Characterizations of limit laws of residual life time distributions by generalizations of the lack of memory property

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Table of Contents

Introduction

It is known that, residual life time (R.L.T.) distributions, and their behavior for $t \to \infty$ play an important role in probability theory, such as, survival analysis, renewal theory, and queueing processes. The starting work was Balkema and De.Haan [1], that initiated investigations into the asymptotic behavior of R.L.T. (for $t \to \infty$) on the real line. They derived the types of possible limit distribution and their domain of attraction. See also, the monograph P. Embrechts, C. Klueppelberg, T. Mikosch [6].

In a second work Balkema and Yong-Cheng Qi [2] investigated limit laws in the multivariate setting and introduced concepts of stability and strict stability of R.L.T. distribution in the multivariate setup. For recent investigations for $\mathbb{R}^2$ see M.V.Wuethrich [27].

The (semi-) stability of R.L.T. (for $d = 1$) are characterized by the general lack of memory property (G.L.M.P). On the other hand, the G.L.M.P turns out to be a functional equation which we call "(semi-) stability" functional equation. It is found that, the limit laws of R.L.T. distributions fulfil also these functional equation. (The R.L.T. semi-stable laws are slightly different from the class of discrete limit laws in [1]). We obtained all possible R.L.T. limit distribution by obtaining the general solutions of these functional equation (in the continuous and discrete cases). For $\mathbb{R}^d$, $d > 1$, there exist various generalizations of the L.M.P.
Most of these are not suitable for investigations of R.L.T. distributions in the multivariate case. We introduce a concept of (semi-) stability for R.L.T. distributions which differs from the approach [2] mentioned above.

The paper is organized as follows:
Beginning with preparations where results of the structure of affine transformations and convergence of types theorem are collected. Chapter 1 is concerned with the one-dimensional setup. Following the ideas in [1] on the one hand, and having in mind the theory of stable and semi-stable laws (in the usual sense) it is proved that the possible limit laws satisfy a " stability" functional equation. These laws are called R.L.T. stable resp. R.L.T. semi-stable. Parallel to the investigations of (semi-) stable laws (in the "additive scheme", i.e. in the usual sense) the R.L.T. (semi-) stability is characterized by the "decomposability semigroup" and by the existence of domains of

attraction. Finally, in 1.8 and 1.9 we emphasis the similarities between R.L.T. (semi-) stability, (semi-) stability in the "additive scheme", and in the "max-scheme".

Chapter 2 is concerned with analogous investigations for the multivariate case $\mathbb{R}^d$, $d \geq 2$. The investigations are parallel to Chapter 1 however the results are less complete.
Following [2] the admissible normalizing operators are now "$\mathrm{CAT}'s$" (coordinate wise affine transformations). The structure of one-parameter groups of "$\mathrm{CAT}'s$" is more complicated than for $d = 1$, hence the possible solutions of a "stability functional equation" (for R.L.T.) have – in the general case – no simple representations. Due to the fact that we have no complete overview over the possible solutions of the functional equation the R.L.T. (semi-) stable laws are characterized only in the special cases where the underlying one-parameter group of "$\mathrm{CAT}'s$" is a group of shifts or has a unique fixed point. Again R.L.T. (semi-) stable laws are characterized by their decomposability semigroup and by the existence of domains of attraction. (It should be noted that (for $d \geq 2$)) the investigations [2] and the recently published [27] lead to different generalizations).

Finally, as for $d = 1$ some connections between R.L.T. (semi-) stability and max (semi-) stability are pointed out. These parts are only sketched because (for $d > 1$) the set of max (semi) stable laws is much more complicated as in the one dimensional situation, (and only few investigations are available) where due to the famous result of Gnedenko only three types of max-stable laws exist.

Chapter 0

Preparation

0.1 The structure of affine transformations on a finite dimensional vector space

In this section we reformulate results from Edelstein, Tan [5] for the special case of finite dimensional vector spaces. Since for $\dim \mathbb{V} < \infty$ strong and weak convergence coincide the results and proofs in [5] are simplified. Therefore we reformulate some results and include sketches of proofs. We start with the following notations.

Definition 0.1.1. *(Affine transformation): Let $\mathbb{V} = \mathbb{R}^d$ be a finite dimensional vector space, and T be of $\mathbb{V}$ defined by*

$$T : x \mapsto T(x) = A \cdot x + b \ \ \forall\, x \in \mathbb{V}, \tag{0.1.1}$$

$A \in \mathrm{End}(\mathbb{V})$, $b \in \mathbb{V}$. *Then T is said to be an affine transformation of* $\mathbb{V}$.

Notation 0.1.2. Let $\mathcal{A}(\mathbb{V})$ denote the semigroup of affine transformations and $\mathrm{Aff}(\mathbb{V}) \subseteq \mathcal{A}(\mathbb{V})$ denote the subgroup of invertible transformations.

Note that $T : x \mapsto A \cdot x + b$ invertible iff $A \in \mathrm{GL}(\mathbb{V})$. Later we shall restrict the consideration to subgroups of $\mathrm{Aff}(\mathbb{V})$.
In particular for $d = 1$ let

$$\mathrm{Aff}_0(\mathbb{R}, 1) := \{x \mapsto A \cdot x + b : A > 0\} \tag{0.1.2}$$

Theorem 0.1.3. Let $\mathbb{V} \equiv \mathbb{R}^d$, let $T : x \mapsto A \cdot x + b$, $b \in \mathbb{V}$ be affine transformation defined as in definition 0.1.1. If $b \notin (A - I)(\mathbb{V})$ then $\{T^n\}_{n \geq 1}$ has no bounded subsequence.

Proof. There exist $\mathbb{V} \ni y \perp (A-I)(\mathbb{V})$ such that $\langle y, b\rangle = 1$ (since $b \notin (A-I)(\mathbb{V})$ i.e. $\langle y, b\rangle \neq 0$). According to Edelstein, Tan [5] lemma(1)we have

$$T^n x = (A-I)(\sum_{k=0}^{n-1} T^k x) + x + n \cdot b \tag{0.1.3}$$

Therefore

$$\langle y, T^n x\rangle = \langle y, (A-I)(\sum_{k=0}^{n-1} T^k x)\rangle + \langle y, x\rangle + n \cdot \langle y, b\rangle \tag{0.1.4}$$

Since $(A-I)(\sum_{0}^{k-1} T^k x) \subseteq (A-I)(\mathbb{V})$ hence $\langle y, (A-I)(\sum_{k=0}^{n-1} T^k x)\rangle = 0$, then we have

$$\langle y, T^n x\rangle =: c + n$$

with $c = \langle y, x\rangle$. Hence $\langle y, T^n x\rangle$ has no bounded subsequence, and the assertion follows.(See [5] theorem 2.2) $\square$

Theorem 0.1.4. Let $\mathbb{V} = \mathbb{R}^d$, let T be an affine transformation as before. Then the following are equivalent

(i) There exists a fixed point $x_\star$ (i.e. $Tx_\star = x_\star$).

(ii) For some $x \in \mathbb{V}$, $\{T^n x\}$ contains a bounded subsequence.

(iii) For some $x \in \mathbb{V}$, $\{T^n x\}$ is bounded.

(iv) For some $x \in \mathbb{V}$, $\{T^n x\}$ is convergent.

(v) For some $x \in \mathbb{V}$, $\{T^n x\}$ contains a convergent subsequence.

Proof. $(i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (ii)$, $(i) \Rightarrow (iii) \Rightarrow (ii)$ are obvious.
$(ii) \Rightarrow (i)$: From (ii), and according to theorem 0.1.3 we have $b \in (A-I)(\mathbb{V})$. Let $x_\star \in \mathbb{V}$: $(A-I)x_\star = -b \iff Ax_\star = x_\star - b$. Then we have

$$Tx_\star = Ax_\star + b = (x_\star - b) + b = x_\star$$

hence $x_\star \in \mathrm{Fix}(T)$. (See e.g. [5] theorem 2.3) $\square$

Theorem 0.1.5. Let $\mathbb{V} = \mathbb{R}^d$ as in theorem 0.1.4. Let $\mathcal{S} \subseteq \mathcal{A}(\mathbb{V})$ be a commutative semigroup. If for some x_0 the orbit

$$\mathcal{S}x_0 = \{Tx_0 : T \in \mathcal{S}\}$$

is bounded then $\mathcal{S}$ has a common fixed point $x_\star$.

Proof. $\mathcal{S}x_0 = \{Tx_0 : T \in \mathcal{S}\}$, is bounded, hence relatively compact. Then the closed convex hull $\overline{\mathrm{co}}\,(\mathcal{S}x_0) := K$ is compact

Claim : $T(K) \subseteq K, \forall\, T \in \mathcal{S}$.

T is affine $\Longrightarrow T(\mathrm{co}(\mathcal{S}x_0)) \subseteq \mathrm{co}(\mathcal{S}x_0)$. Since $\mathbb{V}$ is finite-dimensional, T is continuous therefore $T(K) \subseteq K$ follows. Hence $\mathcal{S}$ is a commutative semigroup of continuous affine transformation acting on a compact convex set. Therefore, by $Kakuntani's$ fixed point theorem see e.g. Rudin [**?**], there exist a common fixed point $x_\star$ for $\mathcal{S}$ □

Corollary 0.1.6. Let $\mathbb{V} = \mathbb{R}^d$, let T be affine. If $\{T^n x_0\}_{n\in\mathbb{N}}$ is bounded or has at least a bounded subsequence for some $x_0 \in \mathbb{V}$, then $\{T^n\}_{n\in\mathbb{N}}$, has a common fixed point $x_\star$, i.e. $T^n x_\star = x_\star$ for all n $\in \mathbb{N}$

Proof. Apply theorem 0.1.5 to the semigroup $\{T^n\}_{n\in\mathbb{N}}$ □

Remark 0.1.7. By induction we obtain

a) For $T : x \mapsto A \cdot x + b$, if $A = A_T$, $b = b_T$ we have

$$T^n(x) = A^n \cdot x + \sum_{k=0}^{n-1} A^k \cdot b$$

with $A^0 := I$ hence $A_{T^n} = A_T^n$, $b_{T^n} = \sum\limits_{k=0}^{n-1} A^k \cdot b_T$

b) T, and hence $\{T^n\}$, has a fixed point $x_\star$ if

$$Tx_\star = Ax_\star + b = x_\star \Longleftrightarrow (I - A)x_\star = b$$

Hence, given A, and $x_\star$ we obtain

$$Tx = A \cdot x + (I - A) \cdot x_\star = A \cdot (x - x_\star) + x_\star.$$

Therefore by induction we obtain

$$T^n(x) = A^n(x - x_\star) + x_\star$$

Corollary 0.1.8. Let $\{T_t\}_{t>0}$, be a continuous one parameter semigroup $\subseteq \mathrm{Aff}(\mathbb{V})$, with $T_tT_s = T_{t+s}$, $T_0 = I$ (additive parameterization). Hence $\{T_t\}_{t>0}$ is extendable to a group $\{T_s\}_{s\in\mathbb{R}}$. If $\{T_t x_0\}_{t>0}$, is bounded for some x_0 then $\{T_t\}_{t>0}$ has a common fixed point $x_\star$

Remark 0.1.9. We have in corollary 0.1.8

$$T_t x = A_t x + b_t. \tag{0.1.5}$$

If $T_t x_\star = x_\star$ we obtain for all $t > 0$ $A_t x_\star + b_t = x_\star$. Hence as in the discrete case $T_t x = A_t x + b_t = A_t x + (I - A_t)x_\star$. Therefore we have

$$T_t x = A_t(x - x_\star) + x_\star. \tag{0.1.6}$$

The semigroup property (additive parameterization)

$$T_{t+s} = T_t T_s \ \text{ for } \ t, s \geq 0$$

and continuity yield that

$$A_t(A_s x + b_s) + b_t = A_t A_s x + A_t b_s + b_t = A_{t+s} x + b_{t+s}.$$

Hence $A_{t+s} = A_t A_s \Rightarrow A_t = e^{tQ}$ for some linear Q. Therefore

$$T_t(x) = e^{tQ}(x - x_\star) + x_\star \tag{0.1.7}$$

and $b_t = (\mathrm{I} - e^{tQ})x_\star$

Remark 0.1.10. We can switch from the additive parameterization to the multiplicative parameterization by defining
$\widetilde{T}_u := T_{\log u}$ for $t = \log u$ (resp. $u = e^t,\ u \geq 1$). Then we have

$$T_{t+s} = T_t T_s \ \text{ for } t, s \geq 0 \quad \Longleftrightarrow \quad \widetilde{T}_{uv} = \widetilde{T}_u \widetilde{T}_v \ \text{ for } u, v \geq 1 \tag{0.1.8}$$

Remark 0.1.11. Let $T : x \mapsto Ax + b$ be affine as above

(a) Put $x = 0$: If $\{T^n(0) = \sum\limits_{k=0}^{n-1} A^k \cdot b\}$ is bounded, then T has a fixed point

(b) Assume that A^{-1} exists (i.e. $T \in \mathrm{Aff}(\mathbb{V})$). If $\{T^{-n}(0) = -\sum\limits_{k=0}^{n-1} A^{-k} A^{-1} \cdot b = -\sum\limits_{k=0}^{n} A^{-(k+1)} \cdot b\}$ is bounded, then T has a fixed point

The special case $d = 1$: We have $T : x \mapsto Ax + b$, $A > 0$

Proposition 0.1.12. (a)
- If $0 < A < 1$: Then $\{T^n(0) = \sum\limits_{k=0}^{n-1} A^k \cdot b\}$ is convergent, hence bounded. Hence T has a fixed point.
- If $A > 1$: Then $\{T^{-n}(0) = -\sum\limits_{k=0}^{n-1} \frac{1}{A^{k+1}} \cdot b\}$ is bounded, hence T^{-1} has a fixed point. We obtain : $T : x \mapsto Ax + b$ has a fixed point iff $A \neq 1$. Then $x_\star = \frac{b}{1-A}$.

- Assume $b \neq 0$. If $A = 1$: Then $T : x \mapsto x + b$, has no fixed point. We always assume $T(\mathbb{R}_+) \subseteq \mathbb{R}_+$. Hence $b \geq 0, A > 0$. Therefore $x_\star \geq 0$ iff $0 < A < 1$, and $x_\star \leq 0$ iff $A > 1$
- The case $x_\star = 0$ appears iff $b = 0$, i.e. iff T is linear, $Tx = A \cdot x$

(b) (Continuous one-parameter groups for $d = 1$): Let $T_t : x \mapsto A(t)x + b(t)$ be a continuous one-parameter group in $\mathrm{Aff}(\mathbb{R})$ then $A(t) = e^{tQ}$ for some real Q. (T_t) has a fixed point $x_\star \iff T_t(x) = e^{tQ} \cdot (x - x_\star) + x_\star$. This is the case iff $Q \neq 0$, (i.e. $e^{tQ} \neq 1$).

Proof. This follows immediately if we consider the discrete sub-semigroups $(T_{t_0 \cdot n})_{n \geq 1}$ for some $t_0 > 0$. As in the discrete case we obtain assuming $T_t(\mathbb{R}_+) \subseteq \mathbb{R}_+$:

$$\begin{aligned}
x_\star \geq 0 &\iff A(t) < 1 \,(\text{i.e.} \iff Q < 0), \\
x_\star \leq 0 &\iff A(t) > 1 \,(\text{i.e.} \iff Q > 0), \text{ and} \\
x_\star = 0 &\iff b(t) = 0 \;\; (\text{i.e.} \iff T_t(x) = e^{tQ} \cdot x).
\end{aligned}$$

□

Lemma 0.1.13. $d = 1$. Let $\gamma \in \mathrm{Aff}_0(\mathbb{R})$ and $c > 0$, $c \neq 1$. Then there exist a continuous one parameter group $(T_t)_{t>0}$ with multiplicative parameterization, such that $T_c = \gamma$.

Proof. Put $\gamma : x \mapsto a \cdot x + b$.

Case 1: If $a = 1$. Put $T_t : x \mapsto x + \log(t) \cdot \frac{b}{\log c}$. Then, obviously $T_{ts}(x) = x + \frac{b}{\log c}(\log(t) + \log(s)) = T_t(T_s(x)) = T_t(\frac{b}{\log c} \cdot s + x) = x + \frac{b}{\log c} \cdot s + \frac{b}{\log c} \cdot t$. Furthermore, $T_c(x) = x + b = \gamma$.

Case 2: If $a > 0$, $a \neq 1$. Then, according to proposition 0.1.12 above there exist a fixed point $x_\star$ and $\gamma(x) = a \cdot (x - x_\star) + x_\star$. Then we define $T_t : x \mapsto a^t(x - x_\star) + x_\star$, and obtain $T_{\log_a(c)} = \gamma$.
And then using remark 0.1.10 to verify the existence of a group $(\widetilde{T_t})$ with multiplicative parameterization.

□

Lemma 0.1.14. Let $\gamma \in \mathrm{Aff}_0(\mathbb{R})$. Then there exist a one-parameter group $(T_t)_{t>0}$ (with additive parameterization), such that $T_1 = \gamma$

Proof. As above let $\gamma : x \mapsto a \cdot x + b$. Then we have

Case 1: Assume that $a = 1$. Put $\gamma(t) : x \mapsto x + b \cdot t$. Therefore, again as in lemma 0.1.13 (case 1) $T_{t+s} = T_t \cdot T_s$ (additive parameterization). Furthermore $T = \gamma$

Case 2: Assume $a \neq 1$. Then according to proposition 0.1.12 there exists a fixed point $x_\star$ with $\gamma(x_\star) = x_\star$, $\gamma(x) = a \cdot (x - x_\star) + x_\star$.
Put $T_t : x \mapsto a^t(x - x_\star) + x_\star$. We have $T_1 = \gamma$.

□

Remark 0.1.15. Let γ be as above, $c > 0$. Then there exists a one-parameter group (with additive parameterization) (T'_t) with $T'_c = \gamma$

Proof. Put $T'_t = T_{t/c}$. Later we use this notation in section 1.7 to obtain some reformulations. □

Remark 0.1.16. In the sequel we shall frequently assume that affine transformation γ have the following properties :

(i) γ is strictly increasing,

(ii) $\gamma(x) > 0$ for all $x > 0$,

(iii) $\gamma^n(x) \stackrel{n\to\infty}{\longrightarrow} \infty$ for all $x > 0$.

We denote this set by $\mathrm{Aff}_0^+(\mathbb{R}, 1)$. Let $\mathbb{S}^+ := \{\gamma_{1,b} : b > 0\}$ and

$$\begin{aligned} \mathbb{U}^+ : &= \{\gamma_{a,b} : a > 1, \text{ with fixed point } x_\star \leq 0\} \\ &= \{\gamma_{a,b} : a > 1, b = (1 - a) \cdot x_\star\} \end{aligned}$$

As easily seen, the semigroup generated by $\mathbb{S}^+$ and $\mathbb{U}^+$ is Aff_0^+, whence we obtain

$$\begin{aligned} \mathrm{Aff}_0^+(\mathbb{R}, 1) &= \{\gamma_{a,b} : a \geq 1,\ b > 0 \text{ or } a > 1,\ b \geq 0\} \\ &= \{\gamma_{a,b} : (a, b) \in [1, \infty) \times [0, \infty) \setminus \{(1, 0)\}\} \\ &= \{\gamma = \gamma_{a,b} :\ (i),\ (ii) \text{ and } (iii) \text{ hold}\} \end{aligned}$$

0.2 Convergence of types theorems (C.T.T.)

The central object of this section is the convergence of types theorem which connects weak convergence in the set of probabilities $M^1(\mathbb{R})$ and convergence of affine transformation of $\mathbb{R}_+$. Note that non- degenerate distribution means that, the distribution is not a point measure. Convergence of types is a simple but powerful limit theorem that is useful in many branches of probability theory (see e.g. Hazod [14]). In this section, we present the one dimension version.

Proposition 0.2.1. The classic version of C.T.T. for $d = \dim \mathbb{V} = 1$:
Let μ_n, μ, λ be probability measures on $\mathbb{R}$, μ, λ non-degenerate. Let $\gamma_n \in \mathrm{Aff}_0(\mathbb{R}) : x \mapsto A_n \cdot x + b_n$ with $A_n > 0$. Assume

(1) $\mu_n \longrightarrow \mu$, and

(2) $\gamma_n(\mu_n) \longrightarrow \lambda$.
Then the sequence $\{\gamma_n\}_{n\geq 1}$ is relatively compact in $\mathrm{Aff}_0(\mathbb{R})$. And for all accumulation points γ (i.e. $\gamma_n \xrightarrow{\widetilde{n}} \gamma$ for some subsequence $\widetilde{n}$) we have

(3) $\gamma(\mu) = \lambda$.
Here we will be concerned with probabilities μ_n, μ, λ concentrated on $\mathbb{R}_+$ hence we assume $\gamma_n(\mathbb{R}_+) \subseteq \mathbb{R}_+$, in particular $A_n > 0$ for all n. Therefore we obtain in this particular case:
$\lim\limits_{n\to\infty} \gamma_n =: \gamma$ exists and (3) holds. (See e.g. Letta [17]).

We shall continue with general C.T.T. for $\dim \mathbb{V} > 1$ in Chapter 2. Here we continue with an equivalent form of the C.T.T. in the context of distribution functions.

Proposition 0.2.2. Let F_n, F and G denote the distribution functions of μ_n, μ and λ, (of prop. 0.2.1) respectively. Then (1), (2) and (3) in proposition 0.2.1 are equivalent to the following

$(1)'$ $F_n(x) \longrightarrow F(x)$, for all continuity points of F.

$(2)'$ $F_n(\gamma_n^{-1}(x)) = F_n(\frac{1}{a_n}(x - b_n)) \to G(x)$ for all continuity points of G.

$(3)'$ $F(\gamma^{-1}(x)) = G(x)$ for all x.

Note that F and G are right continuous. Therefore, if the relation $(3)'$ holds true for all continuity points, it holds true for all $x \in \mathbb{R}_+$. In the following we shall use the notation $F_n(\cdot) \xrightarrow{w} F(\cdot)$ iff $F_n(x) \longrightarrow F(x)$ for all continuity points of F.

Proposition 0.2.3. Another useful version of C.T.T. for $(d = 1)$: Let F be a non-degenerate distribution function, let $\alpha_n, \beta_n \in \mathrm{Aff}_0(\mathbb{R})$ be affine transformations of $\mathbb{R}$. Assume in addition that

$$F(\alpha_n(x)) \xrightarrow{w} F(x) \tag{0.2.1}$$

and

$$F(\beta_n(x)) \xrightarrow{w} \widetilde{F}(x) \tag{0.2.2}$$

where F and $\widetilde{F}$ are non-degenerate distribution functions. Then the sequence, $\{\gamma_n\}_{n\geq 1} := \{\beta_n \alpha_n^{-1}\}_{n\geq 1}$ is relatively compact and for all accumulation points γ we have

$$\widetilde{F}(x) = F(\gamma(x)),\ x \in \mathbb{R}. \tag{0.2.3}$$

(In fact, since $\gamma_n, \beta_n \in \mathrm{Aff}_0(\mathbb{R})$ we obtain $\gamma_n \to \gamma$)

Proof. The proof is an obvious consequence of proposition 0.2.2 if we put $F_n := F \circ \alpha_n$. Then we have $F_n(x) \to F(x)$, and hence $F_n \circ \gamma_n(x) \to \widetilde{F}(x)$. $\square$

For the next version of the C.T.T. we first note

Lemma 0.2.4. Let $c_n \to \infty$, $0 < x_n < 1$. Then

$$\begin{aligned} c_n \cdot (1 - x_n) \to y \quad &\Longleftrightarrow \quad c_n \cdot (x_n - 1) \to -y \\ &\Longleftrightarrow \quad e^{c_n \cdot (x_n - 1)} \to e^{-y} \\ &\Longleftrightarrow \quad x_n^{c_n} \to e^{-y} \end{aligned}$$

Proof. Without loss of generality we take $c_n \in \mathbb{N}$, otherwise we take $[c_n]$. Then

$$\begin{aligned} \mid e^{c_n \cdot (x_n - 1)} - x_n^{c_n} \mid &= \mid (e^{x_n - 1})^{c_n} - x_n^{c_n} \mid \\ &\leq \sum_{n=1}^{c_n} \mid e^{x_n - 1} - x_n \mid \quad (\text{since } \mid x_n \mid \leq 1 \Longrightarrow \mid e^{x_n - 1} \mid \leq 1) \\ &\leq c_n \cdot \mid x_n - 1 \mid . \underbrace{\mid x_n - 1 \mid}_{\to 0} \cdot \sum_{k=2}^{\infty} \frac{(x_n - 1)^k}{k!} \overset{n \to \infty}{\longrightarrow} 0 \end{aligned}$$

$\square$

This yields immediately the following new version of the C.T.T. (See e.g. Balkema, de Haan [1]).

Proposition 0.2.5. Let $S, \widetilde{S}$ be decreasing real functions, $c_n \to \infty$, γ_n, β_n, as in proposition 0.2.3. Let F be a distribution function and assume that
$c_n \cdot (1 - F(\gamma_n(x)) \xrightarrow{w} S(x)$, and
$c_n \cdot (1 - F(\beta_n(x))) \xrightarrow{w} \widetilde{S}(x)$, when S, $\widetilde{S} \geq 0$, are non-degenerate functions. Then

$$\exp(c_n \cdot (F(\gamma_n(x))) - 1) \xrightarrow{w} \exp(-S(x)) \Longleftrightarrow F^{c_n}(\gamma_n(x)) \xrightarrow{w} \exp(-S(x)) \quad (0.2.4)$$

and

$$\exp(c_n \cdot (F(\beta_n(x))) - 1) \xrightarrow{w} \exp(-\widetilde{S}(x)) \Longleftrightarrow F^{c_n}(\beta_n(x)) \xrightarrow{w} \exp(-\widetilde{S}(x)) \quad (0.2.5)$$

(Where $\xrightarrow{w}$ means weak convergence as above mentioned.)

Proposition 0.2.6. Let S, $\widetilde{S}$, β_n, γ_n, and c_n, be as in proposition 0.2.5. Assume that

$$c_n \cdot (1 - F(\gamma_n(x)) \xrightarrow{w} S(x) \quad (0.2.6)$$

and

$$c_n \cdot (1 - F(\beta_n(x))) \xrightarrow{w} \widetilde{S}(x) \quad (0.2.7)$$

$x \geq x_0$, for some $x_0 \geq 0$. Then $\{\Gamma_n = \beta_n \gamma_n^{-1}\}_{n \geq 1}$ is relatively compact, and for all accumulation points Γ we have $\widetilde{S} = S \circ \Gamma$. (In fact, we have $\Gamma_n \longrightarrow \Gamma$).

0.3 The set of the probability measures $M^1(\mathbb{R}_+)$

Notation 0.3.1. Let $(\Omega,\ \Sigma,\ P)$ be a probability space, let $X_n,\ X : \Omega \to \mathbb{R}$ be a real random variables, with distributions $\mu_n,\ \mu$ (resp. distribution functions $F_n,\ F$, and the tails functions $R_n,\ R$) respectively, where
$F(x) := P(X \leq x)$ and $R = 1 - F \quad (R(x) := P(X > x))$, and let Y be the set of all continuity points of F (resp. R) for which $F(x) < 1$ (resp. R(x) > 0). Then we have

$$\mu_n \xrightarrow{w} \mu \quad (\mathrm{resp.\ F_n} \xrightarrow{\mathrm{w}} \mathrm{F}) \iff F_n(x) \to F(x)\ \forall\ x \in Y.$$

Equivalently formulated for the tail R as:

$$R_n \overset{w}{\to} R \iff R_n(x) \to R(x)\ \forall\ x \in Y.$$

Definition 0.3.2. *We define the R.L.T. distribution for* $t,\ x \geq 0$ *by:*

$$F_t(x) := P(X \leq x + t | X > t). \tag{0.3.1}$$

Hence if $F(t) < 1$ *we have*

$$F_t(x) = (F(x+t) - F(t))/(1 - F(t)) = \mu(t, x+t]/\mu(t, \infty). \tag{0.3.2}$$

Analogously, if $R(t) > 0$ *we define the corresponding tail function by*

$$R_t(x) := P(X > x + t | X > t) = \mu(x+t, \infty)/\mu(t, \infty). \tag{0.3.3}$$

Which is expressed analytically by:

$$R_t(x) := \min(1, R(x+t)/R(t)). \tag{0.3.4}$$

Remark 0.3.3. R.L.T. distributions may be defined by transformations acting on the set of probabilities $M^1(\mathbb{R}_+)$ as:
If $\mu(t, \infty) > 0$ define $\tau_t : M^1(\mathbb{R}_+) \to M^1(\mathbb{R}_+)$ by

$$\tau_t(\mu)(0, x] := \mu(t, x+t]/\mu(t, \infty). \tag{0.3.5}$$

(I.e. the distribution function of $\tau_t(\mu)$ is F_t if F is the distribution function of μ).
As easily seen we have

$$\tau_t\tau_s(\mu) = \tau_{t+s}(\mu) \tag{0.3.6}$$

hence $(\tau_t)_{t\geq 0}$ is a continuous one parameter semigroup.
Put $\gamma(t) : x \mapsto x + t$. Then the tail $R(t)$ may be written as

$$R_t(x) = R(\gamma(t)(x))/R(t) \tag{0.3.7}$$

Proposition 0.3.4. For later use we note: Let F be a distribution function with tail R. Let $t \mapsto \Gamma(t) : x \mapsto a(t) \cdot x + b(t)$ be a function $\mathbb{R}_+ \to \mathrm{Aff}_0(\mathbb{R})$. Assume that $\Gamma(t)(x) \geq x \; \forall \, x \geq 0$. Define $\widetilde{\Gamma}(t) : x \mapsto a(t) \cdot x + b(t) - t$. Then for all $x, \, t \geq 0$ we have

$$R_t(\widetilde{\Gamma}(t)(x)) = R(x) \iff R(\Gamma(t)(x)) = R(t) \cdot R(x). \tag{0.3.8}$$

Functional equations of this type,

$$R(\Gamma(t)(x)) = R(t) \cdot R(x) \tag{0.3.9}$$

more generally

$$R(\Gamma(t)(x)) = c(t) \cdot R(x) \tag{0.3.10}$$

for some function $c(\cdot)$.

0.4 Examples

Example 0.4.1. *Exponential distribution :* Let $\alpha > 0$.
The distribution function is defined by: $E_\alpha(x) = 1 - e^{-\alpha x}, \; x \geq 0$
And the tail distribution is: $\overline{E}_\alpha(x) = e^{-\alpha x}, \; x \geq 0$

Example 0.4.2. *Shifted exponential distribution :* Fix $x_0 \in \mathbb{R}$.
The distribution function is defined by: $E_{\alpha,\, x_0}(x) = 1 - e^{-\alpha(x - x_0)} = E_\alpha(x - x_0), \; x \geq x_0$.
And the tail distribution is: $\overline{E}_{\alpha, x_0}(x) = e^{-\alpha(x-x_0)} = \overline{E}_\alpha(x - x_0), \; x \geq x_0$.
By re-scaling the example 0.4.2 above we obtain again a shifted exponential distribution as in the following example

Example 0.4.3. *Re-scaled exponential distribution.* Let $\beta > 0$ and put
$E_{\alpha,\, x_0, \beta}(x) := E_{\alpha,\, x_0}(\beta \cdot x), \; x \geq 0$.
Then the distribution function is:

$$E_{\alpha,\, x_0}(\beta \cdot x) = 1 - e^{-\alpha(\beta x - x_0)} = 1 - e^{-\alpha\beta(x - x_0/\beta)} = E_{\alpha\beta, x_0/\beta}(x), \; x \geq 0.$$

And the tail distribution is: $\overline{E}_{\alpha\beta,\, x_0/\beta}(x) = e^{-\alpha\beta(x - x_0/\beta)}$

Example 0.4.4. *Standard Pareto distribution:* Let $\alpha > 0$. Then the distribution function is defined by: $P_{\alpha,1}(x) = 1 - x^{-\alpha}, \; x \geq 1$.
And the tail distribution is: $\overline{P}_{\alpha,1}(x) = x^{-\alpha}, \; x \geq 1$

Example 0.4.5. *Shifted Pareto distribution:* Let $c \in \mathbb{R}$, consider the Pareto distribution shifted by $1 - c$, then the distribution function is defined by:
$P_{\alpha,\, c}(x) = P_{\alpha,1}(x - (1 - c)) = 1 - (x - (1 - c))^{-\alpha}, \; c + x - 1 \geq 1 \Leftrightarrow x \geq 2 - c$.
And the tail distribution is: $\overline{P}_{\alpha,\, c}(x) = (x - (1 - c))^{-\alpha}, \;\; x \geq 2 - c$

Notation 0.4.6. *:* It is may be better in the previous example to replace $1-c$ by x_0, then the distribution function is: $P_{\alpha,\,x_0}(x) = P_{\alpha,\,1}(x-x_0) = 1-(x-x_0)^{-\alpha}$ for $x - x_0 \geq 1$.
And the tail distribution is: $\overline{P}_{\alpha,\,x_0}(x) = (x-x_0)^{-\alpha},\ x - x_0 \geq 1$. Therefore it is better to define the re-scaled Pareto distribution as the following example

Example 0.4.7. *Re-scaled Pareto distribution :* Let $\beta > 0$.
Define $P_{\alpha,\beta,x_0}(x)$ as: $P_{\alpha,\beta,x_0}(x) = P_{\alpha,x_0}(\beta\cdot x)$. Therefore the distribution function is: $P_{\alpha,\,\beta,\,x_0}(x) = 1-(\beta\cdot x - x_0)^{-\alpha},\ \beta\cdot x - x_0 \geq 1 \Leftrightarrow x \geq \frac{1+x_0}{\beta}$
$= 1-\beta^{-\alpha}(x-\frac{x_0}{\beta})^{-\alpha},\ x \geq \frac{1+x_0}{\beta}$.
And the tail distribution is: $\overline{P}_{\alpha,\,\beta,\,x_0}(x) = \beta^{-\alpha}(x-\frac{x_0}{\beta})^{-\alpha},\ x \geq \frac{1+x_0}{\beta}$

Notation 0.4.8. By using the re-scaled Pareto distribution 0.4.7 above. we obtain the following: $P_{\alpha,x_0}(x) = 1-(x+1-x_0)^{-\alpha},\ x - x_0 \geq 0 \iff x \geq x_0$.
Put $x_0 = 0$. Then we obtain the distribution function: $P_{\alpha,0}(x) := 1-(x+1)^{-\alpha}$, $x \geq 0$.
And the tail is: $\overline{P}_{\alpha,\,0}(x) = (1+x)^{-\alpha},\ x \geq 0$
Put $x_0 = 1$. Then the distribution function is $P_{\alpha,1}(x) := P_{\alpha,0}(x-1) = 1-x^{-\alpha},\ x \geq 1$.
And the tail distribution is: $\overline{P}_{\alpha,\,1}(x) = x^{-\alpha},\ x \geq 1$

Example 0.4.9. *Re-Parameterization of the Pareto distribution:* From 0.4.7 we obtained the following $P_{\alpha,\,\beta,\,x_0}(x) = P_{\alpha,\,1}(\beta\cdot x - x_0),\ x \geq \frac{1+x_0}{\beta}$. Then a useful re-parameterization of the continuous Pareto distribution is obtained in the following:
$P_{\alpha,\beta,x_0}(x) = P_{\alpha,1}(\beta\cdot x - x_0) = 1 - e^{-\alpha(\log\beta + \log(x-\frac{x_0}{\beta}))},\ x \geq \frac{1+x_0}{\beta}$.
In particular for $c = 1$ (i.e. $x_0 = 0$) we have
$P_{\alpha,1}(x) \equiv 1-(\beta\cdot x)^{-\alpha} = 1 - e^{-\alpha\log\beta x},\ \beta\cdot x \geq 1$.
And the tail distribution is: $\overline{P}_{\alpha,1}(x) = e^{-\alpha\cdot\log\beta\cdot x},\ \beta\cdot x \geq 1$

Example 0.4.10. *Weibull distribution :*
The distribution function is: $W_{\lambda,\,\alpha}(x) = 1-e^{-\lambda x^{\alpha}},\ \lambda > 0,\ \alpha > 0,\ x > 0$.
And the tail distribution is: $\overline{W}_{\lambda,\,\alpha}(x) = e^{-\lambda x^{\alpha}},\ \lambda > 0,\ \alpha > 0,\ x > 0$.

Example 0.4.11. The following distribution will appear later: Let $\beta > 0$.
The distribution function: $B_\beta(x) = 1-(1-x)_+^{\beta}\ ,\ 0 \leq x \leq 1$
And the tail distribution is: $\overline{B}_\beta(x) = (1-x)_+^{\beta}\ ,\ 0 \leq x \leq 1$

Example 0.4.12. *Geometric distribution:* We have to consider two types of geometric distributions. Let $0 < q < 1$, $p := 1-q$. We put

$(a)\ \ \mu = \mu_q = \sum_{k\geq 0} p\cdot q^k \cdot \varepsilon_k\ ,\ k \in \mathbb{Z}_+$ or

$(a)'$ (Shifted μ_q): $\quad \mu = \xi_q := \sum_{k\geq 0} p\cdot q^k\cdot\varepsilon_{k+1} = \mu_q * \varepsilon_1\ ,\ k \in \mathbb{Z}_+$

Consequently, if we consider the type (a), then
the distribution function is: $G_q(x) = \sum_{k=0}^{[x]} p \cdot q^k \cdot \varepsilon_k = 1 - q^{[x+1]}$, $x \geq 0$.
And the tail distribution is: $\overline{G}_q(x) = q^{[x+1]}$, $x \geq 0$

Example 0.4.13. *Shifted geometric distribution:* Let $x_0 \in \mathbb{R}$ then we have the probability measure is: $\mu_{q,\,x_0} = \sum_{k\geq 0} p \cdot q^k \cdot \varepsilon_{k-x_0}$. Therefore the distribution function is:

$$G_{q,\,x_0} = G_q(x - x_0) = \sum_{k=0}^{[x-x_0]} p \cdot q^k \cdot \varepsilon_{k-x_0} = 1 - q^{[x-x_0+1]}, \; x \geq x_0$$

And the tail is: $\overline{G}_{q,\,x_0} = \overline{G}_q(x - x_0) = q^{[x-x_0+1]}$, $x \geq x_0$

Example 0.4.14. *Re-scaled geometric distribution:* Let p, q be as above, then we have $\mu_{q,\,\tau} = \sum_{k\geq 0} pq^k \varepsilon_{k\tau}$, $\tau > 0$ fixed, $x > 0$.

The distribution function is: $G_{q,\tau}(x) = \sum_{k=0}^{[\frac{x}{\tau}]} pq^k \cdot \varepsilon_{k\tau} = G_q(\frac{x}{\tau}) = 1 - q^{[1+\frac{x}{\tau}]}$, $x > 0$.
And the tail distribution is: $\overline{G}_{q,\,\tau}(x) = q^{[1+\frac{x}{\tau}]}$, $x > 0$

Example 0.4.15. *Shifted re-scaled geometric distribution:* Let τ be fixed. Then we have $\mu_{q,\,\tau,\,x_0} = \sum_{k\geq 0} pq^k \cdot \varepsilon_{k\tau - x_0}$. Then
the distribution function is : $G_{q,\,\tau,\,x_0}(x) = 1 - q^{[\frac{x-x_0}{\tau}]}$, $x \geq x_0$, $\tau > 0$
And the tail distribution is : $\overline{G}_{q,\,\tau,\,x_0}(x) = q^{[\frac{x-x_0}{\tau}]}$, $x \geq x_0$

Example 0.4.16. Some times a re-parameterization of the geometric distribution is useful: Take $\gamma < 0$, $q = e^\gamma$, $(\gamma = \log q)$. Then
the distribution function is: $G_{e^\gamma}(x) = \sum_{k=0}^{[x]} (1 - e^\gamma) \cdot e^{k\gamma} = 1 - e^{-\gamma[x+1]}$.
If we take $\gamma > 0$, $q = e^{-\gamma}$ hence we obtain
the distribution function $G_{e^{-\gamma},\tau}(x) = 1 - e^{-\gamma(1+[\frac{x}{\tau}])}$, $x > 0$.
And the tail distribution is: $\overline{G}_{e^{-\gamma},\,\tau}(x) = e^{-\gamma\cdot(1+[\frac{x}{\tau}])}$, $x > 0$,
and in particular if $\tau = 1$ we have

$$\overline{G}_{e^{-\gamma},\,1}(x) = e^{-\gamma\cdot[1+x]}$$

Example 0.4.17. *Discrete Pareto distribution :* We define a discrete version of 0.4.4 where the function in the exponent is constant between lattice points.
The distribution function is: $DP_{\alpha,\,1}(x) = 1 - e^{-\alpha\cdot[\log x]}$, $x \geq 1$
And the tail: $\overline{DP}_{\alpha,\,1}(x) = e^{-\alpha\cdot[\log x]}$, $x \geq 1$

More generally we have the following Pareto distribution:

Example 0.4.18. The distribution function is:
$DP_{\alpha,\,\beta,\,x_0}(x) = 1 - e^{-\alpha\cdot[\log\beta + \log(x - \frac{x_0}{\beta})]},\ \beta(x - \frac{x_0}{\beta}) \geq 1 \iff x \geq \frac{1+x_0}{\beta}$.
And the tail: $\overline{DP}_{\alpha,\,\beta,\,x_0}(x) = e^{-\alpha\cdot[\log\beta + \log(x - \frac{x_0}{\beta})]}$, $\beta\cdot(x - \frac{x_0}{\beta}) \geq 1 \iff x \geq \frac{1+x_0}{\beta}$.
In particular

Example 0.4.19. *Shifted discrete Pareto distribution :* Let $c \in \mathbb{R}$. Then we have the distribution function is: $DP_{\alpha,\,c}(x) = 1 - \beta^{-\alpha}e^{-\alpha\cdot[\log(x+c)]},\ x + c \geq \beta$.
And the tail is: $\overline{DP}_{\alpha,c}(x) = \beta^{-\alpha}e^{-\alpha\cdot[\log(x+c)]}$

Example 0.4.20. *Re-scaled discrete Pareto distribution.*

The distribution function is: $DP_{\alpha,\,1}(\frac{x}{\beta}) = 1 - e^{-\alpha\cdot[\log\frac{x}{\beta}]}\ ,\ x \geq \beta$.
And the tail is: $\overline{DP}_{\alpha,\,1}(x) = e^{-\alpha\cdot[\log\frac{x}{\beta}]}\ ,\ x \geq \beta$.

Chapter 1

(Semi-) stability of residual life time (R.L.T.) distributions in the one dimensional case

1.1 The lack of the memory property (L.M.P.)

It is convenient to describe the lack of memory property (L.M.P.) of a random variable or of its distribution in terms of residual life time. Firstly, in this situation we know, the classical version of the (L.M.P.) of the exponential distribution, described by Galambos [8]. In this section we formulate the general definition of the L.M.P. for R.L.T. distributions, which will be important in the sequel, to characterize the (semi-) stability of the limit laws, and we begin with the following definition:

The classical case of the lack of memory property

Definition 1.1.1. *We say that the probability measure μ possesses the lack of memory property for R.L.T. if*

$$\mu(x+t,\infty)/\mu(t,\infty) = \mu(x,\infty),\ x \geq 0,\ t > 0,\ \mu(t,\infty) > 0. \tag{1.1.1}$$

Equivalently: Let F be the distribution function, with the tail function R. Then we say that F possesses the lack of memory property if

$$R(x+t)/R(t) = R(x),\ x \geq 0, t \geq 0,\ R(t) > 0 \tag{1.1.2}$$

(*See Galambos*[*8*]).

Definition 1.1.2. *For a distribution function F with the tail R, we define*

$$LM(F) := \{t : \text{ (1.1.2) } holds\} \tag{1.1.3}$$

to be the set of all points t which satisfies the L.M.P. with respect to the distribution function F

Example 1.1.3. In example 0.4.1 we obtain

$$\overline{E}_\alpha(x+t)/\overline{E}_\alpha(t) = e^{-\alpha(x+t)}/e^{-\alpha t} = e^{-\alpha x} = \overline{E}_\alpha(x), \forall x \geq 0.$$

Hence the exponential distribution possesses the L.M.P., that is, $\mathrm{LM}(E_\alpha) = \mathbb{R}_+$.

Example 1.1.4. In example 0.4.2 we obtain

$$\overline{E}_{\alpha,x_0}(x+t)/\overline{E}_{\alpha,x_0}(t) = e^{-\alpha(x-x_0+t)}/e^{-\alpha(t-x_0)} = e^{-\alpha x} \neq \overline{E}_{\alpha,x_0}(x)$$

$\forall\, x \geq x_0,\ \alpha > 0,\ t \geq 0$. Hence the shifted exponential distribution does not possess the L.M.P. And we write

$$\mathrm{LM}(E_{\alpha,x_0}) = \emptyset.$$

On the other hand, these distributions satisfy a similar equation

$$R(\gamma(t)(x)) = c(t) \cdot R(x) \tag{1.1.4}$$

with $R(x) = \overline{E}_{\alpha,x_0}(x)$, for some $\gamma(t) \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$, and for some function $c : \mathbb{R}_+ \longrightarrow [0, 1]$. In fact, for this example. Put $\gamma(t) : x \mapsto x + t$ then

$$R(\gamma(t)(x)) = e^{-\alpha t} \cdot R(x),\ t > 0,\ x \geq x_0$$

Note that, if $R(0) = 1$, then $c(t) = R(\gamma(t)(0))$. This will be important in the sequence.

Example 1.1.5. In example 0.4.4 similarly we obtain that the standard Pareto distribution does not possess the L.M.P. But this distribution satisfies an equation similar to (1.1.4). Examples 0.4.4 - 0.4.20 can be treated in a similar way. For this reason the equation (1.1.4) will be studied in the next sections.

Definition 1.1.6. *Let $\mu \in M^1(\mathbb{R}_+)$ with distribution function F and tail function R. Let $(\gamma(t))_{t\in\mathbb{R}}$ be a continuous one parameter group in $\mathrm{Aff}_0^+(\mathbb{R}, 1)$ (see remark 0.1.16) and $\gamma \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$ with $\gamma^n(x) \to \infty$, as $n \to \infty$ for $x > 0$. Then μ (resp. F resp. R) is called R.L.T. stable w.r.t. $\gamma(\cdot)$ if*

$$\frac{R(\gamma(t)(x))}{R(\gamma(t)(0))} = R(x),\ x \geq 0,\ t \geq 0 \tag{1.1.5}$$

Analogously, μ (resp. F, resp. R) is called R.L.T. semi-stable w.r.t. $\gamma(\cdot)$ if

$$\frac{R(\gamma(x))}{R(\gamma(0))} = R(x),\ x \geq 0 \tag{1.1.6}$$

The general case of the lack of memory property 1.1.6 may be considered as a generalization of the L.M.P:

Definition 1.1.7. *A further generalization of the lack of memory property: We say that the probability μ (resp. the distribution function F) possesses the generalized L.M.P. if there exist a path $\gamma : t \longmapsto \gamma(t) \in \mathrm{Aff}_0^+(\mathbb{R},\ 1)$, $\gamma(t)(x) = a(t) \cdot x + b(t)$ such that $\gamma(t)(x) > x$ that is $x \geq x_0$, $\gamma(t) \nearrow \infty$, and in addition*

$$R(\gamma(t)(x))/R(\gamma(t)(0)) = R(x) \tag{1.1.7}$$

for all $x \geq 0$ and all $t \geq 0$ with $R(\gamma(t)(0)) > 0$. We do not assume that $\gamma(\cdot)$ is a semigroup. Hence equation (1.1.4) holds for some $c(t) > 0$.
If $x = x_0,\ \ R(x_0) > 0$ such that $\gamma(t)(x_0) \overset{t\to\infty}{\longrightarrow} \infty$ then we have

$$R(\gamma(t)(x_0)) = c(t) \cdot R(x_0),\ t > 0 \tag{1.1.8}$$

Remark 1.1.8. For a distribution μ (resp. F) which possesses the generalized L.M.P. (i.e. if (1.1.7) holds), i.e. μ is R.L.T. stable (in short, R.L.T.stable). Then, for a fixed x_0 such that $\gamma(t)(x_0) \overset{t\to\infty}{\longrightarrow} \infty$ we observe:
Put $y := \gamma(t)(x_0)$ in (1.1.8) then we have

$$R(y) = c(t) \cdot R(x_0) = c(t(y)) \cdot R(x_0). \tag{1.1.9}$$

with $t = t(y) = f^{-1}(y)$, where f denotes $f : t \mapsto \gamma(t)(x_0)$. Hence R (resp. F) is uniquely determined by $\gamma(\cdot)$ and $c(\cdot)$

1.2 (Semi-) stable R.L.T. distributions

<u>Stable R.L.T. distributions:</u> We called in Definition 1.1.6 distributions which possess the generalized L.M.P. (1.1.5) R.L.T. stable.

Example 1.2.1. Consider the distribution obtained by using the re-scaled Pareto distribution in example 0.4.7. In particular the case at $x_0 = 0$ in notation 0.4.8. We have

$$\overline{P}_{\alpha,0}(x) = (1+x)^{-\alpha},\ x \geq 0, \alpha > 0 \tag{1.2.1}$$

Claim: The Pareto distribution is R.L.T. stable. Indeed, we have

$$\overline{P}_{\alpha,0}(x+t) = (1+x+t)^{-\alpha}. \tag{1.2.2}$$

At the same time

$$\overline{P}_{\alpha,0}(x) \cdot \overline{P}_{\alpha,0}(t) = (1+x)^{-\alpha} \cdot (1+t)^{-\alpha} = (1+t+x+xt)^{-\alpha}. \tag{1.2.3}$$

Hence we can define affine transformations

$$\Gamma(t) : x \to t + x + xt = (1+t)x + t \tag{1.2.4}$$

with $a(t) = 1 + t, b(t) = t$, then

$$\overline{P}_{\alpha,0}(\Gamma(t)(x)) = \overline{P}_{\alpha,0}(\Gamma(t)(0))\overline{P}_{\alpha,0}(x),\ x \geq 0. \tag{1.2.5}$$

And in the case of $x_0 = 1$ with $x \geq 1$, the same affine $\Gamma(t)(x)$ will be taken to prove the stability condition as above.

Remark 1.2.2. Another view of example 1.2.1: Instead of the relation $\overline{P}_{\alpha,0}(\Gamma(t)(x)) = \overline{P}_{\alpha,0}(\Gamma(t)(0))\overline{P}_{\alpha,0}(x)$ we can write the equivalent form

$$(\overline{P}_{\alpha,0})_{\Gamma(t)(0)}(\widetilde{\Gamma}(t)(x)) = \overline{P}_{\alpha,0}(x) \tag{1.2.6}$$

with $\widetilde{\Gamma}(t)(\cdot) = \Gamma(t)(\cdot) - \Gamma(t)(0)$. Now define the affine mappings

$$\beta(t) : x \mapsto \frac{1}{1+t} \cdot x$$

with $\widetilde{\Gamma}(t) = \beta^{-1}(t)$. Then we observe $\Gamma(t)(0) = t$. Therefore, for the Pareto distributions we have $(\overline{P}_{\alpha,0})_t(\beta^{-1}(t)(x)) = (\overline{P}_{\alpha,0})(x)$ which coincides with

$$(\overline{P}_{\alpha,0})_t(\widetilde{\Gamma}(t)(x)) = \overline{P}_{\alpha,0}(x)$$

Notation 1.2.3. The distribution in the above example does not possess the L.M.P. but it is R.L.T. stable. The other examples have similar properties. Hence, we can say that, the L.M.P. is not suitable to characterize R.L.T. stability. That is the reason why R.L.T. stability is defined. At the same time we obtain a generalization of the stability condition, to cover and characterize all R.L.T. limit distributions. Now similar to Prop. 0.3.4 (in particular for the discrete case distribution) we have:

Proposition 1.2.4. Let Γ_k be affine $\Gamma_k : x \mapsto a_k \cdot x + b_k,\ x \geq 0, a_k > 0$, such that $\Gamma_k \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$ (i.e. $\Gamma_k(x) > x$ for all $x > 0$ and $\Gamma_k(x) \nearrow \infty$) then

$$R(\Gamma_k(x)) = c_k \cdot R(x) \Longleftrightarrow R_{\Gamma_k(0)}(\widetilde{\Gamma}_k(x)) = R(x)$$

where $\widetilde{\Gamma}_k(x) = \Gamma_k(x) - \Gamma_k(0)$

Proof. Define $\widetilde{\Gamma}_k(x) := \Gamma_k(x) - b_k = a_k \cdot x,\ x \geq 0,\ a_k > 0$. Then we have

$$\begin{aligned} R_{\Gamma_k(0)}(\widetilde{\Gamma}_k(x)) = R(x) \quad &\Longleftrightarrow \quad \frac{R(\widetilde{\Gamma}_k(x) + \Gamma_k(0))}{R(\Gamma_k(0))} = R(x) \\ &\Longleftrightarrow \quad \frac{R(\widetilde{\Gamma}_k(x) + b_k)}{R(\Gamma_k(0))} = R(x) \\ &\Longleftrightarrow \quad \frac{R(\Gamma_k(x))}{R(\Gamma_k(0))} = R(x) \\ &\Longleftrightarrow \quad R(\Gamma_k(x)) = R(\Gamma_k(0)) \cdot R(x) \\ &\Longleftrightarrow \quad R(\Gamma_k(x)) = c_k \cdot R(x), \quad c_k := R(\Gamma_k(0)), x \geq 0. \end{aligned}$$

□

(Semi-)stable R.L.T. distributions: We defined in 1.1.6 the R.L.T. semi-stability distributions as: Let μ be a probability distribution, then μ is said to be R.L.T. semi-stable if there exist affine mappings $\Gamma_k : x \longmapsto \Gamma_k(x)$ defined by
$\Gamma_k(x) = a_k \cdot x + b_k = \Gamma^k(x)$ with $\Gamma_k(x) \geq x\ \forall\ x \geq 0,\ k \in \mathbb{N}$ and in addition

$$R(\Gamma_k(x)) = c_k \cdot R(x),\ x \geq 0 \tag{1.2.7}$$

Similar to the continuous Pareto distribution 1.2.1 we verify the R.L.T. semi-stability for discrete Pareto distributions 0.4.17

Example 1.2.5. Put (as in 0.4.17)
$\overline{DP}_{\alpha,\,1}(x) = e^{-\alpha \cdot [\log x]} =: q^{[\log x]},\ x \geq 1$ (with $q = e^{-\alpha}$).
Define $\gamma_k : x \mapsto e^k \cdot x$. Then for $x \geq 1$ we have

$$R(\gamma_k(x)) = q^{[\log(e^k x)]} = q^{[k + \log x]} = q^{[\log x]} \cdot q^k,\ k \in \mathbb{Z}_+.$$

(Note that $q^k = R(\gamma_k(1))$). Hence $\overline{DP}_{\alpha,\,1}$ is R.L.T. semi-stable. Note that $\gamma_k = \gamma^k$ with $\gamma = \gamma_1$.

Now, we will prove that the set of distributions satisfying the generalized L.M.P. is closed, that is, the limit law is also characterized by a functional equation, and hence satisfies the condition of the (semi-) stability (resp. the general L.M.P.)

Proposition 1.2.6. Let F_n be R.L.T. stable continuous distribution functions with tails R_n such that $R_n(0) = 1$, and $R_n(\gamma_n(t)(x)) = c_n(t) \cdot R_n(x)\ \forall\, x, t, \forall\ n$ with $\gamma_n(\cdot) \in \mathrm{Aff}_0(\mathbb{R})$, and $c_n : \mathbb{R}_+ \mapsto [0, 1]$. Assume that $F_n \xrightarrow{w} F \not\equiv 1$. Then
$\gamma_n(\cdot) \overset{n\to\infty}{\longrightarrow} \gamma(\cdot)$, $c_n(\cdot) \overset{n\to\infty}{\longrightarrow} c_\infty(\cdot)$, and if $c_\infty \not\equiv 0, 1$, F is R.L.T. stable,
i.e. $F(\gamma(t)(x)) = c_\infty(t) \cdot F(x)$

Proof. Assume $x = 0$ is a continuity point of F, then, with $\gamma_n : x \mapsto a_n x + b_n$

$$R_n(\gamma_n(t)(0)) = c_n(t) \Longrightarrow R_n(b_n(t)) = c_n(t),\ 0 \leq c_n(\cdot) \leq 1$$

For fixed t there exist a subsequence (n') of (n) such that

$$c_n(t) \stackrel{n' \to \infty}{\longrightarrow} c_\infty(t) \in [0,1].$$

If $\inf\limits_n c_n(t_0) > 0$ for some $t_0 > 0$. Then we can apply the convergence of types (Prop. 0.2.3). Since $c_\infty(t) \neq 0$, we have $\{\gamma_n(t)\}$ is relatively compact, then there exist a subsequence $(\widetilde{n})$ of (n') such that

$$\gamma_n(t) \stackrel{\widetilde{n} \to \infty}{\longrightarrow} \widetilde{\gamma}(t)(x) = \widetilde{a}(t) \cdot x + \widetilde{b}(t).$$

In fact, since $\gamma_n \in \mathrm{Aff}_0(\mathbb{R})$, $(\widetilde{n}) = \mathbb{N}$, and hence $(n') = \mathbb{N}$. Therefore

$$c_n(t) \cdot R_n(x) = R_n(\gamma_n(t)(x)) \stackrel{n \to \infty}{\longrightarrow} R(\widetilde{\gamma}(t)(x))$$

weakly, and therefore $R(\widetilde{\gamma}(t)(x)) = c_\infty(t) \cdot R(x)$. We have

- If $c_\infty(t) = 1 \Longrightarrow R(\widetilde{\gamma}(t)(x)) = R(x)\ \forall\, x$ therefore $\widetilde{\gamma}(t)(x) = \mathrm{id}\ \forall\, x$, a contradiction to $R(\widetilde{\gamma}(t)(x)) = c(t) \cdot R(x)$ for all continuity points $x,\ x \geq 0,\ \forall\, t$.

Hence the limit laws fulfil the functional equation for all continuity points. Since R is right continuous, this relation holds true for all x. And then we have $t \mapsto c(t)$ is a continuous homomorphism, whence $c(t) = c^t$, $c = c(1)$. □

1.3 Limit laws: characterization by "stability functional equations"

In this section we find that R.L.T. limit laws satisfy a functional equation which turns out to be the condition of R.L.T. stability (semi-stability). Functional equation of the R.L.T. semi-stability will be solved in the next section in its general form, in both, continuous and discrete cases, in order to find the possible limit distributions of R.L.T. (We follow here the investigations of Balkema, de Haan [1]).

Notation 1.3.1. Let X be a random variable, let μ (resp. F) denote the corresponding distribution (resp. distribution function), and let R denote the tail. Assume that

$\Gamma(t) \in \mathrm{Aff}_0^+(\mathbb{R},1)$, $\Gamma(t)(x) \stackrel{t\to\infty}{\longrightarrow} \infty$.
Assume further

$$R_t(\widetilde{\Gamma}(t)(x)) = \frac{R(\widetilde{\Gamma}(t)(x)+t)}{R(t)} = \frac{R(\Gamma(t)(x))}{R(t)} \stackrel{t\to\infty}{\longrightarrow} S(x)$$

for all continuity points x of $S(x)$ such that $S(x) < 1$ where $1-S$ is a non-degenerate distribution function, and $\widetilde{\Gamma}(t)(x)$ be as in proposition 0.3.4.

Lemma 1.3.2. There exist a function $Y \ni y \mapsto \Gamma(y) \in \mathrm{Aff}_0^+(\mathbb{R},1)$ such that

$$S(\Gamma(y)(x)) = S(y) \cdot S(x) \tag{1.3.1}$$

$\forall\, x,\ y \in Y$, where Y denotes the set of all continuity points of $S(x)$ such that $S(x) < 1$

Proof. Assume (as in the above notation) that,

$$\frac{R(\Gamma(t)(x))}{R(t)} = \frac{R(a(t)\cdot x + b(t))}{R(t)} \stackrel{t\to\infty}{\longrightarrow} S(x) \tag{1.3.2}$$

for all $x \in Y,\ \ x \geq 0$.

Let $Y \ni y \geq 0$, put $t := \Gamma(s)(y) = a(s)\cdot y + b(s)$ for all $y \in Y,\ s > 0$ with $\Gamma(s)(y) \stackrel{s\to\infty}{\longrightarrow} \infty$ then we obtain in (1.3.2)

$$\frac{R(a(\Gamma(s)(y))\cdot x + b(\Gamma(s)(y)))}{R(\Gamma(s)(y))} \stackrel{s\to\infty}{\longrightarrow} S(x) \tag{1.3.3}$$

Therefore

$$\frac{R(a(\Gamma(s)(y))\cdot x + b(\Gamma(s)(y)))}{R(\Gamma(s)(y))} \cdot \frac{R(\Gamma(s)(y))}{R(s)} \stackrel{s\to\infty}{\longrightarrow} S(x)\cdot S(y) \tag{1.3.4}$$

for all $Y \ni x, y \geq 0$.

The left hand side is equal

$$\frac{R(a(\Gamma(s)(y))\cdot x + b(\Gamma(s)(y)))}{R(s)} \quad = \quad \frac{R(a(s)\cdot(\frac{a(\Gamma(s)(y))\cdot x + b(\Gamma(s)(y)) - b(s)}{a(s)}) + b(s))}{R(s)}$$

with

$$(\tfrac{a(\Gamma(s)(y))\cdot x + b(\Gamma(s)(y)) - b(s)}{a(s)}) = \tfrac{a(\Gamma(s)(y))}{a(s)}\cdot x + \tfrac{b(\Gamma(s)(y)) - b(s)}{a(s)} = \Gamma^{-1}(s)(\Gamma(\Gamma(s)(y))(x)).$$

(note that $\frac{a(\Gamma(s)(y))}{a(s)} \overset{s\to\infty}{\longrightarrow} A(y)$ and $\frac{b(\Gamma(s)(y))-b(s)}{a(s)} \overset{s\to\infty}{\longrightarrow} B(y)$). Hence we obtain in 1.3.4

$$\frac{R(a(\Gamma(s)(y))\cdot x + b(\Gamma(s)(y)))}{R(\Gamma(s)(y))} \cdot \frac{R(\Gamma(s)(y))}{R(s)} \overset{s\to\infty}{\longrightarrow} S(\Gamma(y)(x)). \tag{1.3.5}$$

where $\Gamma(y)(x) = A(y)\cdot x + B(y)$.
Applying C.T.T. to (1.3.4) and (1.3.5) we obtain

$$S(\Gamma(y)(x)) = S(y)\cdot S(x)$$

for all points $x, y \in Y$, $x, y \geq 0$. Hence the assertion follows. □

Remark 1.3.3. Assume $S(0) = 1$ and $0 \in Y$. Then
$R_t(\widetilde{\Gamma}(t)(x)) \overset{w}{\longrightarrow} S(x)$ implies that there exists a function $s \mapsto \gamma(s) \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$, hence $\gamma(s)(x) \to \infty$ for all $x > 0$ as $s \to \infty$ and

$$\frac{R(\gamma(s)(x))}{R(\gamma(s)(0))} \overset{w}{\longrightarrow} S(x) \text{ (for } s \to \infty) \tag{1.3.6}$$

Proof. With the notation of the proof of lemma 1.3.2 we have for $y = 0$

1. $\frac{R(\Gamma(\Gamma(s)(0))(x))}{R(\Gamma(s)(0))} \overset{w}{\longrightarrow} S(x)$. Then for $x = 0$ we have
2. $\frac{R(\Gamma(\Gamma(s)(0))(0))}{R(\Gamma(s)(0))} \overset{w}{\longrightarrow} S(0) = 1$

Whence we have $\frac{R(\gamma(s)(x))}{R(\gamma(s)(0))} \overset{w}{\longrightarrow} S(x)$ with $\gamma(s) = \Gamma(\Gamma(s)(0))$. □

1.4 Solutions of the stability functional equation

Here we solve a general functional equation, special cases of which appeared in the preceding sections.

Let $\Lambda : \mathbb{R} \longmapsto \mathbb{R}$ be a non constant decreasing right continuous function. Let $\{\gamma(t)\}_{t\in T} \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1)$ be a semigroup of affine transformations. Let

$$T = \begin{cases} \mathbb{R}_+ & \text{in the continuous case,} \\ \mathbb{Z}_+ & \text{in the discrete case} \end{cases}$$

We always assume $\gamma(t)(0) \geq 0$. Assume

$$\Lambda(\gamma(t)(x)) = c(t)\cdot\Lambda(x) \tag{1.4.1}$$

$\forall\, x \in \mathbb{R}_+$, $t \in T$ where $c(t) \geq 0$.

SOLUTION Since $t \mapsto \gamma(t)$ is a homomorphism, we obtain immediately that $t \mapsto c(t)$ is a homomorphism $(T,+) \longrightarrow ([0,1],\cdot)$. If $t \mapsto \gamma(t)$ is continuous, we obtain that $t \mapsto c(t)$ is continuous.
In the discrete case, we obtain $c(t) = (c(1))^t = q^t := e^{-\beta t}$ $(q = e^{-\beta}$ resp. $\beta = -\log c(1) \geq 0)$, $t \in \mathbb{Z}_+$. The trivial cases $c(\cdot) \equiv 0$, $c(\cdot) \equiv 1$ are excluded in the sequel.
In the continuous case we obtain the solution $c(t) = e^{-\beta t} = q^t$, $t \in \mathbb{R}_+$. According to the structure of affine semigroups obtained in section 0.1 we have the following two cases:

Case(1) $(Q = 0)$, $\gamma(t)(x) = x + t \cdot \alpha$

Case(2) $(Q \neq 0)$, $\gamma(t)(x) = e^{tQ}(x - x_\star) + x_\star$.

Note that in case (2): $x_\star < 0 \iff Q > 0$, and $x_\star > 0 \iff Q < 0$. See prop. 0.1.12. Hence the functional equation 1.4.1 can be formulated as:

Case(1) $\Lambda(x + t\cdot\alpha) = e^{-\beta t} \cdot \Lambda(x)$

Case(2) $\Lambda(e^{tQ} \cdot (x - x_\star) + x_\star) = e^{-\beta t} \cdot \Lambda(x)$

The solution in the continuous case ($T = \mathbb{R}_+$):

A1) According to case(1) we have $\gamma(t)(x) = x + t \cdot \alpha$.
Put $x = 0$, $\Lambda(0) =: c$. Therefore we obtain

$$\Lambda(t \cdot \alpha) = e^{-\beta \cdot t} \cdot \Lambda(0) = c \cdot e^{-\beta \cdot t}. \tag{1.4.2}$$

Hence we obtain

$$\Lambda(z) = c \cdot e^{-\frac{\beta}{\alpha}\cdot z}, \; z \geq 0 \tag{1.4.3}$$

where $z := t \cdot \alpha = \gamma(t)(0)$ and $\Lambda(t\cdot\alpha) = \Lambda(\gamma(t)(0)) = c \cdot e^{-\beta\cdot t}$. (In particular Λ is continuous on $\mathbb{R}_+$.)

A2) According to case(2), it is more convenient to use multiplicative parameterization (in Remark 0.1.10). Therefore

$$\Lambda(u^Q(x - x_\star) + x_\star) = u^{-\beta} \cdot \Lambda(x), \; u = e^t \geq 1.$$

For $x = 0$, $\Lambda(0) =: c$ we obtain again :

$$\Lambda((u^Q - 1)(-x_\star)) = u^{-\beta} \cdot c \,, u \geq 1.$$

- The case $Q > 0$: (i.e. $x_\star \lneqq 0$). $-x_\star =| x_\star |=: d$ this yields
$$\Lambda(d\cdot(u^Q-1)) = u^{-\beta}\cdot c,\ (c>0, d\geq 0) \tag{1.4.4}$$
Put $z = d\cdot(u^Q - 1)\geq 0$ hence $u = (1+\frac{z}{d})^{\frac{1}{Q}}$. And the solution of the functional equation (using 1.4.4) obtains the form
$$\Lambda(z) = c\cdot(1+\frac{z}{d})^{\frac{-\beta}{Q}}, z\geq 0. \tag{1.4.5}$$
With $C = c^{\frac{\beta}{Q}},\ D = \frac{c^{\frac{\beta}{Q}}}{d},\ \gamma = \frac{\beta}{Q}$ we have
$$\Lambda(z) = (C + D\cdot z)^{-\gamma}. \tag{1.4.6}$$
- The case $Q < 0$ (i.e. $x_\star > 0$), c, d as above. We obtain analogously with $d = x_\star \geq 0$
$$\Lambda(d\cdot(1-u^Q)) = u^{-\beta}\cdot c \tag{1.4.7}$$
Put $z = d\cdot(1-u^Q)$, $0\leq z\leq d$ for $u\geq 1$ then we have
$$u = (1-\frac{z}{d})^{\frac{1}{Q}},\ 0\leq z\leq d.$$
Hence we obtain (using 1.4.7)
$$\Lambda(z) = c\cdot(1-\frac{z}{d})^{\frac{-\beta}{Q}} = (C - D\cdot z)^{-\gamma} \tag{1.4.8}$$
With C, D, γ as above.
- Now we treat the case $x_\star = 0$: We have
$\gamma(t)(x) = e^{tQ}\cdot x \Longrightarrow \Lambda(e^{tQ}\cdot x) = e^{-t\beta}\cdot\Lambda(x)$.
For $x = 1,\ \Lambda(1) =: c$ we obtain $\Lambda(e^{tQ}) = e^{-t\beta}\cdot c$.
Put $z = e^{tQ}$ hence $e^t = z^{\frac{1}{Q}}$, and therefore we obtain
$$\Lambda(z) = z^{\frac{-\beta}{Q}}\cdot c,\ z\geq 1 \tag{1.4.9}$$

The solution in the discrete case ($T = \mathbb{Z}_+$)**:**

A3) According to case(1) we have $\gamma(t)(x) = x + t\alpha,\ t\in\mathbb{Z}_+$ therefore
$$\Lambda(z) = c\cdot e^{\frac{-\beta}{\alpha}.z} \tag{1.4.10}$$
as before with $c = \Lambda(0)$, $z = \alpha\cdot t = \gamma(t)(0)\in\mathbb{Z}_+\cdot\alpha$.
Put $T(x) := \Lambda(x)$, $0\leq x<\alpha$, $q := e^{\frac{-\beta}{\alpha}}$. Then we obtain the solution
$$\Lambda(z) = c\cdot q^k\cdot T(x) \tag{1.4.11}$$
if $k\cdot\alpha\leq z<(k+1)\cdot\alpha,\ x := z-k\cdot\alpha$. For later use, note that $t\cdot\alpha = \gamma(t)(0)$, $t\in\mathbb{Z}_+$, hence
$$\Lambda(z) = c\cdot q^k\cdot T(x) \tag{1.4.12}$$
if $\gamma(k)(0)\leq z<\gamma(k+1)(0),\ x := z-\gamma(k)(0)$

Properties of the function T: $T \searrow$, $T(0)=\Lambda(0)$ w.l.o.g.$=1$, $T(\alpha-)\geq c\cdot q$ (equivalently, $T\searrow,\ T(\alpha-)=\Lambda(\alpha-)\geq\Lambda(0)\cdot q$)

A4) According to case(2), $t\in\mathbb{Z}_+$. As before we obtain with $\gamma(1)=:R>1,\ R=e^Q:$ $\Lambda(R^k(x-x_\star)+x_\star)=e^{-\beta k}\Lambda(x)$. Put $x=0,\ c=\Lambda(0)$, $z=\mid x_\star\mid(R^k-1)=\mid x_\star\mid(e^{kQ}-1)$. Here $e^{kQ}=1+\frac{z}{|x_\star|}$ and $e^{-k}=(1+\frac{z}{|x_\star|})^{\frac{-1}{Q}}$. Therefore, for this discrete set of z's we have

$$\Lambda(z)=c\cdot(1+\frac{z}{\mid x_\star\mid})^{\frac{-\beta}{Q}} \tag{1.4.13}$$

Put for short $v_k=\gamma(k)(0)=\mid x_\star\mid(R^k-1)=\mid x_\star\mid(e^{kQ}-1),\ k\in\mathbb{Z}_+$.
For $z\geq 0\ \exists k=k(z)\in\mathbb{Z}_+$ such that $z=\gamma(k)(x), 0\leq x<\gamma(1)(0)$, i.e. $v_k\leq z<v_{k+1}$ with

$$\begin{aligned} z=\gamma(k)(x) &= e^{kQ}(x-x_\star)+x_\star \\ &= e^{kQ}\cdot x+\mid x_\star\mid(e^{kQ}-1) \\ &= e^{kQ}\cdot x+v_k. \end{aligned}$$

Hence $x=\frac{(z-v_k)}{e^{kQ}}=x(z)=\frac{z-v_k}{R^k}$.
Put again $T(x):=\Lambda(x),\ \ 0\leq x<\gamma(1)(0)$. Then we obtain the solution

$$\Lambda(z)=c\cdot(1+\frac{v_k}{\mid x_\star\mid})^{-\frac{\beta}{Q}}\cdot T(x), \tag{1.4.14}$$

$k=k(z),\ \ x=x(z)$ where $T\searrow$, $T(0)=1$, $T(\gamma(1)(0)-)\geq c\cdot(1+\frac{v_1}{|x_\star|})^{\frac{-\beta}{Q}}$.

A5) In case(2) if $Q<0,\ x_\star>0$, this case is omitted because it is of no importance in the sequel.

A6) In case(2) if $Q>0,\ x_\star=0$ we have:
$\gamma(k)(x)=R^k\cdot x,\ R=e^Q$. Put e.g. $x=1,\ \Lambda(1):=c$. Hence

$$\Lambda(\gamma(k)(1))=\Lambda(R^k)=c\cdot e^{-k\beta}$$

for $z\geq 0,\ \ \gamma(k)(1)=R^k\leq z<R^{k+1}=\gamma(k+1)(1)$. Then $z=\gamma(k)(x),\ 1\leq x<R$ (i.e. $x=x(z)=\frac{z}{R^k},\ k=k(z)\in\mathbb{Z}_+$). Put $T(x)=\Lambda(x)$, $1\leq x<R$. Then we obtain for $z\geq 1:$

$$\Lambda(z)=\Lambda(\gamma(k)(x))=c(k)\cdot\Lambda(x) \tag{1.4.15}$$

Hence

$$\Lambda(z)=e^{-k\beta}\cdot T(x)=q^k\cdot T(x) \tag{1.4.16}$$

with $k=k(z)=[\frac{\log z}{Q}]$ and $x=x(z)=\frac{z}{R^k},\ q=e^{-\beta}$.

Remark 1.4.1. Conversely, as easily seen, all functions obtained in the above cases in equations 1.4.3, 1.4.5, 1.4.8, and 1.4.9 are solutions of the functional equation in the continuous case. And in the discrete case, as easily seen in 1.4.12, 1.4.13, 1.4.14, and 1.4.16 we obtain solutions for any fixed function T. The solution is uniquely determined only if we assume T in addition to be constant $\equiv 1$.

Remark 1.4.2. In most cases we considered solutions of the functional equation to be valid for $x \geq 0$ (resp. $x \geq 1$). Same arguments allows to consider also solutions of $\Lambda(\gamma(t)(x)) = c(t) \cdot \Lambda(x)$, for $t \geq 0$ and $x \geq x_0$, for some x_0.

In fact, in the continuous case solution, define t_0 by $x_0 = \gamma(t_0)(0)$.
Put $\Lambda^\star(x) := \Lambda(\gamma(t_0)(x)) =: \Lambda(y),\ x \geq 0 \Longrightarrow y \geq x_0$. Then $\Lambda^\star$ fulfils the equation

$$\Lambda^\star(\gamma(t)(x)) = c(t) \cdot \Lambda^\star(x), \text{for } t \geq 0,\ x \geq 0 \tag{1.4.17}$$

Similarly in the discrete case: Assume $x_0 = \gamma(k_0)(0)$ then define $\Lambda^\star$ as before. Or for $\gamma(k_0 - 1)(0) < x_0 < \gamma(k_0)(0)$, then define $\Lambda^\star(x) := \Lambda(\gamma(k_0)(x)),\ x \geq 0, y > x_0$ with

$$\Lambda^\star(\gamma(k)(x)) = c(k) \cdot \Lambda^\star(x),\ x \geq 0,\ k \geq 0$$

Remark 1.4.3. If Λ is the solution of the functional equation, and if we look for solutions (for sufficiently large x) which are tail functions, we have to replace Λ by $\widetilde{\Lambda} = \min(1, \Lambda)$. Hence, if Λ fulfils the equation for all $x \geq 0$, $\widetilde{\Lambda}$ fulfils the equation for all $x \geq x_0$, where x_0 is defined by $\Lambda(x_0) = 1$ in the continuous case (resp. $\Lambda(x_0) \leq 1$ in the discrete case).

Theorem 1.4.4. The probability distributions, with tail function R such that $0 < R(x) < 1$ for all $x > 0$ following a R.L.T. stability functional equation are

1. Exponential distributions
2. Pareto distributions
3. Generalized geometric distributions
4. Generalized discrete Pareto distribution

and suitable shifted versions of (1)–(4).

Proof. **The continuous case:**

1) The case $Q = 0$. We obtain in $A1$) equation 1.4.3 the solution
$\Lambda(z) = c \cdot e^{\frac{-\beta}{\alpha} \cdot z}$, $z \geq 0$ with $c = \Lambda(0)$.

2) The case $Q > 0$, $x_\star < 0$. We obtain in A2), in particular if $c := \Lambda(0) = 1$ the following:

a) $\Lambda(z) = (1 + \frac{z}{d})^{\frac{-\beta}{Q}},\ z \geq 0$ (equation 1.4.5)

b) $\Lambda(z) = (1 - \frac{z}{d})^{\frac{-\beta}{Q}},\ 0 \leq z \leq d$ (equation 1.4.8)

c) $\Lambda(z) = z^{\frac{-\beta}{Q}} \cdot c,\ z \geq 1$ (equation 1.4.9).

3) We obtain in $(A2)$, case $Q > 0$, $x_\star = 0$. In particular if $c := \Lambda(1) = 1$:

$$\Lambda(z) = \begin{cases} z^{-\gamma} & z \geq 1 \\ 1 & z < 1. \end{cases}.$$

Hence we obtain the following tail functions

- In step (1) the tail function of $E_{\frac{\beta}{\alpha}}$ if $(c = 1)$, see example 0.4.1.
- In step (2): a) and c) the tail functions of a Pareto distribution which is investigated by using the re-scaled version, see notation 0.4.8.
- In step (3) the tail function of a shifted Pareto distribution, see notation 0.4.6.

Notation 1.4.5. Note that we obtain in step (2):b), (and in addition to the Pareto distribution), the tail of a bounded distribution $B_{\frac{\beta}{Q}}$, see example 0.4.11, which are of no importance in the sequel.

The discrete case:

4) We obtain In A3) (using equation 1.4.12), in particular if c=1: $\Lambda(z) = q^k \cdot T(x)$. Hence $\Lambda(z) = q^t \cdot T(x)$. Hence we obtain the tail of a generalized discrete Pareto distribution, see example 0.4.17.

5) We obtain in A4) (using equation 1.4.14), in particular if c=1:
$\Lambda(z) = (1 + \frac{v_k}{|x_\star|})^{-\frac{\beta}{Q}} \cdot T(x)$. Hence we obtain the tail function of a generalized shifted discrete Pareto distribution, see example 0.4.19.

6) The case $Q > 0, x_\star = 0$. We obtain in A6) equation 1.4.16
$\Lambda(z) = e^{-k\beta} \cdot T(x) = q^k \cdot T(x)$. Hence we obtain the tail of a generalized geometric distribution, see example 0.4.14

Notation 1.4.6. The case $Q < 0, x_\star > 0$ is omitted because it is of no importance in the sequel. □

1.5 The decomposability semigroups of R.L.T. distributions

Here we introduce a concept which has been successfully used for investigations in (operator) semi-stability of vector space – and group– valued random variables.
We give a general definition which will be useful for vector spaces (in chapter 2). For $d = 1$ it turns out that the objects a quite simple.

Definition 1.5.1. *Let R be a non-degenerate tail function and $x_0 = x_0(R) \geq 0$. We define the R.L.T. decomposability semigroup*
$Dec(R) := \{\gamma \in \mathrm{Aff}_0^+(\mathbb{R}, 1) : \gamma = \gamma_{a,b}$ with $a \geq 1$, $b \geq 0$, such that $R(\gamma(x)) = c(\gamma) \cdot R(x)$ for all $x \geq x_0$, $c = c(\gamma) \in (0, 1]\}$.

Remark 1.5.2. Note that with the notations 0.1.16 in 0.1
$\mathrm{Dec}(R) \subseteq \{\{id\} \cup \mathrm{Aff}_0^+(\mathbb{R}, 1)\}$. The assumption $\gamma = \gamma_{a,b} \in \mathrm{Dec}(R)$ implies that either $\gamma = \gamma_{1,0} = \mathrm{id}$, or γ is a shift, $\gamma = \gamma_{1,b}$, $b > 0$ or γ has a fixed point $x_\star \leq 0$ with $\gamma(x) = a \cdot (x - x_\star) + x_\star = a \cdot x + (a - 1)(-x_\star)$, hence $b = (a - 1)(-x_\star) \geq 0$.
In particular, γ is (strictly) increasing on $\mathbb{R}_+$ hence $\gamma(x) \geq x_0$ for all $x \geq x_0$. Obviously we have

Proposition 1.5.3. $\mathrm{Dec}(R)$ is a closed subsemigroup of $\mathrm{Aff}_0^+(\mathbb{R}, 1)$ and $\gamma \mapsto c(\gamma)$ is a continuous homomorphism $c : \mathrm{Dec}(R) \to ((0, 1], \cdot)$

Proof. Let $\gamma, \widetilde{\gamma} \in \mathrm{Dec}(R)$. According to the remark 1.5.2 we have

$$\begin{aligned} R((\gamma \circ \widetilde{\gamma})(x)) &= c(\gamma) \cdot R(\widetilde{\gamma}(x)) \\ &= c(\gamma)c(\widetilde{\gamma}) \cdot R(x) \end{aligned}$$

for $x \geq x_0$ (hence $\widetilde{\gamma}(x) \geq x_0$). Hence $\gamma \mapsto c(\gamma)$ is a homomorphism. Let $x \geq x_0$ with $R(x) > 0$ such that $\gamma(x)$ is a continuity point of R. Then $R(\gamma^{(n)}(x)) \to R(\gamma(x))$ hence $c(\gamma^{(n)})R(x) \to c(\gamma)R(x)$ whence $c(\gamma^{(n)}) \to c(\gamma)$ follows. □

Moreover we have

Proposition 1.5.4. $c : \mathrm{Dec(R)} \to (0, 1]$ is a closed map if R is non-degenerate

Proof. Let $\{\alpha_n\} \subseteq \mathrm{im}(c)$, i.e. $\alpha_n = c(\gamma_n)$ with $\gamma_n \in \mathrm{Dec(R)}$, $0 < \alpha_n \leq 1$ and assume further $\alpha_n \to \alpha \in (0, 1]$. If $x_0 > 0$ replace R by $\widetilde{R}$ such that

$$\widetilde{R}(x) := \begin{cases} R(x) & x \geq x_0 \\ 1 & x < x_0 \end{cases}$$

Then we have $R(\gamma_n(x)) = \alpha_n \cdot R(x) \longrightarrow \alpha \cdot R(x), x > x_0$. On the other hand

$$\widetilde{R}(\gamma_n(x)) = \begin{cases} R(\gamma_n(x)) & \gamma_n(x) > x_0 \iff x > \gamma_n^{-1}(x_0) \\ 1 & x \leq \gamma_n^{-1}(x_0) \end{cases}$$

If we define

$$S(x) := \begin{cases} \alpha \cdot R(x) & x > x_0 \\ 1 & x \leq x_0 \end{cases}$$

we obtain $\widetilde{R}(\gamma_n(\cdot)) \xrightarrow{w} S$, whence by the convergence of types theorem we have (γ_n) is relatively compact, in fact $\gamma_n \to \gamma$ follows, and moreover $\widetilde{R}(\gamma(x)) = \alpha \cdot \widetilde{R}(x)$. Whence $R(\gamma(x)) = \alpha \cdot R(x)$. Therefore, $\gamma \in \mathrm{Dec}(R)$ with $c(\gamma) = \alpha$. Thus $\mathrm{im}(c)$ is closed in $(0, 1]$ □

In analogy to investigations of semi-stable laws on vectors spaces and groups, we define the invariance group. Note that this definition appears at first complicated, due to the fact that $\mathrm{Dec}(R)$ is only a semi group. (See also 1.5.5 below)

Definition 1.5.5. *Let R, x_0 as above.*
$\mathrm{Inv}(R) := \{\gamma \in \mathrm{Aff}_0(\mathbb{R}),\ \gamma \nearrow : \exists\ x_\gamma \geq x_0$ *such that* $R(\gamma(x)) = R(x),\ x \geq x_\gamma\}$.
$\mathrm{Inv}(R)$ *is called the invariance group of R.*

Proposition 1.5.6. For $d = 1$ and for a non-degenerate tail function R with $x_\gamma > x_0$, and $R(x_\gamma) > 0$ we obtain: $\mathrm{Inv}(R) = \{\mathrm{id}\} = \{\gamma : R(\gamma(x)) = R(x),\ x \geq x_0\}$

Proof. We assume $\gamma \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$, hence $\gamma \nearrow$, i.e.$\gamma := \gamma_{a,b} : x \mapsto ax + b$ with $a > 0$.

- Assume first $a \geq 1, b \geq 0$, $\gamma \neq \mathrm{id}$ such that $\gamma^n(x) \nearrow \infty$ for $x \geq 0$. Hence $R(x) = R(\gamma^n(x)) \xrightarrow{n\to\infty} 0$ for all sufficiently large x, (i.e. for $x \geq x_\gamma$, with $\gamma^n(x) \geq x_\gamma$ for all large $n \in \mathbb{N}$). Hence $R(x) = 0$, a contradiction.

- If $a = 1, b < 0$. Then $\gamma^n(x) \to -\infty$ and we obtain $R(x) = R(x - nb) \geq R(x_\gamma + b)$ for all large x and $n \in \mathbb{N}$, such that $x - (n+1)b \leq x_\gamma < x - nb$.
I.e. $0 = \lim\limits_{x\to\infty} R(x) = \lim\limits_{n\to\infty} R(\gamma^n(x)) \geq R(x_\gamma) > 0$, a contradiction.

- If $0 < a < 1$, $\gamma(x) = a(x - x_\star) + x_\star$, we obtain $\gamma^n(x) \to x_\star$. Therefore, for all large $x > x_\gamma$, $n \in \mathbb{N}$ with $\gamma^{n+1}(x) \leq x_\gamma < \gamma^n(x)$ we have $R(x) = R(\gamma^n(x))$. Whence again $\lim\limits_{y\to\infty} R(y) > 0$, a contradiction.

Thus we have proved: $\mathrm{Inv}(R) = \{\mathrm{id}\} = \{\gamma_{1,0}\}$ □

Remark 1.5.7. Evenly the assumption $R(x) > 0$ for all x would not be a serious restriction: Assume there exists $\gamma \in \mathrm{Dec}(R)$, $\gamma \neq \mathrm{id}$, and assume furthermore; $R(x_1) > 0$ for some $x_1 > 0$. Then $R(x) > 0$ on $\mathbb{R}_+$.

Proof. Obvious, since $R(\gamma^n(x_1)) = c(\gamma)^n R(x_1) > 0$, and by assumption we have $\gamma \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$, $\gamma^n(x_1) \nearrow \infty$. Hence $R(x) > 0$ for all $x \geq 0$. ☐

The next result explains the complicated definition of $\mathrm{Inv}(R)$ in 1.5.5 :

Proposition 1.5.8. The semigroup Dec(R) is embeddable into a group

$$\widetilde{\mathrm{Dec}}(\mathrm{R}) \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1).$$

c extends to a continuous injective homomorphism, $\widetilde{c} : \widetilde{\mathrm{Dec}}(\mathrm{R}) \to (\mathbb{R}_+^\times, \cdot)$ with (trivial) kernel $\ker(c) = \mathrm{Inv}(R) = \{\mathrm{id}\}$ or $c \equiv 1$ (if $\mathrm{Dec}(R) = \mathrm{Inv}(R)$)

Proof. Let $\widetilde{\mathrm{Dec}}(R)$ denote the subgroup generated by $\mathrm{Dec}(R)$. $\widetilde{\mathrm{Dec}}(\mathrm{R})$ acts in a suitable way on R: let $\gamma, \tau \in \mathrm{Dec}(R)$ then $\gamma\tau^{-1} \in \widetilde{\mathrm{Dec}}(R)$. And we obtain for $x \geq x_0$, $y = \tau(x) \geq \tau(x_0)$ that

$$R(y) = c(\tau) \cdot R(x) = c(\tau) \cdot R(\tau^{-1}(y))$$

Therefore for all sufficiently large y we have $R(\tau^{-1}(y)) = \frac{1}{c(\tau)} R(y)$ and $\tau^{-1}(y) \geq x_0$. Whence $R(\gamma\tau^{-1}(y)) = \frac{c(\gamma)}{c(\tau)} \cdot R(y)$ follows. Hence, put $\widetilde{c}(\gamma\tau^{-1}) := \frac{c(\gamma)}{c(\tau)}$ we obtain

$$R(\gamma\tau^{-1}(y)) = \widetilde{c}(\gamma\tau^{-1}) \cdot R(y).$$

Assume $c(\tau) = c(\gamma) =: z$. Then, for all sufficiently large y, $R(\gamma\tau^{-1}(y)) = R(y)$, i.e. $\gamma\tau^{-1} \in \mathrm{Inv}(R)$. But $\mathrm{Inv}(R) = \{\mathrm{id}\}$, whence $\gamma = \tau$ follows ☐

Note that we have proved:

$$\begin{aligned} \widetilde{\mathrm{Dec}}(R) &= \{\tau\gamma^{-1} : \tau, \gamma \in \mathrm{Dec}(R)\} \\ &= \mathrm{Dec}(R) \cdot \mathrm{Dec}^{-1}(R) \end{aligned}$$

Therefore, the preceding results extend immediately to the group $\widetilde{\mathrm{Dec}}(R) \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1)$

Theorem 1.5.9. With the notations introduced above we have:

a) Dec(R) is a closed sub-semigroup of the closed subgroup $\widetilde{\mathrm{Dec}}(\mathrm{R}) \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1)$

b) $\gamma \mapsto \widetilde{c}(\gamma)$ is a continuous closed homomorphism $\widetilde{\mathrm{Dec}}(R) \longrightarrow ((0, \infty), \cdot)$

Remark 1.5.10. $\mathrm{im}(\widetilde{c}) \cap (0,1]$ is a multiplicative semigroup of $(0,1]$, which is closed since c is closed map. Therefore either

- $\mathrm{im}(\widetilde{c}) = (0,1]$ or
- $\mathrm{im}(\widetilde{c}) = \{q^k : \ k \in \mathbb{Z}_+ \ \text{ for some } \ 0 < q < 1\}$ or
- $\mathrm{im}(\widetilde{c}) = \{1\}$

Therefore we obtain the following

Theorem 1.5.11. Let R be as above. Then either

a) $\mathrm{im}(\widetilde{c}) = \{1\}$ (i.e. $\mathrm{Dec}(R) = \{\mathrm{id}\} = \mathrm{Inv}(R)$) or

b) there exist $\gamma \in \mathrm{Dec}(R)$ such that $\mathrm{Dec}(R) = \{\gamma^k : k \in \mathbb{Z}_+\}$ or

c) there exists a continuous one-parameter group $\mathbb{R} \ni t \mapsto \gamma(t)$ (w.l.o.g. additive parameterization) such that $\widetilde{\mathrm{Dec}}(R) = \{\gamma(t) : t \in \mathbb{R}\}$, $\mathrm{Dec}(R) = \{\gamma(t) : t \geq 0\}$ and $c(\gamma(t)) = q^t$ $(= e^{-\beta t})$ for some $q \in (0,1)$, $\beta = -\log q > 0$. W.l.o.g. we may assume $\beta = 1$.

Proof.

- In fact, if $\widetilde{\mathrm{Dec}}(R) \neq \{\mathrm{id}\}$, then for any $\gamma \in \widetilde{\mathrm{Dec}}(R)$, $c(\gamma) \in (0,1)$. Obviously $\{\gamma^k\} \subseteq \widetilde{\mathrm{Dec}}(R)$, $c(\gamma^k) = c(\gamma)^k$.
- If $\mathrm{im}(c) = \{q^k : k \in \mathbb{Z}_+\}$, there exists $\gamma \in \widetilde{\mathrm{Dec}}(R)$ with $\widetilde{c}(\gamma) = q$. Now (b) follows.
- If $\mathrm{im}(c) = (0,1]$, $\gamma \mapsto \widetilde{c}(\gamma) \in (0,\infty)$ is a continuous homomorphism of the Lie group $\widetilde{\mathrm{Dec}}(R)$ onto $((0,\infty),\cdot)$, $\widetilde{\mathrm{Dec}}(R)$ is closed in $\mathrm{Aff}_0^+(\mathbb{R},1)$, hence closed in $\mathrm{GL}(\mathbb{R})$. Therefore $\widetilde{\mathrm{Dec}}(R)$ has at most countably many connected components. Therefore there exists a continuous homomorphism

$$((0,\infty),\cdot) \longrightarrow \widetilde{\mathrm{Dec}}(R) : \quad u \mapsto \widetilde{\gamma}(u)$$

such that $\widetilde{c}(\widetilde{\gamma}(u)) = u$, $u > 0$. Passing to additive parameterization, $\gamma(t) := \widetilde{\gamma}(e^t)$, $u = e^t$, yields the assertion (for $q = e^{-1}$). Now it is obvious that $\gamma(t)(x) \xrightarrow{t\to\infty} \infty$ for all $x \geq x_0$ (resp. ≥ 0) (since $R(\gamma(t)(x)) = q^t \cdot R(x) \xrightarrow{t\to\infty} 0$)

$\square$

Corollary 1.5.12. Let R be as above. Let $\mathcal{D} \subseteq \widetilde{\mathrm{Dec}}(R)$, $\mathcal{D} \neq \{\mathrm{id}\}$,

$$\mathcal{C} := \{c(\gamma) : \gamma \in \mathcal{D}\}$$

Then either

a) the semigroup $\langle\mathcal{C}\rangle$ generated by $\mathcal{C}$ is discrete $\{q^k : k \in \mathbb{Z}_+\}$, then R is R.L.T. semi-stable, with $\mathrm{Dec}(R) = \{\gamma^k : k \in \mathbb{Z}_+\}$ or

b) $\langle\mathcal{C}\rangle$ is dense in $(0, 1]$, then $c(\mathrm{Dec}(R)) = (0, 1]$ and there exists a one-parameter group $\gamma(\cdot)$ such that (e.g. with additive parameterization) $c(\gamma(t)) = q^t$, $t \geq 0$. I.e. in this case R is R.L.T. stable.

Theorem 1.5.13. The limit distributions which satisfy the limit relation in section 1.3 are R.L.T.(semi-) stable, and hence belong to the class of distributions characterized in section 1.4 (see1.4.4), i.e. they belong to (shifted) exponential distributions, (shifted) Pareto distributions, generalized geometric distributions or generalized discrete Pareto distributions.

Proof. We obtained in 1.3 that for R.L.T. limit distributions we have affine $\Gamma_y \in \mathrm{Aff}(\mathbb{R})$, $y \in Y$, such that

$$R(\Gamma_y(x)) = R(y) \cdot R(x),\, y \in Y,\; x \geq 0.$$

In other words–with the notation of section 1.5–

$$\mathcal{D} = (\Gamma_y :\; y \in Y) \subseteq \mathrm{Dec}(R)\,\text{with}\, c(\Gamma_y) = R(y)\,\forall\, y \in Y.$$

Therefore, either $\langle R(y) :\; y \in Y\rangle$ is discrete, or dense in $(0, 1]$, whence the assertion follows by the preceding corollary 1.5.12. □

1.6 Domains of attraction of stable R.L.T. distributions

Definition 1.6.1. *Let F, G be a non degenerate distribution functions, such that*

$$F_t(\Gamma(t)(x)) \overset{t\to\infty}{\longrightarrow} G(x) \tag{1.6.1}$$

for all continuity points x of $G(x)$, $x \geq 0$, for some continuous function $\Gamma(t) : x \mapsto \Gamma(t)(x) \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$ of affine transformations such that $\Gamma(t)(x) \overset{t\to\infty}{\longrightarrow} \infty$, $\forall\, x \geq 0$, and $\widetilde{\Gamma}(t)(x)$ be as in proposition 0.3.4. Then the set of all distribution functions F which satisfy (1.6.1) is said to be the domain of R.L.T. attraction of G and we write

$$(DOA)_r(G) = \{F :\; \textit{equation}\,(1.6.1)\;\textit{holds}\} \tag{1.6.2}$$

Equivalently, let $R > 0$ be the tail of F, $L = 1 - G$ be the tail of G then 1.6.1 is equivalent to

$$\frac{R(\Gamma(t)(x))}{R(t)} \overset{t\to\infty}{\longrightarrow} L(x) \tag{1.6.3}$$

for all continuity points x of L. In this case we write

$$(DOA)_r(L) = \{R : \ \textit{equation}\,(1.6.3)\ \textit{holds}\} \tag{1.6.4}$$

More generally, we admit non-continuous $\Gamma(\cdot)$, with right and left limits such that for all sequences $t_n \to \infty$

$$R(\Gamma(t_n+))/R(\Gamma(t_n-)) \overset{n\to\infty}{\longrightarrow} 1 \tag{1.6.5}$$

The shape of the limit distributions :

Remark 1.6.2. From the solution of the functional equation 1.4.1 see 1.4, and the descriptions of limit laws in 1.3, we have according to theorem 1.5.13, that the possible limit distribution G are R.L.T. stable (in short R.L.T. stable), and hence are solutions of the functional equation (1.4.1). Thus we know according to theorem 1.4.4, that G resp. L, has one of the following forms

1) Exponential Laws: $L(x) = \overline{E}_\lambda(x) = \exp(-\lambda x),\ x \geq 0$ for $\lambda > 0$
with $\gamma(t) : x \mapsto x + t,\ \ c(t) = \exp(-\lambda t)\ (= L(\Gamma(t)(0)))$

1a) Shifted exponential laws: $L(x) = \overline{E}_{\lambda,x_0}(x) = \exp(-\lambda(x - x_0)),\ x \geq x_0$
with $\gamma(t)$ as in 1), $c(t) = \exp(-\lambda t)\ (= L(\Gamma(t)(x_0)))$

2) Pareto distribution: $L(x) = \overline{P}_\lambda(x) = (1 + \kappa \cdot x)^{-\lambda},\ x \geq 0$
with $\gamma(t) : x \mapsto e^t(x + \kappa^{-1}) - \kappa^{-1}$, $c(t) = e^{-\lambda t}\ (= L(\Gamma(t)(0))),$ and

2a) Shifted Pareto distribution: $L(x) = \overline{P}_{\lambda,\,x_0}(x) = (1 + \kappa(x - x_0))^{-\lambda},\ x \geq x_0$
with $\gamma(t)$ and $c(t)$ are as in 2) with assumption that $x - x_0 = z$

The domains of attraction of R.L.T. stable laws are non empty. First we show

Proposition 1.6.3. a) The definition of $(\mathrm{DOA})_r(L)$ (1.6.3) or (1.6.4) can be written in the following equivalent form:

$$(\mathrm{DOA})_r(L) = \{R\ :\ \frac{R(\gamma(t)(x))}{R(\gamma(t)(0))} \overset{w}{\longrightarrow} L(x)\} \tag{1.6.6}$$

where R is a tail and $\gamma : \mathbb{R}_+ \longrightarrow \mathrm{Aff}_0^+(\mathbb{R}, 1)$ is a function, $\gamma(t)(0) \overset{t\to\infty}{\longrightarrow} \infty$, which is continuous or at least fulfils (1.6.5) in definition 1.6.1

b) L is R.L.T. stable iff $(\mathrm{DOA})_r(L) \neq \emptyset$

Proof. **a)** Assume $\frac{R(\Gamma(t)(x))}{R(t)} \overset{w}{\longrightarrow} L(x)$. Assume for convenience that $\Gamma(\cdot)$ is continuous. Then define $s = s(t)$ such that $\Gamma(s)(0) = t$. With this notations (applying remark 1.3.3) then equation (1.6.3) means that $\frac{R(\gamma(t)(x))}{R(\gamma(t)(0))} \overset{w}{\longrightarrow} L(x)$.

Conversely, assume (1.6.6) and assume again $\gamma(\cdot)$ to be continuous. Then define $s = s(t)$ such that $\gamma(s(t))(0) = t$. Put $\Gamma(t) := \gamma(s(t))$ to obtain

$$\frac{R(\Gamma(t)(x))}{R(t)} = \frac{R(\gamma(s(t))(x))}{R(\gamma(s(t)(0)))} \xrightarrow{w} L(x).$$

If $\Gamma(\cdot)$ (resp. $\gamma(\cdot)$) is not continuous but satisfies (1.6.5), it is easy to modify the proof above.

b) Let L be a R.L.T. stable tail with continuous one-parameter groups $\gamma(\cdot)$ such that $\frac{L(\gamma(t)(x))}{L(\gamma(t)(0))} = L(x)$. Then with $R := L$ (1.6.6) is fulfilled , hence $L \in (\mathrm{DOA})_r(L)$. Conversely, assume there exists $R \in (\mathrm{DOA})_r(L)$ satisfying (1.6.3). Then, as already mentioned in 1.6.2 L is R.L.T. stable

$\square$

Remark 1.6.4. For the R.L.T. stable tails L in 1.6.2 (1),(2) and the corresponding one parameter groups $\gamma(\cdot)$ we have, as mentioned, $c(t) = L(\gamma(t)(0))$, and

$$L(\gamma(t)(x)) = c(t) \cdot L(x) \tag{1.6.7}$$

Put $\Gamma(t) := \gamma(s(t))$, $\widetilde{\Gamma}(t) : x \mapsto \Gamma(t)(x) - t$ where for $t > 0$ we define $s = s(t)$ such that $c(s(t)) = L(t)$. Then (1.6.7) yields $L(\Gamma(t)(x)) = L(t) \cdot L(x)$, i.e.

$$L_t(\widetilde{\Gamma}(t)(x)) = L(x) \tag{1.6.8}$$

And the converse is true too. Hence (1.6.7) and (1.6.8) are equivalent descriptions of R.L.T. stability. More precise, $E_\lambda \in (\mathrm{DOA})_r(E_\lambda)$, and $P_\lambda \in (\mathrm{DOA})_r(P_\lambda)$ (i.e. both of the $(\mathrm{DOA})_r(E_\lambda)$ and $(\mathrm{DOA})_r(P_\lambda)$ are not empty).

Remark 1.6.5. Max-stable distributions on $\mathbb{R}$ were investigated for the first time by Gnedenko [9]. For a recent survey on extreme value theory see Galambos [8]. There exist three types of max-stable distribution functions Φ_α, Λ, Ψ_α, where Φ_α and Ψ_α, $(\alpha > 0)$ are concentrated on $\mathbb{R}_+$, $\mathbb{R}_-$ respectively:

$$\Phi_\alpha(x) = \exp(-(x^{-\alpha})),\ x > 0$$

(Ψ_α is not important in the sequel), and

$$\Lambda_\alpha(x) = \exp(-\alpha \cdot \exp(-x)),\ x \in \mathbb{R},\ \text{for } \alpha > 0.$$

In particular, $\Lambda = \Lambda_1 = \exp(-\exp(-x))$. We obtain

$$-\log \Phi_\alpha(x)\ |_{\mathbb{R}_+} = x^{-\alpha} \cdot 1_{[1,\infty)}(x)$$

and

$$-\log \Lambda(x)\mid_{\mathbb{R}_+} = \exp(-x)$$

are the tails of the Pareto and of an exponential distribution respectively, see remark 1.4.2 both R.L.T. stable distributions.

Let F denote the distribution function of a max-stable law. Then we denote by $(\mathrm{DOA})_m(F)$, the domain of max-stable attraction of F. We obtain for the domains of attractions of R.L.T. stable resp. of max-stable laws:

Theorem 1.6.6. Let F be a distribution function with tail R, $R(x) > 0,\ x > 0$. Let G be a max-stable distribution function, $G(x) = \exp(-H(x))$ for all $x \geq 0$, assume that $\alpha \cdot G\mid_{\mathbb{R}_+}$ is the tail of a distribution function for some $\alpha > 0$. Then we have

$$F \in (\mathrm{DOA})_m(G) \iff F \in (\mathrm{DOA})_r(\alpha H)$$

Precisely: There exist $\gamma_n \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$ such that

$$F^n(\gamma_n(x)) \xrightarrow{w} G(x),\ x \geq 0$$

iff for a function $\gamma(\cdot)$ fulfilling (1.6.5) in 1.6.1 defined by the sequence $\{\gamma_n\}_{n\geq 1}$ we have

$$R_t(\widetilde{\gamma}(t)(x)) \xrightarrow{w} \alpha H(x),\ x \geq 0$$

and vice versa. Analogous results are obtained for shifted versions, if 0 is replaced by some x_0 (with $R(x_0) < 1$).

Proof. " $\Longrightarrow$ " Let F, G be distribution functions, $G = \exp(-H)$. Assume that there exists a sequence of affine transformations $\{\gamma_n\}_{n\geq 1} \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1)$ such that

$$F^n(\gamma_n(x)) \overset{n\to\infty}{\longrightarrow} G(x),\ x \geq 0.$$

According to lemma 0.2.4 this is equivalent with

$$n \cdot (1 - F(\gamma_n(x))) \xrightarrow{w} H(x)$$

hence

$$n \cdot R(\gamma_n(x)) \xrightarrow{w} H(x).$$

Define $\gamma : \mathbb{R}_+ \to \mathrm{Aff}_0^+(\mathbb{R}, 1)$ by $\gamma(t) := \gamma_{n(t)}$ with $n(t) =: n,\ \frac{1}{n+1} < \alpha R(t) \leq \frac{1}{n}$. Therefore,

$$n \cdot \alpha \cdot R(\gamma_{n(t)}(x)) \leq \frac{R(\gamma(t)(x))}{R(t)} \leq (n+1) \cdot \alpha \cdot R(\gamma_{n(t)}(x))$$

Therefore, since $\lim \frac{n+1}{n} = 1$, we obtain

$$\frac{R(\gamma(t)(x))}{R(t)} = \alpha \cdot R_t(\widetilde{\gamma}(t)(x)) \xrightarrow{w} \alpha \cdot H(x),\ x \geq 0$$

" $\Longleftarrow$ ". Assume the existence of a function $t \longmapsto \gamma(t) \in \mathrm{Aff}_0(\mathbb{R})$ such that

$$R_t(\widetilde{\gamma}(t)(x)) \stackrel{t\to\infty}{\longrightarrow} \alpha \cdot H(x),\ x \geq 0$$

Define $\gamma_n \in \mathrm{Aff}_0^+(\mathbb{R},1)$ by $\gamma_n = \gamma(t_n)$, with $\frac{1}{n+1} < \frac{R(\gamma(t_n(x)))}{\alpha} \leq \frac{1}{n}$,
then $t_n \stackrel{n\to\infty}{\longrightarrow} \infty$ and

$$\frac{R(\gamma(t_n)(x))}{R(t_n)} \stackrel{n\to\infty}{\longrightarrow} \alpha \cdot H(x)$$

yield

$$n \cdot R(\gamma_n(x)) \stackrel{n\to\infty}{\longrightarrow} H(x),\ x \geq 0$$

Therefore as above, according to 0.2.4

$$F(\gamma_n(x))^n \stackrel{n\to\infty}{\longrightarrow} \exp(-H(x)) = G(x)$$

$\square$

Corollary 1.6.7. Let F, R as above, let Λ, Φ_α be defined as in [8]. Then

a) $F \in (\mathrm{DOA})_m(\Lambda)$ iff F belongs to the domain of R.L.T. attraction of (shifted) exponential laws

b) $F \in (\mathrm{DOA})_m(\Phi_\alpha)$ iff F belongs to the domain of R.L.T. attraction of (shifted) Pareto laws.

Proof. **a)** Obvious, since $\Lambda(x) = \exp(-\exp(-x))$ hence $\Lambda\ |_{\{x\geq x_0\}} = e^{-H}$, with

$$H(x) = e^{-x}\ |_{\{x\geq x_0\}} = e^{-x_0} \cdot e^{-(x-x_0)}\ |_{\{x\geq x_0\}} =: \alpha \cdot K$$

with K the tail of a shifted exponential distribution.

b) Analogously, $\overline{P}(x) = x^{-\alpha},\ x \geq 1$, is the tail of a shifted Pareto distribution, and

$$\Phi_\alpha(x) = \exp(-(x^{-\alpha})) = \exp(-\overline{P}(x)),\ x \geq 0.$$

$\square$

Remark 1.6.8. For later use (in Chapter 2) we note that in the definitions of R.L.T. stability and domain of R.L.T. attraction (for $d = 1$) we followed the notations introduced in Balkema and de Haan [1]. But we have seen that under mild conditions these definitions are equivalent with the following ones which turn out to be useful for the multivariate case:

Remark 1.6.9. a) From remark 1.3.3 we have already shown that for a limit tail function S which is continuous at $y = 0$ and fulfils $S(0) = 1$, we have

$$\frac{R(\Gamma(t)(x))}{R(t)} \xrightarrow{w} S(x) \text{ (for } t \to \infty) \tag{1.6.9}$$

iff (for some function $s \mapsto \gamma(s) \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$ with $\gamma(s)(x) \to \infty$, $x > 0$) we have

$$\frac{R(\gamma(s)(x))}{R(\gamma(s)(0))} \xrightarrow{w} S(x) \ \ (as\ s \to \infty) \tag{1.6.10}$$

b) Hence in Definition 1.6.1 if L is continuous at 0 and $L(0) = 1$, equation 1.6.1 (resp. 1.6.3) is fulfilled iff for some function $s \mapsto \gamma(s) \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$, $\gamma(s)(x) \overset{s\to\infty}{\longrightarrow} \infty$, $x > 0$,

$$\frac{R(\gamma(s)(x))}{R(\gamma(s)(0))} \xrightarrow{w} L(x) \text{ , (for } s \to \infty) \tag{1.6.11}$$

Therefore,

$$(\mathrm{DOA})_r(L) = \{R : \text{ 1.6.11 holds}\}.$$

c) If in addition R is continuous then (1.6.11) is equivalent to

$$\frac{R(\gamma_{n+1}(0))}{R(\gamma_n(0))} \overset{n\to\infty}{\longrightarrow} 1 \tag{1.6.12}$$

Indeed, this follows immediately from the proof of theorem 1.6.6 and part (a) of this remark.

d) Thus we can re-define domains of R.L.T. attraction as follows

(i) (Domain of R.L.T. attraction):

$$R \in (\mathrm{DOA})_r(L) \text{ iff } \frac{R(\gamma_n(x))}{R(\gamma_n(0))} \xrightarrow{w} L(x)$$

for some sequence $\gamma_n \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$ such that $\gamma_n(x) \overset{n\to\infty}{\longrightarrow} \infty$, $x > 0$, and

$$\frac{R(\gamma_{n+1}(0))}{R(\gamma_n(0))} \overset{n\to\infty}{\longrightarrow} 1$$

(ii) (Normal domain of R.L.T. attraction):

$$R \in (\mathrm{NDOA})_r(L) \text{ iff } \frac{R(\gamma(t)(x))}{R(\gamma(t)(0))} \xrightarrow{w} L(x)$$

where $(\gamma(t)) \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1)$ is a one parameter group such that L is R.L.T. stable w.r.t. $\gamma(\cdot)$, i.e.

$$L(\gamma(t)(x)) = L(\gamma(t)(0)) \cdot L(x),\ t \geq 0.$$

1.7 Domains of attraction of R.L.T. semi-stable laws

With the notation from 0.1. and according to lemma 0.1.14 and 0.1.15 we define the following:

Definition 1.7.1. *Let* $\gamma \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$, *such that* $\gamma^k(x) \overset{k\to\infty}{\longrightarrow} \infty$, $x > 0$, *then*

Case (1) $\gamma(x) = x + \tau$ *and* $\gamma^k(x) = x + k \cdot \tau$, $k \in \mathbb{Z}_+$, $\tau > 0$. *Put*

$$z_{k,\tau} := \gamma^k(0) = k \cdot \tau$$

Case(2) $\gamma(x) = u_0^\alpha(x - x_\star) + x_\star$ *and* $\gamma^k(x) = u_0^{k\alpha}(x - x_\star) + x_\star$, $x_\star < 0, u_0 > 1$, $\alpha > 0$. *Put*

$$z_{k,\alpha} := \gamma^k(0) =\mid u_0^{k\cdot\alpha} - 1 \mid\mid x_\star \mid$$

Put

$$v_k := \begin{cases} z_{k,\tau} = \gamma^k(0) = k \cdot \tau & \text{in case (1)}, \\ z_{k,\alpha} = \gamma^k(0) =\mid u_0^{k\cdot\alpha} - 1 \mid\mid x_\star \mid & \text{in case (2)}. \end{cases}$$

Let $T : [0, v_1) \mapsto [p, 1)$ *be decreasing with* $T(0) = 1$, $T(v_1-) \geq p$ *where* $0 < p < 1$. *Put*

$$L(x) = p^k \cdot T(y) =: \begin{cases} 1 - G_{p,T,\tau,\gamma}(x) & \text{Case (1)}, \\ 1 - N_{p,T,\alpha,\gamma}(x) & \text{Case (2)}. \end{cases}$$

where $v_k \leq x < v_{k+1}$, *such that* $y := x - v_k \in [0, v_1)$.
$L(x) = 1 - G_{p,T,\tau,\gamma}(x)$ *(in case (1)) is called the generalized geometric distribution, and* $L(x) = 1 - N_{p,T,\alpha,\gamma}(x)$ *(in case (2)) is called the generalized discrete Pareto distribution (cf. 1.4, A3–A4).*

Corollary 1.7.2. Put $p = e^{-\beta}$, and put w.l.o.g. $\tau = 1$.

- For $T(x) = 1, 0 \leq x < 1$: $L(x) = 1 - G_{p,1,1,\gamma}(x)$ is the (usual) "geometric distribution G_p".
- For $T(x) = e^{-\beta\cdot x}$, $0 \leq x < 1$: $L(x) = 1 - G_{p,T,\alpha,\gamma}(x)$ is the "exponential distribution function E_β". In a similar way, Pareto distributions and discrete Pareto distributions are representable as "generalized Pareto distribution".
- If $T(x) = 1$, $0 \leq x < 1$: $L(x) = 1 - N_{p,1,\alpha,\gamma}(x)$ is "a discrete Pareto distribution $\mathrm{D}P_{\alpha,1}(x)$"

Definition 1.7.3. a) *Let Γ be R.L.T. semi-stable with tail $1-\Gamma=\Psi$,*

$$\Psi(\gamma(x))=\Psi(\gamma(0))\cdot\Psi(x),\ x\geq 0.$$

Then a distribution function F with tail R belongs to the R.L.T. domain of semi-stable attraction of Γ ($F\in(\mathrm{DOA})_{r,ss}(\Gamma)$) iff there exists γ_n such that $\gamma_n(0)\to\infty$ furthermore

$$R(\gamma_{n+1}(0))/R(\gamma_n(0))\overset{n\to\infty}{\longrightarrow}\beta\in\mathbb{R}_+\ (\beta\in(0,1)) \tag{1.7.1}$$

and

$$R(\gamma_n(x))/R(\gamma_n(0))\overset{n\to\infty}{\longrightarrow}\Psi(x),\ x\geq 0 \tag{1.7.2}$$

b) *F belongs to the normal domain of R.L.T. semi-stable attraction ($F\in(\mathrm{NDOA})_{r,ss}(\Gamma)$) if $\gamma_n=\gamma^n$ in (a).*

Proposition 1.7.4. We have

a) $(\mathrm{DOA})_{r,ss}(\Psi)\neq\emptyset\Longrightarrow\Psi$ is R.L.T. semi-stable

b) If Ψ is R.L.T. semi-stable then the normal domain of R.L.T. semi-stable attraction is non-empty, in fact $\Psi\in(\mathrm{NDOA})_{r,ss}(\Psi)$

Proof. **a)** Assume for a tail function R that $R(\gamma_n(x))/R(\gamma_n(0))\overset{w}{\longrightarrow}\Psi(x)$. Hence

$$\frac{R(\gamma_n\gamma_n^{-1}\gamma_{n+1}(x))}{R(\gamma_n(0))}\cdot\frac{R(\gamma_n(0))}{R(\gamma_{n+1}(0))}\overset{n\to\infty}{\longrightarrow}\Psi(x)$$

Observing that $\frac{R(\gamma_n(0))}{R(\gamma_{n+1}(0))}\overset{n\to\infty}{\longrightarrow}1/\beta\quad(>1)$, we obtain by the convergence of types theorems that
$\gamma_n^{-1}\gamma_{n+1}\overset{n\to\infty}{\longrightarrow}\gamma\in\mathrm{Aff}_0^+(\mathbb{R},1)$. Therefore, $\Psi(\gamma(x))=\beta\cdot\Psi(x),\ x>0$, follows. I.e., Ψ is R.L.T. semi-stable.

b) follows immediately from the definition: We have by (1.7.1) and (1.7.2)

$$\Psi(\gamma^k(x))=\Psi(\gamma^k(0))\cdot\Psi(x)=:p^k\cdot\Psi(x).$$

Note that $k\mapsto\Psi(\gamma^k(0))=c(\gamma^k)$ is a homomorphism. Hence with $p=\Psi(\gamma(0))\in(0,1)$

$$\Psi(\gamma^k(0))=p^k\in(0,1) \tag{1.7.3}$$

□

Remark 1.7.5. In [1] the authors consider limit laws of the following type:

$$\lim_{t\to\infty} R_t(\Gamma(t)(x)) = S(x) \tag{1.7.4}$$

Where in the *discrete case* $t \mapsto \gamma(t)$ is a jump-function, i.e. there exist $t_k \nearrow \infty$, such that $\gamma(t) = \gamma(t_k),\ t_k \le t < t_{k+1}$. The so defined domains of attraction are not equivalent to $(\mathrm{DOA})_{r,ss}$ defined above. In fact, if the stronger condition (1.7.4) is fulfilled, R is R.L.T. semi-stable, hence $R = G_{p,T,\tau,\gamma}$ or $R = N_{p,T,\alpha,\gamma}$ where $T \equiv 1$ on $[0, v_1)$.

Proof. Note that the discrete set $\{v_k\}$ is of the form

$$v_k \in \begin{cases} D_{k,\tau} = \{k\cdot\tau :\ k \in \mathbb{Z}_+, \tau > 0\} & \text{Case (1),} \\ D_{k,\alpha} = \{\mid x_\star \mid (u_0^{k\cdot\alpha} - 1) :\ k \in \mathbb{Z}_+, u_0 \ge 1\} & \text{Case (2).} \end{cases}$$

Let F be a distribution function with the tail $R > 0$. F belongs to the domain of semi-stable R.L.T. attraction of S. Hence

$$R_{v_k}(\gamma^k(x)) \overset{k\to\infty}{\longrightarrow} S(x) \tag{1.7.5}$$

for all continuity points x, $x \ge 0$ where $\gamma \in \mathrm{Aff}(\mathbb{R})$, and $v_k = \gamma^k(0)$ (resp. $\gamma^k(x_0)$). Define $\Gamma : \mathbb{R}_+ \longrightarrow \mathrm{Aff}_0^+(\mathbb{R}, 1)$, $\Gamma(t) := \gamma^k$ if $v_k \le t < v_{k+1}$. (We write $v_k = v_{k(t)}$ in this case). In other words, with this notation we have

$$R_{v_{k(t)}}(\Gamma(t)(x)) \overset{t\to\infty}{\longrightarrow} S(x). \tag{1.7.6}$$

If $t = v_k$ one could also replace (1.7.6) by

$$R_{v_k}(\Gamma(v_k)(x)) \overset{v_k\to\infty}{\longrightarrow} S(x)$$

But not necessarily $R_t(\Gamma(t)(x)) \overset{t\to\infty}{\longrightarrow} S(x)$ as in Balkema, De-Haan [1].
For $v_k < t < v_{k+1}$ we have (if $\Gamma(t)(x) \ge t$)

$$R_t(\Gamma(t)(x)) = \frac{R(\Gamma(t)(x) + t)}{R(t)} = \frac{R(v_k)}{R(t)} \cdot \underbrace{R_{v_k}(\Gamma(v_k)(x))}_{\to S(x)}.$$

Thus, let $\mathrm{LIM}(\cdot)$ denote the set of accumulation points, then

$$\mathrm{LIM}(R_t(\Gamma(t)(x)))_{t\to\infty} \subseteq \mathrm{LIM}(\frac{R(v_k)}{R(t)}) \cdot S(x)$$

At the same time we have

$$\frac{R(v_k)}{R(t)} = \frac{1}{\frac{R(t)}{R(v_k)}} = \frac{1}{\frac{R(\gamma^k(v_1))}{R(\gamma^k(0))}} \; if \; t = \gamma^k(u), 0 \leq u < v_1 = \gamma(0).$$

Thus by assumption,

$$\mathrm{LIM}(\frac{R(v_k)}{R(t)}) \subseteq [1, \lim \frac{R(\gamma^k(0))}{R(\gamma^k(v_1))}] = [1, \frac{1}{S(v_1)}] \text{ or}$$

$$\mathrm{LIM}(R_t(\Gamma(t)(x))) \subseteq S(x) \cdot [1, \frac{1}{T(\gamma(0)-)}] \subseteq S(x) \cdot [1, \frac{1}{q}]$$

So the defined R.L.T. semi-stability, in condition (1.7.5) is not equivalent to the condition in [1]. $\square$

Therefore we obtain

Corollary 1.7.6. The stronger condition $R_t(\Gamma(t)(x)) \overset{t\to\infty}{\longrightarrow} S(x)$ is fulfilled iff T is constant.

1.8 R.L.T. semi-stability and max-semi-stability

Max stable laws and their domains of attraction are well known, whereas for max-semi-stable laws there exist only a few investigations. see e.g. [4], [10], [11], and [24]. With the notation in 0.1, according to lemma 0.1.14 and 0.1.15 we obtain

Definition 1.8.1. *A distribution μ with distribution function F and tail R is called max-semi-stable if there exist $\gamma \in \mathrm{Aff}_0(\mathbb{R})$, $c \in \mathbb{R}_+ \setminus \{1\}$, such that*

$$\begin{aligned} F^{1/c}(\gamma(x)) &= F(x), \quad \textit{equivalently} \\ F(\gamma(x)) &= F^c(x) \end{aligned}$$

for $x \geq x_0$ ($\geq -\infty$) and $F(x) = 0,\ x \leq x_0$.

As we are interested in distributions concentrated on $\mathbb{R}_+$ we assume $\gamma \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$ and $0 < c < 1$ (since then $\gamma^n(x) \to \infty$ for $x > 0$ and $F(\gamma^n(x)) = F^{c^n}(x) \nearrow 1$). Recall that F is max-stable if there exists a one parameter group $(\gamma(t))_{t>0} \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1)$ such that

$$F(\gamma(t)(x)) = F^t(x) \text{ for } t \geq 0,\ x \geq x_0\ (\geq -\infty)$$

Remark 1.8.2. According to lemma 0.1.13 for given γ and c there exists (a uniquely determined) one parameter semigroup $(\gamma(t))_{t>0}$, and (with multiplicative parameterization) such that $\gamma(c) = \gamma$, $c \in \mathbb{R}_+ \setminus \{1\}$. Therefore we obtain:
F is max-semi-stable $\iff$ for a continuous group $(\gamma(t), t > 0)$ we have

$$F(\gamma(c^k)(x)) = F^{c^k}(x),\ k \in \mathbb{Z},\ x \geq x_0.$$

As mentioned in section 1.6 there exist only three types of max-stable distributions. For max-semi-stable distributions the situation is more complicated. We shall obtain a new characterization of max-semi-stable laws in the sequel.
As in 1.6 we define

Definition 1.8.3. *Let F be a distribution function; $F(x) > 0$ for $x > x_0$ and $F(x) = 0$ if $x \leq x_0$. Then*

$$H(x) := \begin{cases} -\log F(x) & x > x_0 \ (i.e. F(x) > 0) \\ \infty & x \leq x_0. \end{cases}$$

is well defined. We obtain $H \geq 0$ and $H \searrow$. If $H(x_1) < \infty$ then we define

$$\widetilde{H}(x) := \begin{cases} \frac{1}{H(x_1)} \cdot H(x) & x \geq x_1 \\ 1 & x < x_1. \end{cases}$$

Then $\widetilde{H}$ is the tail of a probability distribution function with

$$H(x) = H(x_1) \cdot \widetilde{H}(x),\ x \geq x_1$$

Therefore, putting $H(x_1) = \alpha$. We obtain:

$$F(x) = e^{-\alpha \cdot \widetilde{H}(x)},\ x \geq x_1.$$

Theorem 1.8.4. Let F be a distribution function concentrated on $\mathbb{R}_+$, assume $F(x) = e^{-\alpha \cdot H(x)}$, $x \geq 0$ w.l.o.g. $x_1 = 0$ where H is the tail of a distribution function. Then we have:

$$F \text{ is max-semi-stable} \iff H \text{ is R.L.T. semi-stable}$$

Proof. Let F be a max-semi-stable distribution. Hence we have

$$\begin{aligned} F(\gamma(x)) = F^c(x) \quad &\iff \quad e^{-\alpha \cdot H(\gamma(x))} = e^{-c \cdot \alpha \cdot H(x)} \\ &\iff \quad H(\gamma(x)) = c \cdot H(x) \end{aligned}$$

This is the case if and only if H is a solution of the R.L.T. semi-stability functional equation, resp. H is R.L.T. semi-stable. $\square$

Remark 1.8.5. Let F, and H be as in the theorem 1.8.4 above. Assume that F is max-semi-stable. Then H is either $\overline{G}_{p,T,\tau,\gamma}$ or $\overline{N}_{p,T,\alpha,\gamma}$.

Proof. Applying theorem 1.8.4 above, we have H is R.L.T. semi-stable. (I.e. H is a solution of the semi-stability functional equation). See section 1.4, theorem 1.4.4. □

In view of this characterization we recall the definitions of the domain of max-semi-stable attraction, and we obtain immediately:

Definition 1.8.6. a) *Let G be a max-semi-stable distribution function, $G(\gamma(x)) = G^c(x),\ x \geq 0$. Let F be a distribution function. Then F belong to the domain of max semi-stable attraction of G (in short $F \in (\mathrm{DOA})_{m,ss}(G)$) $\Longleftrightarrow$ there exist $k_n \nearrow \infty, k_n/k_{n+1} \to c$ and $\gamma_n \in \mathrm{Aff}_0^+(\mathbb{R}, 1)$ such that*

$$F(\gamma_n(x))^{k_n} \xrightarrow{w} G(x),\ x \geq 0$$

b) *F belongs to the normal domain of max-semi-stable attraction of G if $\gamma_n = \gamma^n$ and $k_n = [1/c^n]$*

c) *F in $(\mathrm{DOA})_r(G)$ belongs to the normal domain of R.L.T. semi-stable attraction if $\gamma_n = \gamma^n$.*

With this notation we obtain

Theorem 1.8.7. Let $G = e^{-\alpha \cdot H}$ as in theorem 1.8.4. Let F be a distribution function concentrated on $\mathbb{R}_+$, let w.l.o.g. $H(0) = 1$. Then we have:

$$F \in (\mathrm{DOA})_{m,ss}(G) \iff F \in (\mathrm{DOA})_{r,ss}(1 - H).$$

And the same relation holds true for normal domains of semi-stable attraction.

Proof. **1)** According to lemma 0.1.14 we have

$$(1 - R(\gamma_n(x)))^{k_n} = F^{k_n}(\gamma_n(x)) \overset{n\to\infty}{\longrightarrow} G(x)\ (= e^{-\alpha \cdot H(x)}) \text{ iff}$$

$$k_n \cdot R(\gamma_n(x)) \overset{n\to\infty}{\longrightarrow} \alpha \cdot H(x)$$

2) Replacing G by $G^{1/\alpha}$, k_n by $[k_n/\alpha]$, we may assume w.l.o.g. $\alpha = 1$

3) Therefore, since $H(0) = 1$ we obtain

$$\lim_{n\to\infty} \frac{R(\gamma_{n+1}(0))}{R(\gamma_n(0))} = \lim_{n\to\infty} \frac{k_n}{k_{n+1}} = c$$

4) Thus

$$k_n \cdot R(\gamma_n(x)) \overset{n\to\infty}{\longrightarrow} H(x) \iff \frac{R(\gamma_n(x))}{R(\gamma_n(0))} \overset{n\to\infty}{\longrightarrow} H(x)$$

i.e. $1 - R = F \in (\mathrm{DOA})_{r,ss}(1 - H)$

5) Conversely, assume

$$\frac{R(\gamma_{n+1}(0))}{R(\gamma_n(0))} \overset{n\to\infty}{\longrightarrow} c \quad \textit{and}$$

$$\frac{R(\gamma_n(x))}{R(\gamma_n(0))} \overset{n\to\infty}{\longrightarrow} H(x)$$

assume again as above $\alpha = 1$ and put $k_n := [1/R(\gamma_n(0))]$. Then, as above

$$k_n \cdot R(\gamma_n(x)) \overset{n\to\infty}{\longrightarrow} H(x) \text{ follows.}$$

I.e. $F \in (\mathrm{DOA})_{m,ss}(G)$ as asserted.
The coincidence of the domains of normal attraction is now obvious.

1.9 Similarities between (semi-) stability, max-(semi-) stability and R.L.T. (semi-) stability.

Definition 1.9.1. *Let $\mu \in M^1(\mathbb{R})$ with distribution function F. An infinitely divisible measure $\mu \in M^1(\mathbb{R})$ is (strictly) stable if and only if there exists a continuous one-parameter group of linear transformations*
$(\gamma(t) : x \mapsto t^\alpha x)_{t>0}$ *(with multiplicative parameterization) such that*

$$\gamma(t)(\mu^s) = \mu^{ts}, \ t, s > 0 \tag{1.9.1}$$

Recall that a subset $(\mu_t)_{t\geq 0} \subseteq M^1(\mathbb{R})$ is called a continuous convolution semigroup if

(i) $\mathbb{R}_+ \ni t \mapsto \mu_t \in M^1(\mathbb{R})$ is weakly continuous

(ii) $\mu_t * \mu_s = \mu_{t+s}, \ t, s \geq 0$

Here (μ^s) is the convolution semigroup with $\mu^1 = \mu$, the number α is called the index of μ (See e.g. Hazod [15], or Meerschaert, Scheffler [20]).
If μ is stable but not strictly stable, then there exists a continuous function

$\mathbb{R}_+^\times \ni t \mapsto \gamma(t) \in \mathrm{Aff}(\mathbb{R})$ such that (1.9.1) holds. But $(\gamma(\cdot))$ is in general not a one-parameter group.
In fact, let $\tau_t: \ x \mapsto t^\alpha \cdot x$. Then μ is usually defined to be stable iff

$$\tau_t(\mu) = \mu^t * \varepsilon_{-b(t)}$$

for some $b(t) \in \mathbb{R}$.
Equivalently, put $\gamma(t) : x \mapsto \tau_t(x) + b(t)$, then

$$\gamma(t)(\mu) = \mu^t, \ t > 0.$$

As easily shown, that if $(\gamma(\cdot))$ were a group then $b(\cdot)$ fulfills the functional equation

$$b(st) = t^\alpha b(s) + b(t), \ t, s > 0 \tag{1.9.2}$$

hence $\gamma(\cdot)$ is a semigroup iff $b(\cdot) \equiv 0$ (in 1.9.2), hence if $\gamma(t) = \tau_t, \ t > 0$.

Definition 1.9.2. *μ is (strictly) semi-stable iff there exist $\gamma \in \mathrm{Aff}_0(\mathbb{R})$ and $c \in (0,1)$ such that $\gamma(\mu^s) = \mu^{cs}, \ s \geq 0$.*

Since γ is embeddable into a continuous one-parameter group $(\gamma(t))_{t>0}$ such that $\gamma(c) = \gamma$ we obtain:

$$\mu \text{ is semi-stable } \iff \gamma(c)(\mu) = \mu^c \tag{1.9.3}$$

Definition 1.9.3. *μ (resp. F) is max-stable iff for some continuous group with $(\gamma(t))_{t>0} \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1)$*

$$F(\gamma(t)(x)) = F^t(x), \ x \geq x_0, \tag{1.9.4}$$

where $F(z) = 0, \ z \leq x_0$

Remark 1.9.4.
- Note that (in the one-dimensional case) F^t is a distribution function for any $t > 0$.
- Note that as mentioned in 1.6.5 a max-stable distribution is of the type $\Lambda, \ \Phi_\alpha$, or Ψ_α (which will be of no importance in the sequel). The corresponding affine transformations $\gamma(\cdot)$ following (1.9.4) are
$\gamma(t) : \ x \mapsto x + t$ and $x \mapsto t^{1/\alpha} \cdot x$ respectively, hence in both cases $\gamma(\cdot)$ is a group of affine transformations.

Definition 1.9.5. *μ (resp. F) is max-semi-stable iff for some $(\gamma(t))_{t>0}$ and $c \in (0,1)$ we have*

$$F(\gamma(c^n)(x)) = F^{c^n}(x), \ x \geq x_0, n \in \mathbb{Z}_+. \tag{1.9.5}$$

This is a motivation to define operators acting on the set of probabilities similar to τ_t in definition 0.3.3

Definition 1.9.6. *Let $\gamma(\cdot)$ denote continuous functions with values in $\mathrm{Aff}_0(\mathbb{R})$. Let ID denote the set of infinity divisible probabilities on $\mathbb{R}$.*

a) *let* $\gamma(t): \ x \mapsto t^\alpha x + b(t)$ *fulfil (1.9.2),* $S^{\gamma(t)} : ID \longrightarrow ID$ *is defined by*

$$S^{\gamma(t)}(\mu) := (\gamma(t)(\mu))^{1/t}$$

b) *let* $\gamma(\cdot)$ *be a continuous one-parameter group in* $\mathrm{Aff}_0^+(\mathbb{R}, 1)$ *e.g. with multiplicative parameterization.* $M^{\gamma(t)} : M^1(\mathbb{R}) \longrightarrow M^1(\mathbb{R})$ *is defined by*

$$M^{\gamma(t)}(F)(x) = (F(\gamma^{-1}(t)(x)))^{1/t}$$

where F *is the distribution function of* μ

c) *let again* $\gamma(\cdot)$ *be a continuous one-parameter group in* $\mathrm{Aff}_0^+(\mathbb{R}, 1)$.
$T^{\gamma(t)} : M_*^1(\mathbb{R}_+) \longrightarrow M_*^1(\mathbb{R}_+)$ *defined by*

$$T^{\gamma(t)}(\mu(x, \infty)) := \frac{\mu(\gamma(t)(x), \infty)}{\mu(\gamma(t)(0), \infty)}, \ x > 0$$

where $M_*^1(\mathbb{R}_+)) = \{\mu \in M^1(\mathbb{R}_+)$ *with* $R(x) > 0, \ x > 0\}$

With this notations we obtain:

Proposition 1.9.7. $t \mapsto S^{\gamma(t)}$, $t \mapsto M^{\gamma(t)}$ and $t \mapsto T^{\gamma(t)}$ are homomorphisms.

Proof. This follows from the definitions by easy calculations. □

Theorem 1.9.8. a) $\mu \in$ ID is strictly stable (resp. semi-stable) iff for some $\gamma(\cdot)$, (and $c \in (0, 1)$) we have $S^{\gamma(t)}(\mu) = \mu, \ t > 0$ (resp. $S^{\gamma(c)}(\mu) = \mu$)

b) $\mu \in M^1(\mathbb{R})$ with distribution function F is max-stable (resp. max-semi-stable) iff for some $\gamma(\cdot)$ (and $c \in (0, 1)$) $M^{\gamma(t)}(F) = F, \ t > 0$ (resp. $M^{\gamma(c)}(F) = F$)

c) $\mu \in M_*^1(\mathbb{R}_+)$ is residual life time stable (resp. R.L.T. semi-stable) iff for some $\gamma(\cdot)$, (and $c \in (0, 1)$) we have

$$T^{\gamma(t)}(\mu) = \mu, \ t > 0 \text{ (resp. } T^{\gamma(c)}(\mu) = \mu)$$

Proof. **a) and b)** follow immediately by using 1.9.1 resp. 1.9.3 and 1.9.4 resp. 1.9.5 above

c) Let R be the tail of μ, let $\mu(t) = T^{\gamma(t)}(\mu)$ with tail $x \mapsto \frac{R(\gamma(t)(x))}{R(\gamma(t)(0))}$ by definition. Hence, obviously, μ is R.L.T. stable iff $\mu(t) = \mu$, i.e. if

$$T^{\gamma(t)}(\mu) = \mu \text{ for } t > 0$$

and μ is R.L.T. semi-stable if

$$\frac{R(\gamma(x))}{R(\gamma(0))} = R(x), \ x > 0.$$

Hence, if γ is embedded into a group $\gamma(\cdot)$ such that $\gamma(c) = \gamma$, the assertion follows.

$\square$

Note that if we define in 1.9.6-(c) with a continuous function $\gamma(\cdot) \subseteq \mathrm{Aff}_0(\mathbb{R})$ with $\gamma(t) \overset{t\to\infty}{\longrightarrow} \infty$, then $T^{\gamma(t)}(\mu) = \mu$ for $t > 0$ yields that

$$(\gamma(\cdot)) \subseteq \mathrm{Dec}(\mu)$$

Hence, as shown in section 1.5, there exist a continuous one-parameter group $\widetilde{\gamma} \subseteq \mathrm{Dec}(\mu)$. I.e. μ is R.L.T. stable.

1.10 References and comments for Chapter 1

This section contains remarks and comments to Chapter 1, and references to further literature on the (semi-) stability of R.L.T. distributions (on $\mathbb{R}$). Particularly, we are interested in a property called " Lack of Memory property (L.M.P.)", and how to generalize this property to characterize the property of (semi-) stability of R.L.T. distributions and their domains of attraction by limit laws. Finally we investigate similar relations between the (semi-) stability, max-(semi-) stability, and R.L.T. (semi-) stability distributions.

R 0.1 In this section we give an overview of affine transformation. We follow Edelstein, Tan [5].

R 0.2 This section contains a survey of C.T.T., on $\mathbb{R}^1$ in particular for probabilities concentrated on $\mathbb{R}_+$. For general version see Letta [17]; for the particular version the reader is referred to Balkema, de Haan [1].

R 0.3 In this section we collected some notations for probabilities on $\mathbb{R}_+$.

R 0.4 This section contains some standard examples of distributions which are important in the sequel.

R 1.1 In this section, we start collecting definitions, remarks, and examples of well-known classical L.M.P. of the exponential distribution followed from J. Galambos and S. Kotz [8] (cited in § 2.1). See also Balkema and de-Haan [1]. Here, we defined R.L.T. (semi-) stable distributions allowing $(\gamma(t))_{t\in\mathbb{R}} \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1)$ to be a continuous one parameter group (see 1.1.6) which may be considered as a generalization of the L.M.P. A further generalization of the L.M.P. (see 1.1.7 and 1.1.8).

R 1.2 Firstly, it should be noted that the L.M.P. is not suitable to characterize R.L.T. (semi-) stability. That is the reason why R.L.T. (semi-) stability is defined, for this (see 1.2.1–1.2.3). Here we introduced a further definition of the (semi-) stability R.L.T. distributions closely with the general L.M.P. but $\gamma(t) \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1)$ not assumed to be a semi-group. Note that all R.L.T. (semi-) stable distributions are characterized by this generalizations of the L.M.P. (see e.g. 1.2.1 or 1.2.2). Finally we investigated that the R.L.T. distributions (in the limit laws) fulfills the (semi-) stability condition (G.L.M.P.) (i.e. the set of distributions satisfying G.L.M.P. is closed, (see 1.2.6).

R 1.3 In this section we followed investigations of Balkema and de-Haan [1]. In (1.3.2) it is proved (applying the convergence of types theorem (1.3.4, and 1.3.5)) that the limit laws fulfill a functional equation which turns out to be the condition of R.L.T. (semi-) stability.

R 1.4 Here we are mainly concerned with a general form of the introduced stability functional equation, and obtained its general solutions after re-formulating it with respect to the formulation of affine semigroups (according to the existence of common fixed point or not) introduced in §0.1 (see 0.1.12). All these solutions are already R.L.T. stable. It should be noted that the solutions in the discrete case are uniquely determined only if the function T is assumed to be constant (e.g. $T \equiv 1$) see 1.4.1.
In most cases the solutions considered to be valid for $x \geq 0$ (resp. $x \geq 1$), same arguments allows to consider solutions for $x \geq x_0$, $t > 0$ see 1.4.2, 1.4.3. Finally, we obtained a new class of limit distributions with a suitable shifted versions see 1.4.4.

R 1.5 Here we introduced the decomposability semigroup of R.L.T. distributions ($\mathrm{Dec}(\mu)$) similar to the useful concept to characterize (operator) semi-stability for vector space and group valued random variables. Again we characterized the (semi-) stability of the limit laws in§ 1.3 by the decomposability semi group (i.e. they belong to the class of distributions characterized in 1.4.4). See in particularly 1.5.13. Decomposability groups and corresponding canonical homomorphisms are essential tools for investigations of (semi-) stability of vector space– and group– valued random variables. See e.g. Hazod, and Siebert [15] §1.5, 1.12, 2.5 .

R 1.6 With the notations following from Balkema [1], we re-write the domain of R.L.T. attraction in an equivalent form related to a continuous one parameter group $(\gamma(t))_{t\geq 0} \subseteq \mathrm{Aff}_0^+(\mathbb{R}, 1)$ (resp. w.r.t. a right continuous function). Moreover, this domain of attraction characterizes the R.L.T. stability of the limit laws (see 1.6.4). It should be noted that, (as in 1.6.5) the domain of R.L.T. attraction for (shifted) exponential laws and (shifted) Pareto laws, are closely related to the domains of attraction for the Max-stable distribution functions $\Phi_\alpha,\ \Lambda_\alpha$ respectively. (See also Balkema [1] theorems 3,4). Finally we re-defined in 1.6.9 R.L.T. stability in an equivalent way and domain of R.L.T. (for $d = 1$), which will be useful in Chapter(2)(for $d > 1$).

R 1.7 Firstly, in this section we investigated in 1.7.1 the general geometric distribution $G_{p,T,\tau,\gamma}(x)$ (case (1)), and general discrete Pareto distribution $N_{p,T,\alpha,\gamma}(x)$ (case (2)). Again analogously, the R.L.T. semi-stability is characterized by its domain of attraction. Moreover it is proved in 1.7.4, that the defined R.L.T. semi-stability, in condition (1.7.5) is not equivalent to the condition in [1]: This stronger condition $R_t(\Gamma(t)(x)) \overset{t\to\infty}{\longrightarrow} S(x)$ is fulfilled iff T is constant.

R 1.8 It should be noted that, the situation for the max-semi-stable more complicated. As we are interested in distributions concentrated on $\mathbb{R}_+$, we defined max-(semi-) stable distributions related to the existence of a continuous one parameter groups $(\gamma(t))_{t\geq 0} \subseteq \mathrm{Aff}_0^+(\mathbb{R},1)$ (see 1.8.1 and 1.8.2). As in 1.8.4 if the distribution function $F(x) = e^{-\alpha\cdot H(x)},\ x \geq 0$ allow F to be max-semi-stable then $H \in \{\overline{G}_{p,T,\tau,\gamma}\ ,\ \overline{N}_{p,T,\alpha,\gamma}\}$. Finally we investigated the similarity between the max-semi stable domain of attraction (of a distribution function $G = e^{-\alpha\cdot H}$, (as in 1.8.7) and the domain of R.L.T. semi-stable attraction).
(Semistable R.L.T. and similarities to max-semi-stability are not considered in [1].)

R 1.9 In this section we start collecting definitions on (strictly) stable, semi stable, max-stable, and max-semi-stable distributions related to a continuous one parameter group $(\gamma(t))_{t\geq 0} \subseteq \mathrm{Aff}_0^+(\mathbb{R},1)$ (see 1.9.1–1.9.5). We introduced in 0.3.3 and 1.9.7 a motivation to define a homomorphism operators $t \mapsto S^{\gamma(t)}$, $t \mapsto M^{\gamma(t)}$ and $t \mapsto T^{\gamma(t)}$ acting on the set of probabilities. This allowed us to re-define (strictly-) stable, max-(semi-) stable, and (semi-) stable R.L.T. see 1.9.8. (Compare also the characterization of semi-stable processes by invariance under space–time–transformations, See e.g. [15]§3.6.24).

Chapter 2

(Semi-) stability of R.L.T. distributions in the multidimensional case

2.1 The structure of affine transformations on $\mathbb{R}^d$. The subgroups of coordinate-wise affine transformations $\mathbf{CAT}(\mathbb{R}, d))$ $(d > 1)$

In the preparatory section 0.1 affine transformations were introduced. Here in chapter 2 we investigate a subgroup $\mathrm{CAT}(\mathbb{R}, d)$ which will play the role of affine normalization. First we recall and fix some notations.

Notation 2.1.1.

- Let $M(\mathbb{R}, d)$ denote the algebra of real $d \times d$ matrices, $\mathrm{GL}(\mathbb{R}, d)$ the general linear group.
- Let $\mathcal{A}(\mathbb{R}, d)$ denote the set of affine transformations on $\mathbb{R}^d$, $T : \vec{x} \mapsto A\ \vec{x} + \vec{b}$, $A \in M(\mathbb{R}, d)$, $\vec{b} \in \mathbb{R}^d$.
- We always have fixed vector space bases, hence identify linear transformations and matrices, and we use the notation $T = \gamma_{A, \vec{b}}$.

- We shall denote the group of affine transformations by

$$\mathrm{Aff}(\mathbb{R}, d) = \{\gamma_{A,\vec{b}} \in \mathcal{A}(\mathbb{R}, d) : A \in \mathrm{GL}(\mathbb{R}, d)\}$$

and $\mathrm{Aff}_0(\mathbb{R}, d)$ the connected component of the unit element $\gamma_{I,\vec{0}} = \mathrm{Id}$.

Remark 2.1.2. Analogous to the remark 0.1.16 we assume that the normalizing affine transformations $T = \gamma_{A,\vec{b}}$ have the following properties

(i) $\vec{x} \mapsto T(\vec{x})$ is strictly increasing (coordinate wise)

(ii) $T(\vec{x}) > \vec{0}$ for all $\vec{x} > \vec{0}$ and

(iii) $(T^n(\vec{x}))_i \overset{n\to\infty}{\longrightarrow} \infty,\ 1 \leq i \leq d$, for all $\vec{x} > \vec{0}$.

Hence we begin with the following definition

Definition 2.1.3. *(Coordinate-wise affine transformations): We define*

$$\mathrm{CAT}(\mathbb{R}, d) = \{\gamma_{A,\vec{b}} : A \textit{ is diagonal with positive entries}\} \subseteq \mathrm{Aff}_0(\mathbb{R}, d). \tag{2.1.1}$$

For $\vec{u} \in \mathbb{R}^d$ *let* $\mathrm{diag}(\vec{u}) := \begin{pmatrix} u_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & u_d \end{pmatrix}$.

Hence

$$\mathrm{CAT}(\mathbb{R}, d) = \{\gamma_{\mathrm{diag}(\vec{u}),\vec{b}} : \vec{u} > \vec{0}, \vec{b} \in \mathbb{R}^d\}$$

Note that $\mathrm{CAT}(\mathbb{R}, d)$ is a proper subgroup of $\mathrm{Aff}_0(\mathbb{R}, d)$ iff $d > 1$.
We observe for $\gamma_{A,\vec{a}}$, $\gamma_{B,\vec{b}} \in \mathcal{A}(\mathbb{R}, d)$

$$\gamma_{A,\vec{a}} \circ \gamma_{B,\vec{b}} = \gamma_{A\cdot B, A\cdot\vec{b}+\vec{a}} \tag{2.1.2}$$

and for $\gamma_{A,\vec{a}} \in \mathrm{Aff}(\mathbb{R}, d)$

$$\gamma^{-1}_{A,\vec{a}} = \gamma_{A^{-1},-A^{-1}\vec{a}} \tag{2.1.3}$$

$\mathrm{Aff}(\mathbb{R}, d)$ contains subgroups isomorphic to $(\mathbb{R}^d, +)$ and to $\mathrm{GL}(\mathbb{R}, d)$ respectively:

$$\mathbb{S}(\mathbb{R}, d) := \{\gamma_{\mathrm{I},\vec{a}} : \vec{a} \in \mathbb{R}^d\} \cong (\mathbb{R}^d) \tag{2.1.4}$$

the group of translations (or shifts), and

$$\mathbb{L}(\mathbb{R}, d) := \{\gamma_{A,\vec{0}} : A \in \mathrm{GL}(\mathbb{R}, d)\} \cong \mathrm{GL}(\mathbb{R}, d), \tag{2.1.5}$$

the group of linear transformations.
Put furthermore
$$\mathbb{D}(\mathbb{R}, d) := \mathrm{CAT}(\mathbb{R}, d) \cap \mathbb{L}(\mathbb{R}, d), \tag{2.1.6}$$
the group of coordinate-wise linear transformations (resp. diagonal matrices).

Affine transformations on $\mathbb{R}^d$ may be considered as linear transformations on $\mathbb{R}^{d+1}$, in fact
$$\varphi : \gamma_{A, \vec{b}} \mapsto \Gamma_{A, \vec{b}} := \begin{pmatrix} 1 & {}^t\vec{0} \\ \vec{b} & A \end{pmatrix} \in M(\mathbb{R}, d+1) \tag{2.1.7}$$
is a continuous injective homomorphism. In the sequel we use this interpretation frequently. For $(C, \vec{c}) \in M(\mathbb{R}, d) \times \mathbb{R}^d$ let $X_{C, \vec{c}}$ denote the matrix in $M(\mathbb{R}, d+1))$:
$X_{C, \vec{c}} := \begin{pmatrix} 0 & {}^t\vec{0} \\ \vec{c} & C \end{pmatrix}$.

In section 0.1 we started with the description of one-parameter groups
$\mathbb{R} \ni t \mapsto T_t \in \mathrm{Aff}(\mathbb{R}, d)$. Let $T_t = \gamma_{A(t), \vec{b}(t)}$. Recall that
$$T_{t+s} = T_t T_s \iff \begin{cases} A(t+s) = A(t)A(s) & \text{and} \\ \vec{b}\,(t+s) = A(t)\,\vec{b}\,(s) + \vec{b}\,(t) & . \end{cases} \tag{2.1.8}$$
(Analogous relations are obtained if we switch to multiplicative parameterization, putting $S_u := T_{\log u}$, $u > 1$. Hence $S_u S_v = S_{uv}$.) One-parameter groups (T_t) are always assumed to be continuous. Hence according to basic Lie group theory $t \mapsto T_t$ is even analytic and the derivatives $\frac{d}{dt} T_t \mid_{t=0}$ may be considered as elements of the Lie algebra. If we consider the above mentioned matrix representation we obtain for $T_t = \gamma_{A(t), \vec{b}(t)}$ (resp. $\Gamma_{A(t),\, \vec{b}(t)}$) : $A(t) = \exp(tQ)$ for some $Q \in M(\mathbb{R}, d)$, hence $\frac{d}{dt} A(t) \mid_{t=0} = Q$ and $\frac{d}{dt}\, \vec{b}\, (t) \mid_{t=0} =: \vec{d} \in \mathbb{R}^d$. (Note that $T_0 = \mathrm{Id} = \gamma_{I,\, \vec{0}}$, hence $\vec{b}\,(0) = \vec{0}$.) Therefore the infinitesimal generator exists
$$\frac{d}{dt} \Gamma_{A(t), \vec{b}(t)} \mid_{t=0} = X_{Q, \vec{d}} \tag{2.1.9}$$
with the above- mentioned notation. For further use, note
$$[X_{A, \vec{a}}, X_{B, \vec{b}}] = X_{[A,B], A\vec{b} - B\vec{a}} \tag{2.1.10}$$
where $[U, V] := UV - VU$ as usual. We write
$$\Gamma_{A(t), \vec{b}(t)} = \exp(tX_{Q, \vec{d}}) \tag{2.1.11}$$

if $X_{Q,\vec{d}}$ is the infinitesimal generator, and since the matrix representation is injective, we also write (by abuse of language) $T_t = \exp(tX_{Q,\vec{d}})$ in this case.
For the above mentioned subgroups we observe for $T_t = \gamma_{A(t),\vec{b}(t)} = \exp(tX_{Q,\vec{d}})$:

- $(T_t : t \in \mathbb{R}) \subseteq \mathrm{CAT}(\mathbb{R}, d)$ iff $A(t) = \mathrm{diag}(e^{t\cdot q_1}, \ldots, e^{t\cdot q_d}) = \exp(t \cdot Q)$ with $Q = \mathrm{diag}(q_1, \ldots, q_d)$, i.e. iff Q is diagonal.

- $(T_t : t \in \mathbb{R}) \subset \mathbb{S}(\mathbb{R}, d)$ iff $A(t) = \mathrm{Id}$ hence iff $Q = 0$. Then obviously $\vec{b}\,(t) = t\cdot \vec{d}$, $T_t = \gamma_{I,t\cdot\vec{d}}$, and the infinitesimal generator is given by $X_{Q,\vec{d}} = X_{0,\vec{d}}$.

- $(T_t : t \in \mathbb{R}) \subseteq \mathbb{L}(\mathbb{R}, d)$ iff $\vec{d}=\vec{0}$, hence iff $X_{Q,\vec{d}} = X_{Q,\vec{0}}$ and we have
$$T_t = \gamma_{\exp(tQ),\vec{0}}$$

- According to section 0.1 (0.1.3–0.1.7) we observe: Either $\{T_t\ \vec{x}\colon\ t \in \mathbb{R}\}$ is unbounded for all $\vec{x}$ or there exists a common fixed point $\vec{x}_\star$. The first case appears e.g. for shifts, i.e. for $T_t = \gamma_{I,t\cdot\vec{d}}$, in the second case we have
$$T_t\ \vec{x}= e^{tQ} \cdot (\vec{x} - \vec{x}_\star)+ \vec{x}_\star= e^{tQ}\ \vec{x} +(I - e^{tQ})\cdot\ \vec{x}_\star$$
hence $T_t = \gamma_{e^{tQ},(I-e^{tQ})\cdot\vec{x}_\star}$

- Put for short $\Gamma_t := \Gamma_{A(t),\vec{b}(t)} = \exp(tX_{Q,\vec{d}})$. Then
$$\frac{d}{dt}\Gamma_t = \Gamma_t X_{Q,\vec{d}} = X_{Q,\vec{d}}\Gamma_t,\ t \in \mathbb{R}.$$

Hence $\vec{b}\,(\cdot)$ satisfies the differential equation
$$\frac{d}{dt}\ \vec{b}\,(t) =\vec{d} +Q\ \vec{b}\,(t) = e^{tQ}\ \vec{d},\ \ \vec{b}\,(0) =\vec{0} \tag{2.1.12}$$

We are going to investigate the structure of one-parameter subgroups
$$\{T_t, t \in \mathbb{R}\} \subseteq \mathrm{CAT}(\mathbb{R}, d)$$
in more details. (For a slightly different description see Balkema and de Yong-Cheng Qi [2].)

Definition 2.1.4. *Let* $(T_t = \gamma_{e^{tQ}, \vec{b}(t)}) \subseteq \mathrm{CAT}(\mathbb{R}, d)$ *with* $Q = \mathrm{diag}(q_1, \ldots, q_d)$*. Put* $D_0 := \{i : q_i = 0\}$ *and* $D_1 := \complement D_0 = \{i : q_i \neq 0\}$*. Let*

$$\mathbb{V}_i := \{\vec{x} \in \mathbb{R}^d : x_j = 0, j \notin D_i\} = \sum_{j \in D_i} \oplus \mathbb{R}\ ,\ i = 0, 1$$

Let, as above, $\vec{d} := \frac{d}{dt}\ \vec{b}\Big|_{t=0}(t)$*. Furthermore, let* $T_t^{(i)}$ *denote the one-dimensional affine groups* $\mathbb{R} \ni z \mapsto e^{tq_i} \cdot z + b_i(t) (= (T_t z \cdot \vec{e})_i$*,* $\vec{e}_i$ *denoting the* i^{th} *unit vector,* $1 \leq i \leq d$*.) The general considerations above yield for this particular case:*

$$\frac{d}{dt} b_i(t) = d_i + q_i b_i(t) = e^{tq_i} \cdot d_i\ , b_i(0) = 0 \tag{2.1.13}$$

Whence we obtain immediately

Corollary 2.1.5. Assume, as above $T_t \subseteq \mathrm{CAT}(\mathbb{R}, d)$. Then we have :

a) $i \in D_0 \Longrightarrow b_i(t) = t \cdot d_i,\ t \in \mathbb{R}\ ,\ (T_t\ \vec{x})_i = x_i + t \cdot d_i$

b) $i \in D_1 \Longrightarrow b_i(t) = (e^{tq_i} - 1) \cdot \frac{d_i}{q_i}\ ,\ (T_t\ \vec{x})_i = e^{tq_i} \cdot x_i + (e^{tq_i} - 1) \cdot \frac{d_i}{q_i} = e^{tq_i}(x_i + \frac{d_i}{q_i}) - \frac{d_i}{q_i}$.
Hence in this case $T_t^{(i)}$ has a fixed point $(\vec{x}_\star)_i = -\frac{d_i}{q_i}$.

(Note that in a) $(T_t\ \vec{x})_i \equiv x_i$ if $d_i = 0$.)
With this observation we can describe the behavior for $t \to \infty$

Proposition 2.1.6. With the above notations we have:

a) Let $i \in D_0$. Then $(T_t\ \vec{x})_i \nearrow \infty$ for $t \to \infty \iff d_i > 0$.
In this case, $(T_t\ \vec{x})_i > 0$ if $t > 0,\ x_i > 0$

b) Let $i \in D_1$. Then $(T_t\ \vec{x})_i \nearrow \infty$ for $t \to \infty \iff (\vec{x})_i > -\frac{d_i}{q_i}$ and $q_i > 0$.

As in chapter 1 we are interested only in distributions concentrated on $\mathbb{R}_+^d$, therefore in CAT*s* (T_t) with $(T_t\ \vec{x})_i \stackrel{t\to\infty}{\longrightarrow} \infty$ for all i and for all $\vec{x}$ in a suitable region of $\mathbb{R}_+^d$. Therefore we introduce the notations

Definition 2.1.7.

$$\begin{aligned}
\mathfrak{cat}(\mathbb{R}, d) &:= \{X_{Q, \vec{d}} : (\exp(t \cdot X_{Q, \vec{d}}) \in \mathrm{CAT}(\mathbb{R}, d)\ \textit{for}\ t > 0\} \\
&= \{X_{Q, \vec{d}} : Q = \mathrm{diag}(q_1, \ldots q_d)\} \\
\mathrm{CAT}^+(\mathbb{R}, d) &= \{X_{Q, \vec{d}} : Q = \mathrm{diag}(q_1, \ldots, q_d) :\ q_i \geq 0,\ 1 \leq i \leq d \\
&\qquad\qquad \textit{and}\ \ d_i > 0\ \textit{if}\ \ q_i = 0,\ \ \textit{i.e. if}\ \ i \in D_0\}.
\end{aligned}$$

Hence for $X_{Q, \vec{d}} \in \mathrm{CAT}^+(\mathbb{R}, d)$ *the corresponding group* (T_t) *fulfils* $(T_t\ \vec{x})_i \nearrow \infty$ *for* $t \to \infty$ *for all* $\vec{x} \in \mathbb{R}_+^d$ *with* $x_i \geq 0$*,* $i \in D_0$ *and* $x_i \geq -\frac{d_i}{q_i}$*,* $i \in D_1$

Proposition 2.1.8. Let $\gamma_{U,\vec{u}}$, $\gamma_{A,\vec{a}} \in \mathrm{Aff}(\mathbb{R}, d)$ then

$$\gamma_{U,\vec{u}}\gamma_{A,\vec{a}}\gamma^{-1}_{U,\vec{u}} = \gamma_{UAU^{-1},(I-UAU^{-1})\vec{u}+U\vec{a}}.$$

Therefore, the subgroup $\mathbb{S}$ of shifts is a normal subgroup,

$$\mathbb{S} = \{\gamma_{I,\vec{a}} : \vec{a} \in \mathbb{R}^d\} \lhd \mathrm{Aff}(\mathbb{R}, d).$$

In fact, $\gamma_{U,\vec{u}}\gamma_{I,\vec{a}}\gamma^{-1}_{U,\vec{u}} = \gamma_{I,U\vec{a}}$.
Applying this observation to one-parameter groups $T_t = \exp(tX_{Q,\vec{d}}) = \gamma_{\exp(tQ),\vec{b}(t)}$ we obtain

Proposition 2.1.9. Let $T_t = \exp(tX_{Q,\vec{d}})$. Put $T_t^{U,\vec{u}} := \gamma_{U,\vec{u}}T_t\gamma^{-1}_{U,\vec{u}}$. Then $(T_t^{U,\vec{u}})$ is a one parameter group,

$$T_t^{U,\vec{u}} = \gamma_{\exp(tUQU^{-1}),(I-\exp(tUQU^{-1}))\vec{u}+U\vec{b}(t)}$$

with infinitesimal generator $X_{\widetilde{Q},\widetilde{d}}$,

$$\widetilde{Q} = UQU^{-1}, \widetilde{d} = U\vec{d} - UQU^{-1}\vec{u}$$

Proof.

$$\begin{aligned}
\frac{d}{dt}|_{t=0}\,\Gamma_{U,\vec{u}}\Gamma_{\exp(tQ),\vec{b}(t)}\Gamma^{-1}_{U,\vec{u}} &= \Gamma_{U,\vec{u}}X_{Q,\vec{d}}\Gamma^{-1}_{U,\vec{u}} \\
&= \begin{pmatrix} 1 & 0 \\ \vec{u} & U \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \vec{d} & Q \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -U^{-1}\vec{u} & U^{-1} \end{pmatrix}
\end{aligned}$$

yields the assertion. □

In particular we obtain

$$\Gamma_{U,\vec{0}}X_{Q,\vec{d}}\Gamma^{-1}_{U,\vec{0}} = X_{UQU^{-1},U\vec{d}} \tag{2.1.14}$$

and

$$\Gamma_{I,\vec{u}}X_{Q,\vec{d}}\Gamma_{I,-\vec{u}} = X_{Q,-Q\vec{u}+\vec{d}} \tag{2.1.15}$$

With the notations introduced in definition 2.1.4 $\mathbb{R}^d = \mathbb{V}_0 \oplus \mathbb{V}_1$, we obtain a decomposition $\vec{d} = \vec{d}^{(0)} + \vec{d}^{(1)}$, $\vec{d}^{(i)} \in \mathbb{V}_i$, such that $Q|_{\mathbb{V}_0} = 0$, hence $\exp(tQ)|_{\mathbb{V}_0} \equiv \mathrm{id}_{\mathbb{V}_0}$. We obtain

Proposition 2.1.10. For a one-parameter group $(T_t) \subseteq \mathrm{CAT}(\mathbb{R}, d)$ with $Q = \mathrm{diag}(q_1, \ldots, q_d)$, $\vec{d} = \vec{d}^{(0)} + \vec{d}^{(1)}$ there exists a shift $\gamma_{I,\vec{u}}$ such that $(\gamma_{I,\vec{u}} T_t \gamma_{I,-\vec{u}})$ has $X_{Q,\vec{d}^{(0)}}$ as infinitesimal generator.

Proof. In fact, choose $\vec{u} \in \mathbb{V}_1$ such that $Q\,\vec{u} = d^{(1)}$, i.e. $u_i = \frac{d_i}{q_i}$, $i \in D_1$. Then

$$\gamma_{I,\vec{u}} \cdot X_{Q,\vec{d}} \cdot \gamma_{I,-\vec{u}} = X_{Q,-Q\vec{u}+\vec{d}} = X_{Q,\vec{d}^{(0)}}$$

$\square$

This yields

Corollary 2.1.11. Assume $Q\,\vec{u} = \vec{0}$. Then $X_{Q,\vec{u}} = X_{Q,\vec{0}} + X_{0,\vec{u}}$ with $[X_{Q,\vec{0}}, X_{0,\vec{u}}] = 0$. Therefore, if $T_t = \gamma_{A(t),\vec{b}(t)}$ with infinitesimal generator $X_{Q,\vec{u}}$ then

$$T_t = \gamma_{\exp(tQ),0} \cdot \gamma_{I,t\vec{u}} = \gamma_{I,t\vec{u}} \cdot \gamma_{\exp(tQ),0}$$

Proof. As immediately seen $[X_{Q,\vec{0}}, X_{0,\vec{u}}] = 0$. Whence the remanning assertion follows by elementary Lie group theory $\square$

Corollary 2.1.12. Let $T_t = \exp t(X_{Q,\vec{d}^{(0)}})$ as in the above proposition 2.1.10, let $\mathbb{R}^d = \mathbb{V}_0 + \mathbb{V}_1$ as above. Then for $\vec{x} = \vec{x}^{(0)} + \vec{x}^{(1)}$, $\vec{x}^{(i)} \in \mathbb{V}_i$.

$$(T_t\,\vec{x})_i = \begin{cases} x_i + t \cdot d_i & i \in D_0, \\ e^{t \cdot q_i} \cdot x_i & i \in D_1. \end{cases} \tag{2.1.16}$$

We decompose $D_1 = \biguplus_{j=1}^{r} D^{(j)}$ such that $q_k = q_l =: p_j$ if $k, l \in D_j$, $p_i \neq p_j$, $i \neq j$. Hence $Q = \sum_{j=1}^{r} \oplus p_j \cdot \mathrm{Id}_{\mathbb{W}_j}$, $\mathbb{W}_j = \{\vec{x} \colon x_k = 0, k \notin D_j\} \cong \sum_{i \in D^{(j)}} \oplus \mathbb{R}$. With this notations we obtain $\mathbb{V}_1 = \sum_{j=1}^{r} \oplus \mathbb{W}_j$, and $T_t\,|_{\mathbb{V}_1} = \sum_{j=1}^{r} \oplus e^{t \cdot p_j} \cdot \mathrm{Id}_{\mathbb{W}_j}$.
Putting things together we have proved

Theorem 2.1.13. Let $(T_t = \exp t(X_{Q,\vec{d}})) \subseteq \mathrm{CAT}(\mathbb{R}, d)$. Then $\mathbb{R}^d$ is decomposed as a direct sum of lattice ideals

$$\mathbb{R}^d = \mathbb{V}_0 \oplus \sum_{i=1}^{r} \oplus \mathbb{W}_i$$

where $\mathbb{V}_0 = \{\vec{x}\colon x_k = 0, k \notin D_0\}$, $\mathbb{W}_i = \{\vec{x}\colon x_k = 0, k \notin D^{(i)}\}$ and there exists a basis transformation $\gamma_{I,\vec{u}}$ of coordinate-wise shifts such that $\gamma_{I,\vec{u}} T_t \gamma^{-1}_{I,\vec{u}} =: \widetilde{T}_t$ decomposes as a direct sum of a shift and of linear operators

$$\widetilde{T}_t \mid_{\mathbb{V}_0} \oplus \sum_{i=1}^{r} \oplus \widetilde{T}_t \mid_{\mathbb{W}_i}$$

with $\widetilde{T}_t \mid_{\mathbb{V}_0} \colon \vec{z} \mapsto \vec{z} + t \vec{b}^{(0)}$ and $\widetilde{T}_t \mid_{\mathbb{W}_i} \colon \vec{z} \mapsto e^{tp_i} . \vec{z}, \;\; i = 1, \dots r.$

Remark 2.1.14. a) Note that, for all $T_t = \gamma_{A(t), \vec{b}(t)} = \exp t(X_{Q, \vec{d}^{(0)}}) \in \mathrm{CAT}(\mathbb{R}, d)$ with the above assumptions, we have

$$\vec{x} \leq \vec{y} \Longrightarrow T_t \, \vec{x} \leq T_t \, \vec{y} \;\; \forall \, \vec{x}, \vec{y} \in \mathbb{R}^d_+$$

Therefore, we restrict ourselves in the following considerations to one-parameter-subgroups in $\mathrm{CAT}(\mathbb{R}, d)$.

b) Note that in the above considerations we used

$$\mathrm{CAT}(\mathbb{R}, d) = \bigoplus_{i=1}^{d} \mathrm{Aff}_0(\mathbb{R}, 1) \text{ since}$$

$$\gamma_{\mathrm{diag}(q_1, \dots q_d), \vec{b}} = \sum_{i=1}^{d} \oplus \gamma_{q_i, b_i} \;, \; \gamma^{(i)} = \gamma_{q_i, b_i} \text{ acting on } \mathbb{R} \cdot \vec{e}_i \cong \mathbb{R}$$

Define as in the one-dimensional case
$\mathrm{CAT}^+(\mathbb{R}, d) = \bigoplus_{i=1}^{d} \mathrm{Aff}^+_0(\mathbb{R}, 1)$

$= \{\gamma = \sum_{i=1}^{d} \oplus \gamma_{q_i, b_i} : q_i = 1, b_i > 0 \text{ or } q_i > 1, b_i \geq 0\}$. Then it follows easily by reduction to the one dimensional situation (in 0.1 remark 0.1.16) that

$$\mathrm{CAT}^+(\mathbb{R}, d) = \{\gamma \in \mathrm{CAT}(\mathbb{R}, d) : \; (i), \; (ii) \textit{ and } (iii) \text{ in 2.1.2 hold}\} \tag{2.1.17}$$

2.2 Multidimensional versions of the C.T.T.

In this section we note some generalizations of the convergence of types theorems mentioned in section 0.2 to the multidimensional case. See e.g. [14], [15] or [20].

Definition 2.2.1. *A probability measure $\mu \in M^1(\mathbb{R}^d)$ is called full if μ is not supported by a proper affine subspace.*

Proposition 2.2.2. *(Convergence of types theorem)* Let $\mu_n \in M(\mathbb{R}^d)$ be probability measures of $\mathbb{R}^d$, for $n \in \mathbb{N}$, $(d > 1)$, and assume that $\mu_n \to \mu$ for some full measure $\mu \in M^1(\mathbb{R}^d)$.
Let $\{T_n := \gamma_{A_n, \vec{b}_n}\}_{n \geq 1}$ be a sequence in $\mathcal{A}(\mathbb{R}, d)$, assume that

a) $T_n(\mu_n) \to \lambda$ for some full measure $\lambda \in M^1(\mathbb{R}^d)$.
Then the sequence $\{T_n\}_{n \geq 1}$ is relatively compact in $\mathcal{A}(\mathbb{R}, d)$, and for all limit points T we have $T(\mu) = \lambda$.

b) If we assume in addition $T_n \in \mathrm{Aff}(\mathbb{R}, d)$ (resp. $\in \mathrm{CAT}(\mathbb{R}, d)$) and μ, λ are full, then $\{T_n\}_{n \geq 1}$ is relatively compact in $\mathrm{Aff}(\mathbb{R}, d)$ (resp. $\mathrm{CAT}(\mathbb{R}, d)$) and any limit point T is invertible, with $T(\mu) = \lambda$, $T^{-1}(\lambda) = \mu$.

This theorem will be applied in the sequel where we assume

$$T_n \in \mathrm{CAT}(\mathbb{R}, d).$$

In fact, since $\mathrm{CAT}(\mathbb{R}, d) \cong \sum_1^d \oplus \mathrm{Aff}_0(\mathbb{R}, 1)$ it would be sufficient to replace fullness by "CAT-fullness": μ is CAT-full iff all marginals $\pi_i(\mu) \in M^1(\mathbb{R}^1)$ are non degenerate, where $\pi_i : \vec{x} \mapsto x_i$ denote the coordinate projections. (See e.g. [15] §1.13 in particular 1.13.29–30 or [14] §4.5). We shall need the following well known

Definition 2.2.3. *Let* $\mu \in M^1(\mathbb{R}^d_+)$. *Then*

$$\mathrm{Sym}(\mu) := \{T \in \mathrm{Aff}(\mathbb{R}, d) : T(\mu) = \mu\} \tag{2.2.1}$$

called the ***symmetry group*** *of* μ. *By* $\mathrm{Inv}(\mu)$ *we denote the set* $\{\gamma \in \mathrm{CAT}(\mathbb{R}, d) : \gamma(\mu) = \mu\} = \mathrm{Sym}(\mu) \cap \mathrm{CAT}(\mathbb{R}, \mathrm{d})$ *called the* ***invariance group*** *of* μ.

Proposition 2.2.4. *Note that* $\mathrm{Sym}(\mu)$ *is a closed subgroup of* $\mathrm{Aff}(\mathbb{R}, d)$ *(See e.g. Hazod and Siebert [15] page 14,15), and* $\mathrm{Inv}(\mu)$ *is a closed subgroup of* $\mathrm{Sym}(\mu)$.

As in section 0.2 we reformulate the convergence of types theorem in the context of distribution functions. We call F_μ non degenerate iff μ is full.

Proposition 2.2.5. Let $F_n = F_{\mu_n}$ and $F = F_\mu$ denote the distribution functions of μ_n and μ respectively. Assume that

a)' $F_n(\cdot) \xrightarrow{w} F(\cdot)$ for some non-degenerate distribution function F. Let $\{T_n\}_{n \geq 1}$ be as in proposition 2.2.2, and

b)' $F(T_n^{-1}(\cdot)) \xrightarrow{w} \widetilde{F}(\cdot)$ for some non-degenerate distribution function $\widetilde{F}$. Then we have $\{\tau_n := T_n^{-1}\}_{n \geq 1}$ is a relatively compact sequence in $\mathrm{Aff}(\mathbb{R}, d)$ (resp. in $\mathrm{CAT}(\mathbb{R}, d)$) and for all limit points τ we have $F_\mu(\tau(\vec{x})) = \widetilde{F}(\vec{x})$.

The following version is sometimes useful.

Proposition 2.2.6. *Let F and $\widetilde{F}$ be non-degenerate distribution function, let T_n, $\tau_n \in$ Aff$(\mathbb{R}, d)$ (resp. CAT$(\mathbb{R}, d)$.) be defined as before. Assume that*

$$F(T_n(\cdot)) \xrightarrow{w} F(\cdot)$$

and

$$F(\tau_n(\cdot)) \xrightarrow{w} \widetilde{F}(\cdot).$$

Then the sequence $\{M_n := \tau_n T_n^{-1}\}_{n\geq 1}$ is relatively compact and for all limit points M we have

$$F(M(\vec{x})) = \widetilde{F}(\vec{x}) \ , \vec{x} \in \mathbb{R}_+^d$$

Proposition 2.2.7. a) According to 2.2.3 and 2.2.5 we have: μ is full iff Sym(μ) is compact (See e.g. [15] page 15).

b) In this case Inv(μ) is a compact subgroup of CAT$(\mathbb{R}, d)$.

Proof. The assertion follows by applying the convergence of type theorem to 2.2.4 above. □

Lemma 2.2.8. Let $G \subseteq$ CAT$(\mathbb{R}, d)$ be any compact subgroup of CAT$(\mathbb{R}, d)$. Then $G = \{\text{id}\}$

Proof. Let $\gamma \in G$. Since G is compact, $\{\gamma^n : n \in \mathbb{Z}\}$ is bounded. Assume that $\gamma \neq$ id. According to the structure of the subgroup of CAT$(\mathbb{R}, d)$ in section 2.1 (bounded case) γ is given as:
$\gamma(\vec{x}) = A(\vec{x} + \vec{x}_0) - \vec{x}_0$ with a fixed point $x_\star = -\vec{x}_0$, $A = \text{diag}(a_1, \ldots, a_d)$, $a_i > 0$, for $\vec{x} \geq \vec{0}$.

- If there exists i, such that $a_i > 1$ then $(\gamma^n(\vec{x}))_i \to \infty$, $\vec{x} > \vec{0}$, a contradiction to boundedness of $\{\gamma^n\}$.
- If $a_i \leq 1$ for all i, and there exists i_0 such that $a_{i_0} < 1$ then consider γ^{-1} : As above, $\{(\gamma^n)^{-1} : n \in \mathbb{N}\}$ is not bounded, a contradiction.

Hence $a_i = 1$ for all i. I.e. $\gamma = \gamma_{I,\vec{0}} =$ id. Therefore, $G = \{\text{id}\}$. □

Result 2.2.9. Let μ be full. Inv(μ) is a compact subgroup of CAT$(\mathbb{R}, d)$. Applying lemma 2.2.8, we have Inv$(\mu) = \{\text{id}\}$. Hence in the particular case of CAT'*s* we observe:

Theorem 2.2.10. *Convergence of types theorem for* CAT$(\mathbb{R}, d)$: Let $\{\mu_n\}_{n\geq 1}$ be a sequence of probability distributions of $\mathbb{R}^d$, $d > 1$, let μ, ν be full distributions. Assume that $\{\gamma_n\}_{n\geq 1}$ is a sequence in CAT$(\mathbb{R}, d)$. Assume in addition that

1) $\mu_n \to \mu$

2) $\gamma_n(\mu_n) \to \nu$.
Then $\gamma_n \to \gamma \in \mathrm{CAT}(\mathbb{R}, d)$ and $\gamma(\mu) = \nu$

Proof. According to prop. 2.2.2 we have $\{\gamma_n\}_{n\geq 1}$ is relatively compact in $\mathrm{CAT}(\mathbb{R}, d)$ hence $\gamma_n \overset{n'}{\to} \gamma$ for some subsequence $(n)'$ and any limit point γ is invertible with $\gamma(\mu) = \nu$ and $\gamma^{-1}(\nu) = \mu$ (since ν full). Assume that $\{\gamma_n\}$ converges to another limit point i.e. $\gamma_n \overset{(\widetilde{n})}{\to} \widetilde{\gamma}$ for some subsequence $\widetilde{(n)}$ then $\widetilde{\gamma}$ is invertible with $\widetilde{\gamma}(\mu) = \nu$ and $\widetilde{\gamma}^{-1}(\nu) = \mu$. Define $\beta := \widetilde{\gamma}^{-1}\gamma \in \mathrm{CAT}(\mathbb{R}, d)$, we have $\beta(\mu) = \mu \Longrightarrow \beta \in \mathrm{Sym}(\mu)$. $\mathrm{Sym}(\mu) \cap \mathrm{CAT}(\mathbb{R}, d) =: \mathrm{Inv}(\mu)$. Since $\mathrm{Inv}(\mu)$ is a compact subgroup of $\mathrm{CAT}(\mathbb{R}, d)$ (by lemma 2.2.8) we have $\mathrm{Inv}(\mu) = \{\mathrm{id}\}$ follows . Hence $\beta = \mathrm{I}$, and $\widetilde{\gamma} = \gamma$. Hence we obtain $\gamma_n \overset{n\to\infty}{\to} \gamma$ and moreover $\gamma(\mu) = \nu$ for any limit point γ □

2.3 The set of probability measures $M^1(\mathbb{R}^d_+)$

In this section, we reformulate the multivariate case of the notations and theorems given in section 0.3 as a preparatory of chapter 2 In this case a random variable X is usually called random vector and we begin with the following notations. (For details see e.g. [16] or [20])

Notation 2.3.1. Let $(\Omega, \sum, P)$ be a probability space, let $X : \Omega \to \mathbb{R}^d$ be a random vector, the distribution is denoted by μ, and the distribution function is F with the tail function $R = \overline{F}$.

$$F(\vec{x}) := P(X_1 \leq x_1, \ldots, X_d \leq x_d) = \mu(-\infty, \vec{x}] \tag{2.3.1}$$

We are mostly concerned with probabilities on $\mathbb{R}^d_+$. I.e. $F(\vec{x}) > 0$ only for $\vec{x}$ with $x_i \geq 0$, $1 \leq i \leq d$. In short we write

$$F(\vec{x}) = P(X \leq \vec{x})\ \forall\ \vec{x} \in \mathbb{R}^d_+ \tag{2.3.2}$$

and the tail function R is

$$R(\vec{x}) = P(X > \vec{x})\ \forall\ \vec{x} \in \mathbb{R}^d_+ \tag{2.3.3}$$

Note that the relation

$$R(\vec{x}) = 1 - F(\vec{x})\ \forall\ \vec{x} \geq \vec{0}$$

holds only in the one dimensional case, but in general , this simple relation fails to hold (that is not true in the multidimensional case). See [18] page 168.

Remark 2.3.2. With the above notations, if X_n and X are random vectors with distribution μ_n and μ (resp. distribution function F_n and F) respectively then we write again

$$\mu_n \to \mu \; (resp. \; F_n \xrightarrow{w} F) \iff F_n(\vec{x}) \longrightarrow F(\vec{x}) \tag{2.3.4}$$

for all continuity points of F.
This is the case iff

$$R_n(\vec{x}) \longrightarrow R(\vec{x}) \tag{2.3.5}$$

for all continuity points of R.

Since we are interested in the limit behavior of tail functions (distributions) of non-negative random vectors. We begin with the tail functions, which we can obtain from the univariate case.

Definition 2.3.3. *Let X be a random vector, F is the distribution function, and the tail is R. Let Λ denote the tail of a one-dimensional distribution. Then we define $R : \mathbb{R}_+^d \to [0,1]$ by*

$$R(\vec{x}) := \Lambda(x_1 + \ldots + x_d) = \Lambda(\langle \vec{x}, \vec{e} \rangle)$$

for all $\vec{x} \geq \vec{0}$. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product of the vector space $\mathbb{R}^d$,
$\vec{e} = (1, 1, \ldots, 1)$

Notation 2.3.4. Since $\langle \vec{x}, \vec{e} \rangle \geq 0 \; \forall \; \vec{x} \geq \vec{0}$ we can define a continuous positive real valued function $\psi : \mathbb{R}_+^d \to \mathbb{R}_+$ by

$$\psi(\vec{x}) := \langle \vec{x}, \vec{e} \rangle \; \forall \; \vec{x} \geq \vec{0}$$

Then we have

$$R(\vec{x}) = \Lambda(\psi(\vec{x})) \tag{2.3.6}$$

which coincides with the univariate case replacing $\langle \vec{x}, \vec{e} \rangle$ instead of x.
And now we give some illustrative examples to show how we obtain the tails (for $d > 1$)

Example 2.3.5. *Exponential distribution :* The tail is:

$$\overline{E}_\alpha(\vec{x}) = e^{-\alpha \cdot \langle \vec{x}, \vec{e} \rangle} \; , \vec{x} \geq \vec{0}, \alpha > 0$$

Example 2.3.6. *Shifted Exp. distribution:* Put $E_{\alpha, \vec{x}_0}(\vec{x}) = E_\alpha(\vec{x} - \vec{x}_0)$. In a similar way, we obtain the tail $\overline{E}_{\alpha, \vec{x}_0}(\vec{x})$, and we have

$$\overline{E}_{\alpha, \vec{x}_0}(\vec{x}) = \overline{E}_\alpha(\vec{x} - \vec{x}_0) = e^{\alpha \cdot \langle \vec{x}_0, \vec{e} \rangle} \cdot e^{-\alpha \cdot \langle \vec{x}, \vec{e} \rangle} \; \forall \; \vec{x} \geq \vec{x}_0, \alpha > 0$$

Example 2.3.7. <u>*Standard Pareto distribution*</u>*:* The tail is

$$\overline{P}_{\alpha,1}(\overrightarrow{x}) = (1 + \langle \overrightarrow{x}, \overrightarrow{e} \rangle)^{-\alpha} \; \forall \; \overrightarrow{x} \geq \overrightarrow{0}, \alpha > 0$$

By this way we can obtain multidimensional tail functions for all univariate examples given in section 0.4.

Definition 2.3.8. *A first possible generalization of R.L.T. distribution is as follows:*

$$F_{t\overrightarrow{e}}(\overrightarrow{x}) := P(X_1 \leq x_1 + t, \ldots, X_d \leq x_d + t \mid X_1 > t, \ldots, X_d > t)$$

for all $x_i \geq 0, \; 1 \leq i \leq d, \; t > 0$
In short we write

$$F_{t\overrightarrow{e}}(\overrightarrow{x}) = P(X \leq \overrightarrow{x} + t \cdot \overrightarrow{e} \mid X > t \cdot \overrightarrow{e}) \; \forall \; \overrightarrow{x} \in \mathbb{R}^d_+, t > 0 \tag{2.3.7}$$

If $R(t \cdot \overrightarrow{e}) > 0 \; \forall \; t > 0$ then R.L.T. is analytically defined as:

$$R_{t\cdot\overrightarrow{e}}(\overrightarrow{x}) = \min(1, \frac{R(\psi(\overrightarrow{x} + t \cdot \overrightarrow{e}))}{R(\psi(t \cdot \overrightarrow{e}))}) \; \forall \; \overrightarrow{x} \geq \overrightarrow{0}, \; t > 0$$

where ψ is defined as in notation 2.3.4. More generally, we define

Definition 2.3.9. *Let* $\mu \in M^1(\mathbb{R}^d)$ *be any distribution and* F *is the distribution function with the tail function* R*. Hence the R.L.T. distribution may be defined by a transformation acting on a set of probabilities* $M^1(\mathbb{R}^d_+)$ *as*
$\tau_t : M^1(\mathbb{R}^d_+) \to M^1(\mathbb{R}^d_+)$ *defined by*

$$\tau_t(\mu)(\overrightarrow{0}, \overrightarrow{x}] = \frac{\mu(t \cdot \overrightarrow{e}, \overrightarrow{x} + t \cdot \overrightarrow{e}]}{\mu(t \cdot \overrightarrow{e}, \infty)} \quad \textit{if } \mu(t \cdot \overrightarrow{e}, \infty) > 0 \tag{2.3.8}$$

Equivalently, for this distribution function

$$\tau_t(F(\overrightarrow{x})) = F_t(\overrightarrow{x}) \tag{2.3.9}$$

Using equations 2.3.8 and 2.3.9 then:
If F is the distribution function of μ then F_t is the distribution function of $\tau_t(\mu)$. Moreover we have $(\tau_t; t \geq 0)$ is a continuous one parameter semi-group.
Equivalent representation: If $R(t \cdot \overrightarrow{e}) > 0$ then

$$\tau_t(\mu(\Lambda_{\overrightarrow{x}})) := \frac{\mu(\Lambda_{\overrightarrow{x}+t\cdot\overrightarrow{e}})}{\mu(\Lambda_{t\cdot\overrightarrow{e}})} \quad \forall \; \overrightarrow{x} \geq \overrightarrow{0}, t > 0 \tag{2.3.10}$$

where $\Lambda_{\vec{y}}$ is the cuboid defined by

$$\Lambda_{\vec{y}} = \{\vec{x} \in \mathbb{R}^d_+ \ : y_i < x_i \ , 1 \leq i \leq d\}$$

Another point of view: Let F be a distribution function of a non-negative random vector X of $\mathbb{R}^d_+$, and let $t \mapsto \gamma(t) \in \mathrm{Aff}_0(\mathbb{R}, d)$ (or $\in \mathrm{CAT}(\mathbb{R}, d)$) be some function. Put for $\vec{y} > \vec{0}$

$$R_{\vec{y}}(\vec{x}) := \frac{R(\vec{x} + \vec{y})}{R(\vec{y})}, \quad \text{if } R(\vec{y}) > 0.$$

Assume $\gamma(t)(\vec{x}) > \vec{x} \ \ \forall \ \vec{x} \geq \vec{0}$ and $t > 0$. Then we have

$$R_{t\cdot \vec{e}}(\widetilde{\gamma}(t)(\vec{x})) = R(\vec{x}) \iff R(\gamma(t)(\vec{x})) = R(t\cdot \ \vec{e}\ ) \cdot R(\vec{x})$$

where $\widetilde{\gamma}(t) : \vec{x} \to \widetilde{\gamma}(t)(\vec{x}) := \gamma(t)(\vec{x}) - t\cdot \ \vec{e}$. We obtain a functional equation of the type

$$R(\gamma(t)(\vec{x})) = R(t\cdot \ \vec{e}\ ) \cdot R(\vec{x}) = R(\gamma(t)(\vec{0})) \cdot R(\vec{x}) \tag{2.3.11}$$

similar to the one-dimensional situation. More generally we write

$$R(\gamma(t)(\vec{x})) = c(t) \cdot R(\vec{x}) \tag{2.3.12}$$

for some real $c(t)(= R(\gamma(t)(\vec{0}))) > 0$. Functional equations of this type (stability functional equations) will again be important in the sequel.

2.4 The multidimensional lack of memory property (M.L.M.P.)

Firstly we discuss shortly generalizations of the lack of memory property. It will turn out, that for $d > 1$, this concept is not useful to investigate R.L.T. limit distributions. Nevertheless we present a few concepts which had been investigated in the past.

2.4.1 The classical lack of memory property

Various generalizations of the lack of memory property were investigated in the past, such as "extended form of classical L.M.P. property" See e.g. Galambos [8] (cited on p.22), and "strong L.M.P. " See e.g. Kotz [16] (cited on p.69). Firstly we shall discuss shortly these concepts connected to a continuous group of transformations:

Definition 2.4.1. *Let X be a random vector of $\mathbb{R}^d$ with probability measure $\mu \in M^1(\mathbb{R}^d)$. We say that μ possesses the strong L.M.P. iff there exist a continuous group of transformations $(T(t) := \gamma_{I,t\cdot \vec{e}} \in \mathbb{S}(\mathbb{R}, d))_{t\in\mathbb{R}}$ defined by*

$$T(t) : \vec{x} \mapsto \vec{x} + t\cdot \vec{e}$$

for $\vec{x} \in \mathbb{R}_+^d$ where $\vec{e} = (1, \dots, 1)$, $t \in \mathbb{R}_+$ such that

$$\mu(\Lambda_{T(t)(\vec{x})}) = \mu(\Lambda_{T(t)(\vec{0})}) \cdot \mu(\Lambda_{\vec{x}}) \tag{2.4.1}$$

where $\Lambda_{\vec{y}} := \{\vec{x} \in \mathbb{R}_+^d : x_i > y_i, 1 \le i \le d\}$ as before

Definition 2.4.2. *(Equivalent formulation) Let F be the distribution function of μ in 2.4.1. Let R denote the tail function, where*

$$R(\vec{x}) = P(X_1 > x_1, \dots, X_d > x_d)$$

Equation (2.4.1) holds iff

$$R(x_1 + t, \dots, x_d + t) = R(t, \dots, t) \cdot R(x_1, \dots, x_d) \tag{2.4.2}$$

which can be written in the equivalent vector form as

$$R(T(t)(\vec{x})) = R(t\cdot \vec{e}) \cdot R(\vec{x}) = R(T(t)(\vec{0})) \cdot R(\vec{x}) \tag{2.4.3}$$

with $T(t)(\cdot)$ be as in Definition 2.4.1 above.

Definition 2.4.3. *(Weak lack of memory property (W.L.M.P.)): We say that F in 2.4.2 possesses the W.L.M.P. iff for all $\vec{x} \ge \vec{0}$ and all $\vec{a} \in E$ we have*

$$R(\vec{a} \circ \vec{x} + t\,\vec{e}) = R(t\cdot \vec{e}) \cdot R(\vec{a} \circ \vec{x}) \tag{2.4.4}$$

where $\vec{a} \circ \vec{x} = (a_1x_1, \dots, a_dx_d) = diag(\vec{a})\,\vec{x}$ and
$E := \{\vec{a}:$ only one of a_i's is 0 and the others are $1\}$. (See [16] cited on page 406).

Notation 2.4.4. We define the set of all points which satisfy the condition of W.L.M.P. with respect to the distribution function F as:

$$WLM(F) := \{t\,\vec{e} : \ t > 0 : \ (2.4.4) \text{ holds } \forall \ \vec{x} \ge \vec{0} \ \text{ and all } \ \vec{a} \in E\}$$

And the set of all points which satisfy the condition of S.L.M.P. as:

$$SLM(F) := \{t\,\vec{e} : t > 0 : \ (2.4.2) \text{ holds } \forall\, \vec{x} \ge \vec{0}\}$$

It is obvious that $SLM(F) \subset WLM(F)$.

Now for simplicity we consider the bivariate case (d=2), and we give the following illustrating examples.

Example 2.4.5. (2-dimensional exponential distribution)(See example 2.3.5):

$$R(x_1, x_2) = \exp(-(x_1 + x_2)) \ , x_1, x_2 \geq 0.$$

Put $T(t) := \gamma_{I,t\cdot \vec{e}} : (x_1, x_2) \mapsto (x_1, x_2) + t \cdot (1, 1)$. Then we have

$$\begin{aligned} R(T(t)(\vec{x})) &= \exp(-(x_1 + t, x_2 + t)) \\ &= \exp(-(t + t)) \cdot \exp(-(x_1, x_2)) \\ &= \exp(-2 \cdot t) \cdot \exp(-(x_1, x_2)) \\ &= \exp(-2t) \cdot R(\vec{x}). \end{aligned}$$

And as above

$$\exp(-2t) = R(t\cdot \vec{e}) = R(T(t)(\vec{0}))$$

Hence this distribution possesses the strong L.M.P.

Example 2.4.6. (2-dimensional standard Pareto distribution) (See example 2.3.7): Consider the 2-dimensional standard Pareto distribution defined by the tail as

$$\overline{P}(x_1, x_2) = (1 + x_1 + x_2)^{-1} \ , x_1, x_2 > 0.$$

Recall that for $d = 1$, this distribution does not possess the weak L.M.P. (it does not possesses the strong L.M.P.)

Since these distributions play an important role in the limit laws of residual life times, therefore we will obtain a suitable generalization of the L.M.P. in the next section, (stability of R.L.T. distributions) to cover all the limits distributions similar to Pareto distributions, and later we present this example with some details as R.L.T. stable distribution.

2.4.2 A generalization of the multidimensional lack of memory property (G.L.M.P.)

Definition 2.4.7. *We say that, the probability measure $\mu \in M(\mathbb{R}_+^d)$ (resp. the distribution function F or the tail R) possesses the general multidimensional L.M.P. iff there exist a continuous one parameter group*
$t \mapsto T(t) : \vec{x} \mapsto A(t)\cdot \vec{x} + \vec{b}\,(t) \in \mathrm{CAT}(\mathbb{R}, d)$, *$t > 0$, with $(T(t)(\vec{x}))_i \to \infty$ for $t \to \infty$, $1 \leq i \leq d$, $\vec{x} > \vec{0}$ such that*

$$R(T(t)(\vec{x})) = R(T(t)(\vec{0})) \cdot R(\vec{x}), \ x \in \mathbb{R}_+^d \tag{2.4.5}$$

More generally, this can be written as

$$R(T(t)(\vec{x})) \equiv c(t) \cdot R(\vec{x}) \tag{2.4.6}$$

for some $c(t) = R(T(t)(\vec{0})) > \vec{0}$, for all $\vec{x} > \vec{0}$.

If there exists $\vec{x}_0 > \vec{0}$ such that $(T(t)(\vec{x}_0))_i \to \infty$ for all i and $\vec{x} > \vec{x}_0$ and if $R(\vec{x}_0) > \vec{0}$ then

$$c(t) = \frac{R(T(t)(\vec{x}_0))}{R(\vec{x}_0)} \tag{2.4.7}$$

For $\vec{y} = T(t)(\vec{x}_0)$, $t = t(\vec{y}) > 0$ we have

$$R(\vec{y}) = c(t) \cdot R(\vec{x}_0) = c(t(\vec{y})) \cdot R(\vec{x}_0) \tag{2.4.8}$$

Hence for $\vec{y}$ in the orbit $\{T(t)(\vec{x}_0),\ t > 0\}$, $R(\vec{y})$ is uniquely determined by $T(\cdot)$ and the function $c(\cdot)$

Remark 2.4.8. We consider a multidimensional case of example 2.4.5 :
$R(\vec{x}) = \exp(-\langle \vec{x}, \vec{e} \rangle)$, $\vec{x} \in \mathbb{R}^d_+$. As in the case $d = 2$ we have

$$R(T(t)(\vec{x})) = c(t) \cdot R(\vec{x}).$$

with $c(t) = R(T(t)(\vec{0})) = R(t \cdot \vec{e}) = e^{-d \cdot t}$.

Example 2.4.9. Consider the 2-dimensional standard Pareto distribution defined by the tail $R = \overline{P}$ given in example 2.4.6 :

$$R(x_1, x_2) = (1 + x_1 + x_2)^{-1} \ , x_1, x_2 > 0.$$

We define $T(t) \in \mathrm{CAT}(\mathbb{R}, d)$ by $T(t) := \gamma_{e^t \cdot I, (e^t - 1) \cdot \frac{1}{2} \vec{e}}$. Then we have

$$\begin{aligned} R(T(t)(\vec{x})) &= (1 + e^t \langle \vec{x}, \vec{e} \rangle + \frac{e^t - 1}{2} \cdot \langle \vec{e}, \vec{e} \rangle)^{-1} \\ &= (e^t \langle \vec{x}, \vec{e} \rangle + e^t)^{-1} \\ &= e^{-t}(1 + \langle \vec{x}, \vec{e} \rangle)^{-1} \\ &= e^{-t} R(\vec{x}). \end{aligned}$$

Hence the equation (2.4.6) follows with $c(t) = e^{-t}$. Hence this distribution possesses the G.L.M.P. Moreover we observe that

$$R(T(t)(\vec{0})) = (1 + 0 + \frac{e^t - 1}{2} \langle \vec{e}, \vec{e} \rangle)^{-1} = e^{-t} = c(t).$$

Hence the equation 2.4.5 also follows.

Example 2.4.10. (Direct products): Assume μ_i to be distributions with tails R_i on $\mathbb{R}^{d_i}$, $i = 1, 2$ which have the affine groups $T_t^{(i)}$ acting on $\mathbb{R}^{d_i}$. Put $d := d_1 + d_2$ define

$$T_t : \mathbb{R}^d = \mathbb{R}^{d_1} \oplus \mathbb{R}^{d_2} \to \mathbb{R}^d \text{ by } T_t(\vec{x}^{(1)} \oplus \vec{x}^{(2)}) := (T_t^{(1)} \vec{x}^{(1)}) \oplus (T_t^{(2)} \vec{x}^{(2)}).$$

And let F denote the distribution function of $\mu_1 \otimes \mu_2$ with tail R . Hence

$$R(\vec{x}^{(1)} \oplus \vec{x}^{(2)}) = R_1(\vec{x}^{(1)}) \cdot R_2(\vec{x}^{(2)}) \text{ for all } \vec{x} = \vec{x}^{(1)} \oplus \vec{x}^{(2)} \in \mathbb{R}_+^d.$$

Assume that R_i possess the G.L.M.P. w.r.t $T_t^{(i)}$, $i = 1, 2$. Then R possess the G.L.M.P. w.r.t. (T_t). In fact, we have $R_i(T_t^{(i)}(\vec{x}^{(i)})) = c_i(t) \cdot R_i(\vec{x}^{(i)}),\ i = 1, 2.$
Put $\vec{x} = \vec{x}^{(1)} \oplus \vec{x}^{(2)}$. Hence

$$\begin{aligned} R(T_t \vec{x}) &= R_1(T_t^{(1)} \vec{x}^{(1)}) \cdot R_2(T_t^{(2)} \vec{x}^{(2)}) \\ &= c_1(t)c_2(t) \cdot R(\vec{x}^{(1)} \oplus \vec{x}^{(2)}) \end{aligned}$$

Hence F possesses the G.L.M.P. (with $c(t) = c_1(t) \cdot c_2(t)$).
Note that the exponential distribution in 2.4.8 is a direct product of one-dimensional ones, however the Pareto distribution 2.4.9 is not representable as a direct product.

2.5 Solutions of the stability functional equation (Multidimensional case)

In this section we solve a multidimensional general stability functional equation (2.4.6) which appeared in the preceding section. Firstly we consider the continuous case and we begin with the the following assumptions.
Let $\Lambda : A \subseteq \mathbb{R}^d \to (0, 1]$ be a non- constant decreasing continuous function defined on some subset A of $\mathbb{R}^d$. Let $(\gamma(t))_{t\in\mathbb{R}}$ be a one-parameter group of CAT$(\mathbb{R}, d)$. Assume that $\gamma(t + s) = \gamma(t)\gamma(s)$, $t, s \in \mathbb{R}$ (additive parameterization)
resp. $\widetilde{\gamma}(uv) = \widetilde{\gamma}(u) \cdot \widetilde{\gamma}(v), u, v > 1$ (multiplicative parameterization). Assume that

$$\Lambda(\gamma(t)(\vec{x})) = c(t) \cdot \Lambda(\vec{x}) \ , \ \vec{x} > \vec{0} \tag{2.5.1}$$

Assume in 2.5.1 that

$$(\gamma(t)(\vec{x}))_i \overset{t\to\infty}{\longrightarrow} \infty \ , 1 \le i \le d,\ \vec{x} > \vec{0} \tag{2.5.2}$$

Furthermore

$$0 \le \Lambda \le 1 \text{ and } \Lambda(\vec{x}) \to 0 \text{ if } x_i \to \infty, 1 \le i \le d. \tag{2.5.3}$$

The solution in the continuous case: According to the structure of affine transformations and the subgroups of CAT′s of $\mathbb{R}^d$ presented in section 2.1, we consider the following two cases:

A1) $\gamma(t)(\vec{x}) = t\cdot(\vec{x} + \vec{x}_0) - \vec{x}_0$ with a fixed point $\vec{x}_\star = -\vec{x}_0$, $\vec{x}_0 > \vec{0}$ (multiplicative parameterization).

A2) $\gamma(t)(\vec{x}) = \vec{x} + t\,\vec{x}_0$ for $\vec{x}_0 > \vec{0}$ (additive parameterizations).

Firstly we consider the case **A1**), and we begin with the following descriptions

Remark 2.5.1. Let $\gamma(t)(\vec{x})$ be as in case **A1**), and assume Λ to be a solution of (2.5.1). For $A \subseteq \mathbb{R}^d_+$, let

$$A^\gamma := \{\gamma(t)(\vec{x}) : t > 0, \vec{x} \in A\}. \tag{2.5.4}$$

Then there exists a unique extension of Λ to A^γ such that $t \mapsto c(t)$ is a continuous homomorphism $(\mathbb{R}^\times_+, \cdot) \to (\mathbb{R}^\times_+, \cdot)$, and immediately we obtain $c(t) = t^\beta$ for some real β. According to 2.5.2, and 2.5.3 we have $c(t) \stackrel{t\to\infty}{\longrightarrow} 0$, for $t \to \infty$ hence we write $\beta = -\alpha$ for some $\alpha > 0$. Therefore, the extension of Λ is given by

$$\Lambda(\gamma(t)(\vec{x})) = t^{-\alpha} \cdot \Lambda(\vec{x}) \ , \ \vec{x} \in A^\gamma, t > 0. \tag{2.5.5}$$

Note that for $A = \mathbb{R}^d_+$ we obtain the following description:

Proposition 2.5.2. Let $Q = \langle \vec{x}_0 \rangle^\perp = \{\vec{q} : \langle \vec{q}, \vec{x}_0\rangle = 0\}$. For any $\vec{x} \in \mathbb{R}^d_+$ there exists a unique $\vec{q} = \vec{q}_{\vec{x}} \in Q$ such that $\vec{x} = \gamma(t)(\vec{q})$ for some (unique) $t = t(\vec{x}) > 0$. Indeed, $t = t(\vec{x}) = 1 + \frac{\langle \vec{x}, \vec{x}_0\rangle}{\langle \vec{x}_0, \vec{x}_0\rangle}$, $\vec{q} = \vec{q}_{\vec{x}} = \gamma(\frac{1}{t})(\vec{x})$

Proof. If $t(\vec{x})$, $\vec{q}_{\vec{x}}$ exist, then $\vec{q} = \gamma(\frac{1}{t})(\vec{x}) = \frac{1}{t}(\vec{x} + \vec{x}_0) - \vec{x}_0$ and
$\langle \vec{q}, \vec{x}_0\rangle = 0$. Put $c := \langle \vec{x}_0, \vec{x}_0\rangle$. Since $\langle \vec{q}, \vec{x}_0\rangle = 0$ hence
$\frac{1}{t}\langle \vec{x}, \vec{x}_0\rangle + \frac{1}{t}\cdot c - c = 0 \implies t\cdot c = \langle \vec{x}, \vec{x}_0\rangle + c$. Hence $t = t(\vec{x}) = 1 + \frac{1}{c}\langle \vec{x}, \vec{x}_0\rangle$ and hence $\vec{q} = \vec{q}_{\vec{x}} = \frac{1}{t(x)}\cdot(\vec{x} + \vec{x}_0) - \vec{x}_0$.
Conversely, if $t(\vec{x})$, $\vec{q}_{\vec{x}}$ are defined in this way we obtain $\vec{x} = \gamma(t(\vec{x}))(\vec{q}_{\vec{x}})$. □

Remark 2.5.3. Let $\vec{q} \in Q$. Then $\gamma(t)(\vec{q}) \in Q$ iff $t = 1$.

Proof.

$$\begin{aligned}
\gamma(t)(\vec{q}) \in Q,\ \vec{q} \in Q \quad &\Longleftrightarrow \quad \langle \gamma(t)(\vec{q}), \vec{x}_0 \rangle = 0 \text{ and } \langle \vec{q}, \vec{x}_o \rangle = 0 \\
&\Longleftrightarrow \quad \langle t \cdot (\vec{q} + \vec{x}_0) - \vec{x}_0, \vec{x}_0 \rangle = 0 \text{ and } \langle \vec{q}, \vec{x}_o \rangle = 0 \\
&\Longleftrightarrow \quad t \cdot \langle \vec{q}, \vec{x}_0 \rangle + t \cdot \langle \vec{x}_0, \vec{x}_0 \rangle - \langle \vec{x}_0, \vec{x}_0 \rangle = 0 \text{ and } \langle \vec{q}, \vec{x}_0 \rangle = 0 \\
&\Longleftrightarrow \quad (t-1) \cdot \langle \vec{x}_0, \vec{x}_0 \rangle = 0 \\
&\Longleftrightarrow \quad t = 1
\end{aligned}$$

as asserted. $\square$

Corollary 2.5.4. If we normalize $\vec{x}_0$ such that
$c = \langle \vec{x}_0, \vec{x}_0 \rangle = \| \vec{x}_0 \|_2^2 = 1$ then $t = t(\vec{x}) = 1 + \langle \vec{x}, \vec{x}_0 \rangle$ and hence

$$\vec{q} = \vec{q}_{\vec{x}} = \frac{1}{1 + \langle \vec{x}, \vec{x}_0 \rangle} \cdot (\vec{x} + \vec{x}_0) - \vec{x}_0 \, .$$

Note that $\vec{q}_{\vec{x}}$ is well defined as long as $1 + \langle \vec{x}, \vec{x}_0 \rangle > 0$, hence for $\{\vec{x} \colon \langle \vec{x}, \vec{x}_0 \rangle > -1\}$, in general for $\{\vec{x} \colon \frac{1}{c} \langle \vec{x}, \vec{x}_0 \rangle > -1\}$, so at least for

$$\langle \vec{x}_0 \rangle^+ := \{\vec{x} \colon \langle \vec{x}, \vec{x}_0 \rangle \geq 0\}$$

Lemma 2.5.5. *For all $\vec{x} \in \langle \vec{x}_0 \rangle^+$, all $t > 0$ we have $\vec{q}_{\vec{x}} = \vec{q}_{\gamma(t)(\vec{x})}$*

Proof. We have $\vec{q}_{\vec{x}} \in Q$ $(:= \langle \vec{x}_0 \rangle^\perp)$ therefore $\vec{q}_{\gamma(t)(\vec{x})} \in Q$ since

$$\langle \vec{q}_{\gamma(t)(\vec{x})}, \vec{x}_0 \rangle = \langle \vec{q}_{\vec{x}}, \vec{x}_0 \rangle = 0 \tag{2.5.6}$$

Moreover we obtain (using multiplicative parameterization)

$$\vec{q}_{\gamma(t)(\vec{x})} = \gamma\left(\frac{1}{t \cdot t(\vec{x})}\right) \cdot \gamma(t)(\gamma(t(\vec{x}))(\vec{q}_{\vec{x}})) = \vec{q}_{\vec{x}}$$

Hence the assertion. $\square$

Remark 2.5.6. We have proved: If for $\vec{x} \in \langle \vec{x}_0 \rangle^+$ the orbit of $\gamma(\cdot)$ is given by $\{\gamma(t)(\vec{x}) : t > 0\}$. Then $\langle \vec{x}_0 \rangle^+$ is the disjoint union of orbits and any orbit intersects $Q = \{\langle \vec{x}_0 \rangle^\perp\}$ in exactly one point, namely in $\vec{q}_{\vec{x}}$

Theorem 2.5.7. According to A1) with $\gamma(t)(\vec{x}) = t \cdot (\vec{x} + \vec{x}_0) - \vec{x}_0$, for $t > 0$, $\vec{x} \in \langle \vec{x}_0 \rangle^+$ and $c(t) = t^{-\alpha}$ for $\alpha > 0$, $c = \langle \vec{x}_0, \vec{x}_0 \rangle$ we have

$$\Lambda \text{ is a solution of (2.5.1)} \iff \Lambda(\vec{x}) = \varphi(\vec{q}_{\vec{x}}) \cdot (1 + \frac{1}{c}\langle \vec{x}, \vec{x}_0 \rangle)^{-\alpha} \tag{2.5.7}$$

for some function $\varphi : Q \to (0, 1]$.

Proof. " $\Longleftarrow$ " Assume Λ has the form 2.5.7. Then we have

$$\begin{aligned}
\Lambda(\gamma(t)(\vec{x})) &= \varphi(\vec{q}_{\gamma(t)(\vec{x})}) \cdot (1 + \frac{1}{c} \cdot \langle t \cdot (\vec{x} + \vec{x}_0) - \vec{x}_0, \vec{x}_0 \rangle)^{-\alpha} \\
&= \varphi(\vec{q}_{\gamma(t)(\vec{x})}) \cdot (1 + \frac{1}{c} \cdot \langle t \cdot (\vec{x} + \vec{x}_0), \vec{x}_0 \rangle - \frac{1}{c}\langle \vec{x}_0, \vec{x}_0 \rangle)^{-\alpha} \\
&= \varphi(\vec{q}_{\vec{x}}) \cdot (\frac{1}{c} \cdot t \cdot (\langle \vec{x}, \vec{x}_0 \rangle + \langle \vec{x}_0, \vec{x}_0 \rangle))^{-\alpha} \\
&= \varphi(\vec{q}_{\vec{x}}) \cdot t^{-\alpha} \cdot (1 + \frac{1}{c}\langle \vec{x}, \vec{x}_0 \rangle)^{-\alpha} \\
&= t^{-\alpha} \cdot \Lambda(\vec{x})
\end{aligned}$$

" $\Longrightarrow$ " Assume that Λ satisfies 2.5.1. Let $\vec{x} \in \langle \vec{x}_0 \rangle^+$. Hence $\vec{x} = \gamma(t)(\vec{q})$ with $t = t(\vec{x})$, $\vec{q} = \vec{q}_{\vec{x}}$. Then we obtain

$$\begin{aligned}
\Lambda(\vec{x}) &= \Lambda(\gamma(t)(\vec{q}_{\vec{x}})) \\
&= c(t(\vec{x})) \cdot \Lambda(\vec{q}_{\vec{x}}) \\
&= (1 + \frac{1}{c}\langle \vec{x}, \vec{x}_0 \rangle)^{-\alpha} \cdot \Lambda(\vec{q}_{\vec{x}}).
\end{aligned}$$

Hence the assertion with $\varphi(\vec{q}) = \Lambda(\vec{q})$, $\vec{q} \in Q$. □

Now we consider the case **A2**). Let $\gamma(t)(\vec{x}) = \vec{x} + t\,\vec{x}_0$ as in **A2**). Define A^γ be as in 2.5.4 before. Put again $Q = \langle \vec{x}_0 \rangle^\perp$. Hence we obtain $t \mapsto c(t)$ is a continuous homomorphism, and by using the additive parameterization we obtain $c(t) = e^{-\beta \cdot t}$ for some real $\beta > 0$. Analogously, there exists a unique extension of Λ to A^γ (in this case $(\mathbb{R}^d_+)^\gamma = \mathbb{R}^d$), and again for any $\vec{x} \in \langle \vec{x}_0 \rangle^\perp$ there exists a unique representation $\vec{x} = \gamma(t)(\vec{q}_{\vec{x}})$ with $t = t(\vec{x}) \geq 0$, and $\langle \vec{q}_{\vec{x}}, \vec{x}_0 \rangle = 0$. In fact $\gamma(t)(\vec{q}_{\vec{x}}) = \vec{q}_{\vec{x}} + t \cdot \vec{x}_0 = \vec{x}$ hence we have $\vec{q}_{\vec{x}} = \vec{x} - t \cdot \vec{x}_0$ and we obtain

$$\langle \vec{q}_{\vec{x}}, \vec{x}_0 \rangle = \langle \vec{x}, \vec{x}_0 \rangle - t \cdot \langle \vec{x}_0, \vec{x}_0 \rangle = 0 \tag{2.5.8}$$

Therefore

$$t = t(\vec{x}) = \frac{1}{c} \cdot \langle \vec{x}, \vec{x}_0 \rangle \text{ with } \langle \vec{x}_0, \vec{x}_0 \rangle =: c \tag{2.5.9}$$

and

$$\vec{q}_{\vec{x}} = \vec{x} - \frac{1}{c} \cdot \langle \vec{x}, \vec{x}_0 \rangle \cdot \vec{x}_0. \tag{2.5.10}$$

Again, obviously $\vec{q}_{\vec{x}} = \vec{q}_{\gamma(t)(\vec{x})}$ for all t.

Theorem 2.5.8. According to A2) with $\gamma(t) : \vec{x} \mapsto \vec{x} + t \cdot \vec{x}_0$ and with $c(t) = e^{-\beta \cdot t}$, $c = \langle \vec{x}_0, \vec{x}_0 \rangle$, $\vec{x} \in \mathbb{R}^d$, $t \in \mathbb{R}$ we have

$$\Lambda \text{ fulfils the functional equation 2.5.1} \iff \Lambda(\vec{x}) = e^{-\frac{\beta}{c}\langle \vec{x}, \vec{x}_0 \rangle} \cdot \varphi(\vec{q}_{\vec{x}}) \tag{2.5.11}$$

for some function $\varphi : Q \to (0, 1]$, where $\vec{q}_{\vec{x}} \in \langle \vec{x}_0 \rangle^{\perp}$ defined in 2.5.10

Proof. " $\Longrightarrow$ " Assume

$$\Lambda(\gamma(t)(\vec{x})) = e^{-\beta \cdot t} \cdot \Lambda(\vec{x}) \ , \ c(t) = e^{-\beta \cdot t}$$

Then , for $\vec{x} \in \langle \vec{x}_0 \rangle^{+}$, $\vec{x} = \gamma(t(\vec{x}))(\vec{q}_{\vec{x}}) = \vec{q}_{\vec{x}} + \frac{1}{c}\langle \vec{x}, \vec{x}_0 \rangle \cdot \vec{x}_0$. Hence

$$\Lambda(\vec{x}) = e^{-\beta \cdot t(\vec{x})} \cdot \Lambda(\vec{q}_{\vec{x}}) = e^{-\beta \cdot \frac{1}{c}\langle \vec{x}, \vec{x}_0 \rangle} \cdot \Lambda(\vec{q}_{\vec{x}})$$

Whence the assertion with $\varphi(\vec{q}_{\vec{x}}) = \Lambda(\vec{q}_{\vec{x}})$.
" $\Longleftarrow$ " Conversely, for $\Lambda(\vec{x}) = \varphi(\vec{q}_{\vec{x}}) \cdot e^{-\frac{\beta}{c} \cdot \langle \vec{x}, \vec{x}_0 \rangle}$ we obtain

$$\begin{aligned}
\Lambda(\gamma(t)(\vec{x})) &= \varphi(\vec{q}_{\gamma(t)(\vec{x})}) \cdot e^{-\frac{\beta}{c} \cdot \langle \gamma(t)(\vec{x}), \vec{x}_0 \rangle} \\
&= \varphi(\vec{q}_{\vec{x}}) \cdot e^{-\frac{\beta}{c} \cdot \langle \vec{x} + t \cdot \vec{x}_0, \vec{x}_0 \rangle} \\
&= \varphi(\vec{q}_{\vec{x}}) \cdot e^{-\frac{\beta}{c} \cdot \langle \vec{x}, \vec{x}_0 \rangle} \cdot e^{-\frac{\beta}{c}\langle \vec{x}_0, \vec{x}_0 \rangle \cdot t} \\
&= \Lambda(\vec{x}) \cdot e^{-\beta \cdot t}
\end{aligned}$$

as asserted. □

Notation 2.5.9. It is easily seen that the solutions of the stability functional equation obtained in 2.5.7 and 2.5.8 have R.L.T. stable one-dimensional marginals.

Example 2.5.10. <u>Bivariate Pareto distribution</u>
Let $R(\vec{x}) = R(x_1, x_2) = (1 + x_1 + x_2)^{-\alpha}$, for $(x_1, x_2) = \vec{x} \geq \vec{0}$.
Then $R(\vec{x}) = P(\vec{X} > \vec{x})$ for a random vector $\vec{X} = (\xi, \eta)$. Hence

$$\begin{aligned}
P(\xi > x_1) &= P(\xi > x_1, \eta \in \mathbb{R}^1) \\
&= P(\xi > x_1, \eta \geq 0) \ \ \text{(since } \eta \geq 0) \\
&= R((x_1, 0)) \\
&= (1 + x_1)^{-\alpha}, \ \ \text{a tail of a one dimensional Pareto distribution.}
\end{aligned}$$

Analogously

$$
\begin{aligned}
P(\eta > y) &= P(\xi \in \mathbb{R}^1, \eta > y) \\
&= P(\xi \geq 0, \eta > y) \ \ (\text{since } \xi \geq 0) \\
&= R((0, y)) \\
&= (1+y)^{-\alpha} \text{ again a tail of a Pareto distribution.}
\end{aligned}
$$

In fact, we have no complete solution of the functional equation in the multivariate case. We obtained solutions in the cases A1) and A2). Moreover, if $\gamma(\cdot)$ is the direct sum of groups $\gamma^{(i)}(\cdot)$ of these types, then as in example 2.4.10, we obtain a solution if Λ splits as direct product of functions Λ_i fulfilling $\gamma^{(i)}(\cdot)$ on the corresponding subgroups.
If $\gamma(\cdot)$ is a subgroup of $\mathrm{Aff}_0(\mathbb{R}, d)$ as above, not of type A1) or A2) then $\gamma(t)(\mathbb{R}^d_+) \subseteq \mathbb{R}^d_+$. Let $Q \subseteq \mathbb{R}^d_+$ be a cross-section w.r.t. $\gamma(\cdot)$, then, as above, the solutions Λ are given as

$$\Lambda(\vec{x}) = c(t_{\vec{x}}) \cdot f(\vec{q}_{\vec{x}})$$

for $\vec{x} = \gamma(t_{\vec{x}})(\vec{q}_{\vec{x}})$, $\vec{q}_{\vec{x}} \in Q$, $t_{\vec{x}} > 0$.
But in contract to the cases A1), A2) the cross-section will in general not be explicitly known.

2.6 The decomposability semigroup of R.L.T. distributions

Here we introduce a concept which has been successfully used for investigations in (operator) semi-stability for random vectors in the case of multidimensional vector spaces. and for a group valued random variables, as in 1.5, and according to the structure of the subgroup of $\mathrm{CAT}(\mathbb{R}, d)$, $d > 1$, we begin with the general definition of decomposability semigroup of R.L.T. distributions.

Definition 2.6.1. *Let R be a non-degenerate tail function and $\vec{x}_0 = \vec{x}_0\ (R) \geq \vec{0}$, $R(\vec{x}) > 0 \ \forall\ \vec{x} \geq \vec{0}$. We define the R.L.T. decomposability semigroup*
$\mathrm{Dec}(R) := \{\gamma \in \mathrm{CAT}(\mathbb{R}, d) : \gamma = \gamma_{A, \vec{b}}$ *with* $A = \mathrm{diag}(a_1, \ldots, a_d)$*,* $a_i \geq 1$*,* $\vec{b} \geq \vec{0}$ *such that* $R(\gamma(\vec{x})) = c(\gamma) \cdot R(\vec{x})$ *for all* $\vec{x} \geq \vec{x}_0$*,* $c = c(\gamma) \in (0, 1]\}$.

Remark 2.6.2. The assumption on $\gamma = \gamma_{A, \vec{b}} \in \mathrm{Dec}(R)$ implies that either

- $\gamma = \gamma_{I, \vec{0}} = \mathrm{id}$ or

- γ is a shift, $\gamma = \gamma_{I,\vec{b}}$, $\vec{b} > \vec{0}$ or
- γ has a fixed point $\vec{x}_\star \leq \vec{0}$ with

  $$\gamma(\vec{x}) = A \cdot (\vec{x} - \vec{x}_\star) + \vec{x}_\star = A \cdot \vec{x} + (A - I)(-\vec{x}_\star)$$

  hence $\vec{b} = (A - I)(-\vec{x}_\star) \geq \vec{0}$. (See 2.1).

In particular, γ is (strictly) increasing on $\mathbb{R}^d_+$ hence

$$\gamma(\vec{x}) \geq \vec{x}_0 \text{ for all } \vec{x} \geq \vec{x}_0 \tag{2.6.1}$$

Obviously we have

Proposition 2.6.3. Let R be a fixed tail function. Then we have: $\mathrm{Dec}(R)$ is a closed subsemigroup of $\mathrm{CAT}^+(\mathbb{R}, d)$ and $\gamma \mapsto c(\gamma)$ is a continuous homomorphism $c : \mathrm{Dec}(R) \to ((0, 1], \cdot)$

Proof. Let $\gamma, \widetilde{\gamma} \in \mathrm{Dec}(R)$. According to the remark 2.6.2 we have

$$R((\gamma \circ \widetilde{\gamma})(\vec{x})) = c(\gamma) \cdot R(\widetilde{\gamma}(\vec{x})) = c(\gamma)c(\widetilde{\gamma}) \cdot R(\vec{x}) \tag{2.6.2}$$

for $\vec{x} \geq \vec{x}_0$ such that $\widetilde{\gamma}(\vec{x}) \geq \vec{x}_0$. At the same time we have

$$R(\gamma \circ \widetilde{\gamma})(\vec{x}) = c(\gamma \circ \widetilde{\gamma}) \cdot R(\vec{x}) \tag{2.6.3}$$

Hence $\gamma \mapsto c(\gamma)$ is a homomorphism. To prove continuity of c let $\vec{x} \geq \vec{x}_0$ with $R(\vec{x}) > 0$ such that $\gamma(\vec{x})$ is a continuity point of R. Assume $\gamma^n \to \gamma$. Then $R(\gamma^n(\vec{x})) \to R(\gamma(\vec{x}))$. Whence $c(\gamma^n) \to c(\gamma)$ and the continuity of c follows. Moreover we have

$$R(\gamma(\vec{x})) = c(\gamma) \cdot R(\vec{x})$$

□

Moreover we have

Proposition 2.6.4. If R is non-degenerate, then $c : \mathrm{Dec(R)} \to (0, 1]$ is a closed map

Proof. Let $\{\alpha_n\} \subseteq \mathrm{im}(c)$, i.e. $\alpha_n = c(\gamma_n)$ with $\gamma_n \in \mathrm{Dec}(R)$, $0 < \alpha_n \leq 1$ and assume further $\alpha_n \to \alpha \in (0, 1]$. If $\vec{x}_0 > \vec{0}$ replace R by $\widetilde{R}$ such that

$$\widetilde{R}(\vec{x}) := \begin{cases} R(\vec{x}) & \vec{x} \geq \vec{x}_0 \\ 1 & \vec{x} \ngeq \vec{x}_0 \text{ d.h. } \exists i : x_i < (\vec{x}_0)_i \end{cases}$$

Note that $\widetilde{R}$ is a tail function. In fact, assume $R(\vec{x}) = P(X > \vec{x})$ for some random vector X. Put $Y := X \vee \vec{x}_0$. Then $\widetilde{R}(\vec{x}) = P(Y > \vec{x})$. Then we have

$$\widetilde{R}(\gamma_n(\vec{x})) = \alpha_n \cdot R(\vec{x}) \xrightarrow{w} \alpha \cdot R(\vec{x}),\ \vec{x} > \vec{x}_0$$

On the other hand

$$\widetilde{R}(\gamma_n(\vec{x})) = \begin{cases} R(\gamma_n(\vec{x})) & \gamma_n(\vec{x}) \geq \vec{x}_0 \Longleftrightarrow \vec{x} > \gamma_n^{-1}(\vec{x}_0) \\ 1 & \gamma_n(\vec{x}) \not\geq \vec{x}_0 \end{cases}$$

Hence for $\vec{x} \geq \vec{x}_0$ we have $\gamma_n(\vec{x}) \geq \gamma_n(\vec{x}_0) \geq x_0$ (by (2.6.1))

$$\widetilde{R}(\gamma_n(\vec{x})) = c(\gamma_n) \cdot R(\vec{x}) = \alpha_n \cdot R(\vec{x}) \tag{2.6.4}$$

If we define

$$S(\vec{x}) := \begin{cases} \alpha \cdot R(\vec{x}) & \vec{x} \geq \vec{x}_0 \\ 1 & \vec{x} \not\geq \vec{x}_0 \end{cases}$$

we obtain $\widetilde{R}(\gamma_n(\cdot)) \xrightarrow{w} S$. According to equation (2.6.4) we obtain

$$R(\gamma_n(\vec{x})) = \alpha_n \cdot R(\vec{x}) \xrightarrow{w} \alpha \cdot R(\vec{x}) = S(\vec{x}) \tag{2.6.5}$$

Whence by the convergence of types theorem given in section 2.4 we have (γ_n) is relatively compact in $\mathrm{Aff}(\mathbb{R}, d)$. Moreover $R(\gamma(\vec{x})) = \alpha \cdot \widetilde{R}(\vec{x})$ for all accumulation points γ. Moreover

$$R(\gamma_n(\vec{x})) \xrightarrow{w} R(\gamma(\vec{x})) \tag{2.6.6}$$

Therefore, there exist $\gamma \in \mathrm{Dec}(R)$ with $c(\gamma) = \alpha$. (In fact, by 2.2.10, $\gamma_n \to \gamma$ follows). Thus c is a closed map, in particular , $\mathrm{im}(c)$ is closed. Moreover $\mathrm{Dec}(R)$ is a closed sub-semigroup of $\mathrm{CAT}(\mathbb{R}, d)$. $\square$

Note that we have to extend $\mathrm{Dec}(R)$ to a group $\widetilde{\mathrm{Dec}}(R)$. In analogy to investigations of semi-stable laws on vectors spaces, we define the invariance group.

Definition 2.6.5. *Let R be any tail function, $R > 0$. We define*
$\mathrm{Inv}^\star(R) := \{\gamma \in \mathrm{CAT}(\mathbb{R}, d),\ \gamma \nearrow : \exists\ \vec{x}_\gamma$ *such that* $R(\gamma(\vec{x})) = R(\vec{x}),\ \vec{x} \geq \vec{x}_\gamma\}$
$\mathrm{Inv}^\star(R)$ *is called the invariance group of R. Note that by the direct product representation* $\mathrm{CAT}(\mathbb{R}, d) = \bigoplus \mathrm{Aff}_0(\mathbb{R}, 1)$ *it easily follows that*

$$\mathrm{Inv}^\star(R) = \mathrm{Inv}(R) = \{\mathrm{id}\}.\ (\textit{See}\ 2.2.9).$$

Remark 2.6.6. The assumption $R(\vec{x}) > \vec{0}$ for all $\vec{x}$ which will be used in the sequel is not a serious restriction: Assume there exists $\gamma \in \mathrm{Dec}(R)$, $\gamma \neq \mathrm{id}$, and assume furthermore $R(\vec{x}_1) > 0$ for some $\vec{x}_1 > \vec{0}$. Then $R(\vec{x}) > \vec{0}$ on $\mathbb{R}^d_+$.

Proof. Since $R(\gamma^n(\vec{x}_1)) = c(\gamma)^n R(\vec{x}_1)$, and by assumption we have $(\gamma^n(\vec{x}_1))_i \nearrow \infty$ for all i. Hence for all $\vec{x} > \vec{0}$ there exist n such that $\gamma^n(\vec{x}_1) > \vec{x}$. □

Proposition 2.6.7. The group $\mathrm{Dec}(R)$ is embeddable into a closed subgroup $\widetilde{\mathrm{Dec}}(R) \subseteq \mathrm{CAT}(\mathbb{R}, d)$ such that c extends to a continuous injective homomorphism $\widetilde{c} : \widetilde{\mathrm{Dec}}(\mathrm{R}) \to (\mathbb{R}^\times_+, \cdot)$ with $\ker(c) = \mathrm{Inv}^\star(R) = \{\mathrm{id}\}$.

Proof. Let $\widetilde{\mathrm{Dec}}(\mathbb{R})$ denote the closed subgroup generated by $\mathrm{Dec}(R)$.
Let $\gamma, \tau \in \mathrm{Dec}(R)$, $\gamma\tau^{-1} \in \widetilde{\mathrm{Dec}}(R)$. For all sufficiently large $\vec{y}$ we have $R(\tau^{-1}(\vec{y})) = \frac{1}{c(\tau)} R(\vec{y})$ as in the case $d = 1$. Whence $R(\gamma\tau^{-1}(\vec{y})) = \frac{c(\gamma)}{c(\tau)} \cdot R(\vec{y})$ follows. We obtain

$$R(\gamma\tau^{-1}(\vec{y})) = \widetilde{c}(\gamma\tau^{-1}) \cdot R(\vec{y}).$$

with $\widetilde{c}(\gamma\tau^{-1}) := \frac{c(\gamma)}{c(\tau)}$. Hence

$$R(\vec{y}) = c(\tau) \cdot R(\vec{x}) = c(\tau) \cdot R(\tau^{-1}(\vec{y}))$$

for $\vec{x} \geq \vec{x}_0$, for $\vec{y} = \tau(\vec{x}) \geq \tau(\vec{x}_0)$.
As easily seen, $\widetilde{c}$ extends to $\widetilde{\mathrm{Dec}}(R)$, and $\widetilde{c}$ is a closed homomorphism. □

Therefore, the preceding results extend immediately to the group $\widetilde{\mathrm{Dec}}(R) \subseteq \mathrm{Aff}_0(\mathbb{R})$

Theorem 2.6.8. With the notations introduced above we have:

a) Dec(R) is a closed sub-semigroup of the closed subgroup $\widetilde{\mathrm{Dec}}(R) \subseteq \mathrm{CAT}(\mathbb{R}, d)$

b) $\gamma \mapsto \widetilde{c}(\gamma)$ is a closed continuous homomorphism $\widetilde{\mathrm{Dec}}(R) \longrightarrow ((0, \infty), \cdot)$

Remark 2.6.9. $\mathrm{im}(\widetilde{c}) \cap (0, 1]$ is a multiplicative semigroup, which is closed since $\widetilde{c}$ is closed map. Therefore either

- $\mathrm{im}(\widetilde{c}) = (0, 1]$ or
- $\mathrm{im}(\widetilde{c}) = \{q^k : k \in \mathbb{Z}_+\}$ for some $0 < q < 1$ or
- $\mathrm{im}(\widetilde{c}) = \{1\}$

Therefore we obtain the following

Theorem 2.6.10. Let R be as above. Then either

a) $\mathrm{im}(\widetilde{c}) = \{1\}$ (*i.e.* $\mathrm{Dec}(R) = \{\mathrm{id}\} = \mathrm{Inv}(R)$) or

b) there exist $\gamma \in \mathrm{Dec}(R)$ such that $\mathrm{Dec}(R) = \{\gamma^k : k \in \mathbb{Z}_+\}$ or

c) there exists a continuous one-parameter group $\mathbb{R} \ni t \mapsto \gamma(t)$ (additive parameterization) such that
$\{\gamma(t) : t \in \mathbb{R}\} \subseteq \mathrm{Dec}(R)$, $\{\gamma(t) : t \geq 0\} \subseteq \mathrm{Dec}(R)$ and $c(\gamma(t)) = q^t (= e^{-\beta t})$ for some $q \in (0,1)$ ($\beta = -\log q$).

Proof. In fact, if $\widetilde{\mathrm{Dec}}(R) \neq \{\mathrm{id}\}$, then for any $\gamma \in \widetilde{\mathrm{Dec}}(R)$, $c(\gamma) \in (0,1)$, obviously $\{\gamma^k\} \subseteq \widetilde{\mathrm{Dec}}(R)$, $c(\gamma^k) = c(\gamma)^k$.

- If $\mathrm{im}(c) = \{q^k : k \in \mathbb{Z}_+\}$, there exists $\gamma \in \widetilde{\mathrm{Dec}}(R)$ with $\widetilde{c}(\gamma) = q$. Now (b)follows.
- If $\mathrm{im}(c) = (0,1], \gamma \mapsto \widetilde{c}(\gamma) \in (0,\infty)$ is a continuous homomorphism of the Lie group $G := \widetilde{\mathrm{Dec}}(R)$ onto $((0,\infty),\cdot)$. G being closed, we conclude G/G_0 is at most countable. Therefore there exists a continuous homomorphism $((0,\infty),\cdot) \longrightarrow \widetilde{\mathrm{Dec}}(R)$ where $u \mapsto \widetilde{\gamma}(u)$ such that $\widetilde{c}(\widetilde{\gamma}(u)) = u,\ u > 0$. Passing to additive parameterization, $\gamma(t) := \widetilde{\gamma}(e^{-t}),\ u = e^t$ yields the assertion (for $q = e^{-1}$).

  We have to show $\gamma(t) \in \mathrm{Dec}(R)$, $t > 0$: Since $R(\gamma(t)(\vec{x})) = q^t \cdot R(\vec{x}) \overset{t\to\infty}{\longrightarrow} 0$ it is obvious that $(\gamma(t)(x))_i \overset{t\to\infty}{\longrightarrow} \infty$ for all $\vec{x} \geq \vec{x}_0$ (*resp.* $\geq \vec{0}$), $i = 1,\ldots,d.$
  Hence $\gamma(t) \in \mathrm{Dec}(R)$, $t > 0$.

$\square$

Corollary 2.6.11. Let R be as above. Let $\mathcal{D} \subseteq \widetilde{Dec}(R)$, $\mathcal{D} \neq \{\mathrm{id}\}$,

$$\mathcal{C} := \{c(\gamma) : \gamma \in \mathcal{D}\} \subseteq (0,1]$$

Then either

a) the group $\langle \mathcal{C} \rangle$ generated by $\mathcal{C}$ is discrete $\{q^k : k \in \mathbb{Z}_+\}$. (Then R is *residual life time semi-stable*, with $\mathrm{Dec}(R) = \{\gamma^k : k \in \mathbb{Z}_+\}$ (see Def. 2.7.1 and 2.7.2 below) or

b) $\langle \mathcal{C} \rangle$ is dense in $(0,1]$, then $c(\mathrm{Dec}(R)) = (0,1]$ and there exists a one-parameter group $\gamma(\cdot)$ such that (e.g. with additive parameterization) $c(\gamma(t)) = q^t\ , t \geq 0$. (I.e. in this case R is *residual life time stable*, see 2.7.1 and 2.7.2 below).

2.7 Limit theorems for multivariate R.L.T. distribution ($d > 1$)

In view of the discussion in 1.6 and 1.7 for $d = 1$, we define multivariate R.L.T. (semi-) stability:

Definition 2.7.1. *Let $\mu \in M^1(\mathbb{R}_+^d)$ with distribution function F and tail function R. Let $(\gamma(t))_{t\in\mathbb{R}}$ be a continuous one-parameter group in $\mathrm{CAT}^+(\mathbb{R}, d)$ and $\gamma \in \mathrm{CAT}(\mathbb{R}, d)$ with $(\gamma^n(\vec{x}))_i \overset{n\to\infty}{\longrightarrow} \infty$,$1 \le i \le d$, for $\vec{x}>\vec{0}$. Then μ (resp. F, resp. R) is called R.L.T. stable w.r.t. $\gamma(\cdot)$ if*

$$\frac{R(\gamma(t)(\vec{x}))}{R(\gamma(t)(\vec{0}))} = R(\vec{x}),\ \vec{x}\geq\vec{0},\ t > 0 \tag{2.7.1}$$

Analogously, μ (resp. F, resp. R) is called R.L.T. semi stable w.r.t. $\gamma(\cdot)$ if

$$\frac{R(\gamma(\vec{x}))}{R(\gamma(\vec{0}))} = R(\vec{x}),\ \vec{x}\geq\vec{0} \tag{2.7.2}$$

Using the concept of the decomposability semi-group in 2.6 we obtain an equivalent description:

Theorem 2.7.2. Let μ, F, R be as above. Then we have

a) μ (resp. F, resp. R) is R.L.T. semi-stable iff $\mathrm{Dec}(R) \setminus \{\mathrm{id}\} \neq \emptyset$

b) μ (resp. F, resp. R) is R.L.T. stable if there exists a continuous one- parameter group (e.g. with additive parameterization) $\gamma(\cdot) \subseteq \mathrm{CAT}^+(\mathbb{R}, d)$ such that $\gamma(t) \in \mathrm{Dec}(R) \setminus \{\mathrm{id}\}$, $t > 0$.

Proof. As a consequence of definition 2.7.1 above and definition 2.6.1 (if we put $\vec{x}_0=\vec{0}$) □

Moreover, if F is non-degenerate we obtain by proposition 2.6.4 (resp. Theorem 2.6.10):

Proposition 2.7.3. Let μ be non-degenerate then μ (resp. F, resp. R) is R.L.T. stable iff the image $\mathrm{im}(c)$ is dense in $(0, 1]$ where, $c : \mathrm{Dec}(\mu) \to ((0, 1], \cdot)$ is the homomorphism defined in definition 2.6.1 (resp. 2.6.3)

And furthermore we note that

Proposition 2.7.4. Let μ (resp. F, resp. R) be as above. Then μ (resp. F, resp. R) is R.L.T. stable w.r.t. $(\gamma(t), t > 0)$ iff the tail R is a solution of the functional equation (2.5.1)

Analogously, we have

Proposition 2.7.5. μ is R.L.T. semi- stable w.r.t. $(\gamma(t))_{t>0}$ iff the tail R is a solution of the discrete analogue of (2.5.1) i.e

$$R(\gamma(\vec{x})) = c \cdot R(\vec{x}),\ \vec{x} > \vec{0}, \tag{2.7.3}$$
$$\text{(and hence} \quad R(\gamma^k(\vec{x})) = c^k \cdot R(\vec{x}),\ k \in \mathbb{Z}_+) \tag{2.7.4}$$

2.8 Domains of attraction of (semi-) stable R.L.T. distributions ($d > 1$)

Next we define domains of R.L.T. (semi-) stable attraction in the multivariate case.

Definition 2.8.1. *Let $\mu \in M^1(\mathbb{R}_+^d)$ with distribution function F and tail function R. Let λ be non-degenerate with distribution function G and tail function Λ. Then we have the following:*

a) *μ (resp. F, resp. R) belongs to the domain of semi-stable R.L.T. attraction of Λ if there exist $\beta \in (0,1]$, $\gamma_n \in \mathrm{CAT}^+(\mathbb{R}, d)$ such that $(\gamma_n(\vec{x}))_i \overset{n\to\infty}{\longrightarrow} \infty$, $1 \le i \le d$, $\vec{x} > \vec{0}$, and*

1) $\frac{R(\gamma_{n+1}(\vec{0}))}{R(\gamma_n(\vec{0}))} \overset{n\to\infty}{\longrightarrow} \beta$

2) $\frac{R(\gamma_n(\vec{x}))}{R(\gamma_n(\vec{0}))} \overset{w}{\longrightarrow} \Lambda(\vec{x})$, $\vec{x} \ge \vec{0}$

b) *μ (resp. F, resp. R) belongs to the domain of R.L.T. stable attraction of Λ if there exist $\gamma_n \in \mathrm{CAT}^+(\mathbb{R}, d)$ as in (a) such that*

1)′ $\frac{R(\gamma_{n+1}(\vec{0}))}{R(\gamma_n(\vec{0}))} \overset{n\to\infty}{\longrightarrow} 1$, *and*

2)′ $\frac{R(\gamma_n(\vec{x}))}{R(\gamma_n(\vec{0}))} \overset{w}{\longrightarrow} \Lambda(\vec{x})$, $\vec{x} \ge \vec{0}$

c) *μ belongs to the normal domain of semi-stable R.L.T. attraction of Λ if in (a) we have $\gamma_n = \gamma^n$ for some $\gamma \in \mathrm{CAT}^+(\mathbb{R}, d)$*

d) *μ belongs to the normal domain of stable R.L.T. attraction of Λ if for some one-parameter group $(\gamma(t) : t > 0) \subseteq \mathrm{CAT}^+(\mathbb{R}, d)$ with $(\gamma(t)(\vec{x}))_i \overset{t\to\infty}{\longrightarrow} \infty$, $1 \le i \le d$, $\vec{x} > \vec{0}$ and $\frac{R(\gamma(t)(\vec{x}))}{R(\gamma(t)(\vec{0}))} \overset{t\to\infty}{\longrightarrow} \Lambda(\vec{x})$*

Remark 2.8.2. As well known, the situation (b) of the above definition, $\frac{R(\gamma_{n+1}(\overrightarrow{0}))}{R(\gamma_n(\overrightarrow{0}))} \overset{n\to\infty}{\longrightarrow} 1$ implies that for any $\beta \in (0,1)$ there exist a subsequence (n_j) with

$$\frac{R(\gamma_{n_{j+1}}(\overrightarrow{0}))}{R(\gamma_{n_j}(\overrightarrow{0}))} \overset{n\to\infty}{\longrightarrow} \beta.$$

I.e. μ is in the domain of semi-stable R.L.T. attraction of Λ for any $0 < \beta \leq 1$.

Proposition 2.8.3. a) If the domain of semi-stable R.L.T. attraction of Λ is non empty then Λ is R.L.T. semi-stable.

b) If the domain of R.L.T. stable attraction of Λ is non empty , then Λ is R.L.T. stable.

Proof. **a)** As a consequence of the convergence of type theorem we have: Assume in definition 2.8.1(a) that (1) and (2) are satisfied. Then

$$\frac{R(\gamma_{n+1}(\overrightarrow{x}))}{R(\gamma_{n+1}(\overrightarrow{0}))} = \frac{R(\gamma_n(\gamma_n^{-1}\gamma_{n+1})(\overrightarrow{x}))}{R(\gamma_n(\overrightarrow{0}))} \cdot \frac{R(\gamma_n(\overrightarrow{0}))}{R(\gamma_{n+1}(\overrightarrow{0}))} \overset{n\to\infty}{\longrightarrow} \Lambda(\overrightarrow{x}) \tag{2.8.1}$$

Hence we have

$$\frac{R(\gamma_n(\gamma_n^{-1}\gamma_{n+1})(\overrightarrow{x}))}{R(\gamma_n(\overrightarrow{0}))} =: H_n(\overrightarrow{x}) \to \beta \cdot \Lambda(\overrightarrow{x}), \text{ a non degenerate limit.} \tag{2.8.2}$$

According to the convergence of types theorem, $\{\gamma_n^{-1}\gamma_{n+1}\}$ is relatively compact with accumulation points $\{\gamma\} = \gamma \cdot \mathrm{Inv}(\Lambda)$. Since $\mathrm{Inv}(\Lambda) = \{\mathrm{id}\}$,

$$\gamma_n^{-1}\gamma_{n+1} \to \gamma \in \mathrm{CAT}(\mathbb{R}, d)$$

Thus we have

$$H_n(\overrightarrow{x}) \overset{w}{\longrightarrow} \beta \cdot \Lambda(\gamma(\overrightarrow{x})) \tag{2.8.3}$$

therefore, (using C.T.T. in equations 2.8.2 and 2.8.3) we have

$$\Lambda(\gamma(\overrightarrow{x})) = \beta \cdot \Lambda(\overrightarrow{x}) \text{ as asserted.} \tag{2.8.4}$$

b) To prove (b), note that by the above remark 2.8.2 for any $\beta \in (0,1]$ there exists $\gamma_\beta \in \mathrm{Dec}(\Lambda)$ with $\Lambda(\gamma_\beta(\overrightarrow{x})) = \beta \cdot \Lambda(\overrightarrow{x})$. I.e. $\mathrm{im}(c) = (0,1]$, and therefore there exists a one-parameter group $\gamma(\cdot) \subseteq \widetilde{\mathrm{Dec}}(\mu)$ with $c(\gamma(t)) = t$, $0 < t \leq 1$.

$\square$

On the other hand, we obtain immediately

Proposition 2.8.4. Let Λ be R.L.T. stable (resp. semi-stable). Then the normal domains of R.L.T. (semi-) stable attraction are non empty.

Proof. In fact, as for $d = 1$ it is easily shown that Λ itself belongs to the normal domain of attraction. I.e. the assertion. $\square$

2.9 R.L.T. stability and max-stability for $d > 1$

Finally we sketch how the results of section 1.6–1.8 might be generalized to the multivariate case.

Notation 2.9.1. Let $F \in M^1(\mathbb{R}^d)$ be an infinitely divisible distribution function, i.e. F^t is a distribution function for all $t > 0$.
Put $-H := \log F,\ H(\vec{x}) := \infty$ if $F(\vec{x}) = 0$.
$1 - H$ is a distribution function. (Hence H playes the role of a tail of a (possible unbounded) measure), hence $H = \lim\limits_{t\to 0} \frac{1-F^t}{t}$ (resp. $-H = \lim\limits_{t\to 0} \frac{F^t-1}{t}$ is increasing). We observe

$$F^t = \exp(-t \cdot H),\ t > 0.$$

Theorem 2.9.2. Let $\gamma(\cdot) \subseteq \mathrm{CAT}^+(\mathbb{R}, d)$ be a continuous one-parameter group. Then the following are equivalent:

(i) $F(\gamma(t)(\vec{x})) = F^t(\vec{x})$, i.e. F is max stable

(ii) H satisfies the functional equation (2.5.1)

$$H(\gamma(t)(\vec{x})) = t \cdot H(\vec{x})$$

where we define $0^t := 0,\ t \cdot \infty =: \infty,\ e^{-\infty} = 0.$

Proof. Obvious, since $F^t(\vec{x}) = e^{-t \cdot H(\vec{x})}$ $\square$

And the correspondence of the domains of attraction follows by

Theorem 2.9.3. Let G and F be non-degenerate distribution functions in $M^1(\mathbb{R}^d_+)$. Assume $F = \exp(-H)$, $F(\vec{x}) > 0$, $G(\vec{x}) < 1$ for $\vec{x} > \vec{0}$. Let $\gamma_n \in \mathrm{CAT}^+(\mathbb{R}, d)$. Then

$$G(\gamma_n(\vec{x}))^n \overset{n\to\infty}{\longrightarrow} F(\vec{x}),\ \vec{x} > \vec{0} \ \text{ iff } \ n(1 - G(\gamma_n(\vec{x}))) \overset{n\to\infty}{\longrightarrow} H(\vec{x}),\ \vec{x} > \vec{0}\ .$$

Proof. Immediate consequence of 0.2.4. By standard arguments it is easily shown that F is max-stable in this case: In fact, by the convergence of types theorem we obtain for all $s \in (0, 1)$ there exist $\sigma \in \mathrm{CAT}(\mathbb{R}, d)$ such that $F(\sigma(\vec{x})) = F^s(\vec{x}),\ \vec{x} > \vec{0}$. On the other hand, if αH is a tail, $\alpha > 0$, then αH is R.L.T. stable. $\square$

2.10 References and comments for Chapter 2

R 2.1 As a preparation of the following, we investigated a subgroup $\mathrm{CAT}(\mathbb{R}, d),\ d > 1$ (coordinate-wise of affine transformations introduced in 0.1). we restrict ourselves to one-parameter-subgroups in $\mathrm{CAT}(\mathbb{R}, d)$.

$$\mathrm{CAT}(\mathbb{R}, d) = \bigoplus_{i=1}^{d} \mathrm{Aff}_0(\mathbb{R}, 1)$$

(Note that $\mathrm{CAT}(\mathbb{R}, d)$ is a proper subgroup of $\mathrm{Aff}_0(\mathbb{R}, d)$ iff $d > 1$ see 0.1.16). In 2.1.4 and 2.1.14 we investigated the structure of one-parameter subgroups

$$\{T_t, t \in \mathbb{R}\} \subseteq \mathrm{CAT}(\mathbb{R}, d)$$

(For more details and for a slightly different description see Balkema and Yong-Cheng Qi [2]).

R 2.2 Convergence of types theorems (C.T.T.) turned out to be an essential tool in investigations in operator limit laws. Therefore we introduced in this section some generalizations of the introduced versions of C.T.T. mentioned in section 0.2 to the multidimensional case (see 2.2.10 in particular for normalizing operators belonging to $\mathrm{CAT}(\mathbb{R}, d)$. More details in Hazod [14]).

R 2.3 In this section we reformulated notations and theorems introduced in section 0.3. Again the R.L.T. distribution of μ (its distribution function F with tail R) may be defined by groups of transformations
$\tau_t : M^1(\mathbb{R}^d_+) \to M^1(\mathbb{R}^d_+)$ (see 2.3.9). Equivalently, $\tau_t(F(\vec{x})) = F_t(\vec{x})$, i.e. If F is the distribution function of μ then F_t is the distribution function of $\tau_t(\mu)$.

R 2.4 Here we generalized the L.M.P. parallel to the derivations and introduced concepts and remarks in section 1.1 see Galambos [8]. Note that, according to 2.4.10 the exponential distribution in 2.4.8 is a direct product of one-dimensional ones, however the Pareto distribution 2.4.9 is not representable as a direct product. Further reference for generalizations of the L.M.P. See [16](cited on pages 69, 406).

R. 2.5 The possible solutions of a "stability" functional equation (for R.L.T.) 2.4.6 have-in the general case-no simple representations. Due to the fact that we have no complete overview over the possible solutions of the functional equation the R.L.T. (semi-) stable laws will be characterized in the sequel only in the special cases where the underlying one-parameter group of CAT's is a group of shifts or has a unique fixed point. According to 2.5.1–2.5.6 we introduced in theorem 2.5.7 a particular solution in the case, that the group of CAT's has a unique fixed point, and theorem 2.5.8 gives the particular solution in the case of the group of shifts.

R 2.6 We introduced in this section the concept of "decomposability semigroup" of R.L.T. distributions, which is a successful concept for investigations in (operator) semi-stability for random vectors in the case of multidimensional vector spaces. see e.g. [15]. This leads to characterizations of the (semi-) stability R.L.T. distributions in 2.6.11. (More details is in 2.7).

R 2.7 This section contains a definition and a characterization of R.L.T. (semi-) stability by the decomposability semi group (see 2.7.2). In addition, characterizations by the functional equation 2.5.1 (or its discrete analogue See 2.7.5).

R 2.8 In this section we discussed the $\mathrm{DOA}_{r,s}$, $\mathrm{DOA}_{r,ss}$, $\mathrm{NDOA}_{r,ss}$, and $\mathrm{NDOA}_{r,s}$ see 2.8.1. It is known that, the domain of (semi-) stable R.L.T. attraction characterizes the (semi-) stability when it is non empty see 2.8.3.

R 2.9 In this section we sketched how the results of section 1.6–1.8 might be generalized to the multivariate case see 2.9.1–2.9.3. Since in the multidimensional situation the max-domains of attraction are less investigated than for $d = 1$ we did not give more details.

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