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Abstract

We propose an interpretation of modal logic through stochastic relations, providing a probabilistic complement to the usual nondeterministic interpretations using Kripke models. A simple temporal logic and a logic with a countable number of diamonds illustrate the approach. The main technical result of this paper is a probabilistic analogon to the well-known Hennessy-Milner Theorem characterizing models that have the same theories for their states and bisimilarity as equivalent properties. This requires the study of congruences for stochastic relations that underly the interpretation, for which a general bisimilarity result is also established. The results depend on the existence of semi-pullbacks for stochastic relations over analytic spaces.
Keywords: Stochastic relations, stochastic Kripke models, congruences, bisimulations, modal logic, Hennessy-Milner Theorem.

1 Introduction

Modal logics are usually interpreted through relations: given the modal operator $\triangle$ of arity $n > 0$, a relation $R_\triangle \subseteq S \times S^n$ over the state space S defines that satisfaction relation. A move can be made to a state s with $s \models \triangle(\varphi_1, \ldots, \varphi_n)$ from states $s_1, \ldots, s_n$ such that $s_1 \models \varphi_1$, ..., $s_n \models \varphi_n$ provided s and $s_1, \ldots, s_n$ are related through $R_\triangle$, thus provided $\langle s, s_1, \ldots, s_n \rangle \in R_\triangle$ holds. This paper proposes an interpretation of modal logics through stochastic relations.
Working in a countable state space, we could assign $\triangle$ a discrete transition probability $p_\triangle$ from S to S^n; the probability for $s \models \triangle(\varphi_1, \ldots, \varphi_n)$ can then be described as

$$\sum_{s_1 \models \varphi_1, \ldots, s_n \models \varphi_n} p_\triangle(s)(s_1, \ldots, s_n).$$

This entails in particular a description of s not satisfying the formula at all (when the latter probability in zero) or of s satisfying it properly (when the probability is one).
Finite or countable state spaces are rather special cases, and we discuss state spaces in general. A probabilistic interpretation of a given modal logic assigns then to each modal operator $\triangle$ a stochastic relation $K_\triangle$ of a suitable arity, in addition we cater for the validity of propositional variables which are being taken care of through suitable measurable subsets of the state space. We first study the relationship between probabilistic and nondeterministic interpretations and argue that a stochastic interpretation $\mathcal{K}$ is usually more precise than its nondeterministic cousin $\mathcal{R}$, since $\mathcal{R}$ only provides binary arguments (a formula holds in a state or it does not) whereas $\mathcal{K}$ adds quantitative information (a state can satisfy a formula with a probability of more than, say, 0.7). This leads to the notion of probabilistic refinement: $\mathcal{K}$ refines $\mathcal{R}$ iff each state that satisfies an arbitrary formula from the negation free part of the logic probabilistically does satisfy it also nondeterministically. With $\mathcal{K}$ comes a coarser nondeterministic interpretation: we simply list all possibilities offered by $\mathcal{K}$ and package it as an interpretation. The converse does also hold. We can find under some suitable and quite natural assumptions a stochastic interpretation $\mathcal{K}$ for the nondeterministic interpretation $\mathcal{R}$ refining $\mathcal{R}$: we collect all states that we want to have assigned positive probability and construct a stochastic relation from it.
These simple arguments need to be refined, of course, because in a general probabilistic setting it is difficult to make the notion of *possibilities offered* by a probability or *positive probability* precise. That can be done provided a topological assumption of the state space is added: we assume that the state space is Polish, i.e., a topological space that can be made into a complete metric space and that has a countable dense subset (the reals $\mathbb{R}$ yield an important instance). Polish spaces are a natural generalization of finite and countable spaces, and permit most of the constructions which are necessary in the context described here.
The main technical result of this paper is a probabilistic analogon of the Hennessy-Milner Theorem [3, Theorem 2.24]. It relates two models of a given modal logic through bisimulation: they are bisimilar iff given a state in one model one can find another state in the other one having the same theory, and vice versa. This has been known essentially for the special case of a countable number of unary modal operators under different probabilistic assumptions [4, 9, 7, 8]. One cornerstone in all these constructions is the existence of semi-pullbacks in categories of stochastic relations. This could be established in [7, 8] for Polish and for analytic spaces as the spaces underlying these relations. The other cornerstone is the factorization of a direct sum after a suitable equivalence relation.

The equivalence relation for factorization arises naturally from the logic: two states are said to be equivalent iff they satisfy exactly the same formulas, i.e., iff they have the same theory. When analyzing this relation it turns out that its most important property is its having a countable generator: there exists a countable family of measurable sets such that two states are equivalent iff either both are in each set or both are outside each set. The theory of Borel sets (see e.g. [16]) calls equivalence relations like that *smooth*. These relations have a number of pleasant properties, among others, that their factor space is an analytic space, hence is not too ill-behaved. Since we work with a probabilistic interpretation with stochastic relations between two sets, we have to deal in general with pairs of smooth equivalence relations. This leads to the notion of a congruence for stochastic relation, which is studied here briefly: a congruence (α, β) for a stochastic relation K between the Polish spaces X and Y is a smooth equivalence relation α on X and β on Y such that α-equivalent inputs to X behave in exactly the same way on the β-invariant subsets of Y. This is just what we need when we want to establish a probabilistic analogon of the Hennessy-Milner Theorem. But it is not yet enough: we need a characterization for equivalent states which have the same theory. This is formulated in a more general way, and leads to the notion of equivalent congruences, which in turn permit the description of the same behavior. Two congruences are equivalent if, roughly speaking, they may generate each other, and if they display the same behavior.
Congruences are briefly studied, the factor space is constructed (hereby leaving the realm of Polish and entering the world of analytic spaces), it is shown that each morphism gives rise to a congruence. Given two stochastic relations with equivalent congruences, we show how to construct a morphism from each relation to the factor space constructed from the sum, which then leads to the existence of a semi-pullback. When dealing with modal logics, a stochastic interpretation may be viewed as a family of stochastic relations, thus the notion of morphism between relations carries over, and equivalent models are bisimilar. This is a direct translation of the result for stochastic relations. It does not entail, however, the reverse statement: bisimilar models are equivalent. Strengthening the notion of a morphism by introducing a condition of uniformity then gives the desired version of the Hennessy-Milner Theorem for stochastic Kripke models.

Related Work Larsen and Skou introduce in their seminal paper [13] the notion of a probabilistic interpretation for a Hennessy-Milner logic. The interpretation is based on discrete probability spaces. This was generalized in [4] to analytic spaces with universally measurable transition probabilities and in [7, 8] to Polish resp. analytic spaces with Borel measurable transition probabilities. All these different notions of measurability have to do with the possibility of establishing the existence of semi-pullbacks in the corresponding categories. The work so far concentrates on probabilistic interpretations of different aspects of the Hennessy-Milner logic as a modal logic with a countable number of unary modal operators. [5] introduces the notion of a congruence (called *bisimulation* there) for labelled Markov transition systems and investigates approximations through suitable chosen domains. These congruences are, however, not investigated from an algebraic point of view.

Organization Section 2 introduces stochastic relations and collects some measure theoretic results that are useful, an interpretation of modal logic through stochastic Kripke models is proposed in 3, and is illustrated through two well-known logics. Section 4 relates nondeterministic and stochastic Kripke models through refinements, and it is shown under which conditions a nondeterministic Kripke model has a stochastic refinement. Smooth equivalence relations are introduced in Section 5, and factor spaces are introduced there, too; this Section also defines the important notion of equivalent congruences. Section 6 investigates morphisms, and a first bisimulation result is established. Section 7 establishes the paper's main result, Section 8 wraps it all up and indicates some directions for further work.

2 Stochastic Relations

This Section defines stochastic relations and collects some basic facts from topology and measure theory for the reader's convenience and for later reference.

A *Polish space* $(X, \mathcal{G})$ is a topological space which is second countable, i.e., which has a countable dense subset, and which is metrizable through a complete metric. A *measurable space* $(X, \mathcal{A})$ is a set X with a σ-algebra $\mathcal{A}$. The *Borel sets* $\mathcal{B}(X, \mathcal{G})$ for the topology $\mathcal{G}$ is the smallest σ-algebra on X which contains $\mathcal{G}$. Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a map $f : X \to Y$ is $\mathcal{A} - \mathcal{B}$*-measurable* whenever

$$f^{-1}\left[\mathcal{B}\right] \subseteq \mathcal{A}$$

holds, where

$$f^{-1}\left[\mathcal{B}\right] := \{f^{-1}[B] | B \in \mathcal{B}\}$$

is the set of inverse images

$$f^{-1}[B] := \{x \in X | f(x) \in B\}$$

of elements of $\mathcal{B}$. Note that $f^{-1}\left[\mathcal{B}\right]$ is a σ-algebra, provided $\mathcal{B}$ is one. If the σ-algebras are the Borel sets of some topologies on X and Y, resp., then a measurable map is called *Borel measurable* or simply a *Borel* map. The real numbers $\mathbb{R}$ carry always the Borel structure induced by the usual topology which will usually not be mentioned explicitly when talking about Borel maps.

The product $(X_1 \times X_2, \mathcal{A}_1 \otimes \mathcal{A}_2)$ of two measurable spaces $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$ is the Cartesian product $X_1 \times X_2$ endowed with the σ-algebra

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := \sigma\left(\{A_1 \times A_2 | A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}\right).$$

This is the smallest σ-algebra which contains all the measurable rectangles $A_1 \times A_2$, and it is the smallest σ-algebra $\mathcal{E}$ on $X_1 \times X_2$ which makes the projections $\pi_i : X_1 \times X_2 \to X_i$ $\mathcal{E} - \mathcal{A}_i$-measurable for $i = 1, 2$. Given finite measures μ_i on $\mathcal{A}_i$, there exists a unique finite measure $\mu_1 \otimes \mu_2$ on $\mathcal{A}_1 \otimes \mathcal{A}_2$ such that

$$\left(\mu_1 \otimes \mu_2\right)\left(A_1 \times A_2\right) = \mu_1(a_1) \cdot \mu_2(A_2)$$

holds for each $A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2$.

An *analytic set* $X \subseteq Z$ for a Polish space Z is the image $f\left[Y\right]$ of a Polish space Y for some Borel measurable map $f : Y \to Z$. Endow X with the trace $\mathcal{A}$ of $\mathcal{B}(Z)$ on X, i.e.,

$$\mathcal{A} = \mathcal{B}(Z) \cap X := \{B \cap X | B \in \mathcal{B}(Z)\},$$

the elements of which still being called Borel sets. A measurable space $(X', \mathcal{A}')$ which is Borel isomorphic to $(X, \mathcal{A})$ is called an *analytic space* (a Borel isomorphism is a Borel measurable and bijective map the inverse of which is also Borel measurable).

When the context is clear, we will write down topological or measurable spaces without their topologies and σ-algebras, resp., and the Borel sets are always understood with respect to the topology under consideration.
Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a *stochastic relation* $K : (X, \mathcal{A}) \rightsquigarrow (Y, \mathcal{B})$ is a Borel map from X to the set $\mathbf{S}(Y, \mathcal{B})$, the latter denoting the set of all subprobability measures on $(Y, \mathcal{B})$ which carries the *weak*-σ-algebra*. This is the smallest σ-algebra on $\mathbf{S}(Y, \mathcal{B})$ which renders all maps $\mu \mapsto \mu(D)$ measurable, where $D \in \mathcal{B}$. Hence $K : (X, \mathcal{A}) \rightsquigarrow (Y, \mathcal{B})$ is a *stochastic relation* iff

1. $K(x)$ is a subprobability measure on $(Y, \mathcal{B})$ for all $x \in X$,
2. $x \mapsto K(x)(D)$ is a measurable map for each measurable set $D \in \mathcal{B}$.

We will deal usually with stochastic relations between Polish or between analytic spaces. Accordingly, we call then (X, Y, K) a *Polish* respectively an *analytic object*.
An $\mathcal{A} - \mathcal{B}$- measurable map $f : X \to Y$ between the measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ induces a map $\mathbf{S}(f) : \mathbf{S}(X, \mathcal{A}) \to \mathbf{S}(Y, \mathcal{B})$ upon setting

$$\mathbf{S}(f)(\mu)(D) := \mu(f^{-1}[D])$$

(for $\mu \in \mathbf{S}(X, \mathcal{A}), D \in \mathcal{B}$). It is easy to see that $\mathbf{S}(f)$ is measurable.

3 Interpretations of Modal Logic

We show in this section how a probabilistic interpretation of a general modal logic can be constructed, generalizing the well-known interpretation given for the Hennessy-Milner logic through labelled Markov transition systems which are introduced formally in Example 2.
Let P be a countable set of propositional letters which is fixed throughout, $O \neq \emptyset$ is a set of modal operators. Following [3], $\tau = (O, \rho)$ is called a *modal similarity type* iff $O \neq \emptyset$, and if $\rho : O \to \mathbb{N}$ is a map, assigning each modal operator $\triangle$ its arity $\rho(\triangle) \geq 1$. We will not deal with modal operators of arity zero, since as modal constants they do not have to be dealt with in an interpretation. The similarity type τ will be fixed.
We define three modal languages based on τ and P. The formulas of the *basic modal language* $\mathbf{Mod_b}(\tau, P)$ are given by the syntax

$$\varphi ::= p \mid \top \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \triangle(\varphi_1, \ldots \varphi_{\rho(\triangle)}),$$

where $p \in P$. Omitting negation defines the formulas in the *negation free basic modal language* $\mathbf{Mod_1}(\tau, P)$. Finally the *extended modal language* $\mathbf{Mod_s}(\tau, P)$ is defined through the syntax

$$\varphi ::= p \mid \top \mid \varphi_1 \wedge \varphi_2 \mid \neg\varphi \mid \triangle_q(\varphi_1, \ldots \varphi_{\rho(\triangle)}),$$

where $q \in \mathbb{Q} \cap [0, 1]$ is a rational number, and $p \in P$ is a propositional letter.
A *nondeterministic* τ*- Kripke model* $\mathcal{R} = (S, R_\tau, V)$ consists of a state space S, a family $R_\tau = ((R_\triangle)_{\triangle \in O})$ of set valued maps $R_\triangle : S \to \mathcal{P}\left(S^{\rho(\triangle)}\right)$ and a set valued map $V : P \to \mathcal{P}(S)$.
The satisfaction relation for a nondeterministic τ-Kripke model $\mathcal{R}$ is defined as usual for $\mathbf{Mod_b}(\tau, P)$:

- $\mathcal{R}, s \models p \Leftrightarrow s \in V(p)$

- $\mathcal{R}, s \models \neg\varphi \Leftrightarrow \mathcal{R}, s \not\models \varphi$
- $\mathcal{R}, s \models \varphi_1 \wedge \varphi_2 \Leftrightarrow \mathcal{R}, s \models \varphi_1$ and $\mathcal{R}, s \models \varphi_2$
- $\mathcal{R}, s \models \triangle(\varphi_1, \ldots, \varphi_{\rho(\triangle)}) \Leftrightarrow \exists\langle s_1, \ldots, s_{\rho(\triangle)}\rangle \in R_\triangle(s) : \mathcal{R}, s_i \models \varphi_i$ for $1 \leq i \leq \rho(\triangle)$.

Denote by $[\![\varphi]\!]_\mathcal{R} := \{s \in S | \mathcal{R}, s \models \varphi\}$ the set of states for which formula φ in valid, and by

$$\Lambda_{\mathcal{R},s} := \{\varphi \in \mathbf{Mod_b}(\tau, P) | \mathcal{R}, s \models \varphi\}$$

the theory of state s in $\mathcal{R}$.
An easy calculation shows that

$$\mathcal{R}, s \models \triangle(\varphi_1, \ldots, \varphi_{\rho(\triangle)}) \Leftrightarrow R_\triangle(s) \cap [\![\varphi_1]\!]_\mathcal{R} \times \ldots \times [\![\varphi_{\rho(\triangle)}]\!]_\mathcal{R} \neq \emptyset.$$

In analogy, a *stochastic τ-Kripke model* $\mathcal{K} = (S, K_\tau, V)$ has a state space S which is endowed with a σ-algebra, a family $K_\tau = (K_\triangle)_{\triangle \in O}$ of stochastic relations $K_\triangle : S \rightsquigarrow S^{\rho(\triangle)}$ and a set valued map $V : P \to \mathcal{P}(S)$ such that $V(p)$ is always a measurable set (we leave the σ-algebra on the state space anonymous to avoid cluttering the notation; note that $S^{\rho(\triangle)}$ carries the product σ-algebra).
The interpretation of formulas in $\mathbf{Mod_s}(\tau, P)$ for a stochastic τ-Kripke model $\mathcal{K}$ is fairly straightforward, the interesting case arising when a modal operator is involved:

$$\mathcal{K}, s \models \triangle_q(\varphi_1, \ldots, \varphi_{\rho(\triangle)})$$

holds iff there exists measurable subsets $A_1, \ldots, A_{\rho(\triangle)}$ of S such that $\mathcal{K}, s_i \models \varphi_i$ holds for all $s_i \in A_i$ for $1 \leq i \leq \rho(\triangle)$, and

$$K_\triangle(s)(A_1 \times \ldots \times A_{\rho(\triangle)}) \geq q.$$

Arguing from the point of view of state transition systems, this interpretation of validity reflects that upon the move indicated by $\triangle_q$, a state s satisfies $\triangle_q(\varphi_1, \ldots, \varphi_{\rho(\triangle)})$ iff we can find states s_i satisfying φ_i with a $K_\triangle$-probability exceeding q. Note that the usual operators $\triangle$ and ∇ are replaced by a whole spectrum of operators $\triangle_q$ which permit a finer and probabilistically more adequate notion of satisfaction (see [13]).
Again, let $[\![\varphi]\!]_\mathcal{K}$ be the set of all states for which $\varphi \in \mathbf{Mod_s}(\tau, P)$ is satisfied under $\mathcal{K}$, and

$$\Lambda_{\mathcal{K},s} := \{\varphi \in \mathbf{Mod_s}(\tau, P) | \mathcal{K}, s \models \varphi\}$$

the state's theory.
It turns out that the sets $[\![\varphi]\!]_\mathcal{K}$ are measurable, so that they may be used as arguments for the stochastic relations we are working with:

Lemma 1 *$[\![\varphi]\!]_\mathcal{K}$ is a measurable subset of S for each $\varphi \in \mathbf{Mod_s}(\tau, P)$.*

Proof The proof proceeds by induction on φ. If $\varphi = p \in P$, then $[\![\varphi]\!]_\mathcal{K} = V(p)$ holds, which is measurable by assumption. Since the measurable sets are closed under complementation and intersection, the only interesting case is again the one in which a modal operator is involved. Since

$$[\![\triangle_q(\varphi_1, \ldots, \varphi_{\rho(\triangle)})]\!]_\mathcal{K} = \{s \in S | K_\triangle(s)([\![\varphi_1]\!]_\mathcal{K} \times \ldots \times [\![\varphi_{\rho(\triangle)}]\!]_\mathcal{K}) \geq q\},$$

the assertion follows from the induction hypothesis and the fact that $K_\triangle$ is a stochastic relation. $\Box$
A stochastic relation on the state space induces a stochastic τ-Kripke model. This is illustrated through the following example:

Example 1 Let $K : S \rightsquigarrow S$ be a stochastic relation on the state space S, and define for $s \in S$ and for the modal operator $\triangle$

$$K_\triangle(s) := \bigotimes_{i=1}^{\rho(\triangle)} K(s),$$

then $K_\triangle : S \rightsquigarrow S^{\rho(\triangle)}$ is a stochastic relation. If $V(p) \subseteq S$ is a measurable subset of S, then

$$\mathcal{K}_{K,V} := (S, (K_\triangle)_{\triangle \in O}, V)$$

is a stochastic τ-Kripke model such that

$$\mathcal{K}_{K,V}, s \models \triangle_q(\varphi_1, \ldots, \varphi_{\rho(\triangle)}) \Leftrightarrow K(s)([\![\varphi_1]\!]_{\mathcal{K}_{K,V}}) \cdot \ldots \cdot K(s)([\![\varphi_{\rho(\triangle)}]\!]_{\mathcal{K}_{K,V}}) \geq q.$$

Thus the arguments to each modal operator are stochastically independent. —

The concepts of stochastic interpretation introduced above are illustrated with two well known sample logics.

Example 2 Suppose that L is a countable alphabet of actions. Each action $a \in \mathsf{L}$ is associated with a unary modal operator $\langle a \rangle$, so put $\tau := (O, \rho)$ with $O := \{\langle a \rangle |\ a \in \mathsf{L}\}$ and $\rho(\langle a \rangle) := 1$. A nondeterministic τ-Kripke model is based on a labelled transition system $(S, (\rightarrow_a)_{a \in \mathsf{L}})$ which associates with each action a a unary relation $\rightarrow_a \subseteq S \times S$. Thus

$$s \models \langle a \rangle \varphi \Leftrightarrow \exists s' : s \rightarrow_a s' \wedge s' \models \varphi.$$

A stochastic τ-model is based on a labelled Markov transition system [13, 4, 7] $(S, (k_a)_{a \in \mathsf{L}})$ which associates with each action a a stochastic relation $k_a : S \rightsquigarrow S$ Thus

$$s \models \langle a \rangle_q \varphi \Leftrightarrow k_a(s)([\![\varphi]\!]) \geq q,$$

hence making a transition is replaced by a probability with which a transition can happen. Variants of the logic $\mathbf{Mod_s}(\tau, P)$ with $P = \emptyset$ were investigated in [13, 4] with a reference to the logic investigated by Hennessy and Milner; we refer to them also as *Hennessy-Milner logic.* —

Example 3 The basic temporal language has two unary modal operators $\mathbf{F}$ (forward) and $\mathbf{B}$ (backward), so that $O = \{\mathbf{F}, \mathbf{B}\}$. A nondeterministic τ-Kripke model interprets the forward operator $\mathbf{F}$ through a relation $R \subseteq S \times S$ and the backward operator $\mathbf{B}$ through the converse $R^\smile$, so that

$$s \models \mathbf{B}\varphi \Leftrightarrow \exists t \in S : \langle t, s \rangle \in R \wedge t \models \varphi$$

holds.
A probabilistic interpretation interprets $\mathbf{F}$ through a stochastic relation $K : S \rightsquigarrow S$, so that

$$s \models \mathbf{F}_q\varphi \Leftrightarrow K(s)([\![\varphi]\!]) \geq q$$

The backward operator $\mathbf{B}$ is interpreted through the converse $K_\mu^\smile : S \rightsquigarrow S$, provided the state space S is Polish and an initial probability μ is given (the converse $K_\mu^\smile$ of K given μ is the stochastic relation $L : S \rightsquigarrow S$ such that

$$\int_S K(s)(\{s' | \langle s, s' \rangle \in B\})\ \mu(ds) = \int_S L(s')(\{s | \langle s, s' \rangle \in B\})\ \mu(ds')$$

holds for each Borel set $B \subseteq S \times S$, see [1] for algebraic and [6] for relational and measure theoretic properties of the converse). Thus

$$s \models \mathbf{B}_q\varphi \Leftrightarrow K_\mu^\smile(s)([\![\varphi]\!]) \geq q.$$

An easy calculation shows that

$$\begin{aligned} s \models \mathbf{B}_1\mathbf{F}_1\varphi &\Leftrightarrow K_\mu^\smile(s)\left(\{s' \mid K(s')([\![\varphi]\!]) = 1\}\right) = 1 \\ &\Leftrightarrow \int_S K(s')([\![\varphi]\!])\ K_\mu^\smile(s)(ds') = 1 \end{aligned}$$

Note that the definition of the converse requires an initial probability (this is intuitively clear: if the probability for a backward running process is described, one has to say where to start). It is also noteworthy that a topological assumption has been made; if the state space is not a Polish space, then the technical arguments permitting the definition of the converse are not available. —

4 Refinements

Given a nondeterministic and a stochastic interpretation, we want to compare both. Intuitively, the stochastic interpretation is more precise than its nondeterministic cousin: whereas nondeterministically we can only talk about possibilities, we can assign weights to these possibilities using probabilities. To say that after a certain input the output put will be a, b or c conveys certainly less information than saying that the probabilities for these outputs will be, respectively, $p(a) = 1/100$, $p(b) = 1/50$ and $p(c) = 97/100$.
Since negation has its own problems, we will restrict ourselves to the negation free logic $\mathbf{Mod_1}(\tau, P)$, and we will deal with stochastic relations which are based on probabilities (so that the whole space is always assigned probability one).

Definition 1 *Let $\mathcal{R}$ and $\mathcal{K}$ be a nondeterministic and a stochastic τ-Kripke model, and assume that $K_\triangle(s)(S \times \ldots \times S^{\rho(\triangle)}) = 1$ holds for each $s \in S$ (we will call these models* probabilistic*). $\mathcal{K}$ is said to* refine *$\mathcal{R}$ ($\mathcal{K} \vdash \mathcal{R}$) iff*

$$\forall \varphi \in \mathbf{Mod_1}(\tau, P) : [\![\varphi]\!]_\mathcal{K} \subseteq [\![\varphi]\!]_\mathcal{R}.$$

Consequently, given the interpretations $\mathcal{K}$ and $\mathcal{R}$, we have $\mathcal{R} \vdash \mathcal{K}$ if $\mathcal{R}, s \models \varphi$ holds only if $\mathcal{K}, s \models \varphi$ is true for each formula φ in the negation free part of the logic.
We will investigate here the relationship between nondeterministic and stochastic satisfaction by showing that for each stochastic interpretation $\mathcal{K}$ we can find a nondeterministic one $\mathcal{R}$ with $\mathcal{K} \vdash \mathcal{R}$ by simply taking all possible state changes and making it into a Kripke model. Conversely, we will look into the possibility of refining a given nondeterministic Kripke model into a stochastic model. This requires some topological assumptions (for otherwise the notion *all possible states* cannot be made precise). Thus from now on the state space S is a Polish space with its Borel sets as σ-algebra.
The set of all states possible for a probability μ on a Polish space X is captured through the support of a probability μ: Define $\mathsf{supp}(\mu)$ as the smallest closed subset $F \subseteq X$ such that $\mu(F) = 1$, thus

$$\mathsf{supp}(\mu) = \bigcap\{F \subseteq X \mid F \text{ is closed and } \mu(F) = 1\}.$$

It can be shown that $\mu(\mathsf{supp}(\mu)) = 1$, and $x \in \mathsf{supp}(\mu)$ iff $\mu(U) > 0$ for each neighborhood U of x. So this is exactly what we want.

Proposition 1 *Let $\mathcal{K} = \left(S, (K_\triangle)_{\triangle \in O}, V\right)$ be a probabilistic τ-Kripke model. Define for the modal operator $\triangle \in O$ the set valued map*

$$R^{\mathcal{K}}_\triangle(s) := \mathsf{supp}(K_\triangle(s)).$$

Put

$$\mathcal{R}_{\mathcal{K}} := \left(S, \left(R^{\mathcal{K}}_\triangle\right)_{\triangle \in O}, V\right),$$

then $\mathcal{K}$ is a probabilistic refinement of $\mathcal{R}_{\mathcal{K}}$.

Proof The proof proceeds by induction on the structure of the formulas. Assume that $\triangle$ is a modal operator, and that we know

$$[\![\varphi_i]\!]_{\mathcal{K}} \subseteq [\![\varphi_i]\!]_{\mathcal{R}^{\mathcal{K}}}$$

for $1 \leq i \leq \rho(\triangle)$. Now suppose

$$R^{\mathcal{K}}_\triangle, s \not\models \triangle_1(\varphi_1, \ldots, \varphi_{\rho(\triangle)})$$

for some state s. Thus

$$R^{\mathcal{K}}_\triangle(s) \cap [\![\varphi_1]\!]_{\mathcal{R}^{\mathcal{K}}} \times \ldots \times [\![\varphi_{\rho(\triangle)}]\!]_{\mathcal{R}^{\mathcal{K}}} = \emptyset,$$

and, consequently, by the hypothesis,

$$R^{\mathcal{K}}_\triangle(s) \cap [\![\varphi_1]\!]_{\mathcal{R}} \times \ldots \times [\![\varphi_{\rho(\triangle)}]\!]_{\mathcal{R}} = \emptyset.$$

But this means

$$K_\triangle(s)([\![\varphi_1]\!]_{\mathcal{R}} \times \ldots \times [\![\varphi_{\rho(\triangle)}]\!]_{\mathcal{R}}) < 1,$$

hence

$$\mathcal{K}, s \not\models \triangle_1(\varphi_1, \ldots, \varphi_{\rho(\triangle)}).$$

□

Thus each probabilistic Kripke model carries a nondeterministic one with it, and it refines this companion (one is tempted to perceive this as a *nondeterministic shadow*: a shadow as a coarser, black-and-white image of a probably more colorful, picturesque and graphic original). It will be shown now that the converse of Proposition 1 is also true: Given a nondeterministic Kripke model, there exists a stochastic one refining it. Intuitively, and in the finite case, one simply assigns a uniform weight as a probability to all possible outcomes. This is basically what we will do here, too, but we have to be a bit more careful since in an uncountable setting this idea requires some additional underpinning.

Measurable relations will provide a link between non-deterministic and stochastic systems, as we will see. Let us fix some notations first. Assume that Y is a measurable, and that Z is a Polish space. Consider a set valued map $R : Y \to \mathcal{P}(Z)$. If $R(y)$ always takes closed and non-empty values, and if the (weak) inverse

$$(\exists R)(G) := \{y \in Y | R(y) \cap G \neq \emptyset\}$$

is a measurable set, whenever $G \subseteq Z$ is open, then R is called a *measurable relation on* $Y \times Z$. Since Z is Polish, R is a measurable relation iff the strong inverse

$$(\forall R)(F) := \{y \in Y | R(y) \subseteq F\}$$

is measurable, whenever $F \subseteq Z$ is closed [10, Theorem 3.5].
It is immediate that the support yields a measurable relation for a probabilistic relation $K : Y \rightsquigarrow Z$: put

$$R_K := \{\langle y, z\rangle \in Y \times Z | z \in \mathsf{supp}(K(y))\},$$

then

$$(\forall R_K)(F) = \{y \in Y | K(y)(F) = 1\}$$

is true for the closed set $F \subseteq Z$, and

$$(\exists R_K)(G) = \{y \in Y | K(y)(G) > 0\}$$

holds for the open set $G \subseteq Z$. Both sets are measurable.
It is also plain that

$$(*)\ \forall y \in Y : R(y) = \mathsf{supp}(K(y))$$

implies that R has to be a measurable relation.
Given a set-valued relation R, a probabilistic relation K with $(*)$ can be found. For this, R has to take closed values, and a condition of measurability is imposed. We obtain from [6] the following existential statement (which depends on the existence of a sufficient number of measurable selectors for a measurable relation).

Lemma 2 *Let $R \subseteq Y \times Z$ be a measurable relation for Z Polish. There exists a probabilistic relation $K : Y \rightsquigarrow Z$ such that $R(y) = \mathsf{supp}(K(y))$ holds for each $y \in Y$.*

Thus we can find a probabilistic Kripke structure refining a given nondeterministic one, provided we impose a measurability condition:

Proposition 2 *Suppose $\mathcal{R} := \left(S, (R_\triangle)_{\triangle \in O}, V\right)$ is a nondeterministic τ-Kripke model such that*

1. *$V(p) \in \mathcal{B}(S)$ for all $p \in P$,*
2. *$R_\triangle$ is a measurable relation on $S \times S^{\rho(\triangle)}$ for each $\triangle \in O$.*

Then there exists a probabilistic τ-Kripke model $\mathcal{K} = \left(S, (K_\triangle)_{\triangle \in O}, V\right)$ with $\mathcal{K} \vdash \mathcal{R}$.

Proof Applying Lemma 2, find for each modal operator $\triangle \in O$ a transition probability $K_\triangle : S \rightsquigarrow S^{\rho(\triangle)}$ such that

$$\forall s \in S : R_\triangle(s) = \mathsf{supp}(K_\triangle(s))$$

holds. The argumentation in the proof of Lemma 1 establishes the claim. □
It is clear that the probabilistic τ-Kripke model is underspecified by merely requiring to be a refinement to a nondeterministic one. This is supported through the following observation:

Corollary 1 *Let $\mathcal{R}$ be a nondeterministic τ-Kripke model satisfying the conditions of Proposition 2. Assume that $\mathcal{K}_i = \left(S, (K_{\triangle,i})_{\triangle\in O}, V\right)$ is a probabilistic τ-Kripke model with $\mathcal{K}_i \vdash \mathcal{R}$ for each $i \in \mathbb{N}$. Let $(\alpha_i)_{i\in\mathbb{N}}$ be a sequence of positive real numbers such that $\sum_{i\in\mathbb{N}} \alpha_i = 1$, and define for $\triangle \in O$ the stochastic relation*

$$K_\triangle(s) := \sum_{i\in\mathbb{N}} \alpha_i \cdot K_{\triangle,i}(s).$$

Then

$$\left(S, (K_\triangle)_{\triangle\in O}, V\right) \vdash \mathcal{R}.$$

Proof 1. Let $(\mu_i)_{i\in\mathbb{N}}$ be a sequence of probability measures. Since all α_i are positive, the definition of the support function yields that

$$\mathsf{supp}(\sum_{i\in\mathbb{N}} \alpha_i \cdot \mu_i) = \left(\bigcup_{i\in\mathbb{N}} \mathsf{supp}(\mu_i)\right)^{\mathsf{cl}}$$

holds, $(\cdot)^{\mathsf{cl}}$ denoting topological closure. Thus $R_\triangle$ equals $\mathsf{supp}(K_\triangle)$.
2. The assertion now follows from Proposition 2. □

5 Smooth Equivalence Relations

We will begin in this section a discussion of smooth equivalence relations that will lead us to the definition of congruences and of factor objects. In order to keep a probabilistic grip on the equivalence relations, we require them to be compatible with the Borel structure. The natural way to do this is to have a countable set of Borel measurable generators. We will show that the equivalence relation coming from a modal logic in which two states are equivalent iff they satisfy exactly the same formulas has this property. This leads to the definition of equivalent congruences, which will be studied in Section 6 for bisimulations of general stochastic relations, and in Section 7 for stochastic τ-Kripke models where we mainly focus on the special case of the equivalence of bisimulation and satisfaction of the same formulas for a given modal logic.
We fix for this section a Polish space X with its Borel σ-algebra $\mathcal{B}(X)$.

Definition 2 *An equivalence relation $\alpha \subseteq X \times X$ is called* smooth *iff one of the following equivalent conditions is satisfied:*

1. *there exists a Polish space Y and a Borel measurable map $f : X \to Y$ such that*

$$x \; \alpha \; y \Leftrightarrow f(x) = f(y),$$

2. *there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of Borel sets in X such that*

$$x \; \alpha \; y \Leftrightarrow \forall n \in \mathbb{N} : [x \in A_n \Leftrightarrow y \in A_n].$$

It follows immediately that a smooth equivalence relation is a Borel subset of $X \times X$, see [16, Exercise 5.1.10]. The equivalence classes can be expressed in terms of the sequence $(A_n)_{n\in\mathbb{N}}$:

$$[x]_\alpha = \bigcap\{A_n | x \in A_n\} \cap \bigcap\{X \setminus A_n | x \notin A_n\},$$

hence each class is a Borel subset of X.
Smooth relations arise naturally in the context of stochastic τ-Kripke models. We will deal with it later on.

Example 4 Let $\mathcal{K}$ be a stochastic τ-Kripke model. Two states s, s' are equivalent iff $\Lambda_{\mathcal{K},s} = \Lambda_{\mathcal{K},s}$, hence iff $\mathcal{K}, s \models \varphi \Leftrightarrow \mathcal{K}, s' \models \varphi$ holds for all $\varphi \in \mathbf{Mod_s}(\tau, P)$. This equivalence relation is smooth since $\mathbf{Mod_s}(\tau, P)$ is countable. —

A set $A \subseteq X$ is called *α-invariant* iff $x \in A$ and $x\ \alpha\ y$ implies $y \in A$, thus A is α-invariant iff

$$A = \bigcup\{[x]_\alpha \, | x \in A\}$$

holds. The α-invariant Borel subsets of X form a σ-algebra $\mathcal{A}_\alpha$. Blackwell's Lemma characterizes the invariant Borel sets:

Lemma 3 *Let α be a smooth equivalence relation, and $(A_n)_{n\in\mathbb{N}}$ be the sequence of Borel sets that define α according to Definition 2. Then $\mathcal{A}_\alpha = \sigma\left(\{A_n | \ n \in \mathbb{N}\}\right)$. Thus a Borel set A is α-invariant iff $A \in \sigma\left(\{A_n | \ n \in \mathbb{N}\}\right)$.*

Proof [16, Lemma 5.1.16]. □
This Lemma implies that a smooth relation does not depend on the specific sequence of generators (as Definition 2 and the representation of the equivalence classes seem to suggest) but rather on the σ-algebra induced by them. In fact, it is not difficult to see that the relation $x \in A \Leftrightarrow y \in A$ extends from a generator of a σ-algebra to the σ-algebra itself [16, Lemma 3.1.6]. Lemma 3 has moreover the pleasant consequence that it permits the identification of smooth equivalence relations and countably generated sub-σ-algebras of $\mathcal{B}(X)$, as is shown in [8, Proposition 1]:

Lemma 4 *$\alpha \mapsto \mathcal{A}_\alpha$ is a order anti-isomorphism between the set $\mathcal{M}_X$ of smooth equivalence relations on X and set $\mathcal{Z}_X$ of all the countably generated sub-σ-algebras of $\mathcal{B}(X)$, where the order is given in each case through inclusion. $\mathcal{M}_X$ has in particular a smallest and a largest element, and is closed under countable intersections.*

Smoothness is preserved through finite products:

Lemma 5 *Let α and β be smooth equivalence relations on the Polish spaces X respectively Y. Define*

$$\langle x, y\rangle(\alpha \times \beta)\langle x', y'\rangle \Leftrightarrow x\alpha x' \wedge y\beta y'.$$

Then

1. *$\alpha \times \beta$ is a smooth equivalence relation over $X \times Y$,*
2. *$\mathcal{A}_{\alpha\times\beta} = \mathcal{A}_\alpha \otimes \mathcal{A}_\beta$.*

Proof Let $\{A_n|n \in \mathbb{N}\}$ and $\{B_n|n \in \mathbb{N}\}$ be the generator for $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$, resp. Then $\{A_n \times B_n|n \in \mathbb{N}\}$ generates $\mathcal{A}_\alpha \otimes \mathcal{A}_\beta$, and

$$\langle x, y\rangle(\alpha \times \beta)\langle x', y'\rangle \Leftrightarrow \forall n \in \mathbb{N} : \big[\langle x, y\rangle \in A_n \times B_n \Leftrightarrow \langle x', y'\rangle \in A_n \times B_n\big]$$

□

The atoms of $\mathcal{A}_\alpha$ for a smooth equivalence relation α are the equivalence classes $[x]_\alpha$ (recall that an atom A in $\mathcal{A}_\alpha$ has the property that $A \neq \emptyset$, and that each subset of A in $\mathcal{A}_\alpha$ is either empty or equals A): let $\emptyset \neq B \subseteq [x]_\alpha$ and $B \in \mathcal{A}_\alpha$. Thus B is α-invariant by Lemma 3, and we see that $y \in B$ implies $y\ \alpha\ x$, hence $[x]_\alpha = [y]_\alpha \subseteq B$.

We will use Lemma 4 for switching between smooth equivalence relations and countably generated sub-σ-algebras of $\mathcal{B}(X)$. Denote for $\mathcal{D} \in \mathcal{Z}_X$ by $\alpha_\mathcal{D}$ the smooth equivalence generated by $\mathcal{D}$.

Definition 3 *Let $K : X \rightsquigarrow Y$ be a stochastic relation, where X and Y are Polish spaces. Then (α, β) is called a* congruence *for (X, Y, K) iff*

1. *α and β are smooth equivalence relations on X, and Y, resp.*
2. *$x_1\ \alpha\ x_2$ implies $K(x_1)(D) = K(x_2)(D)$ for all $D \in \mathcal{A}_\beta$.*

Thus a congruence relates the inputs, and, separately, the outputs for $K : X \rightsquigarrow Y$ so that α-equivalent inputs display the same behavior on β-invariant output-sets.

Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of Borel sets determining β, and define

$$F : \begin{cases} X & \to [0, 1]^{\mathbb{N}} \\ x & \mapsto (K(x)(D_n))_{n\in\mathbb{N}}, \end{cases}$$

then (α, β) is a congruence for (X, Y, K) iff $\alpha \subseteq \alpha_F$, where α_F is the smooth equivalence relation induced by F; an equivalent characterization is to say that $\mathcal{D}_F \subseteq \mathcal{A}_\alpha$ with $\mathcal{D}_F$ as the σ-algebra induced by α_F.

It is noted that for each $D \in \mathcal{D}$ the map $x \mapsto K(x)(D)$ is actually $\mathcal{A}_\alpha$-measurable. In fact, if $S \subseteq \mathbb{R}$ is a Borel set, then

$$A_0 := \{x \in X|K(x)(D) \in S\}$$

is a Borel set in X, and it is α-invariant by the definition of a congruence. Using Lemma 3 we see now that $A_0 \in \mathcal{C}$.

The following example illustrates the generation of congruences through countably generates σ-algebras on the target space of a stochastic relation; it works because a finite measure is uniquely determined on a set of generators which is closed under finite intersection.

Example 5 Let $K : X \rightsquigarrow Y$ be a stochastic relations over the Polish spaces X and Y, and assume that $\mathcal{B} \subseteq \mathcal{B}(Y)$ is a countably generated sub-σ-algebra of the Borel sets of Y. Put for $x, x' \in X$

$$x\ \alpha\ x' \Leftrightarrow \forall B \in \mathcal{B} : K(x)(B) = K(x')(B),$$

then $(\alpha, \alpha_\mathcal{B})$ is a congruence for K. —

Fix a Polish object (X, Y, K) and a congruence (α, β) for it. We will investigate now the factorization of (X, Y, K) by (α, β), following essentially the construction outlined in [4].

Definition 4 *Let α be a smooth equivalence relation. Then*

$$X/\alpha := \{[x]_\alpha \,|x \in X\}$$

is the set of all equivalence classes, and $\mathcal{B}(X)/\alpha$ is the largest σ-algebra on X/α that makes the natural injection

$$\eta_\alpha : x \mapsto [x]_\alpha$$

measurable.

This yields a canonic construction which, however, leaves the category of Polish spaces as the base category for construction of stochastic relations.

Proposition 3 *Let α and β be smooth equivalence relations on the Polish space X resp. Y. Then*

1. *$(X/\alpha, \mathcal{B}(X)/\alpha)$ is an analytic space,*
2. *If (α, β) is a congruence for the stochastic relation $K : X \rightsquigarrow Y$, define*

   $$K_{\alpha,\beta}([x]_\alpha)(D) := K(x)(\eta_\beta^{-1}[D])$$

   for $D \in \mathcal{B}(Y)/\beta$. Then $K_{\alpha,\beta} : X/\alpha \rightsquigarrow Y/\beta$ is a stochastic relation such that

   $$K_{\alpha,\beta} \circ \eta_\alpha = \mathbf{S}(\eta_\beta) \circ K$$

   holds.

Proof That $(X/\alpha, \mathcal{B}(X)/\alpha)$ is an analytic space is shown exactly as in the proof of [4, Proposition 9.4] using Souslin's Theorem for analytic sets [16, Theorem 4.4.3] together with the Unique Structure Theorem for Borel sets [2, Theorem 3.3.5]. The proof proper is done as in [8, Section 3], following closely the pattern laid out in [4, Section 9]. □
We will use these factor objects over and over later on, so they are given their own name and symbol:

Definition 5 *Endow X/α and Y/β with the σ-algebras $\mathcal{B}(X)/\alpha$ and $\mathcal{B}(Y)/\beta$, respectively, and denote these analytic spaces for short by X_α and Y_β, resp. The analytic object*

$$(X, Y, K)\,/_{(\alpha,\beta)} := (X/\alpha, Y/\beta, K_{\alpha,\beta})$$

is called the factor object *for (X, Y, K) and the congruence (α, β).*

An example for congruences and factor spaces of interest is furnished through equivalent congruences.
As a preparation we will have a quick look at how the atoms of a countably generated σ-algebra are characterized through the generators.

Lemma 6 *Let $\mathcal{E} = \sigma(\{E_n|n \in \mathbb{N}\})$ be a countably generated σ-algebra over a set E. Define $A^1 := A, A^0 := E \setminus A$ for $A \subseteq E$. Then*

$$\{\bigcap_{n\in\mathbb{N}} E_n^{\alpha(n)}|\alpha \in \{0,1\}^{\mathbb{N}}\}$$

are exactly the atoms of $\mathcal{E}$.

Proof The proof to [16, 3.1.15] establishes this representation. □

Definition 6 *Let α_1 and α_2 be smooth equivalence relations on the Polish spaces X_1 resp. X_2, and assume that $T : X_1/\alpha_1 \to X_2/\alpha_2$ is a map between the equivalence classes. We say that α_1* spawns *α_2* via *$(T, \mathcal{A}_0)$ iff $\mathcal{A}_0$ is a countable generator of $\mathcal{A}_{\alpha_1}$ such that*

1. *$\mathcal{A}_0$ is closed under finite intersections,*
2. *$\{T_A | A \in \mathcal{A}_0\}$ is a generator of $\mathcal{A}_{\alpha_2}$, where $T_A := \bigcup\{T\left([x]_{\alpha_1}\right) | x \in A\}$,*
3. *$[x_1]_{\alpha_1} = [x_2]_{\alpha_1}$ implies the equality of*

$$\bigcap\{T_A | x_1 \in A \in \mathcal{A}_0\} \cap \bigcap\{X_2 \setminus T_A | x_1 \notin A \in \mathcal{A}_0\}$$

and

$$\bigcap\{T_A | x_2 \in A \in \mathcal{A}_0\} \cap \bigcap\{X_2 \setminus T_A | x_2 \notin A \in \mathcal{A}_0\}.$$

Thus if α_1 spawns α_2, then the measurable structure induced by α_1 on X_1 is all we need to construct the measurable structure induced by α_2 on X_2: map T can be made to carry over the generator $\mathcal{A}_0$ from $\mathcal{A}_{\alpha_1}$ to $\mathcal{A}_{\alpha_2}$ and — in the light of Lemma 6 — to transport the atoms from one σ-algebra to the other. This is of particular interest since the atoms constitute the equivalence classes. Hence α_1 together with T and the generator $\mathcal{A}_0$ is all we may care to know about α_2.
The first condition reflects a measure-theoretic precaution: we will need to make sure e.g. in the construction of the direct sum of stochastic relations that measures are uniquely determined by their values on a generator. This, however, can only be done if the generator is stable against taking finite intersections. Note that $T_{A_1 \cap A_2} = T_{A_1} \cap T_{A_2}$ also holds, so that closedness under intersections is inherited through T.
We are now in a position to define equivalent congruences through proportional ones:

Definition 7 *Let (X, Y, K) and (X', Y', K') be Polish objects with congruences (α, β) and (α', β'), respectively.*

1. *Call (α, β)* proportional to *(α', β') (symbolically $(\alpha, \beta) \propto (\alpha', \beta')$) iff α spawns α' via $(Q, \mathcal{A}_0)$, β spawns β' via $(T, \mathcal{B}_0)$ such that*

$$\forall x \in X \forall x' \in Q([x]_\alpha) \forall B \in \mathcal{B}_0 : K(x)(B) = K'(x')(T_B).$$

2. *Call these congruences* equivalent *iff both $(\alpha, \beta) \propto (\alpha', \beta')$ and $(\alpha', \beta') \propto (\alpha, \beta)$ hold.*

Thus equivalent congruences behave in exactly the same way. The same behavior is exhibited on each atom, i.e., equivalence class, as far as the input is concerned, and the respective invariant output sets. It becomes clear now that a characterization of equivalent behavior through congruences requires the double face of congruences: it is certainly necessary to use the equivalence relation on the input spaces; but since the behavior on the output spaces is modelled through probabilities, we need also the invariant Borel sets for a characterization.
We will show now how equivalent congruences on stochastic relations give rise to a factor object built on their sum. This construction will be of use in Section 7 for investigating the bisimilarity of stochastic τ-Kripke models.

Assume that (α, β) and (α', β') are equivalent congruences on the Polish objects (X, Y, K), and (X', Y', K'), respectively. Construct for (X, Y, K) and (X', Y', K') the direct sum

$$(X, Y, K) \oplus \left(X', Y', K'\right) := \left(X + X', Y + Y', K \oplus K'\right),$$

where the only non-obvious construction is $K \oplus K'$: put for the Borel set $E \subseteq Y + Y'$

$$(K \oplus K')(z)(E) := \begin{cases} K(z)(E \cap X), & \text{if } z \in X \\ K'(z)(E \cap X'), & \text{if } z \in X', \end{cases}$$

then clearly $K \oplus K' : X + X' \rightsquigarrow Y + Y'$. Define on $X + X'$ resp. $Y + Y'$ the σ-algebras

$$\begin{aligned} \mathcal{G} &:= \{C + C' | C \in \mathcal{A}_\alpha, C' \in \mathcal{A}_{\alpha'}\} \\ \mathcal{H} &:= \{D + D' | D \in \mathcal{A}_\beta, D' \in \mathcal{A}_{\beta'}\}, \end{aligned}$$

then $\mathcal{G}$ and $\mathcal{H}$ are countable generated sub-σ-algebras of the respective Borel sets. Let γ and χ be the respective smooth equivalence relations (so that $\mathcal{G} = \mathcal{A}_\gamma$ and $\mathcal{H} = \mathcal{A}_\chi$ both hold), then we need to show that (γ, χ) is a congruence on the sum $(X + X', Y + Y', K \oplus K')$. Assume α spawns α' via $(Q, \{C_n | n \in \mathbb{N}\})$. One first establishes that

$$\mathcal{G} = \sigma(\{C_n + Q_{C_n} | n \in \mathbb{N}\}).$$

This is so since $F \subseteq X + X'$ is a member of the sum $\mathcal{G}$ iff both $F \cap X \in \mathcal{A}_\alpha$ and $F \cap X' \in \mathcal{A}_{\alpha'}$ hold, and since $\mathcal{A}_{\alpha'} = \sigma(\{Q_{C_n} | n \in \mathbb{N}\})$ due to the properties of Q and $\{C_n | n \in \mathbb{N}\}$. Similarly, χ may be represented through β and T, if β spawns β' via $(T, \{D_n | n \in \mathbb{N}\})$. Because the σ-algebras in question are countably generated, so is their sum, and because the congruences are equivalent, we claim that $z\ \gamma\ z'$ implies that $(K \oplus K')(z)(F) = (K \oplus K')(z')(F)$ holds for all $F \in \mathcal{H}$. To establish this, let $z \in X, z' \in X'$, and consider

$$\mathcal{S} := \{F \in \mathcal{H} | (K \oplus K')(z)(F) = (K \oplus K')(z')(F)\}.$$

This is a σ-algebra containing the generator $\{D_n + T_{D_n} | n \in \mathbb{N}\}$, since the congruences are equivalent. Since the generator is closed under finite intersections, measures are uniquely determined. This implies $\mathcal{H} \subseteq \sigma(\mathcal{S})$.

Denote the congruence constructed in this way from (α, β) and (α', β') by $(\alpha + \alpha', \beta + \beta')$. The factor object

$$\left(X + X', Y + Y', K \oplus K'\right) /_{(\alpha + \alpha', \beta + \beta')}$$

will be investigated more closely in Proposition 5 for establishing that (X, Y, K) and (X', Y', K') are bisimilar, provided they have equivalent congruences.

6 Morphisms

This section introduces morphisms more formally and shows that each morphism gives rise to a congruence in a rather natural way by imitating the usual construction of obtaining congruences from homomorphisms. It turns out that the factor space for a morphism is itself Polish whenever both source and target are Polish objects, whereas we usually end up with analytic factor spaces. Bisimulations are introduced as spans of morphisms, and we have a brief look at the congruence induced by a bisimulation on the mediating object.

The category $\mathfrak{Stoch}$ has as objects stochastic relations (X, Y, K) for measurable spaces X, Y and $K : X \rightsquigarrow Y$. A *morphism*

$$(\phi, \psi) : (X_1, Y_1, K_1) \to (X_2, Y_2, K_2)$$

between the objects (X_i, Y_i, K_i) is a pair of surjective measurable maps $\phi : X_1 \to X_2$ and $\psi : Y_1 \to Y_2$ such that

$$K_1 \circ \phi = \mathbf{S}(\psi) \circ K_2$$

holds, i.e., the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\ \phi\ } & X_2 \\ {\scriptstyle K_1}\downarrow & & \downarrow{\scriptstyle K_2} \\ \mathbf{S}(Y_1) & \xrightarrow[\mathbf{S}(\psi)]{} & \mathbf{S}(Y_2) \end{array}$$

is commutative.
We did encounter morphisms already when constructing the factor object:

Example 6 Let (α, β) be a congruence on the Polish object (X, Y, K). Then

$$(\eta_\alpha, \eta_\beta) : (X, Y, K) \to (X, Y, K) /_{\alpha,\beta}$$

is a morphism. This follows from

$$\mathbf{S}(\eta_\beta) \circ K = K_{\alpha,\beta} \circ \eta_\alpha,$$

see Proposition 3. —

We will usually investigate morphisms for stochastic relations based on Polish or analytic spaces, but it does not hurt to define them generally. Recall an object (X, Y, K) of $\mathfrak{Stoch}$ is dubbed *Polish* or *analytic* iff X and Y are Polish, and analytic spaces, respectively. Accordingly we denote by $\mathfrak{P} - \mathfrak{Stoch}$ and by $\mathfrak{A} - \mathfrak{Stoch}$ the full subcategories having Polish respectively analytic objects as their objects.
Under suitable conditions we can make a pair of surjective and measurable maps into morphisms: Let $M : A \rightsquigarrow B$ be a stochastic relation between the measurable spaces A and B, assume B is separable, and that X and Y are Polish spaces with measurable and surjective maps $\phi : X \to A, \psi : Y \to B$. Then there exists a stochastic relation $K : X \rightsquigarrow Y$ such that

$$(\phi, \psi) : (X, Y, K) \to (A, B, M)$$

is a morphism.
This was shown in [8] helping to establish the existence of semi-pullbacks.
We note for later use [8, Theorem 2]:

Theorem 1 *Both $\mathfrak{P} - \mathfrak{Stoch}$ and $\mathfrak{A} - \mathfrak{Stoch}$ have semi-pullbacks the underlying object of which is Polish.*

Each morphism spawns a congruence in a rather natural way. Suppose $\phi : X_1 \to X_1'$ is a measurable map, and define

$$x\alpha_\phi x' \Leftrightarrow \phi(x) = \phi(x')$$

as the equivalence relation associated with ϕ.

Lemma 7 *Let*

$$(\phi, \psi) : (X_1, Y_1, K_1) \to (X_2, Y_2, K_2)$$

be a morphism for the stochastic relations $K_1 : X_1 \rightsquigarrow Y_1$ and $K_2 : X_2 \rightsquigarrow Y_2$, where the spaces involved are all Polish. Then $(\alpha_\phi, \alpha_\psi)$ is a congruence for (X_1, Y_1, K_1).

Proof 1. The equivalence relation α_ϕ is smooth by definition, so is ψ. It is not difficult to see that $\mathcal{A}_{\alpha_\phi} = \phi^{-1}[\mathcal{B}(X_2)]$, similarly for α_ψ.
2. Now let $x\alpha_\phi x'$, and let $D_1 \in \mathcal{A}_{\alpha_\psi} = \psi^{-1}[\mathcal{B}(Y_2)]$ be arbitrary, so we can find a Borel set $D_2 \in \mathcal{B}(Y_2)$ with $D_1 = \psi^{-1}[D_2]$. Thus

$$\begin{aligned}
K_1(x)(D_1) &= K_1(x)\left(\psi^{-1}[D_2]\right) \\
&= (\mathbf{S}(\psi) \circ K_1)(x)(D_2) \\
&= (K_2 \circ \phi)(x)(D_2) \\
&= K_2(\phi(x))(D_2) \\
&= K_2(\phi(x'))(D_2) \\
&= K_1(x')(D_1).
\end{aligned}$$

This establishes the assertion. □
Call a congruence (α, β) on (X_1, Y_1, K_1) *adapted* to the morphism (ϕ, ψ) iff

$$\begin{aligned}
x\ \alpha\ x' &\Rightarrow \phi(x) = \phi(x') \\
y\ \beta y' &\Rightarrow \psi(y) = \psi(y'),
\end{aligned}$$

then the congruence constructed in Lemma 7 is the largest congruence on (X_1, Y_1, K_1) that is adapted to the morphism, as the proof for the Lemma shows.
As expected, morphisms which are compatible with congruences factor through the factor object:

Proposition 4 *Assume that (α, β) is a congruence on the Polish object (X, Y, K), and let for the analytic object (X_1, Y_1, K_1)*

$$(\phi, \psi) : (X, Y, K) \to (X_1, Y_1, K_1)$$

be a morphism which is adapted to (α, β). Then (ϕ, ψ) factors uniquely in $\mathfrak{A} - \mathfrak{Stoch}$ through $(X, Y, K)/_{(\alpha,\beta)}$.

Proof 1. Because (ϕ, ψ) are adapted to (α, β), the maps

$$\begin{aligned}
\phi_\alpha([x]_\alpha) &:= \phi(x), \\
\psi_\beta([y]_\beta) &:= \psi(y)
\end{aligned}$$

are well defined. Since ϕ is $\mathcal{B}(X)-\mathcal{B}(X_1)$-measurable, and since $\mathcal{B}(X)/\alpha$ is the final σ-algebra on X_α with respect to η_α, $\mathcal{B}(X)/\alpha - \mathcal{B}(X_1)$-measurability of ϕ_α is inferred: we can write

$$\mathcal{A}_\alpha = \{A \subseteq X_\alpha | \eta_\alpha^{-1}[A] \in \mathcal{B}(X)\},$$

thus $\phi_\alpha^{-1}[B_1] \in \mathcal{A}_\alpha$ for $B_1 \in \mathcal{B}(X_1)$, since $\eta_\alpha^{-1}\left[\phi_\alpha^{-1}[B_1]\right] = \phi^{-1}[B_1] \in \mathcal{B}(X)$ due to the measurability of ϕ. A similar argument is used for ψ_β. Clearly, these maps are onto.
2. It remains to show that $(\phi_\alpha, \psi_\beta)$ is a morphism. In fact, let $D_1 \subseteq Y_1$ be a Borel set, then

$$\begin{aligned} K_1(\phi_\alpha([x]_\alpha))(D_1) &= K_1(\phi(x))(D_1) \\ &= K(x)(\psi^{-1}[D_1]) \\ &= K_{\alpha,\beta}([x]_\alpha)(\psi_\beta^{-1}[D_1]), \end{aligned}$$

because $\psi^{-1}[D_1] = \eta_\beta^{-1}\left[\psi_\beta^{-1}[D_1]\right]$, and because $(\eta_\alpha, \eta_\beta)$ is a morphism. Consequently,

$$K_1 \circ \phi_\alpha = \mathbf{S}(\psi_\beta) \circ K_{\alpha,\beta}$$

has been established. Uniqueness of the morphism is obvious. □
It turns out that the factor spaces associated with morphisms are Polish.

Corollary 2 *Denote under the conditions of Lemma 7 by*

$$(X_1, Y_1, K_1)/_{(\phi,\psi)}$$

the factor space associated with the largest congruence $(\alpha_\phi, \alpha_\psi)$ *which is adapted to the morphism* (ϕ, ψ)*. Then* $(X_1, Y_1, K_1)/_{(\phi,\psi)}$ *is a Polish object and isomorphic to* (X_2, Y_2, K_2)*.*

Proof The factor morphism constructed in Proposition 4 is composed of bijections. Since each morphism in $\mathfrak{Stoch}$ is an epi, and since injective maps are underlying exactly the monos, the assertion follows. □
Bisimilarity is introduced as a span of morphisms [11, 15, 7]. For coalgebras based on the category of sets, this definition agrees with the one through relations, originally given by Milner, see [15]. In [5] the authors call a bisimulation what we have introduced as congruence, albeit that paper restricts itself to labelled Markov transition systems, thus technically to stochastic relations $S \rightsquigarrow S$ for some state space S. It seems conceptually to be clearer to distinguish spans of morphisms from equivalence relations, thus we make this distinction here. Later we will see that there are some close connections: equivalent congruences induce bisimilarity, as we will establish in Proposition 5.
The stochastic relations (X_1, Y_1, K_1) and (X_2, Y_2, K_2) are called *bisimilar* iff there exists a span of $\mathfrak{Stoch}$-morphisms

$$(X_1, Y_1, K_1) \xleftarrow{(\phi_1, \psi_1)} (A, B, M) \xrightarrow{(\phi_2, \psi_2)} (X_2, Y_2, K_2)$$

with a suitable stochastic relation (A, B, M); the latter object is said to be *mediating.* If we deal with Polish spaces X_1, X_2, Y_1, Y_2, then we postulate that A and B are Polish spaces, too.
A bisimulation yields the familiar commutative diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{\phi_1} & A & \xrightarrow{\phi_2} & X_2 \\ {\scriptstyle K_1}\downarrow & & {\scriptstyle M}\downarrow & & \downarrow{\scriptstyle K_2} \\ \mathbf{S}(Y_1) & \xleftarrow[\mathbf{S}(\psi_1)]{} & \mathbf{S}(B) & \xrightarrow[\mathbf{S}(\psi_2)]{} & \mathbf{S}(Y_2) \end{array}$$

In terms of measures, this translates to

$$\begin{aligned} K_1(\phi_1(a))(D_1) &= M(a)(\psi_1^{-1}[D_1]) \\ K_2(\phi_2(a))(D_2) &= M(a)(\psi_2^{-1}[D_2]) \end{aligned}$$

for all $a \in A$ and all Borel sets $D_i \subseteq Y_i$.
Recall from Lemma 4 that the set of all smooth equivalence relations is a lower semi-lattice. Denote the infimum of α_1 and α_2 in this semi-lattice by $\alpha_1 \sqcap \alpha_2$ and note that

$$\mathcal{A}_{\alpha_1 \sqcap \alpha_2} = \sigma\left(\mathcal{A}_{\alpha_1} \cup \mathcal{A}_{\alpha_2}\right)$$

holds.

Lemma 8 *Suppose that*

$$(X_1, Y_1, K_1) \xleftarrow{(\phi_1, \psi_1)} (A, B, M) \xrightarrow{(\phi_2, \psi_2)} (X_2, Y_2, K_2)$$

is a bisimulation where all spaces involved are Polish, and assume that (α_i, β_i) is a congruence on (X_i, Y_i, K_i) which is adapted to (ϕ_i, ψ_i) for $i = 1, 2$. Then the congruence $(\alpha_1 \sqcap \alpha_2, \beta_1 \sqcap \beta_2)$ is adapted to both (ϕ_1, ψ_1) and (ϕ_2, ψ_2).

Proof 1. We demonstrate first that $(\alpha_1 \sqcap \alpha_2, \beta_1 \sqcap \beta_2)$ is a congruence for (A, B, M). It is enough to show that $M(x)(D) = M(x')(D)$ holds for each $D \in \sigma\left(\mathcal{A}_{\beta_1} \cup \mathcal{A}_{\beta_2}\right)$, provided $x\ (\alpha_1 \sqcap \alpha_2)\ x'$. Let

$$\mathcal{S}_0 := \{D \in \sigma\left(\mathcal{A}_{\beta_1} \cup \mathcal{A}_{\beta_2}\right) | M(x)(D) = M(x')(D)\}$$

be the σ-algebra of sets for which the assertion is true. Since in particular $x\ \alpha_1\ x'$ holds, we see that $\mathcal{A}_{\beta_1} \subseteq \mathcal{S}_0$ because (α_1, β_1) is a congruence. Similarly we establish $\mathcal{A}_{\beta_2} \subseteq \mathcal{S}_0$, thus $\sigma\left(\mathcal{A}_{\beta_1} \cup \mathcal{A}_{\beta_2}\right) \subseteq \mathcal{S}_0$ follows from the fact that $\mathcal{S}_0$ is a σ-algebra. This establishes the properties of a congruence.
2. From Lemma 4 it is immediate that the congruence under consideration is adapted to both (ϕ_1, ψ_1) and (ϕ_2, ψ_2). □

Corollary 3 *Under the assumptions of Lemma 8, the congruence*

$$(\alpha_{\phi_1} \sqcap \alpha_{\phi_2}, \alpha_{\psi_1} \sqcap \alpha_{\psi_2})$$

is the largest congruence on (A, B, M) which is adapted to both (ϕ_1, ψ_1) and (ϕ_2, ψ_2).

Stochastic relations can have only equivalent congruences when they are bisimilar. This is a rather far-reaching generalization of the by now well-known characterization of bisimilarity of labelled Markov transition systems through mutually equivalent states:

Proposition 5 *If (α_i, β_i) are equivalent congruences on the Polish objects (X_i, Y_i, K_i) for $i = 1, 2$, then (X_1, Y_1, K_1) and (X_2, Y_2, K_2) are bisimilar.*

Proof 1. Construct the sum $(X_1 + X_2, Y_1 + Y_2, K_1 \oplus K_2)$ of the two objects as at the end of Section 5, and let (κ_i, λ_i) be the corresponding injections, which are, however, no morphisms. Let

$$(\eta_{\alpha_1+\alpha_2}, \eta_{\beta_1+\beta_2}) : \quad (X_1 + X_2, Y_1 + Y_2, K_1 \oplus K_2) \\ \to (X_1 + X_2, Y_1 + Y_2, K_1 \oplus K_2) / (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$$

be the canonical injection, then

$$(\eta_{\alpha_1+\alpha_2} \circ \kappa_i, \eta_{\beta_1+\beta_2} \circ \lambda_i)$$

constitutes a morphism $(X_i, Y_i, K_i) \to (X_1 + X_2, Y_1 + Y_2, K_1 \oplus K_2) / (\alpha_1 + \alpha_2, \beta_1 + \beta_2)$, as will be shown now. The crucial step is establishing surjectivity.
2. We will first give a representation of the equivalence classes for the equivalence relations associated with $\alpha_1 + \alpha_2$ and $\beta_1 + \beta_2$. This will then imply that the injections into the sums composed with the factor maps are indeed onto.
Fix a generator $\{A_n | n \in \mathbb{N}\}$ for $\mathcal{A}_{\alpha_1}$, and assume that α_1 spawns α_2 via $(Q, \{A_n | n \in \mathbb{N}\})$. Then $\{Q_{A_n} | n \in \mathbb{N}\}$ is a generator for $\mathcal{A}_{\alpha_2}$, and

$$\mathcal{A}_{\alpha_1} + \mathcal{A}_{\alpha_2} = \sigma(\{A_n + Q_{A_n} | n \in \mathbb{N}\})$$

holds. We claim that each equivalence class $a \in (X_1 + X_2)/(\alpha_1 + \alpha_2)$ can be represented as

$$a = [x_1]_{\alpha_1} + [x_2]_{\alpha_2}$$

for some suitably chosen $x_1 \in X_1, x_2 \in X_2$. In fact, suppose $a = [x_1]_{\alpha_1+\alpha_2}$ for some $x_1 \in X_1$. Then

$$[x_1]_{\alpha_1} = \bigcap\{A_n | x_1 \in A_n\} \cap \bigcap\{X_1 \setminus A_n | x_1 \notin A_n\}$$

holds (see the remark following Definition 2). Consequently,

$$\begin{aligned} Q([x_1]_{\alpha_1}) &= \bigcap\{Q_{A_n} | x_1 \in A_n\} \cap \bigcap\{X_2 \setminus Q_{A_n} | x_1 \notin A_n\} \\ &= [x_2]_{\alpha_2} \end{aligned}$$

for some $x_2 \in X_2$ by Lemma 6. This is so since we know that the atoms of $\mathcal{A}_{\alpha_2}$ are exactly the classes $[\cdot]_{\alpha_2}$. Consequently, we have for each $n \in \mathbb{N}$ that $x_1 \in A_n$ holds iff $x_2 \in Q_{A_n}$ is true. Since $x_1 \in X_1, x_2 \in X_2$, and $X_1 \cap X_2 = \emptyset$, we have established

$$x_1 \in A_n + Q_{A_n} \Leftrightarrow x_2 \in A_n + Q_{A_n}$$

But this means x_1 $(\alpha_1 + \alpha_2)$ x_2. Thus we have shown that indeed

$$a = [x_1]_{\alpha_1} + [x_2]_{\alpha_2}$$

holds. If $a = [x_2]_{\alpha_1+\alpha_2}$ for some $x_2 \in X_2$, the same conclusion would have been reached by interchanging the roles of X_1 and X_2, which is possible because we have equivalent congruences. It is obvious that $[x_1]_{\alpha_1} + [x_2]_{\alpha_2}$ for $x_i \in X_i$ forms a $\alpha_1 + \alpha_2$-class.
In the same way we show that each equivalence class $b \in (Y_1 + Y_2)/\beta_1 + \beta_2$ can be represented as

$$b = [y_1]_{\beta_1} + [y_2]_{\beta_2}$$

for suitably chosen $y_1 \in Y_1, y_2 \in Y_2$, and that the sum of classes is a class again.
3. The pullback of the pair of morphisms with a joint target constructed in the first step is a Polish object which has the desired properties. □
This is the first cut at investigating congruences for stochastic relations. Much is to be done, in particular it is not yet known whether or not there are counterparts to the factorizations from classical algebra. This will be investigated in due course, but not in this paper. We will, however, apply congruences to stochastic interpretations of modal logic.

7 Bisimulations for Kripke Models

This Section investigates morphisms for stochastic τ- Kripke models; we want to know whether bisimilarity and mutually identical theories are equivalent also for this general case. To this end we first discuss morphisms that are based on morphisms for stochastic relations (a τ-Kripke model contains a family of stochastic relations, after all), indicate that this notion of morphism is not adequate for our purposes and propose the notion of a strong morphism. We show that strong morphisms are suitable for our purposes.
Fix a modal similarity type $\tau = (O, \rho)$. Assume first that the set P of propositional letters is empty, rendering the discussion a bit less technical. Then a stochastic τ-Kripke model $\mathcal{K} := (S, (K_\triangle)_{\triangle \in O}))$ is determined through the Polish state space S and the family $K_\triangle : S \rightsquigarrow S^{\rho(\triangle)}$ of stochastic relations. A morphism

$$\Phi : (S, (K_\triangle)_{\triangle \in O})) \to \big(S', (K'_\triangle)_{\triangle \in O})\big)$$

for stochastic τ-Kripke models is then a family

$$\Phi = ((\phi_\triangle, \psi_\triangle)_{\triangle \in O})$$

of morphisms

$$(\phi_\triangle, \psi_\triangle) : (S, S^{\rho(\triangle)}, K_\triangle) \to (S', (S')^{\rho(\triangle)}, K'_\triangle)$$

for the associated relations.
Fix a modal operator $\triangle$. The σ-algebra $\mathcal{A}_\triangle$ generated by

$$\{[\![\varphi_1]\!]_\mathcal{K} \times \ldots \times [\![\varphi_{\rho(\triangle)}]\!]_\mathcal{K} |\ \varphi_1, \ldots, \varphi_{\rho(\triangle)} \in \mathbf{Mod_s}(\tau, P)\}$$

is evidently countably generated, thus gives rise to a smooth equivalence relation $\beta_\triangle$ on $S^{\rho(\triangle)}$, and the relation

$$s \alpha_\triangle s' \Leftrightarrow \forall B \in \mathcal{A}_\triangle : K_\triangle(s)(B) = K_\triangle(s')(B)$$

is smooth due to $\mathcal{A}_\triangle$ being countably generated. Consequently, $(\alpha_\triangle, \beta_\triangle)$ is a congruence for $K_\triangle : S \rightsquigarrow S^{\rho(\triangle)}$.
Let $\mathcal{K}' = \Big(S', (K'_\triangle)_{\triangle \in O})\Big)$ be another τ-Kripke model which is equivalent to the first one in the sense that for the states the corresponding theories mutually coincide, to be more precise:

Definition 8 *The stochastic τ-Kripke models $\mathcal{K}$ and $\mathcal{K}'$ are said to be equivalent ($\mathcal{K} \sim \mathcal{K}'$) iff given $s \in S$ there exists $s' \in S'$ such that $\Lambda_{\mathcal{K},s} = \Lambda_{\mathcal{K}',s'}$ and vice versa.*

Construct for $\mathcal{K}'$ the congruence $(\alpha'_{\triangle}, \beta'_{\triangle})$ for each modal operator $\triangle$ as above, then it can be shown that $\mathcal{K} \sim \mathcal{K}'$ implies that the congruences $(\alpha_{\triangle}, \beta_{\triangle})$ and $(\alpha'_{\triangle}, \beta'_{\triangle})$ are equivalent. From Proposition 5 we see that $K_{\triangle}$ and $K'_{\triangle}$ are bisimilar for each modal operator$\triangle$, so that there exists a span of morphisms

$$(S, S^{\rho(\triangle)}, K_{\triangle}) \xleftarrow{(\phi_{\triangle}, \psi_{\triangle})} (A_{\triangle}, B_{\triangle}, M_{\triangle}) \xrightarrow{(\phi'_{\triangle}, \psi'_{\triangle})} (S', (S')^{\rho(\triangle)}, K'_{\triangle})$$

This is rather satisfying from the point of view of stochastic relations, but not when considering stochastic τ-Kripke models. This is so since in general $((A_{\triangle}, B_{\triangle}, M_{\triangle})_{\triangle \in O})$ fails to be such a model, because there is no way to guarantee that all $A_{\triangle}$ coincide with, say, a Polish space T, and that $B_{\triangle}$ equals $T^{\rho(\triangle)}$.
Consequently, we have to strengthen the requirements for a morphism in order to achieve some uniformity. This will be done now, and we admit propositional letters again.
The basic idea is to have just one map ϕ between the state spaces so that

$$K'_{\triangle}(\phi(s))(A) = K_{\triangle}(s)(\{\langle s_1, \ldots, s_{\rho(\triangle)}\rangle | \langle \phi(s_1), \ldots, \phi(s_{\rho(\triangle)})\rangle \in A\}$$

holds for each state $s \in S$ and each Borel set $A \subseteq (S')^{\rho(\triangle)}$, making the diagram

$$\begin{array}{ccc} S & \xrightarrow{\quad\phi\quad} & S' \\ \Big\downarrow{\scriptstyle K_{\triangle}} & & \Big\downarrow{\scriptstyle K'_{\triangle}} \\ \mathbf{S}\left(S^{\rho(\triangle)}\right) & \xrightarrow[\mathbf{S}\left(\phi^{\rho(\triangle)}\right)]{} & \mathbf{S}\left((S')^{\rho(\triangle)}\right) \end{array}$$

commutative (where $\phi^n : \langle x_1, \ldots, x_n\rangle \mapsto \langle \phi(x_1), \ldots, \phi(x_n)\rangle$ distributes ϕ into the components), and we want to have $s \in V(p)$ iff $\phi(s) \in V'(p)$ for each propositional letter. This leads to

Definition 9 *Let $\mathcal{K} := (S, (K_{\triangle})_{\triangle \in O}, V)$ and $\mathcal{K}' := \left(S', (K'_{\triangle})_{\triangle \in O}, V'\right)$ be stochastic τ-Kripke models. A* strong morphism *$\phi : \mathcal{K} \to \mathcal{K}'$ is determined through a measurable and surjective map $\phi : S \to S'$ so that these conditions are satisfied:*

1. *$\forall p \in P : V(p) = \phi^{-1}[V'(p)]$,*
2. *for each modal operator $\triangle$,*

$$K'_{\triangle} \circ \phi = \mathbf{S}\left(\phi^{\rho(\triangle)}\right) \circ K_{\triangle}$$

holds.

Thus, if $\phi : \mathcal{K} \to \mathcal{K}'$ is a strong morphism, then

$$(\phi, \phi^{\rho(\triangle)}) : (S, S^{\rho(\triangle)}, K_{\triangle}) \to (S', (S')^{\rho(\triangle)}, K'_{\triangle})$$

is a morphism between the corresponding stochastic relations for each modal operator $\triangle \in O$. Note that we take also the propositional letters into account.

It is clear that the stochastic τ-Kripke models over general measurable spaces form a category $\mathfrak{pKripke}$ with this notion of morphism, because the composition of strong morphisms is again a strong morphism, and because the identity is a strong morphism, too. Furthermore, each modal operator $\triangle$ induces a functor $F_\triangle : \mathfrak{pKripke} \to \mathfrak{Stoch}$ which forgets all but $K_\triangle$. We will below make (rather informal) use of this functor.
Because we work on the safe grounds of a category, we have bisimulations at our disposal, which can be defined again as spans of strong morphisms:

Definition 10 *The stochastic τ-Kripke models $\mathcal{K}$ and $\mathcal{K}'$ are called* strongly bisimilar *iff there exists a mediating stochastic τ-Kripke model $\mathcal{M}$ and strong morphisms*

$$\mathcal{K} \xleftarrow{\ \phi\ } \mathcal{M} \xrightarrow{\ \psi\ } \mathcal{K}'.$$

We will show that $\mathcal{K} \sim \mathcal{K}'$ iff $\mathcal{K}$ and $\mathcal{K}'$ are strongly bisimilar, provided the models are based on Polish spaces. Fix the stochastic τ-Kripke models $\mathcal{K} := (S, (K_\triangle)_{\triangle \in O}), V)$ and $\mathcal{K}' := \left(S', (K'_\triangle)_{\triangle \in O}), V'\right)$.
It is well known that morphisms preserve theories for the Hennessy-Milner logic [4]. This is true in general:

Lemma 9 *If $\phi : \mathcal{K} \to \mathcal{K}'$ is a strong morphism, then*

$$\Lambda_{\mathcal{K},s} = \Lambda_{\mathcal{K}',\phi(s)}$$

holds for all states $s \in S$.

Proof 1. We show by induction on the formula $\varphi \in \mathbf{Mod_s}(\tau, P)$ that

$$\mathcal{K}, s \models \varphi \Leftrightarrow \mathcal{K}', \phi(s) \models \varphi$$

holds; putting it slightly different, we want to show

$$(*)\ [\![\varphi]\!]_{\mathcal{K}} = [\![\varphi]\!]_{\mathcal{K}'}$$

for all these φ.
2. If $\varphi = p \in P$, this follows from $V(p) = \phi^{-1}\left[V'(p)\right]$. The interesting case in the induction step is the application of a n-ary modal operator $\triangle_q$ with rational q. Suppose the assertion is true for $[\![\varphi_1]\!]_{\mathcal{K}}, \ldots, [\![\varphi_n]\!]_{\mathcal{K}}$, then

$$\begin{aligned}
\mathcal{K}, s \models \triangle_q(\varphi_1, \ldots, \varphi_n) \quad &\Leftrightarrow \quad K_\triangle(s)([\![\varphi_1]\!]_{\mathcal{K}} \times \ldots \times [\![\varphi_n]\!]_{\mathcal{K}}) \geq q \\
&\Leftrightarrow \quad K_\triangle(s)((\phi^n)^{-1}\left[[\![\varphi_1]\!]_{\mathcal{K}'} \times \ldots \times [\![\varphi_n]\!]_{\mathcal{K}'}\right]) \geq q \ (\mathbf{a}) \\
&\Leftrightarrow \quad \left(\mathbf{S}\left(\phi^n\right) \circ K_\triangle\right)(s)([\![\varphi_1]\!]_{\mathcal{K}'} \times \ldots \times [\![\varphi_n]\!]_{\mathcal{K}'}) \geq q \\
&\Leftrightarrow \quad K'_\triangle(\phi(s))([\![\varphi_1]\!]_{\mathcal{K}'} \times \ldots \times [\![\varphi_n]\!]_{\mathcal{K}'}) \geq q \ (\mathbf{b}) \\
&\Leftrightarrow \quad \mathcal{K}', \phi(s) \models \triangle_q(\varphi_1, \ldots, \varphi_n)
\end{aligned}$$

In (**a**) we use reformulation $(*)$ for the induction hypothesis, in (**b**) we make use of the defining equation of a (strong) morphism. □
Define the equivalence relation α on state space S through

$$s_1\ \alpha\ s_2 \Leftrightarrow \Lambda_{\mathcal{K},s_1} = \Lambda_{\mathcal{K},s_2},$$

thus two states are α-equivalent iff they satisfy exactly the same formulas in $\mathbf{Mod_s}(\tau, P)$; in a similar way α' is defined on S'. Because we have at most countably many formulas, α and α' are smooth equivalence relations. Define the equivalence relation $\beta_\triangle$ on $S^{\rho(\triangle)}$ through

$$\langle s_1, \ldots, s_{\rho(\triangle)}\rangle \; \beta_\triangle \; \langle t_1, \ldots, t_{\rho(\triangle)}\rangle \Leftrightarrow s_1 \; \alpha \; t_1 \wedge \cdots \wedge s_{\rho(\triangle)} \; \alpha \; t_{\rho(\triangle)},$$

then $\beta_\triangle$ is smooth, and we know that the σ-algebra of β-invariant sets can be written in terms of the α-invariant sets, viz., $\mathcal{A}_{\beta_\triangle} = \bigotimes_{i=1}^{\rho(\triangle)} \mathcal{A}_\alpha$ (see Lemma 5). Similarly, $\beta'_\triangle$ is defined. The equivalence of $\mathcal{K}$ and $\mathcal{K}'$ makes these relations into equivalent congruences:

Lemma 10 *$(\alpha, \beta_\triangle)$ and $(\alpha', \beta'_\triangle)$ are equivalent congruences on the stochastic relations $F_\triangle(\mathcal{K})$ and $F_\triangle(\mathcal{K}')$.*

Proof 1. The equivalence relations involved are all smooth, so it first has to be demonstrated that each pair forms indeed a congruence. Assume that $s_1 \alpha s_2$ holds, then

$$K_\triangle(s_1)([\![\varphi_1]\!]_\mathcal{K} \times \ldots \times [\![\varphi_{\rho(\triangle)}]\!]_\mathcal{K}) = K_\triangle(s_2)([\![\varphi_1]\!]_\mathcal{K} \times \ldots \times [\![\varphi_{\rho(\triangle)}]\!]_\mathcal{K})$$

follows (otherwise we could find a rational number q with $\mathcal{K}, s_1 \models \triangle_q(\varphi_1, \ldots, \varphi_{\rho(\triangle)})$ but $\mathcal{K}, s_2 \not\models \triangle_q(\varphi_1, \ldots, \varphi_{\rho(\triangle)})$ or vice versa). Because

$$\mathcal{B}_0 := \{[\![\varphi_1]\!]_\mathcal{K} \times \cdots \times [\![\varphi_{\rho(\triangle)}]\!]_\mathcal{K} | \varphi_1, \ldots, \varphi_{\rho(\triangle)} \in \mathbf{Mod_s}(\tau, P)\}$$

forms a generator for $\mathcal{A}_{\beta_\triangle} = \bigotimes_{i=1}^{\rho(\triangle)} \mathcal{A}_\alpha$, we see that $(\alpha, \beta_\triangle)$ is a congruence for $F_\triangle(\mathcal{K})$. The same arguments show that also $(\alpha', \beta'_\triangle)$ is a congruence for $F_\triangle(\mathcal{K}')$.
2. $\mathcal{A}_0 := \{[\![\varphi]\!]_\mathcal{K} | \varphi \in \mathbf{Mod_s}(\tau, P)\}$ is a countable generator of the σ-algebra $\mathcal{A}_\alpha$, and since the logic is closed under conjunction, $\mathcal{A}_0$ is closed under finite intersections. Given $s \in S$ there exists $s' \in S'$ such that $\Lambda_{\mathcal{K},s} = \Lambda_{\mathcal{K}',s'}$ holds; define $Q([s]_\alpha) := [s']_{\alpha'}$, then $Q : S/\alpha \to S'/\alpha'$ is well defined, and $Q_{[\![\varphi]\!]_\mathcal{K}} = [\![\varphi]\!]_{\mathcal{K}'}$ holds. Consequently, $\{Q_A | A \in \mathcal{A}_0\}$ generates $\mathcal{A}_{\alpha'}$, and the construction implies that

$$\bigcap\{Q_A | s \in A \in \mathcal{A}_0\} \cap \bigcap\{S' \setminus Q_A | s \notin A \in \mathcal{A}_0\} = [s']_{\alpha'}.$$

Hence α spawns α' via $(Q, \mathcal{A}_0)$.
3. The construction of $\beta_\triangle$ implies that

$$[\langle s_1, \ldots, s_{\rho(\triangle)}\rangle]_{\beta_\triangle} = [s_1]_\alpha \times \ldots \times [s_{\rho(\triangle)}]_\alpha$$

holds. An argument very similar to that used above shows that $\beta_\triangle$ spawns $\beta'_\triangle$ via $(T, \mathcal{B}_0)$, where

$$T : [\langle s_1, \ldots, s_{\rho(\triangle)}\rangle]_{\beta_\triangle} \mapsto Q([s_1]_\alpha) \times \ldots \times Q([s_{\rho(\triangle)}]_\alpha),$$

and $\mathcal{B}_0$ is defined above.
4. An argumentation very close to the first part of the proof shows that $\Lambda_{\mathcal{K},s} = \Lambda_{\mathcal{K}',s'}$ for $s \in S, s' \in S'$ implies for all formulas $\varphi_1, \ldots, \varphi_{\rho(\triangle)}$ that

$$K_\triangle(s)([\![\varphi_1]\!]_\mathcal{K} \times \ldots \times [\![\varphi_{\rho(\triangle)}]\!]_\mathcal{K}) = K'_\triangle(s')([\![\varphi_1]\!]_{\mathcal{K}'} \times \ldots \times [\![\varphi_{\rho(\triangle)}]\!]_{\mathcal{K}'})$$

(cp. part 2 of the proof of Lemma 9). Thus $(\alpha, \beta_\triangle) \propto (\alpha', \beta'_\triangle)$, and in the same way, interchanging the roles of $\mathcal{K}$ and $\mathcal{K}'$, we infer $(\alpha', \beta'_\triangle) \propto (\alpha, \beta_\triangle)$. □

Accordingly, we know from Proposition 5 that for equivalent Kripke models $\mathcal{K}$ and $\mathcal{K}'$ and for each modal operator $\triangle$ the stochastic relations $F_\triangle(\mathcal{K})$ and $F_\triangle(\mathcal{K}')$ are bisimilar. All the mediating relations can be collected to form a mediating Kripke model. This requires, however, that we know a wee bit about the internal structure of the semi-pullback which is constructed along the way.
The Hennessy-Milner Theorem for stochastic τ-Kripke models may be established now:

Theorem 2 *Assume that $\mathcal{K}$ and $\mathcal{K}'$ are stochastic τ-Kripke models over Polish spaces, then the following statements are equivalent:*

1. *$\mathcal{K}$ and $\mathcal{K}'$ are strongly bisimilar,*
2. *$\mathcal{K} \sim \mathcal{K}'$.*

Proof 1. Since "1 $\Rightarrow$ 2" follows from Lemma 9, we may concentrate on the proof for "2 $\Rightarrow$ 1".
2. Since $\mathcal{K} \sim \mathcal{K}'$, we know from Lemma 10 that the congruences $(\alpha, \beta_\triangle)$ and $(\alpha', \beta'_\triangle)$ are equivalent for each modal operator $\triangle$. Let $\mathcal{M}_\triangle = (M_\triangle, N_\triangle, L_\triangle)$ be the mediating stochastic relation, which exists by Proposition 5. An analysis of the proof of [7, Theorem 1] shows that $(n := \rho(\triangle))$

$$\begin{aligned} M_\triangle &= \{\langle s, s'\rangle \in S \times S' |\ s\ (\alpha + \alpha')\ s'\rangle\}, \\ N_\triangle &= \{\langle s_1, s'_1, \dots, s_n, s'_n\rangle \in \left(S \times S'\right)^n |\ s_i\ (\alpha + \alpha')\ s'_i \text{ for } 1 \le i \le n\}, \end{aligned}$$

These may be made into Polish spaces. Note that $S'' := M_\triangle$ does not depend at all on the modal operator, and that $N_\triangle$ depends only on its arity. Furthermore, we may infer for the $\mathfrak{P} - \mathfrak{Stoch}$-morphisms

$$F_\triangle(\mathcal{K}) \xleftarrow{(\phi_\triangle, \psi_\triangle)} \mathcal{M}_\triangle \xrightarrow{(\phi'_\triangle, \psi'_\triangle)} F_\triangle(\mathcal{K}')$$

that

$$\phi_\triangle = \pi_1, \psi_\triangle = \pi_1^n, \phi'_\triangle = \pi_2, \psi'_\triangle = \pi_2^n$$

holds, where the π denote the projections. Now define for the propositional letter $p \in P$

$$W(p) := \{\langle s, s'\rangle \in M_\triangle |\ s \in V(p), s' \in V'(p)\},$$

then it is immediate that the equations

$$W(p) = \pi_1^{-1}\left[V(p)\right] = \pi_2^{-1}\left[V'(p)\right]$$

hold. Consequently, $\mathcal{M} := (S'', (L_\triangle)_{\triangle \in O}, W)$ is a stochastic τ-Kripke model with

$$\mathcal{K} \xleftarrow{\pi_1} \mathcal{M} \xrightarrow{\pi_2} \mathcal{K}'$$

in $\mathfrak{pKripke}$. $\Box$

8 Conclusion and Further Work

The main technical result of this paper is establishing a Hennessy-Milner Theorem for stochastic interpretations of modal logic. This paper proposes these contributions:

1. Stochastic Kripke models are introduced as a generalization of the well-known labelled Markov transition systems for general modal logics. A refinement relation between these models and their nondeterministic counterparts is investigated.

2. Properties of stochastic relations applicable to stochastic Kripke models (which are essentially families of these relations indexed through the modal operators) are investigated, in particular congruences, morphisms and their mutual relationship.

3. The notion of equivalent congruences is introduced and studied; equivalent congruences arise in a natural way in modal logic through states which have the same theory. Stochastic relations having equivalent congruences are shown to be bisimilar.

4. For stochastic Kripke models we propose the notion of a strong morphism, and, correspondingly, of strong bisimulations. A stochastic version of the Hennessy-Milner Theorem of the equivalence of bisimilarity and mutually identical theories is established.

Examples show how two popular logics are interpreted through a stochastic Kripke model, and this opens the avenue for further research. It should be interesting to see how other modal logics are interpreted. A prime candidate is PDL with its rich interaction among the modal operators that would have to be reflected in a suitable structure for the stochastic relations interpreting it. Investigating the interrelationship between a probabilistic interpretation, Kozen's semantics of probabilistic programs [12] and probabilistic predicate transformers [14] are expected to give new insights into PDL as well as probabilistic program semantics and, incidentally, the algebraic properties of stochastic relations.

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