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INVITED REVIEW

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Freezing limits for Calogero–Moser–Sutherland particle models

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Abstract

One-dimensional interacting particle models of Calogero–Moser–Sutherland type with N particles can be regarded as diffusion processes on suitable subsets of $\mathbb{R}^N$ like Weyl chambers and alcoves with second-order differential operators as generators of the transition semigroups, where these operators are singular on the boundaries of the state spaces. The most relevant examples are multivariate Bessel processes and Heckman–Opdam processes in a compact and noncompact setting where in all cases, these processes are related to special functions associated with root systems. More precisely, the transition probabilities can be described with the aid of multivariate Bessel functions, Jack and Heckman–Opdam Jacobi polynomials, and Heckman–Opdam hypergeometric functions, respectively. These models, in particular, form dynamic eigenvalue evolutions of the classical random matrix models like β-Hermite, β- Laguerre, and β-Jacobi, that is, MANOVA, ensembles. In particular, Dyson's Brownian motions and multivariate Jacobi processes are included. In all cases, the processes depend on so-called coupling parameters. We review several freezing limit theorems for these diffusions where, for fixed N, one or several of the coupling parameters tend to ∞. In many cases, the limits will be N-dimensional normal distributions and, in the process case, Gauss processes. However, in some cases, normal distributions on half spaces and distributions related to some other ensembles appear as limits. In all cases, the limits are connected with the zeros of the classical one-dimensional orthogonal polynomials of order N.

KEYWORDS
Dyson Brownian motion, freezing, Heckman–Opdam processes, β-Hermite ensembles, Jacobi ensembles, Jacobi processes, β-Laguerre ensembles, MANOVA ensembles, multivariate Bessel processes, time-dependent random matrix models, zeros of classical orthogonal polynomials

1 INTRODUCTION

In this survey, we present several freezing limit theorems for interacting particle models of Calogero–Moser–Sutherland type with N particles in some one-dimensional space I. After some linear transform, we may restrict our attention to $I = \mathbb{R}, [0,\infty[, [-1,1]$, and the torus $I = \mathbb{T}$. Moreover, we assume that the order of the particles in I remains fixed, that is, we use the state spaces

$$C := \{x \in I^N : \quad x_1 \geq x_2 \geq \ldots \geq x_N\} \qquad (I = \mathbb{R}, [0,\infty[, [-1,1]) \tag{1}$$

and

$$C := \{z = (z_1 := e^{ix_1}, \ldots, z_N := e^{ix_N}) : \quad x_1 \geq x_2 \geq \ldots \geq x_N \geq x_1 + 2\pi\} \qquad (I = \mathbb{T}) \tag{2}$$

for the time-homogeneous diffusions $(X_t)_{t\geq 0}$ of the particle models. In all cases, the generators of the transition semigroups are second-order differential operators that are singular on the boundary ∂C of C. These operators have the form

$$Lf := \frac{1}{2}\Delta f + Df \quad \text{with a first-order drift part} \quad Df = D_1 f + kD_2 f, \tag{3}$$

where Δ is the usual Laplacian, D_1, D_2 are fixed first-order drift operators (which are singular on ∂C), $f : \mathbb{R} \to \mathbb{C}$ is a C^2-function satisfying certain symmetry conditions regarding ∂C, and where $k > 0$ is some constant. The symmetry conditions on f correspond to the fact that we assume reflecting boundaries. In most situations, we have $D_1 \equiv 0$, but the case $D_1 \not\equiv 0$ may also appear.

In this survey, we discuss several limit results for $k \to \infty$. For this reason, we write the generators and processes as L_k and $(X_{t,k} := (X_{t,k}^1, \ldots, X_{t,k}^N))_{t\geq 0}$ to emphasize the dependence from k. For many concrete examples relevant in physics, the transition densities of the processes $(X_{t,k})_{t\geq 0}$ are explicitly known and have the form of partition functions of one-dimensional log-gases (see Refs. 5, 35, 44, 61, 65, 67) where the parameter $k > 0$ carries the meaning of an inverse temperature in the Boltzmann stationary case, that is, the limit $k \to \infty$ corresponds to a freezing limit.

These freezing limits will be derived by two different approaches.

In the first approach, we write the transition probabilities of the processes $(X_{t,k})_{t\geq 0}$ in terms of special functions associated with root systems, which are eigenfunctions of L_k. In particular, for multivariate Bessel processes, the transition probabilities can be written in a simple way in terms of the associated Bessel functions. Even nicer, if one starts in these cases in $0 \in C \subset \mathbb{R}^N$, then no Bessel functions appear, and the distributions of the X_t^k for the root systems of types A and B are simply the eigenvalue distributions of the classical β-Hermite and β-Laguerre ensembles.

The situation is more complicated in the Heckman–Opdam case. Here, in the compact cases of types A and BC, the diffusions have the alcoves $C \subset \mathbb{T}^N$ from (2) and $C \subset [-1,1]^N$ from (1) with $I = [-1,1]$, respectively, as state spaces, and the transition densities can be described in terms of series w.r.t. Jack polynomials and Heckman–Opdam Jacobi polynomials, respectively. Similarly, in the noncompact case, the transition densities are integrals w.r.t. Heckman–Opdam hypergeometric functions. Due to these complicated representations of the transition densities, this direct approach is of limited interest only in view of freezing limits.

Anyway, in all these cases, one can try to use limits for the special functions for $k \to \infty$ in order to derive limit theorems for the distributions of the $X_{t,k}$. For instance, it is known that in some cases, the special functions tend to exponential functions, and in some cases for types B and BC, to the corresponding functions of type A (up to some transformed arguments). In fact, this approach works in several cases; see Section 3.

The second approach to limit theorems for $k \to \infty$ is based on the fact that for a start in the interior of C and under additional restrictions (like that all parameters are large enough), $(X_{t,k})_{t\geq 0}$ is the unique solution of the stochastic differential equation (SDE) associated with the generators L_k, where the processes never meet the singular boundary. In several cases, where the starting points depend on k, SDE techniques then lead to limit theorems; see Section 5 for more details. Due to the different starting conditions, these SDE-based results are different from those of the first approach. Moreover, the SDE approach leads to stronger limit results like almost sure convergence that is locally uniform w.r.t. t. The basic idea here is the observation that $\frac{1}{\sqrt{k}}L_k$ is the generator of $(X_{t/\sqrt{k},k})_{t\geq 0}$, and that one can expect that, for deterministic starting points, $(X_{t/\sqrt{k},k})_{t\geq 0}$ tends for $k \to \infty$ to the deterministic solution $(X_{t,\infty})_{t\geq 0}$ of the ordinary differential equation (ODE)

$$\frac{d}{dt}X_{t,\infty} = \big(g_1(X_{t,\infty}), \dots, g_N(X_{t,\infty})\big) \tag{4}$$

when the drift part D_2 in (3) has the form $D_2 f(x) = \sum_{j=1}^N g_j(x)\frac{\partial}{\partial x_i} f(x)$ on the interior of C.

To illustrate both approaches, we now briefly discuss the simplest example, namely, the Bessel processes of type A and β-Hermite ensembles. Here, for $k > 0$, the generators L_k have the form

$$L_k f := \frac{1}{2}\Delta f + k\sum_{i=1}^N \left(\sum_{j\neq i} \frac{1}{x_i - x_j}\right)\frac{\partial}{\partial x_i} f \tag{5}$$

for

$$f \in D(L_k) := \{f \in C^2(\mathbb{R}^N), \quad f \quad \text{invariant under all permutations of coordinates}\}.$$

Please notice that we here use a time normalization that fits to that of classical Brownian motions as usual in probability. The $(X_t^k)_{t\geq 0}$ then are just Dyson's Brownian motions (see Ref. 38 and textbooks like Ref. 3) with the Weyl chambers

$$C_N^A := \{x \in \mathbb{R}^N : \quad x_1 \geq x_2 \geq \ldots \geq x_N\}$$

of type A as state spaces. By Ref. 75 (see also the surveys[10,77,80,81]), the associated transition probabilities are given for $t > 0$, $x \in C_N^A$, $A \subset C_N^A$ a Borel set, by

$$K_t^k(x, A) = c_k^A \int_A \frac{1}{t^{\gamma_A + N/2}} e^{-(\|x\|^2+\|y\|^2)/(2t)} J_k^A\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \cdot w_k^A(y)\, dy \tag{6}$$

with

$$w_k^A(x) := \prod_{i<j}(x_i - x_j)^{2k}, \qquad \gamma_A = kN(N-1)/2, \tag{7}$$

and

$$c_k^A := \left(\int_{C_N^A} e^{-\|y\|^2/2} \cdot \prod_{i<j}(y_i - y_j)^{2k}\, dy\right)^{-1} = \frac{N!}{(2\pi)^{N/2}} \cdot \prod_{j=1}^{N} \frac{\Gamma(1+k)}{\Gamma(1+jk)}; \tag{8}$$

see, for example, Ref. 63 for the constant. Moreover, J_k^A is a multivariate Bessel function of type A_{N-1} with multiplicity k; see, for example, Refs. 75, 77 and references there. We do not need much details about J_k^A here. We only recapitulate that J_k^A is analytic on $\mathbb{C}^N \times \mathbb{C}^N$ and invariant under all permutations in both arguments with $J_k^A(x, y) > 0$, $J_k^A(x, y) = J_k^A(y, x)$, and $J_k^A(0, y) = 1$ for $x, y \in \mathbb{R}^N$. For further properties, see Subsection 2.3. Therefore, if the process starts in $0 \in C_N^A$, then $X_{t,k}$ has the density

$$\frac{c_k^A}{t^{\gamma+N/2}} e^{-\|y\|^2/(2t)} \cdot w_k(y)\, dy \tag{9}$$

on C_N^A for $t > 0$. In particular, for $k = 1/2, 1, 2$, the distributions of the ordered eigenvalues of Gaussian orthogonal, unitary, and symplectic ensembles (see, e.g., Ref. 24) appear in this way. Furthermore, the Bessel processes of type A with generators (5) and transition probabilities (6) describe the time evolutions of the eigenvalues of Brownian motions on the vector spaces $\mathbb{H}_N(\mathbb{F})$ of all Hermitian $N \times N$-matrices over $\mathbb{R}, \mathbb{C}$ and the quaternions with real dimensions $d = 1, 2, 4$ for $k = d/2 = 1/2, 1, 2$.

Moreover, for general $k > 0$, the distributions (9) belong to the β-Hermite ensembles that are the eigenvalue distributions of well-known tridiagonal random matrix models of Dumitriu and Edelman.[34] It is an interesting task whether the Bessel processes $(X_{t,k})_{t\geq 0}$ for general $k > 0$ also admit such tridiagonal matrix representations. It seems that up to now this problem does not have a sufficiently nice solution that can be applied to freezing limits. For possible approaches in this direction, see Refs. 1, 53, 104.

We point out that also the Bessel processes of type B admit corresponding connections to random matrix models like β-Laguerre ensembles and to Wishart processes on the cones

$\mathbb{P}_N(\mathbb{F}) \subset \mathbb{H}_N(\mathbb{F})$ of all positive semidefinite matrices; see Section 3 and in particular Remark 4 for more details.

We next turn to the limit $k \to \infty$ in the first approach according to [4, 5, 7-9, 35, 43]. If we start in $0 \in C_N^A$, we write the densities of $X_{t,k}/\sqrt{tk}$ as

$$const.(k) \cdot \exp\left(k\left(2\sum_{i,j:i<j} \ln(y_i - y_j) - \|y\|^2/2\right)\right) =: const.(k) \cdot \exp\left(k \cdot W_A(y)\right). \tag{10}$$

By a classical result of Stieltjes (see Section 6.7 of Ref. 90), the exponent W_A has a unique maximum on C_N^A that appears for $y = \sqrt{2} \cdot \mathbf{z}$ where $\mathbf{z} \in C_N^A$ is the vector whose coordinates are ordered zeros of the classical Hermite polynomial H_N, where, as in Ref. 90, the polynomials $(H_N)_{N\geq 0}$ are orthogonal w.r.t. the density e^{-x^2}. More precisely, by Refs. 8, 97:

Lemma 1. *For $\mathbf{z} = (z_1, \dots, z_N) \in C_N^A$, the following statements are equivalent:*

(1) The function $x \mapsto \sum_{i,j:i<j} \ln(x_i - x_j) - \|x\|^2/2$ has in $\mathbf{z}$ its unique maximum in $\mathbf{z} \in C_N^A$.
(2) For $i = 1, \dots, N$, $z_i = \sum_{j:j\neq i} \frac{1}{z_i - z_j}$.
(3) $z_1 > \dots > z_N$ are the ordered zeros of H_N.

This characterization and a Taylor expansion of the logarithmic terms in W_A lead to the following central limit theorem (CLT); see Subsection 3.1 below for more details on the proof and Theorem 2.2 in Ref. 97 (please notice that the limit $N(0, t \cdot \Sigma_N)$ in Ref. 97 must be replaced by $N(0, \Sigma_N)$).

Theorem 1. *Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ on C_N^A for $k > 0$ with start in $0 \in C_N^A$. Then, for each $t > 0$,*

$$\frac{X_{t,k}}{\sqrt{t}} - \sqrt{2k} \cdot \mathbf{z}$$

converges in distribution for $k \to \infty$ to the centered N-dimensional normal distribution $N(0, \Sigma_N)$ with the regular covariance matrix Σ_N with $\Sigma_N^{-1} = S_N = (s_{i,j})_{i,j=1}^N$ and

$$s_{i,j} := \begin{cases} 1 + \sum_{l\neq i} (z_i - z_l)^{-2} & \text{for} \quad i = j \\ -(z_i - z_j)^{-2} & \text{for} \quad i \neq j \end{cases}. \tag{11}$$

The CLT 1 can be extended to arbitrary starting points $x \in C_N^A$ by using the densities (6) and an asymptotic result for the Bessel functions J_k^A. As the root systems A_{N-1} are not reduced on $\mathbb{R}^N$, the statement of this result needs some preparation. If we consider the vector $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^N$, then $\mathbb{R}^N$ decomposes into $\mathbb{R} \cdot \mathbf{1}$ and its orthogonal complement

$$\mathbf{1}^\perp = \left\{ x \in \mathbb{R}^N : \sum_i x_i = 0 \right\} \subset \mathbb{R}^N.$$

The associated Weyl group, that is, the symmetric group S_N, acts on both spaces separately. We denote the orthogonal projections from $\mathbb{R}^N$ onto $\mathbb{R}\cdot\mathbf{1}$ and $\mathbf{1}^\perp$ by $\pi_{\mathbf{1}}$ and $\pi_{\mathbf{1}^\perp}$, respectively. Then $\pi_{\mathbf{1}}(x)=\bar{x}\mathbf{1}$ for the center of gravity $\bar{x}:=\frac{1}{N}\sum_{i=1}^N x_i$ of the particles. This decomposition of $\mathbb{R}^N$ appears also on the level of the Bessel functions J_k^A; see, for example, Ref. 15.

Lemma 2. *For all $x,y\in\mathbb{R}^N$,*

$$J_k^A(x,y)=e^{\langle\pi_{\mathbf{1}}(x),\pi_{\mathbf{1}}(y)\rangle}\cdot J_k^A(\pi_{\mathbf{1}^\perp}(x),\pi_{\mathbf{1}^\perp}(y))=e^{N\bar{x}\bar{y}}\cdot J_k^A(\pi_{\mathbf{1}^\perp}(x),\pi_{\mathbf{1}^\perp}(y)). \tag{12}$$

In particular, the J_k^A are quite simple on the center-of-gravity part $\mathbb{R}\cdot\mathbf{1}$. Moreover, by (6), Bessel processes of type A can be decomposed into stochastically independent parts, namely, the center of gravity part and the part on $\mathbf{1}^\perp$, where the distances of the particles are encoded. On the other hand, the following limit appears on $\mathbf{1}^\perp$; see Corollary 8 of Ref. 7 or Theorem 2.5 in Ref. 9:

Theorem 2. *Locally uniformly, for $x,y\in\mathbf{1}^\perp$,*

$$\lim_{k\to\infty} J_k^A(\sqrt{2k}\cdot x,y)=\exp\left(\frac{\|x\|^2\|y\|^2}{N(N-1)}\right). \tag{13}$$

A combination of (13) and (12) with the ideas of the direct proof of the CLT 1 then leads to a CLT for arbitrary starting points $x\in C_N^A$. For this, one observes that by the transition densities (6), the kernels K_t^k of type A admit the same space-time-scaling as Brownian motions, that is, we can assume $t=1$ w.l.o.g. Moreover, by (12), the kernels K_t^k of type A are partially translation invariant, that is,

$$K_t^k(x+c\mathbf{1},S+c\mathbf{1})=K_t^k(x,S)\qquad \text{for } c\in\mathbb{R},\ ,t>0,\ x\in C_N^A,\ S\subset C_N^A. \tag{14}$$

Thus, again w.l.o.g., we assume $x\in\mathbf{1}^\perp$. Under these assumptions, Theorem 2 and a Taylor expansion of the logarithmic terms in W_A lead to the following CLT; see Ref. 9.

Theorem 3. *Consider the Bessel processes $(X_{t,k})_{t\ge0}$ of type A_{N-1} on C_N^A for $k\ge0$ with starting point $x\in C_N^A$. Then, for $t>0$,*

$$\frac{X_{t,k}}{\sqrt{t}}-\sqrt{2k}\cdot\mathbf{z}$$

converges for $k\to\infty$ to the N-dimensional normal distribution $N(\pi_{\mathbf{1}}(x/\sqrt{t}),\Sigma_N)$ with the covariance matrix Σ_N in Theorem 1.

More details on these results and in particular on the covariance matrices Σ_N and their inverses S_N from Refs. 4, 9, 35, 43 will be discussed in Sections 3 and 4. We point out that Theorem 2 with some explicit formulas for S_N was proved first by Dumitriu and Edelman[34,35] via their classical tridiagonal matrix models for β-Hermite ensembles.

In the next step, we review the SDE approach; for the background, see, for example, Refs. 69, 74. Having the generators (5) in mind, we now describe the Bessel processes $(X_{t,k})_{t\ge0}$ of type A_{N-1}

as solutions of the SDEs

$$dX^i_{t,k} = dB^i_t + k\sum_{j\neq i}\frac{1}{X^i_{t,k}-X^j_{t,k}}dt \qquad (i=1,\dots,N), \tag{15}$$

with an N-dimensional Brownian motion $(B^1_t,\dots,B^N_t)_{t\geq 0}$. For this, we recapitulate that initial value problems associated with (15) with start in the interior of C_N^A have unique strong solutions; see, for example, Section 4.3 of [3]. The theory developed in Ref. 22 implies that SDEs as in (15) even admit unique solutions for initial values on the singular boundaries; see, for example, Refs. 45, 88

We now assume $k \geq 1/2$. In this case, the Bessel processes $(X_{t,k})_{t\geq 0}$ never meet the boundary for $t > 0$ a.s.; see Ref. 25 or Section 4.3 of Ref. 3.

Clearly, by (15), the renormalized processes $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k})_{t\geq 0}$ are the unique solutions of

$$d\tilde{X}^i_{t,k} = \frac{1}{\sqrt{k}}dB^i_t + \sum_{j\neq i}\frac{1}{\tilde{X}^i_{t,k}-\tilde{X}^j_{t,k}}dt \qquad (i=1,\dots,N). \tag{16}$$

In the next step, we observe that in the limit $k=\infty$, the SDEs (16) degenerate into the ODEs

$$\frac{dx}{dt}(t) = H(x(t)), \quad \text{with} \quad H(x) := \left(\sum_{j\neq 1}\frac{1}{x_1-x_j},\dots,\sum_{j\neq N}\frac{1}{x_N-x_j}\right). \tag{17}$$

It is known that for all initial values $x(0) = x \in C_N^A$, these ODEs have unique solutions in C_N^A in the sense that $x(t)$ is in the interior of C_N^A and a solution of (17) for $t > 0$, where $t \to x(t)$ is continuous in $t = 0$; see Ref. 102. The ODE (17) is closely related to the vector $\mathbf{z}$ consisting of the zeros of H_N. In fact, Part (2) of Lemma 1 implies that (17) has the particular solutions

$$x(t) = \sqrt{2t+c^2}\cdot\mathbf{z} \qquad \text{for} \quad c\geq 0,\ t\geq 0, \tag{18}$$

which again shows the close connection between our Bessel processes and the zeros of H_N.

We now use the solutions $x(t)$ of (17) for arbitrary starting points in C_N^A in order to describe limit results for the processes $(\tilde{X}_{t,k})_{t\geq 0}$ for $k\to\infty$. We start with the following strong limit law, which follows from a comparison of (17) and (16); see Theorem 2.4 of Ref. 8.

Theorem 4. *Let x_0 be a point in the interior of C_N^A. For $k \geq 1$, let $(\tilde{X}_{t,k})_{t\geq 0}$ be a Bessel process of type A starting in $\sqrt{k}\cdot x_0$, and ϕ the solution of (17) starting in x_0. Then, for all $t > 0$,*

$$\sup_{0\leq s\leq t, k\geq 1}\|X_{s,k}-\sqrt{k}x(s)\| < \infty \qquad a.s. \tag{19}$$

In particular, locally uniformly in t a.s., $X_{t,k}/\sqrt{k}\to x(t)$ for $k\to\infty$.

We now turn to a functional central limit theorem that improves (19). We again fix x_0 in the interior of C_N^A and $x(t)$ as above. We consider the N-dimensional process $(W_t)_{t\geq 0}$ with $W_0 = 0$, which is the unique solution of the inhomogeneous linear SDE

$$dW_t^i = dB_t^i + \sum_{j\neq i} \frac{W_t^j - W_t^i}{(x_i(t) - x_j(t))^2} dt \qquad (i = 1, \dots, N), \tag{20}$$

that is, in matrix notation, $dW_t = dB_t + A(t)W_t dt$ with the matrices $A(t) \in \mathbb{R}^{N\times N}$ with

$$A(t)_{i,j} := \frac{1}{(x_i(t) - x_j(t))^2}, \quad A(t)_{i,i} := -\sum_{j\neq i} \frac{1}{(x_i(t) - x_j(t))^2} \quad (i, j = 1, \dots, N, \ i \neq j), \tag{21}$$

where the Brownian motion $(B_t)_{t\geq 0}$ from (16) is used. By the theory of linear SDEs, $(W_t)_{t\geq 0}$ is Gaussian and has the following form in terms of matrix-valued exponentials:

$$W_t = e^{\int_0^t A(s)ds} \int_0^t e^{-\int_0^s A(u)du} dB_s \quad (t \geq 0). \tag{22}$$

In particular, $(W_t)_{t\geq 0}$ is centered. We have the following functional CLT from.[100] Details of the proof will be given in Subsection 5.1 below.

Theorem 5. *Let x_0 be a point in the interior of C_N^A. For $k \geq 1$, consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type A_{N-1} starting at $\sqrt{k} \cdot x_0$ and let $x(t)$ as above. Then, for $t > 0$,*

$$\sup_{0\leq s\leq t, k\geq 1} \sqrt{k} \cdot \|X_{s,k} - \sqrt{k}x(s) - W_s\| < \infty \qquad a.s. \tag{23}$$

In particular, locally uniformly in t a.s., $X_{s,k} - \sqrt{k}x(s) - W_s \to 0$ for $k \to \infty$.

A weaker version of Theorem 5 can be found in Ref. 43.

Let us compare Theorem 5 with the static CLT 1, and consider the particular solution $x(t) = \sqrt{2t} \cdot \mathbf{z}$ of (17) as in (18) for $c = 0$. If we could apply Theorem 5 here (notice that we would start on the boundary!), we would obtain that $X_{s,k} - \sqrt{2tk} \cdot \mathbf{z}$ tends in distribution to W_t for $t > 0$ and $k \to \infty$. It can be see from (22) that this, in fact, fits perfectly with the CLT 1; see Ref. 100 and also Section 5 below. This continuity appears also in other connected situations and may be regarded as natural at a first glance. On the other hand, we shall see below that in some degenerate cases, such a continuity is not available; see, for instance, Remark 3.

Besides the Bessel processes of type A and the associated β-Hermite ensembles, we discuss in this survey Bessel processes of type B (and D) and the associated β-Laguerre ensembles as well as multivariate Jacobi processes on compact alcoves (which belong to Heckman–Opdam theory of type BC) and the associated β-Jacobi ensembles more closely. For the further classes of examples like Heckman–Opdam processes in the noncompact setting, we only present a few facts.

This survey is organized as follows. In Section 2, we recapitulate some facts on root systems and the associated special functions like Bessel functions and Heckman–Opdam hypergeometric functions. We there also introduce the associated diffusions. Section 3 is devoted to freezing limits, which can be derived via explicit formulas of the transition densities like Theorems 1 and 3 above. In these CLTs, there appear interesting covariance matrices and their inverses that are closely related to the classical one-dimensional orthogonal polynomials (Hermite, Laguerre, Jacobi) and their dual polynomials in the sense of de Boor and Saff; see Refs. 4, 43. We explain this connection briefly in Section 4. Finally, in Section 5, we show how SDEs lead to freezing limits in several situations like in Theorems 4 and 5 above.

2 | SPECIAL FUNCTIONS AND DIFFUSIONS ASSOCIATED WITH ROOT SYSTEMS

We present a short review on the special functions related to Calogero–Moser–Sutherland models, and we introduce the associated diffusions. Regarding multivariate special functions, we restrict our attention to facts that are needed for the transition probabilities and freezing.

We first recapitulate some basic notations on the three classical families of one-dimensional orthogonal polynomials, which will be needed in this survey. For details, see the monographs.[55,90]

2.1 | The classical one-dimensional orthogonal polynomials

As already mentioned above, the Hermite polynomials $(H_N)_{N\geq 0}$ are orthogonal w.r.t. the density e^{-x^2}.

Moreover, for $\alpha > -1$, the Laguerre polynomials $(L_N^{(\alpha)})_{N\geq 0}$ are orthogonal w.r.t. the density $e^{-x}x^\alpha$ on $]0,\infty[$. They have the explicit form

$$L_N^{(\alpha)}(x) = \sum_{j=0}^{N} \binom{N+\alpha}{N-j} \frac{(-x)^j}{j!} \qquad (N \in \mathbb{N}) \tag{24}$$

(see (5.1.6)[90]). By (24), we may form $L_N^{(-1)}$ for $N \geq 0$ where, by (5.2.1) of Ref. 90,

$$L_N^{(-1)}(x) = -\frac{x}{N} L_{N-1}^{(1)}(x) \qquad (N \geq 1). \tag{25}$$

If $z_1 > \dots > z_{N-1}$ are the $N-1$ zeros of $L_{N-1}^{(1)}$, we obtain the N zeros $z_1 > \dots > z_{N-1} > z_N := 0$ of $L_N^{(-1)}$. We shall need the polynomials $L_N^{(-1)}$ for some degenerate case in Subsections 3.3, 5.4, and 5.5 below.

We finally recapitulate that the Jacobi polynomials $(P_N^{(\alpha,\beta)})_{N\geq 0}$ with $\alpha, \beta > -1$ are orthogonal polynomials w.r.t. the weights $(1-x)^\alpha(1+x)^\beta$ on $]-1,1[$.

We next consider some special functions associated with root systems, namely, Bessel functions related to Dunkl theory and Heckman–Opdam hypergeometric functions. The material is taken from Refs. 10, 36, 37, 48, 49, 75, 80, 81 and references therein. As we are interested in concrete examples, we keep the algebraic background to a minimum. We start with root systems.

2.2 | Root systems and Weyl chambers

We equip $\mathbb{R}^N$ with the usual scalar product $\langle\, .\, ,\, .\, \rangle$, the associated 2-norm $\|.\|$, and the standard basis $e_1, \dots, e_N$. Let $R \subset \mathbb{R}^N \setminus \{0\}$ be a possibly not reduced root system on $\mathbb{R}^N$ with associated finite reflection group W, that is, R is a finite set of vectors α, such that the reflections σ_α on the hyperplanes orthogonal to α with

$$\sigma_\alpha(x) = x - \frac{2\langle \alpha, x\rangle}{\|\alpha\|^2}\alpha \qquad (x \in \mathbb{R}^N, \quad \alpha \in R)$$

generate a finite group W of orthogonal transformations on $\mathbb{R}^N$, where for each $\alpha \in R, \sigma_\alpha(R) = R$. We here only consider the following crystallographic root systems in $\mathbb{R}^N$:

$$\begin{aligned} A_{N-1} &= \{\pm(e_i - e_j);\ 1 \le i < j \le N\}, \\ B_N &= \{\pm e_i, \pm(e_i \pm e_j);\ 1 \le i, j \le N,\ i < j\}, \\ BC_N &= \{\pm e_i, \pm 2e_i, \pm(e_i \pm e_j);\ 1 \le i, j \le N,\ i < j\}, \\ D_N &= \{\pm(e_i \pm e_j) : \quad 1 \le i < j \le N\}. \end{aligned} \tag{26}$$

For $R = A_{N-1}$, the Weyl group W is just the symmetric group $\mathcal{S}_N$ permuting all coordinates, and for B_N and BC_N, the Weyl group W is the hyperoctahedral group, where in addition sign changes may appear in all coordinates.

We next fix some positive subsystem $R_+ \subset R$ where we only take the roots in (26) where instead of the first $\pm$-signs, only the $+$-signs appear. Moreover, we fix some multiplicity function $k : R \to [0, \infty[$, that is, a function that is invariant under the action of W on R. For the root systems A_{N-1} and D_N, we have a single parameter $k \in [0, \infty[$. Moreover, for B_N, k is described by two parameters (k_1, k_2) where k_1, k_2 are the values on the roots $\pm e_i, e_i \pm e_j$, respectively. Finally, for BC_N, k is described by three parameters (k_1, k_2, k_3) where k_1, k_2, k_3 are the values on the roots $\pm e_i, \pm 2e_i, e_i \pm e_j$, respectively.

We next fix a Weyl chamber, that is, a closed convex set $C_N \subset \mathbb{R}^N$ consisting of representatives of the orbits under the action of W on $\mathbb{R}^N$. In the case of our examples, we choose these chambers as

$$\begin{aligned} C_N^A &= \{x \in \mathbb{R}^N : \quad x_1 \ge x_2 \ge \dots \ge x_N\}, \\ C_N^B = C_N^{BC} &= \{x \in \mathbb{R}^N : \quad x_1 \ge x_2 \ge \dots \ge x_N \ge 0\}, \\ C_N^D &= \{x \in \mathbb{R}^N : \quad x_1 \ge \dots \ge x_{N-1} \ge |x_N|\}. \end{aligned} \tag{27}$$

We now fix R, R_+, W, k, and the associated Weyl chamber $C_N \subset \mathbb{R}^N$ and consider the associated Dunkl and Heckman–Opdam theories.

2.3 | Dunkl operators, Bessel functions, and Bessel processes

The Dunkl operators associated with R and k are defined as

$$T_\xi(k)f(x) = \partial_\xi f(x) + \frac{1}{2}\sum_{\alpha \in R} k(\alpha) \cdot \langle \alpha, \xi\rangle \frac{1}{\langle \alpha, x\rangle}(f(x) - f(\sigma_\alpha(x))) \tag{28}$$

for $\xi \in \mathbb{R}^N$ and directional derivatives ∂_ξ. The $T_\xi(k)$ form a commuting family of operators that are homogeneous of degree -1 on the space of all polynomials in N variables. Moreover, for each

$w \in \mathbb{C}^N$, the joint eigenvalue problem

$$T_\xi(k)f \;=\; \langle \xi, w\rangle f \quad (\xi \in \mathbb{C}^N) \quad \text{with} \qquad f(0) = 1$$

has a unique holomorphic solution $f(z) = E_k(z, w)$ that is called the Dunkl kernel. The kernel E_k is symmetric in z, w and satisfies

$$E_k(rz, w) = E_k(z, rw) \quad \text{for} \quad r \in \mathbb{C} \quad \text{and} \quad E_k(gz, gw) = E_k(z, w) \quad \text{for} \quad g \in W. \tag{29}$$

The Bessel function

$$J_k(z, w) := \frac{1}{|W|} \sum_{g \in W} E_k(z, gw)$$

then is W-invariant in both arguments, and for $w \in \mathbb{C}^N$, $g(z) := J_k(z, w)$ is the unique holomorphic solution of the "Bessel system"

$$p(T_{e_1}(k), \dots, T_{e_N}(k))g \;=\; p(w)g \quad \forall\, p \in \mathcal{P}^W \quad \text{with} \quad g(0) = 1, \tag{30}$$

where $\mathcal{P}^W$ is the algebra of W-invariant polynomials in N variables. Restricted to sufficiently smooth, W-invariant functions on $\mathbb{R}^N$, the operators $p(T_{e_1}(k), \dots, T_{e_N}(k))$ are differential operators. In particular, for $p(x) = x_1^2 + \dots + x_N^2$, one obtains the Dunkl–Laplacian Δ_k that can be written explicitly (see, e.g., Ref. 37) as

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \left(\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} - \frac{\|\alpha\|^2}{2} \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x\rangle^2} \right). \tag{31}$$

Note that for W-invariant functions, the reflection parts at the end of this formula disappear. These symmetric parts will be used in the next section for the root systems A_{N-1}, B_N, D_N as generators of transition semigroups of Feller diffusions on the associated closed Weyl chambers C_N.

We briefly summarize some properties of the Bessel functions.

Proposition 1. *For $x, y \in \mathbb{C}^N$, $\lambda \in \mathbb{C}$ and $w \in W$,*

(1) $J_k(x, y) = J_k(y, x)$.
(2) $J_k(\lambda x, y) = J_k(x, \lambda y)$ *and* $J_k(x, wy) = J_k(x, y)$.
(3) $\overline{J_k(x, y)} = J_k(\overline{x}, \overline{y})$.
(4) $J_k(x, y) > 0$ *and* $|J_k(x, iy)| \le 1$ *for* $x, y \in \mathbb{R}^N$.

Part (4) follows from the existence of an abstract positive integral representation of J_k in terms of exponential functions in Ref. 76. For the root systems A_{N-1}, there exist explicit recursive formulas for these integral representations that follow from such representations for Jack polynomials; see Refs. 2, 67, 86 for details. This leads to the following sharp estimate in Ref. 46.

Proposition 2. *On $C_N \times C_N$,*

$$J_k^A(x, y) \sim \frac{e^{\langle x, y\rangle}}{\left(\prod_{1 \le i < j \le N} (1 + (x_i - x_j)(y_i - y_j)) \right)^k},$$

where $f \sim g$ means that there exist constants $c_1, c_2 > 0$ (depending on k) with $c_1 \cdot f \leq g \leq c_2 \cdot g$.

There are conjectures about such estimates for general root systems in Ref. 46. However, up to some singular cases for type B in Ref. 78, no explicit integral representations are known; see also Refs. 82, 84 for related integral representations for type B. Moreover, related estimates for the heat kernels in the Dunkl setting can be found in Ref. 39.

We also remark that the estimate in Proposition 2 is an exact equality for $x \in \mathbb{R}\mathbf{1}$ or $y \in \mathbb{R}\mathbf{1}$ with $\mathbf{1} = (1, \dots, 1)$, whereas this estimate is not in accordance with Theorem 2 for $x, y \in \mathbf{1}^\perp$ and $k \to \infty$, that is, the dependence of the constants on k in Proposition 2 in essential.

We next turn to limits for the Bessel functions for large multiplicities, where these limits are needed for freezing limits below, and where these limits can be derived from explicit series representations of these functions in terms of Jack polynomials. We first recapitulate from[5]:

Theorem 6. *For $x, y \in \mathbb{R}^N$ and $k \geq 0$,*

$$J_k^A(x,y) = e^{\langle \pi_{\mathbf{1}}(x), \pi_{\mathbf{1}}(y) \rangle} \cdot J_k^A(\pi_{\mathbf{1}^\perp}(x), \pi_{\mathbf{1}^\perp}(y)) = e^{N\bar{x}\bar{y}} \cdot J_k^A(\pi_{\mathbf{1}^\perp}(x), \pi_{\mathbf{1}^\perp}(y)), \tag{32}$$

and locally uniformly, for $x, y \in \mathbf{1}^\perp$,

$$\lim_{k\to\infty} J_k^A(\sqrt{2k} \cdot x, y) = \exp\left(\frac{\|x\|^2 \|y\|^2}{N(N-1)} \right). \tag{33}$$

For the Bessel functions $J_{(k_1,k_2)}^B$ of type B_N, we have the following limits; see Refs. 6, 7, 9.

Theorem 7.

(1) *For all $x, y \in C_N^B$ and $\nu \geq 0$, locally uniformly,*

$$\lim_{\beta\to\infty} J_{(\nu\cdot\beta,\beta)}^B(\sqrt{\beta} \cdot x, y) = \exp\left(\frac{\|x\|^2 \|y\|^2}{4N(\nu + N - 1)} \right).$$

(2) *Let $k_1 > 0$. Then also locally uniformly for $x, y \in C_N^B$,*

$$\lim_{\beta\to\infty} J_{(k_1,\beta)}^B(\sqrt{\beta} \cdot x, y) = \exp\left(\frac{\|x\|^2 \|y\|^2}{4N(N-1)} \right).$$

(3) *Let $k_2 > 0$. Then,*

$$\lim_{k_1\to\infty} J_{(k_1,k_2)}^B\left(\sqrt{k_1} x, y\right) = J_{k_2}^A(x^2/2, y^2/2)$$

locally uniformly in $x, y \in C_N^B$ where $x^2 := (x_1^2, \dots, x_N^2) \in C_N^B$.

We remark that Theorem 7(3) was derived for $x, y \in i \cdot \mathbb{R}^N$ with precise estimates for the rate of convergence in Ref. 82. We expect that such explicit rates are also available for the limits in Theorems 6 and the further ones in Theorem 7 for $x, y \in i \cdot \mathbb{R}^N$.

We next return to the Dunkl–Laplacian in (31) and consider, up to a multiplication by 1/2 as usual in probability, its W-invariant part

$$L_k f(x) = \frac{1}{2}\Delta f(x) + \sum_{\alpha\in R_+} k(\alpha)\frac{\langle \nabla f(x), \alpha\rangle}{\langle \alpha, x\rangle} \tag{34}$$

on the associated Weyl chamber C_N where f is a Weyl-group invariant C^2-function on $\mathbb{R}^N$. It is well known (see Ref. 75 and also Refs. 80, 81) that L_k, in fact, is the generator of Feller diffusions $(X_{t,k})_{t\geq 0}$ on C_N with reflecting boundaries and with the transition probabilities

$$K_t^k(x, A) = c_k \int_A \frac{1}{t^{\gamma+N/2}} e^{-(\|x\|^2+\|y\|^2)/(2t)} J_k\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \cdot w_k(y)\, dy \tag{35}$$

for $t > 0$, $x \in C_N$, and $A \subset C_N$ a Borel set where the weight function

$$w_k(x) := \prod_{\alpha\in R} |\langle \alpha, x\rangle|^{k(\alpha)}, \tag{36}$$

is Weyl-group invariant on $\mathbb{R}^N$ and homogeneous of degree 2γ with $\gamma := \sum_{\alpha\in R_+} k(\alpha)$, and where $c_k > 0$ is a known suitable normalization. The processes $(X_{t,k})_{t\geq 0}$ are called (multivariate) Bessel processes (or also Dunkl–Bessel processes or invariant Dunkl processes in some references).

For the root system $R = A_{N-1}$, (34) and the transition data correspond to Equations (5)–(7) above. Moreover, for the root systems of types B and D, these data will be made explicit in the Subsections 3.2 and 3.3 below.

The theory of diffusion processes and the generators (34) suggests that the Bessel processes $(X_{t,k})_{t\geq 0}$ are the strong unique solutions of the SDEs

$$dX_{t,k}^i = dB_t^i + k\sum_{j\neq i} \frac{1}{X_{t,k}^i - X_{t,k}^j} dt \qquad (i = 1, \dots, N) \tag{37}$$

for given initial data with an N-dimensional Brownian motion $(B_t^1, \dots, B_t^N)_{t\geq 0}$. In fact, the theory developed in Ref. 22 implies that for all $k \geq 0$ and initial data in C_N (even on the boundary), initial value problems associated with (37) admit unique strong solutions with reflecting boundaries; see Ref. 45. We also remark that for $k \geq 1/2$, the Bessel processes $(X_{t,k})_{t\geq 0}$ never meet the boundary for $t > 0$ a.s.; see Ref. 25 or Section 4.3 of Ref. 3.

We collect some general facts about Bessel processes.

Proposition 3.

(1) For $y \in \mathbb{C}^N$, the functions $x \to J_k(x, y)$ are eigenfunction of L_k and thus of the transition operators e^{tL_k} with the eigenvalues $(x_1^2 + \cdots + x_N^2)/2$ and $e^{t(x_1^2+\cdots+x_N^2)/2}$, respectively.

(2) For each Bessel process $(X_{t,k})_{t\geq 0}$ and each $c > 0$, the process $(X_{\sqrt{c}\cdot t,k}/c)_{t\geq 0}$ is also a Bessel process.

(3) For each Bessel process $(X_{t,k})_{t\geq 0}$ with an arbitrary root system and $k \geq 0$, the process $(\|X_{t,k}\|)_{t\geq 0}$ on $[0, \infty[$ is a classical Bessel process, that is, of type B_1, where the parameter k_2 does not appear, and where $k_1 = \gamma + (N-1)/2$ holds.

In fact, part (1) is obvious by construction, (2) is clear by the explicit transition probabilities in (35), and for (3), we refer to Ref. 80 for an analytic proof. We mention that (2) and (3) also follow from the SDE (37); see, for example, Refs. 23, 46.

We next consider the theory of Heckman and Opdam; cf. Refs. 48, 49, 88.

2.4 | Heckman–Opdam hypergeometric functions and associated diffusions

The Cherednik operators associated with some positive root system R_+ and multiplicity $k \geq 0$ are

$$D_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha\in R_+} \frac{k(\alpha)\langle\alpha,\xi\rangle}{1-e^{-\langle\alpha,x\rangle}}(f(x) - f(\sigma_\alpha(x)) - \langle\rho(k),\xi\rangle f(x) \tag{38}$$

for $\xi \in \mathbb{R}^N$, with the weighted half-sum of positive roots $\rho(k) := \frac{1}{2}\sum_{\alpha\in R_+} k(\alpha)\alpha$. The $D_\xi(k)$, $\xi \in \mathbb{R}^N$, commute, and for each $\lambda \in \mathbb{C}^N$, there is a unique analytic function $G(\lambda, k; .)$ on some W-invariant tubular neighborhood of $\mathbb{R}^N$ in $\mathbb{C}^N$, the so-called Opdam–Cherednik kernel, which satisfies

$$G(\lambda, k; 0) = 1 \quad \text{and} \quad D_\xi(k)G(\lambda, k;\, .\,) = \langle\lambda,\xi\rangle\, G(\lambda, k;\, .\,) \quad \text{for all } \xi \in \mathbb{R}^N. \tag{39}$$

The hypergeometric function associated with R is defined by

$$F(\lambda, k; z) = \frac{1}{|W|}\sum_{w\in W} G\big(\lambda, k; w^{-1}z\big). \tag{40}$$

In contrast to G, the function F does not depend of the choice of R_+. The hypergeometric function has the following properties:

Proposition 4.

(1) For $z \in \mathbb{C}^N$, $F(-\rho(k), k; z) = 1$.
(2) For $z, \lambda \in \mathbb{R}^N$, $F(\lambda, k; z) > 0$.
(3) $|F(\lambda, k; z)| \leq 1$ for $z \in \mathbb{R}^N$ and $\lambda \in i\cdot\mathbb{R}^N \cup [0,1]\cdot\rho(k)$.

Part (3) can be considerably improved; we refer to Refs. 64, 79, 88 and references there. In particular, similar to the A_{N-1}-Bessel case, integral representations for $F(\lambda, k; z)$ in the case A_{N-1} (see Refs. 67, 86) lead to sharp estimates for $F(\lambda, k; z)$ that correspond to Proposition 2; see Ref. 47. Such integral representations are also available for some BC_N-cases; see Refs. 83, 87.

Moreover, Theorem 7(3) also has the following analog; see Theorem 5.1 of Ref. 79.

Theorem 8. *For $t \in \mathbb{R}^N$ and locally uniformly for $\lambda \in \mathbb{C}^N$,*

$$\lim_{\substack{k_1+k_2\to\infty \\ k_1/k_2\to\infty}} F_{BC}(\lambda + \rho_{BC}(k), k; t) = \prod_{i=1}^N \left(\cosh^2\frac{t_i}{2}\right)^{\pi_{\mathbf{1}}(\lambda)} \cdot F_A\Big(\pi_{\mathbf{1}^\perp}(\lambda) + \rho_A(k_3), k_3; \pi_{\mathbf{1}^\perp}\Big(\ln\cosh^2\frac{t}{2}\Big)\Big),$$

where all occurring functions are applied in all components.

It is an interesting question whether the other limits in Theorems 6 and 7 have also analogs for Heckman–Opdam hypergeometric functions.

We next consider the Heckman–Opdam Laplacian

$$\Delta_k f := \sum_{j=1}^{N} D_{\xi_j}(k)^2 f - \|\rho(k)\|^2 f \tag{41}$$

with some orthonormal basis $\xi_1, \dots, \xi_N$ of $\mathbb{R}^N$ where Δ_k is independent of the choice of this basis. By Ref. 88, the restriction L_k of Δ_k to W-invariant functions is the differential operator

$$L_k f(x) = \Delta f(x) + \sum_{\alpha \in R_+} k(\alpha) \coth\left(\frac{\langle \alpha, x\rangle}{2}\right) \cdot \partial_\alpha f(x). \tag{42}$$

L_k is the generator of a Feller diffusion on the associated closed Weyl chamber $C_N \subset \mathbb{R}^N$ with reflecting boundaries as in the Bessel cases. These diffusions are called noncompact Heckman–Opdam diffusions. Notice that by construction, for all $\lambda \in \mathbb{C}^n$, the functions $F(\lambda, k; .)$ are eigenfunctions of L_k with eigenvalues

$$\sum_{j=1}^{N} \langle \lambda, \xi_j\rangle^2 - \|\rho(k)\|^2.$$

Explicit formulas for the transition densities are more involved than in the Bessel case. We here follow Refs. 88, 89 and write these densities by using an integral representation in terms of the eigenfunctions $F(i\lambda, k; .)$ $(\lambda \in \mathbb{R}^N)$ and the associated Plancherel measure

$$d\nu_k(\lambda) = c_k \cdot \prod_{\alpha \in R_+} \frac{\Gamma\left(\langle \lambda, \alpha^\vee\rangle + k(\alpha) + \frac{1}{2}k(\alpha/2)\right) \cdot \Gamma\left(-\langle \lambda, \alpha^\vee\rangle + k(\alpha) + \frac{1}{2}k(\alpha/2)\right)}{\Gamma\left(\langle \lambda, \alpha^\vee\rangle + \frac{1}{2}k(\alpha/2)\right) \cdot \Gamma\left(-\langle \lambda, \alpha^\vee\rangle + \frac{1}{2}k(\alpha/2)\right)}\, d\lambda \tag{43}$$

with $\alpha^\vee := \frac{2\alpha}{\langle \alpha, \alpha\rangle}$ and some constant $c_k > 0$ where we agree that $k(\alpha/2) = 0$ for $\alpha/2 \notin R$. The transition probabilities of the Heckman–Opdam diffusions are then given by

$$K_t^k(x, A) = \int_A \left(\int_{iC_N} e^{-t(\|\lambda\|^2 + \|\rho(k)\|^2)/2} F(\lambda, k; x) F(\lambda, k; -y)\, d\nu_k(\lambda)\right) w_k(y)\, dy \tag{44}$$

with the weight function

$$w_k(x) := \prod_{\alpha \in R_+} \left|2\sinh\left(\frac{\langle \alpha, x\rangle}{2}\right)\right|^{2k(\alpha)}. \tag{45}$$

This representation of the transition probabilities via an inverse spherical Fourier transform is complicated and does not seem to be very useful for freezing limits. However, this representation and Theorem 8 can be used for a CLT for Heckman–Opdam processes of type BC with the distributions of Heckman–Opdam processes of type A as limits in special situations that are connected with noncompact Grassmann manifolds. The details are omitted here; we refer to Section 5 of Ref. 12. A similar and simpler result for Bessel processes will be given in Theorem 15 below.

We next turn to the compact Heckman–Opdam case where the associated special functions are trigonometric polynomials, the so-called Heckman–Opdam polynomials. To introduce them, we need the weight lattice and the set of dominant weights associated with R_+, that is,

$$P = \{\lambda \in \mathbb{R}^N : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\ \forall \alpha \in R\}, \quad P_+ = \{\lambda \in P : \langle \lambda, \alpha^\vee \rangle \geq 0\ \forall \alpha \in R_+\} \supset R_+,$$

where P_+ is equipped with the dominance order, that is, $\mu \leq \lambda$ means that $\lambda - \mu \in \text{span}_{\mathbb{Z}_+}(R_+)$. Let

$$\mathcal{T} := \text{span}_{\mathbb{C}}\{e^{i\lambda}, \lambda \in P\}$$

be the space of trigonometric polynomials associated with R. The orbit sums

$$M_\lambda(x) = \sum_{\mu \in W\lambda} e^{i\langle \mu, x\rangle}, \quad \lambda \in P_+$$

form a basis of the subspace $\mathcal{T}^W$ of W-invariant polynomials in $\mathcal{T}$. For $Q^\vee := \text{span}_{\mathbb{Z}}\{\alpha^\vee, \alpha \in R\}$, consider the compact torus $T = \mathbb{R}^N/2\pi Q^\vee$ with the weight function

$$\hat{w}_k(x) := \prod_{\alpha \in R_+} \left| 2 \sin\left(\frac{\langle \alpha, x\rangle}{2}\right)\right|^{2k(\alpha)}. \tag{46}$$

The Heckman–Opdam polynomials associated with R_+ and k are now defined by

$$P_\lambda(k; z) = M_\lambda(z) + \sum_{\nu < \lambda} c_{\lambda\nu}(k) M_\nu(z) \quad (\lambda \in P_+, z \in \mathbb{C}^N), \tag{47}$$

where the coefficients $c_{\lambda\nu}(k) \in \mathbb{R}$ are uniquely determined by the condition that $P_\lambda(k; .)$ is orthogonal to M_ν in $L^2(T, \hat{w}_k)$ for all $\nu \in P_+$ with $\nu < \lambda$. It is known that $\{P_\lambda(k, .), \lambda \in P_+\}$ is an orthogonal basis of the space $L^2(T, \hat{w}_k)^W$ of all W-invariant functions in $L^2(T, w_k)$. Then, by Ref. 49, the normalized polynomials

$$R_\lambda(k, z) := P_\lambda(k; z)/P_\lambda(k; 0)$$

can be expressed in terms of the hypergeometric function as

$$R_\lambda(k, z) = F(\lambda + \rho(k), k; iz). \tag{48}$$

Note that our notion slightly differs from Refs. 48, 49 where the P_λ are defined as exponential polynomials on the torus $i\mathbb{R}^N/2\pi i Q^\vee$. We now take the factor i in (48) into account and consider the "compact" Heckman–Opdam Laplacian

$$\hat{L}_k f(x) := \Delta f(x) + \sum_{\alpha \in R_+} k(\alpha) \cot\left(\frac{\langle \alpha, x\rangle}{2}\right) \cdot \partial_\alpha f(x). \tag{49}$$

$\widehat{L}_k$ generates a Feller diffusion on any fixed compact fundamental alcove $\mathcal{A} \subset \mathbb{R}^N$ associated with the affine Weyl group $W_{aff} = W \ltimes 2\pi Q^\vee$, where we again assume reflecting boundaries. The R_λ and P_λ have the following properties:

Proposition 5.

1. *For all $\lambda, \nu \in P_+$, the coefficients $c_{\lambda\nu}(k)$ in the representation (47) of P_λ are rational in $k(\alpha)$ and nonnegative for $k \geq 0$.*
2. *$|R_\lambda(x)| \leq R_\lambda(0) = 1$ for all $\lambda \in P_+$ and $x \in \mathbb{R}^N$.*
3. *R_λ is an eigenfunction of $\widehat{L}_k$ with eigenvalue $-\langle\lambda, \lambda + 2\rho(k)\rangle \leq 0$ for $\lambda \in P_+$.*

In fact, (1) is shown in Ref. 62, (1) implies (2), and (3) is clear by construction. We point out that the $c_{\lambda\nu}(k)$ as well as the usual L^2-norms of the polynomials P_λ and R_λ in $L^2(\mathcal{A}, \hat{w}_k)$ are explicitly known; see, in particular, Ref. 48 and also Section 2 of Ref. 71. The L^2-norms of the R_λ appear in the series representation of the transition densities of the Feller diffusions associated with $\widehat{L}_k$. More precisely, by Section 3 of Ref. 71, we have the following transition probabilities that are analog to the noncompact case in (44):

$$\hat{K}_t^k(x, A) = \int_A \sum_{\lambda \in P_+} \frac{1}{\|R_\lambda\|^2} e^{-\langle\lambda,\lambda+2\rho(k)\rangle t} R_\lambda(x) R_\lambda(-y) \cdot \hat{w}_k(y)\, dy \tag{50}$$

for $t > 0$, $x \in \mathcal{A}$, and a Borel set $A \subset \mathcal{A}$ where the series in the integral converges absolutely and uniformly. Moreover, the probability measure $\frac{1}{\int_{\mathcal{A}} \hat{w}_k(x)\, dx} \cdot \hat{w}_k(y) dy$ on $\mathcal{A}$ is the unique stationary measure associated with these transition probabilities, and the kernels in (50) converge uniformly for $t \to \infty$ to that of the stationary measure, that is, we in particular have weak convergence here.

All data of the preceding equations can be made explicit for the root systems of type A and BC. In particular, in the BC-case, the P_λ are multivariate Jacobi polynomials, the associated Feller diffusions are multivariate Jacobi processes as studied, for example, in Refs. 14, and the stationary measures belong to β-Jacobi ensembles after some transformation. For some discrete parameters, these Jacobi processes have an interpretation in terms of Brownian motions on compact Grassmann manifolds over $\mathbb{R}$, $\mathbb{C}$, and the quaternions $\mathbb{H}$. Moreover, like for MANOVA ensembles in the stationary case, they are related to Brownian motions on unitary groups over $\mathbb{R}, \mathbb{C}, \mathbb{H}$; see Ref. 33 for details. More details on Jacobi processes will be given in Subsections 3.4 and 5.8.

For the root system A_{N-1}, the P_λ are symmetric Jack polynomials. As this case involves particle processes on the torus, freezing limits here have a different form. For this reason, we skip this case in this paper. For some related results here, see, for example, Ref. 73.

We finally remark that the limit result (Theorem 8) can be specialized for the corresponding Heckman–Opdam polynomials R_λ. We refer to Ref. 60 and Theorem 4.2 of Ref. 79.

3 | FREEZING LIMITS VIA THE DISTRIBUTIONS OF THE PROCESSES

In this section, we derive freezing limits for Bessel processes of types A, B, and D as well as for the compact stationary BC-case, that is, for β-Jacobi ensembles, by using the explicit distributions.

We first turn to Bessel processes of type A that were already presented in the introduction.

3.1 | Freezing limits for Bessel processes of type A

Let $(X_{t,k})_{t\geq 0}$ be a Bessel process of type A starting in $0 \in C_N^A$. Then the densities of $X_{t,k}$ in (9) and (10) in combination with Lemma 1 on the unique maxima of these densities imply readily that

$$\lim_{k\to\infty} \frac{X_{t,k}}{\sqrt{2tk}} = \mathbf{z} \qquad (k \to \infty) \tag{51}$$

with the vector $\mathbf{z} = (z_1, \dots, z_N)$ consisting of the ordered zeros of H_N; cf. Ref. 5. This result may be seen as a first step to the CLT 1. We next turn to the proof of this CLT in Ref. 97 where w.l.o.g. we may assume $t = 1$. The random variables $X_{1,k} - \sqrt{2k} \cdot \mathbf{z}$ have the density

$$f_k^A(y) = c_k^A \cdot \exp\left(-\|y\|^2/2 - \sqrt{2k}\langle y, \mathbf{z}\rangle - k\|\mathbf{z}\|^2 + 2k\sum_{i<j} \ln\left(\sqrt{2k}(z_i - z_j)\right)\right) \tag{52}$$

$$\times \exp\left(2k \sum_{i<j} \ln\left(1 + \frac{y_i - y_j}{\sqrt{2k}(z_i - z_j)}\right)\right)$$

$$=: \tilde{c}_k^A \cdot h_k(y) \tag{53}$$

on the shifted cone $C_N^A - \sqrt{2k} \cdot \mathbf{z}$ with $f_k^A(y) = 0$ otherwise on $\mathbb{R}^N$ where by the Taylor formula for $\ln(1 + x)$, and by Lemma 1(2),

$$h_k(y) = \exp\left(-\|y\|^2/2 - \sqrt{2k}\langle y, \mathbf{z}\rangle + 2k\sum_{i<j} \ln\left(1 + \frac{y_i - y_j}{\sqrt{2k}(z_i - z_j)}\right)\right) \tag{54}$$

$$= \exp\left(-\|y\|^2/2 - \frac{1}{2}\sum_{i<j} \frac{(y_i - y_j)^2}{(z_i - z_j)^2} + O(k^{-1/2})\right). \tag{55}$$

This yields that these densities tend locally uniformly to the densities of the normal distributions in Theorem 1 up to the positive multiplicative normalization constants. To handle the normalizations, one has to use (8), Stirling's formula, and a dominated convergence argument that finally lead to weak convergence of probability measures. It was recently remarked in Ref. 50 that the weak convergence can be also obtained from the following well-known result on the Laplace method in asymptotic analysis; see Ch. IX of Ref. 103 or Theorem 41 and Lemma 38 in Ref. 19.

Theorem 9. *Let $\phi \in C^2(D)$ and $\psi \in C_0(D)$ be functions with $\phi, \psi \geq 0$ on some domain $D \subset \mathbb{R}^N$ with the following properties:*

(1) ϕ has a unique global maximum at $z \in D$ such that for every neighborhood V of z,

$$\sup\{\phi(x) : x \in D \setminus V\} < \phi(z).$$

(2) ϕ has a negative definite Hessian matrix $H_\phi(z)$ in z.
(3) The Lebesgue integrals $\int_D \psi(x)\phi(x)^k\, dx$ ($k \in \mathbb{N}$) exist and converge for $k \to \infty$.
(4) $\psi(z) > 0$.

Then,

$$\lim_{k\to\infty}\int_D \psi(x)\phi(x)^k\,dx\cdot\left(\frac{2\pi}{k}\right)^{-N/2}\cdot\frac{\det(-H_{\ln\phi}(z))^{1/2}}{\psi(z)\phi(z)^k}=1.$$

The approach above to the CLT 1 implies in addition that the inverse covariance matrices satisfy $\det S_N = N!$. This observation was the motivation to derive the eigenvalues and eigenvectors of S_N in Ref. 9. For this, consider the finite orthogonal polynomials $\{Q_n^{(N)}\}_{n=0}^{N-1}$ that are orthogonal w.r.t. the empirical measures

$$\mu_N := \frac{1}{N}(\delta_{z_1}+\cdots+\delta_{z_N})\in M^1(\mathbb{R}) \tag{56}$$

of the zeros of H_N where these polynomials are determined uniquely up to multiplicative constants; for the theory of (finite) orthogonal polynomials, see, for example, Ref. 55. We then have the following result by Theorem 3.1 of Ref. 9 and Proposition 2.4 of Ref. 4.

Theorem 10. *The matrix S_N from Theorem 1 has the eigenvalues $\lambda_n = n$ with the eigenvectors*

$$\left(Q_{n-1}^{(N)}(z_1),\dots,Q_{n-1}^{(N)}(z_N)\right)^T$$

for $n = 1,\dots,N$. Moreover, the monic orthogonal polynomials $\{\hat{Q}_n^{(N)}\}_{n=0}^{N-1}$ satisfy the three-term recurrence

$$\hat{Q}_0^{(N)}=1,\ \hat{Q}_1^{(N)}(x)=x,\ \hat{Q}_{n+1}^{(N)}(x)=x\hat{Q}_n^{(N)}(x)-\left(\frac{N-n}{2}\right)\hat{Q}_{n-1}^{(N)}(x)\qquad (n=1,\dots,N-2). \tag{57}$$

These results were used in Ref. 4 to compute the entries of the matrices Σ_N from Theorem 1; see also Section 4 for a systematic approach.

Theorem 11. *The covariance matrices $\Sigma_N = (\sigma_{i,j}^2)_{i,j=1,\dots,N}$ from Theorem 1 satisfy*

$$\sigma_{i,j}^2=(-1)^{i+j}\frac{\sum_{k=0}^{N-1}\frac{H_k(z_i)H_k(z_j)}{2^k k!(N-k)}}{\sqrt{\sum_{k=0}^{N-1}\frac{(H_k(z_i))^2}{2^k k!}\sum_{l=0}^{N-1}\frac{(H_l(z_j))^2}{2^l l!}}}. \tag{58}$$

Remark 1.

(1) The CLT 1.2 was proved earlier by Dumitriu and Edelman[35] via their tridiagonal random matrix models in Ref. 34 for β-Hermite ensembles, where in the diagonal iid $N(0,1)$-distributed and on the second diagonal suitable gamma-distributed independent random variables appear. Here the well-known convergence of gamma distributions to normal distributions after normalization leads to Theorem 1.2 where the entries of $\Sigma_N = (\sigma_{i,j}^2)_{i,j=1,\dots,N}$ satisfy

$$\sigma_{i,j}^2=\frac{\sum_{l=0}^{N-1}\tilde{H}_l^2(z_i)\tilde{H}_l^2(z_j)+\sum_{l=0}^{N-2}\tilde{H}_{l+1}(z_i)\tilde{H}_l(z_i)\tilde{H}_{l+1}(z_j)\tilde{H}_l(z_j)}{\sum_{l=0}^{N-1}\tilde{H}_l^2(z_i)\cdot\sum_{l=0}^{N-1}\tilde{H}_l^2(z_j)} \tag{59}$$

with the orthonormal Hermite polynomials

$$\tilde{H}_n(x) = \frac{1}{2^{n/2}\sqrt{n!}} H_n(x) \qquad (n \geq 0) \tag{60}$$

w.r.t. the probability measure $\pi^{-1/2}e^{-x^2}$ on $\mathbb{R}$. For small dimensions N, numerical computations confirm (59) = (58), but to the knowledge of the author, there is no analytic direct proof for this identity for general i, j, N at present.

(2) Theorem 1 is also proved in a different way by Gorin and Kleptsyn.[43] They also obtain (58) by induction on the dimension N and by classical formulas for the distributions of β-Hermite ensembles w.r.t. Gelfand–Tsetlin bases; see also Ref. 48.

(3) The approaches to (58) in Refs. 4 and 43 are based on the orthogonal polynomials $\{Q_n^{(N)}\}_{n=0}^{N-1}$. It turns out that these polynomials are just the dual polynomials of the Hermite polynomials in the sense of de Boor and Saff[16] that are also studied in Refs. 55, 95. It turns out that this duality is a tool that also works for Bessel processes of type B and for β-Jacobi ensembles, where the classical Laguerre and Jacobi polynomials, respectively, appear instead of Hermite polynomials. We explain this in Section 4.

Equations (59) and (58) can be used in combination with the classical formula of Plancherel–Rotach on the largest zeroes of Hermite polynomials to derive limits for $N \to \infty$ in the freezing case; see Refs. 4, 35, 43. For this, consider the Airy function Ai that satisfies $y''(x) = x \cdot y(x)$ and $\lim_{x\to\infty} \mathrm{Ai}(x) = 0$. All zeros of Ai are negative. For details on Ai, we refer to Ref. 93.

Let $a_r < 0$ be the rth largest zero of Ai ($r \in \mathbb{N}$). Then, by a classical formula of Plancherel–Rotach (see, e.g., Ref. 91),

$$\frac{z_{N-r+1}}{\sqrt{2N}} = 1 - \frac{|a_r|}{2N^{\frac{2}{3}}} + O(N^{-1}) \qquad (N \to \infty). \tag{61}$$

Then the following holds.

Theorem 12. *Let $r \in \mathbb{N}$. For $N \geq r$, consider the Bessel processes*

$$(X_{t,k}^N)_{t\geq 0} = (X_{t,k,1}^N), \dots, X_{t,k,N}^N))_{t\geq 0}$$

of type A_{N-1} with start in $0 \in \mathbb{R}^N$. Then, for $t > 0$,

$$\lim_{N\to\infty}\left(\lim_{k\to\infty} N^{\frac{1}{6}}\sqrt{2k}\left(\frac{X_{t,k,N-r+1}^N}{\sqrt{2kt}} - z_{N-r+1}\right)\right) = G_r \tag{62}$$

in distribution with some $N(0, \sigma_{max,r}^2)$-distributed random variable G_r with variance

$$\sigma_{max,r}^2 = \int_0^\infty \frac{\mathrm{Ai}(x+a_r)^2}{\mathrm{Ai}'(a_r)^2 x} dx = \begin{cases} 0.834\dots \textit{for } r = 1 \\ 0.582\dots \textit{for } r = 2 \\ 0.472\dots \textit{for } r = 3 \\ 0.407\dots \textit{for } r = 4 \\ \dots . \end{cases} \tag{63}$$

In particular, the variances $\sigma^2_{max,r}$ *tend to 0 for* $r \to \infty$.

A corresponding result for $r = 1$ was stated in Corollary 3.4 of Ref. 35 based on the covariances (59), where there

$$\sigma^2_{max,1} = 2\frac{\int_0^\infty \mathrm{Ai}^4(x+a_1)dx}{\left(\int_0^\infty \mathrm{Ai}^2(x+a_1)dx\right)^2} = 2\int_0^\infty \left(\frac{\mathrm{Ai}(x+a_1)}{\mathrm{Ai}'(a_1)}\right)^4 dx \tag{64}$$

appears. A numerical computation confirms (63)=(64) for $r = 1$, but we have no direct analytic proof for this.

Theorem 12 should also be compared with the freezing results in Ref. 70, where first $N \to \infty$ and then the freezing limit is taken.

We next turn to Bessel processes of type B and β-Laguerre ensembles.

3.2 | Freezing limits for Bessel processes of type B

We here start with some multiplicity $k = (k_1, k_2)$, the Weyl chamber $C_N^B := \{x \in \mathbb{R}^N : x_1 \geq x_2 \geq \dots \geq x_N \geq 0\}$ of type B as state space, and the generators

$$Lf := \frac{1}{2}\Delta f + k_2 \sum_{i=1}^N \sum_{j\neq i} \left(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\right)\frac{\partial}{\partial x_i} f \quad + k_1 \sum_{i=1}^N \frac{1}{x_i}\frac{\partial}{\partial x_i} f. \tag{65}$$

The transition probabilities are given by

$$K_{t,k}(x, A) = c_k^B \int_A \frac{1}{t^{\gamma_B + N/2}} e^{-(\|x\|^2+\|y\|^2)/(2t)} J_k^B\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \cdot w_k^B(y)\, dy \tag{66}$$

with

$$w_k^B(x) := \prod_{i<j}\left(x_i^2 - x_j^2\right)^{2k_2} \cdot \prod_{i=1}^N x_i^{2k_1}, \tag{67}$$

$\gamma_B = k_2 N(N-1) + k_1 N$, and with the Macdonald–Mehta–Opdam-type normalization

$$\begin{aligned} c_k^B :=& \left(\int_{C_N^B} e^{-\|y\|^2/2} w_k^B(y)\, dy\right)^{-1} \\ =& \frac{N!}{2^{N(k_1+(N-1)k_2-1/2)}} \cdot \prod_{j=1}^N \frac{\Gamma(1+k_2)}{\Gamma(1+jk_2)\Gamma\left(\frac{1}{2}+k_1+(j-1)k_2\right)}. \end{aligned} \tag{68}$$

Please notice that for $x = 0$, the densities in (66) are the well-known densities for β-Laguerre ensembles that can be realized via the tridiagonal random matrix models of Dumitriu and Edelman.[26]

Our first freezing limit concerns the case $(k_1, k_2) = (\nu \cdot \beta, \beta)$ with $\nu > 0$ fixed and $\beta \to \infty$ where the limits are described in terms of the Laguerre polynomial $L_N^{(\nu-1)}$. We start with the following facts about the zeros of $L_N^{(\nu-1)}$, which is analog to Lemma 1; see Section 6.7 of Ref. 90 or 6.

Lemma 3. *Let $\nu > 0$. For $r \in C_N^B$, the following statements are equivalent:*

(1) The function $W_B(y) := 2\sum_{i<j} \ln(y_i^2 - y_j^2) + 2\nu \sum_i \ln y_i - \|y\|^2/2$ is maximal at $r \in C_N^B$.
(2) For $i = 1, \dots, N$, the vector $r = (r_1, \dots, r_N)$ satisfies

$$\frac{1}{2}r_i = \sum_{j:j\neq i} \frac{2r_i}{r_i^2 - r_j^2} + \frac{\nu}{r_i} = \sum_{j:j\neq i}\left(\frac{1}{r_i - r_j} + \frac{1}{r_i + r_j}\right) + \frac{\nu}{r_i} \qquad (i = 1, \dots, N).$$

(3) If $z_1^{(\nu-1)} \geq \dots \geq z_N^{(\nu-1)}$ are the ordered zeros of $L_N^{(\nu-1)}$, then

$$2\left(z_1^{(\nu-1)}, \dots, z_N^{(\nu-1)}\right) = \left(r_1^2, \dots, r_N^2\right). \tag{69}$$

Lemma 3 and Theorem 7(1) lead to the following CLT with arbitrary starting points similar to Theorem 3 where here the statement is even simpler, as no direct product situation appears. For the proof, see Ref. 97 where some details can be simplified by the Laplace method in Theorem 9.

Theorem 13. *Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type B_N on C_N^B for $k = (k_1, k_2) = (\nu \cdot \beta, \beta)$ with $\nu > 0$ fixed and with some fixed starting point $x \in C_N^B$. Then, for each $t > 0$, and for the vector $r \in C_N^B$ from Lemma 3,*

$$\frac{X_{t,(\nu\cdot\beta,\beta)}}{\sqrt{t}} - \sqrt{\beta}\cdot r$$

converges for $\beta \to \infty$ to the N-dimensional normal distribution $N(0, \Sigma_N)$ where $\Sigma_N^{-1} =: S_N = (s_{i,j})_{i,j=1,\dots,N}$ satisfies

$$s_{i,j} := \begin{cases} 1 + \frac{2\nu}{r_i^2} + 2\sum_{l\neq i}(r_i - r_l)^{-2} + 2\sum_{l\neq i}(r_i + r_l)^{-2} & \text{for} \quad i = j \\ 2(r_i + r_j)^{-2} - 2(r_i - r_j)^{-2} & \text{for} \quad i \neq j \end{cases}. \tag{70}$$

Moreover, $\det S_N = N! \cdot 2^N$.

Again, the eigenvalues and eigenvectors of S_N can be determined via finite orthogonal polynomials; see Ref. 9. For this, we introduce the measures

$$\mu_{N,\nu} := \frac{1}{N(N+\nu-1)}\left(z_{1,N}^{(\nu-1)}\delta_{z_{1,N}^{(\nu-1)}} + \dots + z_{N,N}^{(\nu-1)}\delta_{z_{N,N}^{(\nu-1)}}\right). \tag{71}$$

As $\sum_{k=1}^N z_{k,N}^{(\nu-1)} = N(N+\nu-1)$ by Appendix C of Ref. 6, these measures are probability measures. We now define the unique associated orthogonal polynomials $(Q_n^{(N,\nu)})_{n=0,\dots,N-1}$ with deg $Q_n^{(N,\nu)} = n$, positive leading coefficients, and normalization

$$\sum_{i=1}^{N} z_{i,N}^{(\nu-1)} Q_k^{(N,\nu)}\left(z_{i,N}^{(\nu-1)}\right)^2 = 1 \qquad (k = 0, \dots, N-1). \tag{72}$$

Then, by Ref. 9, the matrices $T_N := (r_i \cdot Q_n^{(N,\nu)}(r_i^2))_{i=1,\dots,N,n=0,\dots,N-1}$ are orthogonal, and we get the following.

Theorem 14. *For $N \geq 2$, the matrix S_N in Theorem 13 has the eigenvalues $\lambda_n = 2(n+1)$ with the eigenvectors*

$$\left(r_1 Q_n^{(N,\nu)}(r_1^2), \dots, r_N Q_n^{(N,\nu)}(r_N^2)\right)^T, \ n = 0, 1, \dots, N-1.$$

In particular, $S_N = T_N \cdot diag(2, 4, \dots, 2N) \cdot T_N^T$.

The three-term recurrence relations of these polynomials $(Q_n^{(N,\nu)})_{n=0,\dots,N-1}$ and the entries of Σ_N can now be determined. We here skip details and present a unifying approach in the next section via dual orthogonal polynomials; see in particular Theorem 22.

Remark 2.

(1) Theorem 13 can be used to derive a limit result for $N \to \infty$ for the largest particles for the root systems B_N that is analog to Theorem 12; see Ref. 4. Results on the hard edge, that is, for the smallest particles, are different.
(2) Subordinations of Bessel processes of types A and B starting in $0 \in \mathbb{R}^N$ by the classical convolution semigroup of inverse Gaussian measures on $[0, \infty[$ lead to Feller processes with Cauchy-type distributions of the form

$$\frac{c_k \, t \, \Gamma(\gamma_k + (N+1)/2)}{\sqrt{4\pi}} \cdot \left(\frac{4}{t^2 + 2\|y\|^2}\right)^{\gamma_k + (N+1)/2} \cdot w_k(y) \tag{73}$$

on the Weyl chambers C_N of types A and B with explicit constants $c_k > 0$ and the weight functions w_k from (36); see Section 5 of Ref. 80 with a slightly different t-scaling. In these cases, there exist also freezing limits that are connected with the CLTs 3 and 13 for Bessel processes of types A and B; see Ref. 99. We expect that similar freezing CLTs hold for the Hua–Pickrell measures in Refs. 11, 13, 18, 68 (see also Ref. 54) that are similar to those for the distributions (73).

We next turn to a further freezing limit for Bessel processes of type B where now in the multiplicity $k = (k_1, k_2)$, the parameter $k_2 > 0$ is fixed and $k_1 \to \infty$ holds. In this case, we obtain the following limit result where the limit is not longer a normal distribution as here the part of the densities in (66) that concerns k_2 remains unchanged. Moreover, in the proof, one has to use Theorem 7(3) in order to handle arbitrary starting points; see Ref. 97 for details.

Theorem 15. *For any fixed starting point $x \in C_N^B$, consider the Bessel processes $(X_{t,(k_1,k_2)})_{t\geq 0}$ of type B_N on C_N^B. Then, for $k_2 > 0$, $t > 0$, and the vector $\mathbf{1} := (1, \dots, 1) \in \mathbb{R}^N$,*

$$X_{t,(k_1,k_2)} - \sqrt{2tk_1} \cdot \mathbf{1}$$

converges for $k_1 \to \infty$ in distribution to $X^A_{t/2,k_2}$, where $(X^A_{s,k_2})_{s\geq 0}$ is a Bessel process of type A starting in the origin.

Remark 3. Theorem 15 for the starting point $x = 0$ should be compared with the CLT 28 below for k_2 fixed, $k_1 \to \infty$, and for the starting points of the form $\sqrt{k_1} \cdot x_0$ with x_0 in the interior on C_N^B, where in Theorem 28, an N-dimensional normal distribution appears. As this is quite different from the limit in Theorem 15 for $x_0 = 0$, this means that Theorem 28 cannot be extended "continuously" from the interior of C_N^B to the origin. This is also clear by simple geometric considerations about the support of the limit measure.

For $k_2 = 1/2, 1, 2$, Theorem 15 has a matrix-theoretic background:

Remark 4. Fix one of the (skew-)fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with the real dimension $d = 1, 2, 4$ respectively. For integers $p \in \mathbb{N}$, consider the vector spaces $M_{p,N}(\mathbb{F})$ of all $p \times N$-matrices over $\mathbb{F}$ with the real dimension dpN. Choose the standard bases on these vector spaces such that we have d basis vectors in each entry. Consider the dpN-dimensional associated Brownian motion $(B_t^p)_{t\geq 0}$ on $M_{p,N}(\mathbb{F})$ starting in the origin. If we write $A^* := \overline{A}^T \in M_{N,p}(\mathbb{F})$ for matrices $A \in M_{p,N}(\mathbb{F})$ with the usual conjugation on $\mathbb{F}$, then the process $(Z_t^p := (B_t^p)^* B_t^p)_{t\geq 0}$ becomes a Wishart process on the closed cone $\Pi_N(\mathbb{F})$ of all $N \times N$ positive semidefinite matrices over $\mathbb{F}$ with shape parameter p; see Refs. 20, 21, 32 and references there for details on Wishart processes.

Consider the spectral mapping $\sigma_N : \Pi_N(\mathbb{F}) \to C_N^B$ that assigns to each matrix in $\Pi_N(\mathbb{F})$ its ordered spectrum. It is well known that then $(\sqrt{\sigma_N(Z_t^p)})_{t\geq 0}$ is a Bessel process on C_N^B of type B_N with multiplicities

$$(k_1, k_2) := ((p - N + 1) \cdot d/2, d/2),$$

where the symbol $\sqrt{.}$ means taking square roots in each component.

Therefore, Theorem 15 for $k_2 = 1/2, 1, 2$ corresponds to a CLT for Wishart distributions on $\Pi_N(\mathbb{F})$ with fixed time parameters where the shape parameters p tend to ∞. To explain this on the level of matrices, recall that the distributions $\mu_t^p := P_{Z_t^p} \in M^1(\Pi_N(\mathbb{F}))$ of Z_t^p satisfy $\mu_t^{p_1} * \mu_t^{p_2} = \mu_t^{p_1+p_2}$ for $p_1, p_2 \in \mathbb{N}$ with the usual convolution of measures on the vector space $\mathbb{H}_N(\mathbb{F})$ of all $N \times N$ Hermitian matrices over $\mathbb{F}$ by the construction of the random variables Z_t^p. Moreover, this convolution relation even remains valid for all $p \in]0, \infty[$, which are sufficiently large. Thus, the classical CLT for sums of iid random variables on finitely dimensional vector spaces leads to a CLT for Wishart distributions for $p \to \infty$. A computation in this setting shows that here the centering on $\mathbb{H}_N(\mathbb{F})$ occurs with a multiple of the identity matrix. Also the covariance matrices of the associated centered limit normal distributions on $\mathbb{H}_N(\mathbb{F})$ can be determined explicitly. For this, we notice that the Gaussian limit on $\mathbb{H}_N(\mathbb{F})$ (before or after centering) is invariant under all conjugations with matrices in $U_N(\mathbb{F})$ as this is the case for the Wishart distributions above. Moreover, by Theorem 15, the image measures of the centered limit normal distributions on $\mathbb{H}_N(\mathbb{F})$ under the spectral map $\sigma_N : \mathbb{H}_N(\mathbb{F}) \to C_N^A$ are just the distributions with densities

$$c(k,t,N) \cdot e^{-\|y\|/t} \cdot w_k^A(y)\, dy \tag{74}$$

of type A on C_N^A as in Theorem 15 above with some normalizations $c(k,t,N) > 0$. As these measures are the spectral distributions of Gaussian orthogonal/unitary/symplectic ensembles up to time scaling, it follows that up to this scaling, the centered Gaussian limits on $\mathbb{H}_N(\mathbb{F})$ are just the distributions of Gaussian orthogonal/unitary/symplectic ensembles on the matrix level.

In summary, for $k_2 = 1/2, 1, 2$, Theorem 15 corresponds to a classical CLT for Wishart distributions on $\mathbb{H}_N(\mathbb{F})$ where the shape parameters tend to ∞.

With these comments, we complete the case $k_2 > 0$ fixed with $k_1 \to \infty$. We next study the case $k_1 > 0$ fixed with $k_2 \to \infty$ that is related to limits for Bessel processes of type D_N:

3.3 | Freezing limits for Bessel processes of type D and a connection to type B

Consider the root system $D_N = \{\pm(e_i \pm e_j) : \quad 1 \le i < j \le N\}$ and the associated closed Weyl chamber

$$C_N^D = \{x \in \mathbb{R}^N : \quad x_1 \ge \ldots \ge x_{N-1} \ge |x_N|\}.$$

C_N^D may be seen as a doubling of C_N^B w.r.t. the last coordinate. We have a one-dimensional multiplicity $k \ge 0$ with the generator

$$L_k f := \frac{1}{2}\Delta f + k \sum_{i=1}^N \sum_{j \ne i} \left(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j} \right) \frac{\partial}{\partial x_i} f, \tag{75}$$

of the transition semigroup of the Bessel process $(X_{t,k})_{t\ge0}$ where we again assume reflecting boundaries. By Section 2, we have the transition probabilities

$$K_{t,k}(x,A) = c_k^D \int_A \frac{1}{t^{\gamma_D + N/2}} e^{-(\|x\|^2 + \|y\|^2)/(2t)} J_k^D\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right) \cdot w_k^D(y)\, dy \tag{76}$$

with

$$w_k^D(x) := \prod_{i<j} \left(x_i^2 - x_j^2 \right)^{2k}, \quad \gamma_D := kN(N-1);$$

see Ref. 26 for further details. By the normalization (68) for $k_2 = k, k_1 = 0$, we obtain the normalization

$$c_k^D := \left(\int_{C_N^D} e^{-\|y\|^2/2} w_k^D(y)\, dy \right)^{-1} \tag{77}$$

$$= \frac{N!}{2^{N(N-1)k - N/2 + 1}} \cdot \prod_{j=1}^N \frac{\Gamma(1+k)}{\Gamma(1+jk)\Gamma\left(\frac{1}{2} + (j-1)k\right)}.$$

We now proceed similarly to the first B-case above with $\nu = 0$ and use the Laguerre polynomial $L_N^{(-1)}$ from Subsection 2.1. Motivated by (25) and Lemma 3, we consider the $N-1$ ordered zeros $z_1^{(1)}, \dots, z_{N-1}^{(1)}$ of $L_{N-1}^{(1)}$ and the vector $r = (r_1, \dots, r_N) \in C_N^D$ with

$$2 \cdot \left(z_1^{(1)}, \dots, z_{N-1}^{(1)}, 0\right) = \left(r_1^2, \dots, r_N^2\right). \tag{78}$$

We then have the following analog of the CLTs 1 and 13.

Theorem 16. *Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D_N on C_N^D with $k > 0$ that start in 0. Then,*

$$\frac{X_{t,k}}{\sqrt{t}} - \sqrt{k} \cdot r$$

converges for $k \to \infty$ to the centered N-dimensional distribution $N(0, \Sigma)$ with the regular covariance matrix Σ with $\Sigma^{-1} = (s_{i,j})_{i,j=1,\dots,N}$ with

$$s_{i,j} := \begin{cases} 1 + 2\sum_{l\neq i}(r_i - r_l)^{-2} + 2\sum_{l\neq i}(r_i + r_l)^{-2} & \text{for} \quad i = j \\ 2(r_i + r_j)^{-2} - 2(r_i - r_j)^{-2} & \text{for} \quad i \neq j \end{cases}. \tag{79}$$

Let $(X_{t,k}^D)_{t\geq 0}$ be a Bessel process of type D with $k \geq 0$. Then the process $(X_{t,k}^B)_{t\geq 0}$ with

$$X_{t,k}^{B,i} := X_{t,k}^{D,i} (i = 1, \dots, N-1), \quad X_{t,k}^{B,N} := \left|X_{t,k}^{D,N}\right|$$

is a Bessel process of type B with the multiplicity $(k_1, k_2) := (0, k)$. This follows easily from a comparison of the corresponding generators; see Ref. 8. Theorem 16 thus leads to the following "one-sided CLT" for Bessel processes of type B with the multiplicities $(0, k_2)$ for $k_2 \to \infty$.

Corollary 1. *Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type B_N on C_N^B with $k = (0, k_2)$, which start in 0. Then, for the vector r from (78),*

$$\frac{X_{t,k}}{\sqrt{t}} - \sqrt{k} \cdot r$$

tends for $k \to \infty$ in distribution to the "one-sided normal distribution" on the half space

$$H_N := \{x \in \mathbb{R}^N : x_N \geq 0\},$$

which appears as image of the centered N-dimensional distribution $N(0, \Sigma)$ with the regular covariance matrix Σ of Theorem 16 under the mapping

$$\mathbb{R}^N \longrightarrow H_N, \quad (x_1, \dots, x_N) \mapsto (x_1, \dots, x_{N-1}, |x_N|).$$

This result can be extended to Bessel processes of type B with multiplicities $k = (k_1, k_2)$ with $k_1 > 0$ fixed for $k_2 \to \infty$; see Ref. 97.

Theorem 17. *Consider the Bessel processes $(X_{t,k})_{t\ge 0}$ of type B_N on C_N^B with multiplicities (k_1, k_2) that start in some $x \in C_N^B$. Let $k_1 \ge 0$, $t > 0$, and r as in (78). Then, for $k_2 \to \infty$,*

$$\frac{X_{t,(k_1,k_2)}}{\sqrt{t}} - \sqrt{k_2} \cdot r$$

converges in distribution to the one-sided normal distribution in Corollary 1.

We next turn to the β-Jacobi ensembles, that is, the stationary compact BC-case.

3.4 | Freezing limits for Jacobi ensembles

β-Jacobi ensembles are usually described in an algebraic way by the distributions $\mu_{(k_1,k_2,k_3)}$ with the Lebesgue densities

$$c_{k_1,k_2,k_3} \prod_{1\le i<j\le N} \left(x_j - x_i\right)^{k_3} \prod_{i=1}^{N} (1-x_i)^{\frac{k_1+k_2}{2}-\frac{1}{2}} (1+x_i)^{\frac{k_2}{2}-\frac{1}{2}} \tag{80}$$

for $k_1, k_2, k_3 \ge 0$ on the alcoves $A_N := \{x \in \mathbb{R}^N : -1 \le x_1 \le ... \le x_N \le 1\}$ with some Selberg constants $c_{k_1,k_2,k_3} > 0$ (see, e.g., the survey[41] for explicit formulas) where often other parameters for the powers are used, and where sometimes, a linear transformation is used that transforms $[-1, 1]$ into [0,1]; see Refs. 40, 56, 58, 59, 63. Our notation for the parameters $k = (k_1, k_2, k)$ is motivated by the connection to the Heckman–Opdam theory of type BC in trigonometric form in Subsection 2.4. By Ref. 51, it turns out that the trigonometric version has some advantage in view of the covariance matrices of the CLT below. We thus use the transformation

$$(t_1, \dots, t_N) \longrightarrow (\cos(t_1), \dots, \cos(t_N))$$

such that the distributions $\mu_{(k_1,k_2,k_3)}$ are the transform of the distributions $\tilde{\mu}_{(k_1,k_2,k_3)}$ with the densities

$$\tilde{c}_k \cdot \prod_{1\le i<j\le N} \left(\cos t_j - \cos t_i\right)^{k_3} \prod_{i=1}^{N} \left(\sin^{k_1} \frac{t_i}{2} \cdot \sin^{k_2} t_i\right) \tag{81}$$

on the trigonometric alcove

$$\tilde{A}_N := \{t \in \mathbb{R}^N : \quad \pi \ge t_1 \ge ... \ge t_N \ge 0\}$$

with some normalizations $\tilde{c}_k > 0$. This fits to the densities (46) for type BC in Subsection 2.4 and differs slightly from the notation in Ref. 51. We now present a CLT for

$$(k_1, k_2, k_3) = \kappa \cdot (a, b, 1) \quad \text{for} \quad a \ge 0, b > 0 \quad \text{fixed and} \quad \kappa \to \infty,$$

which corresponds to the CLTs 1, 13, and 16. For this, we need the ordered zeros $z_1 \le ... \le z_N$ of the Jacobi polynomials $P_N^{(\alpha,\beta)}$ with

$$\alpha := a + b - 1 > -1, \quad \beta = b - 1 > -1.$$

Define the vector $z := (z_1, \dots, z_N) \in A_N$. The following CLT is shown in Ref. 51:

Theorem 18. *Let $a \geq 0$, $b > 0$. Let $\tilde{X}_\varkappa$ be $\tilde{A}_N$-valued random variables with the distributions $\tilde{\mu}_{\varkappa\cdot(a,b,1)}$ for $\varkappa > 0$. Then, for $\varkappa \to \infty$,*

$$\frac{\sqrt{\varkappa}}{2}(\tilde{X}_\varkappa - \tilde{z}) \qquad \text{with} \qquad \tilde{z} := (\arccos z_1, \dots, \arccos z_N) \in \tilde{A}_N$$

converges in distribution to $N(0, \tilde{\Sigma}_N)$ where $\tilde{\Sigma}_N^{-1} =: \tilde{S}_N = (\tilde{s}_{i,j})_{i,j=1,\dots,N}$ satisfies

$$\tilde{s}_{i,j} = \begin{cases} 4\sum_{l\neq j} \frac{1-z_j^2}{(z_j-z_l)^2} + 2(a+b)\frac{1+z_j}{1-z_j} + 2b\frac{1-z_j}{1+z_j} & fori = j \\ \frac{-4\sqrt{(1-z_j^2)(1-z_i^2)}}{(z_i-z_j)^2} & fori \neq j \end{cases}.$$

The eigenvalues and eigenvectors of $\tilde{S}_N$ can again be determined via finite orthogonal polynomials that here are orthogonal w.r.t. the measures

$$\mu_{N,\alpha,\beta} := (1-z_1^2)\delta_{z_1} + \dots + (1-z_N^2)\delta_{z_N}. \tag{82}$$

If we use the associated orthonormal polynomials $(Q_l^{(\alpha,\beta,N)})_{l=0,\dots,N-1}$ with positive leading coefficients and the normalization

$$\sum_{i=1}^{N} Q_l^{(\alpha,\beta,N)}(z_i)Q_k^{(\alpha,\beta,N)}(z_i)(1-z_i^2) = \delta_{l,k} \qquad (k,l = 0,\dots,N-1), \tag{83}$$

we obtain the following result; see Ref. 51.

Theorem 19. *The matrix $\tilde{S}_N$ has the eigenvalues $\lambda_k = 2k(2N+\alpha+\beta+1-k) > 0$ $(k = 1,\dots,N)$ with the eigenvectors*

$$v_k := \left(Q_{k-1}^{(\alpha,\beta,N)}(z_1)\sqrt{1-z_1^2}, \dots, Q_{k-1}^{(\alpha,\beta,N)}(z_N)\sqrt{1-z_N^2}\right)^T.$$

In particular, with the orthogonal matrix $T_N := (v_1,\dots,v_N)$,

$$\tilde{S}_N = T_N \cdot \mathrm{diag}(2(2N+\alpha+\beta+1-1), \dots, 2N(2N+\alpha+\beta+1-N)) \cdot T_N^T.$$

As in the preceding Bessel cases, we can now determine the three-term recurrence of the polynomials $(Q_k^{(\alpha,\beta,N)})_{k=0,\dots,N-1}$ as well as the entries of $\tilde{\Sigma}_N$; see the next section.

We point out that Theorems 18 and 19 were stated in trigonometric form as only here the eigenvalues and eigenvectors of the (inverse) covariance matrices of the limit are known. Clearly, these trigonometric results can be easily transferred back to the classical β-Jacobi ensembles with the distributions $\mu_{(k_1,k_2,k_3)}$ on A_N. In these coordinates, Theorem 18 then has the following form by the so-called Delta-method for CLTs of transformed random variables; see Section 3.1 of Ref. 94.

Theorem 20. *Let $a \geq 0$ and $b > 0$. Let $X_\varkappa$ be random variables with the distributions $\mu_{\varkappa\cdot(a,b,1)}$ as above. Then $\sqrt{\varkappa}(X_\varkappa - z)$ converges for $\varkappa \to \infty$ to the normal distribution $N(0, \Sigma_N)$ with the* *covariance matrix* Σ_N *where* $\Sigma_N^{-1} =: S_N = (s_{i,j})_{i,j=1,\dots,N}$ *satisfies*

$$s_{i,j} = \begin{cases} \sum_{l=1,\dots,N;l\neq j} \frac{1}{(z_j-z_l)^2} + \frac{a+b}{2}\frac{1}{(1-z_j)^2} + \frac{b}{2}\frac{1}{(1+z_j)^2} & for\, i = j \\ \frac{-1}{(z_i-z_j)^2} & for\, i \neq j \end{cases}.$$

In fact, the matrices $\tilde{\Sigma}_N^{-1}$ and Σ_N^{-1} from Theorems 18 and 20 are related by $\tilde{S} = DSD$ with the diagonal matrix

$$D = \mathrm{diag}\left(-2\sqrt{1-z_{1,N}^2}, \dots, -2\sqrt{1-z_{N,N}^2}\right)$$

by Ref. 51. Hence, Theorem 22 from the next section implies in the nontrigonometric Jacobi case:

Theorem 21. *The covariance matrix* $\Sigma_N = (\sigma_{i,j}^N)_{i,j=1,\dots,N}$ *in Theorem 20 has entries*

$$\sigma_{i,j}^N = \frac{(-1)^{i+j}4\sqrt{1-z_{i,N}^2}\sqrt{1-z_{j,N}^2}}{\sqrt{\sum_{k,l=0}^{N-1}\left(\tilde{P}_k^{(\alpha,\beta)}(z_{i,N})\tilde{P}_l^{(\alpha,\beta)}(z_{j,N})\right)^2}} \sum_{k=0}^{N-1} \frac{\tilde{P}_k^{(\alpha,\beta)}(z_{i,N})\tilde{P}_k^{(\alpha,\beta)}(z_{j,N})}{\lambda_{N-k}}, \tag{84}$$

where the $\tilde{P}_k^{(\alpha,\beta)}$ are the orthonormal Jacobi polynomials.

Remark 5.

(1) The freezing limits above for β-Jacobi ensembles, that is, the compact stationary BC-case, fit to the corresponding limits for Bessel processes of types A, B, and D. Moreover, the CLT 15 in the Bessel-case of type B can also be stated for β-Jacobi ensembles. More precisely, if we fix constants $a, b, k_3 \in \mathbb{R}$ with $k_3, b, a+b > 0$, and if we consider random variables X_κ with the distributions $\mu_{(\kappa a,\kappa b,k_3)}$ on the alcove A_N, then

$$\sqrt{\kappa}\left(X_\kappa - \frac{-a}{a+2b}\mathbf{1}\right)$$

converges for $\kappa \to \infty$ in distribution to the distribution of a Bessel process of type A as described in (9) with the weight function $w_{k_3}^A$ from (7) with time parameter

$$t = \frac{8(a+b)b}{(a+2b)^3} > 0.$$

(2) It is also possible to state freezing limits for the root system A_{N-1} in the compact stationary case with N particles on the torus $\{z \in \mathbb{C} : |z| = 1\}$. Here, however, due to the torus, the geometric details are more technical; see Ref. 85 for some more details.

(3) It is an interesting task to study freezing limits for Jacobi processes for arbitrary starting points and their dependence on the time t. It is clear by the comments on the transition probabilities (50) in the end of Section 2 that Theorem 20 is still available for the random variables $X_{t(\kappa),\kappa\cdot(a,b,1)}$ of Jacobi processes $(X_{t,\kappa\cdot(a,b,1)})_{t\geq 0}$ with multiplicities $\kappa\cdot(a,b,1)$ for $\kappa \to \infty$

whenever the time parameters $t(\varkappa)$ are large enough, where this "large enough" can be made precise. Clearly, the situation is more involved for general time parameters.

For instance, in the case BC_1, that is, for classical Jacobi processes on $[-1, 1]$ with classical beta distributions as stationary distributions, there exist several limit theorems where the time t has to be renormalized by the freezing constants $\varkappa$; see, for example, Ref. 96 and references there. These limit theorems have one-dimensional normal and Gamma distributions as limits; they are related with the cutoff phenomenon of Diaconis for series of random walks (or Markov processes) on compact (or even finite) state spaces and the question, how close the distributions of these processes are to their stationary distributions for large time parameters with respect to the total variation norm. We do not know whether the explicit analytic methods in Ref. 96 and references there can be transferred to $N \geq 2$. On the other hand, we sketch an approach via SDEs in Subsection 5.8 below.

(4) In the non-compact Heckman–Opdam case, there are only a very few freezing limits. For instance, in Ref. 12, some freezing limit is given on the basis of Theorem 8 where the BC-case tends to the A-case. These limits are interesting from a geometric point of view, as here for some parameters, there are connections to Brownian motions on noncompact Grassmann manifolds over $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

(5) The freezing limits in Theorems 1, 13, and 20 for the classical β-ensembles (Hermite, Laguerre, Jacobi) are connected with results in Refs. 29, 30 where the corresponding distributions of the ensembles for finite parameters are compared with the freezing limits.

4 | DE BOOR–SAFF DUALITY AND THE COVARIANCE MATRICES

In this section, we follow Ref. 4 and show how dual orthogonal polynomials in the sense of de Boor and Saff[16] can be used to compute the covariance matrices Σ_N of the preceding section from Σ_N^{-1} in a unifying way. This is motivated by the observation that the finite monic orthogonal polynomials $(\hat{Q}_k^{(N)})_{k=0,\ldots,N-1}$ in (57) are, in fact, the dual polynomial of the Hermite polynomials in this sense. To explain this, we follow Ref. 95 and Section 2.11 of Ref. 55. Let $(\hat{P}_n)_{n=0}^{\infty}$ be monic orthogonal polynomials where the orthogonality measure is a probability measure μ on $\mathbb{R}$ with

$$\int_{\mathbb{R}} \hat{P}_i(x)\hat{P}_j(x)d\mu(x) = \xi_i \delta_{ij} \quad (i, j = 0, 1, 2, \ldots) \tag{85}$$

with constants $\xi_i > 0$. We have a three-term recurrence

$$\hat{P}_0 = 1,\ \hat{P}_1(x) = x - a_0,\ x\hat{P}_n(x) = \hat{P}_{n+1}(x) + a_n\hat{P}_n(x) + u_n\hat{P}_{n-1}(x) \quad (n \geq 1) \tag{86}$$

with $a_n \in \mathbb{R}, u_n > 0$. We also consider the orthonormal polynomials $(\tilde{P}_n := \xi_n^{-1/2}\hat{P}_n)_{n=0}^{\infty}$ with

$$\tilde{P}_0 = 1,\ \tilde{P}_1(x) = b_1^{-1}(x - a_0),\ x\tilde{P}_n(x) = b_{n+1}\tilde{P}_{n+1}(x) + a_n\tilde{P}_n(x) + b_n\tilde{P}_{n-1}(x) \quad (n \geq 1) \tag{87}$$

with $b_n = u_n\sqrt{\xi_{n-1}/\xi_n} = \sqrt{\xi_n/\xi_{n-1}}$ for $n \geq 1$. In particular,

$$\xi_0 = 1,\ \xi_n = u_n u_{n-1} \cdots u_1 \quad \text{and} \quad b_n = \sqrt{u_n} \quad (n \geq 1). \tag{88}$$

Now fix $N > 0$. Gauss quadrature (see, e.g., Ref. 55) implies that $(\tilde{P}_n)_{n=0}^{N-1}$ has the discrete orthogonality relation

$$\sum_{i=1}^{N} w_i \tilde{P}_m(z_{i,N})\tilde{P}_n(z_{i,N}) = \delta_{mn}, \qquad (n, m = 0, \dots, N-1) \tag{89}$$

with the N ordered zeros $z_{1,N} < \dots < z_{n,N}$ of $\tilde{P}_N$ and the Christoffel numbers

$$w_i := \frac{1}{b_N \tilde{P}_{N-1}(z_{i,N})\tilde{P}'_N(z_{i,N})} > 0 \quad (i = 1, \dots, N) \tag{90}$$

with the normalization $\sum_{i=1}^{N} w_i = 1$.

Definition 1. The monic polynomials $(\hat{Q}_{k,N})_{k=0}^{N-1}$ are called dual (in the de Boor–Saff sense) to $(\hat{P}_n(x))_{n=0}^{N-1}$ if

$$\hat{Q}_{0,N} = 1,\ \hat{Q}_{1,N}(x) = x - a_{N-1}, \tag{91}$$
$$x\hat{Q}_{k,N}(x) = \hat{Q}_{k+1,N}(x) + a_{N-k-1}\hat{Q}_{k,N}(x) + u_{N-k}\hat{Q}_{k-1,N}(x) \quad (k = 1, \dots, N-2).$$

In the case of the Hermite polynomials $(P_n := H_n)_{n\geq 0}$, this definition and the three-term recurrence of the Hermite polynomials ensure that the polynomials $(\hat{Q}_k^{(N)})_{k=0,\dots,N-1}$ from (57) are, in fact, dual to the monic Hermite poynomials.

We now recall the following consequences of this duality from Ref. 95.

Lemma 4. *The dual monic polynomials $(\hat{Q}_{k,N})_{k=0}^{N-1}$ are orthogonal w.r.t. the discrete measure*

$$\sum_{i=1}^{N} w_i^* \delta_{z_{i,N}}$$

with the dual Christoffel numbers

$$w_i^* = \frac{\tilde{P}_{N-1}(z_{i,N})}{b_N \tilde{P}'_N(z_{i,N})} > 0 \quad (i = 1, \dots, N), \tag{92}$$

which again satisfy $\sum_{i=1}^{N} w_i^* = 1$.

In particular, by (88), the normalized dual polynomials $(\tilde{Q}_{k,N})_{k=0}^{N-1}$ with

$$\sum_{i=1}^{N} w_i^* \tilde{Q}_{m,N}(z_{i,N})\tilde{Q}_{n,N}(z_{i,N}) = \delta_{mn} \quad (m, n = 0, \dots, N-1) \tag{93}$$

satisfy

$$\tilde{Q}_{k,N}(x) = \frac{\hat{Q}_{k,N}}{b_N^2 b_{N-1}^2 \cdots b_{N-k}^2}. \tag{94}$$

In summary, (94) and the three-term recurrence of the dual polynomials yield:

Lemma 5. *The orthonormal dual polynomials* $(\tilde{Q}_{k,N})_{k=0}^{N-1}$ *satisfy*

$$\tilde{Q}_{0,N} = 1,\ \tilde{Q}_{1,N}(x) = b_{N-1}^{-1}(x - a_{N-1}), \tag{95}$$
$$x\tilde{Q}_{k,N}(x) = b_{N-k-1}\tilde{Q}_{k+1,N}(x) + a_{N-k-1}\tilde{Q}_{k,N}(x) + b_{N-k}\tilde{Q}_{k-1,N}(x) \quad (k \le N-2).$$

Remark 6. The monic three-term recurrence (91) is also available for $k = N-1$, that is, we obtain a monic polynomial $\hat{Q}_{N,N}$. It can be easily seen (see Ref. 95 or Section 2.11 of Ref. 55) that $\hat{Q}_{N,N} = \hat{P}_N$ holds. Moreover, if we choose $b_0 = 0$ in (95), then the recurrence (5) remains valid for $k = N-1$, arbitrary polynomials $\tilde{Q}_{k+1,N}$, and $x = z_{i,N}$ for $i = 1, \dots, N$.

We now use finite dual orthogonal polynomials for the computation of the covariance matrices Σ_N in the cases of the preceding section where $S_N = \Sigma_N^{-1}$ is known in terms of the Hermite polynomials H_n, Laguerre polynomials $L_n^{(\alpha)}$ with $\alpha = \nu - 1$, and the Jacobi polynomials $P_n^{(\alpha,\beta)}$ with $\alpha = a + b - 1, \beta = b - 1$. For fixed N, we use the associated orthonormal dual polynomials which we denote by $(Q_{k,N})_{k=0}^{N-1}$, $(Q_{k,N}^{(\alpha)})_{k=0}^{N-1}$, and $(Q_{k,N}^{(\alpha,\beta)})_{k=0}^{N-1}$, respectively, In all cases, let $z_{1,N} < \dots < z_{N,N}$ be the ordered zeros of the Nth polynomial. With these notations, the following was shown in Ref. 4:

Lemma 6. *In the Hermite, Laguerre, and Jacobi cases, orthonormal eigenvectors of* $S_N = \Sigma_N^{-1}$ *are given by*

$$\frac{1}{\sqrt{\kappa_N}}\left(\sqrt{\pi(z_{1,N})}\tilde{Q}_{j-1,N}(z_{1,N}), \dots, \sqrt{\pi(z_{N,N})}\tilde{Q}_{j-1,N}(z_{N,N})\right)^T,\ 1 \le j \le N, \tag{96}$$

where the coefficients κ_N *and functions* π *are given by*

$$\pi(x) = 1, \quad \pi^{(\alpha)}(x) = x, \quad \pi^{(\alpha,\beta)}(x) = 1 - x^2, \text{ and} \tag{97}$$
$$\kappa_N = N, \quad \kappa_N^{(\alpha)} = N(N+\alpha), \quad \kappa_N^{(\alpha,\beta)} = \frac{4N(N+\alpha)(N+\beta)(N+\alpha+\beta)}{(2N+\alpha+\beta)^2(2N+\alpha+\beta-1)},$$

respectively.

These results can be used to compute Σ_N from their inverses. For this, we write (87) for general orthonormal polynomials $(\tilde{P}_n)_{n=0,\dots,N}$ at the N zeros $z_{i,N}$ of $\tilde{P}_N$ as

$$\begin{pmatrix} a_0 & b_1 & & & \\ b_1 & a_1 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & a_{N-2} & b_{N-1} \\ & & & b_{N-1} & a_{N-1} \end{pmatrix}\begin{pmatrix} \tilde{P}_0(z_{i,N}) \\ \tilde{P}_1(z_{i,N}) \\ \tilde{P}_2(z_{i,N}) \\ \vdots \\ \tilde{P}_{N-1}(z_{i,N}) \end{pmatrix} = z_{i,N}\begin{pmatrix} \tilde{P}_0(z_{i,N}) \\ \tilde{P}_1(z_{i,N}) \\ \tilde{P}_2(z_{i,N}) \\ \vdots \\ \tilde{P}_{N-1}(z_{i,N}) \end{pmatrix},$$

that is, the $z_{i,N}$ are the eigenvalues of this symmetric matrix, where the eigenvectors of this matrix are orthogonal and unique up to constants. As $(\tilde{Q}_{N-1,N}(z_{i,N}), \dots, \tilde{Q}_{0,N}(z_{i,N}))^T$ is also an eigenvector of this matrix for the eigenvalue $z_{i,N}$ by Lemma 5 and Remark 6, it follows that

$$(\tilde{P}_0(z_{i,N}), \dots, \tilde{P}_{N-1}(z_{i,N})) = c_{i,N} \cdot (\tilde{Q}_{N-1,N}(z_{i,N}), \tilde{Q}_{N-2,N}(z_{i,N}), \dots, \tilde{Q}_{0,N}(z_{i,N}))$$

with $c_{i,N} = \tilde{P}_{N-1}(z_{i,N})$. The constants $c_{i,N}$ usually have the sign $(-1)^{N-i}$ by the well-known interlacing property of the zeros of $\tilde{P}_{N-1}(x)$ and $\tilde{P}_N(x)$ together with the assumption that the leading coefficient of $\tilde{P}_{N-1}(x)$ is positive. This holds, for example, for the Hermite and Jacobi cases.

The $c_{i,N}$ can also be computed from an eigenvalue equation for $S_N = \Sigma_N^{-1}$, as the vectors in Lemma 6 form an orthogonal matrix in each of the cases considered there, we see that all rows and all columns of that matrix are orthogonal. Hence,

$$\frac{\sqrt{\pi(z_{i,N})\pi(z_{k,N})}}{\kappa_N c_{i,N} c_{k,N}} \sum_{j=0}^{N-1} \tilde{P}_j(z_{i,N})\tilde{P}_j(z_{k,N}) = \delta_{i,k} \quad \text{for all} \quad 1 \le i,k \le N.$$

In particular, for $i = k$, and with the sign of $c_{i,N}$ above, we obtain

$$c_{i,N} = (-1)^{N-i} \sqrt{\frac{\pi(z_{i,N})}{\kappa_N} \sum_{j=0}^{N-1} \tilde{P}_j^2(z_{i,N})}. \tag{98}$$

Moreover, as the leading coefficient of $L_{N-1}^{(\alpha)}(x)$ has the sign $(-1)^{N-1}$, we here obtain

$$c_{i,N}^{(\alpha)} = (-1)^{i-1} \sqrt{\frac{\pi^{(\alpha)}\left(z_{i,N}^{(\alpha)}\right)}{\kappa_N^{(\alpha)}} \sum_{j=0}^{N-1} \left(\tilde{L}_j^{(\alpha)}(z_{i,N})\right)^2}. \tag{99}$$

These observations now lead to the following result on Σ_N in the three cases; see Ref. 4.

Theorem 22. *For the Hermite and Laguerre case, $\Sigma_N = (\sigma_{i,j}^N)_{i,j=1,\dots,N}$ is given with the notations of Lemma 6 and with the eigenvalues λ_k from the Theorems 10, 14, and 19 by*

$$\begin{aligned}\sigma_{i,j}^N &= \frac{\sqrt{\pi(z_{i,N})\pi(z_{j,N})}}{\kappa_N \tilde{P}_{N-1}(z_{i,N})\tilde{P}_{N-1}(z_{j,N})} \sum_{k=0}^{N-1} \frac{\tilde{P}_k(z_{i,N})\tilde{P}_k(z_{j,N})}{\lambda_{N-k}} \\ &= \frac{(-1)^{i+j}}{\sqrt{\sum_{k,l=0}^{N-1} \tilde{P}_k^2(z_{i,N})\tilde{P}_l^2(z_{j,N})}} \sum_{k=0}^{N-1} \frac{\tilde{P}_k(z_{i,N})\tilde{P}_k(z_{j,N})}{\lambda_{N-k}}.\end{aligned} \tag{100}$$

Moreover, a corresponding result holds in the trigonometric Jacobi case for $\tilde{\Sigma}_N = (\tilde{\sigma}_{i,j}^N)_{i,j=1,\dots,N}$.

Remark 7. The formulas for the entries of the covariance matrices Σ_N in (100) should be compared with the corresponding results of Dumitriu and Edelman[35] for the Hermite and Laguerre ensembles. In fact, the Hermite case was already discussed in Remark 1(1). In the Laguerre case, we have a corresponding picture, that is, a direct proof of the equality of the entries for general N seems to be unknown. We point out that due to our general approach, our formula (100) has the same structure as in the Hermite case, whereas the corresponding formula in Ref. 35 is more involved and completely different from the Hermite case there.

In the Jacobi case, there do not exist other formulas for the entries of Σ_N as far as we are aware.

5 | FREEZING LIMITS FOR BESSEL PROCESSES VIA STOCHASTIC DIFFERENTIAL EQUATIONS

In this section, we consider freezing limits with the aid of SDEs similar to Theorems 4 and 5 for Bessel processes of type A. The approach and the type of results are similar in all cases. However, the analytic details for the limits are partially quite different. For this reason, we discuss the different cases step by step.

We start with Bessel processes of type A and discuss some details skipped in the introduction. Then we turn to Bessel processes of types B and D similar to Section 3. The Heckman–Opdam processes are discussed briefly in the end.

5.1 | Freezing limits for Bessel processes of type A

As in Section 1, we consider the Bessel processes $(X_{t,k})_{t\geq 0}$ and their renormalizations $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k})_{t\geq 0}$ of type A with $k \geq 1/2$, that is, $(\tilde{X}_{t,k})_{t\geq 0}$ is the unique solution of

$$d\tilde{X}^i_{t,k} = \frac{1}{\sqrt{k}} dB^i_t + \sum_{j\neq i} \frac{1}{\tilde{X}^i_{t,k} - \tilde{X}^j_{t,k}} dt \qquad (i = 1, \dots, N) \tag{101}$$

with some fixed starting point in the interior of C^A_N. Moreover, for the degenerate case $k = \infty$, we consider the associated (unique) deterministic solutions $x(t)$ of the ODE

$$\frac{dx}{dt}(t) = H(x(t)), \quad \text{with} \quad H(x) := \left(\sum_{j\neq 1} \frac{1}{x_1 - x_j}, \dots, \sum_{j\neq N} \frac{1}{x_N - x_j}\right). \tag{102}$$

We collect some features of the solutions of (102) that were skipped in the introduction. For this, we consider the vector $\mathbf{z}$ from Lemma 1 consisting of the ordered zeros of the Hermite polynomial H_N. Lemma 1 implies that for each $c \geq 0$, $x(t) := \sqrt{2t + c^2} \cdot \mathbf{z}$ is the solution of (102) with $x(0) = c\mathbf{z}$. These solutions are typical in some way, and it is possible to decompose all solutions of (102) into an easy radial part and a more complicated spherical; see Refs. 100, 102.

Lemma 7. *For each solution $x(t)$ of (102) with start in the interior of C^A_N,*

$$\|x(t)\|^2 = N(N-1)t + \|x(0)\|^2. \tag{103}$$

Moreover, $x(t)$ can be written as

$$x(t) = \sqrt{N(N-1)t + \|x(0)\|^2} \cdot \phi(t) \qquad (t \geq 0)$$

with $\|\phi(t)\| = 1$ where the spherical part ϕ satisfies

$$\frac{d}{dt}(\phi_i(t)) = \frac{1}{N(N-1)t + \|x(0)\|^2} \left(\sum_{j\neq i} \frac{1}{\phi_i(t) - \phi_j(t)} - \frac{N(N-1)}{2}\phi_i(t) \right), \tag{104}$$

and

$$\lim_{t\to\infty} \phi(t) = \sqrt{\frac{2}{N(N-1)}} \cdot \mathbf{z}. \tag{105}$$

We point out that (104) can be transformed by a time change into a gradient system such that (105) then follows from standard result on gradient systems in Section 9.4 of Ref. 52; see Lemma 2.3 of Ref. 102. We also remark that the simple separation of the solution in Lemma 7 into a radial and spherical part in the freezing case depends heavily on the simple explicit radial part. For Bessel processes $(X_{t,k})_{t\geq 0}$, this decomposition is more involved, as here the radial part $(\|X_{t,k}\|)_{t\geq 0}$ is a one-dimensional Bessel process, which has some influence on the spherical part; see some SDE computations in Ref. 23 in this direction.

We cannot describe explicitly all solutions $x(t)$ of (102). However, it is possible to compute $t \mapsto p(x(t))$ for symmetric polynomials p in N variables in an iterative way. For this, we use the elementary symmetric polynomials e_k $(k = 0, \dots, N)$ in N variables that satisfy

$$\prod_{k=1}^{N}(z - x_k) = \sum_{k=0}^{N}(-1)^{N-k} e_{N-k}(x) z^k \qquad (z \in \mathbb{C},\ x = (x_1, \dots, x_N)). \tag{106}$$

In particular, $e_0 = 1,\ e_1(x) = \sum_{k=1}^{N} x_k, \dots, e_N(x) = \prod_{k=1}^{N} x_k$. As each symmetric polynomial q in N variables is a polynomial in $e_1, \dots, e_N$ by a classical result, the following lemma shows that here $q(x(t))$ is a polynomial in t that can be computed for each initial condition; see Ref. 102.

Lemma 8. *Let x be a solution of (102). Then, for $k = 0, \dots, N$, $t \mapsto e_k(x(t))$ is a polynomial in t of degree at most $\lfloor \frac{k}{2} \rfloor$. More precisely, for $t \geq 0$,*

$$\begin{aligned} &e_0(x(t)) \equiv N, \quad e_1(x(t)) \equiv x(0), \\ &\frac{d}{dt} e_l(x(t)) = -\frac{1}{2}(N - l + 2)(N - l + 1) e_{l-2}(x(t)) \qquad (l = 2, \dots, N). \end{aligned} \tag{107}$$

This transformation of (102) into the system (107) via $e_0, \dots, e_N$ can be used to show that (102) has also a unique solution for starting points on the singular boundary of C_N^A; see Ref. 102. This result may be seen as the deterministic analog of the corresponding stochastic results in Ref. 45 mentioned in the introduction.

Theorem 23. *Let $N \geq 2$. Then, for each starting point $x_0 \in C_N^A$, (102) has a unique solution for all $t \geq 0$ in the following sense: For each $x_0 \in C_N^A$, there is a unique continuous function $x : [0, \infty[\to C_N^A$ with $x(0) = x_0$ such that for $t > 0$, $x(t)$ is in the interior of C_N^A and satisfies (102).*

We now turn to the functional CLT 5. Fix some starting point x_0 in the interior of C_N^A and study the processes $(\tilde{X}_{t,k})_{t\geq 0}$ of type A_{N-1} and the associated solution $x(t)$ of (102). We also use the centered Gaussian process $(W_t)_{t\geq 0}$ with $W_0 = 0$ that satisfies

$$dW_t = dB_t + A(t) W_t dt \tag{108}$$

with the matrices $A(t) \in \mathbb{R}^{N\times N}$ with

$$A(t)_{i,j} = \frac{1}{(x_i(t)-x_j(t))^2}, \quad A(t)_{i,i} = -\sum_{j\neq i} \frac{1}{(x_i(t)-x_j(t))^2} \quad (i,j=1,\dots,N,\ \ i\neq j), \tag{109}$$

as in the introduction. By Theorem 5, the processes $(X_{t,k})_{t\geq 0}$ starting at $\sqrt{k}\cdot x_0$ then satisfy

$$\sup_{0\leq s\leq t, k\geq 1} \sqrt{k}\cdot \|(X_{s,k} - \sqrt{k}x(s) - W_s\| < \infty \qquad a.s. \qquad (t>0). \tag{110}$$

Proof of the CLT 5. We use the SDEs (15) and (108) and the ODE for $x(t)$ and study

$$R^i_{t,k} = k\int_0^t \sum_{j\neq i}\Bigg(\frac{1}{X^i_{s,k}-X^j_{s,k}} - \frac{1}{\sqrt{k}(x_i(s)-x_j(s))} - \frac{W^j_s - W^i_s}{\big(\sqrt{k}(x_i(s)-x_j(s))\big)^2}\Bigg)ds$$

for $i=1,\dots,N$. A Taylor expansion for $x\mapsto 1/x$ with Lagrange remainder, that is,

$$\frac{1}{x} = \frac{1}{x_0} - \frac{x-x_0}{x_0^2} + \frac{(x-x_0)^2}{\tilde{x}^3}$$

with some $\tilde{x}$ between x and x_0, then yields

$$\begin{aligned} R^i_{t,k} &= -\int_0^t\left(\sum_{j\neq i}\frac{(X^i_{s,k}-\sqrt{k}x_i(s)-W^i_s)-(X^j_{s,k}-\sqrt{k}x_j(s)-W^j_s)}{(x_i(s)-x_j(s))^2} + H^i_{s,k}\right)ds \\ &= -\int_0^t\left(\sum_{j\neq i}\frac{R^i_{s,k}-R^j_{s,k}}{(x_i(s)-x_j(s))^2} + H^i_{s,k}\right)ds \end{aligned}$$

with the error terms

$$H^i_{s,k} = k\sum_{j\neq i}\frac{\Big((X^i_{s,k}-\sqrt{k}x_i(s))-(X^j_{s,k}-\sqrt{k}x_j(s))\Big)^2}{\Big(\sqrt{k}(x_i(s)-x_j(s))+D_{i,j}(s)\Big)^3}$$

where, by the Lagrange remainder,

$$|D_{i,j}(s)| \leq |(X^j_{s,k}-\sqrt{k}x_j(s)) - (X^j_{s,k}-\sqrt{k}x_j(s))|.$$

By Theorem 4, this can be bounded by some a.s. finite random variable D that only depends on x,y,t. Hence, $|H^i_{s,k}|\leq H/\sqrt{k}$ for $k\geq k_0$, $s\in[0,t]$, $i=1,\dots,N$ with some a.s. finite random variable H. Hence,

$$R_{t,k} = -\int_0^t (A(s,x)R_{s,k} + H_{s,k})\,ds, \quad R_{0,k}=0,$$

and for $u \in [0,t]$,

$$\|R_{u,k}\| \le A\int_0^u \|R_{s,k}\|\,ds + \frac{t\cdot\|H\|}{\sqrt{k}}$$

with $A := \sup_{s\in[0,t]} \|A(s,x)\| < \infty$. Hence, by the classical lemma of Gronwall,

$$\|R_{u,k}\| \le \frac{t\|H\|}{\sqrt{k}} e^{tA}$$

for all $u \in [0,t]$. This yields (110) as claimed. □

It is an interesting task to compute the covariance matrices $\Sigma_t \in \mathbb{R}^{N\times N}$ of W_t for $t > 0$. This can be done easily for the special starting points $x_0 = c\cdot \mathbf{z}$ with $c > 0$ and the vector $\mathbf{z} = (z_1,\dots,z_N)$ consisting of the zeros of the Hermite polynomial H_N. In this case, $x(t) = \sqrt{2t+c^2}\cdot \mathbf{z}$ for $t \ge 0$, that is, we consider the matrices $A \in \mathbb{R}^{N\times N}$ with

$$A_{i,i} := -\sum_{j\neq i} \frac{1}{(z_i - z_j)^2}, \quad A_{i,j} := \frac{1}{(z_i - z_j)^2} \quad \text{for} \quad i \ne j. \tag{111}$$

Hence, in this case, by a straightforward computation,

$$\Sigma_t = \left(t + \frac{c^2}{2}\right)(E-A)^{-1}\left(E - e^{\left(\ln\frac{c^2}{2t+c^2}\right)(E-A)}\right) \tag{112}$$

with the identity matrix $E \in \mathbb{R}^{N\times N}$; see Ref. 100. Moreover, as the matrix $E - A$ has the eigenvalues $1, 2, \dots, N$ by Theorem 10, the matrix Σ_t has the eigenvalues

$$\lambda_j^A(t,c) = \frac{1}{2j}\,\frac{(2t+c^2)^j - c^{2j}}{(2t+c^2)^{j-1}} \qquad (j = 1,\dots,N),$$

where, in particular, $\lambda_1^A(t,c) = t$.

Notice that the matrix $(E-A)^{-1}$ also appears in the CLT 1 where the process starts in the origin $0 \in C_N^A$, and that the limit $c \to 0$ of Equation (112) yields $\Sigma_t = t(E-A)^{-1}$, that is, we here obtain a continuous dependence of the covariance matrices Σ_t in our CLTs at $c = 0$.

We next turn to freezing limits for Bessel processes of type B.

5.2 | Some freezing limits for Bessel processes of type B

We now study Bessel processes $(X_{t,k})_{t\ge 0}$ of type B_N with multiplicities $k = (k_1, k_2) = (\nu\cdot\beta, \beta)$ with $\nu > 0$ fixed and $\beta \to \infty$. The degenerated case $\nu = 0$ will be considered in Subsection 5.5. Here, the SDE has the form

$$dX_{t,k}^i = dB_t^i + \beta\sum_{j\neq i}\left(\frac{1}{X_{t,k}^i - X_{t,k}^j} + \frac{1}{X_{t,k}^i + X_{t,k}^j}\right)dt + \frac{\nu\cdot\beta}{X_{t,k}^i}dt \qquad (i = 1,\dots,N) \tag{113}$$

with an N-dimensional Brownian motion $(B_t^1, \dots, B_t^N)_{t\geq 0}$. The renormalized processes $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{\beta})_{t\geq 0}$ then satisfy

$$d\tilde{X}_{t,k}^i = \frac{1}{\sqrt{\beta}} dB_t^i + \sum_{j\neq i}\left(\frac{1}{\tilde{X}_{t,k}^i - \tilde{X}_{t,k}^j} + \frac{1}{\tilde{X}_{t,k}^i + \tilde{X}_{t,k}^j}\right)dt + \frac{\nu}{\tilde{X}_{t,k}^i}dt \qquad (i = 1, \dots, N). \tag{114}$$

The SDE (114) degenerates for $\beta = \infty$ into the ODE

$$\frac{dx}{dt}(t) = H(x(t)), \quad \text{with} \tag{115}$$

$$H(x) := \left(\sum_{j\neq 1}\left(\frac{1}{x_1 - x_j} + \frac{1}{x_1 + x_j}\right) + \frac{\nu}{x_1}, \dots, \sum_{j\neq N}\left(\frac{1}{x_N - x_j} + \frac{1}{x_N + x_j}\right) + \frac{\nu}{x_N}\right).$$

We now collect several features of the solutions of (115) for $\nu > 0$ and $N \geq 2$ from Refs. 100, 102 that are analog to the preceding results:

Theorem 24. *For each starting point $x_0 \in C_N^B$, the ODE (115) has a unique solution for all $t \geq 0$ in the sense of Theorem 23.*

For the next results, we use the ordered zeros $z_1^{(\nu-1)}, \dots, z_N^{(\nu-1)}$ of $L_N^{(\nu-1)}$ and the vector $y \in C_N^B$ with

$$2\left(z_1^{(\nu-1)}, \dots, z_N^{(\nu-1)}\right) = (y_1^2, \dots, y_N^2). \tag{116}$$

Lemma 9. *For each $c \geq 0$, $x(t) := \sqrt{t + c^2} \cdot y$ is a solution of (115).*

Moreover, for each starting point x_0 in the interior of C_N^B, the solution x of (115) has the form

$$x(t) = \sqrt{2N(N + \nu - 1)t + \|x_0\|^2} \cdot \phi(t) \qquad (t \geq 0)$$

with $\|\phi(t)\| = 1$. Furthermore, $\lim_{t\to\infty} \phi(t) = \frac{y}{\sqrt{2N(N+\nu-1)}}$.

We remark that the spherical part ϕ in Lemma 9 satisfies some ODE similar to (104) in the A-case; see the proof of Lemma 3.4 in Ref. 102.

As in the preceding subsection, we need the solutions of (115) for limit theorems for $\beta \to \infty$. In fact, the following strong limit law from Ref. 8 is analog to Theorem 4:

Theorem 25. *Let $\nu > 0$ and x_0 be a point in the interior of C_N^B. Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type B with $k = (k_1, k_2) = (\beta \cdot \nu, \beta)$ and start in $\sqrt{\beta} \cdot x_0$ as well as the solution $x(t)$ of the ODE (115) with start in x_0. Then, for all $t > 0$,*

$$\sup_{0\leq s\leq t, \beta\geq 1} \|X_{s,k} - \sqrt{\beta}x(s)\| < \infty \qquad a.s.$$

In particular, $X_{t,(\nu\cdot\beta,\beta)}/\sqrt{\beta} \to x(t)$ for $\beta \to \infty$ locally uniformly in t a.s.

For an associated functional CLT, we again use the solution $x(t)$ of the ODE (115) and use the centered Gaussian process $(W_t)_{t\geq 0}$ that is the unique solution of the inhomogeneous linear SDE

$$dW_t^i = dB_t^i + \left(\sum_{j\neq i}\left(\frac{W_t^j - W_t^i}{(x_i(t)-x_j(t))^2} - \frac{W_t^j + W_t^i}{(x_i(t)+x_j(t))^2}\right) - \frac{\nu\cdot W_t^i}{x_i(t)^2}\right)dt \tag{117}$$

for $i = 1,\dots,N$ with $W_0 = 0$. Hence, in matrix notation, $dW_t = dB_t + A_\nu(t)W_t dt$ with

$$A_\nu(t)_{i,j} := \frac{1}{(x_i(t,x)-x_j(t,x))^2} - \frac{1}{(x_i(t)+x_j(t))^2} \quad (i\neq j),$$
$$A_\nu(t)_{i,i} := \sum_{j\neq i}\left(\frac{-1}{(x_i(t)-x_j(t))^2} - \frac{1}{(x_i(t)+x_j(t))^2}\right) - \frac{\nu}{x_i(t)^2} \qquad (i,j = 1,\dots,N). \tag{118}$$

Moreover,

$$W_t = e^{\int_0^t A_\nu(s,x)ds}\int_0^t e^{-\int_0^s A_\nu(u,x)du}dB_s \quad (t\geq 0). \tag{119}$$

The functional CLT is now as follows; see Ref. 100:

Theorem 26. *Let $\nu > 0$ and x_0 a point in the interior of C_N^B. Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type B_N with $k = (\nu\beta,\beta)$ starting at $\sqrt{\beta}\cdot x$. Then, for all $t > 0$,*

$$\sup_{0\leq s\leq t,\beta\geq 1}\sqrt{\beta}\cdot\|X_{s,k} - \sqrt{\beta}x(s) - W_s\| < \infty \qquad a.s., \tag{120}$$

that is, $X_{t,k} - \sqrt{\beta}x(t) \longrightarrow W_t$ for $\beta\to\infty$ locally uniformly in t a.s. with rate $O(1/\sqrt{\beta})$.

We now calculate the covariance matrices of W_t for starting points of the form $x_0 = c\cdot y$ with $c > 0$. Motivated by Lemma 9, we define the matrices $A_\nu = (A_{\nu,i,j})_{i,j}\in\mathbb{R}^{N\times N}$ with

$$A_{\nu,i,j} := \frac{1}{(y_i-y_j)^2} - \frac{1}{(y_i+y_j)^2},\quad A_{\nu,i,i} := \sum_{j\neq i}\left(\frac{-1}{(y_i-y_j)^2} - \frac{1}{(y_i+y_j)^2}\right) - \frac{\nu}{y_i^2} \tag{121}$$

for $i,j = 1,\dots,N$, $i\neq j$ and y as in (116). By Theorem 14, $E - 2A_\nu$ has the eigenvalues $2,4,\dots,2N$ that are independent of ν. With these notations, we obtain (see Ref. 100) that for $t > 0$, the covariance matrices $\Sigma_{\nu,t}\in\mathbb{R}^{N\times N}$ of W_t satisfy

$$\Sigma_{\nu,t} = (t+c^2)(E-2A_\nu)^{-1}\left(E - e^{\ln\frac{c^2}{t+c^2}(E-2A_\nu)}\right)$$

with the eigenvalues $\lambda_j^B(t,c) = \frac{1}{2j}\frac{(t+c^2)^{2j}-c^{4j}}{(t+c^2)^{2j-1}}$ with $k = 1,\dots,N$.

Remark 8. The eigenvalues of Σ_t in the cases A_{2N-1} and B_N satisfy

$$\lambda_i^B(t,c) = \lambda_{2i}^A(t,c\cdot\sqrt{2}) \qquad (i = 1,\dots,N)$$

independent of ν. This is related to the classical relation $H_{2N}(x) = c_N \cdot L_N^{(-1/2)}(x^2)$ for Hermite and Laguerre polynomials with suitable normalizations c_N.

5.3 | A further freezing limit for Bessel processes of type B

We now turn to freezing limits for Bessel processes $(X_{t,k})_{t\geq 0}$ of type B with multiplicities $k = (k_1, k_2)$ with $k_2 > 0$ fixed and $k_1 \to \infty$. We again consider the normalized processes $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k_1})_{t\geq 0}$ with

$$d\tilde{X}^i_{t,k} = \frac{1}{\sqrt{k_1}} dB^i_t + \frac{k_2}{k_1} \sum_{j\neq i} \left(\frac{1}{\tilde{X}^i_{t,k} - \tilde{X}^j_{t,k}} + \frac{1}{\tilde{X}^i_{t,k} + \tilde{X}^j_{t,k}} \right) dt + \frac{1}{\tilde{X}^i_{t,k}} dt \tag{122}$$

for $i = 1, \dots, N$. In the limit $k_1 = \infty$, we obtain

$$\frac{d\phi_i}{dt}(t) = 1/\phi_i(t) \qquad (i = 1, \dots, N), \tag{123}$$

that is, the solutions with start in $\phi(0) = x \in C_N^B$ are

$$\phi(t) = \left(\sqrt{2t + x_1^2}, \dots, \sqrt{2t + x_N^2} \right).$$

A comparison of (122) and (123) leads to the following strong limit result; see Ref. 8.

Theorem 27. *Let $k_1, k_2 > 0$, and let x be a point in the interior of C_N^B. Let $(X_{t,k})_{t\geq 0}$ be the associated Bessel processes $(X_{t,k})_{t\geq 0}$ of type B_N with $k = (k_1, k_2)$ starting in $\sqrt{k_1} \cdot x$, and let ϕ the solution of (123) starting in x. Then, for all $t > 0$,*

$$\sup_{0\leq s\leq t, k_1 \geq 1/2} \| X_{s,k} - \sqrt{k_1}\phi(s) \| < \infty \qquad a.s.$$

In particular, $X_{t,(k_1,k_2)}/\sqrt{k_1} \to \phi(t)$ for $k_1 \to \infty$ locally uniformly in t a.s.

Again, this can be extended to a functional CLT; see Ref. 8.

Theorem 28. *Fix some x in the interior of C_N^B. For $k_1, k_2 > 0$, consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type B with $k = (k_1, k_2)$, which start in $\sqrt{k_1} \cdot x$. Then, for all $k_2 > 0$ and $t > 0$,*

$$X_{t,(k_1,k_2)} - \sqrt{k_1} \cdot \left(\sqrt{2t + x_1^2}, \dots, \sqrt{2t + x_N^2} \right)$$

tends in distribution for $k_1 \to \infty$ to the normal distribution

$$N\left(0, \mathrm{diag}\left(\frac{t^2 + t x_1^2}{2t + x_1^2}, \dots, \frac{t^2 + t x_N^2}{2t + x_N^2} \right) \right).$$

In the next step, we turn to freezing limits for $k = (0, k_2)$ for $k_2 \to \infty$, that is, the case of Subsection 5.2 for $\nu = 0$. As in Section 3, we handle this case by using results for the root systems of type D that are analog to Subsections 5.1 and 5.2.

5.4 | Freezing limits for Bessel processes of type D

We here have a multiplicity $k \in [0, \infty[$, and the SDE for the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D is

$$dX^i_{t,k} = dB^i_t + k \sum_{j\neq i} \left(\frac{1}{X^i_{t,k} - X^j_{t,k}} + \frac{1}{X^i_{t,k} + X^j_{t,k}} \right) dt \quad (i = 1, \dots, N). \tag{124}$$

The renormalized processes $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k})_{t\geq 0}$ thus satisfy

$$d\tilde{X}^i_{t,k} = \frac{1}{\sqrt{k}} dB^i_t + \sum_{j\neq i} \left(\frac{1}{\tilde{X}^i_{t,k} - \tilde{X}^j_{t,k}} + \frac{1}{\tilde{X}^i_{t,k} + \tilde{X}^j_{t,k}} \right) dt \quad (i = 1, \dots, N). \tag{125}$$

This SDE degenerates for $k = \infty$ into the ODE

$$\frac{dx}{dt}(t) = H(x(t)), \quad \text{with} \tag{126}$$

$$H(x) := \left(\sum_{j\neq 1} \left(\frac{1}{x_1 - x_j} + \frac{1}{x_1 + x_j} \right), \dots, \sum_{j\neq N} \left(\frac{1}{x_N - x_j} + \frac{1}{x_N + x_j} \right) \right).$$

We now collect several features of the solutions of (126) from Refs. 100, 102 where most of them are analogs to the preceding results. For this, we use the N zeros $z_1 > \dots > z_{N-1} > z_N = 0$ of $L_N^{(-1)}$ where $z_1, \dots z_{N-1}$ are the zeros of $L_{N-1}^{(1)}$; see Subsection 2.1. Moreover, let $r \in C_N^D$ be the vector defined in (78).

Proposition 6.

(1) *For each starting point $x_0 \in C_N^D$, the ODE (126) has a unique solution $x(t)$ for $t \geq 0$ in the sense of Theorem 23 where $x(t)$ is in the interior of C_N^D for $t > 0$.*

(2) *For $c \geq 0$ and $x_0 = cr$, this solution is $x(t) := \sqrt{t + c^2} \cdot r$.*

(3) *For all starting points $x_0 \in C_N^D$, the solution of (126) satisfies*

$$x(t) = \sqrt{2N(N-1)t + \|x_0\|^2} \cdot \phi(t) \qquad (t \geq 0)$$

with some function ϕ with

$$\|\phi(t)\| = 1 \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = \frac{r}{\sqrt{2N(N-1)}} \quad \text{for} \quad t \geq 0.$$

(4) *If $x_0 = (x_{0,1}, \dots, x_{0,N})$ with $x_{0,N} = 0$, then the solution $x(t)$ satisfies $x(t)_N = 0$ for all t, and $(x(t)_1, \dots, x(t)_{N-1})$ solves (115) with dimension $N - 1$ and $\nu = 2$. In particular, if $x_{0,N} > 0$ or < 0, then for all t, $x(t)_N > 0$ or < 0, respectively.*

The following strong limit law from Ref. 8 is analog to Theorems 4 and 25:

Theorem 29. *Let x_0 be in the interior of C_N^B. Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D with start in $\sqrt{k}\cdot x_0$ as well as the solution $x(t)$ of (126) with start in x_0. Then, for all $t>0$,*

$$\sup_{0\leq s\leq t,\beta\geq\beta_0} \|X_{s,k}-\sqrt{\beta}x(s)\| < \infty \qquad a.s.$$

For an associated functional CLT, we again use the solution $x(t)$ of the ODE (126) and use the centered Gaussian process $(W_t)_{t\geq 0}$ that is the unique solution of the inhomogeneous linear SDE

$$dW_t^i = dB_t^i + \sum_{j\neq i}\left(\frac{W_t^j - W_t^i}{(x_i(t)-x_j(t))^2} - \frac{W_t^j + W_t^i}{(x_i(t)+x_j(t))^2}\right)dt \qquad (i=1,\dots,N) \tag{127}$$

with $W_0=0$, that is, in matrix notation, $dW_t = dB_t + A_\nu(t)W_t dt$ with

$$A(t)_{i,j} := \frac{1}{(x_i(t,x)-x_j(t,x))^2} - \frac{1}{(x_i(t)+x_j(t))^2} \quad (i\neq j),$$
$$A(t)_{i,i} := \sum_{j\neq i}\left(\frac{-1}{(x_i(t)-x_j(t))^2} - \frac{1}{(x_i(t)+x_j(t))^2}\right) \tag{128}$$

for $i,j=1,\dots,N$. The functional CLT is now as follows; see Ref. 100.

Theorem 30. *Let x_0 be in the interior of C_N^D. For $k\geq 1$, consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D starting at $\sqrt{k}\cdot x_0$, and let $x(t)$ be the solution of (126) starting in x_0. Then, for $t>0$,*

$$\sup_{0\leq s\leq t,k\geq 1} \sqrt{k}\cdot\|X_{s,k}-\sqrt{k}x(s)-W_s\| < \infty \qquad a.s. \tag{129}$$

We next consider the covariance matrix of W_t for the special solutions $x(t) := \sqrt{t+c^2}\cdot y$. For this, let $A\in\mathbb{R}^{N\times N}$ with

$$A_{i,j} := \frac{1}{(r_i-r_j)^2} - \frac{1}{(r_i+r_j)^2}, \quad A_{i,i} := \sum_{j\neq i}\left(\frac{-1}{(r_i-r_j)^2} - \frac{1}{(r_i+r_j)^2}\right) \tag{130}$$

for $i,j=1,\dots,N$, $i\neq j$ with r as in (78). Then $E-2A$ has the eigenvalues $2,4,\dots,2N$. Hence:

Lemma 10. *Assume that the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D_N start in the points $\sqrt{k}\cdot cr$ with $c>0$ and the vector r above. Then, the covariance matrices Σ_t of W_t are*

$$\Sigma_t = (t+c^2)(E-2A)^{-1}\left(E - e^{\ln\frac{c^2}{t+c^2}(E-2A)}\right).$$

Remark 9. Assume that in the situation of Theorem 30, the Nth coordinate of the starting point x_0 satisfies $x_{0,N}=0$. Then, $x(t)_N=0$ for all $t\geq 0$, that is, the matrices $A(t)$ in Equation (128) satisfy $A(t)_{N,j} = A(t,x)_{j,N} = 0$ for $j\neq N$ and $t>0$. This means that the Nth component of $(W_t^{(N)})_{t\geq 0}$ is independent from $(W_t^{(1)},\dots,W_t^{(N-1)})_{t\geq 0}$.

This situation appears in particular in the setting of Lemma 10.

We now apply Subsection 5.4 to the B-case for $k = (0, k_2)$ for $k_2 \to \infty$.

5.5 | A further freezing limit in the B-case

We recapitulate from Section 3 that for $k_2 \geq 0$ and a Bessel process $(X^D_{t,k_2})_{t\geq 0}$ of type D with multiplicity $k \geq 0$ on C^D_N with starting point $x \in C^D_N$, the process $(X^B_{t,k})_{t\geq 0}$ with

$$X^{B,i}_{t,k} := X^{D,i}_{t,k} \quad (i = 1, \dots, N-1), \quad X^{B,N}_{t,k} := |X^{D,N}_{t,k}|$$

is a Bessel process of type B with $k = (0, k_2)$ with starting point $(x_1, \dots, x_{N-1}, |x_N|) \in C^B_N$ with $x_1 > \dots > x_{N-1} > |x_N| \geq 0$. where $(X^B_{t,k})_{t\geq 0}$ is a diffusion with reflecting boundaries where now the boundary parts with the Nth coordinate equal to zero are attained.

We now translate the results of Subsection 5.4 and consider the solutions $x(t)$ of the ODE (126) with start in x in the following two particular cases:

(1) If x is in the interior of C^B_N, then $x(t)$ will be also in the interior of C^B_N for all $t \geq 0$ by Proposition 6(4).
(2) If $x_1 > \dots > x_{N-1} > x_N = 0$, then we have $x(t)_1 > \dots > x(t)_{N-1} > x(t)_N = 0$ for all $t \geq 0$.

Case (2) appears in particular for the special solutions $x(t) = \sqrt{t + c^2} \cdot r$ for $c > 0$ and the vector r from (78) with $r_N = 0$.

Theorem 29 now reads as follows for the B-case with $(k_1, k_2) = (0, k)$ for $k \to \infty$:

Theorem 31. *Let x be a point as described above in (1) or (2). For $k \geq 1/2$, consider the Bessel processes $(X_{t,(0,k)})_{t\geq 0}$ of type B_N starting in $\sqrt{k} \cdot x$. Then, for all $t > 0$,*

$$\sup_{0\leq s\leq t, k\geq 1/2} \|X_{s,k} - \sqrt{k}x(s)\| < \infty \qquad a.s..$$

We next use the Gaussian processes $(W_t)_{t\geq 0}$ satisfying (127). Theorem 30 now leads to functional CLTs where the cases (1) and (2) have to be treated separately for geometric reasons. For the case (1), we have the following result; see Ref. 100.

Theorem 32. *Let x be a point in the interior of C^B_N. For $k \geq 1/2$, consider Bessel processes $(X_{t,(0,k)})_{t\geq 0}$ of type B starting at $\sqrt{k} \cdot x$. Then, for all $t > 0$,*

$$\sup_{0\leq s\leq t, k\geq k_0} \sqrt{k} \cdot \|X_{s,(0,k)} - \sqrt{k}x(s) - W_s\| < \infty \qquad a.s. \tag{131}$$

Theorem 32 corresponds to Theorem 26 for $\nu = 0$. We finally turn to case (2); see Ref. 100.

Theorem 33. *Let $x \in C_N^B$ with $x_1 > \dots > x_{N-1} > x_N = 0$. For $k \geq 1/2$, consider Bessel processes $(X_{t,(0,k)})_{t\geq 0}$ of type B starting at $\sqrt{k}\cdot x$. Then, with the process*

$$(\tilde{W}_t := (W_t^{(1)}, \dots, W_t^{(N-1)}, |W_t^{(N)}|))_{t\geq 0},$$

and all $t > 0$,

$$\sup_{0\leq s\leq t, k\geq 1} \sqrt{k}\cdot \|X_{s,(0,k)} - \sqrt{k}x(s) - \tilde{W}_s\| < \infty \qquad a.s. \tag{132}$$

In the end of this section, we briefly discuss some freezing limits in the Heckman–Opdam case. As this is still work in progress, we only discuss a few ideas. We begin with the noncompact Heckman–Opdam case of type A.

5.6 | Freezing limits for the noncompact Hockman–Opdam case of type A

Following Subsection 2.4, we here start with the W-invariant Heckman–Opdam Laplacians

$$L_k f(x) = \Delta f(x) + k\sum_{j=1}^N \sum_{l\neq j} \coth\left(\frac{x_j - x_l}{2}\right)\frac{\partial}{\partial x_j} f(x) \tag{133}$$

on the Weyl chamber C_N^A for $k \in [0,\infty[$. As in the preceding section, we also consider the renormalized generators $\widetilde{L}_k := \frac{1}{k}L_k$ that degenerate for $k \to \infty$ into

$$\widetilde{L}_\infty = \sum_{j=1}^N \sum_{l\neq j} \coth\left(\frac{x_j - x_l}{2}\right)\frac{\partial}{\partial x_j}.$$

The process $(\widetilde{X}_{t,k} := X_{t/k,k})_{t\geq 0}$ for $k \in]0,\infty[$ then solves the SDE

$$d\widetilde{X}_{t,k,j} = \frac{\sqrt{2}}{\sqrt{k}} dB_{t,j} + \sum_{l\neq j} \coth\left(\frac{\widetilde{X}_{t,k,j} - \widetilde{X}_{t,k,l}}{2}\right) dt \qquad (j = 1, \dots, N), \tag{134}$$

which degenerates for $k = \infty$ into the ODE

$$\frac{dx_j}{dt}(t) = \sum_{l\neq j} \coth\left(\frac{x_j(t) - x_l(t)}{2}\right) \qquad (j = 1, \dots, N). \tag{135}$$

Again, for initial data in the interior of the chamber C_N^A, the solution $(\widetilde{X}_{t,\infty})_{t\geq 0}$ of these differential equations exists for all $t \geq 0$ in the interior of C_N^A.

Like Bessel processes of type A (see the comments after Lemma 2), the diffusions $\widetilde{X}_k$ can be decomposed into the center-of-gravity process $\widetilde{X}_k^{cg} := (\widetilde{X}_{t,k}^{cg})_{t\geq 0}$ with

$$\widetilde{X}_{t,k}^{cg} := \frac{1}{N}(\widetilde{X}_{t,k,1} + \dots + \widetilde{X}_{t,k,N}) \cdot \mathbf{1}$$

and the process $\widetilde{X}_k^{diff} := \widetilde{X}_k - \widetilde{X}_k^{cg}$ on $C_{N,0}^A := \{x \in C_N^A : x_1 + \cdots + x_N = 0\}$, which describes the distances of the particles. The processes $\widetilde{X}_k^{diff}$ and $\widetilde{X}_k^{cg}$ are stochastically independent.

Using the ideas of the proof of Theorem 2.4 and related results in Ref. 8, it is easy to transfer Theorem 4 as follows:

Theorem 34. *Let x_0 be a in the interior of C_N^A. For $k \geq 1$, let $(\tilde{X}_{t,k})_{t\geq 0}$ be a normalized Heckman–Opdam process of type A satisfying (134) with start in x_0, and ϕ the solution of (135) starting in x_0. Then, locally uniformly in t, $\tilde{X}_{t,k} \to x(t)$ for $k \to \infty$ a.s.*

Unfortunately, no solutions of the ODEs (134) are known except for $N = 2, 3$. It is likely that there exists an associated CLT similar to Theorem 5.

Notice that Theorem 34 is concerned with time rescalings of Heckman–Opdam processes, and that limit theorems concerning space rescalings here have a different form.

5.7 | Freezing limits for the noncompact Hockman–Opdam case of type BC

We here start with the nonreduced root system BC_N as in (26) with $k = (k_1, k_2, k_3)$, where $k_1, k_2 \geq 0$, $k_3 > 0$ are the values on the roots $e_i, 2e_i, e_i \pm e_j$. We now follow[28,85,98] and reparameterize the multiplicity k via

$$\kappa = k_3, \quad q = N - 1 + \frac{1 + 2k_1 + 2k_2}{2k_3}, \quad p = N - 1 + \frac{1 + 2k_2}{2k_3} \tag{136}$$

with $\kappa > 0$, $q \geq p \geq N - 1 + 1/2\kappa$, and

$$k = (k_1, k_2, k_3) = \kappa \cdot \big(q - p, k_{0,2}, 1\big) \quad \text{with} \quad k_{0,2} := p - (N-1) - \frac{1}{2\kappa}. \tag{137}$$

We now fix p, q and consider $\kappa \to \infty$. For this, we write the multiplicity k in (137) by k_κ. The associated Heckman–Opdam Laplacian (42) is

$$L_\kappa = \Delta + \sum_{i=1}^N \kappa\Bigg((q-p)\coth\left(\frac{x_i}{2}\right) + 2k_{0,2}\coth(x_i) + \sum_{j: j\neq i}\left(\coth\left(\frac{x_i - x_j}{2}\right) + \coth\left(\frac{x_i + x_j}{2}\right)\right)\Bigg)\partial_i \tag{138}$$

on the Weyl chamber C_N^B. The operators L_κ are the generators of diffusions $(X_{t,\kappa})_{t\geq 0}$ on C_N^B, and the renormalized operators

$$\widetilde{L}_\kappa := \frac{1}{\kappa} L_\kappa$$

then are the generators of the diffusions $(\widetilde{X}_{t,\kappa} := X_{t/\kappa,\kappa})_{t\geq 0}$ that may be seen as solutions of the SDEs

$$d\widetilde{X}_{t,\kappa,j} = \frac{\sqrt{2}}{\sqrt{\kappa}} dB_{t,j} + \sum_{l\neq j}\left(\coth\left(\frac{\widetilde{X}_{t,\kappa,j}-\widetilde{X}_{t,\kappa,l}}{2}\right) + \coth\left(\frac{\widetilde{X}_{t,\kappa,j}+\widetilde{X}_{t,\kappa,l}}{2}\right)\right)dt$$
$$+\left((q-p)\coth\left(\frac{\widetilde{X}_{t,\kappa,j}}{2}\right) + 2k_{0,2}\coth(\widetilde{X}_{t,\kappa,j})\right)dt \qquad (j=1,\dots,N). \tag{139}$$

For $\kappa\to\infty$, these SDEs degenerate into the ODE

$$\frac{dx_j}{dt}(t) = \sum_{l\neq j}\left(\coth\left(\frac{x_j(t)-x_l(t)}{2}\right) + \coth\left(\frac{x_j(t)+x_l(t)}{2}\right)\right)$$
$$+(q-p)\coth\left(\frac{x_j(t)}{2}\right) + 2(p-(N-1))\coth x_j(t) \qquad (j=1,\dots,N). \tag{140}$$

Again, for initial data in the interior of the chamber C_N^B, the solutions $(\widetilde{X}_{t,\kappa})_{t\geq 0}$ and $((x(t))_{t\geq 0}$ of (139) and (140) exist for $t\geq 0$ in the interior of C_N^B.

It is now possible to restate Theorem 34 in this setting. We point out that also an analog of Theorems 23 and 24 is available; see Ref. 14.

Theorem 35. *For each starting point $x_0\in C_N^B$, the ODE (140) has a unique solution for all $t\geq 0$ in the sense of Theorem 23.*

It is likely that a corresponding result holds in the A-case in Subsection 5.6.

In the end of the article, we briefly consider the compact BC-case, that is, Jacobi processes.

5.8 | Freezing limits for the compact Hockman–Opdam case of type BC

For the multiplicity $k=(k_1,k_2,k_3)$, the trigonometric Heckman–Opdam Laplacian (49) here has the form

$$\widehat{L}_k f(t) := \Delta f(t) + \sum_{i=1}^N\left(k_1\cot\left(\frac{t_i}{2}\right) + 2k_2\cot(t_i)\right.$$
$$\left. +k_3\sum_{j:j\neq i}\left(\cot\left(\frac{t_i-t_j}{2}\right)+\cot\left(\frac{t_i+t_j}{2}\right)\right)\right)\partial_i f(t). \tag{141}$$

We now follow Refs. 14, 28, 98 and use the transformation $x_i=\cos t_i$ $(i=1,\dots,N)$ from Subsection 3.4, which transforms this into the algebraic Heckman–Opdam Laplacian

$$L_k f(x) := \sum_{i=1}^N (1-x_i^2) f_{x_i,x_i}(x) + \sum_{i=1}^N\left(-k_1-(1+k_1+2k_2)x_i + 2k_3\sum_{j:j\neq i}\frac{1-x_i^2}{x_i-x_j}\right)f_{x_i}(x), \tag{142}$$

which is the generator of a Feller diffusion on the alcove

$$A_N := \{x \in \mathbb{R}^N \mid -1 \le x_1 \le \ldots \le x_N \le 1\}.$$

As before, the associated diffusions $(X_t)_{t\ge 0}$ can also be seen as solutions of the SDEs

$$dX_{t,i} = \sqrt{2(1-X_{t,i}^2)}\, dB_{t,i} + \left(-k_1 - (1+k_1+2k_2)X_{t,i} + 2k_3 \sum_{j:\, j\ne i} \frac{1-X_{t,i}^2}{X_{t,i}-X_{t,j}}\right)dt, \tag{143}$$

for $i = 1, \ldots, N$ and an N-dimensional Brownian motion $(B_t)_{t\ge 0}$. For freezing limits, we again follow Refs. 14, 28, 98 and change the parameters by

$$\kappa = k_3, \quad q = N - 1 + \frac{1+2k_1+2k_2}{2k_3}, \quad p = N - 1 + \frac{1+2k_2}{2k_3}. \tag{144}$$

The SDE (143) then has the form

$$dX_{t,i} = \sqrt{2(1-X_{t,i}^2)}\, dB_{t,i} + \kappa\Bigg((p-q) + (2(N-1)-(p+q))X_{t,i} + 2\sum_{j:\, j\ne i} \frac{1-X_{t,i}^2}{X_{t,i}-X_{t,j}}\Bigg)dt.$$

The transformed diffusions $(\tilde{X}_t := X_{t/\kappa})_{t\ge 0}$ with the generators $\tilde{L}_k := \frac{1}{\kappa} L_k$ then satisfy

$$d\tilde{X}_{t,i} = \frac{\sqrt{2}}{\sqrt{\kappa}} \sqrt{1-\tilde{X}_{t,i}^2}\, d\tilde{B}_{t,i} + \left((p-q) - (p+q)\tilde{X}_{t,i} + 2\sum_{j\ne i} \frac{1-\tilde{X}_{t,i}\tilde{X}_{t,j}}{\tilde{X}_{t,i}-\tilde{X}_{t,j}}\right)dt \tag{145}$$

for $i = 1, \ldots, N$. For the freezing case $\kappa = \infty$, (145) degenerates into the ODE

$$\frac{dx_i}{dt}(t) = (p-q) - (p+q)x_i(t) + 2\sum_{j\ne i} \frac{1-x_i(t)x_j(t)}{x_i(t)-x_j(t)} \qquad (i = 1, \ldots, N). \tag{146}$$

By Ref. 14, this ODE has the following properties that correspond to Theorems 23 and 24 and Proposition 6.

Proposition 7.

(1) *The ODE (146) has a unique stationary solution in the interior of A_N. This solution is given by the vector z with the ordered zeros of the classical Jacobi polynomial $P_N^{(q-N,p-N)}$ as entries.*
(2) *For each starting point $x_0 \in A_N$, the ODE (146) has a unique solution $x(t)$ for $t \ge 0$ in the sense of Theorem 23 with $x(t)$ in the interior of A_N for $t > 0$. Moreover, $\lim_{t\to\infty} x(t) = z$.*

Also Theorem 34 can be stated in this context.

We finally consider a further limit that is connected with Remark 5(1). We fix constants $a, b, k_3 \in \mathbb{R}$ with $k_3, b, a+b > 0$, and study for $\kappa > 0$, diffusions $(X_{t,\kappa})_{t\ge 0}$ that solve (143) for the parameter (k_1, k_2, k_3) with $k_1 = \kappa a$, $k_2 = \kappa b$. A short computation yields that then the processes

$(\hat{X}_{t,\kappa})_{t\geq 0}$ with

$$\hat{X}_{t,\kappa} := \sqrt{\kappa}\left(X_{t/\kappa,\kappa} - \frac{-a}{a+2b}\mathbf{1}\right) \tag{147}$$

satisfy

$$\begin{aligned} d\hat{X}_{t,\kappa,i} =& \sqrt{2}\sqrt{1-\left(\frac{-a}{a+2b}+\frac{\hat{X}_{t,\kappa,i}}{\sqrt{\kappa}}\right)^2}\, dB_{t,i} \\ &+\left(\frac{a}{\sqrt{\kappa}(a+2b)} - \left(\frac{1}{\kappa}+a+2b\right)\hat{X}_{t,\kappa,i} + 2k_3 \sum_{j:j\neq i} \frac{1-\left(\frac{-a}{a+2b}+\frac{\hat{X}_{t,\kappa,i}}{\sqrt{\kappa}}\right)^2}{X_{t,\kappa,i}-X_{t,\kappa,j}}\right)dt, \end{aligned} \tag{148}$$

which degenerates for $\kappa = \infty$ into

$$d\hat{X}_{t,\infty,i} = \sqrt{2}\sqrt{1-\frac{a^2}{(a+2b)^2}}\, dB_{t,i} + \left(-(a+2b)\hat{X}_{t,\infty,i} + 2k_3\sum_{j:j\neq i}\frac{1-\frac{a^2}{(a+2b)^2}}{X_{t,\infty,i}-X_{t,\infty,j}}\right)dt. \tag{149}$$

The solutions of (149) are Ornstein–Uhlenbeck-type asymptotically stationary versions of Bessel processes of type A (see, e.g., Refs. 80, 101) with transition densities of the form (6) (after suitable transformations of all parameters). This fits to Remark 5(1) where in the limit $\kappa \to \infty$ and with the transformations in (147), the stationary measures of our Jacobi processes tend to the stationary distributions of these Ornstein–Uhlenbeck-type Bessel processes of type A. This idea also works for other limits, and in this way, this SDE approach offers a method to derive several limit theorems.

In general, the SDE approach to particle processes is quite powerful. In fact, in this article, we discussed exclusively limits for fixed numbers N of particles except for Theorem 12 on the largest particles in the freezing regime for $N \to \infty$. The SDE techniques and the underlying symmetries can also be used to derive almost sure limit theorems for the empirical measures

$$\frac{1}{N}(\delta_{\tilde{X}_t^1} + \cdots + \delta_{\tilde{X}_t^N}) \tag{150}$$

of possibly suitably transformed particle processes $(\tilde{X}_t = (\tilde{X}_t^1, \dots, \tilde{X}_t^N))_{t\geq 0}$ for $N \to \infty$ under suitable assumptions on the starting configurations for $t = 0$ where then free convolutions (see, e.g., NS) and distributions like Wigner and Marchenko–Pastur distributions appear. Results in this directions can be found in Refs. 3, 14, 24, 27, 101 and references there.

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