^{(1,\beta)}}{(z_{i,N-1}^{(1,\beta)} - z_{l,N-1}^{(1,\beta)})^2} & \\ +\dfrac{8}{1 - z_{i,N-1}^{(1,\beta)}} + \dfrac{4(\beta+1)}{1 + z_{i,N-1}^{(1,\beta)}} & \textit{for} \quad 1 \leq i = j \leq N-1, \\ -8\frac{\sqrt{1-(z_{i,N-1}^{(1,\beta)})^2}\sqrt{1-(z_{j,N-1}^{(1,\beta)})^2}}{(z_{i,N-1}^{(1,\beta)} - z_{j,N-1}^{(1,\beta)})^2} & \textit{for} \quad i \neq j \textit{ and } i,j \geq 2, \\ 2(\beta+1) + 8\sum_{l=1}^{N-1} \frac{1}{1 - z_{l,N-1}^{(1,\beta)}} & \textit{for} \quad i = j = 1, \\ 0 & \textit{for} \quad i \neq j \textit{ and } i = 1 \textit{ or } j = 1. \end{cases}$$

*Proof.* The goal is to apply Theorem 3.8. Here we have $l = 1$, $m = N + 1$ and

$$\begin{aligned} \phi(t) &= 2\sum_{1\leq i<j\leq N} \log\left(\cos(2t_j) - \cos(2t_i)\right) + 2(\beta+1)\sum_{i=1}^{N}\log(\cos(t_i)), \psi(x) \equiv 1, \\ a_1 &= 0 - e_N, a_2 = e_N - e_{N-1}, ..., a_N = e_2 - e_1, a_{N+1} = e_1, \\ b_1 &= b_2 = ... = b_N = 0, b_{N+1} = -\frac{\pi}{2}, \\ \Omega^* &= \{t \in \mathbb{R}^N : t_N \geq 0\}. \end{aligned}$$

It remains to show that $\tilde{z}_N^{(-1,\beta)}$ is the unique global maximum of $\phi$. First, observe that $\phi$ is monotonically increasing in $t_N$. The maximum of $\tilde{\phi} : \mathbb{R}^{N-1} \to \mathbb{R}$ with

$$\begin{aligned} &\tilde{\phi}(t_1, ..., t_{N-1}) = \phi(t_1, ..., t_{N-1}, 0) \\ =&2\sum_{1\leq i<j\leq N-1} \log(\cos(2t_j) - \cos(2t_i)) + (N-1)\log(2) \\ &+ \sum_{i=1}^{N-1} \left(2(\beta+1)\log(\cos(t_i)) + 4\log(\sin(t_i))\right) \end{aligned}$$

with respect to

$$\tilde{\Omega} := \{0 \leq t_{N-1} \leq ... \leq t_1 \leq \frac{\pi}{2}\}$$

is given by $\frac{1}{2}\left(\arccos z_{1,N-1}^{(1,\beta)}, ..., \arccos z_{N-1,N-1}^{(1,\beta)}\right)$ because $\tilde{\phi}$ is, up to the correct additive constant, just the logarithm of the density function (4.21) in Theorem 4.12 for $\alpha = 1$ and $\beta, N$ fitting. Furthermore, $\phi$ is a concave function because the Hessian at every point is negative definite. This holds because $-H_\phi(t)$ is given by (4.26) for $\alpha = -1$. Using the identity $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x-y)$ shows that this matrix is diagonally dominant with positive entries. Thus $-H_\phi(t)$ is positive definite. In particular, $-H_\phi(z_N^{-1,\beta})$ is positive definite, and therefore the first condition of Theorem 3.8 is also fulfilled. Now Theorem 3.8 yields the central limit theorem and the inverse covariance matrix is given by

$$(-H(\tilde{z}_N^{(-1,\beta)}))_{i,j} := \begin{cases} 8\sum_{l\neq i} \frac{1-\cos(2t_i)\cos(2t_j)}{(\cos(2t_j)-\cos(2t_i))^2} + \frac{2(\beta+1)}{\cos^2(t_i)} & \text{for } i=j \\ -8\frac{\sin(2t_i)\sin(2t_j)}{(\cos(2t_j)-\cos(2t_i))^2} & \text{for } i\neq j \end{cases}\Bigg|_{t=\tilde{z}_N^{(-1,\beta)}}$$

$$= \begin{cases} 8\sum_{l\neq i} \frac{1-z_{i,N}^{(-1,\beta)}z_{l,N}^{(-1,\beta)}}{(z_{i,N}^{(-1,\beta)}-z_{l,N}^{(-1,\beta)})^2} + \frac{4(\beta+1)}{z_{i,N}^{(-1,\beta)}+1} & \text{for } i=j \\ -8\frac{\sqrt{1-(z_{i,N}^{(-1,\beta)})^2}\sqrt{1-(z_{j,N}^{(-1,\beta)})^2}}{(z_{i,N}^{(-1,\beta)}-z_{j,N}^{(-1,\beta)})^2} & \text{for } i\neq j \end{cases}$$

$$= \begin{cases} 8\sum_{l=1,l\neq i}^{N-1} \frac{1-z_{i,N-1}^{(1,\beta)}z_{l,N-1}^{(1,\beta)}}{(z_{i,N-1}^{(1,\beta)}-z_{l,N-1}^{(1,\beta)})^2} \\ +\frac{4(1+1)}{1-z_{i,N-1}^{(1,\beta)}} + \frac{4(\beta+1)}{1+z_{i,N-1}^{(1,\beta)}} & \text{for} \quad 1\leq i=j\leq N-1 \\ -8\frac{\sqrt{1-(z_{i,N-1}^{(1,\beta)})^2}\sqrt{1-(z_{j,N-1}^{(1,\beta)})^2}}{(z_{i,N-1}^{(1,\beta)}-z_{j,N-1}^{(1,\beta)})^2} & \text{for} \quad i\neq j \text{ and } i,j\geq 2 \\ 2(\beta+1) + 8\sum_{l=1}^{N-1}\frac{1}{1-z_{l,N-1}^{(1,\beta)}} & \text{for} \quad i=j=1 \\ 0 & \text{for} \quad i\neq j \text{ and } i=1 \text{ or } j=1. \end{cases}$$

□

The two-sided edge case for the trigonometric Jacobi ensemble if both $\alpha = -1$ and $\beta = -1$ is as follows:

**Theorem 4.14.** *Let $X_\kappa$ be $N$-dimensional random variables with densities*

$$\tilde{c}_{\kappa,-1,-1} \prod_{1\leq i<j\leq N} (\cos(2t_j) - \cos(2t_i))^{2\kappa}$$

*on $\Omega = \{0 \leq t_N \leq ... \leq t_1 \leq \frac{\pi}{2}\}$ with*

$$\tilde{c}_{\kappa,-1,-1}^{-1} := \int_\Omega \prod_{1\leq i<j\leq N} (\cos(2t_j) - \cos(2t_i))^{2\kappa}.$$

*Furthermore define*

$$z_N^{(-1,-1)} := (z_{1,N}^{(-1,-1)}, z_{2,N}^{(-1,-1)}, ..., z_{N,N}^{(-1,-1)}) = (-1, z_{1,N-2}^{(1,1)}, ..., z_{N-2,N-2}^{(1,1)}, 1) \text{ and}$$

$$\begin{aligned}\tilde{z}_N^{(-1,-1)} &:= (\tilde{z}_{1,N}^{(-1,-1)}, \tilde{z}_{2,N}^{(-1,-1)}, ..., \tilde{z}_{N,N}^{(-1,-1)}) \\ &:= \frac{1}{2}\left(\frac{\pi}{2}, \arccos z_{1,N-2}^{(1,1)}, ..., \arccos z_{N-2,N-2}^{(1,1)}, 0\right).\end{aligned}$$

*Then*

$$\sqrt{\kappa}\left(X_\kappa - \tilde{z}_N^{(-1,-1)}\right)$$

*converges weakly for $\kappa \to \infty$ to the "normal distribution on a quarterspace", which is given by the density*

$$\frac{4\det(-H(\tilde{z}_N^{(-1,-1)}))}{(2\pi)^{\frac{N}{2}}} e^{\frac{1}{2}x^T H(\tilde{z}_N^{(-1,-1)})x} \mathbb{1}_{\{x\in\mathbb{R}^N : x_1 \le 0, x_N \ge 0\}}(x).$$

*Here $H(\tilde{z}_N^{(-1,-1)})$ is the Hessian of*

$$t \mapsto 2\sum_{1\le i<j\le N} \log\left(\cos(2t_j) - \cos(2t_i)\right)$$

*at the point $x = \tilde{z}_N^{(-1,-1)}$, i.e. $(-H(\tilde{z}_N^{(-1,-1)}))_{i,j}$ is given by*

$$\begin{cases} 4 + 8\sum_{l=1}^{N-2} \frac{1}{1+z_{l,N-1}^{(1,1)}} & \text{for} \quad i=j=1 \text{ or } i=j=N \\ 8\sum\limits_{l=1,l\neq i}^{N-1} \dfrac{1 - z_{i,N-2}^{(1,1)} z_{l,N-2}^{(1,1)}}{(z_{i,N-2}^{(1,1)} - z_{l,N-2}^{(1,1)})^2} & \\ + \dfrac{8}{1 - z_{i,N-2}^{(1,1)}} + \dfrac{8}{1 + z_{i,N-2}^{(1,1)}} & \text{for} \quad 2 \le i = j \le N-1 \\ -8\frac{\sqrt{1-(z_{i,N-2}^{(1,1)})^2}\sqrt{1-(z_{j,N-2}^{(1,1)})^2}}{(z_{i,N-2}^{(1,1)} - z_{j,N-2}^{(1,1)})^2} & \text{for} \quad i \neq j \text{ and } 2 \le i,j \le N-1 \\ 0 & \text{for} \quad i \neq j \text{ and } i = 1, N \text{ or } j = 1, N. \end{cases}$$

*Proof.* The goal is to apply Theorem 3.8. Here we have $l = 2$, $m = N + 1$ and

$$\begin{aligned} \phi(t) &= 2\sum_{1\le i<j\le N} \log\left(\cos(2t_j) - \cos(2t_i)\right), \psi(x) \equiv 1, \\ a_1 &= e_1, a_2 = 0 - e_N, a_3 = e_N - e_{N-1}, ..., a_{N+1} = e_2 - e_1, \\ b_1 &= -\frac{\pi}{2}, b_2 = b_3 = ... = b_{N+1} = 0, \\ \Omega^* &= \{t \in \mathbb{R}^N : t_1 \le 0, t_N \ge 0\}. \end{aligned}$$

We will now show that $\tilde{z}_N^{(-1,-1)}$ is the unique global maximum of $\phi$. First, we see that $\phi$ is monotonically increasing in $t_N$ and monotonically decreasing in $t_1$ and thus the global maximum of $\tilde{\phi} : \mathbb{R}^{N-2} \to \mathbb{R}$ with $\tilde{\phi}(t_2, ..., t_{N-1}) = \phi(\frac{\pi}{2}, t_2, ..., t_{N-1}, 0)$ needs to be determined. We have

$$\begin{aligned}
&\tilde{\phi}(t_2, ..., t_{N-1}) = \phi(\frac{\pi}{2}, t_2, ..., t_{N-1}, 0) \\
=&2 \sum_{2 \le i < j \le N-1} \log\left(\cos(2t_j) - \cos(2t_i)\right) + 2 \sum_{j=2}^{N-1} \log\left(\cos(2t_j) - \cos(2t_1)\right) \\
&+ 2 \sum_{i=2}^{N-1} \log\left(\cos(2t_N) - \cos(2t_i)\right) + 2 \log\left(\cos(2t_N) - \cos(2t_1)\right)
\end{aligned}$$

and

$$\begin{aligned}
&2 \sum_{j=2}^{N-1} \log\left(\cos(2t_j) - \cos(2t_1)\right) + 2 \sum_{i=2}^{N-1} \log\left(\cos(2t_N) - \cos(2t_i)\right) \\
&+ 2 \log\left(\cos(2t_N) - \cos(2t_1)\right) \\
=&2 \sum_{j=2}^{N-1} \log\left(\cos(2t_j) + 1\right) + 2 \sum_{i=2}^{N-1} \log\left(1 - \cos(2t_i)\right) + 2 \log(2) .
\end{aligned}$$

By (4.18) we see that the maximum of

$$2 \sum_{2 \le i < j \le N-1} \log(x_j - x_i) + 2 \sum_{i=2}^{N-2} \big( \log(1 - x_i) + \log(1 + x_i) \big)$$

with respect to

$$\tilde{\Omega} := \{0 \le t_{N-1} \le ... \le t_2 \le \frac{\pi}{2}\}$$

is achieved at $z_{N-2}^{(1,1)}$. Therefore, $\tilde{\phi}$ has its maximum at $\tilde{z}_{N-2}^{(1,1)} = \frac{1}{2} \arccos z_{N-2}^{(1,1)}$, and the maximum of $\phi$ is at $\tilde{z}_N^{(-1,-1)} = (\frac{\pi}{2}, z_{N-2}, 0)$. This maximum is global and unique with respect to $\Omega$. $\phi$ is a concave function because the Hessian at every point is negative definite. This holds because $-H_\phi(t)$ is given by (4.26) for $\alpha = \beta = -1$. Using the identity $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x - y)$ shows that this matrix is diagonally dominant with positive entries. Thus $-H_\phi(t)$ is positive definite. In particular, $-H_\phi(z_N^{-1,-1})$ is positive definite, and therefore the first condition of Theorem 3.8 is fulfilled. Now Theorem 3.8 yields the central limit theorem and the inverse covariance matrix is given by

$$(-H(\tilde{z}_N^{(-1,\beta)}))_{i,j} := \begin{cases} 8\sum_{l\neq i} \frac{1-\cos(2t_i)\cos(2t_j)}{(\cos(2t_j)-\cos(2t_i))^2} & \text{for } i=j \\ -8\frac{\sin(2t_i)\sin(2t_j)}{(\cos(2t_j)-\cos(2t_i))^2} & \text{for } i\neq j \end{cases}\Bigg|_{t=\tilde{z}_N^{(-1,-1)}}$$

$$= \begin{cases} 8\sum_{l\neq i} \frac{1-z_{i,N}^{(-1,-1)}z_{l,N}^{(-1,-1)}}{(z_{i,N}^{(-1,-1)}-z_{l,N}^{(-1,-1)})^2} & \text{for } i=j \\ -8\frac{\sqrt{1-(z_{i,N}^{(-1,-1)})^2}\sqrt{1-(z_{j,N}^{(-1,-1)})^2}}{(z_{i,N}^{(-1,-1)}-z_{j,N}^{(-1,-1)})^2} & \text{for } i\neq j \end{cases}$$

$$= \begin{cases} 4+8\sum_{l=1}^{N-2}\frac{1}{1+z_{l,N-1}^{(1,1)}} & \text{for} \quad i=j=1 \text{ or } i=j=N \\ 8\sum_{l=1,l\neq i}^{N-1}\frac{1-z_{i,N-2}^{(1,1)}z_{l,N-2}^{(1,1)}}{(z_{i,N-2}^{(1,1)}-z_{l,N-2}^{(1,1)})^2} & \text{for} \quad 2\leq i=j\leq N-1 \\ +\frac{8}{1-z_{i,N-2}^{(1,1)}}+\frac{8}{1+z_{i,N-2}^{(1,1)}} & \\ -8\frac{\sqrt{1-(z_{i,N-2}^{(1,1)})^2}\sqrt{1-(z_{j,N-2}^{(1,1)})^2}}{(z_{i,N-2}^{(1,1)}-z_{j,N-2}^{(1,1)})^2} & \text{for} \quad i\neq j \text{ and } 2\leq i,j\leq N-1 \\ 0 & \text{for} \quad i\neq j \text{ and } i=1,N \text{ or } j=1,N. \end{cases}$$

□

We will now treat the algebraic Jacobi ensembles and will investigate the edge cases $\alpha,\beta\to -1$. The algebraic Jacobi ensemble is defined by the densities

$$c_{\kappa,\alpha,\beta}\prod_{1\leq i<j\leq N}(x_j-x_i)^{2\kappa}\prod_{i=1}^N(1-x_i)^{(\alpha+1)\kappa-\frac{1}{2}}(1+x_i)^{(\beta+1)\kappa-\frac{1}{2}}, \qquad (4.27)$$

compare Theorem 4.11. Here, the gradient at the maximum is not zero and we will apply Theorem 3.10. Note that the term $\prod_{i=1}^N\sqrt{(1-x_i)(1+x_i)}$ causes some technical problems. For $\alpha>-1$ and $\beta>-1$ in Theorem 4.5, this was irrelevant because for $\psi(x)=\prod_{i=1}^N\sqrt{(1-x_i)(1+x_i)}$ it held that $\psi(z)\neq 0$. This is no longer the case here. Therefore we will investigate

$$c_{\kappa,\alpha,\beta}\prod_{1\leq i<j\leq N}(x_j-x_i)^{2\kappa}\prod_{i=1}^N(1-x_i)^{(\alpha+1)\kappa}(1+x_i)^{(\beta+1)\kappa}, \qquad (4.28)$$

with $c_{\kappa,\alpha,\beta}>0$ accordingly, instead. The behavior in the limit $\kappa\to\infty$ will very likely be the same. As we have two parameters $\alpha$ and $\beta$, there are several edge cases to be considered. The one-sided edge case for $-1=\alpha<\beta$ in the algebraic setting is as follows:

**Theorem 4.15.** *For $\beta>-1$ let $X_\kappa=(X_\kappa^1,...,X_\kappa^N)$ be $N$-dimensional random variables with the densities*

$$c_{\kappa,-1,\beta}\prod_{1\leq i<j\leq N}(x_j-x_i)^{2\kappa}\prod_{i=1}^N(1+x_i)^{(\beta+1)\kappa}$$

*on* $\Omega = \{-1 \leq x_1 \leq ... \leq x_N \leq 1\}$ *with*

$$c_{\kappa,-1,\beta}^{-1} := \int_\Omega \prod_{1\leq i<j\leq N} (x_j - x_i)^{2\kappa} \prod_{i=1}^N (1+x_i)^{(\beta+1)\kappa}$$

*and define*

$$z_N^{(-1,\beta)} := (z_{1,N}^{(-1,\beta)}, z_{2,N}^{(-1,\beta)}, ..., z_{N,N}^{(-1,\beta)}) = (z_{1,N-1}^{(1,\beta)}, ..., z_{N-1,N-1}^{(1,\beta)}, 1).$$

*Then*

$$\left(\sqrt{k}(X_\kappa^1 - z_{1,N-1}^{(1,\beta)}), ..., \sqrt{\kappa}(X_\kappa^{N-1} - z_{N-1,N-1}^{(1,\beta)}), \kappa(1 - X_\kappa^N)\right)$$

*converges in distribution for* $\kappa \to \infty$ *to a measure whose density is given by*

$$\exp\left(-\gamma_1 x_N\right) \frac{\sqrt{|\det H^*(z_N^{(-1,\beta)})|}}{(2\pi)^{\frac{N-1}{2}}}$$
$$\cdot \exp\left(\frac{1}{2}(x_1, ..., x_{N-1}) H^*(z_N^{(-1,\beta)})(x_1, ..., x_{N-1})^T\right)$$

*on* $\{(x_1, ..., x_n) \in \mathbb{R} : x_n \geq 0\}$. *Here,* $\gamma_1 := \left(\frac{\beta+1}{2} + 2\sum_{j=1}^{N-1} \frac{1}{1-z_{j,N-1}^{(1,\beta)}}\right)$ *and* $H^*(z_N^{(-1,\beta)}) = \left(H_\phi(z_N^{(-1,\beta)})_{i,j}\right)_{i,j=1,...,N-1}$ *is a submatrix of the Hessian of*

$$\phi(x) = 2 \sum_{1\leq i<j\leq N} \log(x_j - x_i) + (\beta+1) \sum_{i=1}^N (1+x_i)$$

*at* $z_N^{(-1,\beta)}$, *i.e*

$$(-H^*(z_N^{(-1,\beta)}))_{i,j}$$
$$= \begin{cases} 2\sum_{l=1,l\neq i}^{N-1} (z_{i,N-1}^{(1,\beta)} - z_{l,N-1}^{(1,\beta)})^{-2} + \frac{\beta+1}{1+z_{i,N-1}^{(1,\beta)}} + \frac{2}{1-z_{i,N-1}^{(1,\beta)}} & \textit{for} \quad i = j \\ -2(z_{i,N-1}^{(1,\beta)} - z_{j,N-1}^{(1,\beta)})^{-2} & \textit{for} \quad i \neq j. \end{cases}$$

*Proof.* The goal is to apply Theorem 3.10. Here, we have $l = 1$, $m = N+1$ and

$$\begin{aligned}
\phi(x) &= 2 \sum_{1\le i<j\le N} \log\left(x_j - x_i\right) + (\beta+1)\sum_{i=1}^{N} \log(1+x_i), \psi(x) \equiv 1,\\
\Omega &= \{-1 \le x_1 \le ... \le x_N \le 1\},\\
a_1 &= e_N, a_2 = e_{N-1} - e_N, ..., a_N = e_1 - e_2, a_{N+1} = -e_1,\\
b_2 &= b_3 = ... = b_N = 0, b_{N+1} = b_1 = -1,\\
z &= (z_N^{(-1,\beta)})^T,\\
M &= (e_N, e_{N-1}, ..., e_1), M^{-1} = M = (e_N, e_{N-1}, ..., e_1),\\
G_I &= I_N, K = \mathrm{diag}(\kappa, \sqrt{\kappa}, ..., \sqrt{\kappa}),\\
\Omega^* &= \{x \in \mathbb{R}^N : x_1 \le 0\},\\
\nabla\phi(z) &= \left(\frac{\beta+1}{2} + 2\sum_{j=1}^{N-1} \frac{1}{1 - z_{j,N-1}^{(1,\beta)}}\right)\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix} = \gamma_1 a_1.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
KG_I M^{-1} X_\kappa^T &= (\kappa X_\kappa^N, \sqrt{\kappa} X_\kappa^{N-1}, ..., X_\kappa^1)^T \text{ and}\\
KG_I M^{-1} z &= \left(\kappa, \sqrt{\kappa} z_{N-1,N-1}^{(1,\beta)}, ..., \sqrt{\kappa} z_{1,N-1}^{(1,\beta)}\right)^T.
\end{aligned}$$

Furthermore $z \in \partial\Omega$, and $\phi$ attains its global maximum on $\Omega$ in $z$. This holds true because $\phi$ is monotonically increasing in $x_N$, and the only maximum of $\tilde{\phi} : \mathbb{R}^{N-1} \to \mathbb{R}$,

$$\begin{aligned}
&\tilde{\phi}(x_1, ..., x_{N-1}) := \phi(x_1, x_2, ...x_{N-1}, 1)\\
=&2 \sum_{1\le i<j\le N-1} \log(x_j - x_i)\\
+&\sum_{i=1}^{N-1}\Big((\beta+1)\log(1+x_i) + 2\log(1-x_i)\Big) + (\beta+1)\log(2)
\end{aligned}$$

with respect to

$$\tilde{\Omega} := \{x \in \mathbb{R}^{N-1} : -1 \le x_1 \le .... \le x_{N-1} \le 1\}$$

is given by $\left(z_{1,N-1}^{(1,\beta)}, z_{2,N-1}^{(1,\beta)}, ..., z_{N-1,N-1}^{(1,\beta)}\right)$. $\tilde{\phi}$ is a strictly concave function and hence, for $C = (e_{N-1}, e_{N-2}, .., e_1) = (e_1, ..., e_{N-1}) \cdot (e_{N-1,N-1}, ..., e_{1,N-1})$, the matrix

$$\begin{aligned}
\tilde{H}^*(z_N^{(-1,\beta)}) &= C^T H_\phi(z_N^{(-1,\beta)}) C\\
&= \left(e_{N-1,N-1}, ..., e_{1,N-1}\right) H_{\tilde{\phi}} \left(e_{N-1,N-1}, ..., e_{1,N-1}\right)
\end{aligned}$$

is negative definite. Thus all conditions of Theorem 3.10 are fulfilled. Therefore

$$\left(k(X_\kappa^N - 1), \sqrt{k}(X_\kappa^{N-1} - z_{N-1,N-1}^{(1,\beta)}), ..., \sqrt{\kappa}(X_\kappa^1 - z_{1,N-1}^{(1,\beta)})\right)$$

converges weakly to a random variable with the density

$$\mathbb{1}_{\{x_1 \leq 0\}}(x) \exp(\gamma_1 x_1) \frac{\sqrt{|\det \tilde{H}^*(z_N^{(-1,\beta)})|}}{(2\pi)^{\frac{N-1}{2}}} \cdot \exp\left(\frac{1}{2}(x_2, ..., x_N)\tilde{H}^*(z_N^{(-1,\beta)})(x_2, ..., x_N)^T\right).$$

Now, calculating the push-forward measure under the mapping

$$(x_1 ...., x_N) \mapsto (x_N, x_{N-1}, .., x_2, -x_1)$$

yields the claim. □

In the above theorem, the one-sided limiting edge case $-1 = \alpha < \beta$ was considered, in which one parameter $\alpha$ takes the value -1. The other one-sided limiting edge case $-1 = \beta < \alpha$, in which $\beta$ takes the value -1, is identical due to symmetry. Next, the two-sided limiting edge case is considered, in which both parameters $\alpha$ and $\beta$ take the value -1.

**Theorem 4.16.** *For $\alpha = \beta = -1$ let $X_\kappa = (X_\kappa^1, ..., X_\kappa^N)$ be $N$-dimensional random variables with the densities*

$$c_{\kappa,-1,-1} \prod_{1 \leq i < j \leq N} (x_j - x_i)^{2\kappa}$$

*on $\Omega = \{-1 \leq x_1 \leq ... \leq x_N \leq 1\}$ with*

$$c_{\kappa,-1,-1}^{-1} := \int_\Omega \prod_{1 \leq i < j \leq N} (x_j - x_i)^{2\kappa},$$

*and define*

$$z_N^{(-1,-1)} := (z_{1,N}^{(-1,-1)}, z_{2,N}^{(-1,-1)}, ..., z_{N,N}^{(-1,\beta)}) = (-1, z_{1,N-2}^{(1,1)}, ..., z_{N-2,N-2}^{(1,1)}, 1).$$

*Then*

$$\left(\kappa(X_\kappa^1 + 1), \sqrt{k}(X_\kappa^2 - z_{1,N-2}^{(1,1)}), ..., \sqrt{\kappa}(X_\kappa^{N-1} - z_{N-2,N-2}^{(1,1)}), \kappa(1 - X_\kappa^N)\right)$$

*converges weakly for $\kappa \to \infty$ to a random variable with the density*

$$e^{-\gamma_1 x_1} e^{-\gamma_N x_N} \frac{\sqrt{|\det H^*(z_N^{(-1,-1)})|}}{(2\pi)^{\frac{N-2}{2}}} \cdot \exp\left(\frac{1}{2}(x_2, ..., x_{N-1})H^*(z_N^{(-1,-1)})(x_2, ..., x_{N-1})^T\right)$$

*on* $\{x \in \mathbb{R} : x_1, x_N \geq 0\}$.
*Here* $\gamma_1 = \gamma_2 := 2\sum_{j=2}^{N-1} \frac{1}{1-z_{j+1,N-2}^{(1,1)}} + 1$, *and*

$$H^*(z_N^{(-1,-1)}) = \left(H_\phi(z_N^{(-1,-1)})_{i,j}\right)_{i,j=2,\dots,N-1}$$

*is a submatrix of the Hessian of*

$$\phi(x) = 2\sum_{1\leq i<j\leq N} \log(x_j - x_i)$$

*at* $z_N^{(-1,-1)}$, *i.e*

$$\begin{aligned}&(-H^*(z_N^{(-1,-1)}))_{i,j}\\ &= \begin{cases} 2\sum_{l=2,l\neq i}^{N-1}(z_{i,N-2}^{(1,1)} - z_{l,N-2}^{(1,1)})^{-2} + \frac{2}{1+z_{i,N-2}^{(1,1)}} + \frac{2}{1-z_{i,N-2}^{(1,1)}} & \textit{for} \quad i = j \\ -2(z_{i,N-2}^{(1,1)} - z_{j,N-2}^{(1,1)})^{-2} & \textit{for} \quad i \neq j. \end{cases}\end{aligned}$$

*Proof.* This goal is to apply Theorem 3.10. Here we have $l = 2$, $m = N+1$ and

$$\begin{aligned}\phi(x) &= 2\sum_{1\leq i<j\leq N} \log(x_j - x_i), \psi(x) \equiv 1,\\ \Omega &= \{-1 \leq x_1 \leq \dots \leq x_N \leq 1\},\\ a_1 &= e_N, a_2 = -e_1, a_3 = e_1 - e_2, \dots, a_{N+1} = e_{N-1} - e_N,\\ b_1 &= b_2 = -1, b_3 = \dots = b_{N+1} = 0,\\ z &= (z_N^{(-1,-1)})^T,\\ M &= (e_N, -e_1, e_2, \dots, e_{N-1}), M^{-1} = M^T,\\ G_I &= I_N, K = \mathrm{diag}(\kappa, \kappa, \sqrt{\kappa}, \dots, \sqrt{\kappa}),\\ \Omega^* &= \{x \in \mathbb{R}^N : x_1 \leq 0, x_N \leq 0\},\\ \nabla\phi(z) &= \left(2\sum_{j=2}^{N-1} \frac{1}{1 - z_{j+1,N-2}^{(1,1)}} + 1\right)\begin{pmatrix}-1\\0\\ \vdots \\0\\1\end{pmatrix} = \gamma_1 a_1 + \gamma_2 a_2.\end{aligned}$$

Therefore we have

$$\begin{aligned}KG_IM^{-1}X_\kappa^T &= (\kappa X_\kappa^N, -\kappa X_\kappa^1, \sqrt{\kappa}X_\kappa^2, \dots, \sqrt{\kappa}X_\kappa^{N-1})^T \text{ and}\\ KG_IM^{-1}z &= (\kappa, \kappa, \sqrt{\kappa}z_{1,N-2}^{(1,1)}, \dots, \sqrt{\kappa}z_{N-2,N-2}^{(1,1)})^T.\end{aligned}$$

Furthermore, $z \in \partial\Omega$ and $\phi$ achieves its global maximum on $\Omega$ in $z$. This holds true because $\phi$ is monotonically increasing in $x_N$, monotonically decreasing in

$x_1$, and the only maximum of $\tilde{\phi} : \mathbb{R}^{N-1} \to \mathbb{R}$,

$$\begin{aligned} &\tilde{\phi}(x_2, ..., x_{N-1}) = \phi(-1, x_2, .., x_{N-1}, 1) \\ =&2 \sum_{2 \le i < j \le N-1} \log(x_j - x_i) + 2 \sum_{j=2}^{N-1} \log(1 + x_j) + +2 \sum_{j=2}^{N-1} \log(1 - x_j) + 2\log(2) \end{aligned}$$

with respect to

$$\tilde{\Omega} := \{-1 \le x_2 \le ... \le x_{N-1} \le 1\}$$

is given by $(z^{(1,1)}_{1,N-2}, z^{(1,1)}_{2,N-2}, ..., z^{(1,1)}_{N-2,N-2})$. Because we have $C = (e_2, e_3, .., e_{N-1})$, the matrix

$$H^*(z_N^{(-1,-1)}) = C^T H_\phi(z_N^{-1,-1}) C = H_{\tilde{\phi}}(z^{(1,1)}_{1,N-2}, z^{(1,1)}_{2,N-2}, ..., z^{(1,1)}_{N-2,N-2})$$

is negative definite. Thus all conditions of Theorem 3.10 are fulfilled and

$$\Big(\kappa(X_\kappa^N - 1), -\kappa(X_\kappa^1 + 1), \sqrt{\kappa}(X_\kappa^2 - z^{(1,1)}_{1,N-2}), ..., \sqrt{\kappa}(X_\kappa^{N-1} - z^{(1,1)}_{N-2,N-2})\Big)$$

converges weakly to a random variable with the density

$$e^{\gamma_1 x_1} e^{\gamma_1 x_2} \frac{\sqrt{|\det H^*(z_N^{(-1,-1)})|}}{(2\pi)^{\frac{N-2}{2}}} \exp\left(\frac{1}{2}(x_3, ..., x_N) H^*(z_N^{(-1,-1)})(x_3, .., x_N)^T\right)$$

on

$$\{x_1 \le 0, x_2 \le 0\}.$$

Now, calculating the push-forward measure under the mapping

$$(x_1, .., x_N) \mapsto (-x_2, x_3, .., x_{N-1}, -x_1)$$

yields the claim. $\square$

# 5 Dual polynomials and diagonalization

## 5.1 Diagonalizations of covariance matrices

In this section we analyze some of the covariance matrices $\Sigma_N$ from the central limit theorems in Section 4. Afterwards, we will use the theory of dual orthogonal polynomials of de Boor and Saff [BS86] to describe the eigenvectors of the covariance matrices. Therefore we will first summarize some results from [AV19; HV21; AHV21; DE05].

**Theorem 5.1** (Andraus,Voit(2019)). *Let $z_N = (z_{1,N}, .., z_{N,N})$ be the vector of ordered zeros of the $N$-th Hermite polynomial, and let*

$$(\Sigma_N^{-1})_{i,j} := \begin{cases} 1 + \sum_{l \neq i}(z_{i,N} - z_{l,N})^{-2} & \text{for} \quad i = j \\ -(z_{i,N} - z_{j,N})^{-2} & \text{for} \quad i \neq j \end{cases}$$

*be the inverse covariance matrix from Theorem 4.1. Then the eigenvalues of $\Sigma_N^{-1}$ are given by $\lambda_k = k$ for $k = 1, ..., N$. Moreover, denote the finite system of polynomials, orthonormal with respect to the measure*

$$\frac{1}{N}(\delta_{z_{1,N}} + \ldots + \delta_{z_{N,N}}) \in M^1(\mathbb{R}) \tag{5.1}$$

*by $(\tilde{Q}^H_{k,N})_{k=0}^{N-1}$. Then the vector*

$$(\tilde{Q}^H_{k-1,N}(z_{1,N}), ..., \tilde{Q}^H_{k-1,N}(z_{N,N}))^T$$

*is an eigenvector of $\Sigma_N^{-1}$ to the eigenvalue $k$.*

In the Laguerre case, the situation is as follows:

**Theorem 5.2** (Andraus,Voit(2019)). *For $\alpha > -1$ let*

$$z_N^\alpha = (z_{1,N}^\alpha, ..., z_{N,N}^\alpha)$$

*be the vector of ordered zeros $z_{1,N}^\alpha < ... < z_{N,N}^\alpha$ of the $N$-th Laguerre polynomial. Furthermore denote the inverse covariance matrix from Theorem 4.6 by*

$$(\Sigma_N^{-1})_{i,j} := \begin{cases} 8 z_{i,N}^\alpha \sum_{l \neq i} \frac{1}{(z_{i,N}^\alpha - z_{l,N}^\alpha)^2} + 4 \frac{\alpha+1}{z_{i,N}^\alpha} & \text{for} \quad i = j \\ -8 \frac{\sqrt{z_{i,N}^\alpha z_{j,N}^\alpha}}{(z_{i,N}^\alpha - z_{j,N}^\alpha)^2} & \text{for} \quad i \neq j. \end{cases}$$

*Then the eigenvalues of $\Sigma_N^{-1}$ are given by $\lambda_k = 4k$ for $k = 1, ..., N$. Moreover, let $(\tilde{Q}^L_{k,N})_{k=0}^{N-1}$ be the finite system of polynomials orthonormal with respect to the measure*

$$\frac{1}{N(N+\alpha)} \left( z_{1,N}^\alpha \delta_{z_{1,N}^\alpha} + ... + z_{N,N}^\alpha \delta_{z_{N,N}^\alpha} \right). \tag{5.2}$$

*Then the vector*

$$\frac{1}{\sqrt{N(N+\alpha)}}\left(\sqrt{z_{1,N}^{\alpha}}\tilde{Q}_{k-1,N}^{L}(z_{1,N}^{\alpha}),...,\sqrt{z_{N,N}^{\alpha}}\tilde{Q}_{k-1,N}^{L}(z_{N,N}^{\alpha})\right)^{T}$$

*is an eigenvector of* $\Sigma_N^{-1}$ *to the eigenvalue* $4k$.

It is possible to derive similar results for the Jacobi case. See [HV21] for details. One main Theorem there is as follows:

**Theorem 5.3** (H.,Voit(2020))**.** *For* $\alpha, \beta > -1$ *let*

$$z_N^{(\alpha,\beta)} = (z_{1,N}^{(\alpha,\beta)}, ..., z_{N,N}^{(\alpha,\beta)})$$

*be the vector of ordered zeros* $z_{1,N}^{(\alpha,\beta)} < ... < z_{N,N}^{(\alpha,\beta)}$ *of the* $N$*-th Jacobi polynomial. Furhtermore, denote the inverse covariance matrix*

$$\begin{cases} 8\sum_{l\neq i}\frac{1-(z_{i,N}^{(\alpha,\beta)})^2}{\left(z_{i,N}^{(\alpha,\beta)}-z_{l,N}^{(\alpha,\beta)}\right)^2} + 4(\alpha+1)\frac{1+z_{i,N}^{(\alpha,\beta)}}{1-z_{i,N}^{(\alpha,\beta)}} + 4(\beta+1)\frac{1-z_{i,N}^{(\alpha,\beta)}}{1+z_{i,N}^{(\alpha,\beta)}} & \text{for } i=j \\ -8\frac{\sqrt{1-(z_{i,N}^{(\alpha,\beta)})^2}\sqrt{1-(z_{j,N}^{(\alpha,\beta)})^2}}{(z_{i,N}^{\alpha}-z_{j,N}^{\alpha})^2} & \text{for } i\neq j. \end{cases}$$

*from Theorem 4.12 by* $\Sigma_N^{-1}$. *Then for* $k = 1, ..., N$ *the eigenvalues of* $\Sigma_N^{-1}$ *are given by*

$$4k(2N+\alpha+\beta+1-k) > 0.$$

*Moreover, for*

$$\zeta_{N,\alpha,\beta} := \frac{(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)}{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}$$

*let* $(\tilde{Q}_{k,N}^{J})_{k=0}^{N-1}$ *be the finite system of polynomials orthonormal with respect to the measure*

$$\zeta_{N,\alpha,\beta}\left(\left(1-(z_{1,N}^{(\alpha,\beta)})^2\right)\delta_{z_{1,N}^{(\alpha,\beta)}} + ... + \left(1-(z_{N,N}^{(\alpha,\beta)})^2\right)\delta_{z_{N,N}^{(\alpha,\beta)}}\right). \tag{5.3}$$

*Then the vector*

$$\sqrt{\frac{(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)}{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}}\begin{pmatrix}\sqrt{1-(z_{1,N}^{(\alpha,\beta)})^2}\tilde{Q}_{k,N}^{J}(z_{1,N}^{(\alpha,\beta)}) \\ \vdots \\ \sqrt{1-(z_{N,N}^{(\alpha,\beta)})^2}\tilde{Q}_{k,N}^{J}(z_{N,N}^{(\alpha,\beta)})\end{pmatrix}$$

*is an eigenvector of* $\Sigma_N^{-1}$ *for the eigenvalue* $4k(2N+\alpha+\beta+1-k)$.

*Proof.* We will use the abbreviations

$$\tilde{S} := \frac{1}{2}\Sigma_N^{-1},$$
$$z_i := z_{i,N}^{(\alpha,\beta)},$$
$$v_k := \begin{pmatrix} \sqrt{1-z_1^2}z_1^{k-1} \\ \vdots \\ \sqrt{1-z_N^2}z_N^{k-1} \end{pmatrix}.$$

First we will show that for $k = 1, ..., N$ it holds that

$$\tilde{S}v_k = \begin{pmatrix} (\lambda_k z_1^{k-1} + p_{k-2}(z_1))\sqrt{1-z_1^2} \\ \vdots \\ (\lambda_k z_N^{k-1} + p_{k-2}(z_N))\sqrt{1-z_N^2} \end{pmatrix}, \tag{5.4}$$

where $\lambda_k = 2k(2N + \alpha + \beta + 1 - k)$ and $p_{k-2}$ is a polynomial of degree $k - 2$. For $k = 1$ we use the convention $p_{-1} \equiv 0$.
We will start with $k = 1$. In this case, Equation (5.4) gives that $v_1$ is an eigenvector of $\tilde{S}$ to the eigenvalue $\lambda_1$. By definition of $\tilde{S}$, the $i$-th component ($i = 1, \dots, N$) of $\tilde{S}v_1$ is given by

$$\begin{aligned}(\tilde{S}v_1)_i =& 4\sum_{l\neq i}\frac{1-z_i^2}{(z_i-z_l)^2}\sqrt{1-z_i^2} + 2(\alpha+1)\frac{1+z_i}{1-z_i}\sqrt{1-z_i^2} \\ &+ 2(\beta+1)\frac{1-z_i}{1+z_i}\sqrt{1-z_i^2} - 4\sum_{l\neq i}\frac{(1-z_l^2)\sqrt{1-z_i^2}}{(z_i-z_l)^2} \\ =& 4\sqrt{1-z_i^2}\left(\sum_{l\neq i}\frac{z_l^2-z_i^2}{(z_i-z_l)^2} + \frac{\alpha+1}{2}\frac{1+z_i}{1-z_i} + \frac{\beta+1}{2}\frac{1-z_i}{1+z_i}\right).\end{aligned}$$

Hence,

$$\begin{aligned}(\tilde{S}v_1)_i =& 4\sum_{l\neq i}\frac{1-z_i^2}{(z_i-z_l)^2}\sqrt{1-z_i^2} + 2(\alpha+1)\frac{1+z_i}{1-z_i}\sqrt{1-z_i^2} \\ &+ 2(\beta+1)\frac{1-z_i}{1+z_i}\sqrt{1-z_i^2} - 4\sum_{l\neq i}\frac{(1-z_l^2)\sqrt{1-z_i^2}}{(z_i-z_l)^2} \\ =& 4\sqrt{1-z_i^2}\left(\sum_{l\neq i}\frac{z_l^2-z_i^2}{(z_i-z_l)^2} + \frac{\alpha+1}{2}\frac{1+z_i}{1-z_i} + \frac{\beta+1}{2}\frac{1-z_i}{1+z_i}\right) \\ =& 4\sqrt{1-z_i^2}\left(\sum_{l\neq i}\frac{-2z_i+(z_i-z_l)}{z_i-z_l} + \frac{\alpha+1}{2}\frac{1+z_i}{1-z_i} + \frac{\beta+1}{2}\frac{1-z_i}{1+z_i}\right) \\ =& 4\sqrt{1-z_i^2}\left((N-1) - 2z_i\sum_{l\neq i}\frac{1}{z_i-z_l} + \frac{\alpha+1}{2}\frac{1+z_i}{1-z_i} + \frac{\beta+1}{2}\frac{1-z_i}{1+z_i}\right).\end{aligned}$$

Inserting Equation (4.19) now leads to

$$(\tilde{S}v_1)_i = 4\sqrt{1-z_i^2}\left((N-1)+\frac{\alpha+1}{2}+\frac{\beta+1}{2}\right) \qquad (i=1,\ldots,N).$$

This proves that $v_1$ is an eigenvector to eigenvalue $\lambda_1$ as claimed.
We now consider the case $k=2$. We now have

$$\begin{aligned}(\tilde{S}v_2)_i =&4\sum_{l\neq i}\frac{1-z_i^2}{(z_i-z_l)^2}z_i\sqrt{1-z_i^2}+2(\alpha+1)\frac{1+z_i}{1-z_i}z_i\sqrt{1-z_i^2}\\ &+2(\beta+1)\frac{1-z_i}{1+z_i}z_i\sqrt{1-z_i^2}-4\sum_{l\neq i}\frac{z_l(1-z_l^2)\sqrt{1-z_i^2}}{(z_i-z_l)^2}.\end{aligned}$$

Hence,

$$\begin{aligned}&(\tilde{S}v_2)_i\\ =&4\sqrt{1-z_i^2}\left(\sum_{l\neq i}\frac{(1-z_i^2)z_i-(1-z_l^2)z_l}{(z_i-z_l)^2}+\frac{\alpha+1}{2}\frac{1+z_i}{1-z_i}z_i+\frac{\beta+1}{2}\frac{1-z_i}{1+z_i}z_i\right)\\ =&4\sqrt{1-z_i^2}\left(\sum_{l\neq i}\frac{1-z_l^2-z_iz_l-z_i^2}{z_i-z_l}+\frac{\alpha+1}{2}\frac{1+z_i}{1-z_i}z_i+\frac{\beta+1}{2}\frac{1-z_i}{1+z_i}z_i\right)\\ =&4\sqrt{1-z_i^2}\Bigg(\sum_{l\neq i}\frac{1+z_l(z_i-z_l)+2z_i(z_i-z_l)-3z_i^2}{z_i-z_l}\\ &\qquad+\frac{\alpha+1}{2}\frac{1+z_i}{1-z_i}z_i+\frac{\beta+1}{2}\frac{1-z_i}{1+z_i}z_i\Bigg)\\ =&4\sqrt{1-z_i^2}\Bigg((c-z_i)+2z_i(N-1)+(1-3z_i^2)\sum_{l\neq i}\frac{1}{z_i-z_l}\\ &\qquad+\frac{\alpha+1}{2}\frac{1+z_i}{1-z_i}z_i+\frac{\beta+1}{2}\frac{1-z_i}{1+z_i}z_i\Bigg)\end{aligned}$$

with $c:=\sum_{j=1}^N z_j$. Equation (4.19) and a short computation now lead to

$$(\tilde{S}v_2)_i = 4\sqrt{1-z_i^2}\left((c-z_i)+2z_i(N-1)+\frac{\alpha+1}{2}(2z_i+1)+\frac{\beta+1}{2}(2z_i-1)\right)$$

for $i=1,\ldots,N$. This proves that $\tilde{S}v_2$ has the form as claimed in (5.4) with some constant polynomial $p_2$.

We now turn to the case $k \geq 3$. Here we have

$$\begin{aligned}(\tilde{S}v_k)_i =&4\sum_{l\neq i}\frac{1-z_i^2}{(z_i-z_l)^2}z_i^{k-1}\sqrt{1-z_i^2}+2(\alpha+1)\frac{1+z_i}{1-z_i}z_i^{k-1}\sqrt{1-z_i^2}\\ &+2(\beta+1)\frac{1-z_i}{1+z_i}z_i^{k-1}\sqrt{1-z_i^2}-4\sum_{l\neq i}\frac{z_l^{k-1}(1-z_l^2)\sqrt{1-z_i^2}}{(z_i-z_l)^2}\\ =&4\sqrt{1-z_i^2}\Bigg(\sum_{l\neq i}\frac{z_i^{k-1}-z_i^{k+1}-z_l^{k-1}+z_l^{k+1}}{(z_i-z_l)^2}+\\ &+\frac{\alpha+1}{2}\frac{1+z_i}{1-z_i}z_i^{k-1}+\frac{\beta+1}{2}\frac{1-z_i}{1+z_i}z_i^{k-1}\Bigg)\end{aligned}$$

with

$$\begin{aligned}&z_i^{k-1}-z_i^{k+1}-z_l^{k-1}+z_l^{k+1}=\\ &=(z_i-z_l)\Bigg(z_i^{k-2}+z_i^{k-3}z_l+\ldots+z_l^{k-2}-z_i^k-z_i^{k-1}z_l-\ldots-z_l^k\Bigg)\\ &=(z_i-z_l)\Bigg((z_l-z_i)\Big(z_l^{k-3}+2z_l^{k-4}z_i+\ldots+(k-2)z_i^{k-3}\Big)+(k-1)z_i^{k-2}\\ &\qquad-(z_l-z_i)\Big(z_l^{k-1}+2z_l^{k-2}z_i+\ldots+kz_i^{k-1}\Big)-(k+1)z_i^k\Bigg).\end{aligned}$$

We thus conclude that

$$\begin{aligned}(\tilde{S}v_k)_i\frac{1}{4\sqrt{1-z_i^2}}=&\sum_{l\neq i}\Big(z_l^{k-1}+2z_l^{k-2}z_i+\ldots+kz_i^{k-1}\\ &-z_l^{k-3}-2z_l^{k-4}z_i-\ldots-(k-2)z_i^{k-3}\\ &+\sum_{l\neq i}\frac{(k-1)z_i^{k-2}-(k+1)z_i^k}{z_i-z_l}\\ &+\frac{\alpha+1}{2}\frac{1+z_i}{1-z_i}z_i^{k-1}+\frac{\beta+1}{2}\frac{1-z_i}{1+z_i}z_i^{k-1}\Bigg).\end{aligned}$$

With Equation (4.19), a suitable constant $C$, and with suitable polynomials

$r_k, r_k^{(1)}, r_k^{(2)}, r_k^{(3)}, r_k^{(4)}$ of order at most $k-2$, we thus obtain

$$\begin{aligned}(\tilde{S}v_k)_i \frac{1}{4\sqrt{1-z_i^2}} &= C - z_i^{k-1} - 2z_i^{k-1} - \ldots - (k-1)z_i^{k-1} \\ &+ k(N-1)z_i^{k-1} + r_k(z_i) \\ &+ \frac{\alpha+1}{2} \frac{z_i^k + z_i^{k-1} + (k-1)z_i^{k-2} - (k+1)z_i^k}{1-z_i} \\ &+ \frac{\beta+1}{2} \frac{z_i^{k-1} - z_i^k - (k-1)z_i^{k-2} + (k+1)z_i^k}{1+z_i}.\end{aligned}$$

Hence

$$\begin{aligned}(\tilde{S}v_k)_i =& 4\sqrt{1-z_i^2}\Bigg(\Big(k(N-1) - \frac{(k-1)k}{2}\Big)z_i^{k-1} + r_k^{(1)}(z_i) \\ &+ \frac{\alpha+1}{2}(kz_i^{k-1} + r_k^{(2)}(z_i)) + \frac{\beta+1}{2}(kz_i^{k-1} + r_k^{(3)}(z_i))\Bigg) \\ =& \sqrt{1-z_i^2}\Big(2k(2N+\alpha+\beta+1-k)z_i^{k-1} + r_k^{(4)}(z_i)\Big).\end{aligned}$$

This shows Equation (5.4) for $k \geq 3$. Using (5.4) and an induction on $k = 1, .., N$ shows that there exist polynomials $q_{k-1}$ of degree $k-1$ such that

$$\begin{pmatrix} q_{k-1}(z_1)\sqrt{1-z_1^2} \\ \vdots \\ q_{k-1}(z_N)\sqrt{1-z_N^2} \end{pmatrix}$$

is an eigenvector of $\Sigma_N^{-1}$ to the eigenvalue $4k(2N+\alpha+\beta+1-k)$.
Because $\Sigma_N^{-1}$ is a symmetric matrix and the eigenspaces are orthogonal, the system $(q_k)_{k=0,..,N-1}$ is orthogonal with respect to the measure given in (5.3). Therefore, for $k = 0, ..., N-1$, $q_k$ and $\tilde{Q}_{k,N}^J$ are identical up to some multiplicative constant. Thus the vectors

$$\begin{pmatrix} \sqrt{1-(z_1)^2}Q_{k,N}^J(z_1) \\ \vdots \\ \sqrt{1-(z_N)^2}Q_{k,N}^J(z_N) \end{pmatrix} \text{ and } \begin{pmatrix} q_{k-1}(z_1)\sqrt{1-z_1^2} \\ \vdots \\ q_{k-1}(z_N)\sqrt{1-z_N^2} \end{pmatrix}$$

are in the same eigenspace. □

**Remark 5.4.**

- The above results also yield the eigenvalues and vectors of appearing covariance matrices in the edge cases of the trigonometric Jacobi ensemble (Theorem 4.13 and Theorem 4.14) and the squared Laguerre ensemble (Theorem 4.7).

- This proof does not seem to be transferable from the trigonometric Jacobi ensemble to the algebraic Jacobi ensemble. The same holds true for the Laguerre ensemble, where it does not seem to be transferable to the parametrization of Theorem 4.5. The situation is similar for the edge cases of the algebraic Jacobi ensemble (Theorem 4.15 and Theorem 4.16) and the Laguerre ensemble (Theorem 4.8). Obviously, because the matrices are positive definite and symmetric, those eigenvalues and vectors exist. However, it seems more complicated to describe them.

## 5.2 De Boor-Saff duality

We will now use the de Boor-Saff [BS86] dual polynomials to describe the occurring orthogonal polynomials. To explain this, we first review some theory from [VZ04] and Section 2.11 of [Ism05]. This section follows Section 4 of [AHV21]. Let $(\hat{P}_n)_{n=0}^{\infty}$ be a sequence of monic orthogonal polynomials, where the orthogonality measure is a probability measure $\mu$ on $\mathbb{R}$ which admits all moments, i.e.,

$$\int_{\mathbb{R}} \hat{P}_i(x)\hat{P}_j(x)d\mu(x) = \xi_i\delta_{ij} \quad (i,j = 0,1,2,\ldots) \tag{5.5}$$

with some constants $\xi_i > 0$ $(i \geq 0)$. We also have a three-term recurrence relation

$$\hat{P}_0 = 1,\ \hat{P}_1(x) = x - b_0,\ x\hat{P}_n(x) = \hat{P}_{n+1}(x) + b_n\hat{P}_n(x) + u_n\hat{P}_{n-1}(x) \quad (n \geq 1) \tag{5.6}$$

with coefficients $b_n \in \mathbb{R}$ and $u_n > 0$. We also consider the associated orthonormal polynomials $(\tilde{P}_n := \xi_n^{-1/2}\hat{P}_n)_{n=0}^{\infty}$ with $\int_{\mathbb{R}} \tilde{P}_i(x)\tilde{P}_j(x)d\mu(x) = \delta_{ij}$. For $n \geq 1$ these polynomials then satisfy the three-term recurrence

$$\tilde{P}_0 = 1,\ \tilde{P}_1(x) = a_1^{-1}(x - b_0),\ x\tilde{P}_n(x) = a_{n+1}\tilde{P}_{n+1}(x) + b_n\tilde{P}_n(x) + a_n\tilde{P}_{n-1}(x). \tag{5.7}$$

Here, $a_n = u_n\sqrt{\xi_{n-1}/\xi_n} = \sqrt{\xi_n/\xi_{n-1}}$. In particular, we have

$$\xi_0 = 1,\ \xi_n = u_n u_{n-1}\cdots u_1 \quad \text{and} \quad a_n = \sqrt{u_n} \quad (n \geq 1). \tag{5.8}$$

Now fix $N > 0$ arbitrarily. Gauss quadrature implies that the finite set of polynomials $(\tilde{P}_n)_{n=0}^{N-1}$ obeys the discrete orthogonality relation

$$\sum_{i=1}^{N} w_i\tilde{P}_m(z_{i,N})\tilde{P}_n(z_{i,N}) = \delta_{mn}, \tag{5.9}$$

with the $N$ ordered zeros $z_{1,N} < \ldots < z_{N,N}$ of $\tilde{P}_N$ and the Christoffel numbers

$$w_i := \frac{1}{a_N\tilde{P}_{N-1}(z_{i,N})\tilde{P}'_N(z_{i,N})} > 0 \quad (i = 1,\ldots,N), \tag{5.10}$$

which satisfy the normalization $\sum_{i=1}^{N} w_i = 1$.

**Definition 5.5.** Let $N > 0$. The monic polynomials $(\hat{Q}_{k,N})_{k=0}^{N-1}$ are called dual (in the de Boor-Saff sense) to $(\hat{P}_n(x))_{n=0}^{N-1}$ if they satisfy the three-term recurrence

$$\begin{aligned} &\hat{Q}_{0,N} = 1,\ \hat{Q}_{1,N}(x) = x - b_{N-1}, \\ &x\hat{Q}_{k,N}(x) = \hat{Q}_{k+1,N}(x) + b_{N-k-1}\hat{Q}_{k,N}(x) + u_{N-k}\hat{Q}_{k-1,N}(x) \end{aligned} \tag{5.11}$$

for $k = 1, .., N-2$.

We now recall some consequences of this duality from [VZ04]:

**Lemma 5.6.** *The dual monic polynomials $(\hat{Q}_{k,N})_{k=0}^{N-1}$ are orthogonal with respect to the discrete measure*

$$\sum_{i=1}^{N} w_i^* \delta_{z_{i,N}}$$

*with the dual Christoffel numbers*

$$w_i^* = \frac{\tilde{P}_{N-1}(z_{i,N})}{a_N \tilde{P}'_N(z_{i,N})} > 0 \quad (i = 1, \ldots, N), \tag{5.12}$$

*which again satisfy*

$$\sum_{i=1}^{N} w_i^* = 1. \tag{5.13}$$

In particular, by (5.8), the normalized dual polynomials $(\tilde{Q}_{k,N})_{k=0}^{N-1}$ with

$$\sum_{i=1}^{N} w_i^* \tilde{Q}_{m,N}(z_{i,N}) \tilde{Q}_{n,N}(z_{i,N}) = \delta_{mn} \quad (m, n = 0, \ldots, N-1) \tag{5.14}$$

satisfy

$$\tilde{Q}_{k,N}(x) = \frac{\hat{Q}_{k,N}}{a_N^2 a_{N-1}^2 \cdots a_{N-k}^2}. \tag{5.15}$$

In summary, we obtain from (5.15) and the three-term-recurrence in Definition (5.5):

**Lemma 5.7.** *The orthonormal dual polynomials $(\tilde{Q}_{k,N})_{k=0}^{N-1}$ satisfy the three-term-recurrence relation*

$$\begin{aligned} &\tilde{Q}_{0,N} = 1,\ \tilde{Q}_{1,N}(x) = a_{N-1}^{-1}(x - b_{N-1}), \\ &x\tilde{Q}_{k,N}(x) = a_{N-k-1}\tilde{Q}_{k+1,N}(x) + b_{N-k-1}\tilde{Q}_{k,N}(x) + a_{N-k}\tilde{Q}_{k-1,N}(x) \end{aligned} \tag{5.16}$$

*for $k = 0, ..., N-2$.*

**Remark 5.8.** The monic three-term-recurrence (5.11) is also available for $k = N-1$, i.e., we obtain a monic polynomial $\hat{Q}_{N,N}$. It can easily be seen (see [VZ04] or Section 2.11 of [Ism05]) that $\hat{Q}_{N,N} = \hat{P}_N$ holds. Moreover, if we choose $a_0 = 0$ in (5.16), then the recurrence (5.7) remains valid for $k = N-1$, arbitrary polynomials $\tilde{Q}_{k+1,N}$, and $x = z_{i,N}$ for $i = 1, \dots, N$.

**Lemma 5.9.** *The polynomials occurring in Theorem 5.1, Theorem 5.2 and Theorem 5.3 are the dual polynomials of the Hermite, Laguerre and Jacobi polynomials. In other words, the normalized eigenvectors of the corresponding inverse covariance matrices* $\Sigma_N^{-1}$ *have the form*

$$\left(\sqrt{w_1^*}\tilde{Q}_{1,N}, ..., \sqrt{w_N^*}\tilde{Q}_{N,N}\right)^T, \tag{5.17}$$

*and the dual weights* $(w_i^*)_{i=1}^N$ *have the form*

$$\begin{aligned} &\textit{Hermite:} && w_i^* = \frac{1}{N}, \\ &\textit{Laguerre:} && w_i^* = \frac{z_{i,N}^{\alpha}}{N(N+\alpha)}, \textit{ and} \\ &\textit{Jacobi:} && w_i^* = \frac{(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)}{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}(1-(z_{i,N}^{(\alpha,\beta)})^2). \end{aligned}$$

*Proof.* Equation (5.14) implies that the vector (5.17) has length 1. Therefore it remains to show that this vector is an eigenvector of the corresponding inverse covariance matrix $\Sigma_N^{-1}$.
We first consider the Hermite case. Here we have for the monic Hermite polynomials $\hat{H}_N$ by Section 5.5 of [Sze75] that $\hat{H}_N'(x) = N\hat{H}_{N-1}(x)$. Hence, by (5.12) and (5.14), for the orthonormal dual Hermite polynomials $\tilde{Q}_{k,N}^H$ we have

$$\sum_{i=1}^{N} \frac{1}{N}\tilde{Q}_{k,N}^H(z_{i,N})\tilde{Q}_{l,N}^H(z_{i,N}) = \delta_{kl}. \tag{5.18}$$

This means that these polynomials are orthonormal with respect to the measure given in (5.1). By Theorem 5.1 we now have that (5.17) is an eigenvector of the corresponding covariance matrix $\Sigma_N^{-1}$.

We now turn to the Laguerre case. By Section 4.6 of [Ism05], the monic Laguerre polynomials satisfy

$$x\hat{L}_n^{(\alpha)\prime}(x) = n\hat{L}_n^{(\alpha)}(x) + n(n+\alpha)\hat{L}_{n-1}^{(\alpha)}(x). \tag{5.19}$$

In particular, $z_{i,N}^{(\alpha)}\hat{L}_N^{(\alpha)\prime}(z_{i,N}^{(\alpha)}) = N(N+\alpha)\hat{L}_{N-1}^{(\alpha)}(z_{i,N}^{(\alpha)})$. This, (5.12), and (5.14) yield

$$\sum_{i=1}^{N} \frac{z_{i,N}^{(\alpha)}}{N(N+\alpha)}\tilde{Q}_{k,N}^{(\alpha)}(z_{i,N}^{(\alpha)}) \cdot \tilde{Q}_{l,N}^{(\alpha)}(z_{i,N}^{(\alpha)}) = \delta_{kl} \tag{5.20}$$

for the normalized dual Laguerre polynomials $(\tilde{Q}_{k,N}^{(\alpha)})_{k=1}^{N-1}$. This means that these polynomials are orthonormal with respect to the measure given in (5.2). By Theorem 5.2 we now have that (5.17) is an eigenvector of the corresponding covariance matrix $\Sigma_N^{-1}$.

Finally, the monic Jacobi polynomials $\hat{R}_N := \hat{P}_N^{(\alpha,\beta)}$ satisfy

$$(1-(z_{i,N}^{(\alpha,\beta)})^2)\hat{R}_N'(z_{i,N}^{(\alpha,\beta)}) = \frac{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}{(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)}\hat{R}_{N-1}(z_{i,N}^{(\alpha,\beta)}).$$

This, (5.12), and (5.14) show that

$$\sum_{i=1}^{N} \frac{(1-(z_{i,N}^{(\alpha,\beta)})^2)(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)}{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}\tilde{Q}_{k,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)})\tilde{Q}_{l,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)}) = \delta_{kl} \tag{5.21}$$

for the normalized dual Jacobi polynomials $(\tilde{Q}_{k,N}^{(\alpha,\beta)})_{k=1}^{N-1}$. This means that these polynomials are orthonormal with respect to the measure given in (5.3). By Theorem 5.3 we now have that (5.17) is an eigenvector of the corresponding covariance matrix $\Sigma_N^{-1}$. □

**Remark 5.10.** Note that by the proof of Lemma 5.9 in the Hermite, Laguerre, and Jacobi case, the dual Christoffel numbers from (5.12) have the form

$$w_i^* = \frac{\hat{P}_{N-1}(z_{i,N})}{\hat{P}_N'(z_{i,N})} = \frac{\pi(z_{i,N})}{\kappa_N} \tag{5.22}$$

with suitable constants $\kappa_N$ and polynomials $\pi$ of degrees 0,1,and 2, respectively. By [VZ04], such simple relations for the dual Christoffel numbers are available only for the classical orthogonal polynomials. This also includes the Bessel polynomials, which are limits of Jacobi polynomials; see [Ism05, p. 124, (4.10.10) and (4.10.13)].

In the next step we use the preceding results on dual orthogonal polynomials to compute the covariance matrices $\Sigma_N$ from their inverses. For this we write the recurrence (5.7) for general orthonormal polynomials $(\tilde{P}_n)_{n\geq 0}$ for $n \leq N$ at the $N$ ordered zeros $z_{i,N}$ of $\tilde{P}_N$ as the eigenvalue equation

$$\begin{pmatrix} b_0 & a_1 & & & \\ a_1 & b_1 & a_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & b_{N-2} & a_{N-1} \\ & & & a_{N-1} & b_{N-1} \end{pmatrix} \begin{pmatrix} \tilde{P}_0(z_{i,N}) \\ \tilde{P}_1(z_{i,N}) \\ \tilde{P}_2(z_{i,N}) \\ \vdots \\ \tilde{P}_{N-1}(z_{i,N}) \end{pmatrix} = z_{i,N} \begin{pmatrix} \tilde{P}_0(z_{i,N}) \\ \tilde{P}_1(z_{i,N}) \\ \tilde{P}_2(z_{i,N}) \\ \vdots \\ \tilde{P}_{N-1}(z_{i,N}) \end{pmatrix}$$

of an $N \times N$-dimensional matrix. The zeros $\{z_{i,N}\}_{i=1}^{N}$ are the eigenvalues of this symmetric matrix and are distinct; this yields that the eigenvectors of this matrix are orthogonal and unique up to a constant coefficient. On the other hand, Lemma 5.7 and Remark 5.8 show that

$$(\tilde{Q}_{N-1,N}(z_{i,N}), \ldots, \tilde{Q}_{0,N}(z_{i,N}))^T$$

is also an eigenvector of this matrix to the eigenvalue $z_{i,N}$. It follows that

$$\begin{pmatrix} \tilde{P}_0(z_{i,N}) \\ \tilde{P}_1(z_{i,N}) \\ \tilde{P}_2(z_{i,N}) \\ \vdots \\ \tilde{P}_{N-1}(z_{i,N}) \end{pmatrix} = c_{i,N} \begin{pmatrix} \tilde{Q}_{N-1,N}(z_{i,N}) \\ \tilde{Q}_{N-2,N}(z_{i,N}) \\ \tilde{Q}_{N-3,N}(z_{i,N}) \\ \vdots \\ \tilde{Q}_{0,N}(z_{i,N}) \end{pmatrix} \tag{5.23}$$

with a constant $c_{i,N} \neq 0$. The last row of this equation and $\tilde{Q}_{0,N}(x) = 1$ give

$$c_{i,N} = \tilde{P}_{N-1}(z_{i,N}). \tag{5.24}$$

We remark that $c_{i,N}$ usually has the sign $(-1)^{N-i}$. This follows from the well-known interlacing property of the zeros of $\tilde{P}_{N-1}(x)$ and $\tilde{P}_N(x)$ together with the assumption that the leading coefficient of $\tilde{P}_{N-1}(x)$ is positive. This assumption holds for the Hermite and Jacobi cases. In the Laguerre case, the leading coefficient of the classical Laguerre polynomial $L_N^{(\alpha)}(x)$ has the sign $(-1)^N$. However, we stick to the convention $\sqrt{\xi_N}\tilde{L}_N^{(\alpha)} = \hat{L}_N^{(\alpha)}$. Therefore, the leading coefficient of $\tilde{L}_{N-1}^{(\alpha)}$ is positive. Multiplying (5.10) and (5.12) yields

$$c_{i,N}^2 = \frac{w_i^*}{w_i},$$

compare [VZ04] (1.23) or [BS86]. Using the sign of $c_{i,N}$ above, we conclude that

$$c_{i,N} = (-1)^{N-i}\sqrt{\frac{w_i^*}{w_i}}. \tag{5.25}$$

Now this can be used to rewrite Equation (5.23) for $k = 0, ..., N-1$ to

$$(-1)^{N-i}\sqrt{w_i}\tilde{P}_k(z_{i,N}) = \sqrt{w_i^*}\tilde{Q}_{N-k-1,N}. \tag{5.26}$$

This, together with the well-known fact that

$$\frac{1}{w_i} = \sum_{k=0}^{N-1} (\tilde{P}_k(z_{i,N}))^2, \tag{5.27}$$

yields the following representation of $\Sigma_N$:

**Theorem 5.11** (Andraus,H.,Voit(2021))**.** *For the Hermite, Laguerre and Jacobi cases, the covariance matrices $\Sigma_N = (\sigma_{i,j}^N)_{i,j=1,\ldots,N}$ are given with the* *notations of Lemma 5.9 and with the eigenvalues $\lambda_k$ from the Theorems 5.1, 5.2, and 5.3 by*

$$\begin{aligned}
\sigma_{i,j}^N &= \frac{\sqrt{w_i^* w_j^*}}{\tilde{P}_{N-1}(z_{i,N})\tilde{P}_{N-1}(z_{j,N})} \sum_{k=0}^{N-1} \frac{\tilde{P}_k(z_{i,N})\tilde{P}_k(z_{j,N})}{\lambda_{N-k}} \\
&= (-1)^{i+j} \sum_{k=0}^{N-1} \frac{\sqrt{w_i}\tilde{P}_k(z_{i,N})\sqrt{w_j}\tilde{P}_k(z_{j,N})}{\lambda_{N-k}} \\
&= \frac{(-1)^{i+j}}{\sqrt{\sum_{k,l=0}^{N-1} \tilde{P}_k^2(z_{i,N})\tilde{P}_l^2(z_{j,N})}} \sum_{k=0}^{N-1} \frac{\tilde{P}_k(z_{i,N})\tilde{P}_k(z_{j,N})}{\lambda_{N-k}}.
\end{aligned} \tag{5.28}$$

*Proof.* In all cases,

$$T_N^T \Sigma_N T_N = \operatorname{diag}(\lambda_1^{-1}, \ldots, \lambda_N^{-1}),$$

where the orthogonal matrix $T$ has entries

$$\begin{aligned}
[T_N]_{i,j} &= \sqrt{w_i^*}\tilde{Q}_{j-1,N}(z_{i,N}) \\
&= \frac{1}{c_{i,N}} \sqrt{w_i^*}\tilde{P}_{N-j}(z_{i,N}) \\
&= (-1)^{N-i}\sqrt{w_i}\tilde{P}_{N-j}(z_{i,N}).
\end{aligned}$$

Hence,

$$\sigma_{i,j}^N = \frac{\sqrt{w_i^* w_j^*}}{c_{i,N}c_{j,N}} \sum_{k=0}^{N-1} \frac{\tilde{P}_{N-1-k}(z_{i,N})\tilde{P}_{N-1-k}(z_{j,N})}{\lambda_{k+1}}.$$

The substitution $N-1-k \to k$, (5.24), (5.25) and (5.27) yield the result. □

Even though some results for the eigenvalues and eigenvectors are not given in every case, compare remark 5.4, it is still possible to describe the entries of the covariance matrix in a way similar to (5.28). In the following, we will present these formulas as well as the formulas from Theorem 5.11 spelled out for the concrete cases. Consider first the Hermite case:

**Corollary 5.12.** *Consider the classical Hermite polynomials $H_k = \sqrt{2^k k!}\tilde{H}_k$, the vector $z_N = (z_{1,N}, ..., z_{N,N})$ of its ordered zeros and random variables $X_\kappa$ with the densities*

$$c_\kappa \prod_{1 \le i < j \le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} e^{-\frac{\kappa}{2}x_i^2}.$$

*Then the entries of the covariance matrix $\Sigma_N^H = (\sigma_{i,j}^{N,H})_{i,j=1,\ldots,N}$ from the limit*

$$\sqrt{\kappa}(X_\kappa - \sqrt{2}z_N) \to \mathcal{N}(0, \Sigma_N^H), \qquad (\kappa \to \infty, \text{ in distribution})$$

*compare Theorem 4.1, are given by*

$$\sigma_{i,j}^{N,H} = (-1)^{i+j} \left( \sum_{k=0}^{N-1} \frac{(H_k(z_{i,N}))^2}{2^k k!} \sum_{l=0}^{N-1} \frac{(H_l(z_{j,N}))^2}{2^l l!} \right)^{-1/2} \sum_{k=0}^{N-1} \frac{H_k(z_{i,N}) H_k(z_{j,N})}{2^k k!(N-k)}.$$

Next, consider the formulation in the Laguerre case:

**Corollary 5.13.** *Consider the classical Laguerre polynomials*

$$L_k^{(\alpha)} = (-1)^k \tilde{L}_k \sqrt{\frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)}},$$

*the vector* $z_N^{(\alpha)} = (z_{1,N}^{(\alpha)}, ..., z_{N,N}^{(\alpha)})$ *of their ordered zeros and random variables* $X_\kappa$ *with the densities*

$$\tilde{c}_{\kappa,\alpha} \prod_{1 \le i < j \le N} (x_j^2 - x_i^2)^{2\kappa} \prod_{i=1}^{N} x_i^{2(\alpha+1)\kappa} e^{-x_i^2}$$

*Then the entries of the covariance matrix* $\tilde{\Sigma}_N^L = (\tilde{\sigma}_{i,j}^{N,L})_{i,j=1,\dots,N}$ *from the limit*

$$\sqrt{\kappa} \left( \frac{X_\kappa}{\sqrt{\kappa}} - \sqrt{z_N^\alpha} \right) \to \mathcal{N}(0, \tilde{\Sigma}_N^L), \qquad (\kappa \to \infty, \text{ in distribution})$$

*compare Theorem 4.6, are given by*

$$\tilde{\sigma}_{i,j}^{N,L} = \frac{(-1)^{i+j} \sum_{k=0}^{N-1} \left( \frac{k!\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} \frac{L_k(z_{i,N}^{(\alpha)}) L_k(z_{j,N}^{(\alpha)})}{4(N-k)} \right)}{\sqrt{\left( \sum_{k=0}^{N-1} \frac{k!\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} (L_k^{(\alpha)}(z_{i,N}))^2 \right) \left( \sum_{k=0}^{N-1} \frac{k!\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} (L_k^{(\alpha)}(z_{j,N}))^2 \right)}}.$$

As mentioned before, for the covariance matrix in Theorem 4.5 and its inverse, the eigenvalue decomposition seems to be much harder to obtain. Nevertheless, with the help of Theorem 5.11, one can derive nice formulas for the covariance matrix. The result is as follows:

**Corollary 5.14.** *Consider the classical Laguerre polynomials*

$$L_k^{(\alpha)} = (-1)^k \tilde{L}_k \sqrt{\frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)}},$$

*the vector* $z_N^{(\alpha)} = (z_{1,N}^{(\alpha)}, ..., z_{N,N}^{(\alpha)})$ *of their ordered zeros and random variables* $X_\kappa$ *with the densities*

$$c_{\kappa,\alpha} \prod_{1 \le i < j \le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} x_i^{(\alpha+1)\kappa - 1} e^{-x_i}$$

*Then the entries of covariance matrix* $\Sigma_N^L = (\sigma_{i,j}^{N,L})_{i,j=1,\dots,N}$ *from the limit*

$$\sqrt{\kappa}\left(\frac{X_\kappa}{\kappa} - z_N^{(\alpha)}\right) \to \mathcal{N}(0, \Sigma_N^L), \qquad (\kappa \to \infty, \text{ in distribution})$$

*compare Theorem 4.5, are given by*

$$\sigma_{i,j}^{N,L} = \frac{(-1)^{i+j} 4\sqrt{z_{i,N}^{(\alpha)} z_{i,N}^{(\alpha)}} \sum_{k=0}^{N-1}\left(\frac{k!\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} \frac{L_k(z_{i,N}^{(\alpha)}) L_k(z_{j,N}^{(\alpha)})}{4(N-k)}\right)}{\sqrt{\left(\sum_{k=0}^{N-1} \frac{k!\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} (L_k^{(\alpha)}(z_{i,N}))^2\right)\left(\sum_{k=0}^{N-1} \frac{k!\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)} (L_k^{(\alpha)}(z_{j,N}))^2\right)}}.$$

Next, consider the formulation in the Jacobi case:

**Corollary 5.15.** *Consider the classical Jacobi polynomials*

$$\begin{aligned} P_k^{(\alpha,\beta)} &= \sqrt{\frac{1}{2k+\alpha+\beta} \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)} \frac{\Gamma(k+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(k+\alpha+\beta+2)}{\Gamma(\alpha+\beta+1)}}\, \tilde{P}_k^{(\alpha,\beta)} \\ &=: \zeta_k^{(\alpha,\beta)} \tilde{P}_k^{(\alpha,\beta)}, \end{aligned}$$

*the vector* $z_N^{(\alpha,\beta)} = (z_{1,N}^{(\alpha,\beta)}, \dots, z_{N,N}^{(\alpha,\beta)})$ *of their ordered zeros and random variables* $X_\kappa$ *with the densities*

$$\tilde{c}_{\kappa,\alpha,\beta} \prod_{1 \le i < j \le N} (\cos(2t_j) - \cos(2t_i))^{2\kappa} \prod_{i=1}^{N} \sin(t_i)^{2(\alpha+1)\kappa} \cos(t_i)^{2(\beta+1)\kappa}.$$

*Then the entries of covariance matrix* $\tilde{\Sigma}_N^J = (\sigma_{i,j}^{N,J})_{i,j=1,\dots,N}$ *from the limit*

$$\sqrt{\kappa}\left(X_\kappa - \frac{1}{2}\arccos(z_N^{(\alpha,\beta)})\right) \to \mathcal{N}(0, \tilde{\Sigma}_N^J), \qquad (\kappa \to \infty, \text{ in distribution})$$

*compare Theorem 4.12, are given by*

$$\tilde{\sigma}_{i,j}^{N,J} = (-1)^{i+j} \frac{\sum_{k=0}^{N-1}\left(\frac{P_k^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)}) P_k^{(\alpha,\beta)}(z_{j,N}^{(\alpha,\beta)})}{\zeta_k^{(\alpha,\beta)} 4(N-k)(N+k+\alpha+\beta+1)}\right)}{\sqrt{\left(\sum_{k=0}^{N-1} \frac{(P_k^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)}))^2}{\zeta_k^{(\alpha,\beta)}}\right)\left(\sum_{k=0}^{N-1} \frac{(P_k^{(\alpha,\beta)}(z_{j,N}^{(\alpha,\beta)}))^2}{\zeta_k^{(\alpha,\beta)}}\right)}}.$$

As mentioned before, for the covariance matrix in Theorem 4.11 and its inverse, the eigenvalue decomposition seems to be much harder to obtain. Nevertheless, with the help of Theorem 5.11, one can derive nice formulas for the covariance matrix. The result is as follows:

**Corollary 5.16.** *Consider the classical Jacobi polynomials*

$$\begin{aligned} P_k^{(\alpha,\beta)} &= \sqrt{\frac{1}{2k+\alpha+\beta} \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)} \frac{\Gamma(k+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(k+\alpha+\beta+2)}{\Gamma(\alpha+\beta+1)}}\, \tilde{P}_k^{(\alpha,\beta)} \\ &=: \zeta_k^{(\alpha,\beta)} \tilde{P}_k^{(\alpha,\beta)}, \end{aligned}$$

*the vector* $z_N^{(\alpha,\beta)} = (z_{1,N}^{(\alpha,\beta)}, ..., z_{N,N}^{(\alpha,\beta)})$ *of their ordered zeros and random variables* $X_\kappa$ *with the densities*

$$c_{\kappa,\alpha,\beta} \prod_{1 \le i < j \le N} (x_j - x_i)^{2\kappa} \prod_{i=1}^{N} (1 - x_i)^{(\alpha+1)\kappa - \frac{1}{2}} (1 + x_i)^{(\beta+1)\kappa - \frac{1}{2}}.$$

*Then the entries of covariance matrix* $\Sigma_N^J = (\sigma_{i,j}^{N,J})_{i,j=1,\dots,N}$ *from the limit*

$$\sqrt{\kappa}\left(X_\kappa - z_N^{(\alpha,\beta)}\right) \to \mathcal{N}(0, \Sigma_N^J), \qquad (\kappa \to \infty, \text{ in distribution})$$

*compare Theorem 4.11, are given by*

$$\sigma_{i,j}^{N,J} = \frac{\sqrt{1 - (z_{i,N}^{(\alpha,\beta)^2})}\sqrt{1 - (z_{j,N}^{(\alpha,\beta)^2})} \sum_{k=0}^{N-1} \left( \frac{P_k^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)}) P_k^{(\alpha,\beta)}(z_{j,N}^{(\alpha,\beta)})}{\zeta_k^{(\alpha,\beta)}(N-k)(N+k+\alpha+\beta+1)} \right)}{(-1)^{i+j} \sqrt{\left( \sum_{k=0}^{N-1} \frac{(P_k^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)}))^2}{\zeta_k^{(\alpha,\beta)}} \right) \left( \sum_{k=0}^{N-1} \frac{(P_k^{(\alpha,\beta)}(z_{j,N}^{(\alpha,\beta)}))^2}{\zeta_k^{(\alpha,\beta)}} \right)}}.$$

**Remark 5.17.** The formulas for the entries of the covariance matrices $\Sigma_N$ from Theorem 5.11 should be compared with the corresponding results of Dumitriu and Edelman [DE05] for the Hermite and Laguerre ensembles. In the Hermite case, the entries of $\Sigma_N^H$ in Corollary 5.12 must be equal to entries of Theorem 3.1 in[DE05]. Therefore, the following equation holds:

$$\begin{aligned}
\sigma_{i,j}^{N,H} &= (-1)^{i+j} \left( \sum_{k=0}^{N-1} (\tilde{H}_k(z_{i,N}))^2 \sum_{l=0}^{N-1} (\tilde{H}_l(z_{j,N}))^2 \right)^{-1/2} \sum_{k=0}^{N-1} \frac{\tilde{H}_k(z_{i,N}) \tilde{H}_k(z_{j,N})}{N-k} \\
&= \frac{\sum_{l=0}^{N-1} \tilde{H}_l^2(z_{i,N}) \tilde{H}_l^2(z_{j,N}) + \sum_{l=0}^{N-2} \tilde{H}_{l+1}(z_{i,N}) \tilde{H}_l(z_{i,N}) \tilde{H}_{l+1}(z_{j,N}) \tilde{H}_l(z_{j,N})}{\sum_{l=0}^{N-1} \tilde{H}_l^2(z_{i,N}) \cdot \sum_{l=0}^{N-1} \tilde{H}_l^2(z_{j,N})}.
\end{aligned}$$

Unfortunately, a direct proof for arbitrary dimensions $N$ is, to the best of the authors knowledge, not available yet. In the Laguerre case we have a similar situation. Here the formula from Corollary 5.14 equals the one given in Theorem 4.1 from [DE05]. However, Corollary 5.14 has the same structure as in the Hermite case, while the corresponding formula in [DE05] is much more involved.

In the Jacobi case, there do not exist, as far as the author is aware of, formulas for the entries of $\Sigma_N$ in the literature besides the ones from [AHV21], which are summarized here.

**Remark 5.18.** The fact that in all three cases the eigenvectors are given by $(\sqrt{w_i^*}\tilde{Q}_{k,N}(z_{i,N}))_{i=1,..,N}$ and that there are central limit theorems for more general polynomials, see Theorem 3.6, raises the question whether there exist more polynomial-induced $\beta$-ensembles for which the covariance matrix of the central limit theorem has similarly structured eigenvectors. Note that in the Laguerre and Jacobi cases, a transformation to a different parametrization has been used, compare Remark 5.4. We thus pose the following question: Is there a 'canonical' transformation $d : I \to \mathbb{R}$ such that the covariance matrix of the central limit theorem transformed by Theorem 3.5 has $(\sqrt{w_i^*}\tilde{Q}_{k,N}(z_{i,N}))_{i=1,..,N}$ as eigenvectors? In other words, is there a diagonal matrix $D = \mathrm{diag}(d_1, ..., d_N)$ such that $D\Sigma^{-1}D$ has $(\sqrt{w_i^*}\tilde{Q}_{k,N}(z_{i,N}))_{i=1,..,N}$ as eigenvectors? The obvious guess, that this holds true for $D = \mathrm{diag}(\sqrt{w_1^*}, ..., \sqrt{w_N^*})$, is not correct in general. To sketch a possible counterexample, consider $v(x) = x^2 + \alpha \log(x)$ on $I = (0, \infty)$ as in Theorem 4.9. Here we have

$$(\Sigma^{-1})_{i,j} := \begin{cases} 2\sum_{l\neq i}(z_{i,N} - z_{l,N})^{-2} + 2 + \frac{\alpha+1}{z_{i,N}^2} - \frac{1}{(z_{i,N}+b_N)^2} & \text{for} \quad i = j \\ -2(z_{i,N} - z_{j,N})^{-2} & \text{for} \quad i \neq j \end{cases}$$

and therefore

$$(D\Sigma^{-1}D)_{i,j} = \begin{cases} 2\sum_{l\neq i} \frac{w_i^*}{(z_{i,N}-z_{l,N})^2} + w_i^* \left(2 + \frac{\alpha+1}{z_{i,N}^2} - \frac{1}{(z_{i,N}+b_N)^2}\right) & \text{for} \quad i = j \\ -2\frac{\sqrt{w_i^* w_j^*}}{(z_{i,N}-z_{j,N})^2} & \text{for} \quad i \neq j. \end{cases}$$

Thus, for $i = 1, ..., N$ we get

$$\begin{aligned} &\left( D\Sigma^{-1}D \begin{pmatrix} \sqrt{w_1^*} \\ \vdots \\ \sqrt{w_N^*} \end{pmatrix} \right)_i \\ =&2\sum_{j\neq i} \frac{\sqrt{w_i^*}w_i^* - \sqrt{w_i^*}w_j^*}{(z_{i,N} - z_{j,N})^2} + \sqrt{w_i^*}w_i^* \left(2 + \frac{\alpha+1}{z_{i,N}^2} - \frac{1}{(z_{i,N}+b_N)^2}\right) \qquad (5.29) \\ =&\sqrt{w_i^*}\left(2\sum_{j\neq i} \frac{w_i^* - w_j^*}{(z_{i,N} - z_{j,N})^2} + w_i^* \left(2 + \frac{\alpha+1}{z_{i,N}^2} - \frac{1}{(z_{i,N}+b_N)^2}\right)\right). \end{aligned}$$

By (5.12), (3.17) and (4.13) we get

$$w_i^* = \frac{\tilde{P}_{N-1}(z_{i,N})}{a_N \tilde{P}_N'(z_{i,N})} = \frac{1}{a_N A_N(z_{i,N})} = \frac{z_{i,N}}{2z_{i,N} + 2b_N} \frac{1}{a_N^2}. \qquad (5.30)$$

Because multiplicative constants are irrelevant for the question whether a vector is an eigenvector or not, we will disregard them here and continue the calculation with

$$w_i^* = \frac{z_{i,N}}{z_{i,N} + b_N}$$

instead of (5.30). Thus, $w_i^* - w_j^*$ becomes

$$\begin{aligned} w_i^* - w_j^* &= \left( \frac{z_{i,N}}{z_{i,N} + b_N} - \frac{z_{j,N}}{z_{j,N} + b_N} \right) \\ &= \left( \frac{z_i(z_j + b_N) - z_j(z_i + b_N)}{(z_i + b_N)(z_j + b_N)} \right) \\ &= \frac{b_N(z_i - z_j)}{(z_i + b_N)(z_j + b_N)} \\ &= b_N(z_i - z_j)\frac{w_j^*}{z_j}\frac{w_i^*}{z_i}. \end{aligned}$$

This yields

$$\begin{aligned} 2\sum_{j \neq i} \frac{w_i^* - w_j^*}{(z_{i,N} - z_{j,N})^2} &= 2\frac{w_i^*}{z_i}\sum_{j \neq i}\frac{w_j^*}{z_j}\frac{1}{(z_i - z_j)} \\ &= 2\sum_{j \neq i}\frac{\frac{w_j^*}{z_j} - \frac{w_i^*}{z_i}}{(z_i - z_j)} + 2\frac{w_i^*}{z_i}\sum_{j \neq i}\frac{1}{(z_i - z_j)}. \end{aligned}$$

Now, by

$$\begin{aligned} \frac{w_j^*}{z_j} - \frac{w_i^*}{z_i} &= \left( \frac{1}{z_j + b_N} - \frac{1}{z_i + b_N} \right) \\ &= \left( \frac{z_i - z_j}{(z_i + b_N)(z_j + b_N)} \right) \\ &= w_i^* w_j^* (z_i - z_j), \end{aligned}$$

we get

$$2\sum_{j \neq i}\frac{\frac{w_j^*}{z_j} - \frac{w_i^*}{z_i}}{(z_i - z_j)} = w_i^* 2\sum_{j \neq i} w_j^*$$

and

$$\begin{aligned} 2\frac{w_i^*}{z_i}\sum_{j \neq i}\frac{1}{(z_i - z_j)} &= \frac{w_i^*}{z_i}\left(2z_i - \frac{\alpha + 1}{z_i} + \frac{1}{z_i + b_N}\right) \\ &= \frac{w_i^*}{z_i}\left(2z_i - \frac{\alpha + 1}{z_i} + \frac{w_i^*}{z_i}\right). \end{aligned}$$

Overall, we have

$$\begin{aligned} 2\sum_{j \neq i} \frac{w_i^* - w_j^*}{(z_{i,N} - z_{j,N})^2} &= w_i^* 2\sum_{j \neq i} w_j^* + \frac{w_i^*}{z_i}\left(2z_i - \frac{\alpha + 1}{z_i} + \frac{w_i^*}{z_i}\right) \\ &= w_i^*\left(2 - \frac{\alpha + 1}{z_i^2} + \frac{w_i^*}{z_i^2} + 2\sum_{j \neq i} w_j^*\right). \end{aligned}$$

This leads to

$$
\begin{aligned}
&2\sum_{j\neq i}\frac{w_i^*-w_j^*}{(z_{i,N}-z_{j,N})^2}+w_i^*\left(2+\frac{\alpha+1}{z_{i,N}^2}-\frac{1}{(z_{i,N}+b_N)^2}\right)\\
=&2\sum_{j\neq i}\frac{w_i^*-w_j^*}{(z_{i,N}-z_{j,N})^2}+w_i^*\left(2+\frac{\alpha+1}{z_{i,N}^2}-\left(\frac{w_i^*}{z_i}\right)^2\right)\\
=&w_i^*\left(4+\frac{w_i}{z_i^2}-\frac{w_i^2}{z_i^2}+2(2a_N^2-w_i)\right)\\
=&w_i^*\left(4+4a_N^2+\frac{w_i}{z_i^2}-\frac{w_i^2}{z_i^2}-2w_i)\right). \qquad (5.31)
\end{aligned}
$$

A necessary condition for $(\sqrt{w_1^*},...,\sqrt{w_N^*})$ to be an eigenvalue of $D\Sigma^{-1}D$ is that 5.31 is independent of $i$. This can be seen in Equation 5.29. Although (5.31) simplifies the equation in a quite elegant way, it is not independent of $i$. Thus it can be seen that the assumption described above does not hold here.

# 6 Limit results for the largest eigenvalue for $N \to \infty$

## 6.1 Limit results for the largest eigenvalue in the Hermite case

In this section we discuss the soft edge statistics in the Hermite case in the freezing regime. This means that we analyze the limit behavior of the largest entry of the vector in the freezing regime in Theorem 4.1 for $N \to \infty$. This will be done on the basis of Theorem 5.11. We will follow Chapter 5 from [AHV21] here. This problem is also discussed in [GK22], where similar results occur, but viewpoint and method of proof are different. Furthermore, the problem is discussed in [DE05]. However, the shown approach via Theorem 5.11 leads to a limit with a different form from that in [DE05]. In this section let $z_{1,N} < ..., z_{N,N}$ be the the ordered zeros of the Hermite polynomial $H_N$. Moreover, for each $N$, let $(Q_{k,N})_{k=0,..,N-1}$ be the dual polynomials associated with $(H_k)_{k=0,...,N}$, normalized with

$$Q_{k,N} = \sqrt{w_i^*}\tilde{Q}_{k,N}^H = \frac{1}{\sqrt{N}}\tilde{Q}_{k,N}^H$$

where $\tilde{Q}_{k,N}^H$ are given as in Theorem 5.1. This means that

$$T_N := (Q_{j-1,N}(z_{i,N}))_{i,j=1,...,N}$$

is an orthogonal matrix with $T_N^T \Sigma_N T_N = \text{diag}(1, ..., \frac{1}{N})$ as in the proof of Theorem 5.11. These polynomials satisfy the three-term-recurrence

$$xQ_{k,N}(x) = \sqrt{\frac{N-k-1}{2}}Q_{k+1,N}(x) + \sqrt{\frac{N-k}{2}}Q_{k-1,N}(x) \quad (k < N) \tag{6.1}$$

with the initial conditions $Q_{-1,N} = 0$ and $Q_{0,N} = \frac{1}{\sqrt{N}}$.

We now derive a limit result for $N \to \infty$ which involves the Airy function $\mathsf{Ai}$. For this, we recall some well-known facts about $\mathsf{Ai}$; see e.g. Section 9 of [Olv+10] or the monograph [VS04]. $\mathsf{Ai}$ is the unique solution of

$$y''(z) = z \cdot y(z) \quad (z \in \mathbb{R}) \quad \text{with} \quad \lim_{z\to\infty} y(z) = 0 \tag{6.2}$$

and with $y(0) = \frac{1}{3^{2/3}\Gamma(2/3)} = 0.355028\ldots$. The Airy function $\mathsf{Ai}$ has a unique largest zero at $a_1 = -2.338\ldots$ with $\mathsf{Ai}(z) > 0$ for $z > a_1$. Moreover, $\mathsf{Ai}$ has infinitely many isolated, simple zeros in $]-\infty, a_1]$. For $r \in \mathbb{N}$, the $r$-th largest zero $a_r$ of $\mathsf{Ai}$ satisfies

$$a_r \simeq -\Big(\frac{3\pi}{2}(r - 1/4)\Big)^{2/3} \qquad \text{for} \quad r \to \infty. \tag{6.3}$$

In addition, we have the asymptotic behavior as $z \to -\infty$

$$\mathsf{Ai}(-z) \simeq \frac{1}{\sqrt{\pi}z^{1/4}} \cos\Big(\frac{2}{3}z^{3/2} - \frac{\pi}{4}\Big), \tag{6.4}$$

as well as

$$\mathsf{Ai}'(a_r) \simeq \frac{(-1)^{r-1}}{\sqrt{\pi}} \Big(\frac{3\pi}{2}(r-1/4)\Big)^{1/6} \qquad \text{for} \quad r \to \infty. \tag{6.5}$$

The following Theorem is the central step for our limit results for $N \to \infty$:

**Theorem 6.1** (Andraus,H.,Voit(2021))**.** *Consider the functions*

$$f_N(y) := N^{\frac{1}{6}} Q_{\lfloor N^{\frac{1}{3}} y \rfloor, N}(z_{N,N}) \quad \textit{for} \quad y \in [0, N^{\frac{2}{3}}[ \tag{6.6}$$

*and* $f_N(y) = 0$ *otherwise. Then* $(f_N)_{N \geq 1}$ *tends for* $N \to \infty$ *locally uniformly to*

$$f(y) = \frac{\mathsf{Ai}(y + a_1)}{\mathsf{Ai}'(a_1)} \qquad \textit{for} \quad y \in [0, \infty[.$$

We split the proof into three lemmas and use the abbreviation

$$q_k := Q_{k,N}(z_{N,N}),$$

where we suppress the dependency on $N$. We start with the following result:

**Lemma 6.2.** *The functions* $f_N$ *satisfy the equation*

$$f_N(y) = \int_0^y \int_0^s (t - |a_1|) f_N(t)\, dt\, ds + y + \mathrm{err}(y, N)$$

*for* $y \in [0, N^{\frac{2}{3}}[$*. The error term* $\mathrm{err}(y, N)$ *is specified in Equation (6.19) at the end of the proof.*

*Proof.* Let $y \geq 0$. We divide the recurrence (6.1) by $x := z_{N,N}$ by $\sqrt{N}$ and get

$$\sqrt{1 - \frac{k+1}{N}}\, q_{k+1} = \frac{2 z_{N,N}}{\sqrt{2N}} q_k - \sqrt{1 - \frac{k}{N}}\, q_{k-1} \quad (k \leq N) \tag{6.7}$$

with $q_{-1} = 0$, $q_0 = N^{-1/2}$. We next observe that by the Lagrange remainder in Taylor's formula, for $k = 0, ..., \lfloor y N^{\frac{1}{3}} \rfloor$,

$$\sqrt{1 - \frac{k}{N}} = 1 - \frac{k}{2N} - \frac{1}{8(1-\xi_k)^{\frac{3}{2}}} \left(\frac{k}{N}\right)^2 \quad \text{with} \quad \xi_k \in (0, \frac{k}{N}). \tag{6.8}$$

We now define

$$\alpha(k, N) := \frac{1}{8(1-\xi_k)^{\frac{3}{2}}} \left(\frac{k}{N}\right)^2$$

and conclude from (6.8) that for $k = 0, ..., \lfloor y N^{\frac{1}{3}} \rfloor$ it holds that

$$0 < \alpha(k, N) < \left(\frac{N}{N-k}\right)^{\frac{3}{2}} \left(\frac{k}{N}\right)^2. \tag{6.9}$$

Moreover, we obtain from the sharp Plancherel-Rotach theorem by Ricci [Ric95] that

$$\frac{z_{N,N}}{\sqrt{2N}} = 1 - \frac{|a_1|}{2N^{\frac{2}{3}}} + O(N^{-1}). \tag{6.10}$$

Using (6.10) we rewrite the recurrence (6.7) as

$$\begin{aligned} & q_{k+1} - q_k - (q_k - q_{k-1}) \\ =& \frac{k+1}{2N} q_{k+1} - \frac{|a_1|}{N^{\frac{2}{3}}} q_k + \frac{k}{2N} q_{k-1} + \alpha(k+1,N) q_{k+1} + \alpha(k,N) q_{k-1} + O(N^{-1}) q_k. \end{aligned} \tag{6.11}$$

Summation over $k = 0, ..., l$ now yields

$$\begin{aligned} q_{l+1} - q_l - \frac{1}{\sqrt{N}} = q_{l+1} - q_l - (q_0 - q_{-1}) &= \sum_{k=0}^{l} \Big( q_{k+1} - q_k - (q_k - q_{k-1}) \Big) \\ &= \sum_{k=0}^{l} \Big( \frac{k+1}{2N} q_{k+1} - \frac{|a_1|}{N^{\frac{2}{3}}} q_k + \frac{k}{2N} q_{k-1} \Big) + \\ &\quad + \sum_{k=0}^{l} \Big( \alpha(k+1,N) q_{k+1} + \alpha(k,N) q_{k-1} + O(N^{-1}) q_k \Big). \end{aligned}$$

A second summation over $l = 0, ..., \lfloor yN^{\frac{1}{3}} \rfloor - 1$ now leads to

$$\begin{aligned} q_{\lfloor yN^{\frac{1}{3}} \rfloor} - \frac{\lfloor yN^{\frac{1}{3}} \rfloor + 1}{\sqrt{N}} &= \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \left( q_l - q_{l-1} - \frac{1}{\sqrt{N}} \right) = \\ = \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \sum_{k=0}^{l} & \left( \frac{k+1}{2N} q_{k+1} - \frac{|a_1|}{N^{\frac{2}{3}}} q_k + \frac{k}{2N} q_{k-1} \right) + \rho(y,N) \end{aligned}$$

with

$$\rho(y,N) := \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \sum_{k=0}^{l} \Big( \alpha(k+1,N) q_{k+1} + \alpha(k,N) q_{k-1} + O(N^{-1}) q_k \Big). \tag{6.12}$$

If we multiply this by $N^{\frac{1}{6}}$, we get

$$\begin{aligned}
&f_N(y) - \frac{\lfloor yN^{\frac{1}{3}} \rfloor + 1}{N^{\frac{1}{3}}} - N^{\frac{1}{6}}\rho(y, N) \\
&= \frac{1}{N^{\frac{1}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{l} \left( \frac{k+1}{2N^{\frac{1}{3}}} N^{\frac{1}{6}} q_{k+1} - |a_1| N^{\frac{1}{6}} q_k + \frac{k}{2N^{\frac{1}{3}}} N^{\frac{1}{6}} q_{k-1} \right) \\
&= \frac{1}{N^{\frac{1}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{l} \left( N^{\frac{1}{6}} q_k \left( \frac{k + \frac{1}{2}}{N^{\frac{1}{3}}} - |a_1| \right) \right) \\
&\quad + \frac{1}{N^{\frac{2}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \left( \frac{l+1}{2N^{\frac{1}{3}}} N^{\frac{1}{6}} \left( q_{l+1} - q_l \right) \right). \qquad (6.13)
\end{aligned}$$

Note that the last equation was obtained from the shifts $k+1 \mapsto k$ and $k-1 \mapsto k$. We now compare the right-hand side of (6.13) with

$$\int_0^y \int_0^s (x - |a_1|) f_N(t) dt ds. \qquad (6.14)$$

For this we use the functions

$$g_N(t) := \sum_{k=0}^{N-1} t_{k,N} \mathbf{1}_{\left[\frac{k}{N^{\frac{1}{3}}}, \frac{k+1}{N^{\frac{1}{3}}}\right]}(t) \quad \text{with} \quad t_{k,N} := N^{\frac{1}{6}} q_k \left( \frac{k + \frac{1}{2}}{N^{\frac{1}{3}}} - |a_1| \right). \qquad (6.15)$$

An elementary calculation yields

$$\begin{aligned}
&\int_0^y \int_0^s g_N(t)\, dt ds = \int_0^y \int_0^s \sum_{k=0}^{N-1} t_{k,N} \mathbf{1}_{\left[\frac{k}{N^{\frac{1}{3}}}, \frac{k+1}{N^{\frac{1}{3}}}\right]}(t)\, dt\, ds \\
&\quad = \frac{1}{2} \left( \frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}} \rfloor}{N^{\frac{1}{3}}} \right)^2 t_{\lfloor yN^{\frac{1}{3}} \rfloor, N} + \frac{1}{N^{\frac{2}{3}}} \sum_{k=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} (\lfloor yN^{\frac{1}{3}} \rfloor - k) t_{k,N} \\
&\qquad + \frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}} \rfloor - \frac{1}{2}}{N^{\frac{1}{3}}} \cdot \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} t_{k,N}.
\end{aligned}$$

Moreover,

$$\sum_{l=0}^{L} \sum_{k=0}^{l} t_{k,N} = \sum_{k=0}^{L} (L - k + 1) t_{k,N} \qquad (L \in \mathbb{N}). \qquad (6.16)$$

Hence,

$$\int_0^y \int_0^s g_N(t)dtds - \frac{1}{N^{\frac{2}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor -1} \sum_{k=0}^{l} t_{k,N} \tag{6.17}$$

$$= \frac{1}{2}\left(\frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}} \rfloor}{N^{\frac{1}{3}}}\right)^2 t_{\lfloor yN^{\frac{1}{3}} \rfloor, N} + \frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}} \rfloor - \frac{1}{2}}{N^{\frac{1}{3}}} \cdot \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{\lfloor yN^{\frac{1}{3}} \rfloor -1} t_{k,N}.$$

(6.13), (6.14), and (6.17) now show that

$$f_N(y) = \int_0^y \int_0^s f_N(t)\left(\frac{\lfloor tN^{\frac{1}{3}} \rfloor + \frac{1}{2}}{N^{\frac{1}{3}}} - |a_1|\right) dtds + \frac{\lfloor yN^{\frac{1}{3}} \rfloor + 1}{N^{\frac{1}{3}}} + \widetilde{\mathrm{err}}(N,y) \tag{6.18}$$

with the error term

$$\begin{aligned}\widetilde{\mathrm{err}}(N,y) := N^{\frac{1}{6}}\rho(y,N) + \frac{1}{N^{\frac{2}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor -1} \frac{l+1}{2N^{\frac{1}{3}}}(q_{l+1} - q_l) \\ - \frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}} \rfloor - \frac{1}{2}}{N^{\frac{1}{3}}} \frac{1}{N^{\frac{1}{3}}} \sum_{k=0}^{\lfloor yN^{\frac{1}{3}} \rfloor -1} N^{\frac{1}{6}} q_k \left(\frac{k + \frac{1}{2}}{N^{\frac{1}{3}}} - |a_1|\right) \\ - \frac{1}{2}\left(\frac{yN^{\frac{1}{3}} - \lfloor yN^{\frac{1}{3}} \rfloor}{N^{\frac{1}{3}}}\right)^2 N^{\frac{1}{6}} q_{\lfloor yN^{\frac{1}{3}} \rfloor} \left(\frac{\lfloor yN^{\frac{1}{3}} \rfloor + \frac{1}{2}}{N^{\frac{1}{3}}} - |a_1|\right).\end{aligned}$$

As

$$\frac{\lfloor yN^{\frac{1}{3}} \rfloor + 1}{N^{\frac{1}{3}}} = y + O(N^{-\frac{1}{3}})$$

and

$$\frac{\lfloor tN^{\frac{1}{3}} \rfloor + \frac{1}{2}}{N^{\frac{1}{3}}} = t + \frac{\lfloor tN^{\frac{1}{3}} \rfloor - tN^{\frac{1}{3}} + \frac{1}{2}}{N^{\frac{1}{3}}} = t + O(N^{-\frac{1}{3}}),$$

we get

$$f_N(y) = \int_0^y \int_0^s (t - |a_1|) f_N(t)dtds + y + \mathrm{err}(y,N)$$

with the error term

$$\operatorname{err}(N,y) = N^{\frac{1}{6}}\rho(y,N) + \frac{1}{N^{\frac{2}{3}}}\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\frac{l+1}{2N^{\frac{1}{3}}}\left(q_{l+1}-q_l\right)$$

$$-\frac{yN^{\frac{1}{3}}-\lfloor yN^{\frac{1}{3}}\rfloor-\frac{1}{2}}{N^{\frac{1}{3}}}\frac{1}{N^{\frac{1}{3}}}\sum_{k=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}N^{\frac{1}{6}}q_k\left(\frac{k+\frac{1}{2}}{N^{\frac{1}{3}}}-|a_1|\right)$$

$$-\frac{1}{2}\left(\frac{yN^{\frac{1}{3}}-\lfloor yN^{\frac{1}{3}}\rfloor}{N^{\frac{1}{3}}}\right)^2 N^{\frac{1}{6}}q_{\lfloor yN^{\frac{1}{3}}\rfloor}\left(\frac{\lfloor yN^{\frac{1}{3}}\rfloor+\frac{1}{2}}{N^{\frac{1}{3}}}-|a_1|\right)$$

$$+\frac{\lfloor yN^{\frac{1}{3}}\rfloor-yN^{\frac{1}{3}}+1}{N^{\frac{1}{3}}}+\int_0^y\int_0^s\frac{\lfloor tN^{\frac{1}{3}}\rfloor-tN^{\frac{1}{3}}+\frac{1}{2}}{N^{\frac{1}{3}}}f_N(t)dtds. \quad (6.19)$$

□

**Lemma 6.3.** *The error term in (6.19) satisfies* $\operatorname{err}(N,y) = O(N^{-\frac{1}{3}})$ *locally uniformly in* $y \in [0,\infty[$.

*Proof.* Fix some $M > 0$ and consider $y \in [0,M]$. We recall that the matrices $T_N = (Q_{k-1,N}(z_{i,N}))_{k,i=1,\dots,N}$ are orthogonal, which implies that for all $N \in \mathbb{N}$

$$1 = \sum_{k=0}^{N-1}(Q_{k,N}(z_{N,N}))^2 = \frac{1}{N^{\frac{1}{3}}}\sum_{k=0}^{N-1}(N^{\frac{1}{6}}Q_{k,N}(z_{N,N}))^2 = \int_0^\infty f_N^2(t)dt. \quad (6.20)$$

We next prove

$$\int_0^y f_N(t)dt = O(1) \quad \text{for} \quad N\to\infty. \quad (6.21)$$

For this we recall that by the definition of $f_N$,

$$f_N(t) = \sum_{k=0}^{N-1}N^{\frac{1}{6}}Q_{k,N}(z_{N,N})\mathbf{1}_{\left[\frac{k}{N^{1/3}},\frac{k+1}{N^{1/3}}\right]}(t)$$

with $Q_{k,N}(z_{N,N}) > 0$ for all $k$. This follows from the fact that the polynomials $Q_{k,N}$ have a positive leading coefficient and are orthogonal with respect to some measure with support $\{z_{1,N},\dots,z_{N,N}\}$, which implies that their zeros are contained in $]z_{1,N}, z_{N,N}[$; see e.g. [Chi78]. We thus see that $f_N(t) \geq 0$ for $t \geq 0$. Hence, for $y \in [0,M]$,

$$\int_0^y f_N(t)dt \leq \int_0^M f_N(t)dt$$

$$= \frac{1}{N^{\frac{1}{3}}}\sum_{k=0}^{\lfloor MN^{\frac{1}{3}}\rfloor-1}N^{\frac{1}{6}}q_k + \frac{MN^{\frac{1}{3}}-\lfloor MN^{\frac{1}{3}}\rfloor}{N^{\frac{1}{3}}}N^{\frac{1}{6}}q_{\lfloor MN^{\frac{1}{3}}\rfloor}$$

$$\leq \frac{1}{N^{\frac{1}{3}}}\sum_{k=0}^{\lfloor MN^{\frac{1}{3}}\rfloor}N^{\frac{1}{6}}q_k.$$

Hölder's inequality and (6.20) now imply that for $y \in [0, M]$ and $N \in \mathbb{N}$,

$$\int_0^y f_N(t)dt \leq \frac{1}{N^{\frac{1}{3}}} \left( \sum_{k=0}^{\lfloor MN^{\frac{1}{3}} \rfloor} q_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\lfloor MN^{\frac{1}{3}} \rfloor} N^{\frac{1}{3}} \right)^{\frac{1}{2}} \tag{6.22}$$

$$\leq \frac{1}{N^{\frac{1}{3}}} \sqrt{N^{\frac{1}{3}}(\lfloor MN^{\frac{1}{3}} \rfloor + 1)} \leq \sqrt{M + \frac{2}{N^{\frac{1}{3}}}} \leq \sqrt{M+2}.$$

This shows (6.21). In an analogous way we prove that for $y \in [0, M]$ and $\theta \in [0, 1]$,

$$\frac{1}{N^{\frac{1}{3}}} \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \frac{l+\theta}{N^{\frac{1}{3}}} N^{\frac{1}{6}} q_l = O(1). \tag{6.23}$$

For this we observe that

$$\sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \frac{l+\theta}{N^{\frac{1}{3}}} N^{\frac{1}{6}} q_l \leq \sum_{l=0}^{\lfloor MN^{\frac{1}{3}} \rfloor - 1} \frac{MN^{\frac{1}{3}}+1}{N^{\frac{1}{3}}} N^{\frac{1}{6}} q_l \leq (M+1) \sum_{l=0}^{\lfloor MN^{\frac{1}{3}} \rfloor - 1} N^{\frac{1}{6}} q_l.$$

This together with (6.21) shows (6.23).

Moreover, (6.22) leads to the following estimate for the last term in (6.19):

$$\left| \int_0^y \int_0^s \frac{\lfloor tN^{\frac{1}{3}} \rfloor - tN^{\frac{1}{3}} + \frac{1}{2}}{N^{\frac{1}{3}}} f_N(t)dtds \right| \leq \frac{1}{2N^{\frac{1}{3}}} \int_0^y \sqrt{M+2}ds$$

$$\leq \frac{M\sqrt{M+2}}{N^{\frac{1}{3}}} = O(N^{-\frac{1}{3}}). \tag{6.24}$$

We now turn to the estimation of $N^{\frac{1}{6}}\rho(y, N)$. For $y \in [0, M]$ and $k = 0, ..., \lfloor yN^{\frac{1}{3}} \rfloor$, we obtain that $N/(N-k)$ remains bounded for large $N$. Therefore, (6.9) implies that $\alpha(k, N) = O(N^{-\frac{4}{3}})$ and thus, by (6.12),

$$|N^{\frac{1}{6}}\rho(y, N)| \tag{6.25}$$

$$\leq \sum_{l=0}^{\lfloor yN^{\frac{1}{3}} \rfloor - 1} \sum_{k=0}^{l} \left( |O(N^{-\frac{4}{3}})| N^{\frac{1}{6}} q_{k+1} + |O(N^{-\frac{4}{3}})| N^{\frac{1}{6}} q_{k-1} + |O(N^{-1})| N^{\frac{1}{6}} q_k \right)$$

$$\leq \sum_{l=0}^{\lfloor MN^{\frac{1}{3}} \rfloor - 1} \sum_{k=0}^{l} \left( |O(N^{-\frac{7}{6}})| q_{k+1} + |O(N^{-\frac{7}{6}})| q_{k-1} + |O(N^{-\frac{5}{6}})| q_k \right).$$

If we use the summation formula (6.16) and Hölder's inequality, we see that the third summand on the right-hand side of (6.25) satisfies

$$
\begin{aligned}
&|O(N^{-\frac{5}{6}})| \sum_{l=0}^{\lfloor MN^{\frac{1}{3}}\rfloor-1} \sum_{k=0}^{l} q_k = |O(N^{-5/6})| \sum_{k=0}^{\lfloor MN^{\frac{1}{3}}\rfloor-1} (\lfloor MN^{\frac{1}{3}}\rfloor - k) q_k \\
\leq & |O(N^{-\frac{5}{6}})| \left( \sum_{k=0}^{\lfloor MN^{\frac{1}{3}}\rfloor-1} q_k^2 \right)^{\frac{1}{2}} \left( \sum_{k=0}^{\lfloor MN^{\frac{1}{3}}\rfloor-1} \underbrace{(\lfloor MN^{\frac{1}{3}}\rfloor - k)^2}_{\leq M^2 N^{\frac{2}{3}}} \right)^{\frac{1}{2}} \\
\leq & |O(N^{-\frac{5}{6}})| \left( \lfloor MN^{\frac{1}{3}}\rfloor M^2 N^{\frac{2}{3}} \right)^{\frac{1}{2}} = O(N^{-\frac{1}{3}}).
\end{aligned}
$$

If we keep in mind that $q_0 = \frac{1}{\sqrt{N}}$, we can estimate the other two sums in the same way. In summary, we conclude for the first term in (6.19) that

$$N^{\frac{1}{6}} \rho(y, N) = O(N^{-\frac{1}{3}}).$$

Furthermore, the second term in (6.19) can be estimated by a corresponding bound by (6.23) with $\theta = 1/2$ and $\theta = 1$ and with an index shift together with $Q_{0,N} = \frac{1}{\sqrt{N}}$. Moreover, the third term in (6.19) can be estimated in the same way by splitting the sum there and using (6.23) for the first and (6.20) for the second sum. Finally, the fourth and fifth term in (6.19) obviously have order $O(N^{-\frac{1}{3}})$, while this follows for the last term from (6.24). This completes the proof. □

We now complete the proof of Theorem 6.1 by proving the following statement:

**Lemma 6.4.** *For $N \to \infty$, $|f_N(y) - f(y)| = O(N^{-\frac{1}{3}})$ locally uniformly for $y \in [0, \infty[$.*

*Proof.* Again, fix $M > 0$, let $y \in [0, M]$, and assume that $N^{\frac{2}{3}} > M$. The ordinary differential equation (ODE) (6.2) yields that the function $f(y) = \frac{\mathsf{Ai}(y+a_1)}{\mathsf{Ai}'(a_1)}$ satisfies

$$f''(y) = (y + a_1) f(y) \quad \text{with} \quad f(0) = 0, \quad f'(0) = 1. \tag{6.26}$$

This ODE leads to the integral equation

$$f(y) = \int_0^y \int_0^s (t - |a_1|) f(t) dt ds + y = \int_0^y (t - |a_1|)(y - t) f(t)\, dt + y. \tag{6.27}$$

Note that the second equality in (6.27) follows by partial integration. Moreover, by Lemma 6.2,

$$f_N(y) = \int_0^y (t - |a_1|)(y - t) f_N(t)\, dt + y + \mathrm{err}(y, N).$$

We thus obtain

$$\begin{aligned}|f(y)-f_N(y)| &= \left|\int_0^y (t-|a_1|)(y-t)(f(t)-f_N(t))dt - \operatorname{err}(y,N)\right| \\ &\leq \int_0^y |t-|a_1||\cdot|y-t|\cdot|f(t)-f_N(t)|dt + |\operatorname{err}(y,N)|,\end{aligned}$$

where we know from Lemma 6.3 that there exists a constant $M' = M'(M) > 0$ with

$$|\operatorname{err}(y,N)| \leq \frac{M'}{N^{\frac{1}{3}}} \quad \text{for} \quad y \in [0,M]$$

and $N$ sufficiently large.

As $t \mapsto |(t-|a_1|)(y-t)|$ is the absolute value of a second-order polynomial, we find a constant $M'' > 0$ with $|(t-|a_1|)(y-t)| < M''$ for all $t \in [0,y]$ and $y \in [0,M]$. Hence,

$$|f(y)-f_N(y)| \leq \int_0^y M''|f(t)-f_N(t)|dt + \frac{M'}{N^{\frac{1}{3}}}.$$

Gronwall's lemma now implies our claim that

$$|f(y)-f_N(y)| \leq \frac{M'}{N^{\frac{1}{3}}} e^{M''y} \leq \frac{M'}{N^{\frac{1}{3}}} e^{M''M} = O(N^{-\frac{1}{3}}).$$

□

We now apply Lemma 6.4 to the $(N,N)$-entries of the covariance matrices $\Sigma_N$ for $\beta$-Hermite ensembles in the freezing regime for $N \to \infty$, which are described in Theorem 5.11 and Corollary 5.12.

**Theorem 6.5** (Andraus,H.,Voit(2021))**.** *Consider the covariance matrices* $\Sigma_N =: (\sigma_{i,j})_{i,j=1,\dots,N}$ *of* $\beta$*-Hermite ensembles in the freezing regime, i.e Theorem 4.1. Then*

$$\lim_{N\to\infty} N^{\frac{1}{3}} \sigma_{N,N} = \int_0^\infty \frac{\mathsf{Ai}(x+a_1)^2}{\mathsf{Ai}'(a_1)^2 x} dx = 0.834\dots.$$

*Proof.* We recall that $\Sigma_N = T_N \operatorname{diag}(1, 1/2, ..., 1/N) T_N^T$. Therefore,

$$\sigma_{N,N} = \sum_{k=1}^N \frac{1}{k}(Q_{k-1}^N(z_{N,N}))^2 = \frac{1}{N^{\frac{2}{3}}} \sum_{k=0}^{N-1} \frac{N^{\frac{1}{3}}}{k+1} \left(N^{\frac{1}{6}} Q_k^N(z_{N,N})\right)^2.$$

Define the functions

$$h_N(y) := \sum_{k=0}^{N-1} \frac{N^{\frac{1}{3}}}{k+1} \mathbf{1}_{\left[\frac{k}{N^{\frac{1}{3}}}, \frac{k+1}{N^{\frac{1}{3}}}\right)}(y),$$

which are approximations of the function $y \mapsto \frac{1}{y}$ with

$$0 \leq \frac{1}{y} - h_N(y) \leq \frac{N^{\frac{1}{3}}}{k(k+1)} \leq \frac{1}{y}\frac{1}{k} \quad \text{for} \quad k = \lfloor yN^{\frac{1}{3}} \rfloor,\ y > 0. \tag{6.28}$$

With this notation we have

$$N^{\frac{1}{3}}\sigma_{N,N} = \frac{1}{N^{\frac{1}{3}}}\sum_{k=0}^{N-1}\frac{N^{\frac{1}{3}}}{k+1}\left(N^{\frac{1}{6}}Q_k^N(z_{N,N})\right)^2 = \int_0^\infty (f_N(y))^2 h_N(y)dy.$$

With $f_N$ as defined in 6.6.The statement of the theorem is now equivalent to

$$\lim_{N\to\infty}\int_0^\infty (f_N(y))^2 h_N(y)dy = \int_0^\infty \frac{\mathsf{Ai}(x+a_1)^2}{\mathsf{Ai}'(a_1)^2 y}dy = \int_0^\infty \frac{f(y)^2}{y}dy.$$

To show this, we can equivalently show that

$$\lim_{N\to\infty}\int_0^1 f_N(y)^2 h_N(y)\,dy = \int_0^1 \frac{f(y)^2}{y}\,dy \tag{6.29}$$

and

$$\lim_{N\to\infty}\int_1^\infty f_N(y)^2 h_N(y)\,dy = \int_1^\infty \frac{f(y)^2}{y}\,dy. \tag{6.30}$$

For this, we first recall from (6.20) that

$$\int_0^\infty (f_N(y))^2\,dy = 1. \tag{6.31}$$

Furthermore, as $f''(y) = (y - |a_1|)f(y)$, we know that

$$\int_0^\infty f(y)^2\,dy = \left[(y-|a_1|)f(y)^2 + f'(y)^2\right]_{y=0}^\infty = 1. \tag{6.32}$$

We next observe that Theorem 6.1 implies that the measures $f_N^2 d\lambda$ with Lebesgue densities $f_N^2$ converge vaguely to the measure $f^2 d\lambda$ on $[0,\infty[$. As all measures are probability measures by (6.31) and (6.32), we conclude from a standard result in probability (see e.g. [Bil99]) that these measures converge even weakly, i.e, for all bounded continuous functions $g : [0,\infty[ \to \mathbb{R}$ we have

$$\lim_{N\to\infty}\int_0^\infty g(y)f_N(y)^2\,dy = \int_0^\infty g(y)f(y)^2\,dy. \tag{6.33}$$

Moreover, as all probability measures have Lebesgue densities, we again conclude from a standard result in probability (see e.g. [Bil99]) that (6.33) remains correct on $[1,\infty[$, i.e., for the bounded continuous function $g(y) := \frac{1}{y}$ on $[1,\infty[$ we have

$$\lim_{N\to\infty}\int_1^\infty \frac{f_N(y)^2}{y}\,dy = \int_1^\infty \frac{f(y)^2}{y}\,dy =: R. \tag{6.34}$$

On the other hand, (6.28) shows that for any $\varepsilon > 0$, there is some sufficiently large $N(\varepsilon)$ with

$$\left|\frac{1}{y} - h_N(y)\right| \leq \frac{\varepsilon}{y} \quad \text{for} \quad y \geq 1,\ N \geq N(\varepsilon).$$

Therefore,

$$\int_1^\infty \left|\frac{1}{y} - h_N(y)\right| f_N(y)^2\, dy \leq \varepsilon \int_1^\infty \frac{1}{y} f_N(y)^2\, dy, \tag{6.35}$$

where, by (6.34), the right-hand side converges for $N \to \infty$ to $\varepsilon R$. (6.35), (6.34), and the triangle inequality now yield (6.30).

We lastly verify (6.29). We recall that

$$\lim_{N\to\infty} f_N(y)^2 h_N(y) = \frac{(f(y))^2}{y} \quad \text{for} \quad y \in [0,1[.$$

Note that this formula also holds for $y = 0$, as $f$ is analytic in 0 with $f(0) = 0$. Moreover, (6.28), the fact that $h_N(y) \leq N^{1/3}$, and Lemma 6.4 show that for $N$ sufficiently large

$$\begin{aligned} |f_N(y)^2 h_N(y)| &\leq |f_N(y)^2 - f(y)^2| h_N(y) + f(y)^2 h_N(y) \\ &\leq (f_N(y) + f(y))|f_N(y) - f(y)| N^{\frac{1}{3}} + \frac{f(y)^2}{y} \\ &\leq (1 + 2f(y))O(1) + \frac{f(y)^2}{y}. \end{aligned}$$

As this is a bounded continuous function for $y \in [0,1]$, we conclude from dominated convergence that (6.29) holds. This completes the proof. □

If we combine Theorem 6.5 with Theorem 4.1, we finally obtain:

**Theorem 6.6** (Andraus,H.,Voit(2021),Gorin,Kleptsyn(2021))**.** *Consider the Gaussian $\beta$-ensemble, i.e. random variables*

$$X_\kappa^N = (X_{\kappa,1}^N, \ldots, X_{\kappa,N}^N)$$

*with densities*

$$c_\kappa \prod_{1\leq i<j\leq N} (x_j - x_i)^{2\kappa} \prod_{i=1}^N e^{-\frac{\kappa}{2}x_i^2}.$$

*Then*

$$\lim_{N\to\infty} \left( \lim_{k\to\infty} N^{\frac{1}{6}} \sqrt{2\kappa} \left( \frac{X_{\kappa,N}^N}{\sqrt{2\kappa}} - z_{N,N} \right) \right) = G \tag{6.36}$$

*in distribution with some $\mathcal{N}(0, \sigma_{max}^2)$-distributed random variable $G$ with variance*

$$\sigma_{max}^2 := \int_0^\infty \frac{\mathsf{Ai}(x + a_1)^2}{(\mathsf{Ai}'(a_1))^2 x} dx = 0.834\ldots. \tag{6.37}$$

*Remarks.* (1) If we combine Theorem 6.6 with the formula of Plancherel-Rotach

$$\frac{z_{N,N}}{\sqrt{2N}} = 1 - \frac{|a_1|}{2N^{\frac{2}{3}}} + r_N \quad \text{with} \quad r_N = O(N^{-1}),$$

we can state (6.36) as

$$\lim_{N\to\infty}\left(\lim_{k\to\infty}\left(N^{\frac{2}{3}}\left(\frac{X^N_{t,k,N}}{\sqrt{tN}} - 2\sqrt{k}\right) + 2\sqrt{k}(|a_1| - N^{\frac{2}{3}} r_N)\right)\right) = G. \tag{6.38}$$

Please note that in this limit, the term $2\sqrt{k}N^{\frac{2}{3}}r_N$ cannot be neglected.

(3) Theorem 6.6 was stated by Dumitriu and Edelman (Corollary 3.4 in [DE05]), where the numerical value of $\sigma^2_{max}$ contains a misprint and the proof is sketched only. Moreover, the proof in [DE05] is based on the representation of the covariance matrix $\Sigma_N$ in Theorem 3.1 in [DE05]. This representation of $\Sigma_N$ with essentially the same proof as above also leads to Theorem 6.6, where one then obtains the formula

$$\sigma^2_{max} = 2\frac{\int_0^\infty \mathsf{Ai}^4(x+a_1)dx}{\left(\int_0^\infty \mathsf{Ai}^2(x+a_1)dx\right)^2} = 2\int_0^\infty \left(\frac{\mathsf{Ai}(x+a_1)}{\mathsf{Ai}'(a_1)}\right)^4 dx \tag{6.39}$$

with the aid of (6.32). A numerical computation shows that the value of (6.39) seems to be equal to that in (6.37). Unfortunately, we are not able to verify this equality in an analytical way, as our suggested identity does not fit to identities for integrals of the Airy function in the literature, as e.g. in [VS04]. The proof of [DE05] can also be extended to the $r$-th largest eigenvalue, which then leads to the identity

$$2\int_0^\infty \left(\frac{\mathsf{Ai}(x+a_r)}{\mathsf{Ai}'(a_r)}\right)^4 dx = \int_0^\infty \frac{\mathsf{Ai}(x+a_r)^2}{(\mathsf{Ai}'(a_r))^2 x}dx.$$

(2) In [GK22], a similar result was obtained by using different methods. However, the formulas here match with those derived in [AHV21].

(4) In [RRV11], Ramirez, Rider, and Virag studied the largest eigenvalues of $\beta$-Hermite ensembles where they first took the limit $N\to\infty$ and then $\beta\to\infty$, i.e., $k\to\infty$ here. From the results in [RRV11] one obtains that

$$\lim_{k\to\infty}\left(\lim_{N\to\infty}\left(N^{\frac{2}{3}}\left(\frac{X^N_{t,k,N}}{\sqrt{tN}} - 2\sqrt{k}\right) + 2\sqrt{k}|a_1|\right)\right) = G \tag{6.40}$$

in distribution, where $G$ is $\mathcal{N}(0,\sigma^2_{max})$-distributed with $\sigma^2_{max}$ as in (6.39).

**Remark 6.7.** Clearly, the preceding limit results for the largest eigenvalue in the Hermite case can be transferred to the smallest eigenvalue by symmetry.

We next use the following formula by Plancherel-Rotach

$$\frac{z_{N-r+1,N}}{\sqrt{2N}} = 1 - \frac{|a_r|}{2N^{\frac{2}{3}}} + O(N^{-1}), \tag{6.41}$$

where $a_r$ is the $r$-th largest zero of the Airy function. This formula is derived from the well-known relationship between Hermite and Laguerre polynomials [Olv+10]

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{(-1/2)}(x^2),$$
$$H_{2n+1}(x) = (-1)^n 2^{2n+1} n! L_n^{(1/2)}(x^2),$$

and a corresponding Plancherel-Rotach formula for the Laguerre zeros given by (5) in [Tri49]. This leads to the following result for the $r$-th largest eigenvalue:

**Theorem 6.8.** *For $r \in \mathbb{N}$ consider the functions*

$$f_N(y) := N^{\frac{1}{6}} Q_{\lfloor N^{\frac{1}{3}} y \rfloor, N}(z_{N-r+1,N}) \quad \text{for} \quad y \in [0, N^{\frac{2}{3}}[$$

*and $f_N(y) = 0$ otherwise. Then $(f_N)_{N \geq 1}$ tends locally uniformly to*

$$f(y) := \frac{\mathsf{Ai}(y + a_r)}{\mathsf{Ai}'(a_r)} \qquad \text{for} \quad y \in [0, \infty[$$

*for $N \to \infty$. Moreover, the covariance matrices $\Sigma_N =: (\sigma_{i,j})_{i,j=1,\ldots,N}$ of the freezing $\beta$-Hermite ensembles satisfy*

$$\lim_{N \to \infty} N^{\frac{1}{3}} \sigma_{N-r+1,N-r+1} = \sigma^2_{max,r},$$

*with*

$$\sigma^2_{max,r} = \int_0^\infty \frac{\mathsf{Ai}(x + a_r)^2}{\mathsf{Ai}'(a_r)^2 x} dx = \begin{cases} 0.582\ldots \text{for } r = 2 \\ 0.472\ldots \text{for } r = 3 \\ 0.407\ldots \text{for } r = 4 \\ \vdots \end{cases}.$$

*Proof.* The proof is similar to those of Theorems 6.1 and 6.5, where $f_N$ and $f$ are now those in Theorem 6.8. In particular, for $f$ we now have

$$f''(y) = (y + a_r) f(y) \quad \text{with} \quad f(0) = 0, \quad f'(0) = 1.$$

Moreover, $a_1$ has to be replaced by $a_r$, and (6.10) by (6.41). We notice that now $f_N(t) \geq 0$ for $t \geq 0$ does not hold; we can however still estimate

$$\left| \int_0^y f_N(t) dt \right| \leq \int_0^y |f_N(t)| dt$$

with triangle inequality. We can then proceed precisely as in (6.22). □

**Remark 6.9.** For the first few values of $r$, the integral

$$\int_0^\infty \frac{\mathsf{Ai}(x+a_r)^2}{\mathsf{Ai}'(a_r)^2 x}dx = \int_{a_r}^\infty \frac{\mathsf{Ai}(x)^2}{\mathsf{Ai}'(a_r)^2(x-a_r)}dx$$

seems to be decreasing in $r$. This is indeed the case as $r \to \infty$. To see this, we decompose the last integral into the regions $[a_r, a_{r-1}[$, $[a_{r-1}, a_1[$ and $[a_1, \infty[$, and estimate it in each case. Let us first note that by (6.5),

$$\mathsf{Ai}'(a_r)^{-2} = \Big(\frac{2\pi^2}{3}\Big)^{1/3} r^{-1/3} + O(r^{-4/3}).$$

Now, for the first region we use (6.4), (6.3), the substitution $y = 2(-x)^{3/2}/3\pi - r + 3/4$, and obtain for $r \to \infty$ that

$$\int_{a_r}^{a_{r-1}} \frac{\mathsf{Ai}(x)^2}{(x-a_r)}dx = \Big(\frac{2}{3\pi}\Big)^{4/3} r^{-1/3} \int_{-1/2}^{1/2} \frac{3\cos(\pi y)^2}{1-2y}dy + O(r^{-4/3}).$$

Because $|\mathsf{Ai}(x)| < 1$, (6.3) leads to

$$\begin{aligned}\int_{a_{r-1}}^{a_1} \frac{\mathsf{Ai}(x)^2}{(x-a_r)}dx \le \log\frac{a_1-a_r}{a_{r-1}-a_r} &= \log\Big(r\Big(1+\Big(\frac{2}{3\pi}\Big)^{2/3} a_1 r^{-2/3} + O(r^{-5/3})\Big)\Big) \\ &= \log r + O(r^{-2/3}).\end{aligned}$$

Finally, Theorem 6.5 and $a_r < a_1$ yield the bound

$$\int_{a_1}^\infty \frac{\mathsf{Ai}(x)^2}{(x-a_r)}dx \le \int_{a_1}^\infty \frac{\mathsf{Ai}(x)^2}{(x-a_1)}dx < 1.$$

Putting everything together, we see that for a sufficiently large $r$ there exists a constant $C > 0$ such that

$$\int_0^\infty \frac{\mathsf{Ai}(x+a_r)^2}{\mathsf{Ai}'(a_r)^2 x}dx \le C r^{-1/3}\log r.$$

This stresses the fact that $r \to \infty$ means that we go from the edge into the bulk, where repulsion interactions are stronger, i.e., all variances there are much smaller than at the edge.

## 6.2 Limit results for the largest eigenvalue in the Laguerre case

We now discuss the soft-edge statistics in the freezing Laguerre case similar to Subsection 6.1 for the Hermite case. This means that we analyze the limit behaviour of the largest entry of the vector in the freezing regime in Theorem 4.5 for $N \to \infty$. This will be done on the basis of Theorem 5.11. We follow [AHV21] Chapter 6 here. This problem is also discussed in [Ler23], where similar results occur, but viewpoint and method of proof are different. We here again use the ordered zeros $z_{1,N}^{(\alpha)} < ..., z_{N,N}^{(\alpha)}$ of the $N$-th Laguerre polynomial $L_N^{(\alpha)}$. Moreover, for each $N$, let $(Q_{k,N}^{(\alpha)})_{k=0,..,N-1}$ be the dual polynomials associated with $(L_k^{(\alpha)})_{k=0,...,N}$, normalized with

$$\sqrt{z_{i,N}^{(\alpha)}}Q_{k,N}^{(\alpha)}(z_{i,N}^{(\alpha)}) = \sqrt{w_i^*}\tilde{Q}_{k,N}^L(z_{i,N}^{(\alpha)}) = \sqrt{z_{i,N}^{(\alpha)}}\frac{\tilde{Q}_{k,N}^L(z_{i,N}^{(\alpha)})}{\sqrt{N(N+\alpha)}}$$

where $\tilde{Q}_{k,N}^H$ are given as in Theorem 5.2. This means that the matrices

$$T_N := (\sqrt{z_{i,N}^{(\alpha)}}Q_{j-1,N}^{(\alpha)}(z_{i,N}^{(\alpha)}))_{i,j=1,...,N}$$

are orthogonal, with $T_N^T\Sigma_N T_N = \text{diag}(\frac{1}{4}, \frac{1}{8}, ..., \frac{1}{4N})$ as in the proof of Theorem 5.11. The $Q_{k,N}^{(\alpha)}$ have the three-term-recurrence

$$\begin{aligned} xQ_{k,N}^{(\alpha)}(x) =& \sqrt{(N-k)(N-k+\alpha)}Q_{k-1,N}^{(\alpha)}(x) + (2(N-k)+\alpha-1)Q_{k,N}^{(\alpha)} \\ &+ \sqrt{(N-k-1)(N-k-1+\alpha)}Q_{k+1,N}^{(\alpha)}(x) \quad (k < N) \end{aligned} \tag{6.42}$$

with the initial conditions $Q_{-1,N}^{(\alpha)} = 0$ and $Q_{0,N}^{(\alpha)} = \frac{1}{\sqrt{N(N+\alpha)}}$.

In the Laguerre case we have the following analogy of Theorem 6.1:

**Theorem 6.10.** *Let $\alpha > -1$ and define the functions*

$$f_N(y) := N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}Q_{\lfloor N^{\frac{1}{3}}y\rfloor,N}(z_{N,N}^{(\alpha)}) \quad \textit{for} \quad y \in [0, N^{\frac{2}{3}}[$$

*and $f_N(y) = 0$ otherwise. Then $(f_N)_{N\geq 1}$ tends locally uniformly to*

$$f(y) = \frac{2^{\frac{1}{3}}\mathsf{Ai}(2^{\frac{2}{3}}y + a_1)}{\mathsf{Ai}'(a_1)} \quad (y \in [0, \infty[) \tag{6.43}$$

*for $N \to \infty$.*

*Proof.* We put $q_k^{(\alpha)} := Q_{k,N}^{(\alpha)}(z_{N,N}^{(\alpha)})$ for $k = 0, \ldots, N-1$. The Landau symbol $O$ will be always used for $N \to \infty$ and will be locally uniform with respect to $y \in [0, \infty[$.

The sharp Plancherel-Rotach formula for the zeros $z_{N,N}^{(\alpha)}$ was studied in [Tri49], see [Erd+81] for an english version or compare Theorem 1.2 in [Gat88]. It is as follows:

$$\frac{z_{N,N}^{(\alpha)}}{4N} = 1 + \frac{a_1}{(2N)^{\frac{2}{3}}} + O(N^{-1}). \tag{6.44}$$

We will also use the Taylor expansion

$$\sqrt{1 + \frac{\alpha}{N-k}} = 1 + \frac{\alpha}{2(N-k)} + O(N^{-2}) \qquad \text{for} \quad 0 \leq k \leq yN^{\frac{1}{3}}. \tag{6.45}$$

The recurrence relation (6.42) for $x = z_{N,N}^{(\alpha)}$ and a division by $N$ yield

$$\left(\frac{z_{N,N}^{(\alpha)}}{N} - 2\left(1-\frac{k}{N}\right) + \frac{\alpha-1}{N}\right) q_k^{(\alpha)} =$$
$$\left(1-\frac{k+1}{N}\right)\sqrt{1+\frac{\alpha}{N-k-1}}q_{k+1}^{(\alpha)} + \left(1-\frac{k}{N}\right)\sqrt{1+\frac{\alpha}{N-k}}q_{k-1}^{(\alpha)}.$$

Using (6.44) and (6.45), we obtain

$$\begin{aligned}
&q_k^{(\alpha)}\left(2+\frac{2^{\frac{4}{3}}a_1}{N^{\frac{2}{3}}}+\frac{2k}{N}+O(N^{-1})\right)\\
=&q_{k-1}^{(\alpha)}\left(1-\frac{k}{N}+\frac{\alpha}{2(N-k)}\left(1-\frac{k}{N}\right)+O(N^{-1})\left(1-\frac{k}{N}\right)\right)\\
&+q_{k+1}^{(\alpha)}\left(1-\frac{k+1}{N}+\frac{\alpha}{2(N-k-1)}\left(1-\frac{k+1}{N}\right)+O(N^{-1})\left(1-\frac{k+1}{N}\right)\right)\\
=&q_{k-1}^{(\alpha)}\left(1-\frac{k}{N}+O(N^{-1})\right)+q_{k+1}^{(\alpha)}\left(1-\frac{k+1}{N}+O(N^{-1})\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
q_{k+1}^{(\alpha)}+q_{k-1}^{(\alpha)}-2q_k^{(\alpha)} = \frac{k+1}{N}q_{k+1}^{(\alpha)}+\frac{k}{N}q_{k-1}^{(\alpha)}+\left(\frac{2^{\frac{4}{3}}a_1}{N^{\frac{2}{3}}}+\frac{2k}{N}\right)q_k^{(\alpha)} \qquad (6.46)\\
+O(N^{-1})q_{k+1}^{(\alpha)}+O(N^{-1})q_k^{(\alpha)}+O(N^{-1})q_{k-1}^{(\alpha)}.
\end{aligned}$$

Equation (6.46) is very similar to Equation (6.11), so we skip some details as the calculation below will be very similar to Subsection 6.1. We sum (6.46) over $k = 0, ..., l$ and then over $l = 0, ..., \lfloor yN^{\frac{1}{3}} \rfloor - 1$. After multiplying with $N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}$, we obtain from (6.44) that the left-hand side is equal to

$$\begin{aligned}
&f_N(y) - \frac{(1+\lfloor yN^{\frac{1}{3}}\rfloor)N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}}{\sqrt{N(N+\alpha)}}\\
=&f_N(y) - \frac{(1+\lfloor yN^{\frac{1}{3}}\rfloor)N^{\frac{1}{6}}2\sqrt{N}(1+O(N^{-2/3}))}{\sqrt{N(N+\alpha)}}\\
=&f_N(y) - 2y + O(N^{-1/3})
\end{aligned}$$

and the right-hand side to

$$\begin{aligned}
\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\sum_{k=0}^{l}\Bigg(\frac{k+1}{N}N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_{k+1}^{(\alpha)}+\frac{k}{N}N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_{k-1}^{(\alpha)}\\
+N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}\left(\frac{2^{\frac{4}{3}}a_1}{N^{\frac{2}{3}}}+\frac{2k}{N}\right)q_k^{(\alpha)}\Bigg)+O(N^{-1/3}). \qquad (6.47)
\end{aligned}$$

Note that in the second case, we used the estimation

$$\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\sum_{k=0}^{l}\Big(O(N^{-1})q_{k+1}^{(\alpha)}+O(N^{-1})q_{k}^{(\alpha)}+O(N^{-1})q_{k-1}^{(\alpha)}\Big)=O(N^{-\frac{1}{3}}),$$

which can be proven precisely as Equation (6.25) in the proof of Lemma 6.3. We next use the index shifts $k+1\mapsto k$ and $k-1\mapsto k$ in (6.47) and obtain that the right-hand side above is

$$\begin{aligned}
&\frac{1}{N^{\frac{1}{3}}}\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\frac{1}{N^{\frac{1}{3}}}\sum_{k=0}^{l}\left(N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_k^{(\alpha)}\left(\frac{k+1}{N^{\frac{1}{3}}}+\frac{k}{N^{\frac{1}{3}}}+2^{\frac{4}{3}}a_1+\frac{2k}{N^{\frac{1}{3}}}\right)\right)\\
&+\frac{1}{N^{\frac{2}{3}}}\sum_{k=0}^{\lfloor yN^{\frac{1}{3}}\rfloor}\frac{k}{N^{\frac{1}{3}}}N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_k^{(\alpha)}-\frac{1}{N^{\frac{2}{3}}}\sum_{k=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\frac{k+1}{N^{\frac{1}{3}}}N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_k^{(\alpha)}\\
&=\frac{1}{N^{\frac{1}{3}}}\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\frac{1}{N^{\frac{1}{3}}}\sum_{k=0}^{l}\left(N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_k^{(\alpha)}\left(\frac{k+1}{N^{\frac{1}{3}}}+\frac{k}{N^{\frac{1}{3}}}+2^{\frac{4}{3}}a_1+\frac{2k}{N^{\frac{1}{3}}}\right)\right)\\
&+O(N^{-\frac{1}{3}}),
\end{aligned}$$

where for the last equation an analogous estimation to that in (6.21) has been used.

In summary, we have proven that

$$f_N(y)-2y+O(N^{-1/3})=\frac{1}{N^{\frac{2}{3}}}\sum_{l=0}^{\lfloor yN^{\frac{1}{3}}\rfloor-1}\sum_{k=0}^{l}\left(N^{\frac{1}{6}}\sqrt{z_{N,N}^{(\alpha)}}q_k^{(\alpha)}\left(\frac{4k+1}{N^{\frac{1}{3}}}+2^{\frac{4}{3}}a_1\right)\right).$$

If we use (6.17), we see that this leads to the integral equation

$$f_N(y)=\int_0^y\int_0^s f_N(s)(4t+2^{\frac{4}{3}}a_1)dtds+2y+O(N^{-\frac{1}{3}}).$$

As the function $f$ defined in (6.43) satisfies $f(0)=0$, $f'(0)=2$ and $f''(x)=(4x+2^{\frac{4}{3}}a_1)f(x)$, we obtain from Gronwall's lemma (see also Lemma 6.4) that

$$|f_N(y)-f(y)|=O(N^{-1/3}).$$

□

We now apply Theorem 6.10 to the $(N,N)$-entries of the covariance matrices $\Sigma_N$ for $\beta$-Laguerre ensembles in the freezing regime for $N\to\infty$, which are described in Theorem 5.11 and Corollary 5.14.

**Corollary 6.11.** *Consider the covariance matrices $\Sigma_N=:(\sigma_{i,j})_{i,j=1,\ldots,N}$ of $\beta$-Laguerre ensembles in the freezing regime. Then*

$$\lim_{N\to\infty}N^{\frac{1}{3}}\sigma_{N,N}=\int_0^\infty\frac{\mathsf{Ai}(x+a_1)^2}{\mathsf{Ai}'(a_1)^2x}dx=0.834\ldots.$$

*Proof.* The proof is completely analogous to the proof of Theorem 6.5. □

Similar to Theorem 6.5 and Theorem 6.6, the above theorem can be combined with Theorem 4.6to obtain:

**Theorem 6.12** (Andraus,H.,Voit(2021))**.** *Consider the Laguerre $\beta$-ensemble, i.e. random variables*

$$X_\kappa^{N,\alpha} = (X_{\kappa,1}^{N,\alpha}, \dots, X_{\kappa,N}^{N,\alpha})$$

*with densities*

$$\tilde{c}_{\kappa,\alpha} \prod_{1\le i<j\le N} (x_j^2 - x_i^2)^{2\kappa} \prod_{i=1}^{N} x_i^{2(\alpha+1)\kappa} e^{-x_i^2}.$$

*Then*

$$\lim_{N\to\infty} \left( \lim_{\kappa\to\infty} N^{\frac{1}{6}} \sqrt{\kappa} \left( \frac{X_{\kappa,N}^{N,\alpha}}{\sqrt{\kappa}} - \sqrt{z_{N,N}^{(\alpha)}} \right) \right) = G$$

*in distribution with some $\mathcal{N}(0, \sigma_{max}^2)$-distributed random variable $G$ with the variance*

$$\sigma_{max}^2 := \int_0^\infty \frac{\mathsf{Ai}(x+a_1)^2}{(\mathsf{Ai}'(a_1))^2 x} dx = 0.834\dots.$$

## 6.3 Limit results for the smallest eigenvalue in the Laguerre case

In this subsection, we consider the hard-edge statistics in the freezing Laguerre case. Specifically, we analyze the limiting behavior of the smallest entry in the vector within the freezing regime, as described in Theorem 4.5, for $N \to \infty$. This analysis is based on Theorem 5.11. The outlined approach is initially presented in [And21] and will be rigorously proven here. The proof in that work employs spline polynomials to transform the difference equation into a differential equation. While this improves readability, it leaves some convergence issues unresolved. Therefore, we will use integral equations and Gronwall's Lemma to address these concerns. Additionally, similar results are presented in [Ler23], though the perspectives and proof techniques differ.
We will again use the ordered zeros $z_{1,N}^{(\alpha)} < \dots < z_{N,N}^{(\alpha)}$ of the $N$-th Laguerre polynomial $L_N^{(\alpha)}$. Furthermore, for each $N$, let $\{Q_{k,N}^{(\alpha)}\}_{k=0}^{N-1}$ denote the dual polynomials associated with $\{L_k^{(\alpha)}\}_{k=0}^{N}$, normalized as follows:

$$\sqrt{z_{i,N}^{(\alpha)}} Q_{k,N}^{(\alpha)}(z_{i,N}^{(\alpha)}) = \sqrt{w_i^*} \tilde{Q}_{k,N}^{L}(z_{i,N}^{(\alpha)}) = \sqrt{z_{i,N}^{(\alpha)}} \frac{\tilde{Q}_{k,N}^{L}(z_{i,N}^{(\alpha)})}{\sqrt{N(N+\alpha)}}.$$

Here $\tilde{Q}^L_{k,N}$ are given as in Theorem 5.2. This means that the matrices

$$T_N := \left(\sqrt{z^{(\alpha)}_{i,N}}Q^{(\alpha)}_{j-1,N}(z^{(\alpha)}_{i,N})\right)_{i,j=1,\dots,N} \tag{6.48}$$

are orthogonal with $T_N^T\Sigma_N T_N = \mathrm{diag}(\frac{1}{4},\frac{1}{8},\dots,\frac{1}{4N})$ as in the proof of Theorem 5.11. The $Q^{(\alpha)}_{k,N}$ have the three-term-recurrence

$$\begin{aligned}xQ^{(\alpha)}_{k,N}(x) =& \sqrt{(N-k)(N-k+\alpha)}Q^{(\alpha)}_{k-1,N}(x) + (2(N-k)+\alpha-1)Q^{(\alpha)}_{k,N}\\ &+\sqrt{(N-k-1)(N-k-1+\alpha)}Q^{(\alpha)}_{k+1,N}(x) \quad (k<N)\end{aligned} \tag{6.49}$$

with the initial conditions $Q^{(\alpha)}_{-1,N}=0$ and $Q^{(\alpha)}_{0,N}=\frac{1}{\sqrt{N(N+\alpha)}}$. Note that here the convention $\sqrt{\xi_N}\tilde{L}^{(\alpha)}_N = \hat{L}^{(\alpha)}_N$ from Section 5 is used. This means that $\tilde{L}^{(\alpha)}_N$ and $Q^{(\alpha)}_{k,N}$ do not have alternating positive and negative leading coefficients. To study the behavior of the hard-edge (smallest eigenvalue) case, it is however useful to use the naturally occurring signs of the Laguerre polynomials for their dual polynomials. Precisely, multiplying (6.49) with $(-1)^k$ yields

$$\begin{aligned}x(-1)^kQ^{(\alpha)}_{k,N}(x) =& -\sqrt{(N-k)(N-k+\alpha)}(-1)^{k-1}Q^{(\alpha)}_{k-1,N}(x)\\ &+(2(N-k)+\alpha-1)(-1)^kQ^{(\alpha)}_{k,N}\\ &-\sqrt{(N-k-1)(N-k-1+\alpha)}(-1)^{k+1}Q^{(\alpha)}_{k+1,N}(x) \quad (k<N)\end{aligned} \tag{6.50}$$

with the initial conditions $Q^{(\alpha)}_{-1,N}=0$ and $Q^{(\alpha)}_{0,N}=\frac{1}{\sqrt{N(N+\alpha)}}$. The limit theorem for $N\to\infty$ that will be derived in this subsection involves the Bessel function $J_\alpha$ of parameter $\alpha\in\mathbb{C}$. For this, recall some well-known facts about $J_\alpha$, see e.g [Mil45]. For a complex parameter $\alpha\in\mathbb{C}$ the function $z\mapsto J_\alpha(z)$ is defined by

$$J_\alpha(z) := \sum_{m=0}^{\infty}\frac{(-1)^m(\frac{z}{2})^{\alpha+2m}}{m!\Gamma(\alpha+m+1)}$$

and solves the differential equation

$$z^2\frac{d^2}{dz^2}y(z) + z\frac{d}{dz}y(z) + (z^2-\alpha^2)y(z) = 0. \tag{6.51}$$

For $\alpha\in\mathbb{R}, r\in\mathbb{N}$ the function $J_\alpha$ has infinitely many positive zeros $j_{\alpha,r}$ with

$$0<j_{\alpha,1}<j_{\alpha,2}<\dots.$$

In analogy with Theorem 6.1 the following result can be stated:

**Theorem 6.13.** *Let $\alpha>-1$, $N,r\in\mathbb{N}$ and for $y\in[0,1)$ define*

$$f_N(y) := \sum_{k=0}^{N-1}\sqrt{N}\sqrt{z^{(\alpha)}_{r,N}}(-1)^kQ^{(\alpha)}_{k,N}(z^{(\alpha)}_{r,N})\mathbb{1}_{[\frac{k}{N},\frac{k+1}{N})}(y).$$

*Then the sequence* $(f_N)_{N\geq 1}$ *tends local uniformly to*

$$f(y) = -\frac{J_\alpha(j_{\alpha,r}\sqrt{1-y})}{J'_\alpha(j_{\alpha,r})} \tag{6.52}$$

*for* $N \to \infty$.

*Proof.* The main idea of this proof is to rewrite the three-term recursion (6.50) into an ordinary differential equation (ODE) of second order and apply Gronwall's lemma. To avoid issues with ill-defined derivatives and other technicalities, the ODE will be written as an integral equation. Let $y \in (0,1)$ and $0 \leq k \leq \lfloor yN \rfloor$. Using the abbreviations

$$\begin{aligned} q_k &:= (-1)^k Q^{(\alpha)}_{k,N}(z^{(\alpha)}_{1,N}), \\ z_r &:= z^{(\alpha)}_{r,N} \end{aligned}$$

and setting $x = z_r$ in Equation (6.50), multiplied with $\frac{1}{N}$, yields

$$\begin{aligned} \frac{1}{N} z_r q_k = &- \sqrt{1-\frac{k}{N}}\sqrt{1-\frac{k}{N}+\frac{\alpha}{N}}\,q_{k-1} \\ &+ \big(2(1-\frac{k}{N}) + \frac{\alpha-1}{N}\big) q_k \\ &- \sqrt{1-\frac{k+1}{N}}\sqrt{1-\frac{k+1}{N}+\frac{\alpha}{N}}\,q_{k+1}. \end{aligned} \tag{6.53}$$

The asymptotic behavior for $z_r$ was studied in [Tri49], see [Erd+81] for an English version or compare Theorem 1.1 in [Gat88]. It is as follows:

$$z_r = \frac{j^2_{\alpha,r}}{4N + 2(\alpha+1)} + O(N^{-3}) = \frac{j^2_{\alpha,r}}{4N} + O(N^{-2}). \tag{6.54}$$

For $0 \leq k \leq \lfloor yN \rfloor$ there is

$$1 \geq 1 - \frac{k}{N} \geq 1 - y \Rightarrow \frac{1}{1-\frac{k}{N}} = O(1).$$

Using the following three Taylor expansions for $0 \leq k \leq \lfloor yN \rfloor$

$$\begin{aligned}
\sqrt{1-\frac{k+1}{N}+\frac{\alpha}{N}} =& \sqrt{1-\frac{k+1}{N}}+\frac{\alpha}{2N}\left(1-\frac{k+1}{N}\right)^{-\frac{1}{2}} \\
&-\frac{\alpha^2}{8N^2}\left(1-\frac{k+1}{N}\right)^{-\frac{3}{2}}+O(N^{-3}) \\
\sqrt{1-\frac{k-1}{N}+\frac{\alpha-1}{N}} =& \sqrt{1-\frac{k-1}{N}}+\frac{\alpha-1}{2N}\left(1-\frac{k-1}{N}\right)^{-\frac{1}{2}} \\
&-\frac{(\alpha-1)^2}{8N^2}\left(1-\frac{k-1}{N}\right)^{-\frac{3}{2}}+O(N^{-3}) \\
\sqrt{1-\frac{k-1}{N}-\frac{1}{N}} =& \sqrt{1-\frac{k-1}{N}}+\frac{1}{2N}\left(1-\frac{k-1}{N}\right)^{-\frac{1}{2}} \\
&-\frac{1}{8N^2}\left(1-\frac{k-1}{N}\right)^{-\frac{3}{2}}+O(N^{-3})
\end{aligned}$$

yields

$$\begin{aligned}
\sqrt{1-\frac{k+1}{N}}\sqrt{1-\frac{k+1}{N}+\frac{\alpha}{N}} =& 1-\frac{k+1}{N}+\frac{\alpha}{2N} \\
&-\frac{\alpha^2}{8N^2}\left(1-\frac{k+1}{N}\right)^{-1}+O(N^{-3}) \\
\sqrt{1-\frac{k}{N}}\sqrt{1-\frac{k}{N}+\frac{\alpha}{N}} =& 1-\frac{k-1}{N}+\frac{\alpha-2}{2N} \\
&-\frac{\alpha^2}{8N^2}\left(1-\frac{k-1}{N}\right)^{-1}+O(N^{-3}).
\end{aligned} \tag{6.55}$$

Inserting the asymptotic expansions (6.55) and (6.54) into the three-term recurrence relation (6.53) gives

$$\begin{aligned}
&q_k\left(\frac{j_{\alpha,r}^2}{4N^2}+O(N^{-2})\right) \\
&=-q_{k-1}\left(1-\frac{k-1}{N}+\frac{\alpha-2}{2N}-\frac{\alpha^2}{8N^2}\left(1-\frac{k-1}{N}\right)^{-1}+O(N^{-3})\right) \\
&\quad+q_k\left(2\left(1-\frac{k}{N}\right)+\frac{\alpha-1}{N}\right) \\
&\quad-q_{k+1}\left(1-\frac{k+1}{N}+\frac{\alpha}{2N}-\frac{\alpha^2}{8N^2}\left(1-\frac{k+1}{N}\right)^{-1}+O(N^{-3})\right).
\end{aligned}$$

The above equation can be rearranged to

$$q_{k+1}\left(1-\frac{k+1}{N}\right)-2q_k\left(1-\frac{k}{N}\right)+q_{k-1}\left(1-\frac{k-1}{N}\right) \tag{6.56}$$

$$=\frac{\alpha-1}{N}q_k-\frac{\alpha-2}{2N}q_{k-1}-\frac{\alpha}{2N}q_{k+1} \tag{6.57}$$

$$+\frac{\alpha^2}{8N^2}\left(1-\frac{k-1}{N}\right)^{-1}q_{k-1}-\frac{j_{\alpha,r}^2}{4N^2}q_k+\frac{\alpha^2}{8N^2}\left(1-\frac{k+1}{N}\right)^{-1}q_{k+1} \tag{6.58}$$

$$+O(N^{-3})(q_{k-1}+q_{k+1}+q_{k-1}). \tag{6.59}$$

The left-hand side, line 6.56, is a second-order difference equation and thus telescope summation over $k=0,...,l-1$ gives

$$\begin{aligned}
&\sum_{k=0}^{l-1}\left(q_{k+1}\left(1-\frac{k+1}{N}\right)-2q_k\left(1-\frac{k}{N}\right)+q_{k-1}\left(1-\frac{k-1}{N}\right)\right)\\
=&\sum_{k=0}^{l-1}\left[q_{k+1}\left(1-\frac{k+1}{N}\right)-q_k\left(1-\frac{k}{N}\right)-\left[q_k\left(1-\frac{k}{N}\right)-q_{k-1}\left(1-\frac{k-1}{N}\right]\right)\right]\\
=&q_l\left(1-\frac{l}{N}\right)-q_{l-1}\left(1-\frac{l-1}{N}\right)-\left(q_0-\left(1-\frac{1}{N}\right)q_{-1}\right)\\
=&q_l\left(1-\frac{l}{N}\right)-q_{l-1}\left(1-\frac{l-1}{N}\right)-\frac{1}{\sqrt{N(N+\alpha)}}.
\end{aligned} \tag{6.60}$$

For the summation of the right hand side, it is useful to study some estimations of $q_k$ first. The Cauchy-Schwarz inequality, (6.54) and the orthogonality of (6.48) can be used to obtain

$$\sum_{k=0}^{N-1}q_k \leq \sqrt{\sum_{k=0}^{N-1}q_k^2}\sqrt{\sum_{k=0}^{N-1}1}=\sqrt{\frac{1}{z_r}}\sqrt{N}=O(N). \tag{6.61}$$

For a direct estimation of $q_k$ consider (6.54) and the orthogonality of (6.48) to obtain

$$\begin{aligned}
1&=z_r\sum_{k=0}^{N-1}q_k^2\\
\Rightarrow& z_rq_k^2\leq 1\\
\Rightarrow& q_k^2\leq CN \text{ for some } C>0\\
\Rightarrow& |q_k|\leq\sqrt{C}\sqrt{N} \text{ for some } C>0\\
\Rightarrow& q_k=O(\sqrt{N}).
\end{aligned} \tag{6.62}$$

The summation over the right-hand side will be split into three parts. Every line ((6.57), (6.58), (6.59)) is one part. Summation over $k = 0, ..., l$ of the second line (6.57) gives

$$
\begin{aligned}
&\frac{\alpha-1}{N}\sum_{k=0}^{l-1} q_k - \frac{\alpha-2}{2N}\sum_{k=0}^{l-1} q_{k-1} - \frac{\alpha}{2N}\sum_{k=0}^{l-1} q_{k+1} \\
=&\frac{\alpha-1}{N}\sum_{k=0}^{l-1} q_k - \frac{\alpha-2}{2N}\sum_{k=0}^{l-2} q_k - \frac{\alpha}{2N}\sum_{k=1}^{l} q_k \\
=&\sum_{k=0}^{l-1} q_k \underbrace{\left(\frac{\alpha-1}{N} - \frac{\alpha-2}{2N} - \frac{\alpha}{2N}\right)}_{=0} + \frac{\alpha-2}{2N} q_{l-1} - \frac{\alpha}{2N} q_l + \frac{\alpha}{2N} q_0 \\
=&\frac{\alpha-2}{2N} q_{l-1} - \frac{\alpha}{2N} q_l + \frac{\alpha}{2N}\frac{1}{\sqrt{N(N+\alpha)}} \\
=&\frac{\alpha-2}{2N} q_{l-1} - \frac{\alpha}{2N} q_l + O(N^{-2}).
\end{aligned}
\tag{6.63}
$$

The summation over $k = 0, ..., l-1$ of the third line (6.58) gives

$$
\begin{aligned}
&\frac{\alpha^2}{8N^2}\sum_{k=0}^{l-1} \frac{q_{k-1}}{1-\frac{k-1}{N}} - \frac{j^2_{\alpha,r}}{4N^2}\sum_{k=0}^{l-1} q_k + \frac{\alpha^2}{8N^2}\sum_{k=0}^{l-1} \frac{q_{k+1}}{(1-\frac{k+1}{N})} \\
=&\frac{\alpha^2}{8N^2}\sum_{k=0}^{l-2} \frac{q_k}{1-\frac{k}{N}} - \frac{j^2_{\alpha,r}}{4N^2}\sum_{k=0}^{l-1} q_k + \frac{\alpha^2}{8N^2}\sum_{k=1}^{l} \frac{q_k}{(1-\frac{k}{N})} \\
=&\frac{1}{N^2}\sum_{k=0}^{l-1} \frac{q_k}{4}\left(\frac{\alpha^2}{1-\frac{k}{N}} - j^2_{\alpha,r}\right) - \frac{\alpha^2}{8N^2}\left(\frac{q_{l-1}}{1-\frac{l-1}{N}} + \frac{q_l}{1-\frac{l}{N}} - q_0\right) \\
=&\frac{1}{N^2}\sum_{k=0}^{l-1} \frac{q_k}{4}\left(\frac{\alpha^2}{1-\frac{k}{N}} - j^2_{\alpha,r}\right) + O(N^{-\frac{3}{2}}).
\end{aligned}
\tag{6.64}
$$

The summation over $k = 0, ..., l-1$ of the fourth line (6.59) gives

$$
O(N^{-3})\sum_{k=0}^{l-1}\big(q_{k-1} + q_k + q_{k+1}\big) = O(N^{-2}). \tag{6.65}
$$

Note that the notation is difficult here (and in (6.59)), because Landau symbols can generally not be factored out. The calculation above should be understood in the way that (6.61) is applied to every term.

The combination of (6.60), (6.63), (6.64), (6.65) and $q_0 = \frac{1}{\sqrt{N(N+\alpha)}}$ yields

$$
\begin{aligned}
&q_l\left(1-\frac{l}{N}\right) - q_{l-1}\left(1-\frac{l-1}{N}\right) - \frac{1}{\sqrt{N(N+\alpha)}} \\
=&\ \frac{\alpha-2}{2N} q_{l-1} - \frac{\alpha}{2N} q_l + \frac{1}{N^2}\sum_{k=0}^{l-1}\frac{q_k}{4}\left(\frac{\alpha^2}{1-\frac{k}{N}} - j^2_{\alpha,r}\right) + O(N^{-\frac{3}{2}}).
\end{aligned}
\tag{6.66}
$$

A second summation of (6.66) over $l = 0, ..., m$ leads to the left-hand side

$$\begin{aligned}&\sum_{l=0}^{m}\left[q_l\left(1-\frac{l}{N}\right)-q_{l-1}\left(1-\frac{l-1}{N}\right)-\frac{1}{\sqrt{N(N+\alpha)}}\right]\\&=q_m\left(1-\frac{m}{N}\right)-\frac{m+1}{N}+O(N^{-1})\end{aligned}\tag{6.67}$$

and to the right-hand side

$$\begin{aligned}&\frac{\alpha-2}{2N}\sum_{l=0}^{m}q_{l-1}-\frac{\alpha}{2N}\sum_{l=0}^{m}q_l+\frac{1}{N^2}\sum_{l=0}^{m}\sum_{k=0}^{l-1}\frac{q_k}{4}\left(\frac{\alpha^2}{1-\frac{k}{N}}-j_{\alpha,r}^2\right)+O(N^{-\frac{1}{2}})\\=&\frac{1}{N^2}\sum_{l=0}^{m-1}\sum_{k=0}^{l}\frac{q_k}{4}\left(\frac{\alpha^2}{1-\frac{k}{N}}-j_{\alpha,r}^2\right)-\frac{1}{N}\sum_{l=0}^{m}q_l-\frac{\alpha-2}{2N}q_m+O(N^{-\frac{1}{2}})\\=&\frac{1}{N^2}\sum_{l=0}^{m-1}\sum_{k=0}^{l}\frac{q_k}{4}\left(\frac{\alpha^2}{1-\frac{k}{N}}-j_{\alpha,r}^2\right)-\frac{1}{N}\sum_{l=0}^{m}q_l+O(N^{-\frac{1}{2}}).\end{aligned}\tag{6.68}$$

The next step is to rewrite (6.67) and (6.68) into an integral equation for $f_N(y)$. For fixed $y \in (0,1)$ consider the sequence of integers $m = m(y,N)$ such that

$$\frac{m(y,N)}{N} \le y < \frac{m(y,N)+1}{N}.$$

In the following the Landau symbol $O(N^{-j})$ will be used with the limit $N \to \infty$ and local uniform convergence with respect to $y$. With the help of (6.62) and (6.54) the following estimations can be made:

$$\begin{aligned}\frac{m+1}{N}&=y+\left(\frac{m+1}{N}-y\right)=y+O(N^{-1}),\\1-\frac{m}{N}&=1-y+y-\frac{m}{N}=1-y+O(N^{-1}),\\\frac{1}{N}\sum_{l=0}^{m}\sqrt{Nz_r}q_l&=\int_0^y f_N(s)ds+f_N(y)\left(\frac{m+1-yN}{N}\right)\\&=\int_0^y f_N(s)ds+\frac{f_N(y)}{N}O(1),\\\frac{1}{1-\frac{m}{N}}&=\frac{1}{1-y+O(N^{-1})}=\frac{1}{1-y}+O(N^{-1}).\end{aligned}\tag{6.69}$$

$$\sqrt{N}\sqrt{z_r}=\frac{j_{\alpha,r}}{2}+O(N^{-1})\tag{6.70}$$

If in Equation (6.15) the function

$$g_N(t) := \sum_{k=0}^{N-1} t_{k,N}\mathbf{1}_{\left[\frac{k}{N},\frac{k+1}{N}\right]}(t) \quad \text{with} \quad t_{k,N} := \sqrt{Nz_r}q_k\left(\frac{\alpha^2}{1-\frac{k}{N}}-j_{\alpha,r}^2\right)$$

is considered, the calculations after (6.15) are the same and result in

$$\begin{aligned}
&\frac{1}{N^2}\sum_{l=0}^{m-1}\sum_{k=0}^{l}\frac{\sqrt{Nz_r}q_k}{4}\left(\frac{\alpha^2}{1-\frac{k}{N}}-j_{\alpha,r}^2\right)-\int_0^y\int_0^s\frac{f_N(t)}{4}\left(\frac{\alpha^2}{1-t}-j_{\alpha,r}^2\right)dtds\\
&=\frac{1}{2}\left(\frac{yN-\lfloor yN\rfloor}{N}\right)^2\sqrt{Nz_r}q_{\lfloor yN\rfloor}\left(\frac{\alpha^2}{1-\frac{\lfloor yN\rfloor}{N}}-j_\alpha^2\right)\\
&+\frac{yN-\lfloor yN\rfloor-\frac{1}{2}}{N}\cdot\frac{1}{N}\sum_{k=0}^{\lfloor yN\rfloor-1}\sqrt{Nz_r}q_k\left(\frac{\alpha^2}{1-\frac{k}{N}}-j_\alpha^2\right)\\
&-\int_0^y\int_0^s\frac{f_N(t)}{4}\left(\frac{\alpha^2}{1-t}-\frac{\alpha^2}{1-\frac{\lfloor tN\rfloor}{N}}\right)dtds\\
&=O(N^{-1}).
\end{aligned}\tag{6.71}$$

For the estimation in the last line (6.61), (6.62) and the fact that convergence in $O(N^{-1})$ is locally uniform with respect to $y$ were used. It is now possible to rewrite (6.67) and (6.68) into an integral equation. Therefore multiply both sides, (6.67) and (6.68), with $\sqrt{N}\sqrt{z_r}$ and apply the error bounds (6.69) and (6.71). This gives

$$f_N(y)(1-y)-\frac{j_{\alpha,r}}{2}y=\int_0^y\int_0^s\frac{f_N(t)}{4}\left(\frac{\alpha^2}{1-t}-j_{\alpha,r}^2\right)dtds-\int_0^y f_N(t)dt+O(N^{-\frac{1}{2}}).\tag{6.72}$$

By partial integration it holds that

$$f_N(y)(1-y)-\frac{j_{\alpha,r}}{2}y=\int_0^y\left((y-t)\frac{f_N(t)}{4}\left(\frac{\alpha^2}{1-t}-j_{\alpha,r}^2\right)-f_N(t)\right)dt+O(N^{-\frac{1}{2}}).$$

Proceeding similarly to Lemma 6.4 and applying Gronwall's lemma then leads to

$$|f_N(y)-f(y)|=O(N^{-\frac{1}{2}}),\tag{6.73}$$

where $f$ is the solution of the integral equation

$$f(y)(1-y)-\frac{j_{\alpha,r}}{2}y=\int_0^y\left((y-t)\frac{f(t)}{4}\left(\frac{\alpha^2}{1-t}-j_{\alpha,r}^2\right)-f(t)\right)dt.\tag{6.74}$$

Note that now (6.62) can be improved to

$$q_k=O(1).$$

If the proof is repeated with this estimation, (6.73) can be improved to

$$|f_N(y)-f(y)|=O(N^{-1}).\tag{6.75}$$

The integral equation (6.74) corresponds to the differential equation

$$f''(y)(1-y)^2 - (1-y)f'(y) + f(y)\frac{1}{4}(j_{\alpha,r}^2(1-y) - \alpha^2) = 0$$

with the initial values $f(0) = 0$ and $f'(0) = \frac{j_{\alpha,r}}{2}$. From this point on we will follow [And21] again. By Lemma 3.1 of [And21] the solution of this differential equation is given by

$$f(y) = \frac{2J_\alpha(j_{\alpha,r}\sqrt{1-y})}{J_{\alpha+1}(j_{\alpha,r}) - J_{\alpha-1}(j_{\alpha,r})}.$$

Now, after applying the identity 10.6.1 from [Olv+10], this becomes

$$f(y) = -\frac{J_\alpha(j_{\alpha,r}\sqrt{1-y})}{J'_\alpha(j_{\alpha,r})},$$

which is exactly as given in (6.52). □

It is now possible to apply Theorem 6.13 introduced above to the $(N-r, N-r)$-entries of the covariance matrices $\Sigma_N$ for $\beta$-Laguerre ensembles in the freezing regime for $N \to \infty$, which are described in Theorem 5.11 and Corollary 5.14.

**Corollary 6.14.** *Consider the covariance matrices $\Sigma_N =: (\sigma_{i,j})_{i,j=1,\dots,N}$ of $\beta$-Laguerre ensembles in the freezing regime. Then for $r \in \mathbb{N}$ it holds that*

$$\lim_{N\to\infty} N\sigma_{N-r+1,N-r+1} = \int_0^1 \frac{(J_\alpha(j_{\alpha,r}\sqrt{1-y}))^2}{(J'_\alpha(j_{\alpha,r}))^2} dy.$$

*Proof.* This proof is analogous to that of Theorem 6.5 above. This corollary was also presented in [And21] as Theorem 1.3. Since the proof there left some questions about convergence open, it is presented in detail here, using the notations of Theorem 6.13. Because

$$T_N^T \Sigma_N T_N = \operatorname{diag}\left(\frac{1}{4}, \frac{1}{8}, \dots, \frac{1}{4N}\right),$$

the covariance $\sigma_{N-r+1,N-r+1}$ can be written as

$$\begin{aligned} N\sigma_{N-r+1,N-r+1} &= \sum_{k=1}^{N} \frac{1}{4k}\left(\sqrt{N}\sqrt{z_{r,N}^{(\alpha)}} Q_{k-1}^N(z_{r,N}^{(\alpha)})\right)^2 \\ &= \frac{1}{4N}\sum_{k=0}^{N-1} \frac{N}{k+1}(f_N(k/N))^2. \end{aligned}$$

This can be identified as integrals of step functions. According to Theorem 6.13, $f_N(k/N)$ is a step function converging to the solution $f$ of (6.52) and

$$h_N(y) := \frac{N}{(k+1)}\mathbb{1}_{[\frac{k}{N}, \frac{k+1}{N})}(y)$$

is a step function converging to $\frac{1}{y}$.
Similar to (6.28) there is

$$0 \leq \frac{1}{y} - h_N(y) \leq \frac{N}{k(k+1)} \leq \frac{1}{y}\frac{1}{k} \quad \text{for} \quad k = \lfloor yN \rfloor, \ 0 < y < 1.$$

Because $f_N$ converges locally uniformly, for every $x \in [0,1)$ there is a neighborhood $V_x$ of $x$ and a constant $C_x > 0$ such that for all $y \in V_x$ it holds that

$$|f_N(y)| \leq C_x(1 + f(y)).$$

Thus for every compact set $[0, M] \subseteq [0, 1)$ there is a $C_M > 0$ such that

$$|f_N(y)| \leq C_M(1 + f(y)).$$

Therefore, with (6.75), it holds that

$$\begin{aligned} |f_N(y)^2 h_N(y)| &\leq |f_N(y)^2 - f(y)^2| h_N(y) + f(y)^2 |h_N(y)| \\ &\leq (f_N(y) + f(y))|f_N(y) - f(y)| N^{-1} + \frac{f(y)^2}{y} \\ &\leq (1 + 2f(y))O(1) + \frac{f(y)^2}{y}. \end{aligned}$$

Because $f$ is continuously differentiable, this gives an integrable upper bound for every compact set $[0, M] \subseteq [0, 1)$.
Next, an integrable upper bound for the interval $[M, 1)$ is found. Due to the orthogonality of (6.48), $\int_0^1 (f_N(y))^2 dy = 1$ holds. Moreover, $\int_0^1 (f(y))^2 dy = 1$ also holds. This can be seen with a simple transformation of the identity

$$\int_0^1 y(J_\alpha(j_{\alpha,r} y))^2 dy = \frac{1}{2}(J'_\alpha(j_{\alpha,r}))^2,$$

which comes from [Olv+10] 10.22.37.
Theorem 6.13 implies that the measures $f_N^2 d\lambda$ with Lebesgue densities $f_N^2$ converge vaguely to the measure $f d\lambda$ on $(0, 1)$. Since all measures are probability measures, they also converge weakly. Because $h_N$ is a bounded function on $(1, M)$, the same argument as in Theorem 6.5 yields an integrable upper bound on $(1, M)$. Now, by dominated convergence, it holds that

$$\lim_{N\to\infty} N\sigma_{N-r+1,N-r+1} = \lim_{N\to\infty} \int_0^1 f_N^2(y) h_N(y) dy = \int_0^1 \frac{(J_\alpha(j_{\alpha,r}\sqrt{1-y}))^2}{(J'_\alpha(j_{\alpha,r}))^2} dy.$$

□

## 6.4 Limit results for the largest eigenvalue in the Jacobi case

In this section the hard-edge statistics in the freezing Jacobi case will be discussed. This means that we analyze the limit behavior of the smallest entry of the vector in the freezing regime in Theorem 4.11 for $N \to \infty$. For fixed $\beta$ the behavior of the largest eigenvalue is widely studied in the literature, see for example [Joh08; DK08; For12; Jia13] and many more. This section can be seen as an extension to the case $\beta = \infty$.

The used methods are similar to those used in the previous subsections, especially to those seen in the Laguerre hard-edge case, i.e., the limit results for the smallest eigenvalue in the Laguerre case. We again use the ordered zeros $z_{1,N}^{(\alpha,\beta)} < ... < z_{N,N}^{(\alpha,\beta)}$ of the $N$-th Jacobi polynomial $J_N^{(\alpha,\beta)}$. Moreover, for each $N$, let $(Q_{k,N}^{(\alpha,\beta)})_{k=0,..,N-1}$ be the dual polynomials associated with $(J_k^{(\alpha,\beta)})_{k=0,...,N}$, normalized with

$$\begin{aligned}&\sqrt{1-(z_{i,N}^{(\alpha,\beta)})^2}Q_{k,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)}) = \sqrt{w_i^*}\tilde{Q}_{k,N}^J(z_{i,N}^{(\alpha,\beta)})\\ =&\sqrt{1-(z_{i,N}^{(\alpha,\beta)})^2}\frac{(2N+\alpha+\beta)\sqrt{2N+\alpha+\beta-1}}{\sqrt{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}}\tilde{Q}_{k,N}^J(z_{i,N}^{(\alpha)})\end{aligned}$$

where $\tilde{Q}_{k,N}^J$ are given as in Theorem 5.3. This means that the matrices

$$T_N := \left(\sqrt{1-(z_{i,N}^{(\alpha,\beta)})^2}Q_{j-1,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)})\right)_{i,j=1,...,N} \tag{6.76}$$

are orthogonal with

$$T_N^T \Sigma_N T_N = \operatorname{diag}\left(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_N}\right)$$

and

$$\lambda_k = 4k(2N+\alpha+\beta+1-k)$$

as in the proof of Theorem 5.11. Consider the abbreviations

$$\begin{aligned}h_N = h_{N,\alpha,\beta} &:= \frac{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}{(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)},\\ a_N = a_{N,\alpha,\beta} &:= \frac{\sqrt{4N(N+\alpha+\beta)(N+\alpha)(N+\beta)}}{\sqrt{(2N+\alpha+\beta+1)(2N+\alpha+\beta-1)}(2N+\alpha+\beta)},\\ b_N = b_{N,\alpha,\beta} &:= -\frac{\alpha^2-\beta^2}{(2N+\alpha+\beta)(2N+\alpha+\beta+2)}.\end{aligned}$$

Then, for $k = 0, ..., N-1$, the three-term recurrence of $Q_{k,N}^{(\alpha,\beta)}$ is given by

$$xQ_{k,N}^{(\alpha)}(x) = a_{N-k}Q_{k-1,N}^{(\alpha)}(x) + b_{N-k-1}Q_{k,N}^{(\alpha)} + a_{N-k-1}Q_{k+1,N}^{(\alpha)}(x) \tag{6.77}$$

with the initial conditions $Q^{(\alpha)}_{-1,N} = 0$ and $Q^{(\alpha)}_{0,N} = \frac{1}{\sqrt{h_{N,\alpha,\beta}}}$. The limit theorem for $N \to \infty$ that will be derived in this subsection involves the Bessel function $J_\alpha$ of parameter $\alpha \in \mathbb{C}$. Some well-known facts about $J_\alpha$ are summarized in Subsection 6.3. Let

$$0 < j_{\alpha,1} < j_{\alpha,2} < \dots.$$

be the positive zeros of $J_\alpha$. In analogy with Theorem 6.1, the following result can be stated:

**Theorem 6.15.** *Let $\alpha, \beta > -1$, $N, r \in \mathbb{N}$ and for $y \in [0,1)$ define*

$$f_N(y) := \sum_{k=0}^{N-1} \sqrt{N}\sqrt{1-(z^{(\alpha,\beta)}_{N-r+1,N})^2}\, Q^{(\alpha,\beta)}_{k,N}(z^{(\alpha,\beta)}_{N-r+1,N})\mathbb{1}_{[\frac{k}{N},\frac{k+1}{N})}(y). \tag{6.78}$$

*Then for $N \to \infty$ the sequence $(f_N)_{N\geq 1}$ tends locally uniformly to the solution $f$ of the differential equation*

$$f''(y)(1-y)^2 + f(y)\left(j^2_{\alpha,r}(1-y)^2 - \alpha^2 + \frac{1}{4}\right) = 0 \tag{6.79}$$

*with the initial values $f(0) = 0$ and $f'(0) = \sqrt{2} j_{\alpha,r}$.*

*Proof.* The main idea of this proof is to rewrite the three-term recursion (6.77) into an ordinary differential equation (ODE) of second order and apply Gronwall's lemma. To avoid issues with ill-defined derivatives and other technicalities, the ODE will be written as an integral equation. Let $y \in (0,1)$ and $0 \leq k \leq \lfloor yN \rfloor$. The following abbreviations will be used:

$$\begin{aligned} q_k &:= Q^{(\alpha,\beta)}_{k,N}(z^{(\alpha,\beta)}_{N-r+1,N}), \\ z_r &:= z^{(\alpha,\beta)}_{N-r+1,N}. \end{aligned}$$

The next step ist to find the Taylor expansion of $a_{N-k}, a_{N-k-1}, b_{N-k-1}, z_r$ from (6.77). The zeros

$$\vartheta_{1,N} < \vartheta_{2,N} < \dots$$

of

$$P^{(\alpha,\beta)}_N(\cos(\vartheta))$$

are studied for example in [Gat94]. The following result from there will be useful: Let

$$\nu = \sqrt{\left(N + \frac{\alpha+\beta+1}{2}\right)^2 + \frac{1-\alpha^2-3\beta^2}{12}} = N + O(1),$$

then by (1.9) of [Gat94] and for $N \to \infty$ it holds that

$$\vartheta_{r,N} = \frac{j_{\alpha,r}}{\nu} + O(N^{-5}).$$

Therefore, with the Taylor expansion of cos, it holds that

$$\begin{aligned}
z_{N-r+1,N}^{(\alpha,\beta)} = \cos(\vartheta_{r,N}) &= 1 - \frac{\vartheta_{r,N}^2}{2} + O(N^{-4}) \\
&= 1 - \frac{j_{\alpha,r}^2}{2\nu^2} + O(N^{-4}) \\
&= 1 - \frac{j_{\alpha,r}^2}{2N^2} + O(N^{-3}) \\
\Rightarrow 1 - (z_{N-r+1,N}^{(\alpha,\beta)})^2 &= \frac{j_{\alpha,r}^2}{N^2} + O(N^{-3}) \\
\Rightarrow \sqrt{1 - z_r^2} = \sqrt{1 - (z_{N-r+1,N}^{(\alpha,\beta)})^2} &= \frac{j_{\alpha,r}}{N} + O(N^{-2}).
\end{aligned} \tag{6.80}$$

For $0 \leq k \leq \lfloor yN \rfloor$ it holds that

$$1 \geq 1 - \frac{k}{N} \geq 1 - y \Rightarrow \frac{1}{1 - \frac{k}{N}} = O(1).$$

To analyze the asymptotic behavior of

$$\begin{aligned}
a_{N-k} &= \frac{\sqrt{4(N-k)(N-k+\alpha+\beta)(N-k+\alpha)(N-k+\beta)}}{\sqrt{(2(N-k)+\alpha+\beta+1)(2(N-k)+\alpha+\beta-1)}(2(N-k)+\alpha+\beta)} \\
&= \frac{1}{2} \frac{\sqrt{1-\frac{k}{N}}\sqrt{1-\frac{k}{N}+\frac{\alpha+\beta}{N}}\sqrt{1-\frac{k}{N}+\frac{\alpha}{N}}\sqrt{1-\frac{k}{N}+\frac{\beta}{N}}}{\sqrt{1-\frac{k}{N}+\frac{\alpha+\beta+1}{2N}}\sqrt{1-\frac{k}{N}+\frac{\alpha+\beta-1}{2N}}\left(1-\frac{k}{N}+\frac{\alpha+\beta}{2N}\right)},
\end{aligned}$$

some Taylor expansions will be used. For any $x \in \mathbb{R}$ it holds that

$$\begin{aligned}
\sqrt{1 - \frac{k}{N} + \frac{x}{N}} &= \sqrt{1 - \frac{k}{N}} + \frac{x}{2N\sqrt{1-\frac{k}{N}}} - \frac{x^2}{8N^2\left(1-\frac{k}{N}\right)^{\frac{3}{2}}} + O(N^{-3}) \\
\sqrt{\left(1 - \frac{k}{N}\right) + \frac{x}{2N}} &= \sqrt{1 - \frac{k}{N}} + \frac{x}{4N\sqrt{1-\frac{k}{N}}} - \frac{x^2}{32\left(1-\frac{k}{N}\right)^{\frac{3}{2}}} + O(N^{-3}).
\end{aligned} \tag{6.81}$$

For $x, y \in \mathbb{R}$ this yields

$$\begin{aligned}
&\sqrt{1-\frac{k}{N}+\frac{x}{N}}\sqrt{1-\frac{k}{N}+\frac{y}{N}} \\
=&1-\frac{k}{N}+\frac{x+y}{2N}+\frac{xy}{4N^2\left(1-\frac{k}{N}\right)}-\frac{x^2+y^2}{8N^2\left(1-\frac{k}{N}\right)}+O(N^{-3}) \qquad (6.82)\\
=&1-\frac{k}{N}+\frac{x+y}{2N}-\frac{(x-y)^2}{8N^2\left(1-\frac{k}{N}\right)}+O(N^{-3}).
\end{aligned}$$

Furthermore, by the geometric series and Taylor's formula (6.81), for any $x \in \mathbb{R}$ the following holds true:

$$\frac{1}{\sqrt{\left(1-\frac{k}{N}\right)+\frac{x}{2N}}}=\frac{1}{\sqrt{1-\frac{k}{N}}}-\frac{x}{4N\left(1-\frac{k}{N}\right)^{\frac{3}{2}}}+\frac{3x^2}{32N^2\left(1-\frac{k}{N}\right)^{\frac{5}{2}}}+O(N^{-3}) \qquad (6.83)$$

$$\frac{1}{1-\frac{k}{N}+\frac{x}{2N}}=\frac{1}{1-\frac{k}{N}}-\frac{x}{2N\left(1-\frac{k}{N}\right)^2}+\frac{x^2}{4N^2\left(1-\frac{k}{N}\right)^3}+O(N^{-3}). \qquad (6.84)$$

For $x, y \in \mathbb{R}$ this yields

$$\begin{aligned}
&\frac{1}{\sqrt{\left(1-\frac{k}{N}\right)+\frac{x}{2N}}}\frac{1}{\sqrt{\left(1-\frac{k}{N}\right)+\frac{y}{2N}}} \\
=&\frac{1}{1-\frac{k}{N}}-\frac{x+y}{4N\left(1-\frac{k}{N}\right)^2}+\frac{3x^2+3y^2+2xy}{32N^2\left(1-\frac{k}{N}\right)^3}+O(N^{-3}) \qquad (6.85)\\
=&\frac{1}{1-\frac{k}{N}}-\frac{x+y}{4N\left(1-\frac{k}{N}\right)^2}+\frac{2(x^2+y^2)+(x+y)^2}{32N^2\left(1-\frac{k}{N}\right)^3}+O(N^{-3}).
\end{aligned}$$

Using (6.82) twice, for $x=\alpha, y=\beta$ and for $x=0, y=\alpha+\beta$, gives an approximation of the nominator of $a_{N-k}$. This is

$$\begin{aligned}
&\prod_{i\in\{0,\alpha,\beta,\alpha+\beta\}}\sqrt{1-\frac{k}{N}+\frac{i}{N}} \\
=&\sqrt{1-\frac{k}{N}}\sqrt{1-\frac{k}{N}+\frac{\alpha+\beta}{N}}\sqrt{1-\frac{k}{N}+\frac{\alpha}{N}}\sqrt{1-\frac{k}{N}+\frac{\beta}{N}} \\
=&\left(1-\frac{k}{N}+\frac{\alpha+\beta}{2N}-\frac{(\alpha+\beta)^2}{8N^2\left(1-\frac{k}{N}\right)}+O(N^{-3})\right) \\
&\cdot\left(1-\frac{k}{N}+\frac{\alpha+\beta}{2N}-\frac{(\alpha-\beta)^2}{8N^2\left(1-\frac{k}{N}\right)}+O(N^{-3})\right) \\
=&\left(1-\frac{k}{N}\right)^2+\frac{\alpha+\beta}{N}\left(1-\frac{k}{N}\right)+\frac{\alpha\beta}{2N^2}+O(N^{-3}). \qquad (6.86)
\end{aligned}$$

For an estimation of the denominator, use (6.85) for $x=\alpha+\beta-1, y=\alpha+\beta+1$ to get

$$\begin{aligned}
&\frac{1}{\sqrt{\left(1-\frac{k}{N}\right)+\frac{\alpha+\beta-1}{2N}}}\frac{1}{\sqrt{\left(1-\frac{k}{N}\right)+\frac{\alpha+\beta+1}{2N}}}\\
=&\frac{1}{1-\frac{k}{N}}-\frac{\alpha+\beta}{2N\left(1-\frac{k}{N}\right)^2}\\
+&\frac{2(\alpha+\beta-1)^2+2(\alpha+\beta+1)^2+4(\alpha+\beta)^2}{32N^2\left(1-\frac{k}{N}\right)^3}+O(N^{-3})\\
=&\frac{1}{1-\frac{k}{N}}-\frac{\alpha+\beta}{2N\left(1-\frac{k}{N}\right)^2}+\frac{2(\alpha+\beta)^2+1}{8N^2\left(1-\frac{k}{N}\right)^3}+O(N^{-3})
\end{aligned}\tag{6.87}$$

and (6.84) for $x=\alpha+\beta$ to get

$$\frac{1}{1-\frac{k}{N}+\frac{\alpha+\beta}{2N}}=\frac{1}{1-\frac{k}{N}}-\frac{\alpha+\beta}{2N\left(1-\frac{k}{N}\right)^2}+\frac{(\alpha+\beta)^2}{4N^2\left(1-\frac{k}{N}\right)^3}+O(N^{-3}).\tag{6.88}$$

The combination of (6.87) and (6.88) gives an approximation for the denominator of $a_{N-k}$, which is

$$\begin{aligned}
&\frac{1}{\sqrt{\left(1-\frac{k}{N}\right)+\frac{\alpha+\beta-1}{2N}}}\frac{1}{\sqrt{\left(1-\frac{k}{N}\right)+\frac{\alpha+\beta+1}{2N}}}\frac{1}{1-\frac{k}{N}+\frac{\alpha+\beta}{2N}}\\
=&\frac{1}{\left(1-\frac{k}{N}\right)^2}-\frac{\alpha+\beta}{N\left(1-\frac{k}{N}\right)^3}+\frac{2(\alpha+\beta)^2+1+2(\alpha+\beta)^2+2(\alpha+\beta)^2}{8N^2\left(1-\frac{k}{N}\right)^4}+O(N^{-3})\\
=&\frac{1}{\left(1-\frac{k}{N}\right)^2}-\frac{\alpha+\beta}{N\left(1-\frac{k}{N}\right)^3}+\frac{6(\alpha+\beta)^2+1}{8N^2\left(1-\frac{k}{N}\right)^4}+O(N^{-3}).
\end{aligned}\tag{6.89}$$

Combining (6.86) and (6.89) yields the follwing for $a_{N-k}$:

$$\begin{aligned}
2a_{N-k}=&\left(\left(1-\frac{k}{N}\right)^2+\frac{\alpha+\beta}{N}\left(1-\frac{k}{N}\right)+\frac{\alpha\beta}{2N^2}\right)\\
&\cdot\left(\frac{1}{\left(1-\frac{k}{N}\right)^2}-\frac{\alpha+\beta}{N\left(1-\frac{k}{N}\right)^3}+\frac{6(\alpha+\beta)^2+1}{8N^2\left(1-\frac{k}{N}\right)^4}\right)+O(N^{-3})\\
=&1+0\frac{1}{\left(1-\frac{k}{N}\right)}+\frac{6(\alpha+\beta)^2+1-8(\alpha+\beta)^2+4\alpha\beta}{8N^2\left(1-\frac{k}{N}\right)^2}+O(N^{-3})\\
=&1-\frac{2(\alpha^2+\beta^2)-1}{8N^2\left(1-\frac{k}{N}\right)^2}+O(N^{-3}).
\end{aligned}\tag{6.90}$$

Thus for $a_{N-k-1}$ it holds that

$$2a_{N-k-1}=1-\frac{2(\alpha^2+\beta^2)-1}{8N^2\left(1-\frac{k+1}{N}\right)^2}+O(N^{-3}).\tag{6.91}$$

The asymptotic behavior of $b_{N-k-1}$ can be described by using (6.84) for $x = \alpha + \beta$ and for $x = \alpha + \beta - 2$. Here only the first-order term is necessary. This yields

$$2b_{N-k-1} = -\frac{\alpha^2 - \beta^2}{2N^2} \frac{1}{\left(1 - \frac{k}{N}\right)^2} + O(N^{-3}). \tag{6.92}$$

Combining (6.80),(6.90),(6.91) and (6.92) yields that the three-term recurrence relation (6.77) at $x = z_r$ is given by

$$\begin{aligned} 2\left(1 - \frac{j_{\alpha,r}^2}{2N^2} + O(N^{-3})\right) q_k = {} & q_{k-1}\left(1 - \frac{2(\alpha^2+\beta^2)-1}{8N^2\left(1-\frac{k+1}{N}\right)^2} + O(N^{-3})\right) \\ & - q_k\left(\frac{\alpha^2-\beta^2}{2N^2}\frac{1}{\left(1-\frac{k}{N}\right)^2} + O(N^{-3})\right) \\ & + q_{k+1}\left(1 - \frac{2(\alpha^2+\beta^2)-1}{8N^2\left(1-\frac{k+1}{N}\right)^2} + O(N^{-3})\right). \end{aligned}$$

Rearranging the terms in the equation above leads to

$$\begin{aligned} q_{k+1} - 2q_k + q_{k-1} = {} & \frac{q_{k-1}}{8N^2}\frac{2(\alpha^2+\beta^2)-1}{\left(1-\frac{k}{N}\right)^2} + \frac{q_k}{2N^2}\frac{\alpha^2-\beta^2}{\left(1-\frac{k}{N}\right)^2} \\ & + \frac{q_{k+1}}{8N^2}\frac{2(\alpha^2+\beta^2)-1}{\left(1-\frac{k+1}{N}\right)^2} - \frac{q_k}{N^2} j_{\alpha,r}^2 \\ & + O(N^{-3})q_{k-1} + O(N^{-3})q_k + O(N^{-3})q_k. \end{aligned} \tag{6.93}$$

The next step is to sum up (6.93) twice and transform it into an integral equation. Before this can be done, some estimations are necessary. The Cauchy-Schwarz inequality, (6.80) and the orthogonality of (6.76) can be used to obtain

$$\sum_{k=0}^{N-1} q_k \leq \sqrt{\sum_{k=0}^{N-1} q_k^2}\sqrt{\sum_{k=0}^{N-1} 1} = \frac{1}{\sqrt{1-z_r^2}}\sqrt{N} = O(N^{\frac{3}{2}}). \tag{6.94}$$

For a direct estimation of $q_k$ consider (6.80) and the orthogonality of (6.76) to obtain

$$\begin{aligned} 1 = {} & \sum_{i=1}^{N}(1-z_r^2)q_k^2 \\ \Rightarrow & (1-z_r^2)q_k^2 \leq 1 \\ \Rightarrow & q_k^2 \leq CN^2 \text{ for some } C > 0 \\ \Rightarrow & |q_k| \leq \sqrt{C}N \text{ for some } C > 0 \\ \Rightarrow & q_k = O(N). \end{aligned} \tag{6.95}$$

The estimation (6.94) yields

$$O(N^{-3})\sum_{k=0}^{l}\big(q_{k-1}+q_k+q_{k+1}\big)=O(N^{-\frac{3}{2}}),$$
$$O(N^{-3})\sum_{l=0}^{m-1}\sum_{k=0}^{l}\big(q_{k-1}+q_k+q_{k+1}\big)=O(N^{-\frac{1}{2}}). \tag{6.96}$$

Note that notation is difficult here (and in (6.93)), because Landau symbols can generally not be factored out. The calculation above should be understood in the way that (6.94) is applied to the absolute value of every term.
For $0\le m\le\lfloor yN\rfloor$, a double summation over $k=0,...,l$ and $l=0,...,m-1$ afterwards yields the following left-hand side in (6.93):

$$\begin{aligned}\sum_{l=0}^{m-1}\sum_{k=0}^{l}(q_{k+1}-2q_k+q_{k-1})&=\sum_{l=0}^{m-1}((q_{l+1}-q_l)-(q_0-q_{-1}))\\&=q_m-(m+1)q_0\\&=q_m-(m+1)\frac{\sqrt{2}}{\sqrt{N}}+O(1).\end{aligned}\tag{6.97}$$

Before the right-hand side is summarized, note that by (6.84), it holds that

$$\frac{1}{\big(1-\frac{k+1}{N}\big)^2}=\frac{1}{\big(1-\frac{k}{N}\big)^2}+O(N^{-1})$$

and therefore for $0\le l\le\lfloor yN\rfloor$:

$$\begin{aligned}&\sum_{k=0}^{l}\frac{q_{k-1}}{8N^2}\frac{2(\alpha^2+\beta^2)-1}{\big(1-\frac{k}{N}\big)^2}\\&=\sum_{k=0}^{l}\left(\frac{q_{k-1}}{8N^2}\frac{2(\alpha^2+\beta^2)-1}{\big(1-\frac{k-1}{N}\big)^2}+O(N^{-2})\right)\\&=\sum_{k=0}^{l}\left(\frac{q_k}{8N^2}\frac{2(\alpha^2+\beta^2)-1}{\big(1-\frac{k}{N}\big)^2}\right)-\frac{q_l}{8N^2}\frac{2(\alpha^2+\beta^2)-1}{\big(1-\frac{l}{N}\big)^2}+O(N^{-1})\\&=\sum_{k=0}^{l}\left(\frac{q_k}{8N^2}\frac{2(\alpha^2+\beta^2)-1}{\big(1-\frac{k}{N}\big)^2}\right)+O(N^{-1}).\end{aligned}\tag{6.98}$$

Furthermore it holds that

$$\begin{aligned}
&\sum_{k=0}^{l} \frac{q_{k+1}}{8N^2} \frac{2(\alpha^2+\beta^2)-1}{\left(1-\frac{k+1}{N}\right)^2} \\
=&\sum_{k=0}^{l} \frac{q_k}{8N^2} \frac{2(\alpha^2+\beta^2)-1}{\left(1-\frac{k}{N}\right)^2} + \frac{q_{l+1}}{8N^2} \frac{2(\alpha^2+\beta^2)-1}{\left(1-\frac{k+1}{N}\right)^2} - \frac{q_0}{8N^2} \frac{2(\alpha^2+\beta^2)-1}{\left(1-\frac{0}{N}\right)^2} \\
=&\sum_{k=0}^{l} \frac{q_k}{8N^2} \frac{2(\alpha^2+\beta^2)-1}{\left(1-\frac{k}{N}\right)^2} + O(N^{-1}).
\end{aligned} \tag{6.99}$$

With (6.96),(6.98) and (6.99) in mind, the double summation over $k = 0, .., l$ and $l = 0, ..., m-1$ afterwards of (6.93) is given by

$$\begin{aligned}
&\sum_{l=0}^{m-1} \sum_{k=0}^{l} \frac{q_k}{8N^2} \frac{2(\alpha^2+\beta^2)-1}{\left(1-\frac{k}{N}\right)^2} + \sum_{l=0}^{m-1} \sum_{k=0}^{l} \frac{q_k}{2} N^2 \frac{\alpha^2-\beta^2}{\left(1-\frac{k}{N}\right)^2} \\
&+ \sum_{l=0}^{m-1} \sum_{k=0}^{l} \frac{q_k}{8N^2} \frac{2(\alpha^2+\beta^2)-1}{\left(1-\frac{k}{N}\right)^2} - \sum_{l=0}^{m-1} \sum_{k=0}^{l} \frac{q_k}{N^2} j_{\alpha,r}^2 + O(1) \\
&= \sum_{l=0}^{m-1} \sum_{k=0}^{l} \frac{q_k}{N^2} \left( \frac{\alpha^2-\frac{1}{4}}{\left(1-\frac{k}{N}\right)^2} - j_{\alpha,r}^2 \right) + O(1).
\end{aligned} \tag{6.100}$$

To align the three-term recurrence relation (6.93) with (6.78), i.e., arranging the appearing terms to have the same form as the values of the step function $f_N$, the left-hand side (6.97) and the right-hand side (6.100) are multiplied with

$$\sqrt{N}\sqrt{1-z_r^2} = \frac{j_{\alpha,r}}{\sqrt{N}} + O(N^{-\frac{3}{2}}),$$

compare (6.80). This results in

$$\begin{aligned}
&\sqrt{N}\sqrt{1-z_r^2} q_m - j_{\alpha,r}\sqrt{2}\frac{m+1}{N} + O(N^{-\frac{1}{2}}) \\
&= \sum_{l=0}^{m-1} \sum_{k=0}^{l} \frac{\sqrt{N}\sqrt{1-z_r^2} q_k}{N^2} \left( \frac{\alpha^2-\frac{1}{4}}{\left(1-\frac{k}{N}\right)^2} - j_{\alpha,r}^2 \right).
\end{aligned} \tag{6.101}$$

The next step is to rewrite (6.101) into an integral equation for $f_N(y)$. For fixed $y \in (0,1)$ consider the sequence of integers $m = m(y, N)$ such that

$$\frac{m(y,N)}{N} \leq y < \frac{m(y,N)+1}{N}.$$

In the following the Landau symbol $O(N^{-j})$ will be used with the limit $N \to \infty$ and local uniform convergence with respect to $y$. With the help of (6.95) the following estimations can be made:

$$\begin{aligned}
\frac{m+1}{N} &= y + \left(\frac{m+1}{N} - y\right) = y + O(N^{-1}), \\
1 - \frac{m}{N} &= 1 - y + y - \frac{m}{N} = 1 - y + O(N^{-1}), \\
\frac{1}{1-\frac{m}{N}} &= \frac{1}{1-y+O(N^{-1})} = \frac{1}{1-y} + O(N^{-1}).
\end{aligned} \tag{6.102}$$

If in Equation (6.15) the function

$$g_N(t) := \sum_{k=0}^{N-1} t_{k,N} \mathbf{1}_{\left[\frac{k}{N}, \frac{k+1}{N}\right]}(t)$$

with

$$t_{k,N} := \frac{\sqrt{N}\sqrt{1-z_r^2}q_k}{N^2} \left( \frac{\alpha^2 - \frac{1}{4}}{\left(1-\frac{k}{N}\right)^2} - j_{\alpha,r}^2 \right)$$

is considered, the calculations after (6.15) are the same and result in

$$\begin{aligned}
&\frac{1}{N^2} \sum_{l=0}^{m-1} \sum_{k=0}^{l} \frac{\sqrt{N}\sqrt{1-z_r^2}q_k}{N^2} \left( \frac{\alpha^2 - \frac{1}{4}}{\left(1-\frac{k}{N}\right)^2} - j_{\alpha,r}^2 \right) \\
&= \int_0^y \int_0^s f_N(t) \left( \frac{\alpha^2 - \frac{1}{4}}{(1-t)^2} - j_{\alpha,r}^2 \right) dt ds + O(N^{-1}).
\end{aligned} \tag{6.103}$$

With the error bounds (6.102) and (6.103), Equation (6.101) becomes the integral equation

$$f_N(y) - \sqrt{2} j_{\alpha,r} y = \int_0^y \int_0^s f_N(t) \left( \frac{\alpha^2 - \frac{1}{4}}{(1-t)^2} - j_{\alpha,r}^2 \right) dt ds + O(N^{-\frac{1}{2}}). \tag{6.104}$$

Proceeding similarly to Lemma 6.4 and applying Gronwall's lemma then leads to

$$|f_N(y) - f(y)| = O(N^{-\frac{1}{2}}), \tag{6.105}$$

where $f$ is the solution of the integral equation

$$f(y) - \sqrt{2} j_{\alpha,r} y = \int_0^y \int_0^s f(t) \left( \frac{\alpha^2 - \frac{1}{4}}{(1-t)^2} - j_{\alpha,r}^2 \right) dt ds. \tag{6.106}$$

Note that now (6.95) can be improved to

$$q_k = O(\sqrt{N}).$$

If the proof is repeated with this estimation, (6.105) can be improved to

$$|f_N(y) - f(y)| = O(N^{-1}). \tag{6.107}$$

The integral equation (6.106) corresponds to the differential equation

$$f''(y)(1-y)^2 + f(y)\left(j_{\alpha,r}^2(1-y)^2 - \alpha^2 + \frac{1}{4}\right) = 0 \tag{6.108}$$

with the initial values $f(0) = 0$ and $f'(0) = \sqrt{2} j_{\alpha,r}$. □

**Remark 6.16.**

- The shape of (6.108) looks similar to Bessel's differential equation (6.51). However, the term containing the first derivative is missing, and instead there is a shift of $\frac{1}{4}$. For the Laguerre hard-edge, the differential equation of corresponding function $f$ from Theorem 6.13 could be transformed into Bessel's differential equation. It is not clear whether (6.108) can be transformed into Bessel's differential equation.

- The calculations in the proof of Theorem 6.15 above are quite long and susceptible to calculation errors. It makes sense to additionally verify Theorem 6.15 numerically. Furthermore, by (5.6.3) in [Sze75], the Hermite polynomials $H_n$ can be written as the limit of Jacobi polynomials $P_n^{(\alpha,\alpha)}$,

  $$\frac{H_n(x)}{n!} = \lim_{\alpha\to\infty} \alpha^{-\frac{n}{2}} P_n^{(\alpha,\alpha)}\left(\frac{x}{\sqrt{\alpha}}\right),$$

  and by (5.3.4) of [Sze75] the Laguerre polynomials $L_n^{(\alpha)}$ can be written as the limit

  $$L_n^{(\alpha)} = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right).$$

  It might be interesting to investigate what happens to Theorem 6.15 and the following Theorem 6.17 when using these limits.

- For parameters $\mu, k$, consider Whittaker's equation

  $$z^2 w''(z) = \left(\frac{1}{4}z^2 + kz - \left(\mu^2 - \frac{1}{4}\right)\right) w(z),$$

  see for example [WW21] Chapter 16. It looks promising to transform (6.79), the differential equation for $f$, into Whittaker's equation. Afterwards an investigation of the conjecture

  $$\int_0^1 (f(y))^2 dy = 1 \tag{6.109}$$

  can be made. If the conjecture is true, it can be used to prove Theorem 6.17 introduced below.

Given the conjecture (6.109), it is possible to apply Theorem 6.15 introduced above to the $(N-r, N-r)$-entries of the covariance matrices $\Sigma_N$ for $\beta$-Jacobi ensembles in the freezing regime for $N \to \infty$, which are described in Theorem 5.11 and Corollary 5.15.

**Theorem 6.17.** *Consider the covariance matrices $\Sigma_N =: (\sigma_{i,j})_{i,j=1,...,N}$ of $\beta$-Jacobi ensembles in the freezing regime and let $f$ be the solution of* (6.79). *Then for $r \in \mathbb{N}$*

$$\lim_{N\to\infty} N\sigma_{N-r+1,N-r+1} = \frac{1}{4}\int_0^1 \frac{1}{y(2-y)} f(y)^2 dy$$

*Proof.* This proof is analogous to that of Theorem 6.5 above. The notations of Theorem 6.15 will be used here. Using the notations

$$\begin{aligned}
\lambda_k &= 4k(2N+\alpha+\beta+1-k),\\
T_N &= \left(\sqrt{1-(z_{i,N}^{(\alpha,\beta)})^2}Q_{j-1,N}^{(\alpha,\beta)}(z_{i,N}^{(\alpha,\beta)})\right)_{i,j=1,...,N},\\
\Sigma_N &= (\sigma_{i,j})_{i,j=1,...,N},
\end{aligned}$$

the statement

$$T_N^T\Sigma_N T_N = \operatorname{diag}\left(\frac{1}{\lambda_1},\frac{1}{\lambda_2},...,\frac{1}{\lambda_N}\right)$$

applies and the covariance $\sigma_{N-r+1,N-r+1}$ can be written as

$$\begin{aligned}
N^2\sigma_{N-r+1,N-r+1} &= \sum_{k=1}^{N}\frac{N}{\lambda_k}\left(\sqrt{N}\sqrt{1-\left(z_{N-r+1}^{(\alpha,\beta)}\right)^2}Q_{k-1}^N(z_{N-r+1}^{(\alpha,\beta)})\right)^2\\
&= \frac{1}{N}\sum_{k=0}^{N-1}\frac{N^2}{\lambda_{k+1}}\left(f_N\left(\frac{k}{N}\right)\right)^2\\
&= \frac{1}{N}\sum_{k=0}^{N-1}\frac{N^2}{4(k+1)(2N+\alpha+\beta-k)}\left(f_N\left(\frac{k}{N}\right)\right)^2.
\end{aligned}$$

This can be identified as integrals of step functions. According to Theorem 6.15, $f_N(k/N)$ is a step function converging to the solution $f$ of (6.79) and

$$\begin{aligned}
h_N(y) :&= \frac{N^2}{4(k+1)(2N+\alpha+\beta-k)}\mathbb{1}_{[\frac{k}{N},\frac{k+1}{N})}(y)\\
&= \frac{1}{4\frac{k+1}{N}\left(2+\frac{\alpha+\beta}{N}-\frac{k}{N}\right)}\mathbb{1}_{[\frac{k}{N},\frac{k+1}{N})}(y)
\end{aligned}$$

is a step function converging to

$$\frac{1}{4y(2-y)}.$$

Because $\frac{k}{N} \le y < \frac{k+1}{N}$ and $0 \le k < N$, the following estimations can be made:

$$\begin{aligned}
|h_N(y)| &= \frac{1}{4\frac{k+1}{N}\left(2+\frac{\alpha+\beta}{N}-\frac{k}{N}\right)} \le \frac{N}{4},\\
|h_N(y)| &= \frac{1}{4\frac{k+1}{N}\left(2+\frac{\alpha+\beta}{N}-\frac{k}{N}\right)} \le \frac{1}{4y}.
\end{aligned}$$

Because $f_N$ converges locally uniformly, for every $x \in [0,1)$ there is a neighborhood $V_x$ of $x$ and a constant $C_x > 0$ such that for all $y \in V_x$ it holds that

$$|f_N(y)| \leq C_x(1+f(y)).$$

Thus for every compact set $[0,M] \subseteq [0,1)$ there is a $C_M > 0$ such that

$$|f_N(y)| \leq C_M(1+f(y)).$$

Therefore, with (6.107), it holds that

$$\begin{aligned} |f_N(y)^2 h_N(y)| &\leq |f_N(y)^2 - f(y)^2| h_N(y) + f(y)^2 |h_N(y)| \\ &\leq (f_N(y)+f(y))|f_N(y)-f(y)|N^{-1} + \frac{f(y)^2}{y} \\ &\leq (1+2f(y))O(1) + \frac{f(y)^2}{y}. \end{aligned}$$

Because $f$ is continuously differentiable, this gives an integrable upper bound for every compact set $[0,M] \subseteq [0,1)$. On the interval $[M,1)$ the function $h_N$ is bounded and because of the conjecture (6.109), the measures with Lebesgue densities $f_N^2$ converge not only vaguely but weakly to the measure with Lebesgue density $f^2$. Thus an integrable upper bound is given and the above integrals converge. This leads to

$$\lim_{N\to\infty} N^2 \sigma_{N-r+1,N-r+1} = \frac{1}{4}\int_0^1 \frac{1}{y(2-y)} f(y)^2 dy.$$

□

# 7 List of symbols

| | |
|---|---|
| tr | trace of a matrix |
| det | determinant of a matrix |
| $I_n$ | $n \times n$-dimensional identity matrix |
| $\perp$ | orthogonal complement |
| $H_f(x)$ | Hessian of a function $f$ at the point $x$ |
| $\mathcal{H}_n(\mathbb{R})$ | Hilbert space over $\mathbb{R}$ of $n \times n$- dimensional symmetric matrices |
| $\mathcal{H}_n(\mathbb{C})$ | Hilbert space over $\mathbb{R}$ of $n \times n$- dimensional symmetric matrices |
| $C^n(\Omega)$ | $n$-times continuously differentiable functions $f : \Omega \to \mathbb{R}$ |
| $C_b(\Omega)$ | bounded functions $f : \Omega \to \mathbb{R}$ |
| $C_c(\Omega)$ | functions $f : \Omega \to \mathbb{R}$ with a compact support |
| $\mathbb{N}$ | $\{1, 2, ....\}$ |
| $\mathbb{N}_0$ | $\{0, 1, 2, ....\}$ |
| $f \sim g$ | $f, g$ functions with $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = 1$ |
| $X \sim \mu$ | $X$ random variable, $\mu$ measure with $P_X = \mu$ |
| $O(f(x))$ | $g(x) = O(f(x))$ for $x \to x_0$ $:\Leftrightarrow \exists C, \delta > 0 : \lvert x - x_0 \rvert < \delta \Rightarrow \left\lvert \frac{g(x)}{f(x)} \right\rvert \leq C$ |
| $\mathsf{Ai}$ | Airy function |
| $J_\alpha$ | Bessel function |
| $\Gamma$ | Gamma function |
| $H_n$ | Hermite polynomials |
| $L_n^{(\alpha)}$ | Laguerre polynomials |
| $J_n^{(\alpha,\beta)}$ | Jacobi polynomials |
| $z_{i,N}^{(\alpha)}$ | $i$-th zero of the $N$-th Laguerre polynomial |
| $z_{i,N}^{(\alpha,\beta)}$ | $i$-th zero of the $N$-th Jacobi polynomial |
| $\hat{P}_n$ | monic orthogonal polynomials |
| $\tilde{P}_n$ | orthonormal polynomials |
| $Q_{k,N}$ | dual polynomials, Section 5 |
| $\hat{Q}_{k,N}$ | monic dual polynomials, Section 5 |
| $\tilde{Q}_{k,N}$ | orthonormal dual polynomials, Section 5 |

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