Limit Theorems of the Power Variation of Fractional Lévy Processes

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Für meine kleine Familie

Ramona und Niklas.

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Notations

$\mathbb{N}$	$\{1, 2, \dots\}$
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\mathbb{1}$	indicator function
$a \wedge b$	$\min\{a, b\}$ for $a, b \in \mathbb{R}$
$a \vee b$	$\max\{a, b\}$ for $a, b \in \mathbb{R}$
$\operatorname{sign}(x)$	sign-function with $\operatorname{sign}(0) = 0$
f_+ resp. f_-	positive resp. negative part of a real valued function f
	defined as $f_+(x) := f(x) \vee 0$ resp. $f_-(x) := -f(x) \vee 0$
X_{t^-}	$\lim_{s \nearrow t} X_s$
ΔX_s	$X_s - X_{s^-}$
$f(x) = \mathbf{O}(g(x))$ as $x \to x_0$	Landau-symbol, $\exists C \geq 0$ such that $\frac{f(x)}{g(x)} \to C$ as $x \to x_0$
$f(x) = \mathbf{o}(g(x))$ as $x \to x_0$	Landau-symbol, $\frac{f(x)}{g(x)} \to 0$ as $x \to x_0$
$f(x) \sim g(x)$	$\exists c > 0$ such that $f(x) = cg(x)$
$f(x) \lesssim g(x)$	$\exists c > 0$ such that $f(x) \leq cg(x)$
$\mathcal{C}_b^0(\mathbb{R})$	$\{f \colon \mathbb{R} \to \mathbb{R} \mid f$ continuous, bounded$\}$
$\|f\|_\infty$	uniform norm of f
$\mathbb{P}$	probability measure
$\mathbb{E}$	expectation
φ_μ	characteristic function of some distribution μ
$\mu * \nu$	convolution of two distributions μ and ν
$X_n \to X \ \mathbb{P} - a.s.$	X_n converges to X almost surely
$X_n \xrightarrow{\mathbb{P}} X$	X_n converges in probability to X
$X_n \xrightarrow{\mathcal{D}} X$	X_n converges in distribution to X
$\mathfrak{S}_m$	group of all permutations of the set $\{1, \dots, m\}$
$\bigoplus$	direct sum

Introduction

The theory of power variation has been developed out of questions raised in mathematical finance. One quantity of interest in this theory is the integrated volatility which is important for pricing and risk assessment. It can be used for example for pricing constant maturity swap options in stochastic volatility models as considered in [**KL10**]. As the link between the mathematical concept of quadratic variation and integrated volatility was established this was the starting point for the use of power variation. The realized power variation was introduced by Barndorff-Nielsen and Shephard [**BNS02, BNS03, BNS04a, BNS04b**] in the context of stochastic volatility models as an estimator of the integrated volatility. The articles [**BNS02, BNS03**] provide limit theorems of power variations when the underlying model is some continuous time semi-martingale of the form $A_t + \int_0^t \sigma_s \, dB_s$, where A satisfies some regularity conditions and is stochastically independent of the Brownian motion B.

In various articles the limit behaviour of the realized power variation is analysed in different models, e.g. for stochastic volatility models in [**Woe05**], for functionals of semi-martingales in [**Jac08**] and for Gaussian processes with non-stationary increments in [**MN14**]. There are also limit theorems for the bipower variation e.g. for semi-martingales in [**BNGJ$^+$06**]. Both concepts are investigated in [**BNCP09, BNCPW09**] for Gaussian processes with stationary increments and in [**Pod14**] for ambit fields.

Our ideas are based on the article [**CNW06**] where limit theorems for the power variation for so-called integrated fractional processes are developed. These are processes of the form

$$Z_t := \int_0^t u_s \, dB_s^H,$$

where B^H denotes a fractional Brownian motion with Hurst parameter $H \in (0, 1)$. When $u \equiv 1$ limit theorems for fractional Brownian motions are obtained. Our goal in this thesis is to go one step further than the last cited article and drop the restriction of having Gaussian marginal distributions. Instead of fractional Brownian motions we use its generalisation, the so-called fractional Lévy processes. Fractional Brownian motions are introduced as Gaussian processes with a certain covariance structure. They also possess a so-called moving average representation, this is

$$B_t^H = C \int_{-\infty}^{\infty} a\left((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right) + b\left((t-s)_-^{H-1/2} - (-s)_-^{H-1/2}\right) dB_s,$$

where a, b and C are real valued constants and B is a two-sided Brownian motion. Fractional Lévy processes are obtained by replacing the Brownian motion B in the above representation by a two-sided Lévy process L. For simplicity we consider the case $a = 1$ and $b = 0$. By linearity of the integral and symmetry of the integrand the proofs of our results presented in Chapters 2 and 3 are the same as in the general case. This means that the representation of fractional Lévy processes reduces to

$$X_t^\gamma := \int_{-\infty}^{\infty} (t-s)_+^\gamma - (-s)_+^\gamma \, dL_s, \quad t \in \mathbb{R},$$

where we also replace the exponent $H - \frac{1}{2}$ in the definition of fractional Brownian motions by γ. The integral is defined in the sense of [**RR89**]. This article contains the construction of integrals with respect to random measures, a representation of the characteristic function of such integrals and it determines functions which are integrable with respect to random measures. The article [**EW13**] applies those techniques to fractional Lévy processes and provides a good overview about the properties of those processes. In [**EW13**, Corollaries 2, 3 and 4] it was shown under which conditions fractional Lévy processes are well defined. Since we need to restrict ourselves to the subclass of local self-similar fractional Lévy processes, which is defined later, we only state existence of local self-similar fractional Lévy processes. We derive the existence of this subclass of fractional Lévy processes from the existence of linear fractional stable motions. These processes were introduced by [**ST00**] and we will see that they are indeed fractional Lévy processes whose integrators are symmetric, α-stable Lévy processes.

In this thesis we develop a consistency theorem for integrated fractional processes in a pure jump model, that is, the fractional Brownian motion in the definition of the process Z above is replaced by a pure jump fractional Lévy process L^H satisfying the local self-similarity property. We will see that similar to the Gaussian models developed in [**BNCP09, CNW06**] we can use Bernstein's blocking technique to deduce consistency of the realized power variation as an estimator of integrated volatility. A limit theorem for the distributional theory is only developed for the power variation of linear fractional stable motions. The reason why we restrict ourselves to this special case of a fractional Lévy process is that the distributional theory is much more involved than in non-Gaussian models. In Gaussian models central and non central limit theorems are deduced with the help of very powerful results developed in the context of Wiener/Itô/Malliavin calculus (see e.g. [**HN05**]). Instead of this we use the technique of subordination to find an elegant way to reduce the proof of a limit theorem for the power variation of linear fractional stable motions (Theorem 3.1) to a Malliavin based limit theorem (Theorem 1.27). The technique we use to apply this theorem is similar to the Gaussian limit theorem provided by [**MN14**, Theorem 1] for the power variation of non-stationary Gaussian processes.

This thesis is structured as follows: In the first chapter we introduce all basic notations and additionally processes we are working with and their properties. Here, we focus on fractional Lévy processes and subordination for α-stable Lévy processes. The second chapter contains the development of a consistency theorem for integrated fractional processes including an analogue statement for local self-similar fractional Lévy processes. The last chapter includes the main result, this is the distributional limit theorem for linear fractional stable motions including a representation of a linear fractional stable motion as a conditionally Gaussian process G and the application of a Malliavin based limit theorem to the power variation of this process G.

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CHAPTER 1

Basics of Fractional Lévy processes

The scope of this chapter is to introduce the basic definitions and notations which are important for this thesis. Mainly these are fractional Lévy processes and subordination. In order to make the thesis self contained we also introduce ordinary Lévy processes and infinitely divisible laws. In addition to that we give a brief survey of stable distributions since they play a central role in this thesis. Also we introduce a limit theorem based on Malliavin calculus which is used later. Since definitions and results presented in this chapter are well known we restrict ourselves to a brief introduction.

We mainly consider real valued stochastic processes except for the explicit construction of two-sided Lévy processes in Chapter 3. Since in this case the two-dimensional process is given by a pair of two one-dimensional processes we restrict ourselves to the one-dimensional case in the definitions and formulation of results in this chapter. This is only for the sake of simplicity and most of the results can easily be generalised to d-dimensional random variables, distributions or stochastic processes.

We assume that all random variables are defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A *stochastic process* X is a family of random variables $X = (X_t)_{t \in I}$ with parameter $t \in I \subseteq \mathbb{R}$. We only consider the cases $I = [0, \infty)$ and $I = \mathbb{R}$. In the latter case we call the stochastic process $X = (X_t)_{t \in \mathbb{R}}$ *two-sided process*. We use the notations X_t, $X_t(\omega)$, $X(t)$ and $X(t, \omega)$ interchangeably. The mapping $t \mapsto X_t(\omega)$ is called *path* of the stochastic process X. A function $f \colon [0, \infty) \to \mathbb{R}$ is called *càdlàg*, if $f(t)$ is right continuous in $t \geq 0$ and has left limits in $t > 0$. A stochastic process has some path property if $\mathbb{P}$-almost every path has the stated property (e.g. a stochastic process is continuous, Hölder-continuous, càdlàg if almost every path is continuous, Hölder-continuous, càdlàg).

Let μ be a probability distribution on $\mathbb{R}$. It is called *trivial* if it is a Dirac distribution. Otherwise it is called *non-trivial*. We denote by $\varphi_\mu \colon \mathbb{R} \to \mathbb{C}$,

$$\varphi_\mu(u) = \int_{\mathbb{R}} e^{iux}\, d\mu(x)$$

the *characteristic function* of μ. The characteristic function of a random variable X is given by $\varphi_X = \varphi_{\mathbb{P}^X}$, where $\mathbb{P}^X$ denotes the law of X. We assume that the reader is familiar with characteristic functions and its properties (see e.g. [**Sat99**]).

The index of notations which can be found on page ii contains other often used notations.

1. Infinitely Divisible and Stable Distributions

In this section we introduce infinitely divisible distributions and describe their characteristic functions. As a particular instance of these distributions we define stable distributions and state some of their basic properties. We start with the definition of infinitely divisible distributions.

DEFINITION 1.1. A distribution μ is called *infinitely divisible* if for any positive integer $n \in \mathbb{N}$ there is a probability measure μ_n such that μ is the nth convolution of μ_n, that is

$$\mu = \underbrace{\mu_n * \ldots * \mu_n}_{n\text{-times}}.$$

The next proposition, the so-called *Lévy-Khintchine formula*, characterises infinitely divisible distributions.

PROPOSITION 1.2. *Let $h \colon \mathbb{R} \to \mathbb{R}$ be a bounded measurable function satisfying*

$$\begin{aligned} h(x) &= 1 + \mathbf{o}(|x|) \quad \textit{as } |x| \to 0, \\ h(x) &= \mathbf{O}\left(\tfrac{1}{|x|}\right) \quad \textit{as } |x| \to \infty. \end{aligned}$$

i) *If μ is an infinite divisible distribution, then there exist $\sigma^2 \geq 0$, a measure ν satisfying*

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} (|x|^2 \wedge 1)\, d\nu(x) < \infty, \tag{1.1}$$

and a real number $b \in \mathbb{R}$ such that

$$\varphi_\mu(u) = \exp\left(-\frac{1}{2}u^2\sigma^2 + ibu + \int_{\mathbb{R}} e^{iux} - 1 - iuxh(x)\, d\nu(x)\right) =: e^{\psi(u)}, \quad u \in \mathbb{R}. \tag{1.2}$$

ii) *The representation of φ_μ as given above by σ^2, ν and b is unique (but depends on h, c.f. below).*

iii) *Conversely, if $\sigma^2 \geq 0$, ν is a measure satisfying (1.1) and $b \in \mathbb{R}$, then there exists an infinitely divisible distribution μ whose characteristic function is given by (1.2).*

PROOF. This proposition is an immediate consequence of [**Sat99**, Theorem 8.1], where the proof is carried out in the d-dimensional case. □

DEFINITION 1.3. The triplet (b, σ^2, ν) as given in the above proposition is called the *generating triplet* of μ. The measure ν is called *Lévy measure* of μ. In the case $\sigma^2 = 0$ the distribution is called *purely non-Gaussian*. The function ψ defined in Equation (1.2) is called *characteristic exponent* of the measure μ.

The function h in the above proposition is called *truncation function* and it is obviously not unique. Also the parameter b depends on h. If we use another truncation function $\tilde{h}$ the parameter $b_{\tilde{h}}$ with respect to $\tilde{h}$ is obtained as follows:

$$b_{\tilde{h}} = b + \int_{\mathbb{R}} x(\tilde{h}(x) - h(x))\, d\nu(x).$$

If in addition the Lévy measure ν satisfies $\int_{[-1,1]} |x|\, d\nu(x) < \infty$, then the function $h \equiv 0$ is a valid choice and the parameter b_0 with respect to the zero function is called *drift*. On the other hand, if $\int_{|x|>1} x\, d\nu(x) < \infty$, the corresponding parameter b_1 to the truncation function $h \equiv 1$ is called the *centre* of μ and equals the mean of μ.

Next we introduce α-stable distributions.

DEFINITION 1.4. Let μ be a non-trivial, infinitely divisible probability measure and $\alpha \in (0,2]$. The measure μ is called *α-stable* if for any $r > 0$ there exists $c \in \mathbb{R}$ such that

$$\varphi_\mu(z)^r = \varphi_\mu\left(r^{\frac{1}{\alpha}} z\right) e^{icz}.$$

It is called *strictly α-stable* if for any $r > 0$

$$\varphi_\mu(z)^r = \varphi_\mu\left(r^{\frac{1}{\alpha}} z\right).$$

In the one-dimensional case the characteristic function can be represented as stated in the next proposition. After this we derive the generating triplet.

PROPOSITION 1.5. *Let $0 < \alpha < 2$. If a distribution μ is non-trivial and α-stable, then there exist $c > 0$, $\beta \in [-1,1]$ and $\tau \in \mathbb{R}$ such that*

$$\varphi_\mu(z) = \begin{cases} e^{-c|z|^\alpha\left(1 - i\beta \tan\frac{\alpha\pi}{2} \operatorname{sign} z\right) + i\tau z} & \text{for } \alpha \neq 1, \\ e^{-c|z|\left(1 + i\beta\frac{2}{\pi} \log|z| \operatorname{sign} z\right) + i\tau z} & \text{for } \alpha = 1. \end{cases} \tag{1.3}$$

Conversely, for every $c > 0$, $\beta \in [-1,1]$ and $\tau \in \mathbb{R}$, there is a non-trivial α-stable distribution μ satisfying (1.3). An α-stable distribution μ is strictly α-stable, iff $\tau = 0$ or $\beta = 0$ according as $\alpha \neq 1$ or $\alpha = 1$.

PROOF. The proof is carried out in [**Sat99**, Theorem 14.15] □

The generating triplet (b, σ^2, ν) of an α-stable distribution is obtained as follows: in the case $\alpha = 2$ the distribution is Gaussian and the triplet is $(m, \sigma^2, 0)$, where $m = \mathbb{E}[\mu]$ and $\sigma^2 = Var(\mu)$. For $\alpha \in (0,2)$ the Lévy measure ν of an α-stable distribution μ is absolutely continuous with respect to the Lebesgue measure and it holds

$$d\nu(x) = \begin{cases} c_1 x^{-1-\alpha} dx & \text{on } (0,\infty), \\ c_2 |x|^{-1-\alpha} dx & \text{on } (-\infty, 0), \end{cases}$$

where $c_1,\ c_2 \geq 0$ with $c_1 + c_2 > 0$ are obtained by $c = c_1 + c_2$ and $\beta = \frac{c_1 - c_2}{c}$. Concerning τ, if $0 < \alpha < 1$ then $\tau = b_0$, the drift of μ, and in the case $1 < \alpha < 2$ it holds $\tau = b_1$, the centre of μ. If $\alpha < 2$, it holds $\sigma^2 = 0$.

An α-stable distribution is called *symmetric* if $c_1 = c_2$ or equivalent $\beta = 0$.

In the next section we introduce Lévy processes.

2. Lévy Processes

Lévy processes are tightly related to infinitely divisible distributions. This will be the content of Proposition 1.7. We define Lévy processes as follows:

DEFINITION 1.6. A stochastic process $L = (L_t)_{t\geq 0}$ is called *Lévy process* if it satisfies the following conditions:

1) $L_0 = 0$ a.s.
2) For any $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \cdots < t_n$ the random variables

$$L_{t_0}, L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}$$

are independent (independent increments).
3) For $s, t \geq 0$ the distribution of $L_{s+t} - L_s$ does not depend on s (stationary increments).
4) The paths of L are almost surely càdlàg.

The following proposition characterises the one-dimensional marginal distributions of Lévy processes.

PROPOSITION 1.7. *If L is a Lévy process, then L_1 has infinitely divisible law. Let $\mu = \mathbb{P}^{L_1}$. If μ has generating triplet (b, σ^2, ν), then there exists a measure μ^t whose generating triplet is given by $(tb, t\sigma^2, t\nu)$. The measure μ^t corresponds to the t th convolution of μ and the law of L_t is given by μ^t. In particular this means that if the law of L_1 has characteristic exponent ψ, then the law of L_t possesses the characteristic exponent $t\psi$.*

PROOF. By [**Sat99**, Example 7.3] μ is infinitely divisible and the tth convolution μ^t of μ is obtained by applying [**Sat99**, Lemma 7.9]. The representation of the generating triplet of μ^t is a consequence of the Lévy-Khintchine formula and is stated in [**Sat99**, Corollary 8.3]. By [**Sat99**, Theorem 7.10 (i)] the law of L_t is given by μ^t. □

Stable Lévy processes are defined as follows:

DEFINITION 1.8. Let $0 < \alpha < 2$ and μ be a measure satisfying (1.3). A process L^α with $\mu = \mathbb{P}^{L_1^\alpha}$ is called a *stable Lévy process with parameters* (α, β, τ, c). Often it is simply called α-stable Lévy process.

We also introduce the notation of spectral negative processes:

DEFINITION 1.9. A Lévy process is called *spectral negative* if $\nu((0,\infty)) = 0$.

For stable processes this is the case iff $\beta = -1$.

In the next section we introduce fractional Lévy processes as processes that still possess the property of stationary increments but no longer have independent increments.

3. Local Self-Similar Fractional Lévy Processes

Fractional Lévy processes are defined as integrals of deterministic integrands (the so-called kernel-functions or kernels) with respect to two-sided Lévy processes. Linear fractional stable motions are one example of fractional Lévy processes. In this case the integrator, the so-called driver, is a symmetric α-stable Lévy process. From their existence we can derive the existence of so-called local self-similar fractional Lévy processes. Since the property of local self-similarity is crucial in the proof of our consistency theorem we will only introduce this subclass of fractional Lévy processes. For a more detailed introduction to fractional Lévy processes we refer to [**EW13**].

In order to define fractional Lévy processes we first define two-sided Lévy processes:

DEFINITION 1.10. Let $L^{(1)}$ and $L^{(2)}$ are independent and identically distributed Lévy processes. The two-sided version of $L^{(1)}$ (*two-sided Lévy process*) $L = (L_t)_{t\in\mathbb{R}}$ is defined by

$$L_t := \begin{cases} L_t^{(1)} & t \geq 0, \\ -L_{-t^-}^{(2)} & t < 0. \end{cases}$$

Note that $L_{-t^-} = L_{(-t)^-}$.

Let L^α be a two-sided symmetric α-stable Lévy process in the sense of Definition 1.8 with $\alpha \in (0,2)$, $\beta = 0$, $c \in \mathbb{R}$ and τ is the drift respectively the centre according as $0 < \alpha < 1$ or $1 < \alpha < 2$. In the case $\alpha = 1$ we assume $\tau = 0$ and use the truncation function $h(x) = \mathbb{1}_{|x|\leq 1}(x)$. This means that the Lévy-Khintchine formula of L^α reduces to

$$\mathbb{E}\left[e^{iuL_t^\alpha}\right] = \exp\left(ibut + t\int_{\mathbb{R}}\left(e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1}(x)\right)\,d\nu(x)\right),$$

where in this case for the generating triplet $(b, 0, \nu)$ it holds

$$b = \begin{cases} \int_{|x|\leq 1} x\,d\nu(x) & \text{if } \alpha < 1, \\ \int_{|x|>1} x\,d\nu(x) & \text{if } \alpha > 1, \\ 0 & \text{if } \alpha = 1 \end{cases} \tag{1.4}$$

and the Lévy measure ν is absolutely continuous with respect to the Lebesgue measure with density $g(x) = \frac{c}{|x|^{1+\alpha}}$.

As mentioned in the introduction fractional Lévy processes are defined as integrals of some deterministic functions, the so-called kernel functions, with respect to two-sided Lévy processes, where the integral is given in the sense of [**RR89**, Definition 2.5]. In this calculus the integral of a (deterministic) function f with respect to a two-sided Lévy process is defined as follows (see e.g. [**EW13**]): if $a < b$ and $f = \mathbb{1}_{(a,b]}$, then

$$\int_{\mathbb{R}} f(s)\,dL_s = L(b) - L(a).$$

Linear-combinations of functions of this type (simple functions) are treated as usual by the linearity of the integral. The integral of a measurable function f with respect to L

exists if f is the almost sure limit of approximating simple functions whose integrals converge in probability. Additionally, the integral does not depend on the approximating sequence. We later state the existence result for linear fractional stable motions.

In order to have the property of stationary increments we can choose the same kernel functions which are used for fractional Brownian motions. These are

- $f_\gamma^+(t,s) := (t-s)_+^\gamma - (-s)_+^\gamma$,
- $f_\gamma^-(t,s) := (t-s)_-^\gamma - (-s)_-^\gamma$ and
- $f_\gamma(t,s) := af_\gamma^+(t,s) + bf_\gamma^-(t,s), \quad a,b \in \mathbb{R}$,

where $\gamma \in \mathbb{R}$ and we always exclude the case $\gamma = 0$. Since all kernel functions can be treated along similar lines we restrict ourselves to $f_\gamma^+(t,s)$ and $\gamma \in \left(-\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right)$. The last condition ensures the existence of the corresponding local self-similar fractional Lévy process (c.f. Remark 1.13 and 1.16 below). We also state an integrability condition for the function $s \mapsto f_\gamma^+(t,s),\ s \in \mathbb{R}$:

LEMMA 1.11. *The function $s \mapsto |f_\gamma^+(t,s)|^\delta$ is integrable at 0 and t with respect to Lebesgue-measure iff either $\gamma > 0$ or both of $\gamma < 0$ and $\delta < -\frac{1}{\gamma}$ are satisfied. It is integrable at $-\infty$ iff $\delta > \frac{1}{1-\gamma}$.*

PROOF. See [**EW13**, Proposition 2]. □

Next we define linear fractional stable motions as one example of fractional Lévy processes. They were introduced in [**ST00**] as an extension of fractional Brownian motions as a self-similar process with non-Gaussian marginal distributions. We define them as follows:

DEFINITION 1.12. Let $\alpha \in (0,2)$, L^α defined as above, $\gamma \in \left(-\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right)$ and $H = \gamma + \frac{1}{\alpha}$. We define the *linear fractional stable motion* X^H by

$$X_t^H := \int_{-\infty}^{\infty} f_\gamma^+(t,s)\, dL_s^\alpha, \quad t \in \mathbb{R},$$

where the integral is defined as described above. In some literature linear fractional stable motions are also called *fractional α-stable motions*.

REMARK 1.13. The parameter H in the above definition indicates the self-similarity index of the process X^H. The existence of linear fractional stable motions defined as above is an immediate consequence of [**EW13**, Corollary 2, 3 or 4] according as $\alpha = 1$, $0 < \alpha < 1$ or $1 < \alpha < 2$. Is the driver an α-stable Lévy processes L^α then the integral

$$\int_{-\infty}^{\infty} f_\gamma^+(t,s)\, dL_s^\alpha$$

exists for all $t \in \mathbb{R}$ in the sense of [**RR89**, Definition 2.5] iff $\left|f_\gamma^+(t,s)\right|^\alpha$ is integrable with respect to the Lebesgue measure. This condition is equivalent to $\gamma \in \left(-\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right)$.

REMARK 1.14. For $0 < \alpha < 2$ an alternative definition of linear fractional stable motions is introduced in [**ST00**, Example 3.6.5 and Section 7.4] as processes of the form

$$Y_t^{\alpha,H} = \int_{-\infty}^{\infty} f_{H-\frac{1}{\alpha}}^+(t,s)\, dM_s, \quad t \in \mathbb{R},$$

where M is an α-stable random measure with Lebesgue control measure, a, b are real constants with $|a| + |b| > 0$ and $0 < H < 1$, $H \neq \frac{1}{\alpha}$. Since $\int_{-\infty}^{\infty} |f_{H-\frac{1}{\alpha}}(t,s)|^\alpha\, ds < \infty$ the process is well defined. If we restrict ourselves to the symmetric case and if we consider w.l.o.g. $f_{H-\frac{1}{\alpha}}^+(t,s)$ as the integrand, then this process is equal in distribution to the process X^H in the above definition. This is a direct consequence of the proof of Proposition 1.18 below.

As mentioned above linear fractional stable motions are self-similar processes with parameter H, which means that for all $a > 0$ the finite dimensional distributions of $\left(X_{at}^H\right)_{t\in\mathbb{R}}$ are the same as those of $\left(a^H X_t^H\right)_{t\in\mathbb{R}}$. General fractional Lévy processes (not coming from stable processes) do not satisfy this condition which plays a central role when studying consistency theorems for the power variation. In Theorem 2.3 we show that for fractional Lévy processes this condition can be relaxed to the so-called local self-similarity. We now define the class of local self-similar fractional Lévy processes and derive a representation of the characteristic function of their marginal distributions. Then we prove the local self-similarity property of such processes (c.f. Proposition 1.18 below).

DEFINITION 1.15. Let $\alpha \in (0,2)$ and L be a Lévy process with generating triplet $(b, 0, \nu)$ whose Lévy measure ν has a Lebesgue-density g such that

(1) $\lim_{x \to 0} \frac{g(x)}{|x|^{-1-\alpha}} = C_1$,
(2) $g(x) \leq C_2 |x|^{-1-\alpha} \qquad \forall x \in \mathbb{R}$,

where C_1, $C_2 > 0$, and b depending on α is given in (1.4). For $\gamma \in \left(-\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right)$ the process $L_t^H = \int_{\mathbb{R}} f_\gamma^+(t,s)\, dL_s$, $t \in \mathbb{R}$, is called *local self-similar fractional Lévy process*. The process L is called *driving Lévy process* or *driver* of L^H.

REMARK 1.16. Since the existence of integrals in the sense of [**RR89**] depends on the Lévy measure and the second property of the Lévy measure of the driving Lévy process L in the above definition is satisfied, the existence of the process L^H is a direct consequence of the existence of the linear fractional stable motions. Obviously linear fractional stable motions are one example of local self-similar fractional Lévy processes. In this case it holds $g(x) = \frac{c}{|x|^{1+\alpha}}$.

From [**RR89**, Proposition 2.6] we can deduce the characteristic function of the marginal distributions of general fractional Lévy processes and in particular for any process taken from the subclass of local self-similar fractional Lévy processes as follows.

PROPOSITION 1.17. *The process L^H as defined in Definition 1.15 has stationary increments. Moreover, for $m \in \mathbb{N}$, $t_1, \dots, t_m \in \mathbb{R}$ and $u_1, \dots, u_m \in \mathbb{R}$ its finite dimensional distributions exhibit the characteristic function given by*

$$\mathbb{E}\left[\exp\left\{i \sum_{j=1}^m u_j L_{t_j}^H\right\}\right] = \exp\left\{\int_{\mathbb{R}} \psi\left(\sum_{j=1}^m u_j f_\gamma^+(t_j, s)\right) ds\right\},$$

where

$$\psi(y) = iyb + \int_{\mathbb{R}} \left(e^{ixy} - 1 - ixy\mathbb{1}_{|x|\leq 1}(x) \right) d\nu(x), \quad y \in \mathbb{R}.$$

Additionally, the distribution of L_t^H is infinitely divisible for all $t \in \mathbb{R}$.

PROOF. The statement is a consequence of [**RR89**, Proposition 2.6] and the proof is worked out in detail in [**EW13**, Proposition 4] for fractional Lévy processes. □

In the next proposition we deduce the local self-similarity property of local self-similar fractional Lévy processes introduced in Definition 1.15. It is essentially the same as [**Mar06**, Theorem 4.7]. The difference is that it considers $\varepsilon^{-H}\left(L_{t+\varepsilon a}^H - L_t^H\right)$ instead of $\varepsilon^{-H}\left(L_{\varepsilon(t+a)}^H - L_{\varepsilon t}^H\right)$ which means that we keep the representation as an increment. This is important for the proof of our consistency theorem.

PROPOSITION 1.18. *Let L^H be as in Definition 1.15. Then L^H is locally self-similar with parameter $H = \gamma + 1/\alpha$, i.e. for each $a \in \mathbb{R}$, it holds*

$$\lim_{\varepsilon \searrow 0} \left(\varepsilon^{-H} \left(L_{\varepsilon(t+a)}^H - L_{\varepsilon t}^H \right) \right)_{t\in\mathbb{R}} \overset{d}{=} \left(X_{t+a}^H - X_t^H \right)_{t\in\mathbb{R}},$$

where the limit is in distribution for all finite dimensional marginals and the process X^H is the linear fractional stable motion with parameter α as in Definition 1.12. Additionally the processes X^H and $Y^{\alpha,H}$ (introduced in Remark 1.14) are equal in distribution.

REMARK 1.19. In the proof of the proposition above we use the distinguished logarithm in the sense of [**Sat99**, Lemma 7.6] in order to show that the finite dimensional marginals have the same characteristic function.

PROOF. The proof is similar to the one in [**Mar06**] and is done in two steps, first, we calculate the limit of the characteristic functions of $\varepsilon^{-H}\left(L_{\varepsilon(t+a)}^H - L_{\varepsilon t}^H\right)$ as $\varepsilon \searrow 0$. Then we show that this limit is indeed the characteristic function of the linear fractional stable motion we introduced in Remark 1.14.

Starting with the first step let $u_1, \dots, u_m \in \mathbb{R}$, $-\infty < t_1 < \dots < t_m < \infty \in \mathbb{R}$ and $m \in \mathbb{N}$. Then we calculate:

$$\log \mathbb{E}\left[exp\left\{i\sum_{k=1}^{m} u_k \varepsilon^{-H}\left(L^H_{\varepsilon(t_k+a)} - L^H_{\varepsilon t_k}\right)\right\}\right]$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \Bigg(e^{ix\sum_{k=1}^{m} u_k \varepsilon^{-H}\left((\varepsilon(t_k+a)-s)_+^{\gamma} - (\varepsilon t_k - s)_+^{\gamma}\right)} - 1$$

$$- ix\sum_{k=1}^{m} u_k \varepsilon^{-H}\left((\varepsilon(t_k+a)-s)_+^{\gamma} - (\varepsilon t_k - s)_+^{\gamma}\right)\mathbb{1}_{|x|\le 1}\Bigg)\, d\nu(x)\, ds$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \Bigg(e^{ix\sum_{k=1}^{m} u_k \varepsilon^{-1/\alpha}\left((t_k+a-s)_+^{\gamma} - (t_k - s)_+^{\gamma}\right)} - 1$$

$$- ix\sum_{k=1}^{m} u_k \varepsilon^{-1/\alpha}\left((t_k+a-s)_+^{\gamma} - (t_k - s)_+^{\gamma}\right)\mathbb{1}_{|x|\le 1}\Bigg)\, \varepsilon\, d\nu(x)\, ds,$$

where the last equation results by substituting s by $s\varepsilon$. Now, we substitute $x = \varepsilon^{1/\alpha} y$ and since

$$\int_{1\le|y|\le\varepsilon^{-1/\alpha}} i\frac{y}{|y|^{1+p}}\sum_{k=1}^{m} u_k\left((t_k+a-s)_+^{\gamma} - (t_k-s)_+^{\gamma}\right) dy = 0$$

we obtain

$$\log \mathbb{E}\left[exp\left\{i\sum_{k=1}^{m} u_k \varepsilon^{-H}\left(L^H_{\varepsilon(t_k+a)} - L^H_{\varepsilon t_k}\right)\right\}\right]$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \Bigg(e^{iy\sum_{k=1}^{m} u_k\left((t_k+a-s)_+^{\gamma} - (t_k - s)_+^{\gamma}\right)} - 1$$

$$- iy\sum_{k=1}^{m} u_k\left((t_k+a-s)_+^{\gamma} - (t_k - s)_+^{\gamma}\right)\mathbb{1}_{|y\varepsilon^{1/\alpha}|\le 1}\Bigg)\, \varepsilon\, d\nu(y\varepsilon^{1/\alpha})\, ds$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \Bigg(e^{iy\sum_{k=1}^{m} u_k\left((t_k+a-s)_+^{\gamma} - (t_k - s)_+^{\gamma}\right)} - 1$$

$$- iy\sum_{k=1}^{m} u_k\left((t_k+a-s)_+^{\gamma} - (t_k - s)_+^{\gamma}\right)\mathbb{1}_{|y|\le 1}\Bigg)\, \varepsilon\, d\nu(y\varepsilon^{1/\alpha})\, ds$$

$$=: \int_{\mathbb{R}}\int_{\mathbb{R}} F(y,s)\varepsilon\, d\nu(y\varepsilon^{1/\alpha})\, ds.$$

By using the asymptotic behaviour of the density g of the Lévy measure ν it holds

$$\varepsilon\, d\nu(y\varepsilon^{1/\alpha}) = \varepsilon g(y\varepsilon^{1/\alpha})\varepsilon^{1/\alpha}\, dy \overset{\varepsilon \text{ small}}{\sim} \varepsilon^{1+1/\alpha}|\varepsilon^{1/\alpha} y|^{-1-\alpha}\, dy = |y|^{-1-\alpha}\, dy,$$

which is the Lévy measure of the process X^H introduced in Definition 1.12. If we now pass to the limit for $\varepsilon \searrow 0$ we can conclude by Lebesgue's theorem

$$\log \mathbb{E}\left[exp\left\{i\sum_{k=1}^{m} u_k \varepsilon^{-H}\left(L^H_{\varepsilon(t_k+a)} - L^H_{\varepsilon(t_k+a)}\right)\right\}\right] \to \int_{\mathbb{R}}\int_{\mathbb{R}} F(y,s)\frac{dy}{|y|^{1+\alpha}}\,ds.$$

This is exactly the representation in Proposition 1.17 of characteristic functions of fractional Lévy processes, in particular this is the representation of the characteristic function of the process X^H.

To prove the second step we use the Euler-representation of the exponential function. For the sake of simplicity we define $z_s := \sum_{k=1}^{m} u_k\left((t_k+a-s)_+^{\gamma} - (t_k-s)_+^{\gamma}\right)$ and use the symmetry of the sine function to calculate

$$\int_{\mathbb{R}} i(\sin yz_s - yz_s \mathbb{1}_{|y|\le 1})\frac{dy}{|y|^{1+\alpha}} = i\int_{-1}^{1}(\sin yz_s - yz_s)\frac{dy}{|y|^{1+\alpha}} + i\int_{|y|>1} \sin yz_s \frac{dy}{|y|^{1+\alpha}} = 0,$$

where all those integrals exist. Hence, we conclude

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} F(y,s)\frac{dy}{|y|^{1+\alpha}}\,ds\\
=&2\int_{\mathbb{R}}\int_{0}^{\infty}(\cos yz_s - 1)\frac{dy}{|y|^{1+\alpha}}\,ds\\
\overset{(*)}{=}&2\int_{\mathbb{R}}\int_{0}^{\infty}(\cos x - 1)\,\mathrm{sign}\,(z_s)\,|z_s|^{\alpha}\,\frac{dx}{|x|^{1+\alpha}}\,ds\\
=&2\int_{0}^{\infty}(\cos x - 1)\frac{dx}{|x|^{1+\alpha}}\int_{\mathbb{R}}\mathrm{sign}\,(z_s)\,|z_s|^{\alpha}\,ds,
\end{aligned}$$

which is the characteristic function of the linear fractional stable motion, see [**ST00**, p.114]. Equation $(*)$ holds by substituting $x = yz_s$. This proves the equality in distribution of the processes X^H and $Y^{\alpha,H}$ as it is stated in Remark 1.14. □

In the next proposition we give conditions under which local self-similar fractional Lévy processes are Hölder-continuous.

PROPOSITION 1.20. *Let $\gamma \in \left(0, 1 - \frac{1}{\alpha}\right)$. Then linear fractional stable motions are Hölder-continuous of order d with $d < \gamma$. Additionally, let L^H be a local self-similar fractional Lévy process such that $\mathbb{E}\left[\left|L_1^H\right|^\eta\right] < \infty$ for some $\eta > \alpha > 1$. Then the process L^H possesses the same order of Hölder-continuity.*

PROOF. The Hölder-continuity of local self-similar fractional Lévy processes is carried out in [**EW13**, Proposition 6]. The result for linear fractional stable motions is proven in [**KM91**, Theorem 2]. □

In this section we only presented those properties of (local self-similar) fractional Lévy processes we use in this thesis. In general the requirements for the Hölder-continuity of fractional Lévy processes are more involved and also depend on the Blumenthal-Getoor-index of the driving Lévy process. Under the assumption of square integrability fractional Lévy processes have the same covariance structure as fractional Brownian motions. Also there are conditions under which fractional Lévy processes are semi-martingales with finite variation although fractional Brownian motions cannot be semi-martingales. For a more detailed insight to fractional Lévy processes we refer to [**EW13**].

In the next section we introduce subordination of Lévy processes.

4. Subordination

Random time changes are one example of transformation of stochastic processes to other stochastic processes. A specific kind of random time changes is subordination. Hereby, the time change is a non-decreasing Lévy process which is independent of the original process. The idea was first introduced by [**Boc49**] and it was expounded in [**Boc55**].

We use this technique to determine the limit distribution of the power variation of linear fractional stable motions. For this purpose we first derive a representation of linear fractional stable motions as conditionally Gaussian processes. This is done by subordination of the driver of these processes. The driver of a linear fractional stable motion is a symmetric α-stable Lévy process. Due to Proposition 1.23 it is sufficient to consider $\frac{\alpha}{2}$-stable subordinators with $1 < \alpha < 2$. In this section we give an introduction to subordination of symmetric α-stable Lévy processes. For a deeper insight into the technique of subordination we refer to [**Sat99**]. Many statements in this section are adopted from this monograph.

We start with the mathematical formulation of subordination.

DEFINITION 1.21. a) A stochastic process Z is called a *subordinator* if it is a non-decreasing real valued Lévy process.

b) Let L be a real valued Lévy process and suppose that Z is an independent subordinator. Then the process Y defined by

$$Y_t(\omega) := L_{Z_t(\omega)}(\omega)$$

is called *subordination* by the subordinator Z. Any process identical in law to Y is said to be *subordinate* to L.

c) Let $1 < \alpha < 2$. A process θ is called $\frac{\alpha}{2}$*-stable subordinator* with parameter c', when its Laplace transform is given by

$$\mathbb{E}\left[e^{-u\theta_t}\right] = e^{-tc'u^{\frac{\alpha}{2}}}, \quad u \geq 0,\, c' > 0.$$

This means θ is a non-decreasing $\frac{\alpha}{2}$-stable Lévy process.

REMARK 1.22. The process Y given as in the definition above is a Lévy process. This statement can be found in [**Sat99**, Theorem 30.1].

The next proposition provides the representation of symmetric α-stable Lévy processes as subordinate to Brownian motions.

PROPOSITION 1.23. *Let $1 < \alpha < 2$. Subordination of a standard Brownian motion B by an independent $\frac{\alpha}{2}$-stable subordinator θ (with parameter c') results in a symmetric α-stable process*

L^α with parameter $c = 2^{-\frac{\alpha}{2}}c'$. Conversely all symmetric α-stable processes are obtained in this way.

PROOF. The first part follows by calculating the characteristic function of the process L^α by conditioning on θ as follows:

$$\mathbb{E}\left[e^{izL_t^\alpha}\right] = \mathbb{E}\left[\mathbb{E}\left[e^{izB(\theta_t)}|\theta\right]\right] = \mathbb{E}\left[e^{-\frac{1}{2}|z|^2\theta_t}\right] = e^{-tc'|z|^\alpha 2^{-\frac{\alpha}{2}}}.$$

The second part is provided by [**Sat99**, Theorem 14.14]. □

The proof of our main result relies on an explicit construction of a two-sided $\frac{\alpha}{2}$-stable subordinator. This will be the two-sided extension of the construction below which is based on the techniques introduced by [**Sat99**]. In this section we also focus on the parameters in order to be able to suppress them in Chapter 3 where this construction is carried our for two-sided processes.

For $1 < \alpha < 2$ an $\frac{\alpha}{2}$-stable subordinator can be obtained in the following way: let C be a spectral negative $\frac{2}{\alpha}$-stable process with parameters $(\frac{2}{\alpha}, -1, 0, \tilde{c})$ and M be its running maximum defined by $M_t := \sup_{0\le s\le t} C_s$. We define the process θ by

$$\theta_u := \inf\{t \ge 0|\, C_t > u\} = \inf\{t \ge 0|\, M_t > u\}, \quad u \ge 0,$$

which is the first passage time of the processes C and M of the level u. Since C has no positive jumps the process M is continuous. The next proposition states that θ is indeed an $\frac{\alpha}{2}$-stable subordinator.

PROPOSITION 1.24. *Let θ be defined as above. Then, if $\tilde{c} \neq 0$ the process θ is an $\frac{\alpha}{2}$-stable subordinator with Laplace transform given by*

$$\mathbb{E}\left[e^{-u\theta_t}\right] = e^{-t\cos(\frac{\alpha\pi}{2})\tilde{c}^{-1}|u|^{\frac{\alpha}{2}}}.$$

PROOF. This is an immediate consequence of [**Sat99**, Theorem 46.3]. □

This section is completed by providing a growth condition on θ. It is the short time behaviour of stable subordinators (c.f. [**Sat99**, Proposition 47.13]).

PROPOSITION 1.25. *Let* $1 < \alpha < 2$ *and* θ *be an* $\frac{\alpha}{2}$*-stable subordinator with parameters* $(\frac{\alpha}{2}, 1, 0, 1)$ *and let* $B_\alpha = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \cos\left(\frac{1}{2}\pi\alpha\right)^{-\frac{1}{1-\alpha}}$. *Then almost surely it holds*

$$\liminf_{t \searrow 0} \frac{\theta_t}{t^{\frac{2}{\alpha}} (2 \log\log(1/t))^{-\frac{2-\alpha}{\alpha}}} = \left(2B_{\frac{\alpha}{2}}\right)^{\frac{2-\alpha}{\alpha}}. \tag{1.5}$$

PROOF. The statement is proven in [**Sat99**, Proposition 47.13]. □

We complete this chapter by a brief introduction to Malliavin calculus and a limit theorem we use to prove our main result.

5. Introduction to Malliavin Calculus for Gaussian Processes

In order to deduce a distributional limit theorem for the power variation of linear fractional stable motions we need a limit theorem based on Malliavin calculus. Hence, we give a short introduction to the Malliavin calculus in order to be able to formulate a central limit theorem for sequences of random variables that admit a Wiener chaos representation. For a more detailed insight to Malliavin calculus based on Wiener chaos decomposition we refer to [**Nua95**].

We start with the Wiener chaos decomposition and generalised multiple Wiener integrals. To this end we first define isonormal Gaussian processes on some Hilbert spaces.

DEFINITION 1.26. Let H be a real, separable Hilbert space with inner product $\langle ., . \rangle_H$ and $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. A family of random variables $W = \{W(h) | h \in H\}$ is called *isonormal Gaussian process* on H if W is a centred Gaussian family of random variables such that for all $g, h \in H$ it holds $\mathbb{E}[W(h)W(g)] = \langle h, g \rangle_H$.

Classically one would start with some given Hilbert space H and construct the Wiener chaos decomposition for square integrable random variables which are measurable with respect to the filtration given be an isonormal Gaussian process. Instead of this we start with a given Gaussian process G and construct a Hilbert space where an isonormal Gaussian process can be defined on. In this way we ensure that the power variation of the given process G satisfies the measurability condition of the Wiener chaos decomposition (c.f. [**Nua95**, Theorem 1.1.1]) and as a consequence it admits a series representation given by a Wiener chaos decomposition. The approach chosen here is based on the appendix of [**MN14**].

Let $T > 0$ and G be a centred, real valued Gaussian process on some complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and let $(\pi^n)_{n\in\mathbb{N}}$ be a sequence of partitions of $[0, T]$, this means

$$\pi^n := \{t_j^n \,|\, 0 \leq t_0^n < t_1^n < \cdots < t_{m_n}^n \leq T\}.$$

We define $\Delta_j^n G := G(t_j^n) - G(t_{j-1}^n)$ and $w_{j,n} := \left(\mathbb{E}\left[\Delta_j^n G^2\right]\right)^{\frac{1}{2}}$. Then

$$W := \left\{ \frac{\Delta_j^n G}{w_{j,n}} \,\middle|\, j = 1, \dots, n,\ n \in \mathbb{N} \right\}$$

is a collection of standard normal random variables. Let $\mathcal{H}$ be the closure of all finite linear combinations of elements of W with respect to the norm of $L^2 := L^2(\Omega, \mathcal{A}, \mathbb{P})$. Under this assumptions the space $\mathcal{H}$ is a Hilbert space with inner product being the covariance of its elements. As a consequence the identity map on $\mathcal{H}$ is an isonormal Gaussian process on $\mathcal{H}$.

Let H_m be the mth Hermite polynomial defined by $H_0(x) \equiv 1$ and

$$H_m(x) := (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m} e^{-\frac{x^2}{2}}, \quad m \geq 1.$$

For each $m \geq 1$ we define $\mathcal{H}_m$ as the closed linear subspace of $L^2(\Omega, \mathcal{A}, \mathbb{P})$ generated by the set of random variables

$$\{H_m(h) \,|\, h \in \mathcal{H} : \|h\|_{\mathcal{H}} = 1\}.$$

For $m = 0$ we define $\mathcal{H}_0$ as the set of constants. For $m \geq 0$ the space $\mathcal{H}_m$ is called *mth Wiener chaos*.

Let $\mathcal{G}$ be the σ-algebra generated by the elements of $\mathcal{H}_1 = \mathcal{H}$. By [**Nua95**, Theorem 1.1.1] the space $L^2(\Omega, \mathcal{G}, \mathbb{P})$ has a decomposition into the infinite orthogonal sum of the subspaces $\mathcal{H}_m$, $m \geq 0$, this means

$$L^2(\Omega, \mathcal{G}, \mathbb{P}) = \bigoplus_{m=0}^{\infty} \mathcal{H}_m.$$

We denote by J_m the projection of $L^2(\Omega, \mathcal{G}, \mathbb{P})$ onto the mth Wiener chaos $\mathcal{H}_m$.

The abstract multiple Wiener integral is defined as follows: if $\{e_k |\ k \geq 1\}$ is a complete orthogonal system of $\mathcal{H}$, then $\{e_{j_1} \otimes \cdots \otimes e_{j_m} |\ j_1, \ldots, j_m \geq 1\}$ is an orthonormal basis of the mth tensor product of $\mathcal{H}$, denoted by $\mathcal{H}^{\otimes m}$. We define the symmetrisation of $e_{j_1} \otimes \cdots \otimes e_{j_m}$ by

$$symm(e_{j_1} \otimes \cdots \otimes e_{j_m}) := \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} e_{\sigma(j_1)} \otimes \cdots \otimes e_{\sigma(j_m)}.$$

Then the set

$$\{symm(e_{j_1} \otimes \cdots \otimes e_{j_m}) |\ j_1, \ldots, j_m \geq 1\}$$

is an orthonormal basis of $\mathcal{H}^{\odot m}$, which is the symmetric mth tensor product of $\mathcal{H}$. The inner product $\langle ., . \rangle_{\mathcal{H} \otimes \mathcal{H}}$ on the tensor product $\mathcal{H} \otimes \mathcal{H}$ is given by the relationship

$$\langle g_1 \otimes h_1, g_2 \otimes h_2 \rangle_{\mathcal{H} \otimes \mathcal{H}} = \langle g_1, g_2 \rangle_{\mathcal{H}} \langle h_1, h_2 \rangle_{\mathcal{H}}.$$

We equip $\mathcal{H}^{\odot m}$ with the norm $\sqrt{m!} \|.\|_{\mathcal{H}^{\otimes m}}$. To a multiindex $d = (d_j)_{j \geq 1} \in \mathbb{N}_0^{\mathbb{N}}$ such that all terms except a finite number of them vanish we define the generalised Hermite polynomial $H_d(x)$, $x \in \mathbb{R}^{\mathbb{N}}$, by

$$H_d(x) = \prod_{j=1}^{\infty} H_{d_j}(x_j).$$

By the above condition on d this is well defined. We also set $d! := \prod_{j=1}^{\infty} d_j$, $|d| = \sum_{j=1}^{\infty} d_j$ and $\Phi_d := \sqrt{d!} \prod_{j=1}^{\infty} H_{d_j}(e_j)$. Note that for the last definition it is involved that the identity is the isonormal Gaussian process used here. The set $\{\Phi_d |\ |d| = m\}$ is a complete orthonormal system of $\mathcal{H}_m$ (c.f. [**Nua95**, Proposition 1.1.1]). As a consequence the mapping

$I_m\colon \mathcal{H}^{\odot m} \to \mathcal{H}_m$ defined by

$$I_m \left(symm \left(\bigotimes_{j=1}^{\infty} e_j^{\otimes d_j} \right) \right) := \sqrt{d!}\Phi_d$$

is an isometry. Consequently, for $h \in \mathcal{H}$ such that $\|h\|_{\mathcal{H}} = 1$ it is

$$I_m(h^{\otimes m}) = H_m(h) \tag{1.6}$$

and it holds

$$\mathbb{E}\left[I_m(f)\right]^2 = m!\|f\|_{\mathcal{H}^{\otimes m}} \tag{1.7}$$

for all $f \in \mathcal{H}^{\odot m}$.

We also define contractions of elements taken from tensor products of Hilbert spaces. Let $m, n \geq 2$ and suppose that $g \in \mathcal{H}^{\otimes m}$ and $h \in \mathcal{H}^{\otimes n}$ have the representation

$$\begin{aligned} g &= \sum_{j_1,\dots,j_m=1}^{\infty} a(j_1,\dots,j_m) e_{j_1} \otimes \dots \otimes e_{j_m} \quad \text{respectively} \\ h &= \sum_{k_1,\dots,k_n=1}^{\infty} b(k_1,\dots,k_n) e_{k_1} \otimes \dots \otimes e_{k_n}, \end{aligned}$$

where $a(j_1,\dots,j_m)$ and $b(k_1,\dots,k_n)$ are real numbers depending on the indices $j_1,\dots,j_m$ respectively $k_1,\dots,k_n$. Then for any $1 \leq \kappa \leq m \wedge n$ we can define the contraction of order κ of g and h by

$$\begin{aligned} g \otimes_\kappa h := \sum_{z_1,\dots,z_{m+n-2\kappa}=1}^{\infty} \sum_{l_1,\dots,l_\kappa=1}^{\infty} & a(l_1,\dots,l_\kappa,z_1,\dots,z_{m-\kappa}) \\ & \cdot b(l_1,\dots,l_\kappa,z_{m-\kappa+1},\dots,z_{m+n-2\kappa}) e_{z_1} \otimes \dots \otimes e_{z_{m+n-2\kappa}}. \end{aligned}$$

Note that $g \otimes_\kappa h \in \mathcal{H}^{\otimes m+n-2\kappa}$.

With these definitions we are able to state the central limit theorem for random variables admitting a Wiener chaos representation. It can be found in [**MN14**, Theorem A.1] which is based on [**HN05**, Theorem 3 and Remark 1].

THEOREM 1.27. *Let $(F_n)_{n\in\mathbb{N}}$ be a sequence of square integrable, centred random variables with Wiener chaos representations given by*

$$F_n = \sum_{m=0}^{\infty} I_m(f_{m,n})$$

with some symmetric functions $f_{m,n} \in \mathcal{H}^{\odot m}$. Under the assumptions

(1) *for every $n \geq 1,\ m \geq 1$ it holds $m!\|f_{m,n}\|^2_{\mathcal{H}^{\otimes m}} \leq \delta_m$, where $\sum\limits_{m=1}^{\infty} \delta_m < \infty$;*
(2) *for every $m \geq 1$ there exists $\lim\limits_{n\to\infty} m!\|f_{m,n}\|^2_{\mathcal{H}^{\otimes m}} =: \sigma_m^2$;*
(3) *for every $m \geq 2$ and $\kappa = 1, \ldots, m-1$ it is $\lim\limits_{n\to\infty} \|f_{m,n} \otimes_\kappa f_{m,n}\|^2_{\mathcal{H}^{\otimes 2(m-\kappa)}} = 0$*

the sequence $(F_n)_{n\in\mathbb{N}}$ converges in distribution to a centred Gaussian random variable with variance given by $\sigma^2 = \sum\limits_{m=1}^{\infty} \sigma_m^2$.

CHAPTER 2

Consistency Theorem for the Power Variation of Integrated Fractional Processes

In this chapter we provide a consistency theorem for the power variation of integrated fractional processes as an estimator for the integrated volatility in a pure jump model. The idea is based on [**CNW06**]. In this article a consistency theorem is deduced for the same class of processes in a Gaussian model, that is, if the integrator is a fractional Brownian motion and the integrand is a stochastic process with finite q-variation where $q < \frac{1}{1-H}$. The last assumption is important to ensure the existence of the integral as a Riemann-Stieltjes integral. To be able to define a Riemann-Stieltjes integral of some stochastic process with respect to a local self-similar fractional Lévy process we have to be more careful. This is because the Hölder regularity of fractional Lévy processes is less than the one of fractional Brownian motions. Therefore, we are only able to prove a consistency theorem for the power variation in our model if $\gamma > 0$ and if the integrand has finite q-variation with $q < \frac{1}{1-\gamma}$. Nevertheless, a technique called Bernstein's blocking technique, which will be highlighted later, can be applied similar to the proof of [**CNW06**, Theorem 1]. This technique is also used in [**BNCP09**] to deduce a consistency theorem for the power variation of Gaussian processes with stationary increments. The results presented in this chapter are already available in [**Gla14**] which has recently appeared in the journal Stochastic Analysis and Applications.

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and L^H be a local self-similar fractional Lévy process. For $p, q > 0$ let $c_{p,q} := \zeta\left(\frac{1}{p} + \frac{1}{q}\right)$ with ζ the Riemann-Zeta function. The article [**You36**] provides the existence of the Riemann-Stieltjes integral of f with respect to g if the functions f and g have finite p-, respectively q-variation and $\frac{1}{p} + \frac{1}{q} > 1$. This is because of the Young inequality

$$\left|\int_a^b f\,dg - f(a)\,(g(b)-g(a))\right| \le c_{p,q} var_p\,(f;[a,b])\,var_q\,(g;[a,b])\,, \tag{2.1}$$

where $var_p\,(f;[a,b])$ is the p-variation of a function f on an interval $[a,b]$, defined as

$$var_p\,(f;[a,b]) := \sup_\pi \left(\sum_{j=1}^n \left|f(t_j)-f(t_{j-1})\right|^p\right)^{\frac{1}{p}},$$

where the supremum is taken over all partitions $\pi = \{a \le t_0 < \cdots < t_n \le b\}$ of the interval $[a,b]$. If a function has β-Hölder-continuous paths it has finite $\frac{1}{\beta}$-variation. In Proposition 1.20 we showed that under the conditions $\alpha > 1$ and $\gamma > 0$ local self-similar fractional Lévy processes are Hölder-continuous of order $\gamma - \varepsilon$ for any $0 < \varepsilon < \gamma$. Then the integral of u with respect to L^H exists if the process u has finite q-variation with $q < \frac{1}{1-\gamma}$. This is where we need to be more restrictive as in the Gaussian model delivered by **[CNW06]** because fractional Brownian motions are Hölder-continuous of order H. Since $H > \gamma$ the process u needs to be more regular in our model.

Let u be a stochastic process with finite q-variation and $q < \frac{1}{1-\gamma}$. We define

$$Z_t := \int_0^t u_s\,dL_s^H$$

and the power variation of the process Z by

$$V_p^n(Z)_t := \sum_{j=1}^{\lfloor nt \rfloor} \left|Z_{\frac{j}{n}} - Z_{\frac{j-1}{n}}\right|^p.$$

For the sake of completeness for $f\colon [a,b] \to \mathbb{R}$ we set

$$\|f\|_{\gamma-\varepsilon;[a,b]} := \sup_{a\le s<t\le b} \frac{|f_t - f_s|}{|t-s|^{\gamma-\varepsilon}}.$$

Before we start considering the consistency theorem we need to prove the following integral representation for the power-function (it is used e.g. in **[BCI04]**).

LEMMA 2.1. *Let $x \in \mathbb{R}$. Then for all $p \in (0,2)$:*

$$|x|^p = \frac{\int_{\mathbb{R}} \left(e^{iyx} - 1 - iyx\mathbb{1}_{|y|\leq 1}(y)\right) |y|^{-(1+p)}\, dy}{\int_{\mathbb{R}} \left(e^{iy} - 1 - iy\mathbb{1}_{|y|\leq 1}(y)\right) |y|^{-(1+p)}\, dy}.$$

PROOF. The result is derived by substituting $z = |x|y$ in the upper integral and observing, that the integral in the numerator does not depend on the sign of x. □

We also need the concept of uniform convergence in probability.

DEFINITION 2.2. A sequence of jointly measurable stochastic processes X^n converges to the limit X uniformly on compacts in probability if for each $t, K > 0$

$$\mathbb{P}\left(\sup_{s\leq t} |X_s^n - X_s| > K\right) \to 0 \quad \text{as } n \to \infty.$$

With these tools we are now able to state the next theorem. This is a generalisation of [**CNW06**, Theorem 1] where the result is shown in the Gaussian model described above.

THEOREM 2.3. *Let $0 < p < \alpha$ and L^H be a local self-similar fractional Lévy process such that the requirements of Proposition 1.20 are satisfied. Suppose that X^H is a linear fractional α-stable motion as in Definition 1.12 and let $u = (u_t)_{t\in[0,T]} \in L^p([0,T])$ be a stochastic process with a.s. finite q-variation, where $q < \frac{1}{1-\gamma}$. Consider the process*

$$Z_t := \int_0^t u_s\, dL_s^H.$$

Then, if n tends to infinity, the following holds:

$$n^{-1+pH} V_p^n(Z)_t \xrightarrow{u.c.p} \mathbb{E}[|X_1^H|^p] \int_0^t |u_s|^p\, ds,$$

where the convergence is the uniform convergence in probability.

Before proving this theorem we consider the case $u \equiv 1$. In contrast to the Gaussian case in [**CNW06**] this is the most complicated step in our proof. That is why we consider this case separated from our consistency theorem stated above. Additionally, in this case there is no restriction on H respectively γ. For the sake of simplicity we define

$$V_t^n := n^{-1+pH} V_p^n(L^H)_t = \frac{1}{n} \sum_{j=1}^{\lfloor nt \rfloor} \left| n^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right) \right|^p .$$

THEOREM 2.4. *For the power variation V_t^n of a local self-similar fractional Lévy process L^H the following convergence is satisfied: for all $0 < p < \alpha$ it holds*

$$V_t^n \xrightarrow{\mathbb{P}} t \cdot \mathbb{E}\left[\left| X_1^H \right|^p \right] \quad \textit{as } n \to \infty,$$

where the process X^H is the linear fractional stable motion.

PROOF. For the proof we will proceed in two steps. At first we show that

$$\mathbb{E}\left[V_t^n\right] \to t\mathbb{E}\left[|X_1^H|^p \right] \quad \text{as } n \to \infty.$$

After this step we prove that V_t^n converges in probability to its expectation as $n \to \infty$.

We first consider the expectation of V_t^n. Let therefore $p < \alpha$ and $q > 1$ such that $pq < \alpha$. Then the random variables $\left| n^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right) \right|^{pq}$, $1 \leq j \leq n$, $n \in \mathbb{N}$ are integrable and since the process L^H has stationary increments we obtain for all $n \in \mathbb{N}$ and $1 \leq j \leq n$

$$\mathbb{E}\left[\left| n^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right) \right|^{pq} \right] = \mathbb{E}\left[\left| n^H L^H_{\frac{1}{n}} \right|^{pq} \right] < \infty.$$

Combining this together with the fact that for any $r > 0$ it holds

$$\mathbb{E}\left[\left| n^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right) \right|^p \mathbb{1}_{\left\{ \left| n^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right) \right|^p > r \right\}} \right] \leq r^{-q+1} \mathbb{E}\left[\left| n^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right) \right|^{pq} \right]$$

the uniform integrability of the sequence $\left(\left| n^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right) \right|^p \right)_{n \in \mathbb{N}}$ is satisfied and by [**Kal10**, Lemma 4.11] the following convergence holds:

$$\mathbb{E}\left[\left| n^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right) \right|^p \right] \xrightarrow{n \to \infty} \mathbb{E}\left[|X_1^H|^p \right].$$

With Proposition 1.18 we can conclude

$$\begin{aligned}\mathbb{E}\left[V_t^n\right] &= \frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor}\mathbb{E}\left[\left|n^H\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}}\right)\right|^p\right] \\ &= \frac{\lfloor nt\rfloor}{n}\mathbb{E}\left[\left|n^H L^H_{\frac{1}{n}}\right|^p\right] \\ &\overset{n\to\infty}{\longrightarrow} t\mathbb{E}\left[|X_1^H|^p\right].\end{aligned}$$

Now, we start with the second step and use Lemma 2.1 to conclude

$$V_t^n = \frac{1}{N}\int\limits_{\mathbb{R}}\left(\frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor} e^{iyn^H\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}}\right)} - \frac{\lfloor nt\rfloor}{n} - iyn^H\frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor}\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}}\right)\mathbb{1}_{|y|\le 1}\right)\frac{dy}{|y|^{p+1}},$$

where N denotes the numerator in the integral representation of Lemma 2.1, that is

$$N := \int\limits_{\mathbb{R}}\left(e^{iy} - 1 - iy\mathbb{1}_{|y|\le 1}(y)\right)|y|^{-(1+p)}\,dy.$$

We consider the two cases $p < 1$ and $p \ge 1$ separated. In the first case the integral $\int\limits_{-1}^{1}\frac{iyx}{|y|^{p+1}}\,dy$ exists for any $x \in \mathbb{R}$ and its value is zero because of the symmetry of the integrand. Hence,

$$V_t^n = \frac{1}{N}\int\limits_{\mathbb{R}}\left(\frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor} e^{iyn^H\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}}\right)} - \frac{\lfloor nt\rfloor}{n}\right)\frac{dy}{|y|^{p+1}}$$

and

$$\mathbb{E}\left[V_t^n\right] = \mathbb{E}\left[\frac{1}{N}\int\limits_{\mathbb{R}}\left(\frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor} e^{iyn^H\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}}\right)} - \frac{\lfloor nt\rfloor}{n}\right)\frac{dy}{|y|^{p+1}}\right].$$

By Fubini's theorem it holds

$$V_t^n - \mathbb{E}[V_t^n] = \frac{1}{N}\int\limits_{\mathbb{R}}\frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor}\left(e^{iyn^H\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}}\right)} - \mathbb{E}\left[e^{iyn^H\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}}\right)}\right]\right)\frac{dy}{|y|^{p+1}}. \tag{2.2}$$

In the case $p \ge 1$ the process L^H has finite first moment and we can immediately use Fubini's theorem to calculate

$$V_t^n - \mathbb{E}[V_t^n]$$

$$= \frac{1}{N} \int_{\mathbb{R}} \left(\frac{1}{n} \sum_{j=1}^{\lfloor nt \rfloor} \left(e^{iyn^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right)} - \mathbb{E} \left[e^{iyn^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right)} \right] \right) \right.$$

$$\left. - iyn^H \frac{1}{n} \sum_{j=1}^{\lfloor nt \rfloor} \left(\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right) - \mathbb{E} \left[L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right] \right) \mathbb{1}_{|y| \le 1} \right) \frac{dy}{|y|^{p+1}}.$$

The last term is a telescopic sum and since $L_0 = 0\, a.s.$ and $H < 1$ we conclude

$$iyn^H \frac{1}{n} \sum_{j=1}^{\lfloor nt \rfloor} \left(\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right) - \mathbb{E} \left[L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right] \right) = iy \frac{n^H}{n} \left(L^H_{\frac{\lfloor nt \rfloor}{n}} - \mathbb{E} \left[L^H_{\frac{\lfloor nt \rfloor}{n}} \right] \right) \to 0$$

$\mathbb{P}$-almost surely as $n \to \infty$. So in the case $p \ge 1$ the difference $V^n_t - \mathbb{E}[V^n_t]$ becomes the same as in the case $p < 1$ and is given by Equation (2.2).

Unfortunately, the following convergence does not hold as $n \to \infty$

$$\frac{1}{n} \sum_{j=1}^{\lfloor nt \rfloor} e^{iyn^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right)} - \mathbb{E} \left[e^{iyn^H \left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}} \right)} \right] \to 0 \quad \mathbb{P} - a.s.. \tag{2.3}$$

If this statement were true, we would be able to use Lebesgue's theorem to show the almost sure convergence of V^n_t to its expectation. But if we can show convergence in probability in Equation (2.3) we can conclude $V^n_t \xrightarrow{\mathbb{P}} \mathbb{E}[V^n_t]$ as $n \to \infty$. This holds by the fact that convergence in probability of a sequence $(\xi_n)_{n \in \mathbb{N}}$ of random variables to a random variable ξ is equivalent to the following condition: for all subsequences $(\xi_{n_k})_{k \in \mathbb{N}}$ of $(\xi_n)_{n \in \mathbb{N}}$ there exists a subsubsequence $(\xi_{n_{k_l}})_{l \in \mathbb{N}}$ of $(\xi_n)_{n \in \mathbb{N}}$ such that $\xi_{n_{k_l}} \to \xi$ $\mathbb{P}$-a.s. for $l \to \infty$. If we take an arbitrary subsequence $\left(V^{n_k}_t - \mathbb{E}\left[V^{n_k}_t \right] \right)_{k \in \mathbb{N}}$ of $\left(V^n_t - \mathbb{E}\left[V^n_t \right] \right)_{n \in \mathbb{N}}$ we can take an almost sure convergent subsubsequence of the resulting term of the left side of Equation (2.3), namely

$$\left(\frac{1}{n_{k_l}} \sum_{j=1}^{\lfloor n_{k_l} t \rfloor} e^{iyn^H_{k_l} \left(L^H_{\frac{j}{n_{k_l}}} - L^H_{\frac{j-1}{n_{k_l}}} \right)} - \mathbb{E} \left[e^{iyn^H_{k_l} \left(L^H_{\frac{j}{n_{k_l}}} - L^H_{\frac{j-1}{n_{k_l}}} \right)} \right] \right)_{l \in \mathbb{N}},$$

and apply Lebesgue's theorem for this subsubsequence. Then the convergence in probability of V^n_t to its expectation as $n \to \infty$ is satisfied.

Hence, it remains to show convergence in probability in Equation (2.3). To show this we prove L^2-convergence. We use the stationarity of increments of fractional Lévy processes and conclude

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{n}\sum_{j=1}^{\lfloor nt\rfloor}\left(e^{iyn^H\left(L^H_{\frac{j}{n}}-L^H_{\frac{j-1}{n}}\right)}-\mathbb{E}\left[e^{iyn^H\left(L^H_{\frac{j}{n}}-L^H_{\frac{j-1}{n}}\right)}\right]\right)\right|^2\right]\\
&=\frac{1}{n^2}\sum_{j=1}^{\lfloor nt\rfloor}\sum_{k=1}^{\lfloor nt\rfloor} Cov\left(e^{iyn^H\left(L^H_{\frac{j}{n}}-L^H_{\frac{j-1}{n}}\right)},e^{iyn^H\left(L^H_{\frac{k}{n}}-L^H_{\frac{k-1}{n}}\right)}\right)\\
&\leq\frac{2}{n^2}\sum_{j=1}^{\lfloor nt\rfloor}\sum_{k=1}^{j} Cov\left(e^{iyn^H\left(L^H_{\frac{j-k+1}{n}}-L^H_{\frac{j-k}{n}}\right)},e^{iyn^H L^H_{\frac{1}{n}}}\right).
\end{aligned}$$

We define $a := j-k$. If we show that

$$Cov\left(e^{iyn^H\left(L^H_{\frac{a+1}{n}}-L^H_{\frac{a}{n}}\right)},e^{iyn^H L^H_{\frac{1}{n}}}\right)=\mathbf{O}(a^{-\delta})$$

for some $\delta > 0$ the L^2-convergence in Equation (2.3) is satisfied. To this end we use the characteristic functions of the process L^H (cf. Proposition 1.17) and similar substitutions as in the proof of Proposition 1.18 to calculate

$$\begin{aligned}
&Cov\left(e^{iyn^H\left(L^H_{\frac{a+1}{n}}-L^H_{\frac{a}{n}}\right)},e^{iyn^H L^H_{\frac{1}{n}}}\right)\\
&=\mathbb{E}\left[e^{iyn^H\left(L^H_{\frac{a+1}{n}}-L^H_{\frac{a}{n}}+L^H_{\frac{1}{n}}\right)}\right]-\mathbb{E}\left[e^{iyn^H\left(L^H_{\frac{a+1}{n}}-L^H_{\frac{a}{n}}\right)}\right]\mathbb{E}\left[e^{iyn^H L^H_{\frac{1}{n}}}\right]\\
&=\exp\left\{\int_{\mathbb{R}^2}\left(e^{iyxn^H\left(\left(\frac{a+1}{n}-s\right)_+^\gamma-\left(\frac{a}{n}-s\right)_+^\gamma+\left(\frac{1}{n}-s\right)_+^\gamma-(-s)_+^\gamma\right)}-1\right.\right.\\
&\qquad\left.\left.-\,iyxn^H\left(\left(\frac{a+1}{n}-s\right)_+^\gamma-\left(\frac{a}{n}-s\right)_+^\gamma+\left(\frac{1}{n}-s\right)_+^\gamma-(-s)_+^\gamma\right)\mathbb{1}_{|x|\leq 1}\right)d\nu(x)\,ds\right\}
\end{aligned}$$

$$- \exp\left\{ \int\limits_{\mathbb{R}^2} \left(e^{iyxn^H\left(\left(\frac{a+1}{n}-s\right)_+^\gamma - \left(\frac{a}{n}-s\right)_+^\gamma\right)} + e^{\left(\left(\frac{1}{n}-s\right)_+^\gamma - (-s)_+^\gamma\right)} - 2 \right.\right.$$

$$\left.\left. - \; iyxn^H \left(\left(\tfrac{a+1}{n} - s\right)_+^\gamma - \left(\tfrac{a}{n} - s\right)_+^\gamma + \left(\tfrac{1}{n} - s\right)_+^\gamma - (-s)_+^\gamma \right) \mathbb{1}_{|x|\le 1} \right) d\nu(x)\, ds \right\}$$

$$= \exp\left\{ \int\limits_{\mathbb{R}^2} \left(e^{iyx\left((a+1-s)_+^\gamma - (a-s)_+^\gamma + (1-s)_+^\gamma - (-s)_+^\gamma\right)} - 1 \right.\right.$$

$$\left.\left. - \; iyx \left((a+1-s)_+^\gamma - (a-s)_+^\gamma + (1-s)_+^\gamma - (-s)_+^\gamma \right) \mathbb{1}_{|x|\le 1} \right) \tfrac{1}{n}\, d\nu(xn^{-1/\alpha})\, ds \right\}$$

$$- \exp\left\{ \int\limits_{\mathbb{R}^2} \left(e^{iyx\left((a+1-s)_+^\gamma - (a-s)_+^\gamma\right)} + e^{\left((1-s)_+^\gamma - (-s)_+^\gamma\right)} - 2 \right.\right.$$

$$\left.\left. - \; iyx \left((a+1-s)_+^\gamma - (a-s)_+^\gamma + (1-s)_+^\gamma - (-s)_+^\gamma \right) \mathbb{1}_{|x|\le 1} \right) \frac{1}{n}\, d\nu(xn^{-1/\alpha})\, ds \right\},$$

where ν is the Lévy measure of the driving Lévy process L of L^H. The measure ν is absolutely continuous with respect to Lebesgue-measure and its density g satisfies the properties given in Definition 1.15.

For the sake of simplicity we also define $z_a(s) := (a+1-s)_+^\gamma - (a-s)_+^\gamma$. To go on with the proof we use the continuity of the exponential-function so we can consider the exponents of the last expression. Also we use the second property of the density g, this is $g(x) \le C \frac{1}{|x|^{1+\alpha}}$.

$$\left| \int\limits_{\mathbb{R}^2} \left(e^{iyx(z_a(s)+z_0(s))} - 1 - iyx\,(z_a(s)+z_0(s))\, \mathbb{1}_{|x|\le 1} \right) \frac{1}{n}\, d\nu(xn^{-1/\alpha})\, ds \right.$$

$$\left. - \int\limits_{\mathbb{R}^2} \left(e^{iyxz_a(s)} + e^{iyxz_0(s)} - 2 - iyx\,(z_a(s)+z_0(s))\, \mathbb{1}_{|x|\le 1} \right) \frac{1}{n}\, d\nu(xn^{-1/\alpha})\, ds \right|$$

$$\le \int\limits_{\mathbb{R}^2} \left| e^{iyx(z_a(s)+z_0(s))} - e^{iyxz_a(s)} - e^{iyxz_0(s)} + 1 \right| \frac{1}{n}\, d\nu(xn^{-1/\alpha})\, ds$$

$$\le C \int\limits_{\mathbb{R}^2} \left| e^{iyx(z_a(s)+z_0(s))} - e^{iyxz_a(s)} - e^{iyxz_0(s)} + 1 \right| \frac{1}{|x|^{1+\alpha}}\, dx\, ds$$

$$= C \int\limits_{\mathbb{R}^2} \left| \cos\left(yx\,(z_a(s)+z_0(s))\right) - \cos\left(yxz_a(s)\right) - \cos\left(yxz_0(s)\right) + 1 \right.$$

$$\left. + \; i\left[\sin\left(yx\,(z_a(s)+z_0(s))\right) - \sin\left(yxz_a(s)\right) - \sin\left(yxz_0(s)\right)\right] \right| \frac{1}{|x|^{1+\alpha}}\, dx\, ds$$

We can clearly see that the integrands are the same if $s > 1$, so the expression is zero for $s > 1$. Using the standard addition theorems for sine and cosine functions we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\cos yx\,(z_a(s)+z_0(s)) - \cos yxz_a(s) - \cos yxz_0(s) + 1 \\
&\qquad + i\,[\sin yx\,(z_a(s)+z_0(s)) - \sin yxz_a(s) - \sin yxz_0(s)]|\,\frac{1}{|x|^{1+\alpha}}\,dx\,ds \\
&= \int_{-\infty}^{1}\int_{\mathbb{R}} |1 - \cos yxz_a(s) + \cos yxz_0(s)(\cos yxz_a(s) - 1) - \sin yxz_a(s)\sin yxz_0(s) \\
&\qquad + i\,[\sin yxz_a(s)(\cos yxz_0(s) - 1) - \sin yxz_0(s)(\cos yxz_a(s) - 1)]|\,\frac{1}{|x|^{1+\alpha}}\,dx\,ds \\
&\le \int_{-\infty}^{1}\int_{\mathbb{R}} (3\,|\cos yxz_a(s) - 1| + |\sin yxz_a(s)|\,|\sin yxz_0(s)| \\
&\qquad + |\sin yxz_a(s)|\,|1 - \cos yxz_0(s)|)\,\frac{1}{|x|^{1+\alpha}}\,dx\,ds.
\end{aligned}$$

Now, we use the standard estimations for sine and cosine functions. These are

$$|\sin x| \le |x| \wedge 1 \quad \text{and} \quad |\cos x - 1| \le \tfrac{|x|^2}{2} \wedge 2$$

for all $x \in \mathbb{R}$. Then we decompose the integrals as follows:

For the first summand we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} |\cos yxz_a(s) - 1|\,\frac{dx}{|x|^{1+\alpha}} \\
&\le \int_{\mathbb{R}} \left(\frac{|yxz_a(s)|^2}{2} \wedge 2\right)\frac{dx}{|x|^{1+\alpha}} \\
&= 2\frac{|yz_a(s)|^2}{2}\int_0^{\frac{2}{|yz_a(s)|}} x^{1-\alpha}\,dx + 2\int_{\frac{2}{|yz_a(s)|}}^{\infty} 2x^{-1-\alpha}\,dx \\
&= \frac{|yz_a(s)|^2}{2-\alpha}\left(\frac{2}{|yz_a(s)|}\right)^{2-\alpha} + \frac{4}{\alpha}\left(\frac{2}{|yz_a(s)|}\right)^{-\alpha} \\
&= const|yz_a(s)|^{\alpha}
\end{aligned}$$

For the second term we use similar techniques and conclude

$$\begin{aligned}
&\int_{\mathbb{R}} |\sin yxz_a(s)|\,|\sin yxz_0(s)|\,\frac{dx}{|x|^{1+\alpha}}\\
&\leq \int_{\mathbb{R}} \left(|yxz_a(s)|\wedge 1\right)\left(|yxz_0(s)|\wedge 1\right)\frac{dx}{|x|^{1+\alpha}}\\
&= 2\int_0^{|yz_0(s)|^{-1}} |yxz_a(s)|\,|yxz_0(s)|\,\frac{dx}{|x|^{1+\alpha}} + 2\int_{|yz_0(s)|^{-1}}^{|yz_a(s)|^{-1}} |yxz_a(s)|\,\frac{dx}{|x|^{1+\alpha}} + 2\int_{|yz_a(s)|^{-1}}^{\infty} \frac{dx}{|x|^{1+\alpha}}\\
&= \frac{2|y|^2|z_a(s)z_0(s)|}{2-\alpha}|yz_0(s)|^{-2+\alpha} + \frac{2|yz_a(s)|}{1-\alpha}\left(|yz_a(s)|^{\alpha-1} - |yz_0(s)|^{\alpha-1}\right) + \frac{2}{\alpha}|yz_a(s)|^{\alpha}\\
&= \left(\frac{2}{2-\alpha} - \frac{2}{1-\alpha}\right)|z_a(s)||y|^{\alpha}|z_0(s)|^{\alpha-1} + \left(\frac{2}{1-\alpha} + \frac{2}{\alpha}\right)|yz_a(s)|^{\alpha}.
\end{aligned}$$

The last term is easy to handle, because

$$|1-\cos yxz_0(s)| \leq \frac{|yxz_0(s)|^2}{2}\wedge 2 = 2\left(\left(\frac{|yxz_0(s)|}{2}\right)^2\wedge 1\right) \leq 2\left(|yxz_0(s)|\wedge 1\right),$$

which is smaller than a constant times the estimation of the second integrand.

It remains to show that both $\int_{-\infty}^{1} |yz_a(s)|^{\alpha}\,ds$ and $\int_{-\infty}^{1} \frac{|z_a(s)|}{|z_0(s)|}|yz_0(s)|^{\alpha}\,ds$ are of order $\mathbf{O}(a^{-\delta})$ for $a \to \infty$. Starting with the second integral the first observation is that $z_a(s)$ is monotone increasing in s on the interval $(-\infty, a)$ and because of the behaviour on the interval $[a, a+1]$, it is $\frac{|z_a(s)|}{|z_0(s)|} \leq 1$ for all $s \in (-\infty, 1]$, $\gamma \in \left(-\frac{1}{\alpha}, 1-\frac{1}{\alpha}\right)$ and $a \geq 2$, and since Lemma 1.11 holds for those γ both integrals are finite. Let $\delta > 0$ such that $\gamma + 2\delta < 1 - \frac{1}{\alpha}$. If we replace γ by $\gamma + 2\delta$, the integrals are still finite and we calculate for the first term and for all $s \leq 1$

$$\begin{aligned}
z_a(s) &= (a+1-s)_+^{-2\delta}\,(a+1-s)_+^{\gamma+2\delta} - (a-s)_+^{-2\delta}\,(a-s)_+^{\gamma+2\delta}\\
&\leq (a-s)^{-2\delta}\left((a+1-s)^{\gamma+2\delta} - (a-s)^{\gamma+2\delta}\right). \qquad (2.4)
\end{aligned}$$

On the other hand

$$z_0(s) \geq (1-s)^{-2\delta}\left((1-s)^{\gamma+2\delta} - (-s)_+^{\gamma+2\delta}\right).$$

For $s \in (-\infty, 1]$ we use both of these inequalities and conclude

$$\frac{|z_a(s)|}{|z_0(s)|} = \frac{(a+1-s)_+^{-2\delta}(a+1-s)_+^{\gamma+2\delta} - (a-s)_+^{-2\delta}(a-s)_+^{\gamma+2\delta}}{(1-s)_+^{-2\delta}(1-s)_+^{\gamma+2\delta} - (-s)_+^{-2\delta}(-s)_+^{\gamma+2\delta}}$$

$$\leq \frac{(a-s)^{-2\delta}}{(1-s)^{-2\delta}} \cdot \frac{(a+1-s)^{\gamma+2\delta} - (a-s)^{\gamma+2\delta}}{(1-s)^{\gamma+2\delta} - (-s)_+^{\gamma+2\delta}}$$

$$\leq \left(1 + \frac{a-1}{1-s}\right)^{-2\delta}. \tag{2.5}$$

Unfortunately, it is not $\frac{|z_a(s)|}{|z_0(s)|} = \mathbf{O}(a^{-\delta})$, so we need to split the integral

$$\int_{-\infty}^{1} \frac{|z_a(s)|}{|z_0(s)|} |yz_0(s)|^\alpha \, ds$$

into two parts, the part on the interval $(-\infty, -R)$ and on the interval $[-R, 1]$ for some $R > 0$ determined later. On the second interval we use Estimation (2.5). On the interval $(-\infty, -R]$ we use $\frac{|z_a(s)|}{|z_0(s)|} \leq 1$ and the mean value theorem: For all $s \in (-\infty, 0)$ there exists $\xi \in [0, 1]$, such that $(1-s)_+^\gamma - (-s)_+^\gamma = \gamma(\xi - s)^{\gamma-1}$. Since $\gamma - 1 < 0$ we have the estimate

$$(1-s)_+^\gamma - (-s)_+^\gamma \leq \gamma(1-s)^{\gamma-1}$$

for all $s \in (-\infty, 0)$. By combining both it holds

$$\int_{-\infty}^{1} |z_0(s)|^\alpha \left|\frac{z_a(s)}{z_0(s)}\right| ds = \int_{-\infty}^{-R} |z_0(s)|^\alpha \left|\frac{z_a(s)}{z_0(s)}\right| ds + \int_{-R}^{1} |z_0(s)|^\alpha \left|\frac{z_a(s)}{z_0(s)}\right| ds$$

$$\leq \left|\frac{\gamma}{\gamma\alpha - \alpha + 1}\right| (1+R)^{\gamma\alpha-\alpha+1} + \left(1 + \frac{a-1}{1+R}\right)^{-2\delta} \int_{-R}^{1} |z_0(s)|^\alpha \, ds.$$

We observe that $\gamma\alpha - \alpha + 1 < -2\delta$. Hence by the choice of $R > 0$ such that

$$1 + R = (a-1)^{\frac{1}{2}}$$

the following estimation holds for any $c_1,\ c_2 > 0$:

$$c_1(1+R)^{\gamma\alpha-\alpha+1} + c_2\left(1 + \frac{a-1}{1+R}\right)^{-2\delta} \leq c_1(a-1)^{-\delta} + c_2\left(1 + (a-1)^{\frac{1}{2}}\right)^{-2\delta} = \mathbf{O}(a^{-\delta}).$$

The first integral is easier to handle. By Equation (2.4) we have

$$\int_{-\infty}^{1} |yz_a(s)|^\alpha \, ds \le |y|^\alpha (a-1)^{-2\delta} \int_{-\infty}^{1} (a+1-s)^{\gamma+2\delta} - (a-s)^{\gamma+2\delta} \, ds = \mathbf{O}(a^{-2\delta}) = \mathbf{O}(a^{-\delta}).$$

Hence we can conclude

$$Cov\left(e^{iy\left(X^H_{a+1}-X^H_a\right)}, e^{iy\left(X^H_1-X^H_0\right)}\right) = \mathbf{O}(a^{-\delta}) = \mathbf{O}((j-k)^{-\delta}) \quad \text{for } a = j-k \to \infty \tag{2.6}$$

which proves the L^2-convergence of

$$\frac{1}{n}\sum_{j=1}^{\lfloor nt \rfloor} e^{iyn^H\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}}\right)} - \mathbb{E}\left[e^{iyn^H\left(L^H_{\frac{j}{n}} - L^H_{\frac{j-1}{n}}\right)}\right] \to 0$$

as $n \to \infty$, which implies convergence in probability in (2.3) and the proof is completed.

□

Now, we can start with the proof of Theorem 2.3.

Proof of Theorem 2.3. For the proof of our consistency theorem we can proceed analogously to the proof of [**CNW06**, Theorem 1].

Fix $T \in \mathbb{R}$ and let $t \in [0, T]$. We define $c_p := \mathbb{E}[|X^H_1|^p]$ and start with the case $p \le 1$. The following decomposition is known as Bernstein's blocking technique and is also used in [**BNCP09**]. This is the following: for all $m \ge n$, it holds

$$\begin{aligned}
&m^{-1+pH} V^m_p(Z)_t - c_p \int_0^t |u_s|^p \, ds \\
=&m^{-1+pH} \sum_{j=1}^{\lfloor mt \rfloor} \left(\left| \int_{\frac{j-1}{m}}^{\frac{j}{m}} u_s \, dL^H_s \right|^p - \left| u_{\frac{j-1}{m}} \left(L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right) \right|^p \right) \\
&+ m^{-1+pH} \left(\sum_{j=1}^{\lfloor mt \rfloor} \left| u_{\frac{j-1}{m}} \left(L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right) \right|^p - \sum_{k=1}^{\lfloor nt \rfloor} \left| u_{\frac{k-1}{n}} \right|^p \sum_{j \in I_n(k)} \left| L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right|^p \right) \\
&+ m^{-1+pH} \sum_{k=1}^{\lfloor nt \rfloor} \left| u_{\frac{k-1}{n}} \right|^p \sum_{j \in I_n(k)} \left| L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right|^p - c_p n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \left| u_{\frac{k-1}{n}} \right|^p
\end{aligned}$$

$$
\begin{aligned}
&+ c_p \left(n^{-1} \sum_{k=1}^{\lfloor nt \rfloor} \left| u_{\frac{k-1}{n}} \right|^p - \int_0^t |u_s|^p \, ds \right) \\
=:& A_t^{(m)} + B_t^{(n,m)} + C_t^{(n,m)} + D_t^{(n)},
\end{aligned}
$$

where

$$
I_n(k) = \left\{ j \in \mathbb{N} \,\middle|\, \frac{j}{m} \in \left(\frac{k-1}{n}, \frac{k}{n} \right] \right\}, \quad 1 \le k \le \lfloor nt \rfloor .
$$

For any fixed $n \in \mathbb{N}$, the summand $C_t^{(n,m)}$ converges to 0 in probability as $m \to \infty$ by observing

$$
\left\| C_t^{(n,m)} \right\|_\infty \le \sum_{k=1}^{\lfloor nt \rfloor} |u_{\frac{k-1}{n}}|^p \left| m^{-1+pH} \sum_{j \in I_n(k)} \left| L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right|^p - c_p n^{-1} \right|
$$

and applying Theorem 2.4. For the term $B_t^{(n,m)}$ we get

$$
\begin{aligned}
\|B^{(n,m)}\|_\infty \le & m^{-1+pH} \sum_{k=1}^{\lfloor nt \rfloor} \sum_{j \in I_n(k)} \left| \left| u_{\frac{k-1}{n}} \right|^p - \left| u_{\frac{j-1}{m}} \right|^p \right| \left| L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right|^p \\
& + \| |u|^p \|_\infty \sup_{0 \le t \le T} m^{-1+pH} \sum_{mn^{-1} \lfloor nt \rfloor \le j \le mn^{-1}(\lfloor nt \rfloor + 1)} \left| L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right|^p \\
\le & m^{-1+pH} \sum_{k=1}^{\lfloor nt \rfloor} \sup_{s \in \mathcal{I}_n(k) \cup \mathcal{I}_n(k-1)} \left| \left| u_{\frac{k-1}{n}} \right|^p - |u_s|^p \right| \sum_{j \in I_n(k)} \left| L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right|^p \\
& + \| |u|^p \|_\infty \sup_{0 \le t \le T} m^{-1+pH} \sum_{mn^{-1} \lfloor nt \rfloor \le j \le mn^{-1}(\lfloor nt \rfloor + 1)} \left| L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right|^p ,
\end{aligned}
$$

where we denote

$$
\mathcal{I}_n(k) := \left(\frac{k-1}{n}, \frac{k}{n} \right], \quad 1 \le k \le \lfloor nt \rfloor .
$$

By applying Theorem 2.4 again we can conclude that this expression converges in probability to

$$
E_n := \frac{c_p}{n} \left(\sum_{k=1}^{\lfloor nt \rfloor} \sup_{s \in \mathcal{I}_n(k) \cup \mathcal{I}_n(k-1)} \left| |u_{\frac{k-1}{n}}|^p - |u_s|^p \right| + \| |u|^p \|_\infty \right) \text{ as } m \to \infty.
$$

With exactly the same arguments as in the proof of [**CNW06**, Theorem 1] this term converges to zero almost surely. Likewise, the convergence of $\|D^{(n)}\|_\infty \to 0$ as $n \to \infty$ is already shown in **[CNW06]**. For the last part, namely $A_t^{(m)}$, we can use the Young inequality and obtain for any $p \leq 1$

$$
\begin{aligned}
|A_t^{(m)}| &\leq m^{-1+pH} \left| \sum_{j=1}^{\lfloor mt \rfloor} \left(\left| \int_{\frac{j-1}{m}}^{\frac{j}{m}} u_s \, dL_s^H \right|^p - \left| u_{\frac{j-1}{m}} \left(L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right) \right|^p \right) \right| \\
&\leq m^{-1+pH} \sum_{j=1}^{\lfloor mt \rfloor} \left| \int_{\frac{j-1}{m}}^{\frac{j}{m}} u_s \, dL_s^H - u_{\frac{j-1}{m}} \left(L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}} \right) \right|^p \\
&\leq c_{p^*,q} m^{-1+pH} \sum_{j=1}^{\lfloor mt \rfloor} \left(var_q(u; \mathcal{I}_m(j)) var_{\frac{1}{\gamma-\varepsilon}} \left(L^H; \mathcal{I}_m(j) \right) \right)^p \\
&=: c_{p^*,q} F_m,
\end{aligned}
$$

where $p^* := \gamma - \varepsilon$ for $0 < \varepsilon < \gamma$. For $\delta > 0$ we now consider the decomposition

$$
\begin{aligned}
F_m \leq & m^{-1+pH} \sum_{\left\{ j \mid \ var_q(u; \mathcal{I}_m(j)) > \delta \right\}} \left(var_q(u; \mathcal{I}_m(j)) var_{\frac{1}{\gamma-\varepsilon}} \left(L^H; \mathcal{I}_m(j) \right) \right)^p \\
& + \delta^p m^{-1+pH} \sum_{j=1}^{\lfloor mt \rfloor} \left(var_{\frac{1}{\gamma-\varepsilon}} \left(L^H; \mathcal{I}_m(j) \right) \right)^p .
\end{aligned}
$$

Since

$$
\sum_{j=1}^{\lfloor mt \rfloor} var_q(u; \mathcal{I}_m(j)) \leq var_q(u; [0,T]) < \infty
$$

we can conclude that the number of indices j for which $var_q(u; \mathcal{I}_m(j)) > \delta$ holds is bounded by $\left\lfloor \frac{var_q(u;[0,T])}{\delta} \right\rfloor + 1 =: M$ and hence

$$
\begin{aligned}
F_m \leq & M m^{-1+pH} \max_{1 \leq j \leq \lfloor mT \rfloor} \left(var_q(u; \mathcal{I}_m(j)) var_{\frac{1}{\gamma-\varepsilon}} \left(L^H; \mathcal{I}_m(j) \right) \right)^p \\
& + \delta^p m^{-1+pH} \sum_{j=1}^{\lfloor mt \rfloor} \left(var_{\frac{1}{\gamma-\varepsilon}} \left(L^H; \mathcal{I}_m(j) \right) \right)^p .
\end{aligned}
$$

For the first term we use the Hölder-continuity of the paths of the process L^H to show that for all $\varepsilon > 0$ such that $-1 + p\varepsilon + \frac{p}{\alpha} < 0$ it holds almost surely

$$
\begin{aligned}
m^{-1+pH} \left(var_{\frac{1}{\gamma-\varepsilon}} \left(L^H; \mathcal{I}_m(j) \right) \right)^p &\leq m^{-1+pH} \| L^H \|^p_{\gamma-\varepsilon} m^{p(\varepsilon-\gamma)} \\
&= m^{-1+p\varepsilon+\frac{p}{\alpha}} \| L^H \|^p_{\gamma-\varepsilon} \overset{m\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

The selection of ε with the condition above is possible because $p < \alpha$. For the second summand it remains to show that

$$\lim_{m\to\infty} m^{-1+pH} \sum_{j=1}^{\lfloor mt\rfloor} \left(var_{\frac{1}{\gamma-\varepsilon}} \left(L^H; \mathcal{I}_m(j) \right) \right)^p < \infty \quad \text{in } L^1. \tag{2.7}$$

Then we can take the limit for $\delta \to 0$ which finishes the proof. To this end we observe

$$\sum_{j=1}^{\lfloor mt\rfloor} \left(var_{\frac{1}{\gamma-\varepsilon}} \left(L^H; \mathcal{I}_m(j) \right) \right)^p \leq \lfloor mt\rfloor \max_{j=1,\ldots,\lfloor mt\rfloor} \left(var_{\frac{1}{\gamma-\varepsilon}} (L^H; \mathcal{I}_m(j)) \right)^p$$

and conclude

$$\begin{aligned}
&\mathbb{E}\left[\lim_{m\to\infty} m^{-1+pH} \sum_{j=1}^{\lfloor mt\rfloor} \left(var_{\frac{1}{\gamma-\varepsilon}} \left(L^H; \mathcal{I}_m(j) \right) \right)^p \right] \\
&\leq \mathbb{E}\left[\lim_{m\to\infty} m^{-1+pH} \lfloor mt\rfloor \max_{j=1,\ldots,\lfloor mt\rfloor} \left(var_{\frac{1}{\gamma-\varepsilon}} (L^H; \mathcal{I}_m(j)) \right)^p \right] \\
&= \mathbb{E}\left[\lim_{m\to\infty} m^{-1+pH} \lfloor mt\rfloor \left(var_{\frac{1}{\gamma-\varepsilon}} (L^H; \mathcal{I}_m(1)) \right)^p \right] \\
&\leq T\mathbb{E}\left[\lim_{m\to\infty} m^{p(H-\gamma+\varepsilon)} \| L^H \|^p_{\gamma-\varepsilon;\left[0,\frac{1}{m}\right]} \right] \\
&= T\mathbb{E}\left[\| X^H \|^p_{\gamma-\varepsilon;[0,1]} \right],
\end{aligned}$$

where the last equation is an application of Proposition 1.18 and X^H is the linear fractional stable motion, introduced in Definition 1.12.

Before finishing the proof we need to gather some information about the linear fractional stable motion. To this end we recall the definition

$$\| X^H \|_{\gamma-\varepsilon;[a,b]} = \sup_{a\leq s<t\leq b} \frac{|X^H_t - X^H_s|}{|t-s|^{\gamma-\varepsilon}}. \tag{2.8}$$

This is finite because of the Hölder-continuity of linear fractional stable motions. By the definition of the supremum there are sequences $(s_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$, such that $s_n,\ t_n \in [0,1]$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \frac{|X^H_{t_n} - X^H_{s_n}|}{|t_n - s_n|^{\gamma-\varepsilon}} = \| X^H \|_{\gamma-\varepsilon;[0,1]}. \tag{2.9}$$

From the self-similarity of the process X^H we can conclude that

$$\frac{|X^H_t - X^H_s|}{|t-s|^{\gamma-\varepsilon}} \overset{\mathcal{D}}{=} |t-s|^{\frac{1}{\alpha}+\varepsilon} |X^H_1|. \tag{2.10}$$

Now, we can finish the proof of (2.7) as follows:

$$\begin{aligned}
&\mathbb{E}\left[\|X^H\|^p_{\gamma-\varepsilon;[0,1]}\right] \\
&\overset{(2.9)}{=} \mathbb{E}\left[\left(\lim_{n\to\infty}\frac{|X^H_{t_n}-X^H_{s_n}|}{|t_n-s_n|^{\gamma-\varepsilon}}\right)^p\right] \\
&\overset{(*)}{\leq} \lim_{n\to\infty}\mathbb{E}\left[\left(\frac{|X^H_{t_n}-X^H_{s_n}|}{|t_n-s_n|^{\gamma-\varepsilon}}\right)^p\right] \\
&\overset{(2.10)}{=} \lim_{n\to\infty}|t_n-s_n|^{p(\frac{1}{\alpha}+\varepsilon)}\mathbb{E}\left[|X^H_1|^p\right] \\
&\leq \mathbb{E}\left[|X^H_1|^p\right] < \infty,
\end{aligned}$$

where Equation $(*)$ follows by the Fatou's lemma. Finally $\|F_m\|_\infty \to 0$ holds by taking the limit for $\delta \to 0$ as mentioned above.

To complete the proof we consider the case $p > 1$. Analogous to **[CNW06]** we use Minkowski's inequality to obtain

$$\begin{aligned}
&\left|\left(m^{-1+pH}V^m_p(Z)_t\right)^{\frac{1}{p}} - \left(c_p\int_0^t |u_s|^p\,ds\right)^{\frac{1}{p}}\right| \\
\leq& m^{-1+pH}\left(\sum_{j=1}^{\lfloor mt\rfloor}\left|\int_{\frac{j-1}{m}}^{\frac{j}{m}} u_s\,dL^H_s - u_{\frac{j-1}{m}}\left(L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}}\right)\right|^p\right)^{\frac{1}{p}} \\
&+ m^{-1+pH}\left(\sum_{k=1}^{\lfloor nt\rfloor}\sum_{j\in I_n(k)}\left|(u_{\frac{j-1}{m}} - u_{\frac{k-1}{m}})\left(L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}}\right)\right|^p\right)^{\frac{1}{p}} \\
&+ \left|m^{-1+pH}\left(\sum_{k=1}^{\lfloor nt\rfloor}\left|u_{\frac{k-1}{n}}\right|^p\sum_{j\in I_n(k)}\left|L^H_{\frac{j}{m}} - L^H_{\frac{j-1}{m}}\right|^p\right)^{\frac{1}{p}} - \left(c_p n^{-1}\sum_{k=1}^{\lfloor nt\rfloor}\left|u_{\frac{k-1}{n}}\right|^p\right)^{\frac{1}{p}}\right| \\
&+ c_p^{\frac{1}{p}}\left|\left(n^{-1}\sum_{k=1}^{\lfloor nt\rfloor}\left|u_{\frac{k-1}{n}}\right|^p\right)^{\frac{1}{p}} - \left(\int_0^t |u_s|^p\,ds\right)^{\frac{1}{p}}\right|.
\end{aligned}$$

Now, we can proceed similar to the case $p \leq 1$ and the proof is completed. □

Clearly, the law of large numbers provides us with an estimate, but without the corresponding central limit theorem we have no information regarding the quality of this estimate. For this reason we investigate in a distributional limit theorem for the power variation of linear fractional stable motions in the next chapter. Some simulation studies published in [**Gla14**] draw the conclusion that the limit distribution of the power variation of fractional Lévy processes is not Gaussian.

CHAPTER 3

Distributional Theory

As we saw in the last chapter the techniques which were used to prove the consistency theorem in the Gaussian case carry over to the more general setting we consider in the present thesis. Now, we are dealing with distributional limit theorems. In Gaussian models distributional limit theorems for the power variation are proven with the help of Malliavin calculus (c.f. e.g. [**CNW06, BNCP09, MN14**]). The article [**CNW06**] develops limit theorems for the power variation of fractional Brownian motions B^H. The results are the following: The expression

$$c(n) \cdot n^{pH} V_p^n(B^H)_t,$$

$c(n)$ suitable as explained below, appropriately centred converges in law to a random variable X^{lim}, and it holds:

- for $0 < H < \frac{3}{4}$: $c(n) = n^{-\frac{1}{2}}$ and X^{lim} is a Gaussian random variable,
- for $H = \frac{3}{4}$: $c(n) = n^{-\frac{1}{2}} \log(n)^{-\frac{1}{2}}$ and X^{lim} is a Gaussian random variable and
- for $\frac{3}{4} < H < 1$: $c(n) = n^{1-2H}$ and X^{lim} is Rosenblatt distributed.

In the case $0 < H < \frac{3}{4}$ it also establishes a limit theorem for $n^{pH-\frac{1}{2}} V_p^n(Z)_t$ appropriately centred to a Gaussian random variable, where Z is an integrated fractional process.

The article [**GI15**] provides a limit theorem for the power variation of stable Lévy processes. Let L be a stable Lévy process with parameters $(\alpha, \beta, 0, c)$. We define

$$C_n(\alpha, p) := \begin{cases} n^{-\frac{p}{\alpha}} \mathbb{E}\left[|L_1|^p\right] & \frac{\alpha}{2} < p < \alpha, \\ \mathbb{E}\left[\sin\left(n^{-1}|L_1|^\alpha\right)\right] & p = \alpha, \\ 0 & p > \alpha. \end{cases}$$

Then for $p > \frac{\alpha}{2}$

$$V_p^n(L)_t - ntC_n(\alpha, p) \xrightarrow{\mathcal{D}} L'_t \quad \text{as } n \to \infty,$$

where L' is an $\frac{\alpha}{p}$-stable process which is independent of L and whose Lévy measure is concentrated on $(0, \infty)$. In the case $p < \frac{\alpha}{2}$ the result can be deduced from the standard central limit theorem since $|L_1|^p$ has finite second moment. Under this condition it holds:

$$n^{-\frac{1}{2}+\frac{p}{\alpha}} V_p^n(L)_t - t\sqrt{n}\mathbb{E}\left[|L_1|^p\right] \xrightarrow{\mathcal{D}} Var\left(|L_1|^p\right) B_t \quad \text{as } n \to \infty,$$

where B is a Brownian motion and independent of L.

In this chapter we combine the properties of both classes of processes and consider linear fractional stable motions which have α-stable marginal distributions and whose dependence structure is the same as the one of fractional Brownian motions. Our goal is to prove the following limit theorem:

THEOREM 3.1. *Let* $1 < \alpha < 2$, $0 < p < \alpha$ *and* X^H *be a linear fractional* α*-stable motion with* $\gamma \in \left(-\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right)$*. If*

$$H < \begin{cases} \frac{3}{4} & \text{for } \gamma > 0, \\ \frac{1}{2} & \text{for } \gamma < 0, \end{cases}$$

the following limit theorem holds:

$$\sqrt{n}\left(n^{-1+pH} V_p^n(X^H)_1 - \mathbb{E}\left[|X_1^H|^p\right]\right) \xrightarrow{\mathcal{D}} \Xi, \tag{3.1}$$

where Ξ *is a non-trivial random variable whose law is obtained as a mixture of Gaussian distributions.*

To achieve this goal we choose an elegant way to reduce the proof of the above mentioned theorem to a Malliavin based limit theorem (c.f. Theorem 1.27) by using the technique of subordination. To apply Theorem 1.27 we follow the strategy developed in [**MN14**]. This article is the first one which provides a distributional limit theorem for the power variation of Gaussian processes relaxing the assumption of stationary increments to processes with locally stationary increments which is defined later.

We proceed in the following steps: in the first section we construct a specific probability space to identify a representation of a linear fractional stable motion X^H as a conditionally Gaussian process G. The limit theorem [**MN14**, Theorem 1] is provided in Section 2. Unfortunately, this limit theorem cannot be applied to the Gaussian process constructed in the first section but the statement of [**MN14**, Theorem 1] still holds true for our conditionally Gaussian process. This will be shown in the third section of this chapter. In Section 4 we prove our main result by applying the limit theorem for the power variation of the process G we deduced in the section before. We finish the chapter with a short conclusion.

1. Representation of Linear Fractional Stable Motions as Conditionally Gaussian Processes

In the following we give an explicit construction of how a linear fractional stable motion can be represented as some conditionally Gaussian process G. We proceed as follows: since it is well known that a Brownian motion subordinated by an $\frac{\alpha}{2}$-stable subordinator yields a symmetric α-stable Lévy process (c.f. chapter 1.3) we start with two-sided analogues of the mentioned processes and observe that also a two-sided Brownian motion subordinated by a two-sided $\frac{\alpha}{2}$-stable subordinator yields a two-sided symmetric α-stable Lévy process. After that we use this result to see that linear fractional stable motions are conditionally Gaussian processes.

Let $\widetilde{B}$ be a two-sided standard Brownian motion on a filtered probability space $(\Omega_1, \mathcal{A}_1, \mathcal{G}^1, \mathbb{P}_1)$, where the filtration $\mathcal{G}^1 = (\mathcal{G}^1_t)_{t\in\mathbb{R}}$ is generated by $\widetilde{B}$. We also assume that the filtration satisfies the usual hypotheses (i.e. it is complete and right continuous).

Let $1 < \alpha < 2$ and $\widetilde{C} = (\widetilde{C}^{(1)}, \widetilde{C}^{(2)})$ be a two-dimensional $\frac{2}{\alpha}$-stable, spectral negative Lévy process with independent components and start in 0 defined on a probability space $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$. In particular the processes $\widetilde{C}^{(1)}$ and $\widetilde{C}^{(2)}$ have no positive jumps. We define the two-sided process $\widetilde{M}$ by

$$\widetilde{M}_t := \begin{cases} \sup\limits_{0\le s\le t} \widetilde{C}_s^{(1)} & t \ge 0, \\ -\sup\limits_{0\le s\le -t^-} \widetilde{C}_s^{(2)} & t < 0. \end{cases}$$

Let the filtration $\mathcal{G}^2$ on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ be the filtration generated by $\widetilde{M}$. We also assume that it fulfils the usual hypotheses. We define the two-sided process $\widetilde{\theta}$ by

$$\widetilde{\theta}_u := \begin{cases} \inf\left\{t \ge 0 |\, \widetilde{C}_t^{(1)} > u\right\} & u \ge 0, \\ -\inf\left\{t \ge 0 |\, \widetilde{C}_t^{(2)} \ge -u\right\} & u < 0 \end{cases}$$

with the convention that the infimum of the empty set is ∞. Then by Proposition 1.23 the process $\widetilde{\theta}$ is a two-sided $\frac{\alpha}{2}$-stable subordinator.

Let $(\Omega, \mathcal{A}, \mathbb{P}) = (\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ be the product space equipped with the filtration $\mathcal{F}_t := \left(\bigcap\limits_{s>t} \mathcal{G}_s^1 \otimes \mathcal{G}_s^2\right)^{\mathbb{P}}$. Hence, the filtration $\mathcal{F} = (\mathcal{F}_t)_{t\in\mathbb{R}}$ is complete and right-continuous (i.e. it fulfils the usual hypotheses). For $\omega = (\omega_1, \omega_2)$ we define the processes B, θ and M by

$$B_t(\omega) := \widetilde{B}_t(\omega_1), \quad \theta_t(\omega) := \widetilde{\theta}_t(\omega_2) \quad \text{and} \quad M_t(\omega) := \widetilde{M}_t(\omega_2).$$

Then B is a two-sided standard $\mathcal{F}$-Brownian motion and θ is a two-sided $\frac{\alpha}{2}$-stable subordinator independent of B. Applying Proposition 1.23 yields that the process L^α defined by

$$L_t^\alpha = B(\theta_t), \quad t \in \mathbb{R}$$

is a two-sided symmetric α-stable Lévy process.

Note that for θ it also holds

$$\theta_u = \inf\left\{t \in \mathbb{R} | M_t > u\right\},$$

which means that for all $u \in \mathbb{R}$ the random variable θ_u is a stopping time with respect to the filtration $\mathcal{F}$.

We now apply this technique to linear fractional stable motions. Let X^H be the linear fractional stable motion driven by L^α. Hence, it can be represented as

$$X_t^H = \int_{-\infty}^{t} (t-s)_+^\gamma - (-s)_+^\gamma \, dL_s^\alpha = \int_{-\infty}^{t} (t-s)_+^\gamma - (-s)_+^\gamma \, dB(\theta_s).$$

This means that the linear fractional stable motion constructed above is a conditionally Gaussian process with conditional covariance structure given by

$$\mathbb{E}\left[X_t^H X_s^H | \theta\right] = \int_{-\infty}^{s\wedge t} \left((t-r)_+^\gamma - (-r)_+^\gamma\right) \left((s-r)_+^\gamma - (-r)_+^\gamma\right) d\theta_r,$$

where the integral is a Lebesgue-Stieltjes integral which is well defined since θ is almost surely increasing. In the following we consider the process X^H under the measure $\mathbb{P}_1$ so we introduce the process G which for fixed $\omega_2 \in \Omega_2$ is defined by

$$G(t,\omega_2) := \int_{-\infty}^{t} (t-s)_+^\gamma - (-s)_+^\gamma \, d\widetilde{B}(\widetilde{\theta}_s(\omega_2)), \quad t \in \mathbb{R}. \tag{3.2}$$

The process G is a Gaussian process defined on $(\Omega_1, \mathcal{A}_1, \mathbb{P}_1)$, has the covariance structure

$$\mathbb{E}_1\left[G_t G_s\right] = \int_{-\infty}^{s\wedge t} \left((t-r)_+^\gamma - (-r)_+^\gamma\right) \left((s-r)_+^\gamma - (-r)_+^\gamma\right) d\widetilde{\theta}_r$$

and we will - as usual - suppress $\omega_2 \in \Omega_2$.

REMARK 3.2. Since θ_s is a stopping time we are able to use substitution of θ_s by r in the representation of X^H above. Then we also have

$$X_t^H = \int_{-\infty}^{t} (t-s)_+^\gamma - (-s)_+^\gamma \, dB(\theta_s) \overset{(*)}{=} \int_{-\infty}^{\theta_t} (t-M(r))_+^\gamma - (-M(r))_+^\gamma \, dB(r),$$

which means that $X_t^H = \widetilde{G}(\theta_t)$, where

$$\widetilde{G}(u) := \int_{-\infty}^{u} (M(u)-M(r))_+^\gamma - (-M(r))_+^\gamma \, dB(r)$$

is also a conditionally Gaussian process since B and M are independent processes. We state this because it might be helpful for other observations which are not in the scope of this thesis. Equation $(*)$ can be observed by approximating the integrand by step functions.

Observe that it is crucial for the construction made above that the driving Lévy process needs to have a representation as a subordinated Brownian motion in order to draw back the proof of our main result to a Gaussian limit theorem. Additionally, we give an example of a driving Lévy process L where this construction cannot be applied even if this process is closely related to L^α. Also the corresponding fractional Lévy process driven by L has the local self-similarity property. The process L arises from L^α by removing all jumps which are bigger than 1.

EXAMPLE 3.3. *Let $1 < \alpha < 2$ and consider the Lévy process L defined by $L = L^\alpha - X$, where X is a stochastic process with $X_t := \sum_{\substack{0 \le s \le t \\ \Delta L_s^\alpha > 1}} L_s^\alpha$. The fractional Lévy process driven by this process L is obviously a local self-similar process (since the Lévy measure of L is $d\nu(x) = \frac{1}{|x|^{1+\alpha}} \mathbb{1}_{|x| \le 1}\, dx$) but L cannot be obtained as a subordination of a Brownian motion by any subordinator Z. This is given because if the subordinator Z has jumps, the process $B(Z)$ has unbounded jumps. On the other hand if the subordinator is continuous the process $B(Z)$ is continuous as well.*

In the next section we state a limit theorem for the power variation of so-called locally stationary Gaussian processes. From the proof of this theorem the same result for the process G defined in (3.2) can be deduced. We will show this in the third section.

2. Limit Theorem for the Power Variation of Gaussian Processes with Locally Stationary Increments

The content of this section is taken from **[MN14]**.

First we introduce the notation to state the limit theorem [**MN14**, Theorem 1]. Let $G = \{G(t) | t \in [0,1]\}$ be a zero mean Gaussian process defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The covariance function of G is the function $\Gamma_G \colon [0,1]^2 \to \mathbb{R}$ defined by

$$\Gamma_G(s,t) := \mathbb{E}\left[G(s)G(t)\right], \quad s,t \in [0,1]$$

We denote the incremental variance function $\sigma_G^2\colon [0,1]^2 \to \mathbb{R}_+$ by

$$\sigma_G^2(s,t) := \mathbb{E}\left[(G(t)-G(s))^2\right], \quad s,t \in [0,1].$$

Let

$$\pi^n = \{0 \le t_0^n < t_1^n < \cdots < t_n^n \le 1\}$$

be a partition of $[0,1]$. Its mesh size is denoted by

$$\Delta_n := \sup\left\{t_j^n - t_{j-1}^n \mid j = 1,\dots,n\right\}.$$

For a function $F\colon [0,1] \to \mathbb{R}$ we define by

$$\Delta_j^n F := F(t_j^n) - F(t_{j-1}^n)$$

its increment over the interval $[t_{j-1}^n, t_j^n]$. For a two-variable function $F\colon [0,1]^2 \to \mathbb{R}$ its double increment over the rectangle $[t_{j-1}^n, t_j^n] \times [t_{k-1}^n, t_k^n]$ is denoted by

$$\diamond_{j,k}^n F := F(t_j^n, t_k^n) - F(t_j^n, t_{k-1}^n) - F(t_{j-1}^n, t_k^n) + F(t_{j-1}^n, t_{k-1}^n).$$

In order to define Gaussian processes of locally stationary increments we need the following class of functions: let $R[0,1]$ be a set of functions $\rho\colon [0,1] \to \mathbb{R}_+$ such that ρ is continuous at zero, $\rho(0) = 0$ and for each $\delta \in (0,1)$, it holds

$$0 < \inf\{\rho(u) \mid u \in [\delta,1]\} \le \sup\{\rho(u) \mid u \in [\delta,1]\} < \infty.$$

DEFINITION 3.4. Let $G = \{G(t) | t \in [0,1]\}$ be a zero-mean Gaussian stochastic process. We say G has *locally stationary increments* if there is a function $\rho \in R[0,1]$ such that the following holds:

(A1) there is a finite constant $c_1 > 0$ such that for all $s,t \in [0,1]$

$$\sigma_G(s,t) \le c_1 \rho\left(|t-s|\right);$$

(A2) for each $\varepsilon > 0$

$$\lim_{\delta \searrow 0} \sup\left\{\left|\frac{\sigma_G(s,s+h)}{\rho(h)} - 1\right| \;\middle|\; s \in [\varepsilon,1),\ h \in (0, \delta \wedge (1-s)]\right\} = 0.$$

The interpretation of the limit theorem [**MN14**, Theorem 1] is the following: the function ρ approximates the local standard deviation. The process G is compared to a stationary, centred Gaussian process $\widetilde{G}$ whose incremental variance is given by

$$\sigma^2_{\widetilde{G}}(s,t) = \rho(|t-s|)^2.$$

If the process $\widetilde{G}$ fulfils a convergence condition (c.f. Condition (b) of Theorem 3.5 below), it satisfies a limit theorem for the power variation . If additionally the difference of the incremental variance of both processes G and $\widetilde{G}$ converges to zero as it is stated in Condition (c) of Theorem 3.5 below, then G satisfies a central limit theorem for the power variation.

We state [**MN14**, Theorem 1] after introducing the pth weighted power variation V_n of G, defined as

$$V_n := \Delta_n \sum_{j=1}^{n} \left(\frac{\left| G(t_j^n) - G(t_{j-1}^n) \right|}{\rho(\Delta_n)} \right)^p$$

and

$$\eta(k,\Delta_n) := \frac{\rho\left((k+1)\Delta_n\right)^2 + \rho\left((k-1)\Delta_n\right)^2 - 2\rho\left(k\Delta_n\right)^2}{2\rho\left(\Delta_n\right)^2} \tag{3.3}$$

THEOREM 3.5. *Let $p > 0$ and let $G = \{G(t) | t \in [0,1]\}$ be a Gaussian process of locally stationary increments with $\rho \in R[0,1]$. Let $(\pi^n)_{n\in\mathbb{N}}$ be a sequence of partitions such that its mesh size Δ_n converges to zero as n tends to infinity. Suppose that*

(a) there is a constant $C_1 > 0$, such that $\sigma_G(s,t) \geq C_1 \rho(|t-s|)$ for all $s,t \in [0,1]$;

(b) for every integer $m \geq 2$, there is a real number Ψ_m such that

$$\lim_{n\to\infty} \sum_{k=1}^{y_n} \left(\eta(k,\Delta_n)\right)^m = \Psi_m \tag{3.4}$$

for every increasing and unbounded sequence of positive integers $(y_n)_{n\in\mathbb{N}}$ with values $y_n \leq n-1$ for each $n \geq 1$;

(c) for every integer $m \geq 2$,

$$\lim_{n\to\infty} \frac{\Delta_n}{[\rho(\Delta_n)]^2} \sum_{j,k=1}^{n} \left| \diamond_{j,k}^n \left[\Gamma_G - \tfrac{1}{2}\tilde{\rho}\right] \right|^m = 0,$$

where $\tilde{\rho}(s,t) := -\rho(|t-s|)^2$ *for* $s,t \in [0,1]$.

Then the central limit theorem

$$\Delta_n^{-1/2}\left(V_n - \mathbb{E}\left[V_n\right]\right) = \sqrt{\Delta_n}\sum_{j=1}^{n}\left[\left(\frac{|\Delta_j^n G|}{\rho(\Delta_n)}\right)^p - \mathbb{E}\left(\frac{|\Delta_j^n G|}{\rho(\Delta_n)}\right)^p\right] \xrightarrow{\mathcal{D}} \xi \text{ as } n \to \infty \tag{3.5}$$

holds, where ξ *is a zero mean Gaussian random variable with variance*

$$\mathbb{E}\xi^2 = \sum_{m=2}^{\infty} a_{p,m}^2 m!\left(1 + 2\Psi_m\right),$$

where Ψ_m *is defined by (3.4), and the coefficients* $a_{p,m}$ *are given by*

$$a_{p,m} := (m!)^{-1}\,\mathbb{E}\left[\left(|Z|^p - \mathbb{E}|Z|^p\right) H_m(Z)\right]$$

with H_m, $m \geq 2$, *being the Hermite polynomials as in Chapter 1.5 and* Z *being a standard normal random variable.*

In order to prove our main result we would like to apply this theorem to the conditionally Gaussian process G we constructed in the last section. Unfortunately, it cannot be applied as stated above but in the next section we will see that under some slight modifications the statement of the last theorem still holds true for our process G constructed in the last section.

3. Application of the Gaussian Limit Theorem

In this section we use the notations and constructions we introduced in the first section in order to apply a modified version of Theorem 3.5 (c.f. Corollary 3.6) for fixed $\omega_2 \in \Omega_2$ to the process $G = (G_t)_{t\in\mathbb{R}}$ which is constructed in section 1. The process G is defined in Equation (3.2) by

$$G(t) = \int_{-\infty}^{t} (t-r)_+^{\gamma} - (-r)_+^{\gamma}\, d\widetilde{B}(\widetilde{\theta}_r).$$

The natural idea to apply Theorem 3.5 to the process G is using the function

$$\rho(u) := \mathbb{E}_1\left[G(u)^2\right]^{\frac{1}{2}}.$$

It turns out that under this assumption Condition (c) of Theorem 3.5 and Assumption (A2) are not satisfied in our model. But we found out that we can proceed analogously to the proof of Theorem 3.5 to deduce the same result (c.f. Corollary 3.6) for our process G. This is the goal of this section. Therefore, we introduce some notations.

The sequence of partitions $(\pi^n)_{n\in\mathbb{N}}$ is given by

$$\pi^n := \left\{ t_j^n = \tfrac{j}{n} \middle| \ j = 0, \dots, n \right\}.$$

For $1 \leq j,k \leq n$ we define $\Delta_j^n G := G(t_j^n) - G(t_{j-1}^n)$ and $r_n(j,k) := \frac{\mathbb{E}_1\left[\Delta_j^n G \Delta_k^n G\right]}{w_{j,n} w_{k,n}}$, where

$$w_{j,n} := \mathbb{E}_1\left[\left(\Delta_j^n G\right)^2\right]^{\frac{1}{2}}. \tag{3.6}$$

The choice of the function ρ in Theorem 3.5 is not unique. As it is described in [**MN14**, Remark 3] we can replace $\rho(\Delta_n)$ in Equation (3.5) by $w_{j,n}$. By doing this there is no need for introducing the function ρ, Assumptions (A1) and (A2) and Condition (a) of Theorem 3.5. The drawback is that we need to find alternatives to Hypotheses (b) and (c) of Theorem 3.5. Then the result of Theorem 3.5 reduces to

$$\sqrt{\frac{1}{n}} \sum_{j=1}^{n} \left[\left(\frac{|\Delta_j^n G|}{w_{j,n}} \right)^p - c_p \right] \xrightarrow{\mathcal{D}} \xi \text{ as } n \to \infty,$$

where $c_p = \mathbb{E}(|Z|^p)$ and Z is standard normal. This is the statement of [**MN14**, Remark 3].

The proof of Theorem 3.5 is reduced to exactly this case and is worked out in detail in [**MN14**]. It is based on Malliavin calculus, the corresponding limit theorem (Theorem 1.27) and on a decomposition of $r_n(j,k)$ into two parts. This is

$$r_n(j,k) = \frac{1}{v_{j,n} v_{k,n}} \left(\eta_n \left(|k-j| \right) + z_n(j,k) \right), \tag{3.7}$$

where $v_{j,n} := \frac{w_{j,n}}{\rho(\Delta_n)}$, η_n is given by (3.3) and the term $z_n(j,k)$ is defined by

$$z_n(j,k) := \frac{\diamond_{j,k}^n \left[\Gamma_G - \frac{1}{2}\tilde{\rho}\right]}{\rho(\Delta_n)^2},$$

where $\tilde{\rho}(s,t) := -\rho(|t-s|)^2$ (c.f. Condition (c) of Theorem 3.5). The interpretation is the following: assume that Condition (b) is true. Then, if it can be shown that the process G is *'almost stationary'* in the sense that the above mentioned decomposition holds and $z_n(j,k)$ satisfies the convergence condition (c) of Theorem 3.5, a central limit theorem holds for the power variation of G. In our case we do not have an analogous decomposition. Instead we show that $r_n(j,k)$ directly satisfies an equivalent condition to Condition (b) of Theorem 3.5.

We now state a corollary of the proof of Theorem 3.5 which provides the central limit theorem for the power variation of our process G. Note that the process G determines $1 < \alpha < 2, \gamma \in \left(-\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right)$ and $H = \gamma + \frac{1}{\alpha}$. In the remaining part of this section we prove this corollary. In the proof we focus on the changes compared to the proof of Theorem 3.5.

COROLLARY 3.6. *Let* $1 < \alpha < 2$, $0 < p < \alpha$ *and*

$$V_n := \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\left|G(t_j^n) - G(t_{j-1}^n)\right|}{w_{j,n}}\right)^p .$$

Under the condition

$$H < \begin{cases} \frac{3}{4} & \text{for } \gamma > 0, \\ \frac{1}{2} & \text{for } \gamma < 0, \end{cases} \tag{3.8}$$

it holds that for any integer $m \geq 2$ *there is a real number* Ψ_m, *such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{2\leq j<k\leq n}\left(r_n(j,k)\right)^m = \Psi_m. \tag{3.9}$$

Additionally, the following convergence holds under the measure $\mathbb{P}_1$ *and* $\mathbb{P}_2$*-almost surely:*

$$\sqrt{n}\left(V_n - \mathbb{E}_1\left[V_n\right]\right) = \sqrt{\tfrac{1}{n}}\sum_{j=1}^{n}\left[\left(\frac{|\Delta_j^n G|}{w_{j,n}}\right)^p - c_p\right] \xrightarrow{\mathcal{D}_1} \xi \text{ as } n\to\infty, \tag{3.10}$$

where under the measure $\mathbb{P}_1$ *the law of the random variable* ξ *is centred Gaussian with variance given by*

$$\mathbb{E}_1\xi^2 = \sum_{m=2}^{\infty} a_{p,m}^2 m!\left(1 + 2\Psi_m\right). \tag{3.11}$$

The real number Ψ_m is defined by (3.9), and the coefficients $a_{p,m}$ are given by

$$a_{p,m} := (m!)^{-1}\,\mathbb{E}\left[\left(|Z|^p - \mathbb{E}|Z|^p\right) H_m(Z)\right]$$

with H_m, $m \geq 2$, being the Hermite polynomials and Z being a standard normal random variable on $(\Omega, \mathcal{A}, \mathbb{P})$.

Before we are able to prove this corollary we need two lemmas. The second one is a crucial detail for the proof of the above corollary. It uses a fact about stable random variables which is stated in the first lemma.

LEMMA 3.7. *Let $\alpha < 1$ and consider the Lévy measure $d\nu(x) = \frac{1}{x^{1+\alpha}}\mathbb{1}_{x\geq 0}\,dx$. Let $c_X, c_Y > 0$ and X and Y be two α-stable random variables on $(\Omega, \mathcal{A}, \mathbb{P})$ with the same Lévy measure ν and characteristic functions given by*

$$\varphi_X(u) = e^{|u|^\alpha \int_{\mathbb{R}} e^{ix}-1\,d\nu(x) c_X},$$

respectively

$$\varphi_Y(u) = e^{|u|^\alpha \int_{\mathbb{R}} e^{ix}-1\,d\nu(x) c_Y}.$$

If $c_X > c_Y$, then for any $\delta > 0$ it holds

$$\mathbb{P}(X > \delta) \geq \mathbb{P}(Y > \delta).$$

PROOF. Let $a = \left(\frac{c_X}{c_Y}\right)^{\frac{1}{\alpha}} > 1$. Then $X \overset{\mathcal{D}}{=} aY$ and

$$\mathbb{P}(X > \delta) = \mathbb{P}(aY > \delta) \geq \mathbb{P}(Y > \delta).$$ □

LEMMA 3.8. *Let G be defined as in* (3.2) *and $H = \gamma + \frac{1}{\alpha}$. For $1 \leq j < k \leq n$ we define*

$$\tau_n(k-j) := n^{-2H} \begin{cases} \left(\frac{k-j+1}{2}\right)^{2H-2} & \text{for } \gamma > 0, \\ \left(\frac{k-j+1}{2}\right)^{H-1} & \text{for } \gamma < 0. \end{cases} \tag{3.12}$$

and

$$Y_n^{j,k} := \mathbb{E}_1\left[\left(G\left(\tfrac{j}{n}\right) - G\left(\tfrac{j-1}{n}\right)\right)\left(G\left(\tfrac{k}{n}\right) - G\left(\tfrac{k-1}{n}\right)\right)\right].$$

Then for any $\varepsilon > 0$ *it holds* $\mathbb{P}_2$*-almost surely*

$$n^{-\varepsilon}\tau_n(k-j)^{-1}Y_n^{j,k} \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

PROOF. From the definition of G we can conclude that

$$\begin{aligned} Y_n^{j,k} &:= \mathbb{E}_1\left[\left(G\left(\tfrac{j}{n}\right) - G\left(\tfrac{j-1}{n}\right)\right)\left(G\left(\tfrac{k}{n}\right) - G\left(\tfrac{k-1}{n}\right)\right)\right] \\ &= \int\limits_{\mathbb{R}} \left(\left(\tfrac{j}{n} - s\right)_+^\gamma - \left(\tfrac{j-1}{n} - s\right)_+^\gamma\right)\left(\left(\tfrac{k}{n} - s\right)_+^\gamma - \left(\tfrac{k-1}{n} - s\right)_+^\gamma\right) d\widetilde{\theta}_s, \end{aligned}$$

where this integral is a Lebesgue-Stieltjes integral. Since $\gamma \in \left(-\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right)$ it also exists in the sense of [**RR89**, Definition 2.5] and both integrals coincide. Thus Proposition 1.17 can be applied to determine its characteristic function.

To prove the statement of the lemma we calculate the characteristic function of $Y_n^{j,k}$ and show that $\tau_n(k-j)^{-1}Y_n^{j,k}$ can be estimated in the sense of Lemma 3.7 by a (stable) random variable X on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ which is independent of j, k and n. The proof then finishes as follows: we define a family of sets $(A_n)_{n\in\mathbb{N}}$ by

$$A_n := \left\{\omega \in \Omega \middle| \ \left|n^{-\varepsilon}X\right| \geq \delta\right\}.$$

Then for all $n \in \mathbb{N}$ and any $\delta > 0$ it holds $A_{n+1} \subseteq A_n$. Additionally, for any $\delta > 0$ it holds $\lim_{n\to\infty} \mathbb{P}_2(A_n) = 0$ which includes that for any $\delta > 0$

$$\mathbb{P}_2\left(\bigcap_{n_0\in\mathbb{N}}\bigcup_{n\geq n_0} n^{-\varepsilon}X \geq \delta\right) = \mathbb{P}_2\left(\bigcap_{n_0\in\mathbb{N}} A_{n_0}\right) = \lim_{n_0\to\infty}\mathbb{P}_2(A_{n_0}) = 0.$$

Then the following ensures the convergence in (3.13):

$$\begin{aligned} &\mathbb{P}_2\left(\lim_{n\to\infty} n^{-\varepsilon}\tau_n(k-j)^{-1}Y_n^{j,k} = 0\right) \\ =&\mathbb{P}_2\left(\forall\delta > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 :\ n^{-\varepsilon}\tau_n(k-j)^{-1}Y_n^{j,k} < \delta\right) \\ =&\mathbb{P}_2\left(\bigcap_{\delta>0}\bigcup_{n_0\in\mathbb{N}}\bigcap_{n\geq n_0} n^{-\varepsilon}\tau_n(k-j)^{-1}Y_n^{j,k} < \delta\right) \\ =&1 - \mathbb{P}_2\left(\bigcup_{\delta>0}\bigcap_{n_0\in\mathbb{N}}\bigcup_{n\geq n_0} n^{-\varepsilon}\tau_n(k-j)^{-1}Y_n^{j,k} \geq \delta\right) \end{aligned}$$

$$\geq 1 - \mathbb{P}_2\left(\bigcup_{\delta\in\mathbb{Q}\cap(0,\infty)}\bigcap_{n_0\in\mathbb{N}}\bigcup_{n\geq n_0} n^{-\varepsilon}X \geq \delta\right)$$
$$=1.$$

To finish the proof we estimate the characteristic function of $Y_n^{j,k}$. Note that Proposition 1.17 can be applied to this random variable and its characteristic function is given by

$$\mathbb{E}_2\left[e^{iuY_n^{j,k}}\right] = \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} e^{ixu\left(\left(\frac{j}{n}-s\right)_+^\gamma-\left(\frac{j-1}{n}-s\right)_+^\gamma\right)\left(\left(\frac{k}{n}-s\right)_+^\gamma-\left(\frac{k-1}{n}-s\right)_+^\gamma\right)} - 1\,d\nu(x)\,ds\right\}.$$

By the substitution

$$y = xu\left(\left(\tfrac{j}{n}-s\right)_+^\gamma - \left(\tfrac{j-1}{n}-s\right)_+^\gamma\right)\left(\left(\tfrac{k}{n}-s\right)_+^\gamma - \left(\tfrac{k-1}{n}-s\right)_+^\gamma\right)$$

and since $d\nu(x) = x^{-1-\frac{\alpha}{2}}\mathbb{1}_{x\geq 0}\,dx$ it follows that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{ixu\left(\left(\frac{j}{n}-s\right)_+^\gamma-\left(\frac{j-1}{n}-s\right)_+^\gamma\right)\left(\left(\frac{k}{n}-s\right)_+^\gamma-\left(\frac{k-1}{n}-s\right)_+^\gamma\right)} - 1\,d\nu(x)\,ds$$
$$= |u|^{\frac{\alpha}{2}}\int_{\mathbb{R}} e^{iy}-1\,d\nu(y)\int_{\mathbb{R}}\left|\left(\left(\tfrac{j}{n}-s\right)_+^\gamma - \left(\tfrac{j-1}{n}-s\right)_+^\gamma\right)\left(\left(\tfrac{k}{n}-s\right)_+^\gamma - \left(\tfrac{k-1}{n}-s\right)_+^\gamma\right)\right|^{\frac{\alpha}{2}}ds.$$

Note that only the last integral depends on j,k and n. To handle this term we proceed as follows:

$$\int_{\mathbb{R}}\left|\left(\left(\tfrac{j}{n}-s\right)_+^\gamma - \left(\tfrac{j-1}{n}-s\right)_+^\gamma\right)\left(\left(\tfrac{k}{n}-s\right)_+^\gamma - \left(\tfrac{k-1}{n}-s\right)_+^\gamma\right)\right|^{\frac{\alpha}{2}}ds$$
$$= \int_{\mathbb{R}}\left|\left(\left(\tfrac{k-j+1}{2n}-s\right)_+^\gamma - \left(\tfrac{k-j-1}{2n}-s\right)_+^\gamma\right)\left(\left(-\tfrac{k-j-1}{2n}-s\right)_+^\gamma - \left(-\tfrac{k-j+1}{2n}-s\right)_+^\gamma\right)\right|^{\frac{\alpha}{2}}ds$$
$$= \left(\tfrac{k-j+1}{2n}\right)^{\alpha\gamma+1}\int_{\mathbb{R}}\left|\left((1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma\right)\left(\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^\gamma - (-1-r)_+^\gamma\right)\right|^{\frac{\alpha}{2}}dr,$$

where the last equation holds by substituting $r = \frac{2ns}{k-j+1}$. The statement of the lemma is already shown if $k-j=1$ since $\frac{k-j+1}{2}=1$. Then, the integral no longer depends on j,k

and n, which means that $\tau_n(k-j)^{-1}Y_n^{j,k} = n^{2H}Y_n^{j,k}$ has the same distribution as the stable random variable

$$\int_{\mathbb{R}} \left((1-r)_+^\gamma - (-r)_+^\gamma\right)\left((-r)_+^\gamma - (-1-r)_+^\gamma\right) d\widetilde{\theta}_s.$$

To see this one easily observes that $\left(n^{2H}\right)^{\frac{\alpha}{2}} = n^{\alpha H} = n^{\alpha\gamma+1}$.

For the remaining part of the proof let $k-j \geq 2$ (this is only needed in Estimation (3.15)). To estimate the integral

$$\int_{\mathbb{R}} \left|\left((1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1} - r\right)_+^\gamma\right)\left(\left(-\tfrac{k-j-1}{k-j+1} - r\right)_+^\gamma - (-1-r)_+^\gamma\right)\right|^{\frac{\alpha}{2}} dr \tag{3.14}$$

the idea is to apply a Taylor expansion of the function $t \mapsto (t-r)_+^\gamma$ for both factors of the integrand. This can be done on the interval $(-\infty,-1)$. On the interval $\left[-1, -\frac{k-j-1}{k-j+1}\right]$ this can only be applied to the first factor and on the interval $\left(-\frac{k-j-1}{k-j+1}, \infty\right)$ the integrand is zero. For the first factor we apply the Taylor expansion as follows: for any $r < 0$ there exists $\xi \in \left(\frac{k-j-1}{k-j+1}, 1\right)$ such that

$$(1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1} - r\right)_+^\gamma = \gamma\left(1 - \tfrac{k-j-1}{k-j+1}\right)(\xi - r)^{\gamma-1}.$$

Since $\gamma - 1 < 0$ its absolute value can be estimated from above by $|\gamma|\left(\frac{2}{k-j+1}\right)(-r)^{\gamma-1}$. With the same arguments it holds for $r < -1$

$$\left|\left(-\tfrac{k-j-1}{k-j+1} - r\right)_+^\gamma - (-1-r)_+^\gamma\right| \leq |\gamma|\left(\tfrac{2}{k-j+1}\right)\left|(-1-r)^{\gamma-1}\right|.$$

Note that the last term is not integrable at $r = -1$ iff $\gamma < 0$. This means that for $\gamma < 0$ the Taylor expansion can only be applied on the first factor in Equation (3.14). We split the integral in (3.14) into the integrals over the intervals

$$(-\infty,-1) \text{ and } \left(-1, -\tfrac{k-j-1}{k-j+1}\right).$$

On the interval $\left(-1, -\frac{k-j-1}{k-j+1}\right)$ it holds for any choice of γ:

$$\int_{-1}^{-\frac{k-j-1}{k-j+1}} \left|\left((1-r)_+^\gamma - \left(\tfrac{k-j-1}{k-j+1} - r\right)_+^\gamma\right)\left(-\tfrac{k-j-1}{k-j+1} - r\right)_+^\gamma\right|^{\frac{\alpha}{2}} dr$$

$$\leq \int_{-1}^{-\frac{k-j-1}{k-j+1}} \left| \left(\gamma \left(\tfrac{2}{k-j+1}\right) (-r)^{\gamma-1}\right) \left(-\tfrac{k-j-1}{k-j+1} - r\right)^{\gamma} \right|^{\frac{\alpha}{2}} dr$$

$$\leq |\gamma| \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}} \left(\tfrac{k-j-1}{k-j+1}\right)^{\gamma-1} \int_{-1}^{-\frac{k-j-1}{k-j+1}} \left| -\tfrac{k-j-1}{k-j+1} - r \right|^{\gamma\frac{\alpha}{2}} dr$$

$$\leq \left| \frac{\gamma}{\gamma\frac{\alpha}{2}+1} \right| \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}} \left(\tfrac{1}{3}\right)^{\gamma-1} \left(\tfrac{2}{k-j+1}\right)^{\gamma\frac{\alpha}{2}+1} =: \tilde{c}_1 \cdot \left(\tfrac{2}{k-j+1}\right)^{\gamma\frac{\alpha}{2}+1+\frac{\alpha}{2}}. \tag{3.15}$$

Under the assumption $\gamma > 0$ we estimate

$$\int_{-\infty}^{-1} \left| \left((1-r)_+^{\gamma} - \left(\tfrac{k-j-1}{k-j+1} - r\right)_+^{\gamma}\right) \left(\left(-\tfrac{k-j-1}{k-j+1} - r\right)_+^{\gamma} - (-1-r)_+^{\gamma}\right) \right|^{\frac{\alpha}{2}} dr$$

$$\leq \gamma^2 \left(\tfrac{2}{k-j+1}\right)^{\alpha} \int_{-\infty}^{-1} \left| (-r)^{\gamma-1} (-1-r)^{\gamma-1} \right|^{\frac{\alpha}{2}} dr =: \tilde{c}_2 \cdot \left(\tfrac{2}{k-j+1}\right)^{\alpha},$$

where the last integral is finite. Since $\frac{2}{k-j+1} \leq 1$ we have

$$\left(\tfrac{2}{k-j+1}\right)^{\alpha} \geq \left(\tfrac{2}{k-j+1}\right)^{\gamma\frac{\alpha}{2}+1+\frac{\alpha}{2}}$$

if and only if $\alpha \leq \gamma\frac{\alpha}{2} + 1 + \frac{\alpha}{2}$ which is equivalent to $\gamma \geq 1 - \frac{2}{\alpha}$. Since $\frac{2}{\alpha} > 1$ and $\gamma > 0$ the condition $\gamma \geq 1 - \frac{2}{\alpha}$ is always satisfied for $\gamma > 0$.

If $\gamma < 0$ the first factor can be estimated by

$$\left| \left((1-r)_+^{\gamma} - \left(\tfrac{k-j-1}{k-j+1} - r\right)_+^{\gamma}\right) \right|^{\frac{\alpha}{2}} \leq |\gamma|^{\frac{\alpha}{2}} \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}} (-r)^{(\gamma-1)\frac{\alpha}{2}}.$$

Then we use the Cauchy-Schwarz inequality to calculate

$$\int_{-\infty}^{-1} \left| \left((1-r)_+^{\gamma} - \left(\tfrac{k-j-1}{k-j+1} - r\right)_+^{\gamma}\right) \left(\left(-\tfrac{k-j-1}{k-j+1} - r\right)_+^{\gamma} - (-1-r)_+^{\gamma}\right) \right|^{\frac{\alpha}{2}} dr$$

$$\leq |\gamma|^{\frac{\alpha}{2}} \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}} \int_{-\infty}^{-1} (-r)^{(\gamma-1)\frac{\alpha}{2}} \left| \left(-\tfrac{k-j-1}{k-j+1} - r\right)^{\gamma} - (-1-r)^{\gamma} \right|^{\frac{\alpha}{2}} dr$$

$$\leq |\gamma|^{\frac{\alpha}{2}} \left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}} \left(\int_{-\infty}^{-1} (-r)^{(\gamma-1)\alpha} dr \right)^{\frac{1}{2}} \left(\int_{-\infty}^{-1} \left| \left(-\tfrac{k-j-1}{k-j+1} - r\right)^{\gamma} - (-1-r)^{\gamma} \right|^{\alpha} dr \right)^{\frac{1}{2}}.$$

By the observations made above the last integral is

$$\left(\int_{-\infty}^{-1}\left|\left(-\tfrac{k-j-1}{k-j+1}-r\right)^{\gamma}-(-1-r)^{\gamma}\right|^{\alpha}dr\right)^{\frac{1}{2}}=\left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha\gamma+1}{2}}\left(\int_{-\infty}^{0}\left|(1-r)_+^{\gamma}-(-r)_+^{\gamma}\right|^{\alpha}\right)^{\frac{1}{2}}$$

and we can conclude

$$\begin{aligned}
&\int_{-\infty}^{-1}\left|\left((1-r)_+^{\gamma}-\left(\tfrac{k-j-1}{k-j+1}-r\right)_+^{\gamma}\right)\left(\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^{\gamma}-(-1-r)_+^{\gamma}\right)\right|^{\frac{\alpha}{2}}dr\\
&\leq|\gamma|^{\frac{\alpha}{2}}\left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}+\frac{\alpha\gamma+1}{2}}\left(\int_{-\infty}^{-1}(-r)^{(\gamma-1)\alpha}\,dr\right)^{\frac{1}{2}}\left(\int_{-\infty}^{0}\left|(1-r)_+^{\gamma}-(-r)_+^{\gamma}\right|^{\alpha}\right)^{\frac{1}{2}}\\
&=:\tilde{c}_3\cdot\left(\tfrac{2}{k-j+1}\right)^{\frac{\alpha}{2}+\frac{\alpha\gamma+1}{2}},
\end{aligned}$$

where both integrals are finite. Since $\frac{\alpha}{2}+\frac{\alpha\gamma+1}{2}<\gamma\frac{\alpha}{2}+1+\frac{\alpha}{2}$ the term $\left(\frac{2}{k-j+1}\right)^{\frac{\alpha}{2}+\frac{\alpha\gamma+1}{2}}$ dominates the term $\left(\frac{2}{k-j+1}\right)^{\gamma\frac{\alpha}{2}+1+\frac{\alpha}{2}}$ given in Equation (3.15).

The characteristic function of $\tau_n(k-j)^{-1}Y_n^{j,k}$ can be calculated by

$$\begin{aligned}
&\varphi_{\tau_n(k-j)^{-1}Y_n^{j,k}}(u)\\
=&\varphi_{Y_n^{j,k}}(\tau_n(k-j)^{-1}u)\\
=&\exp\left\{|\tau_n(k-j)^{-1}u|^{\frac{\alpha}{2}}\int_{\mathbb{R}}e^{iy}-1\,d\nu(y)\left(\tfrac{k-j+1}{2n}\right)^{\alpha\gamma+1}\right.\\
&\left.\cdot\int_{\mathbb{R}}\left|\left((1-r)_+^{\gamma}-\left(\tfrac{k-j-1}{k-j+1}-r\right)_+^{\gamma}\right)\left(\left(-\tfrac{k-j-1}{k-j+1}-r\right)_+^{\gamma}-(-1-r)_+^{\gamma}\right)\right|^{\frac{\alpha}{2}}dr\right\}.
\end{aligned}$$

We first consider the case $\gamma>0$. Then $\tau_n(k-j)^{-\frac{\alpha}{2}}=\left(\frac{k-j+1}{2n}\right)^{-\alpha H}\left(\frac{k-j+1}{2}\right)^{\alpha}$ and since $H=\gamma+\frac{1}{\alpha}$ it holds $|\tau_n(k-j)^{-1}|^{\frac{\alpha}{2}}\left(\frac{k-j+1}{2n}\right)^{\alpha\gamma+1}=\left(\frac{k-j+1}{2}\right)^{\alpha}$. By the calculations made above we have the following representation:

$$\varphi_{\tau_n(k-j)^{-1}Y_n^{j,k}}(u)=\exp\left\{|u|^{\frac{\alpha}{2}}\int_{\mathbb{R}}e^{iy}-1\,d\nu(y)c_1\right\},$$

where

$$c_1 := \left(\tfrac{k-j+1}{2}\right)^{\alpha} \int\limits_{\mathbb{R}} \left| \left((1-r)_+^{\gamma} - \left(\tfrac{k-j-1}{k-j+1} - r\right)_+^{\gamma} \right) \left(\left(-\tfrac{k-j-1}{k-j+1} - r\right)_+^{\gamma} - (-1-r)_+^{\gamma} \right) \right|^{\frac{\alpha}{2}} dr.$$

In the case $\gamma < 0$ it holds $|\tau_n(k-j)|^{-\frac{\alpha}{2}} \left(\frac{k-j+1}{2n}\right)^{\alpha\gamma+1} = \left(\frac{k-j+1}{2}\right)^{\frac{\alpha\gamma+1+\alpha}{2}}$ and the exponent of the term $\frac{k-j+1}{2}$ in the above representation for c_1 changes as follows: instead of α we have $\frac{\alpha\gamma+1+\alpha}{2}$. We have seen that the following estimation holds for the above integral:

$$\begin{aligned}
&\int\limits_{\mathbb{R}} \left| \left((1-r)_+^{\gamma} - \left(\tfrac{k-j-1}{k-j+1} - r\right)_+^{\gamma} \right) \left(\left(-\tfrac{k-j-1}{k-j+1} - r\right)_+^{\gamma} - (-1-r)_+^{\gamma} \right) \right|^{\frac{\alpha}{2}} dr \\
&\leq \begin{cases} \tilde{c}_1 \left(\tfrac{k-j+1}{2}\right)^{-\frac{\alpha\gamma+2+\alpha}{2}} + \tilde{c}_2 \left(\tfrac{k-j+1}{2}\right)^{-\alpha} & \gamma > 0, \\ \tilde{c}_1 \left(\tfrac{k-j+1}{2}\right)^{-\frac{\alpha\gamma+2+\alpha}{2}} + \tilde{c}_3 \left(\tfrac{k-j+1}{2}\right)^{-\frac{\alpha\gamma+1+\alpha}{2}} & \gamma < 0, \end{cases} \\
&\leq \begin{cases} (\tilde{c}_1 + \tilde{c}_2) \left(\tfrac{k-j+1}{2}\right)^{-\alpha} & \gamma > 0, \\ (\tilde{c}_1 + \tilde{c}_3) \left(\tfrac{k-j+1}{2}\right)^{-\frac{\alpha\gamma+1+\alpha}{2}} & \gamma < 0. \end{cases}
\end{aligned}$$

If we define

$$c_2 := \begin{cases} \tilde{c}_1 + \tilde{c}_2 & \gamma > 0, \\ \tilde{c}_1 + \tilde{c}_3 & \gamma < 0, \end{cases}$$

it immediately follows $c_2 > c_1$. Now, we define a random variable X on $(\Omega_2, \mathcal{A}_2, \mathbb{P}_2)$ by its characteristic function given by

$$\mathbb{E}_2\left[e^{iuX}\right] = \exp\left\{ |u|^{\frac{\alpha}{2}} \int\limits_{\mathbb{R}} e^{iy} - 1 \, d\nu(y) c_2 \right\}.$$

Then by Lemma 3.7 it holds

$$\mathbb{P}_2(X \geq \delta) \geq \mathbb{P}_2(\tau_n(k-j)^{-1} Y_n^{j,k} \geq \delta)$$

which finishes the proof. □

Now, we are able to prove Corollary 3.6.

PROOF OF COROLLARY 3.6. We proceed analogously to the proof of [**MN14**, Theorem 1]. We first determine the Wiener chaos representation of $\sqrt{n}\left(V_n - \mathbb{E}\left[V_n\right]\right)$. Then we show that Theorem 1.27 can be applied in our model.

Let $c_p := \mathbb{E}\left[|Z|^p\right]$, where Z is a standard normal random variable and

$$V_n := \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\left|G(t_j^n) - G(t_{j-1}^n)\right|}{w_{j,n}}\right)^p.$$

Then $\mathbb{E}_1\left[V_n\right] = c_p$. Now, we consider the term of interest. This is $\sqrt{n}\left(V_n - \mathbb{E}_1\left[V_n\right]\right)$. By a separation of the first summand we obtain

$$\sqrt{n}\left(V_n - \mathbb{E}_1\left[V_n\right]\right) = \sqrt{\tfrac{1}{n}}\left(\frac{|\Delta_1^n G|^p}{w_{1,n}^p} - c_p\right) + \sqrt{\tfrac{1}{n}}\sum_{j=2}^{n}\left(\frac{\left|\Delta_j^n G\right|^p}{w_{j,n}^p} - c_p\right) =: R_n + Y_n.$$

Analogous to the proof of [**MN14**, Theorem 1] we show that $R_n \overset{\mathbb{P}_1}{\to} 0$ as follows: by applying Tschebyscheff's inequality it holds for any $\delta > 0$

$$\begin{aligned}\mathbb{P}_1\left(|R_n| > \delta\right) &\leq \delta^{-2}\frac{1}{n}\left(\mathbb{E}_1\left[\frac{|\Delta_1^n G|^{2p}}{w_{1,n}^{2p}}\right] - 2c_p\mathbb{E}_1\left[\frac{|\Delta_1^n G|^p}{w_{1,n}^p}\right] + c_p^2\right)\\ &= \delta^{-2}\frac{1}{n}\left(c_{2p} - c_p^2\right) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Hence, by Slutsky's lemma it is sufficient to show that

$$Y_n \overset{\mathcal{D}_1}{\to} \xi \quad \text{as } n \to \infty, \tag{3.16}$$

where ξ is a centred normally distributed random variable with variance given by (3.11). Therefore, we use Malliavin based technique.

Let $\mu = N(0,1)$, then the Hermite-polynomials H_m introduced in Chapter 1.5 are an orthogonal basis of the Hilbert-space $L^2(\mathbb{R},\mu)$. We define a function $H\colon \mathbb{R} \to \mathbb{R}$ by $H(x) := |x|^p - c_p$. Then $H \in L^2(\mathbb{R},\mu)$ which means that H can be expressed by the expansion

$$H = \sum_{m=0}^{\infty} a_m H_m.$$

Then the corresponding expansion of Y_n is given by

$$Y_n = \sum_{m=0}^{\infty}\left(a_m\sqrt{\tfrac{1}{n}}\sum_{j=2}^{n}H_m\left(\frac{\Delta_j^n G}{w_{j,n}}\right)\right) = \sum_{m=2}^{\infty}\left(a_m\sqrt{\tfrac{1}{n}}\sum_{j=2}^{n}H_m\left(\frac{\Delta_j^n G}{w_{j,n}}\right)\right),$$

where the second equality holds since for a standard normal random variable Z it holds

$$a_0 = \mathbb{E}\left[H_0(Z)H(Z)\right] = \mathbb{E}\left[|Z|^p - c_p\right] = 0,$$

$$a_1 = \mathbb{E}\left[H_1(Z)H(Z)\right] = \mathbb{E}\left[Z\left(|Z|^p - c_p\right)\right] = \mathbb{E}\left[|Z|^p Z\right] = 0.$$

Let I_m be the abstract multiple Wiener integral (c.f. Chapter 1.5). By the linearity of I_m and since the L^2-norm of $\frac{\Delta_i^n G}{w_{i,n}}$ equals one we have the following Wiener-chaos representation of Y_n:

$$Y_n = \sum_{m=2}^{\infty} I_m(f_{m,n}),$$

where

$$f_{m,n} := a_m\sqrt{\tfrac{1}{n}}\sum_{j=2}^{n}\left(\frac{\Delta_j^n G}{w_{j,n}}\right)^{\otimes m}.$$

Let J_m be the projection of $\mathcal{H}$ on the mth Wiener chaos $\mathcal{H}_m$. Then

$$J_m Y_n = a_m\sqrt{\tfrac{1}{n}}\sum_{j=2}^{n}H_m\left(\frac{\Delta_j^n G}{w_{j,n}}\right)$$

and by (1.7) it holds for any $n \geq 1$ and $m \geq 2$

$$m!\|f_{m,n}\|_{\mathcal{H}^{\otimes m}} = \mathbb{E}_1\left[(J_m Y_n)^2\right].$$

According to Theorem 1.27 the following conditions imply the convergence of Y_n as it is stated in (3.16):

(1) for every $n \geq 1,\ m \geq 1$ it holds $\mathbb{E}_1\left[(J_m Y_n)^2\right] \leq \delta_m$, where $\sum_{m=1}^{\infty}\delta_m < \infty$;
(2) for every $m \geq 1$, there exists $\lim_{n\to\infty}\mathbb{E}_1\left[(J_m Y_n)^2\right] =: \sigma_m^2$;
(3) for every $m \geq 2$ and $\kappa = 1, \ldots, m-1$ it holds $\lim_{n\to\infty}\|f_{m,n}\otimes_\kappa f_{m,n}\|^2_{\mathcal{H}^{\otimes 2(m-\kappa)}} = 0$.

The variance of ξ is then given by $\mathbb{E}_1\left[\xi^2\right] = \sum_{m=2}^{\infty}\sigma_m^2$. By the orthogonality of the Hermite-polynomials and the resulting orthogonality of the Wiener chaoses it holds $J_1 Y_n = 0$ for each $n \geq 1$, so it suffices to prove Conditions (1) and (2) for $m \geq 2$.

For $n \geq 1$ and $2 \leq j,k \leq n$ we define

$$r_n(j,k) := \mathbb{E}_1\left[\frac{\Delta_j^n G \Delta_k^n G}{w_{j,n} w_{k,n}}\right].$$

By the Cauchy-Schwarz inequality it is $r_n(j,k) \leq 1$ for all $n \geq 1$ and $2 \leq j,k \leq n$. Then for all $m \geq 2$ it holds

$$\left|\sum_{2\leq j,k\leq n} r_n(j,k)^m\right| \leq \sum_{2\leq j,k\leq n} r_n(j,k)^2.$$

By [**Nua95**, Lemma 1.1.1] it is

$$\mathbb{E}_1\left[(J_m Y_n)^2\right] = a_m^2 m! \frac{1}{n} \sum_{2\leq j,k\leq n} r_n(j,k)^m$$

$$= a_m^2 m! \left(1 + 2\frac{1}{n} \sum_{2\leq j<k\leq n} r_n(j,k)^m\right) \tag{3.17}$$

$$\leq a_m^2 m! \left(1 + 2\frac{1}{n} \sum_{2\leq j<k\leq n} r_n(j,k)^2\right). \tag{3.18}$$

In the proof of [**MN14**, Theorem 1] Conditions (1) - (3) are shown by using a decomposition of $r_n(j,k)$ into two terms $r_n(j,k) = \frac{1}{v_{j,n} v_{k,n}} \left(\eta_n\left(|k-j|\right) + z_n(j,k)\right)$ (c.f. (3.7)) and the fact that $r_n(j,k)$ behaves essentially as $\eta_n(|k-j|)$. We claim that Conditions (1) and (2) are satisfied if for each $m \geq 2$ the following limits exist

$$\lim_{n\to\infty} \tfrac{1}{n} \sum_{2\leq j<k\leq n} r_n(j,k)^m =: \psi_m. \tag{3.19}$$

This equation is referred as [**MN14**, Equation (3.15)] and we later show that it is satisfied in our model. Combining this with (3.17) it follows that σ_m^2 in Condition (2) above is given by

$$a_m^2 m! \left(1 + 2\psi_m\right)$$

and additionally by (3.18) it holds that for all $m \geq 2$ the term δ_m can be bounded by $a_m^2 m! \left(1 + 2\psi_2\right)$. Under the hypothesis that (3.19) holds true Conditions (1) and (2) are proven since $\sum_{m=2}^{\infty} a_m^2 m! = \mathbb{E}\left[(H(Z))^2\right] < \infty$, where Z is a standard normal random variable.

On the other hand Condition (3) reduces to the following weaker condition: for $m \geq 2,\ 1 \leq \kappa \leq m-1$ it holds

$$\frac{1}{n^2} \sum_{i,j,k,l=2}^{n} |r_n(i,j)|^{\kappa} |r_n(k,l)|^{\kappa} |r_n(i,k)|^{m-\kappa} |r_n(j,l)|^{m-\kappa} \to 0 \quad \text{as } n \to \infty. \tag{3.20}$$

To see this we observe that by the calculation made in the proof of [**MN14**, Theorem 1] $f_{m,n} \otimes_\kappa f_{m,n}$ is given by

$$f_{m,n} \otimes_\kappa f_{m,n} = \frac{1}{n}\, a_m^2 \sum_{j,k=2}^{n} r_n(j,k)^{\kappa} \left(h_{j,n}^{\otimes(m-\kappa)} \otimes h_{k,n}^{\otimes(m-\kappa)} \right),$$

where $h_{j,n} := \frac{\Delta_j^n G}{w_{j,n}}$. Then the square of the $\mathcal{H}^{\otimes 2(m-\kappa)}$-norm of this term is (again by the calculations done in [**MN14**]) given by

$$\|f_{m,n} \otimes_\kappa f_{m,n}\|^2_{\mathcal{H}^{\otimes 2(m-\kappa)}} = n^{-2} a_m^4 \sum_{i,j,k,l=2}^{n} r_n(i,j)^{\kappa} r_n(k,l)^{\kappa} r_n(i,k)^{m-\kappa} r_n(j,l)^{m-\kappa}.$$

Then, (3.20) implies Condition (3) above.

We later show that there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$ there is a function $\tilde{\eta}_n \colon \mathbb{R} \to \mathbb{R}$ such that

$$|r_n(j,k)| \leq \tilde{\eta}_n(k-j)$$

for any $1 \leq j < k \leq n$ and $\tilde{\eta}_n$ satisfies the following condition: let H be as in (3.8), then for any $n \geq n_0$ and $m \geq 2$ it holds

$$\lim_{n\to\infty} \sum_{k=1}^{n} \tilde{\eta}_n(k)^m < \infty. \tag{3.21}$$

This condition can be identified with Condition (b) of [**MN14**, Theorem 1] and it is sufficient to show that the convergence in (3.20) is satisfied. For this step we proceed exactly as in [**MN14**, Pages 335-337] and it is worked out in the Appendix. Thus Equation (3.21) implies (3.20) and then Condition (3) is satisfied in our model.

It remains to prove the existence of ψ_m given in (3.19). To this end we show that the series $\frac{1}{n} \sum_{2\leq j<k\leq n} r_n(j,k)^m$ is absolutely convergent for any $m \geq 2$. The denominator of $r_n(j,k)$ can be estimated from below as follows: it is

$$w_{j,n}^2 = \int_{\mathbb{R}} \left(\left(\tfrac{j}{n} - s\right)_+^{\gamma} - \left(\tfrac{j-1}{n} - s\right)_+^{\gamma} \right)^2 d\widetilde{\theta}_s,$$

where the integral is defined as Lebesgue-Stieltjes integral. Since $\widetilde{\theta}$ is $\mathbb{P}_2$-almost surely non-decreasing and since the integrand is non-negative it can simply be estimated from below by

$$w_{j,n}^2 \geq \int_{\frac{j-2}{n}}^{\frac{j-1}{n}} \left(\left(\tfrac{j}{n} - s \right)^{\gamma} - \left(\tfrac{j-1}{n} - s \right)^{\gamma} \right)^2 d\widetilde{\theta}_s.$$

The simplest estimation from below is to replace the integrand by its minimum over the interval $\left[\frac{j-2}{n}, \frac{j-1}{n}\right]$. Since the integrand is convex, non-negative and increasing on this interval it has its minimal value at $\frac{j-2}{n}$ so we have the estimate

$$w_{j,n}^2 \geq \frac{(2^{\gamma}-1)^2}{n^{2\gamma}} \Delta_{j-1}^n \widetilde{\theta}.$$

By Proposition 1.25 as $n \to \infty$, we have the following estimate from below for $\Delta_{j-1}^n \widetilde{\theta}$:

$$\Delta_{j-1}^n \widetilde{\theta} \gtrsim n^{-\frac{2}{\alpha}} \quad \mathbb{P}_2 - a.s.,$$

which means that for large n it holds:

$$w_{j,n}^2 \gtrsim n^{-2(\gamma+\frac{1}{\alpha})} = n^{-2H} \quad \mathbb{P}_2 - a.s,$$

which is obviously also satisfied by $w_{k,n}^2$. After all the denominator of $r_n(j,k)$ can be estimated as follows:

$$w_{j,n} w_{k,n} \gtrsim n^{-2H} \quad \mathbb{P}_2 - a.s. \tag{3.22}$$

For the numerator we apply Lemma 3.8 and conclude that there is some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $\varepsilon > 0$ it holds

$$\mathbb{E}_1 \left[\Delta_j^n G \Delta_k^n G \right] \leq n^{\varepsilon} \tau_n(k-j) \quad \mathbb{P}_2 - a.s.,$$

where $\tau_n(k-j)$ is defined in (3.12). This implies

$$r_n(j,k) \leq const \cdot n^{\varepsilon} \left\{ \begin{matrix} (k-j+1)^{2H-2} & \gamma > 0, \\ (k-j+1)^{H-1} & \gamma < 0 \end{matrix} \right\} =: \tilde{\eta}_n(k-j) \quad \mathbb{P}_2 - a.s. \tag{3.23}$$

for all $n \geq n_0$ and for any $\varepsilon > 0$. If $\gamma > 0$ then for any $n \geq n_0$, $H < \frac{3}{4}$ and $0 < \varepsilon < \frac{3-4H}{2}$ it holds

$$\begin{aligned}&\frac{1}{n}\left|\sum_{2\leq j<k\leq n} r_n(j,k)^m\right| \\ \leq&\frac{1}{n}\sum_{2\leq j<k\leq n} r_n(j,k)^2 \\ \lesssim&\frac{1}{n}n^{2\varepsilon}\sum_{2\leq j<k\leq n}(k-j+1)^{4H-4} \\ \sim& n^{2\varepsilon+4H-3} \to 0\end{aligned}$$

$\mathbb{P}_2$-almost surely as $n \to \infty$ which implies the convergence in Equation (3.19).

In the case $\gamma < 0$ it holds

$$r_n(j,k) \leq const \cdot n^{\varepsilon}(k-j+1)^{H-1} = const \cdot n^{\varepsilon}(k-j+1)^{H-1} \quad \mathbb{P}_2 - a.s.$$

for all $n \geq n_0$ and for any $\varepsilon > 0$. Under this condition the convergence in Equation (3.19) holds by the same arguments with the restriction $H < \frac{1}{2}$ and $0 < \varepsilon < \frac{2H-1}{2}$. Note that this is no contradiction to the Gaussian limit theorems developed in **[CNW06, MN14]**. This is given because in the Gaussian model it is $\alpha = 2$ which means that $H = \gamma + \frac{1}{2}$. In this case $\gamma < 0$ implies $H < \frac{1}{2}$. Hence, Conditions (1) and (2) above (c.f. Theorem 1.27) are satisfied in our model.

By those calculations we also conclude that in the case $\gamma > 0$ for any $H < \frac{3}{4}$ for all $m \geq 2$ and $0 < \varepsilon < \frac{3-4H}{m}$ it holds

$$\lim_{n\to\infty}\sum_{k=1}^{n}\tilde{\eta}_n(k)^m < \infty$$

which implies the convergence in Equation (3.21). If $\gamma < 0$ the same result holds for $H < \frac{1}{2}$ and any $0 < \varepsilon < \frac{1-2H}{2}$. Hence, Condition (3) above (c.f. Theorem 1.27) is satisfied in our model which finishes the proof. □

In the next section we apply Corollary 3.6 in our model in order to prove Theorem 3.1.

4. Proof of the Main Result

Now, we are able to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Let $V^n := V_p^n(X^H)_1$. It is

$$V^n = \sum_{j=1}^{n} \left| X^H_{\frac{j}{n}} - X^H_{\frac{j-1}{n}} \right|^p = \sum_{j=1}^{n} \left| \Delta_j^n X^H \right|^p$$

and we define in analogy to Theorem 3.5

$$V_n := \frac{1}{n} \sum_{j=1}^{n} \left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}} \right)^p,$$

where $w_{j,n}$ is defined by (3.6).

Now, we have the following for the left hand side of (3.1):

$$\begin{aligned}
&\sqrt{n}\left(n^{-1+pH} V^n - \mathbb{E}\left[\left|X_1^H\right|^p\right]\right) \\
=&\sqrt{n}\left(n^{-1+pH} \sum_{j=1}^{n} \left| X^H_{\frac{j}{n}} - X^H_{\frac{j-1}{n}} \right|^p - \mathbb{E}\left[\left|X_1^H\right|^p\right]\right) \\
=&\frac{1}{\sqrt{n}} \sum_{j=1}^{n} n^{pH} \left(\left|\Delta_j^n X^H\right|^p - \mathbb{E}\left[\left|\Delta_j^n X^H\right|^p\right]\right) \\
=&\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left(n^{2H} w_{j,n}^2\right)^{\frac{p}{2}} \left(\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}} \right)^p - \mathbb{E}\left[\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}} \right)^p \right] \right)
\end{aligned}$$

By the observations made in the proof of Lemma 3.8 for any $n \in \mathbb{N}$ and any $1 \leq j \leq n$ it holds $n^{2H} w_{j,n}^2 \overset{\mathcal{D}}{=} w_{1,1}^2$ under $\mathbb{P}_2$. Since for any $n \in \mathbb{N}$ and $1 \leq j \leq n$ the random variables $w_{j,n}$ are independent of $\omega_1 \in \Omega_1$ this also holds under the measure $\mathbb{P}$. Additionally, it is

$$\mathbb{E}\left[\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}} \right)^p \right] = \mathbb{E}_2 \mathbb{E}_1 \left[\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}} \right)^p \right] = \mathbb{E}_2 \mathbb{E}_1 \left[|Z|^p \right] =: c_p,$$

where Z is standard normal under $\mathbb{P}_1$. Then the convergence of $\sqrt{n}\left(n^{-1+pH} V^n - \mathbb{E}\left[\left|X_1^H\right|^p\right]\right)$ to a mixture of Gaussian random variables is shown as follows: by Corollary 3.6 the following holds under $\mathbb{P}_1$ and $\mathbb{P}_2$ almost surely (note that the processes G and X^H are the same under $\mathbb{P}$):

$$\sqrt{n}\left(V_n - \mathbb{E}_1\left[V_n\right]\right) = \frac{1}{\sqrt{n}}\left(\sum_{j=1}^{n}\left[\left(\frac{\Delta_j^n X^H}{w_{j,n}}\right)^p - c_p\right]\right) \overset{\mathcal{D}_1}{\to} \xi \text{ as } n \to \infty. \tag{3.24}$$

Then for any continuous, bounded, real valued function $f \in \mathcal{C}_b^0(\mathbb{R})$ it holds

$$\begin{aligned}
&\mathbb{E}\left[f\left(\sqrt{n}\left(n^{-1+pH}V^n - \mathbb{E}\left[\left|X_1^H\right|^p\right]\right)\right)\right] \\
=&\mathbb{E}\left[f\left(w_{1,1}^p \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}}\right)^p - c_p\right)\right)\right] \\
\overset{Fubini}{=}&\mathbb{E}_2\mathbb{E}_1\left[f\left(w_{1,1}^p \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}}\right)^p - c_p\right)\right)\right].
\end{aligned}$$

Since Equation (3.24) holds $\mathbb{P}_2$-almost surely we can apply Lebesgue's theorem to the term above. Then we have the following convergence as $n \to \infty$:

$$\mathbb{E}_2\mathbb{E}_1\left[f\left(w_{1,1}^p \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(\left(\frac{\left|\Delta_j^n X^H\right|}{w_{j,n}}\right)^p - c_p\right)\right)\right] \to \mathbb{E}_2\mathbb{E}_1\left[f\left(w_{1,1}^p\xi\right)\right] = \mathbb{E}\left[f\left(w_{1,1}^p\xi\right)\right].$$

Note that under $\mathbb{P}_1$ the random variable $w_{1,1}^p\xi$ is a Gaussian random variable, which means that under $\mathbb{P}$ it is a mixture of Gaussian random variables and in particular it is non-trivial. This finishes the proof of Theorem 3.1. □

5. Conclusion and Suggested Future Research

We considered the power variation of fractional Lévy processes and its limit behaviour. We have seen that a consistency theorem for integrated fractional processes can be proven with the same techniques as in the Gaussian model introduced by **[CNW06]**. The only difference is that the process u has to be more regular in our model. Probably one can derive a better result if instead of the Riemann-Stieltjes integral another integral calculus is used in the definition of the process Z. But since the Young inequality is involved in the proof of the consistency theorem one has to invent in new techniques of proof in order to use the other calculus.

The technique used for the distributional theory is an elegant way to reduce the proof of limit theorem in non-Gaussian models to those in Gaussian models. But it is limited by the fact that the driving Lévy process needs to admit a representation as Brownian motion subordinated by some other process. As long as the subordinator possesses finite first moment we think that the technique used here can also be applied. On the other hand processes such as those introduced in Example 3.3 could also be handled by studying the difference of those processes and linear fractional stable motions.

Appendix

For the proof of Corollary 3.6 it remains to show

$$\frac{1}{n^2}\sum_{i,j,k,l=2}^{n}|r_n(i,j)|^{\kappa}|r_n(k,l)|^{\kappa}|r_n(i,k)|^{m-\kappa}|r_n(j,l)|^{m-\kappa}\to 0 \quad \text{as } n\to\infty.$$

To this end we proceed exactly as it is worked out in [**MN14**, Pages 335-337]. The term above can be identified with the term B_n of the proof of [**MN14**, Theorem 1]. We have seen that there exists $n_0\in\mathbb{N}$ such that for any $n\geq n_0$ there is some function $\tilde{\eta}_n$ such that $r_n(j,k)\leq\tilde{\eta}_n(|k-j|)$ for almost every $\omega_2\in\Omega_2$ and the limit of the series

$$\lim_{n\to\infty}\sum_{k=1}^{n}\tilde{\eta}_n(k)^m =: \lambda_m$$

exists for any $m\geq 2$. Then we have to show

$$\frac{1}{n^2}\sum_{i,j,k,l=2}^{n}\tilde{\eta}_n(|i-j|)^{\kappa}\tilde{\eta}_n(|k-l|)^{\kappa}\tilde{\eta}_n(|i-k|)^{m-\kappa}\tilde{\eta}_n(|j-l|)^{m-\kappa}\to 0 \quad \text{as } n\to\infty,$$

which is equivalent to

$$E_n := \frac{1}{n}\sum_{i,j,k=2}^{n}\tilde{\eta}_n(|i-j|)^{\kappa}\tilde{\eta}_n(k)^{\kappa}\tilde{\eta}_n(|i-k|)^{m-\kappa}\tilde{\eta}_n(k)^{m-\kappa}\to 0 \quad \text{as } n\to\infty.$$

By Hölder's inequality it holds

$$E_n \leq \left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{k=1}^{n}\tilde{\eta}_n(|i-k|)^{m-\kappa}\tilde{\eta}_n(k)^{\kappa}\right)^2\right)^{\frac{1}{2}}$$
$$\left(\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\tilde{\eta}_n(|i-j|)^{\kappa}\tilde{\eta}_n(j)^{m-\kappa}\right)^2\right)^{\frac{1}{2}} =: U_nW_n.$$

Both factors can be treated similarly. Let $\varepsilon>0$ and let $a,b\geq 1$ be two integers. By using again Hölder's inequality we have the following three bounds for W_n:

$$\begin{aligned}W_{1,n}(a,b) :=& \frac{1}{n}\sum_{i=1}^{\lceil n\varepsilon\rceil}\left(\sum_{j=1}^{n}\tilde{\eta}_n(|i-j|)^{a}\tilde{\eta}_n(j)^{b}\right)^2\\ \leq& \frac{1}{n}\sum_{i=1}^{\lceil n\varepsilon\rceil}\sum_{j=1}^{n}\tilde{\eta}_n(|i-j|)^{2a}\sum_{j=1}^{n}\tilde{\eta}_n(j)^{2b}\\ \leq& 2\frac{1}{n}(\lceil n\varepsilon\rceil+1)\sum_{j=1}^{n}\tilde{\eta}_n(j)^{2a}\sum_{j=1}^{n}\tilde{\eta}_n(j)^{2b}\to 2\varepsilon\lambda_{2a}\lambda_{2b},\end{aligned}$$

as $n \to \infty$,

$$\begin{aligned} W_{2,n}(a,b) :=& \frac{1}{n} \sum_{i=\lceil n\varepsilon \rceil+1}^{n} \left(\sum_{j=1}^{\lceil \frac{n\varepsilon}{2} \rceil} \tilde{\eta}_n(|i-j|)^a \tilde{\eta}_n(j)^b \right)^2 \\ \leq& \frac{1}{n} \sum_{i=\lceil n\varepsilon \rceil+1}^{n} \sum_{j=1}^{\lceil \frac{n\varepsilon}{2} \rceil} \tilde{\eta}_n(|i-j|)^{2a} \sum_{j=1}^{\lceil \frac{n\varepsilon}{2} \rceil} \tilde{\eta}_n(j)^{2b} \\ \leq& 2\frac{1}{n}(n - \lceil n\varepsilon \rceil) \sum_{k=\lceil \frac{n\varepsilon}{2} \rceil}^{n} \tilde{\eta}_n(k)^{2a} \sum_{j=1}^{\lceil \frac{n\varepsilon}{2} \rceil} \tilde{\eta}_n(j)^{2b} \to 0, \end{aligned}$$

as $n \to \infty$, and

$$\begin{aligned} W_{3,n}(a,b) :=& \frac{1}{n} \sum_{i=\lceil n\varepsilon \rceil+1}^{n} \left(\sum_{j=\lceil \frac{n\varepsilon}{2} \rceil+1}^{n} \tilde{\eta}_n(|i-j|)^a \tilde{\eta}_n(j)^b \right)^2 \\ \leq& \frac{1}{n} \sum_{i=\lceil n\varepsilon \rceil+1}^{n} \sum_{j=\lceil \frac{n\varepsilon}{2} \rceil+1}^{n} \tilde{\eta}_n(|i-j|)^{2a} \sum_{j=\lceil \frac{n\varepsilon}{2} \rceil+1}^{n} \tilde{\eta}_n(j)^{2b} \\ \leq& 2\frac{1}{n}(n - \lceil n\varepsilon \rceil) \sum_{k=1}^{n-\lceil n\varepsilon \rceil-1} \tilde{\eta}_n(k)^{2a} \sum_{j=1}^{\lceil \frac{n\varepsilon}{2} \rceil} \tilde{\eta}_n(j)^{2b} \to 0, \end{aligned}$$

as $n \to \infty$. Hence

$$\begin{aligned} &\limsup_{n\to\infty} W_n^2 \\ \leq& \limsup_{n\to\infty} (W_{1,n}(\kappa, m-\kappa) + 2W_{2,n}(\kappa, m-\kappa) + 2W_{3,n}(\kappa, m-\kappa)) \\ \leq& 2\varepsilon\lambda_{2\kappa}\lambda_{2(m-\kappa)} \end{aligned}$$

and, since $\varepsilon > 0$ is arbitrary it holds $W_n \to 0$ as $n \to \infty$. Similarly, $U_n \to 0$ as $n \to \infty$ which implies the convergence in Equation (3.20). This finishes the proof of Corollary 3.6.

Bibliography

[BCI04] A. Benassi, S. Cohen, and J. Istas. On roughness indices for fractional fields. *Bernoulli*, 10(2):357–373, 2004.

[BNCP09] O. E. Barndorff-Nielsen, J. M. Corcuera, and M. Podolskij. Power variation for Gaussian processes with stationary increments. *Stochastic Processes and Their Applications*, 119:1845–1865, 2009.

[BNCPW09] O.E. Barndorff-Nielsen, J.M. Corcuera, M. Podolskij, and J.H.C. Woerner. Bipower variation for gaussian processes with stationary increments. *Journal of Applied Probability*, 46(1):132–150, 2009.

[BNGJ+06] O. E. Barndorff-Nielsen, S. E. Graversen, J. Jacod, M. Podolskij, and N. Shephard. A central limit theorem for realised power and bipower variations of continuous semimartingales. In *From Stochastic Calculus to Mathematical Finance*, pages 33–68, 2006.

[BNS02] O. E. Barndorff-Nielsen and N. Shephard. Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *J. Roy. Statist. Soc. Ser. B*, 64(2):253–280, 2002.

[BNS03] O. E. Barndorff-Nielsen and N. Shephard. Realized power variation and stochastic volatility models. *Bernoulli*, 9(2):243–265, 2003.

[BNS04a] O. E. Barndorff-Nielsen and N. Shephard. Econometric analysis of realized covariation: high frequency based covariance, regression, and correlation in financial economics. *Econometrica*, 72(3):885–925, 2004.

[BNS04b] O. E. Barndorff-Nielsen and N. Shephard. Power and Bipower Variation with Stochastic Volatility and Jumps. *J. Fin. Econometrics*, 2(1):1–48, 2004.

[Boc49] S. Bochner. Diffusion equation and stochastic processes. *Proc. Nat. Acad. Sci. USA*, 35(7):368–370, July 1949.

[Boc55] S. Bochner. *Harmonic Analysis and the Theory of Probability*. California Monographs in mathematical sciences. University of California Press, Berkeley and Los Angeles, 1955.

[CNW06] J.M. Corcuera, D. Nualart, and J.H.C. Woerner. Power variation of some integral fractional processes. *Bernoulli*, 12(4):713–735, 2006.

[EW13] S. Engelke and J.H.C. Woerner. A unifying approach to fractional Lévy processes. *Stochastics and Dynamics*, 13(4), 2013.

[GI15] J.M. Gairing and P. Imkeller. Stable CLTs and Rates for Power Variation of α-Stable Lévy Processes. *Methodology and Computing in Applied Probability*, 17(1):73–90, 2015.

[Gla14] S. Glaser. A law of large numbers for the power variation of fractional Lévy processes. *Stochastic Analysis and Applications*, 2014.

[HN05] Y. Hu and D. Nualart. Renormalized self-intersection local time for Brownian Motion. *Annals of Probability*, 33:948–983, 2005.

[Jac08] J. Jacod. Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Processes and their Applications*, 118(4):517 – 559, 2008.

[Kal10] O. Kallenberg. *Foundations of Modern Probability*. Probability and Its Applications. Springer, 2010.

[KL10] R. Kiesel and M. Lutz. *Efficient Pricing of CMS Spread Options in a Stochastic Volatility LMM*. Available at http://dx.doi.org/10.2139/ssrn.1418571 or SSRN: http://ssrn.com/abstract=1418571, 2010.

[KM91] N. Kôno and M. Maejima. Hölder continuity of sample paths of some self-similar stable processes. *Tokyo Journal of Mathematics*, 14(1):93–100, 06 1991.

[Mar06] T. Marquard. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli*, 12(6):1099–1126, 2006.

[MN14] R. Malukas and R. Norvaiša. A central limit theorem for a weighted power variation of a Gaussian process. *Lithuanian Mathematical Journal*, 54(3):323–344, 2014.

[Nua95] D. Nualart. *The Malliavin Calculus and Related Topics*. Probability and its applications : a series of the applied probability. Springer-Verlag, 1995.

[Pod14] M. Podolskij. *Ambit fields: survey and new challenges*. Available at http://www.podolskij.uni-hd.de/files/p2014.pdf, 2014.

[RR89] B.S. Rajput and J. Rosinski. Spectral representations of infinitely divisible processes. *Probability Theory and Related Fields*, 82:451–487, 1989.

[Sat99] K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1999.

[ST00] G. Samorodnitsky and S. Taqqu. *Stable non-Gaussian random processes*. Stochastic Modeling. Chapman and Hall/CRC, 2000.

[Woe05] J.H.C. Woerner. Estimation of integrated volatility in stochastic volatility models. *Applied Stochastic Models in Business and Industry*, 21:27–44, 2005.

[You36] L.C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Mathematica*, 67(1):251–282, 1936.