Star Products and Geometric Algebra

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For my parents

Thought and science are raising problems which their terms of study can never answer, many of which are doubtless problems only for thought. The trisection of an angle is similarly an insoluble problem only for compass and straight-edge construction, and Achilles cannot overtake the tortoise so long as their progress is considered piecemeal, endlessly halving the distance between them. However, as it is not Achilles but the method of measurement which fails to catch up with the tortoise, so it is not man but his method of thought which fails to find fulfillment in experience. This is by no means to say that science and analytic thought are useless and destructive tools, but rather that the people who use them must be greater than their tools. To be an effective scientist one must be more than a scientist, and a philosopher must be more than a thinker. For the analytic measurement of nature tells us nothing if we cannot see nature in any other way.

Alan W. Watts

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Introduction

Considering physics as it is done today one notices that physics separates into two formally and conceptually different parts. There is on the one hand classical physics that deals with macroscopic phenomena and there is on the other hand quantum physics that deals with microscopic phenomena. Penrose [96] described this situation as a disturbing analogy to ancient Greece, where two different sets of laws for earth and heaven were applied. So immediately the question arises if one really has to use different formalisms on different scales or if it is possible to describe physics in a unified way. In order to investigate this question one first has to consider how classical and quantum physics are related. Quantum mechanics results from classical physics by a procedure called quantization and classical physics is reobtained by taking the classical limit. Both procedures are heavily plagued by problems [61, 100]. In the first chapter the mathematical problems of quantization will be addressed and it will be described how the star product formalism circumvents these problems. The star product formalism has the advantage that there is no formal break if one goes from classical physics to quantum physics. This formal advantage and the resulting beauty is then used as a guiding principle for the further development.

The first question that arises is if the spin can be described in the star product formalism. That this is indeed possible is shown in the second chapter. As a starting point the spin description with pseudoclassical mechanics as it was developed by Berezin is used. One can then construct a fermionic star product and apply it for deformation quantization of pseudoclassical mechanics. In analogy to the bosonic star product formalism one obtains spin Wigner functions that act as spin projectors. Besides the nonrelativistic case it is also possible to formulate Dirac theory with star products. The Clifford algebra of the gamma matrices is hereby described as a deformed version of a four dimensional Grassmann algebra. The fermionic star product in combination with the bosonic Moyal product leads to a supersymmetric star product formalism that can be used to describe supersymmetric quantum mechanics and in the relativistic case to describe the supersymmetric structure of Dirac theory. The other direction of generalization of the star product formalism is the application of star products in quantum field theory that is described in chapter three. After constructing a suitable normal product it is shown that the algebraic structures of perturbative quantum field theory appear also in the star product formalism, which is an expression of the algebra morphism of the operator and the star product formalism. But moreover the quantum group structure that was recently found in perturbative quantum field theory is shown to be a natural algebraic structure of the star product.

The essential advantage of deformation quantization is that the classical limit has a well defined meaning. In the context of the spin description with star products this leads to the question of the classical limit of spin, or equivalently to the question of the physical status of pseudoclassical mechanics. In chapter four it is shown that this question is solved if one realizes that the fermionic sector together with the fermionic star product describe the underlying geometric structure. The deformation of a Grassmann algebra leads to a Clifford calculus that is equivalent to geometric algebra. The formulation of geometric algebra in the star product formalism is given in chapter four. One sees there that geometric algebra as the most fundamental geometric formalism that unifies all geometric structures that appear in physics can be described in a supersymmetric manner that parallels the bosonic star product structures.

Having obtained a formulation of geometry with fermionic star products this formalism is then applied in the fifth chapter to physical problems. As examples for the application of geometric algebra in classical physics the rigid body and the Kepler problem are considered. In both cases the formalism of geometric algebra gives the most elegant formulation of the problem. In the quantum case one can then combine the fermionic star product formalism that describes the underlying geometric structure with the bosonic star product that describes the noncommuative structure of quantum mechanics. The result is a noncommutative version of geometric algebra that leads to a natural appearance of spin terms. The same idea applied on the phase space leads to the split in supersymmetric partner systems. Geometric algebra gives in this way a natural geometric foundation of supersymmetric quantum mechanics. Similarly one can interpret the hidden BRST-structure of classical mechanics that was found by E. Gozzi and M. Reuter in the path integral formalism from a star product point of view.

Chapter 1

The Star Product Formalism

The first chapter should give a short introduction to the star product formalism which allows a formulation of quantum mechanics on the phase space. After giving the motivation for doing quantum theory on the phase space the star product will be constructed with the help of the operator formalism. With the star product it is then possible to formulate an autonomous approach to quantum theory. As examples the harmonic oscillator and angular momentum are then discussed.

1.1 Quantization and its Problems

The standard approach to quantum theory relies on a procedure called canonical quantization, which was first formulated by Dirac in [28]. Starting point is a classical Hamiltonian system with d degrees of freedom, which can be described on a $2d$-dimensional phase space. Scalar functions on the phase space can be multiplied pointwise, i.e. $(fg)(x) = f(x)g(x)$, where the multiplication fulfills for $f, g, h \in C^\infty(P)$ and $\lambda \in \mathbb{R}$ the following conditions:

$$\begin{array}{ll} fg = gf & \text{commutativity,} \\ f(g+\lambda h) = fg + \lambda fh & \text{linearity,} \\ f(gh) = (fg)h & \text{associativity,} \end{array}$$

so that the functions together with the addition and the pointwise multiplication form a commutative and associative algebra. Moreover there exists a differential-geometric structure on the phase space, called the Poisson bracket, which fulfills for $f, g, h \in C^\infty(P)$ and $\lambda \in \mathbb{R}$ the following conditions:

$$\begin{array}{ll} \{f,g\}_{PB} = -\{g,f\}_{PB} & \text{antisymmetry,} \\ \{f,g+\lambda h\}_{PB} = \{f,g\}_{PB} + \lambda\{f,h\}_{PB} & \text{linearity,} \\ \{f,\{g,h\}_{PB}\}_{PB} + \{g,\{h,f\}_{PB}\}_{PB} + \{h,\{f,g\}_{PB}\}_{PB} = 0 & \text{Jacobi identity,} \end{array}$$

so that the functions together with the addition and the Poisson bracket form a Lie algebra. For the special coordinates (q_i, p_i) the Poisson bracket can be written explicitly as

$$\{f,g\}_{PB} = \sum_{n=1}^{d} \left(\frac{\partial f}{\partial q_n}\frac{\partial g}{\partial p_n} - \frac{\partial f}{\partial p_n}\frac{\partial g}{\partial q_n} \right). \tag{1.1}$$

The pointwise multiplication and the Poisson bracket are intertwined by the following relation

$$\{f,gh\}_{PB} = \{f,g\}_{PB}h + g\{f,h\}_{PB}, \tag{1.2}$$

so that both structures together with the addition constitute a Poisson algebra.

The new mathematical feature that enters physics in quantum theory is non-commutativity. The non-commutativity can be described if one notices that the above mentioned classical structures have analogs in the space $Op(\mathcal{H},\mathcal{D})$ of formally self adjoint operators on a Hilbert space $\mathcal{H}$, and common invariant dense domain $\mathcal{D}\subset\mathcal{H}$. It is then possible to multiply the operators of $Op(\mathcal{H},\mathcal{D})$, so that a associative, but not commutative algebra of operators is created, and taking the Lie product as $\frac{1}{\mathrm{i}\hbar}[\cdot,\cdot]$ one obtains a Lie-algebra. These two structures are intertwined for $\hat{A},\ \hat{B},\ \hat{C}\in Op(\mathcal{H},\mathcal{D})$ by

$$\left[\hat{A},\hat{B}\hat{C}\right]=\left[\hat{A},\hat{B}\right]\hat{C}+\hat{B}\left[\hat{A},\hat{C}\right]. \tag{1.3}$$

In order to go over from classical theory to quantum theory one needs then a map $\mathcal{Q}$ that maps the set $Pol(P)$ of phase space polynomials in (q_i,p_i) to $Op(\mathcal{H},\mathcal{D})$:

$$\mathcal{Q}:Pol(P)\to Op(\mathcal{H},\mathcal{D}),\qquad f\mapsto\mathcal{Q}(f)=\hat{f}. \tag{1.4}$$

This map should fulfill the following conditions:

1) The constant function 1 should be mapped on the unit operator:

$$\mathcal{Q}(1)=\hat{1}. \tag{1.5}$$

2) $\mathcal{Q}$ should be linear:

$$\mathcal{Q}(f+\lambda g)=\mathcal{Q}(f)+\lambda\mathcal{Q}(g). \tag{1.6}$$

3) The Lie structures on $Pol(P)$ and on $Op(\mathcal{H},\mathcal{D})$ should be compatible:

$$\mathcal{Q}(\{f,g\}_{PB})=\frac{1}{\mathrm{i}\hbar}\left[\mathcal{Q}(f),\mathcal{Q}(g)\right]. \tag{1.7}$$

4) $\mathcal{Q}$ should be consistent with Schrödinger quantization (which means that $\mathcal{Q}(q_i)$ and $\mathcal{Q}(p_i)$ act irreducibly up to at most finite multiplicity of internal quantum numbers):

$$(\mathcal{Q}(q_i)\psi)(q)=q_i\psi(q)\qquad\text{and}\qquad(\mathcal{Q}(p_i)\psi)(q)=-\mathrm{i}\hbar\partial_{q_i}\psi(q). \tag{1.8}$$

Groenewold and van Hove showed in [71] and [108] that such a map $\mathcal{Q}$ does not exist. The proof of this no-go-theorem will be sketched for a two dimensional phase space in the following steps (for more detailed proofs that include internal quantum numbers or curved phase spaces see [61], [62], [63], [64]):

In the first step one proves the following identities:

$$\widehat{q^n}=\hat{q}^n,\qquad\widehat{p^n}=\hat{p}^n\qquad\text{and}\qquad\widehat{qp}=\frac{1}{2}(\hat{q}\hat{p}+\hat{p}\hat{q}). \tag{1.9}$$

The first equation can be proved if one considers (1.7) for $f=q^n$ and $g=q$. (1.7) gives then $0=\frac{1}{\mathrm{i}\hbar}\left[\widehat{q^n},\hat{q}\right]$, so that $\widehat{q^n}$ has to be a polynomial h_n in $\hat{q}=q$. Setting then in (1.7) $f=q^n$ and $g=p$ gives with $\{q^n,p\}_{PB}=nq^{n-1}$ and (1.8):

$$n\widehat{q^{n-1}}=\frac{1}{\mathrm{i}\hbar}\left[\widehat{q^n},\hat{p}\right]\quad\Rightarrow\quad nh_{n-1}(q)=\partial_q h_n(q), \tag{1.10}$$

so that $\widehat{q^n}=\hat{q}^n$ follows (an integration constant can be calculated as zero [61]). The second equation of (1.9) follows analogously. The third equation of (1.9) can be proved by setting in (1.7) $f=q^2$ and $g=p^2$. With $\{q^2,p^2\}_{PB}=4qp$ this gives

$$\widehat{qp}=\frac{1}{4\mathrm{i}\hbar}\left[\widehat{q^2},\widehat{p^2}\right]=\frac{1}{4\mathrm{i}\hbar}\left[\hat{q}^2,\hat{p}^2\right]=\frac{1}{4\mathrm{i}\hbar}\left[\hat{q}^2,\hat{p}^2\right]=\frac{1}{2}(\hat{q}\hat{p}+\hat{p}\hat{q}), \tag{1.11}$$

where in the last step (1.3) was used.
In the second step one proves the following identities:

$$\widehat{q^2 p} = \frac{1}{2}\left(\widehat{q^2}\hat{p} + \hat{p}\widehat{q^2}\right) \qquad \text{and} \qquad \widehat{p^2 q} = \frac{1}{2}\left(\widehat{p^2}\hat{q} + \hat{q}\widehat{p^2}\right). \tag{1.12}$$

The proof is analogous to the ones in the last step. The first equation of (1.12) is for example obtained by setting $f = q^3$ and $g = p^2$ and using then (1.3).
The third step is now to take the classical identity

$$\frac{1}{9}\{q^3, p^3\} = \frac{1}{3}\{q^2 p, p^2 q\} \tag{1.13}$$

and to quantize with (1.7) both sides of this equality. Using (1.9) and applying (1.3) leads for the left hand side of (1.13) to

$$\frac{1}{9\mathrm{i}\hbar}\left[\hat{q}^3, \hat{p}^3\right] = \hat{q}^2\hat{p}^2 - 2\mathrm{i}\hbar\hat{q}\hat{p} - \frac{2}{3}\hbar^2\hat{1}, \tag{1.14}$$

while quantizing the right hand side of (1.13) gives with (1.12)

$$\frac{1}{3\mathrm{i}\hbar}\left[\widehat{q^2 p}, \widehat{p^2 q}\right] = \hat{q}^2\hat{p}^2 - 2\mathrm{i}\hbar\hat{q}\hat{p} - \frac{1}{3}\hbar^2\hat{1}, \tag{1.15}$$

which differs from the left hand side by $-\frac{1}{3}\hbar^2\hat{1}$. This contradiction finishes the proof. □

The Groenewold-van Hove theorem showed that a map $\mathcal{Q}$ fulfilling the requirements 1) to 4) does in general not exist, it only exists if one restricts to polynomials of second order

$$\{1,\, q,\, p,\, q^2,\, p^2,\, qp\} \quad \xrightarrow{\mathcal{Q}} \quad \{\hat{1},\, \hat{q},\, \hat{p},\, \hat{q}^2,\, \hat{p}^2,\, \frac{1}{2}\left(\hat{q}\hat{p} + \hat{p}\hat{q}\right)\} \tag{1.16}$$

or the classes

$$\{f(q)p + g(q)\} \quad \xrightarrow{\mathcal{Q}} \quad \{\frac{1}{2}\left(f(\hat{q})\hat{p} + \hat{p}f(\hat{q})\right) + g(\hat{q})\} \tag{1.17}$$

$$\text{and} \qquad \{f(p)q + g(p)\} \quad \xrightarrow{\mathcal{Q}} \quad \{\frac{1}{2}\left(f(\hat{p})\hat{q} + \hat{q}f(\hat{p})\right) + g(\hat{p})\}, \tag{1.18}$$

where f and g are arbitrary functions.

In order to quantize more general expressions one could then try to relax one of the conditions 1) to 4). For example quantization without the irreducibility postulate 4) is the so called prequantization. A prequantization exists for all C^∞-functions on $\mathbb{R}^{2n}$, but it leads to physical problems if one wants to calculate spectra (see for example [61]). If one on the other hand abandons condition 3), there is no mathematical constraint that specifies the operator ordering anymore. Which operator ordering one chooses is then a physical question. The rule of operator ordering is called a quantization scheme. The most common quantization schemes are the standard ordering $\mathcal{Q}_S(q^m p^n) = \hat{q}^m\hat{p}^n$, the antistandard ordering $\mathcal{Q}_{AS}(q^m p^n) = \hat{p}^n\hat{q}^m$ and the Weyl ordering $\mathcal{Q}_W(q^m p^n) = (\hat{q}^m\hat{p}^n)_W$, where $(\hat{q}^m\hat{p}^n)_W$ means that one has to sum over all possible products of m operators $\hat{q}$ and n operators $\hat{p}$ and then divide by the number of these products, for example one has $(\hat{q}\hat{p}^2)_W = \frac{1}{3}(\hat{p}^2\hat{q} + \hat{p}\hat{q}\hat{p} + \hat{q}\hat{p}^2)$.

In [110] Weyl showed that $\mathcal{Q}_W(f(q,p))$ can be represented by taking the Fourier transform $\tilde{f}(u,v) = \frac{1}{(2\pi)^2}\int dqdp\, f(q,p)\exp(-\mathrm{i}(uq+vp))$ of $f(q,p)$ and forming an operator valued back-transformation according to

$$\mathcal{Q}_W(f(q,p)) = \int dudv\, \tilde{f}(u,v)\exp(\mathrm{i}(u\hat{q} + v\hat{p})). \tag{1.19}$$

In order to see that (1.19) leads to Weyl-ordering of operators one considers the case $f(q,p) = q^m p^n$. The Fourier transform is then given by

$$\tilde{f}(u,v) = \frac{1}{(2\pi)^2}\int dqdp\, q^m p^n e^{-\mathrm{i}(uq+vp)} = \frac{\partial^{m+n}}{\partial u^m \partial v^n}\frac{\mathrm{i}^{m+n}}{(2\pi)^2}\int dqdp\, e^{-\mathrm{i}(uq+vp)} = \mathrm{i}^{m+n}\delta^{(m)}(u)\delta^{(n)}(v), \quad (1.20)$$

so that (1.19) gives

$$\mathcal{Q}_W(q^m p^n) = \mathrm{i}^{m+n}\int dudv\, \delta^{(m)}(u)\delta^{(n)}(v)e^{\mathrm{i}(u\hat{q}+v\hat{p})} = \sum_{r=0}^{\infty}\frac{\mathrm{i}^{m+n+r}}{r!}\int dudv\, \delta^{(m)}(u)\delta^{(n)}(v)(u\hat{q}+v\hat{p})^r. \quad (1.21)$$

With $\int du\, f(u)\delta^{(m)}(u) = (-1)^m f^{(m)}(0)$ one sees that in the sum only terms with $r = m+n$ contribute and that the result is the sum over all possible products of m operators $\hat{q}$ and n operators $\hat{p}$ divided by the number $\frac{(m+n)!}{m!n!}$ of such products.

Cohen generalized in [21] the integral representation of the Weyl scheme by introducing a filter function

$$\phi_{\mu,\nu,\lambda}(u,v) = \exp\left[\frac{\hbar}{4}\left(\mu u^2 + \nu v^2 + 2\mathrm{i}\lambda uv\right)\right]. \quad (1.22)$$

Different quantization schemes can then be parametrized by the parameters μ, ν and λ and represented as

$$\mathcal{Q}_\phi(f(q,p)) = \int dudv\; \tilde{f}(u,v)\phi_{\mu,\nu,\lambda}(u,v)\exp(\mathrm{i}(u\hat{q}+v\hat{p})). \quad (1.23)$$

The standard ordering is for example given if one chooses $\mu = \nu = 0$ and $\lambda = -1$. The effect of the filter function is that it generates under the integral additional terms in u and v. Proceeding then as described in (1.21) the filter function terms left over after integration turn out to be just the correction terms needed to go over from the Weyl ordering to another scheme. Just as it is possible to order with (1.23) the canonical operators $\hat{q}$ and $\hat{p}$, it is also possible to order creation and annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ [1]. Which values of parameters one has to choose therefore in (1.22) is summarized in the following table. [1]

Canonical coordinates		
Standard ordering	$\mathcal{Q}_S(q^m p^n) = \hat{q}^m \hat{p}^n$	$\mu = 0, \quad \nu = 0, \quad \lambda = -1$
Antistandard ordering	$\mathcal{Q}_{AS}(q^m p^n) = \hat{p}^n \hat{q}^m$	$\mu = 0, \quad \nu = 0, \quad \lambda = 1$
Weyl ordering	$\mathcal{Q}_W(q^m p^n) = (\hat{q}^m \hat{p}^n)_W$	$\mu = 0, \quad \nu = 0, \quad \lambda = 0$
Holomorphic coordinates		
Normal ordering	$\mathcal{Q}_N(a^m \bar{a}^n) = \hat{a}^\dagger \hat{a}$	$\mu = 1, \quad \nu = -1, \lambda = 0$
Antinormal ordering	$\mathcal{Q}_{AN}(a^m \bar{a}^n) = \hat{a}\hat{a}^\dagger$	$\mu = -1, \nu = 1, \quad \lambda = 0$
Weyl ordering	$\mathcal{Q}_W(a^m \bar{a}^n) = (\hat{a}\hat{a}^\dagger)_W$	$\mu = 0, \quad \nu = 0, \quad \lambda = 0$

The integral representation of operator ordering described above works in the following way: One first notices that $(\hat{q}+\hat{p})^r$ is the sum of all possible operator products of order r, i.e. $\mathcal{Q}_W((q+p)^r) = (\hat{q}+\hat{p})^r$. More generally one has for all orders $\mathcal{Q}_W(\exp(q+p)) = \exp(\hat{q}+\hat{p})$. The problem is then to pick out the right terms of the desired order and ordering. This is done by introducing additional variables u and v, so that one has $\exp(u\hat{q}+v\hat{p})$. These variables carry the information of the order of $\hat{q}$ and $\hat{p}$. Picking out the terms of the desired order is then done in the integral representation with δ-functions as described in (1.21).

[1] Note that this parametrization is not unique. The parametrization above is a parametrization with respect to the canonical coordinates, i.e. one acts with the Fourier transformations on the canonical coordinates and the ordering schemes for the holomorphic coordinates follow in a second step. But it is also possible to consider the holomorphic coordinates as primary and do the Fourier transformation in holomorphic coordinates, so that the canonical ordering schemes follow in a second step. Then the normal ordering would be parametrized for example with and $\mu = \nu = 0$, $\lambda = 1$ and the standard ordering would be parametrized with $\mu = 1$, $\nu = -1$ and $\lambda = 0$. Such a parametrization will be used in section 3.3.

This leads to the Weyl ordering and the other schemes can be obtained by introducing an filter function. Picking out the terms of a given order can not only be done with δ-functions but in a much easier way also with differential operators. One uses therefore that the Taylor expansion of $f(q,p)$ around $(q,p) = (0,0)$ can be written as

$$f(q,p) = f(\partial_u, \partial_v) e^{uq+vp}|_{u,v=0} = e^{q\partial_u + p\partial_v} f(u,v)|_{u,v=0}. \tag{1.24}$$

With $\mathcal{Q}_W(\exp(uq+vp)) = \exp(u\hat{q}+v\hat{p})$ one can then write the Weyl quantization as

$$\mathcal{Q}_W(f(q,p)) = f(\partial_u, \partial_v) e^{u\hat{q}+v\hat{p}}|_{u,v=0} = e^{\hat{q}\partial_u + \hat{p}\partial_v} f(u,v)|_{u,v=0}. \tag{1.25}$$

Other quantization schemes can here be obtained analogously to the integral representation with the help of a filter function.

So one sees that the quantization map $\mathcal{Q}$ is in several ways problematic. If one requires that $\mathcal{Q}$ should fulfill the conditions 1) to 4) one sees that only polynomials up to second order can be quantized and relaxing the conditions leads to operator ordering problems. Moreover quantizing with a map $\mathcal{Q}$ induces an conceptual and formal break in physics when one goes over to quantum mechanics. So one could wonder if it is possible to quantize without the problematic quantization map $\mathcal{Q}$. This is indeed possible if one describes the non-commutativity that enters quantum theory not by non-commuting objects like operators, but by a non-commutative product. How such a product can be constructed will be discussed in the next section.

1.2 Star Products

If one wants to circumvent the quantization map $\mathcal{Q}$ by introducing a non-commuting product, called star product, this product should emulate the non-commutativity of the operators. So the star product, denoted by "$*$", should fulfill

$$\mathcal{Q}(f)\mathcal{Q}(g) = \mathcal{Q}(f * g), \tag{1.26}$$

where different quantization schemes would lead to different star products. Equation (1.26) states that the quantum mechanical algebra of observables is a representation of the star product algebra.

Since as shown above the Weyl scheme seems to be the most fundamental quantization scheme in the sense that all other schemes can be constructed out of this scheme with a filter function, one first calculates with (1.26) an explicit expression for the star product in the Weyl scheme. This case was first considered by Moyal [92], so that the star product that corresponds to the Weyl scheme is called Moyal product. With the integral representation (1.19) equation (1.26) can be written as:

$$\begin{aligned} \mathcal{Q}_W(f)\mathcal{Q}_W(g) &= \int du_1 dv_1 du_2 dv_2 \, \tilde{f}(u_1,v_1)\tilde{g}(u_2,v_2) e^{-\mathrm{i}(u_1\hat{q}+v_1\hat{p})} e^{-\mathrm{i}(u_2\hat{q}+v_2\hat{p})} \\ &= \int du_1 dv_1 du_2 dv_2 \, \tilde{f}(u_1,v_1)\tilde{g}(u_2,v_2) e^{-\mathrm{i}((u_1+u_2)\hat{q}+(v_1+v_2)\hat{p})} e^{-\frac{\mathrm{i}\hbar}{2}(u_1 v_2 - v_1 u_2)}, \end{aligned} \tag{1.27}$$

where the truncated Campbell-Baker-Hausdorff formula was used:

$$e^{\hat{A}} e^{\hat{B}} = e^{(\hat{A}+\hat{B})} e^{\frac{1}{2}[\hat{A},\hat{B}]}. \tag{1.28}$$

Expanding now the last exponential in (1.27) and making the substitution $u = u_1 + u_2$ and $v = v_1 + v_2$ gives:

$$\begin{aligned} \mathcal{Q}_W(f)\mathcal{Q}_W(g) = &\int dudv \, e^{-\mathrm{i}(u\hat{q}+v\hat{p})} \\ &\times \int du_1 dv_1 \sum_{m,n=0}^{\infty} \frac{(-1)^m}{m!n!} \left(\frac{\mathrm{i}\hbar}{2}\right)^{m+n} u_1^m v_1^n \tilde{f}(u_1,v_1)(u-u_1)^n (v-v_1)^m \tilde{g}(u-u_1, v-v_1). \end{aligned} \tag{1.29}$$

The expression in the second line of (1.29) is by the Fourier convolution theorem just the Fourier transform of the expression for the Moyal product:

$$\mathcal{Q}_W(f)\mathcal{Q}_W(g) = \int dudv\, e^{-\mathrm{i}(u\hat{q}+v\hat{p})} \widetilde{f *_M g} = \mathcal{Q}_W(f *_M g). \tag{1.30}$$

The Moyal star product can then be read off as:

$$\begin{aligned}(f *_M g)(q,p) &= \sum_{m,n=0}^{\infty} \frac{(-1)^m}{m!n!} \left(\frac{\mathrm{i}\hbar}{2}\right)^{m+n} (\partial_p^m \partial_q^n f)(\partial_p^n \partial_q^m g) \\ &= f(q,p) \exp\left[\frac{\mathrm{i}\hbar}{2}\left(\overleftarrow{\partial}_q \vec{\partial}_p - \overleftarrow{\partial}_p \vec{\partial}_q\right)\right] g(q,p),\end{aligned} \tag{1.31}$$

where the vector arrows indicate in which direction the differentiation acts.

This expression for the Moyal product can be derived even more simply in the differential representation (1.25) for the Weyl scheme:

$$\begin{aligned}\mathcal{Q}_W(f)\mathcal{Q}_W(g) &= f(u_1,v_1) \exp\left[\overleftarrow{\partial}_{u_1}\hat{q} + \overleftarrow{\partial}_{v_1}\hat{p}\right] \exp\left[\hat{q}\vec{\partial}_{u_2} + \hat{p}\vec{\partial}_{v_2}\right] g(u_2,v_2)\Big|_{u_i,v_i=0} \\ &= \exp\left[\hat{q}(\partial_{u_1}+\partial_{u_2}) + \hat{p}(\partial_{v_1}+\partial_{v_2})\right] f(u_1,v_1) \exp\left[\frac{\mathrm{i}\hbar}{2}(\overleftarrow{\partial}_{u_1}\vec{\partial}_{v_2} - \overleftarrow{\partial}_{v_1}\vec{\partial}_{u_1})\right] g(u_2,v_2)\Big|_{u_i,v_i=0} \\ &= \exp\left[\hat{q}\partial_u + \hat{p}\partial_v\right] f(u,v) \exp\left[\frac{\mathrm{i}\hbar}{2}(\overleftarrow{\partial}_u\vec{\partial}_v - \overleftarrow{\partial}_v\vec{\partial}_u)\right] g(u,v)\Big|_{u,v=0},\end{aligned} \tag{1.32}$$

where again (1.28) was used. With (1.24) for the Taylor expansion it is then easy to see that the Moyal product can also be written by a shift formula:

$$(f *_M g)(q,p) = f\left(q + \frac{\mathrm{i}\hbar}{2}\vec{\partial}_p, p - \frac{\mathrm{i}\hbar}{2}\vec{\partial}_q\right) g(q,p). \tag{1.33}$$

Besides the differential representation (1.31) there is also a integral representation for the Moyal product; this can be derived by applying the Fourier convolution theorem backwards, which gives

$$\begin{aligned}f *_M g &= \sum_{m,n=0}^{\infty} \frac{(-1)^m}{m!n!}\left(\frac{\mathrm{i}\hbar}{2}\right)^{m+n} (\partial_p^m\partial_q^n f)(\partial_p^n\partial_q^m g) \\ &= \frac{1}{(2\pi)^2}\int du_1dv_1du_2dv_2dq_1dp_1dq_2dp_2\; e^{\mathrm{i}u_2q}e^{\mathrm{i}v_2p}e^{\frac{\mathrm{i}\hbar}{2}(v_1(u_2-u_1)-u_1(v_2-v_1))} \\ &\qquad \times e^{-\mathrm{i}u_1q_1-\mathrm{i}v_1p_1} f(q_1,p_1) e^{-\mathrm{i}(u_2-u_1)q_2-\mathrm{i}(v_2-v_1)p_2} g(q_2,p_2) \\ &= \frac{1}{(2\pi)^2}\int du_1dv_1du_2dv_2dq_1dp_1dq_2dp_2\; f(q_1,p_1)g(q_2,p_2) \\ &\qquad \times \exp\left[\mathrm{i}u_2\left(q + \frac{\hbar}{2}v_1 - q_2\right) + \mathrm{i}v_2\left(p - \frac{\hbar}{2}u_1 - p_2\right) - \mathrm{i}(u_1q_1 + v_1p_1 - u_1q_2 - v_1p_2)\right].\end{aligned}$$

Rescaling the δ-functions according to $\delta(-q-(\hbar/2)v_1+q_2) = (2/\hbar)\delta(v_1+(2/\hbar)q-(2/\hbar)q_2)$ gives:

$$f *_M g = \frac{1}{\hbar^2\pi^2}\int dq_1dq_2dp_1dp_2 f(q_1,p_1)g(q_2,p_2)\exp\left[\frac{2}{\mathrm{i}\hbar}\left(p(q_1-q_2) + q(p_2-p_1) + (q_2p_1 - q_1p_2)\right)\right],$$

The exponent in the above expression has an interesting geometric interpretation. Denote points in the two dimensional phase space by vectors: $\vec{x} = (q,p)^T$, $\vec{x}_1 = (q_1,p_1)^T$ and $\vec{x}_2 = (q_2,p_2)^T$, the area of the triangle in phase space spanned by $\vec{x}-\vec{x}_1$ and $\vec{x}-\vec{x}_2$ is then given by

$$A_\triangle(\vec{x},\vec{x}_1,\vec{x}_2)=\frac{1}{2}(\vec{x}-\vec{x}_1)\wedge(\vec{x}-\vec{x}_2)=\frac{1}{2}\left[p(q_2-q_1)+q(p_1-p_2)+(q_1p_2-q_2p_1)\right], \tag{1.34}$$

so that the integral representation of the Moyal product is eventually given by

$$(f *_M g)(\vec{x})=\frac{1}{\hbar^2\pi^2}\int d\vec{x}_1 d\vec{x}_2\, f(\vec{x}_1)g(\vec{x}_2)\exp\left[\frac{4\mathrm{i}}{\hbar}A_\triangle(\vec{x},\vec{x}_1,\vec{x}_2)\right]. \tag{1.35}$$

The integral representation can now be used to prove some basic properties of the Moyal product, for example that the Moyal product is associative:

$$(f *_M g) *_M h = f *_M (g *_M h). \tag{1.36}$$

Therefore one writes the left hand side with (1.35) as

$$\begin{aligned}\big((f *_M g) *_M h\big)(\vec{x}) &= \frac{1}{\hbar^2\pi^2}\int d\vec{x}_0 d\vec{x}_3\,(f *_M g)(\vec{x}_0)h(\vec{x}_3)\exp\left[\frac{4\mathrm{i}}{\hbar}A_\triangle(\vec{x},\vec{x}_0,\vec{x}_3)\right]\\ &= \frac{1}{\hbar^4\pi^4}\int d\vec{x}_0 d\vec{x}_1 d\vec{x}_2 d\vec{x}_3\, f(\vec{x}_1)g(\vec{x}_2)h(\vec{x}_3)\\ &\qquad\times\exp\left[\frac{4\mathrm{i}}{\hbar}\big(A_\triangle(\vec{x}_0,\vec{x}_1,\vec{x}_2)+A_\triangle(\vec{x},\vec{x}_0,\vec{x}_3)\big)\right].\end{aligned} \tag{1.37}$$

With (1.34) one obtains for the exponent

$$\begin{aligned}&2\big(A_\triangle(\vec{x}_0,\vec{x}_1,\vec{x}_2)+A_\triangle(\vec{x},\vec{x}_0,\vec{x}_3)\big)\\ &\qquad= q_0(-p+p_1-p_2+p_3)+p_0(q-q_1+q_2-q_3)+p_1q_2+p_2q_1+pq_3-p_3q.\end{aligned} \tag{1.38}$$

The first two terms give together with the q_0- and p_0-integration the δ-functions $\delta(p-p_1+p_2-p_3)$ and $\delta(q-q_1+q_2-q_3)$. The conditions $p=p_1-p_2+p_3$ and $q=q_1-q_2+q_3$, that the δ-functions impose, allow to write the other terms of (1.38) as: $p_1q_2+p_2q_1+pq_3-p_3q=p_1(q_2-q_3)+p_2(q_1-q_3)+p_3(q_2-q_1)=2A_\triangle(\vec{x}_1,\vec{x}_2,\vec{x}_3)$. (1.37) can then be written as

$$\begin{aligned}\big((f *_M g) *_M h\big)(\vec{x}) = \frac{1}{\hbar^2\pi^2}\int d\vec{x}_1 d\vec{x}_2 d\vec{x}_3\, f(\vec{x}_1)g(\vec{x}_2)h(\vec{x}_3)&\\ \times\,\delta(p-p_1+p_2-p_3)\delta(q-q_1+q_2-q_3)\exp\left[\frac{4\mathrm{i}}{\hbar}A_\triangle(\vec{x}_1,\vec{x}_2,\vec{x}_3)\right]&.\end{aligned} \tag{1.39}$$

Since the expression on the right hand side contains no information about the brackets on the left hand side the Moyal product is associative.

Another important property of the Moyal product that can be seen in the integral representation is

$$\int dq dp\, f *_M g = \int dq dp\, fg = \int dq dp\, g *_M f. \tag{1.40}$$

For the proof one considers the phase space integral of (1.35), which gives

$$\begin{aligned}\int dq dp\, f *_M g = \int dq_1 dp_1 dq_2 dp_2\, f(q_1,p_1)g(q_2,p_2)\exp\left(\frac{2}{\mathrm{i}\hbar}(p_1q_2-p_2q_2)\right)&\\ \times\frac{1}{\hbar^2\pi^2}\int dq dp\exp\left(\frac{2}{\mathrm{i}\hbar}\big(q(p_2-p_1)+p(q_1-q_2)\big)\right)&,\end{aligned}$$

where the second line is the product of the δ-functions $\delta(p_2 - p_1)$ and $\delta(q_1 - q_2)$, so that (1.40) follows.

The discussion carried out so far for the star product that corresponds to Weyl ordering can in the same way be repeated for another ordering. For example the star product that corresponds to standard ordering is called the standard product and has the form

$$(f *_S g)(q,p) = f(q,p) \exp\left[\mathrm{i}\hbar \overleftarrow{\partial}_q \overrightarrow{\partial}_p\right] g(q,p). \tag{1.41}$$

Similarly one can discuss the different orderings in the case of holomorphic coordinates. The star product that describes Weyl ordering in holomorphic coordinates is also called Moyal product and results from (1.31) by transformation of variables:

$$(f *_M g)(a,\bar{a}) = f(a,\bar{a}) \exp\left[\frac{\hbar}{2}(\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a)\right] g(a,\bar{a}) \tag{1.42}$$

and for the normal ordering one gets the normal star product as:

$$(f *_N g)(a,\bar{a}) = f(a,\bar{a}) \exp\left[\hbar \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}}\right] g(a,\bar{a}). \tag{1.43}$$

Just as it was possible to relate different orderings in the integral representation by a filter function (1.22), it is also possible to relate different star products. This is done by the concept of c-equivalence. Two star products $*$ and $*'$ are said to be c-equivalent, if there exists an invertible transition operator $T = \sum_{n=0}^{\infty} \hbar^n T_n$, where the T_n are differential operators, so that

$$f *' g = T^{-1}\big((Tf) * (Tg)\big). \tag{1.44}$$

It is known that for flat phase spaces all admissible star products are c-equivalent to the Moyal product. For example the standard product is related to the Moyal product by

$$T(f *_S g) = (Tf) *_M (Tg) \qquad \text{with} \qquad T = \exp\left(-\frac{\mathrm{i}\hbar}{2} \overrightarrow{\partial}_q \overrightarrow{\partial}_p\right). \tag{1.45}$$

One should note that the T-operators are just the filter functions, where one substitutes u by ∂_q and v by ∂_p.

1.3 Quantum Mechanics in the Star Product Formalism

So far one has succeded in describing the operator algebra of observables as a star product algebra of the corresponding phase space functions. But in order to do quantum mechanics with star products on the phase space one also needs phase space functions that correspond to quantum mechanical states. States on the phase space can be constructed with the inverse of the quantization map. The inverse of (1.23) is given by [1]:

$$\mathcal{Q}_\phi^{-1}(\hat{f}(\hat{q},\hat{p})) = 2\pi\hbar \mathrm{Tr}\left[\hat{f}(\hat{q},\hat{p}) \frac{1}{(2\pi)^2} \int du dv\, \phi^{-1}_{\mu,\nu,\lambda}(-u,-v)\, e^{\mathrm{i}[u(\hat{q}-q)+v(\hat{p}-p)]}\right]. \tag{1.46}$$

In the case of Weyl ordering this can easily be calculated as:

$$
\begin{aligned}
\mathcal{Q}_W^{-1}(\hat{f}(\hat{q},\hat{p})) &= \frac{\hbar}{2\pi}\int dudv\, e^{-\mathrm{i}(uq+vp)}\mathrm{Tr}\left[\hat{f}(\hat{q},\hat{p})e^{\mathrm{i}(u\hat{q}+v\hat{p})}\right] && (1.47)\\
&= \frac{\hbar}{2\pi}\int dudv\, e^{-\mathrm{i}(uq+vp)}e^{-\mathrm{i}\frac{\hbar}{2}uv}\mathrm{Tr}\left[\int dq'|q'\rangle\langle q'|\hat{f}(\hat{q},\hat{p})\,e^{\mathrm{i}v\hat{p}}e^{\mathrm{i}u\hat{q}}\right] && (1.48)\\
&= \frac{\hbar}{2\pi}\int dudv\, e^{-\mathrm{i}(uq+vp)}e^{-\mathrm{i}\frac{\hbar}{2}uv}\int dq'\langle q'|\hat{f}(\hat{q},\hat{p})e^{\mathrm{i}v\hat{p}}|q'\rangle e^{\mathrm{i}uq'} && (1.49)\\
&= \frac{\hbar}{2\pi}\int dudvdq'\, e^{-\mathrm{i}u(q-q'-\frac{\hbar}{2}v)}\langle q'|\hat{f}(\hat{q},\hat{p})|q-\hbar v\rangle e^{-\mathrm{i}vp} && (1.50)\\
&= \hbar\int dv\,\langle q+\frac{\hbar}{2}v|\hat{f}(\hat{q},\hat{p})|q-\frac{\hbar}{2}v\rangle e^{-\mathrm{i}vp}. && (1.51)
\end{aligned}
$$

The same calculation can be redone for other operator orderings, for example standard ordering leads to:

$$\mathcal{Q}_S^{-1}(\hat{f}(\hat{q},\hat{p})) = \hbar\int dv\,\langle q|\hat{f}(\hat{q},\hat{p})|q-\hbar v\rangle e^{-\mathrm{i}vp}. \tag{1.52}$$

If one then applies $\mathcal{Q}^{-1}$ to the product of two operators one gets with (1.26) in the case of Weyl ordering:

$$\mathcal{Q}_W^{-1}(\hat{f}\hat{g}) = f *_{\scriptscriptstyle M} g \tag{1.53}$$

and the integrated version of this equation is with (1.40) and (1.51)

$$\int dqdp\,\mathcal{Q}_W^{-1}(\hat{f}\hat{g}) = \int dqdp\, f *_{\scriptscriptstyle M} g = \int dqdp\, fg = 2\pi\hbar\mathrm{Tr}\left[\hat{f}\hat{g}\right]. \tag{1.54}$$

With the map $\mathcal{Q}^{-1}$ one can construct the phase space analogue of a pure state $|\psi\rangle$, simply by calculating the phase space function corresponding to the density matrix $\hat{\rho} = |\psi\rangle\langle\psi|$. This calculation leads to the well known phase space distribution functions (see for example [111] and [87]). For example in the case of Weyl ordering equation (1.51) gives for $\hat{f} = |\psi\rangle\langle\psi|$ the Wigner function

$$\pi(q,p) = \hbar\int dv\,\bar{\psi}(q+\frac{\hbar v}{2})\psi(q-\frac{\hbar v}{2})e^{-\mathrm{i}pv}. \tag{1.55}$$

In contrast to the wave functions the Wigner functions describe a quantum mechanical state on the whole phase space, but the wave functions can be reobtained by integration over one of the phase space coordinates:

$$\int dp\,\pi(q,p) = 2\pi\hbar|\psi(q)|^2 \qquad \text{and} \qquad \int dq\,\pi(q,p) = 2\pi\hbar|\tilde{\psi}(p)|^2. \tag{1.56}$$

The phase space integral gives then the normalization condition for the Wigner functions

$$\int dqdp\,\pi(q,p) = 2\pi\hbar. \tag{1.57}$$

The phase space integral of two Wigner functions $\pi_1(q,p)$ and $\pi_2(q,p)$ is given with (1.54) by

$$\int dqdp\,\pi_1(q,p)\pi_2(q,p) = 2\pi\hbar\mathrm{Tr}\Big[|\psi_1\rangle\langle\psi_1|\psi_2\rangle\langle\psi_2|\Big] = 2\pi\hbar\left|\int dq\,\bar{\psi}_1(q)\psi_2(q)\right|^2. \tag{1.58}$$

For $\psi_1(q) = \psi_2(q)$ this leads to

$$\int dqdp\,\left[\pi(q,p)\right]^2 = \int dqdp\,\pi(q,p), \tag{1.59}$$

while for orthogonal $\psi_1(q)$ and $\psi_2(q)$ one has $\int dqdp\, \pi_1(q,p)\pi_2(q,p) = 0$. Then at least one of the Wigner functions must be negative somewhere. This means that although the Wigner functions have the right marginal probabilities (1.56) they can not be interpreted as probability distributions.

Nevertheless it is possible to do quantum mechanics with the Wigner functions. For example the expectation value of an operator can be obtained with (1.54) as

$$\int dqdp\, f(q,p)\pi(q,p) = 2\pi\hbar \mathrm{Tr}\left[\hat{f}\hat{\rho}\right] = 2\pi\hbar\langle\hat{f}\rangle. \tag{1.60}$$

Similarly the eigenvalue equation of quantum mechanics can be reproduced on the phase space. For example for the Hamilton operator $\hat{H}$ one can calculate $\mathcal{Q}_W^{-1}(\hat{H}\hat{\rho}) = H *_M \pi(q,p)$, which gives for $H = \frac{1}{2m}p^2 + V(q)$ with the shift formula (1.33):

$$\begin{aligned}
H *_M \pi_E &= \left(\frac{1}{2m}\left(p - \mathrm{i}\frac{\hbar}{2}\vec{\partial}_q\right)^2 + V(q)\right)\hbar\int dv\, e^{-\mathrm{i}v(p+i\frac{\hbar}{2}\overleftarrow{\partial}_q)}\psi^*(q-\frac{\hbar}{2}v)\psi(q+\frac{\hbar}{2}v) &(1.61)\\
&= \hbar\int dv\, \left(\frac{1}{2m}\left(p - \mathrm{i}\frac{\hbar}{2}\vec{\partial}_q\right)^2 + V\left(q+\frac{\hbar}{2}v\right)\right)^2 e^{-\mathrm{i}vp}\psi^*(q-\frac{\hbar}{2}v)\psi(q+\frac{\hbar}{2}v) &(1.62)\\
&= \hbar\int dv\, e^{-\mathrm{i}vp}\left(\frac{1}{2m}\left(\mathrm{i}\vec{\partial}_v - \mathrm{i}\frac{\hbar}{2}\vec{\partial}_q\right)^2 + V\left(q+\frac{\hbar}{2}v\right)\right)^2 \psi^*(q-\frac{\hbar}{2}v)\psi(q+\frac{\hbar}{2}v) &(1.63)\\
&= \hbar\int dv\, e^{-\mathrm{i}vp}\psi^*(q-\frac{\hbar}{2}v)E\psi(q+\frac{\hbar}{2}v) &(1.64)\\
&= E *_M \pi_E. &(1.65)
\end{aligned}$$

This equation is the phase space analogue of $\hat{H}\hat{\rho} = E\hat{\rho}$. In the same way one can calculate the time development of the Wigner function by applying $\mathcal{Q}_W^{-1}$ to the von Neumann equation $\mathrm{i}\hbar\frac{\partial}{\partial t}\hat{\rho} = \left[\hat{H},\hat{\rho}\right]$, which gives

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\pi(q,p;t) = \left[H(q,p;t),\pi(q,p;t)\right]_{*_M}. \tag{1.66}$$

The time development of the density matrix can also be calculated with the time development operator, which acts as $|\psi,t\rangle = \hat{U}(t)|\psi,0\rangle = \hat{U}(t)|\psi\rangle_H$, where the index H indicates that $|\psi\rangle_H$ is the time independent state in the Heisenberg picture. The time development of the density matrix is then $\hat{\rho}(t) = \hat{U}(t)|\psi\rangle_H\langle\psi|_H\hat{U}^\dagger(t)$, which translates into the phase space language as

$$\pi(q,p;t) = U(t) *_M \pi_H(q,p) *_M \overline{U}(t), \tag{1.67}$$

where $\pi(q,p;t)$ is the Wigner function in the Schrödinger picture, $\pi_H(q,p) = \pi(q,p;t=0)$ the Wigner function in the Heisenberg picture and $\bar{U}(t)$ the complex conjugate of $U(t)$. The next task is then to find the phase space analogue $U(t)$ of the time development operator $\hat{U}(t)$. The equation that $\hat{U}(t)$ fulfills can be translated directly to the phase space:

$$\mathrm{i}\hbar\frac{d}{dt}\hat{U}(t) = \hat{H}(t)\hat{U}(t) \qquad \Rightarrow \qquad \mathrm{i}\hbar\frac{d}{dt}U(t) = H(t) *_M U(t). \tag{1.68}$$

$\hat{U}(t)$ has then the form $\hat{U}(t) = \mathcal{T}\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^t dt'\,\hat{H}(t')\right)$, where $\mathcal{T}$ is the time ordering operator. In the case of a time independent Hamilton operator one has $\hat{U}(t) = e^{-\mathrm{i}\hat{H}t/\hbar}$, what is expressed on the phase space by the star exponential

$$U(t) = e_{*_M}^{-\mathrm{i}Ht/\hbar} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\mathrm{i}t}{\hbar}\right)^n H^{n*_M} \equiv \mathrm{Exp}_M(Ht), \tag{1.69}$$

where

$$H^{n*_M} = \underbrace{H *_M H *_M \cdots *_M H}_{n \text{ times}}. \tag{1.70}$$

The time development of the observables in the Heisenberg picture can then also be given as

$$\hat{f}_H(t) = \hat{U}^\dagger(t)\hat{f}\hat{U}(t) \qquad \Rightarrow \qquad f_H(t) = \overline{U}(t) *_M f *_M U(t) \tag{1.71}$$

and time-differentiation of these equations gives with (1.68) the von Neumann equation:

$$\frac{d}{dt}\hat{f}_H = \frac{1}{\mathrm{i}\hbar}\left[\hat{f}_H, \hat{H}\right] + \frac{\partial \hat{f}_H}{\partial t} \qquad \Rightarrow \qquad \frac{d}{dt}f_H = \frac{1}{\mathrm{i}\hbar}\left[f_H, H\right]_{*_M} + \frac{\partial f_H}{\partial t}. \tag{1.72}$$

Since in the following only the Heisenberg picture is used the index H will be dropped. One should note that for the observables q and p the von Neumann equation leads to the classical equations of motion

$$\frac{dq}{dt} = \frac{1}{\mathrm{i}\hbar}(q *_M H - H *_M q) = \partial_p H \qquad \text{and} \qquad \frac{dp}{dt} = \frac{1}{\mathrm{i}\hbar}(p *_M H - H *_M p) = -\partial_q H. \tag{1.73}$$

All the translation done above for the case of Weyl ordering can also be done for other orderings. So the star product formalism circumvents the use of the problematic quantization map $\mathcal{Q}$, but one is still plagued by the ordering problem. Choosing different orderings leads to different star products and different phase space distribution functions [87]. The question is then which ordering has to be chosen. This question can be investigated by imposing reasonable requirements a phase space functions has to fulfill, for example it should transform as a scalar function under the transformations of the Galilei group. Such requirements were considered in [85] and it was shown that the Wigner function is the only phase space distribution function that could meet theses requirements. So it seems that at least for nonrelativistic quantum mechanics the Wigner function and the corresponding Moyal product are the canonical structures for doing quantum physics on the phase space.

Having translated quantum mechanics from a version that works on a Hilbert space into a version that works on the phase space suggests to forget about the operator formalism and to describe quantum mechanics directly on the phase space. Starting point is then a classical system that is described in the Hamilton formalism. A state of the system is described as a point of the phase space and the observables of the systems are functions on the phase space. Physical quantities of the system at some time are calculated by evaluating the observables at the corresponding phase space point $x = (q_0, p_0)$ that characterizes the state of the system at this time. The evaluation of the energy can for example be mathematically expressed as

$$E = \int dqdp\, \delta(q - q_0, p - p_0) H(q, p). \tag{1.74}$$

So the observables of the dynamical system are functions on the phase space and the states of the system are positive functionals on the observables (here the Dirac δ-function) and one obtains the value of the observable in a definite state by the above mentioned operation. As described in the beginning of the chapter there are two additional structures. First there is the pointwise product of functions on the phases space, so that a commutative classical algebra of observables is constituted and second there is the Poisson bracket that is used to describe the time development of the system.

Going over to quantum theory means now to implement Heisenberg's uncertainty relation, which implies that the states can no longer be represented as points on the phase space. Moreover the uncertainty is intimately connected to the non-commutativity of the algebra of observables. So uncertainty is realized by describing physical states by phase space distribution functions that are not sharply localized, in contrast to the δ-function that appears in the classical case. If one then evaluates an observable in some definite state according to the quantum analogue of Eq. (1.74), values of the observabels in a whole region contribute to the number that one obtains, which is thus an average value of the observable in the given state. On the other hand non-commutativity is incorporated by introducing a non-commutative product for functions on phase space, so that one gets a non-commutative quantum algebra of observables. One can now make an ansatz for this non-commutative product one is looking for:

$$f * g = fg + (\mathrm{i}\hbar)C_1(f,g) + O(\hbar^2) = \sum_{n=0}^{\infty} (\mathrm{i}\hbar)^n C_n(f,g). \tag{1.75}$$

In mathematics such a modified product was first considered by Gerstenhaber in [60], where the modification was called deformation. The deformation of the pointwise product is here done in a continuous way, which is described by the deformation parameter $(\mathrm{i}\hbar)$. If $\hbar$ is identified with Planck's constant, then what varies is really the magnitude of the action of the dynamical system considered in units of $\hbar$: the classical limit holds for systems with large action. In this limit, which is expressed by $\hbar \to 0$, the star product reduces to the usual pointwise product. The expressions $C_n(f,g)$ denote functions made up of the derivatives of the functions f and g and they should be chosen in a way that the new product in non-commutative. But without further restriction of these coefficients, the star product is too arbitrary to be of any use. Gerstenhaber's discovery was that the simple requirement that the new product be associative imposes such strong requirements on the coefficients C_n that they are essentially unique (up to an equivalence relation that is just the c-equivalence). Formally Gerstenhaber required that the coefficients satisfy the following properties:

(i) $\sum_{j+k=n} C_j(C_k(f,g),h) = \sum_{j+k=n} C_j(f, C_k(g,h))$,

(ii) $C_0(f,g) = fg$,

(iii) $C_1(f,g) - C_1(g,f) = \{f,g\}_{PB}$.

Property (i) guarantees that the star product is associative. Property (ii) means that in the limit $\hbar \to 0$ the star product agrees with the pointwise product. Property (iii) has at least two aspects. Mathematically, it anchors the new product to the given structure of the Poisson manifold. Physically, it provides the connection between the classical and quantum behavior of the dynamical system. Property (iii) can be written with the star commutator as

$$\lim_{\hbar \to 0} \frac{1}{\mathrm{i}\hbar} [f,g]_* = \{f,g\}_{PB}, \tag{1.76}$$

which is the correct form of the correspondence principle. In general, the quantity on the left hand side reduces to the Poisson bracket only in the classical limit. The source of the mathematical difficulties formulating the correspondence principle that the operator formalism encounters is related to trying to enforce equality between the Poisson bracket and the corresponding expressions involving the quantum mechanical commutator. Eq. (1.76) shows that such a relation in general only holds up to corrections of higher order in $\hbar$.

For physical applications one usually also requires the star product to be hermitian:

$$\overline{f * g} = \overline{g} * \overline{f}, \tag{1.77}$$

where $\overline{f}$ denotes the complex conjugate of f. The star products that were constructed from the operator formalism above have this property.

For a given Poisson manifold it is not clear a priori if a star product for the smooth functions on the manifold actually exist, that is, whether it is at all possible to find coefficients C_n that satisfy the above list of properties. Even if one finds such coefficients, it is still not clear that the series they define through (1.75) yields a smooth function. For flat euclidian space such a star product exists. In this case the components of the Poisson tensor Ω^{ij} can be taken to be constants. The coefficients C_1 can then be chosen antisymmetric, so that

$$C_1(f,g) = \frac{1}{2}\Omega^{ij}(\partial_i f)(\partial_j g) = \frac{1}{2}\{f,g\}_{PB}, \tag{1.78}$$

by property (iii) above. The higher order coefficients may be obtained by exponentiation of C_1. This procedure yields then the Moyal star product

$$f *_M g = f \exp\left[\left(\frac{\mathrm{i}\hbar}{2}\right)\Omega^{ij}\overleftarrow{\partial}_i\overrightarrow{\partial}_j\right] g, \tag{1.79}$$

which is in canonical coordinates given by (1.31). If one does not demand according to property (iii) that the coefficients C_1 are antisymmetric one obtains more general products that will be called circle products, denoted by "$\circ$".

Having established the star product on the phase space it is then possible to do quantum physics on the phase space. Starting with the classical system that is described by the Hamiltonian H one proceeds as follows. First one can for example calculate the star exponential $\mathrm{Exp}(Ht)$. For a time independent Hamiltonian this can be done either by direct calculation according to

$$\mathrm{Exp}(Ht) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-\mathrm{i}t}{\hbar}\right)^n H^{n*} \tag{1.80}$$

or by solving the defining time dependent Schrödinger equation

$$\mathrm{i}\hbar\frac{d}{dt}\mathrm{Exp}(Ht) = H * \mathrm{Exp}(Ht). \tag{1.81}$$

Since each state of definite energy E has a time-evolution $e^{-\mathrm{i}Et/\hbar}$, the star exponential as the complete time-evolution function can be written as:

$$\mathrm{Exp}(Ht) = \sum_E \pi_E e^{-\mathrm{i}Et/\hbar}. \tag{1.82}$$

This expansion is called the Fourier-Dirichlet expansion for the time-evolution function. Putting now (1.82) into (1.81) leads to the $*$-eigenvalue equation

$$(H * \pi_E)(q,p) = E\pi_E(q,p), \tag{1.83}$$

which corresponds to the time independent Schrödinger equation. Eq. (1.82) and (1.81) give for $t = 0$ the spectral decomposition of the Hamiltonian:

$$H = \sum_E E\pi_E. \tag{1.84}$$

Substituting this expression for H in (1.83) gives

$$H * \pi_E = \sum_{E'} E'\pi_{E'} * \pi_E = E\pi_E, \tag{1.85}$$

so that the phase space distribution functions fulfill

$$\pi_E * \pi_{E'} = \delta_{E,E'}\pi_E. \tag{1.86}$$

Together with the completeness relation

$$\sum_E \pi_E = 1, \tag{1.87}$$

which follows from (1.82) for $t = 0$, one sees that the phase space distribution functions are projectors. This reflects the fact that the π_E are the phase space analogues of the density matrices $|\psi_E\rangle\langle\psi_E|$. With the phase space distribution functions one can calculate the energy expectation value as

$$\langle H\rangle = \frac{1}{2\pi\hbar}\int dqdp\,(H * \pi_E)(q,p) = \frac{1}{2\pi\hbar}\int dqdp\,H(q,p)\pi_E(q,p) = E, \tag{1.88}$$

which is the quantum mechanical generalization of (1.74).

1.4 The Harmonic Oscillator in the Star Product Formalism

The formalism discussed in the last section will now be used to consider the example of the one dimensional harmonic oscillator. The Hamiltonian is given by

$$H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2. \tag{1.89}$$

While the oscillator was treated in [10] with canonical coordinates, one can alternatively also use holomorphic coordinates

$$a = \sqrt{\frac{m\omega}{2}}\left(q + \mathrm{i}\frac{p}{m\omega}\right) \qquad \text{and} \qquad \bar{a} = \sqrt{\frac{m\omega}{2}}\left(q - \mathrm{i}\frac{p}{m\omega}\right). \tag{1.90}$$

In holomorphic coordinates the Hamiltonian can be written as

$$H = \omega a\bar{a}. \tag{1.91}$$

In order to consider the physical consequence of different orderings first the quantization scheme characterized by the normal star product will be used. For the normal star product (1.43) one has

$$a *_N a = a^2, \qquad \bar{a} *_N \bar{a} = \bar{a}^2, \qquad \bar{a} *_N a = \bar{a}a \qquad \text{and} \qquad a *_N \bar{a} = a\bar{a} + \hbar, \tag{1.92}$$

so that

$$[a,a]_{*_N} = [\bar{a},\bar{a}]_{*_N} = 0 \qquad \text{and} \qquad [a,\bar{a}]_{*_N} = \hbar. \tag{1.93}$$

The defining equation for the starexponential (1.81) is given in the case of normal ordering by

$$\mathrm{i}\hbar\frac{d}{dt}\mathrm{Exp}_N(Ht) = (H + \hbar\omega\bar{a}\partial_{\bar{a}})\,\mathrm{Exp}_N(Ht) \tag{1.94}$$

with the solution

$$\mathrm{Exp}_N(Ht) = e^{-a\bar{a}/\hbar}\exp\left(e^{-\mathrm{i}\omega t}a\bar{a}/\hbar\right). \tag{1.95}$$

Expanding the last exponential one directly obtains the Fourier-Dirichlet expansion:

$$\mathrm{Exp}_N(Ht) = e^{-a\bar{a}/\hbar}\sum_{n=0}^{\infty}\frac{1}{\hbar^n n!}a^n\bar{a}^n e^{-\mathrm{i}n\omega t}. \tag{1.96}$$

Comparing coefficients in (1.95) and (1.82) gives for the phase space distribution functions

$$\pi_n^{(N)} = \frac{1}{\hbar^n n!}a^n\bar{a}^n e^{-a\bar{a}/\hbar} = \frac{1}{n!}\frac{H^n}{(\hbar\omega)^n}e^{-H/\hbar\omega} \tag{1.97}$$

and for the spectrum $E_n = \hbar\omega n$. Note that the spectrum does not include the zero-point energy. With (1.97) and the energy levels one directly verifies the $*$-eigenvalue equation

$$H *_N \pi_n^{(N)} = E_n\pi_n^{(N)} \tag{1.98}$$

and the spectral decomposition (1.84) of the Hamilton function

$$H = \sum_{n=0}^{\infty}\hbar\omega n\left(\frac{1}{\hbar^n n!}a^n\bar{a}^n e^{-a\bar{a}/\hbar}\right) = \omega a\bar{a}. \tag{1.99}$$

Moreover one sees in (1.97) that the holomorphic coordinates work as creation and annihilation functions, i.e.

$$\pi_{n+1}^{(N)} = \frac{1}{\hbar(n+1)}\bar{a} *_N \pi_n^{(N)} *_N a \qquad \text{and} \qquad \pi_{n-1}^{(N)} = \frac{1}{\hbar n}a *_N \pi_n^{(N)} *_N \bar{a} \tag{1.100}$$

and especially for the ground state one has $a *_N \pi_0^{(N)} *_N \bar{a} = a *_N \pi_0^{(N)} = 0$. This allows one to write $\pi_n^{(N)}$ as

$$\pi_n^{(N)} = \frac{1}{\hbar^n n!} \bar{a}^n *_N \pi_0^{(N)} *_N a^n \qquad \text{with} \qquad \pi_0^{(N)} = e^{-a\bar{a}/\hbar}. \tag{1.101}$$

In contrast to the operator formalism one must act in the star product formalism from both sides with the creation and annihilation function $\bar{a}$ and a in order to raise or lower phase space distribution functions, because they contain the wave function and its complex conjugate and both wave functions have to be lowered or raised.

It is easy to show that the $\pi_n^{(N)}$ are projectors. They are normalized according to

$$\frac{1}{2\pi\hbar} \int da d\bar{a}\, \pi_n^{(N)} = 1 \tag{1.102}$$

and with expression (1.97) one can immediately see the completeness relation $\sum_n \pi_n^{(N)} = 1$. The idempotency (1.86) of the phase space distribution functions can be proved as follows. First show with (1.97) that $\pi_0^{(N)} *_N \pi_0^{(N)} = \pi_0^{(N)}$. Then $\pi_m^{(N)} *_N \pi_n^{(N)}$ can be calculated with $[a, \bar{a}^n]_{*_N} = \hbar n \bar{a}^{n-1}$ and the idempotency of $\pi_0^{(N)}$, which gives

$$\pi_m^{(N)} *_N \pi_n^{(N)} = \delta_{mn} \pi_n^{(N)}. \tag{1.103}$$

The projectors $\pi_n^{(N)}$ can also be obtained from the density matrix. In holomorphic coordinates it is convenient to work with the coherent states $\hat{a}|a\rangle = a|a\rangle$ and $\langle \bar{a}|\hat{a}^\dagger = \langle \bar{a}|\bar{a}$, which are related to the energy eigenstates of the harmonic oscillator $|n\rangle = \frac{1}{\sqrt{n!}} \hat{a}^{\dagger n}|0\rangle$ by

$$|a\rangle = e^{-(1/2)a\bar{a}/\hbar} \sum_{n=0}^{\infty} \frac{a^n}{\sqrt{n!}} |n\rangle \qquad \text{and} \qquad \langle \bar{a}| = e^{-(1/2)a\bar{a}/\hbar} \sum_{n=0}^{\infty} \frac{\bar{a}^n}{\sqrt{n!}} \langle n|. \tag{1.104}$$

In normal ordering one obtains the phase space function $f(a, \bar{a})$ corresponding to the operator $\hat{f}$ by just taking the matrix element between coherent states:

$$f(a, \bar{a}) = \langle \bar{a}| \hat{f}(\hat{a}, \hat{a}^\dagger) |a\rangle, \tag{1.105}$$

so that the phase space function corresponding to $|n\rangle\langle n|$ is

$$\pi_n^{(N)} = \frac{1}{\hbar^n} \langle \bar{a}|n\rangle\langle n|a\rangle = \frac{1}{\hbar^n n!} a^n \bar{a}^n e^{-a\bar{a}/\hbar}. \tag{1.106}$$

For the off diagonal Wigner functions $\pi_{mn}^{(N)}$ that correspond to the density matrices $|m\rangle\langle n|$ one gets

$$\pi_{mn}^{(N)} = \frac{1}{\sqrt{\hbar^m m!}\sqrt{\hbar^n n!}} a^n \bar{a}^m e^{-a\bar{a}/\hbar}. \tag{1.107}$$

The Moyal quantization scheme can also be considered in holomorphic coordinates. The Moyal product can be transformed straightforwardly into holomorphic coordinates (1.90), which leads to

$$f *_M g = f \, \exp\left[\frac{\hbar}{2} \left(\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a \right) \right] g. \tag{1.108}$$

Here one has

$$a *_M a = a^2, \qquad \bar{a} *_M \bar{a} = \bar{a}^2, \qquad a *_M \bar{a} = a\bar{a} + \frac{\hbar}{2} \qquad \text{and} \qquad \bar{a} *_M a = \bar{a}a - \frac{\hbar}{2} \tag{1.109}$$

and again as in the case of the normal star product

$$[a,a]_{*_M} = [\bar{a},\bar{a}]_{*_M} = 0 \qquad \text{and} \qquad [a,\bar{a}]_{*_M} = \hbar. \tag{1.110}$$

The value of the commutator of two phase space variables is fixed by the third property of the star product, and cannot change when one goes to a c-equivalent star product. The Moyal star product is c-equivalent to the normal star product with the transition operator

$$T = \exp\left(-\frac{\hbar}{2}\vec{\partial}_a\vec{\partial}_{\bar{a}}\right). \tag{1.111}$$

One can now proceed the same way as in the case of the normal star product, or one can use (1.111) to transform the results already calculated in the case of normal star product into the Moyal star product version. This approach can be advantageous because the calculations in the normal product scheme are easier than the calculations in the Moyal product scheme.

The Moyal Wigner functions can be obtained according to

$$\pi_n^{(M)} = T\pi_n^{(N)} = \frac{1}{\hbar^n n!}T\bar{a}^n *_M T\pi_0^{(N)} *_M Ta^n = \frac{1}{\hbar^n n!}\bar{a}^n *_M \pi_0^{(M)} *_M a^n, \tag{1.112}$$

which shows that the holomorphic coordinates act here also as creation and annihilation functions. Using (1.112), $[a,\bar{a}^n]_{*_M} = \hbar n \bar{a}^{n-1}$ and $a *_M \pi_0^{(M)} = 0$ one can then write the $*$-eigenvalue equation for the Moyal product as

$$H *_M \pi_n^{(M)} = \omega\left(\bar{a} *_M a + \frac{\hbar}{2}\right) *_M \pi_n^{(M)} = \hbar\omega\left(n+\frac{1}{2}\right)\pi_n^{(M)}. \tag{1.113}$$

So in the Moyal-case one obtains the spectrum $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$, which differs from the spectrum in the normal-case by the zero point energy. The physical difference between different orderings or c-equivalent star products is a shift in the spectrum. One may then ask which explicit form the Wigner functions $\pi_n^{(M)}$ have. This can be answered for the ground state by first calculating

$$\begin{aligned}
\pi_0^{(M)} = T\pi_0^{(N)} &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{\hbar}{2}\right)^k \partial_a^k\partial_{\bar{a}}^k e^{-a\bar{a}/\hbar} \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{\hbar}{2}\right)^k \partial_a^k\left(-\frac{a}{\hbar}\right)^k e^{-a\bar{a}/\hbar} \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{2}\right)^k \sum_{l=0}^{k}\binom{k}{l}\left(\partial_a^l a^k\right)\partial_a^{k-l}e^{-a\bar{a}/\hbar} \\
&= e^{-a\bar{a}/\hbar}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{k!}{l!(k-l)!(k-l)!}\left(-\frac{1}{\hbar}a\bar{a}\right)^{k-l} \\
&= e^{-a\bar{a}/\hbar}\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k L_k\left(\frac{1}{\hbar}a\bar{a}\right) \\
&= 2e^{-2a\bar{a}/\hbar},
\end{aligned} \tag{1.114}$$

where one uses in the last step the generating function for the Laguerre polynomials

$$\frac{1}{1+s}\exp\left[\frac{zs}{1+s}\right] = \sum_{n=0}^{\infty}s^n(-1)^n L_n(z). \tag{1.115}$$

The $\pi_n^{(M)}$ can then be obtained via (1.112) or they can be calculated directly as

$$\begin{aligned}
\pi_n^{(M)} = T\pi_n^{(N)} &= e^{-\frac{\hbar}{2}\partial_a\partial_{\bar{a}}}\frac{1}{\hbar^n n!}a^n\bar{a}^n e^{-a\bar{a}/\hbar} \\
&= \frac{1}{\hbar^n n!}a^n\bar{a}^n \exp\left[-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a\overleftarrow{\partial}_{\bar{a}}+\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}}+\overleftarrow{\partial}_{\bar{a}}\overrightarrow{\partial}_a+\overrightarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}}\right)\right]e^{-a\bar{a}/\hbar} \\
&= \frac{2}{\hbar^n n!}a^n\bar{a}^n \exp\left[-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a\overleftarrow{\partial}_{\bar{a}}+\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}}+\overleftarrow{\partial}_{\bar{a}}\overrightarrow{\partial}_a\right)\right]e^{-2a\bar{a}/\hbar}, \qquad (1.116)
\end{aligned}$$

where one uses in first step

$$\exp\left[\partial_a\partial_{\bar{a}}\right]f(a,\bar{a})g(a,\bar{a}) = f(a,\bar{a})\exp\left[(\overleftarrow{\partial}_a+\overleftarrow{\partial}_{\bar{a}})(\overrightarrow{\partial}_a+\overrightarrow{\partial}_{\bar{a}})\right]g(a,\bar{a}) \qquad (1.117)$$

and in the second step (1.114). (1.116) can be further simplified to

$$\begin{aligned}
\pi_n^{(M)} = T\pi_n^{(N)} &= \frac{2}{\hbar^n n!}a^n\bar{a}^n \exp\left[-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a\overleftarrow{\partial}_{\bar{a}}+\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}}\right)\right]e^{\overleftarrow{\partial}_{\bar{a}}\bar{a}}e^{-2a\bar{a}/\hbar} \\
&= \frac{2}{\hbar^n n!}a^n\bar{a}^n e^{\overleftarrow{\partial}_{\bar{a}}\bar{a}} \exp\left[-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a\overleftarrow{\partial}_{\bar{a}}+\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}}\right)\right]e^{-2a\bar{a}/\hbar} \\
&= \frac{2}{\hbar^n n!}2^n a^n\bar{a}^n \exp\left[-\frac{\hbar}{2}\overleftarrow{\partial}_a\overleftarrow{\partial}_{\bar{a}}\right]e^{\overleftarrow{\partial}_a a}e^{-2a\bar{a}/\hbar} \\
&= (-1)^n 2L_n\left(\frac{2}{\hbar}a\bar{a}\right)e^{\overleftarrow{\partial}_a a}e^{-2a\bar{a}/\hbar} \\
&= (-1)^n 2L_n\left(\frac{4H}{\hbar\omega}\right)e^{-2H/\hbar\omega}, \qquad (1.118)
\end{aligned}$$

where $f(x+b) = e^{b\partial_x}f(x)$ and $\left[-\frac{\hbar}{2}\left(\overleftarrow{\partial}_a\overleftarrow{\partial}_{\bar{a}}+\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}}\right), \overleftarrow{\partial}_{\bar{a}}\bar{a}\right] = 0$ was applied. Since the result (1.118) depends only on H one can directly see that it corresponds to the result obtained in canonical coordinates [10]. Furthermore one can also calculate $\pi_n^{(M)}$ in canonical coordinates according to (1.55). The wave functions of the harmonic oscillator contain Hermite polynomials $H_n(q)$ and with

$$\int dx\,\left[H_n(x-a)H_n(x+a)e^{-x^2}\right]e^{-2\mathrm{i}bx} = 2^n\sqrt{\pi}\,n!\,e^{-b^2}L_n(2(a^2+b^2)) \qquad (1.119)$$

the expression (1.118) follows from (1.55). It is also easy to see that the $\pi_n^{(M)}$ satisfy just like the $\pi_n^{(N)}$ the projector conditions [105].

Just as for the Wigner functions there are also several ways to calculate the star exponential in the Moyal case. With the Wigner functions $\pi_n^{(M)}$ in (1.118), the energy levels $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$ and (1.115) one can calculate the star exponential according to the Fourier-Dirichlet expansions (1.82), which gives

$$\mathrm{Exp}_M(Ht) = \frac{1}{\cos(\omega t/2)}\exp\left(\frac{2H}{\mathrm{i}\hbar\omega}\tan\left(\frac{\omega t}{2}\right)\right). \qquad (1.120)$$

The other possibility is to solve the defining differential equation for the star exponential, which with

$$\begin{aligned}
H *_M f(H) &= \sum_{m,n=0}^{\infty}\left(\frac{\hbar}{2}\right)^{m+n}\frac{(-1)^n}{m!n!}\left[\partial_a^m\partial_{\bar{a}}^n H\right]\left[\partial_a^n\partial_{\bar{a}}^m f(H)\right] \\
&= Hf(H) - \left(\frac{\hbar}{2}\right)^2\omega^2\left[\frac{d}{dH}f(H) + H\frac{d^2}{dH^2}f(H)\right] \qquad (1.121)
\end{aligned}$$

reads in the Moyal case for holomorphic coordinates

$$\mathrm{i}\hbar\frac{d}{dt}\mathrm{Exp}_M(Ht) = H\mathrm{Exp}_M(Ht) - \frac{\hbar^2}{4}\omega^2\frac{d}{dH}\mathrm{Exp}_M(Ht) - \frac{\hbar^2}{4}\omega^2 H\frac{d^2}{dH^2}\mathrm{Exp}_M(Ht). \tag{1.122}$$

This differential equation is solved by (1.120).

The third possibility that demonstrates the connection of the star product formalism to the path integral approach is to calculate the Feynman kernel [101], [22]

$$K(q_2, t; q_1, 0) = \langle q_2|e^{-\mathrm{i}\hat{H}t/\hbar}|q_1\rangle. \tag{1.123}$$

The Feynman kernel describes in the conventional approach the time development of a system. Substituting a complete set of energy eigenstates one gets an expression resembling the Fourier-Dirichlet expansion

$$K(q_2, t; q_1, 0) = \sum_{n=0}^{\infty}\langle q_2|n\rangle\langle n|q_1\rangle e^{-\mathrm{i}E_n t/\hbar}. \tag{1.124}$$

Inserting the harmonic oscillator states gives

$$\begin{aligned}\frac{1}{2^n n!}\sqrt{\frac{m\omega}{\pi\hbar}}\exp\left[-\frac{m\omega}{\hbar}(q_1^2+q_2^2)\right]\sum_{n=0}^{\infty}\exp\left[-\mathrm{i}\left(n+\frac{1}{2}\right)\omega t\right]H_n\left(\sqrt{\frac{m\omega}{\hbar}}q_1\right)H_n\left(\sqrt{\frac{m\omega}{\hbar}}q_2\right)\\ = \sqrt{\frac{m\omega}{2\pi\mathrm{i}\hbar\sin\omega t}}\exp\left[\frac{\mathrm{i}m\omega}{2\hbar\sin\omega t}\left((q_1^2+q_2^2)\cos\omega t - 2q_1q_2\right)\right],\end{aligned} \tag{1.125}$$

where one uses

$$\frac{1}{\sqrt{1-s^2}}\exp\left[\frac{2xys - s^2(x^2+y^2)}{1-s^2}\right] = \sum_{n=0}^{\infty}\frac{s^n}{2^n n!}H_n(x)H_n(y). \tag{1.126}$$

Fourier transformation on both sides of (1.125) and applying (1.119) leads then to

$$\sum_{n=0}^{\infty}2(-1)^n L_n\left(\frac{4H}{\hbar\omega}\right)\exp\left(-\frac{2H}{\hbar\omega}\right)\exp\left[-\mathrm{i}\left(n+\frac{1}{2}\right)\omega t\right] = \frac{1}{\cos\omega t/2}\exp\left[\frac{2H}{\mathrm{i}\hbar\omega}\tan\frac{\omega t}{2}\right]. \tag{1.127}$$

The left hand side is just the Fourier-Dirichlet expansion with the Wigner functions found in (1.118) and the right hand side is the expression (1.120) for the star exponential.

1.5 Systems in Higher Dimensions and Angular Momentum

In order to show how the generalization to higher dimensional systems works one first considers the two dimensional harmonic oscillator [10]. The Hamiltonian is given by $H = \frac{p_1^2}{2m} + \frac{m\omega^2}{2}q_1^2 + \frac{p_2^2}{2m} + \frac{m\omega^2}{2}q_2^2 = \omega(a_1\bar{a}_1 + a_2\bar{a}_2) = H_1 + H_2$ with $a_n = \sqrt{\frac{m\omega}{2}}\left(q_n + \frac{\mathrm{i}}{m\omega}p_n\right)$ and $\bar{a}_n = \sqrt{\frac{m\omega}{2}}\left(q_n - \frac{\mathrm{i}}{m\omega}p_n\right)$. The Moyal product generalizes in d dimensions to

$$*_M = \exp\left[\frac{\mathrm{i}\hbar}{2}\sum_{n=1}^{d}\left(\overleftarrow{\partial}_{q_n}\overrightarrow{\partial}_{p_n} - \overleftarrow{\partial}_{p_n}\overrightarrow{\partial}_{q_n}\right)\right] = \exp\left[\frac{\hbar}{2}\sum_{n=1}^{d}\left(\overleftarrow{\partial}_{a_n}\overrightarrow{\partial}_{\bar{a}_n} - \overleftarrow{\partial}_{\bar{a}_n}\overrightarrow{\partial}_{a_n}\right)\right]. \tag{1.128}$$

The Wigner functions of the two dimensional system are the product of the corresponding one dimensional systems:

$$H *_M \pi_{n_1}^{(M)}(a_1,\bar{a}_1)\pi_{n_2}^{(M)}(a_2,\bar{a}_2) = \hbar\omega(n+1)\pi_{n_1}^{(M)}(a_1,\bar{a}_1)\pi_{n_2}^{(M)}(a_2,\bar{a}_2), \tag{1.129}$$

with $n = n_1 + n_2$, which shows the $(n+1)$-fold degeneracy of the system. The two one dimensional Wigner functions can be combined into a two dimensional Wigner function corresponding to the energy level n:

$$\pi_n^{(M)}(a_1, \bar{a}_1, a_2, \bar{a}_2) = \sum_{n_1+n_2=n} \pi_{n_1}^{(M)}(a_1, \bar{a}_1)\pi_{n_2}^{(M)}(a_2, \bar{a}_2) = 4(-1)^n e^{-2H/\hbar\omega} L_n^1\left(\frac{4H}{\hbar\omega}\right). \tag{1.130}$$

The star exponential is given by

$$\begin{aligned}\mathrm{Exp}_M(Ht) &= \sum_{n=0}^{\infty} \pi_n^{(M)}(a_1, \bar{a}_1, a_2, \bar{a}_2) e^{-\mathrm{i}(n+1)\omega t} \\ &= \left(\cos\frac{\omega t}{2}\right)^{-2} \exp\left[\left(\frac{2H}{\mathrm{i}\hbar\omega}\right)\tan\frac{\omega t}{2}\right] = \mathrm{Exp}_M(H_1 t)\mathrm{Exp}_M(H_2 t).\end{aligned} \tag{1.131}$$

This shows that the generalization to higher dimensional systems is just as in conventional quantum mechanics straightforward.

In the two dimensional system described above there exists also an angular momentum. The angular momentum can most easily be described if one defines creation and annihilation operators for positively and negatively rotating quanta:

$$a_+ = \sqrt{\frac{1}{2}}(a_1 - \mathrm{i}a_2)\ ,\ \bar{a}_+ = \sqrt{\frac{1}{2}}(\bar{a}_1 + \mathrm{i}\bar{a}_2) \qquad \text{and} \qquad a_- = \sqrt{\frac{1}{2}}(a_1 + \mathrm{i}a_2)\ ,\ \bar{a}_- = \sqrt{\frac{1}{2}}(\bar{a}_1 - \mathrm{i}\bar{a}_2). \tag{1.132}$$

In these coordinates the Moyal product can be written as:

$$*_M = \exp\left[\frac{\hbar}{2}\sum_{n=1}^{2}\left(\overleftarrow{\partial}_{a_n}\overrightarrow{\partial}_{\bar{a}_n} - \overleftarrow{\partial}_{\bar{a}_n}\overrightarrow{\partial}_{a_n}\right)\right] = \exp\left[\frac{\hbar}{2}\left(\overleftarrow{\partial}_{a_+}\overrightarrow{\partial}_{\bar{a}_+} - \overleftarrow{\partial}_{\bar{a}_+}\overrightarrow{\partial}_{a_+} + \overleftarrow{\partial}_{a_-}\overrightarrow{\partial}_{\bar{a}_-} - \overleftarrow{\partial}_{\bar{a}_-}\overrightarrow{\partial}_{a_-}\right)\right]. \tag{1.133}$$

The Hamiltonian turns into $H = \omega a_+\bar{a}_+ + \omega a_-\bar{a}_- = \omega(N_+ + N_- + \hbar) = H_+ + H_-$ with $N_{+(-)} = \bar{a}_{+(-)} *_M a_{+(-)}$ and $H_{+(-)} = \omega\left(N_{+(-)} + \frac{\hbar}{2}\right)$ and for the angular momentum one obtains:

$$J_3 = q_1 p_2 - p_1 q_2 = \mathrm{i}(a_1\bar{a}_2 - a_2\bar{a}_1) = \bar{a}_+ a_+ - \bar{a}_- a_- = N_+ - N_-. \tag{1.134}$$

The Wigner functions (1.130) should now be turned into a form where they are also $*$-eigenfunctions of J_3. The decomposition of the Hamiltonian is analogous to the decomposition in conventional holomorphic coordinates $\bar{a}_n$ and a_n, moreover the Moyal product has the same structure in both types of coordinates as can be seen in (1.133). So the calculations can be done analogously by substituting the indices (1,2) by (+,–). The one dimensional Wigner functions are then $\pi_{n_{+(-)}}^{(M)}(a_{+(-)}, \bar{a}_{+(-)}) = 2(-1)^{n_{+(-)}} e^{-2H_{+(-)}/\hbar\omega} L_{n_{+(-)}}\left(\frac{4H_{+(-)}}{\hbar\omega}\right)$ and the energy levels are $E_n = \hbar\omega(n+1) = \hbar\omega(n_+ + n_- + 1)$. The two dimensional Wigner functions are

$$\pi_n^{(M)}(a_+, \bar{a}_+, a_-, \bar{a}_-) = \sum_{n_+ + n_- = n} 4(-1)^n e^{-2H/\hbar\omega} L_{n_+}\left(\frac{4H_+}{\hbar\omega}\right) L_{n_-}\left(\frac{4H_-}{\hbar\omega}\right). \tag{1.135}$$

Each term $\pi_{n_+,n_-}^{(M)}$ of (1.135) is hereby a $*$-eigenfunction of J_3: $J_3 *_M \pi_{n_+,n_-}^{(M)} = \hbar(n_+ - n_-)\pi_{n_+,n_-}^{(M)}$. Instead of parametrizing with n_+ and n_- it is also possible to parametrize with $n = n_+ + n_-$ and $m = n_+ - n_-$ so that

$$\pi_{n,m}^{(M)}(a_+, \bar{a}_+, a_-, \bar{a}_-) = 4(-1)^n e^{-2H/\hbar\omega} L_{(n+m)/2}\left(\frac{4H_+}{\hbar\omega}\right) L_{(n-m)/2}\left(\frac{4H_-}{\hbar\omega}\right) \tag{1.136}$$

and $J_3 *_M \pi_{n,m}^{(M)} = m\pi_{n,m}^{(M)}$ with $m = n, n-2, n-4, \ldots, -n$.

The two dimensional harmonic oscillator can now be used as the basis for discussing angular momentum in the star product formalism. One first notices that the angular momentum functions $J_i = \varepsilon_i^{jk} q_j p_k$ form with the Moyal product an $\mathfrak{su}(2)$ algebra

$$[J_i, J_j]_{*_M} = \mathrm{i}\hbar\varepsilon_{ijk} J_k. \tag{1.137}$$

So they can be used to generate rotations. This is done in the star product formalism with the star exponential:

$$\mathrm{Exp}_M(J_i\varphi) *_M \vec{q} *_M \mathrm{Exp}_M(-J_i\varphi) = R(\varphi)\vec{q}, \tag{1.138}$$

where $R(\varphi)$ is the rotation matrix. In order to calculate the star exponential one has to represent J_i with holomorphic coordinates as in (1.134), so that the star exponential corresponds to the one of the harmonic oscillator.

The next question is to find the Wigner functions and the eigenvalues j and m for $\vec{J}^{2*_M}$ and J_3. From the operator formalism it is known that j can have the values $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and m can have the values $m = -j, -j+1, \ldots, j-1, j$. The half integer steps for j cannot be described with the purely bosonic two dimensional harmonic oscillator, because the main quantum number is the energy expressed by the total number of quanta in the system, which is an integer $n = n_+ + n_- = 0, 1, 2, \ldots$; correspondingly the m vary in steps of two. So in order to describe angular momentum one has to introduce a factor $\frac{1}{2}$, i.e.

$$j = \frac{1}{2}(n_+ + n_-) \qquad \text{and} \qquad m = \frac{1}{2}(n_+ - n_-). \tag{1.139}$$

Using these definitions one just obtains the Schwinger representation of angular momentum [99]. With the ladder functions $J_+ = \bar{a}_+ a_-$ and $J_- = \bar{a}_- a_+$ the angular momentum functions are then instead of (1.134) defined by:

$$J_1 = \frac{1}{2}(J_+ + J_-) = \frac{1}{2}(\bar{a}_+ a_- + \bar{a}_- a_+), \qquad J_2 = \frac{1}{2\mathrm{i}}(J_+ - J_-) = \frac{1}{2}(\mathrm{i}\bar{a}_- a_+ - \mathrm{i}\bar{a}_+ a_-)$$

$$\text{and} \qquad J_3 = \frac{1}{2}(N_+ - N_-) = \frac{1}{2}(\bar{a}_+ a_+ - \bar{a}_- a_-). \tag{1.140}$$

For the square of the angular momentum one obtains:

$$\vec{J}^{2*_M} = J_1^{2*_M} + J_2^{2*_M} + J_3^{2*_M} = \frac{1}{2}(N_+ + N_-) *_M \left(\frac{1}{2}(N_+ + N_-) + \hbar\right), \tag{1.141}$$

so that with the Wigner functions $\pi^{(M)}_{n_+,n_-} = \pi^{(M)}_{n_+}\pi^{(M)}_{n_-}$ of the two dimensional harmonic oscillator follows

$$\vec{J}^{2*_M} *_M \pi^{(M)}_{n_+,n_-} = \hbar^2 j(j+1)\pi^{(M)}_{n_+,n_-} \qquad \text{and} \qquad J_3 *_M \pi^{(M)}_{n_+,n_-} = \hbar m \pi^{(M)}_{n_+,n_-}. \tag{1.142}$$

The factor $\frac{1}{2}$ that was introduced in (1.139) stems from the decomposition of the angular momentum into spins that is achieved by the Schwinger representation. How this can be described in the star product formalism will be discussed in the next chapter.

Chapter 2

Fermionic Star Products

After having established the star product formalism for quantum mechanics the next task is to incorporate spin in the formalism. This was done first using the bosonic Moyal product in [109]. But as shown in [11] and [18] spin can be described most elegantly in the context of grassmannian mechanics. This approach will be used here to obtain by deformation quantization of grassmannian mechanics a fermionic star product and a description of spin. The fermionic star products can then in combination with the bosonic ones be used to describe supersymmetric quantum mechanics, spin and Dirac theory.

2.1 Grassmannian Mechanics

Grassmannian mechanics differs from classical mechanics in a fundamental way, because space and velocity are described by Grassmann variables η and $\dot{\eta}$, so that a kinetic term $\frac{1}{2}\dot{\eta}^2$ would be zero. A nontrivial ansatz for a free Lagrangian is $L = \frac{\mathrm{i}}{2}\eta\dot{\eta}$, where the product of two Grassmann variables makes L a bosonic function and the additional i assures that L is real. The equation of motion following from this Lagrangian is $\dot{\eta} = 0$, which means that the dynamical variable η itself is a conserved quantity.

Analogously a quadratic potential term would be zero, so that there is no one dimensional oscillator in grassmannian mechanics. But it is possible to construct a two dimensional oscillator where the two Grassmann variables η^1 and η^2 can be combined to complex Grassmann variables

$$\eta = \frac{1}{\sqrt{2}}\left(\eta^1 + \mathrm{i}\eta^2\right) \qquad \text{and} \qquad \bar{\eta} = \frac{1}{\sqrt{2}}\left(\eta^1 - \mathrm{i}\eta^2\right). \tag{2.1}$$

The Lagrangian for the (twodimensional) grassmannian oscillator is given by

$$L = \mathrm{i}\bar{\eta}\dot{\eta} + \omega\bar{\eta}\eta. \tag{2.2}$$

The Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial^L L}{\partial\dot{\eta}} - \frac{\partial^L L}{\partial\eta} = 0 \qquad \text{and} \qquad \frac{d}{dt}\frac{\partial^L L}{\partial\dot{\bar{\eta}}} - \frac{\partial^L L}{\partial\bar{\eta}} = 0 \tag{2.3}$$

lead then to equations of motions $\dot{\bar{\eta}} = -\mathrm{i}\omega\bar{\eta}$ and $\dot{\eta} = \mathrm{i}\omega\eta$ or in real coordinates $\dot{\eta}^1 = -\omega\eta^2$ and $\dot{\eta}^2 = \omega\eta^1$, which can be combined to $\ddot{\eta}^\alpha = -\omega^2\eta^\alpha$ for $\alpha = 1, 2$. Using the equations of motion the Lagrangian (2.2) can be written in real coordinates as

$$L = \frac{\mathrm{i}}{2}\left(\eta^1\dot{\eta}^1 + \eta^2\dot{\eta}^2\right) + \mathrm{i}\omega\eta^1\eta^2. \tag{2.4}$$

With the complex canonical momentum $\rho = \frac{\partial^L L}{\partial \dot{\eta}} = -\mathrm{i}\bar{\eta}$ the Hamiltonian is

$$H = \dot{\eta}\rho - L = -\omega\bar{\eta}\eta = \mathrm{i}\omega\eta\rho, \tag{2.5}$$

which can be interpreted as a rotation in Grassmann space. In real canonical coordinates η^α and $\rho_\alpha = -\frac{\mathrm{i}}{2}\delta_{\alpha\beta}\eta^\beta$ one gets:

$$H = -\mathrm{i}\omega\eta^1\eta^2. \tag{2.6}$$

The Hamilton equations can be calculated by variation of the action $S = \int dt\,(\dot{\eta}^\alpha\rho_\alpha - H)$ which gives

$$\dot{\eta}^\alpha = -\frac{\partial^L H}{\partial \rho_\alpha} \qquad \text{and} \qquad \dot{\rho}_\alpha = -\frac{\partial^L H}{\partial \eta^\alpha}. \tag{2.7}$$

One sees here that the structure of minus sign differs from the one in classical mechanics. This leads then also to a different sign structure in the grassmannian Poisson bracket, because with the Hamilton equations (2.7) it is possible to write the time derivative of a function $F(\eta, \rho, t)$ as

$$\frac{dF}{dt} = \dot{\eta}^\alpha \frac{\partial^L F}{\partial \eta^\alpha} + \dot{\rho}_\alpha \frac{\partial^L F}{\partial \rho_\alpha} + \frac{\partial F}{\partial t} = (-1)^{\epsilon(F)} \left(\frac{\partial^L F}{\partial \eta^\alpha}\frac{\partial^L H}{\partial \rho_\alpha} + \frac{\partial^L F}{\partial \rho_\alpha}\frac{\partial^L H}{\partial \eta^\alpha} \right) + \frac{\partial F}{\partial t}, \tag{2.8}$$

where $\epsilon(F)$ is the Grassmann grade of F. The grassmannian or fermionic Poisson bracket can then be defined as

$$\{F, G\}_{FPB} = (-1)^{\epsilon(F)} \left(\frac{\partial^L F}{\partial \eta^\alpha}\frac{\partial^L G}{\partial \rho_\alpha} + \frac{\partial^L F}{\partial \rho_\alpha}\frac{\partial^L G}{\partial \eta^\alpha} \right). \tag{2.9}$$

Under the assumption that the canonical coordinates are independent (which is for the grassmannian oscillator not the case) the fundamental fermionic Poisson brackets are

$$\{\eta^\alpha, \eta^\beta\}_{FPB} = 0, \qquad \{\rho_\alpha, \rho_\beta\}_{FPB} = 0 \qquad \text{and} \qquad \{\eta^\alpha, \rho_\beta\}_{FPB} = -\delta^\alpha_\beta. \tag{2.10}$$

The fermionic Poisson bracket (2.9) can be combined with the classical Poisson bracket to a generalized or super Poisson bracket. The derivation is analogous to (2.8), but now one has to calculate a time derivative of a function $F(q, p, \eta, \rho, t)$ that depends on bosonic and fermionic coordinates. This gives then the following expression for the generalized Poisson bracket:

$$\{F, G\}_{GPB} = \left(\frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} \right) + (-1)^{\epsilon(F)} \left(\frac{\partial^L F}{\partial \eta^\alpha}\frac{\partial^L G}{\partial \rho_\alpha} + \frac{\partial^L F}{\partial \rho_\alpha}\frac{\partial^L G}{\partial \eta^\alpha} \right). \tag{2.11}$$

The symmetry of the generalized Poisson bracket (2.11) depends on the Grassmann grade of F and G and is given by: $\{F, G\}_{GPB} = -(-1)^{\epsilon(F)\epsilon(G)}\{G, F\}_{GPB}$. Furthermore one has a generalized Leibniz rule

$$\{F, GH\}_{GPB} = \{F, G\}_{GPB}H + (-1)^{\epsilon(F)\epsilon(G)}G\{F, H\}_{GPB} \tag{2.12}$$

and a generalized Jacobi identity

$$\begin{aligned} \{\{F, G\}_{GPB}, H\}_{GPB} + (-1)^{\epsilon(F)(\epsilon(G)+\epsilon(H))}\{\{G, H\}_{GPB}, F\}_{GPB} \\ + (-1)^{\epsilon(H)(\epsilon(F)+\epsilon(G))}\{\{H, F\}_{GPB}, G\}_{GPB} = 0. \end{aligned} \tag{2.13}$$

In grassmannian mechanics even the simplest systems like the free particle or the oscillator are systems with constraints. The two constraints for the oscillator are

$$\chi_\alpha = \rho_\alpha + \frac{\mathrm{i}}{2}\delta_{\alpha\beta}\eta^\beta = 0 \tag{2.14}$$

with $\alpha = 1, 2$. Calculating the fermionic Poisson brackets for the constraints gives:

$$\{\chi_1, \chi_1\}_{FPB} = \{\chi_2, \chi_2\}_{FPB} = -\mathrm{i} \qquad \text{and} \qquad \{\chi_1, \chi_2\}_{FPB} = \{\chi_2, \chi_1\}_{FPB} = 0. \tag{2.15}$$

Since not all of the Poisson brackets are zero the χ_α are second class constraints. Summarizing the brackets (2.15) in a matrix $C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}_{FPB}$ one gets

$$C_{\alpha\beta} = -\mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad C^{\alpha\beta} = \mathrm{i}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.16}$$

where $C^{\alpha\beta}$ is the inverse of $C_{\alpha\beta}$. With this matrix one defines the fermionic Dirac bracket as

$$\{F, G\}_{FDB} = \{F, G\}_{FPB} - \{F, \chi_\alpha\}_{FPB}\, C^{\alpha\beta}\, \{\chi_\beta, G\}_{FPB}. \tag{2.17}$$

By substituting all Poisson brackets by Dirac brackets one takes into account the constraints and achieves a description that is equivalent to mechanics in a reduced phase space. The Dirac brackets of the constraints are now all zero and the Dirac brackets of the canonical coordinates are

$$\{\eta^\alpha, \eta^\beta\}_{FDB} = -\mathrm{i}\delta^{\alpha\beta}, \qquad \{\rho_\alpha, \rho_\beta\}_{FDB} = -\frac{\mathrm{i}}{4}\delta_{\alpha\beta} \qquad \text{and} \qquad \{\eta^\alpha, \rho_\beta\}_{FDB} = -\frac{1}{2}\delta^\alpha_\beta. \tag{2.18}$$

Just as the bosonic and fermionic Poisson bracket can be generalized to a super Poisson bracket (2.11) one can also combine the bosonic and the fermionic Dirac bracket to a generalized Dirac bracket.

2.2 The Grassmannian Oscillator in the Star Product Formalism

Because of the two constraints (2.14) in the case of the grassmannian oscillator the basis for constructing a star product is not the Poisson bracket but has to be the Dirac bracket (2.17). This bracket can also be written as:

$$\begin{aligned} \{F, G\}_{FDB} = F(-1)^{\epsilon(F)} \Bigg[\Bigg(& \frac{\overleftarrow{\partial}^L}{\partial \eta^\alpha} \frac{\overrightarrow{\partial}^L}{\partial \rho_\alpha} + \frac{\overleftarrow{\partial}^L}{\partial \rho_\alpha} \frac{\overrightarrow{\partial}^L}{\partial \eta^\alpha} \Bigg) \\ & + \left(\frac{\overleftarrow{\partial}^L}{\partial \eta^\alpha} \frac{\partial^L \chi_\gamma}{\partial \rho_\alpha} + \frac{\overleftarrow{\partial}^L}{\partial \rho_\alpha} \frac{\partial^L \chi_\gamma}{\partial \eta^\alpha} \right) i\delta^{\gamma\delta} \left(\frac{\partial^L \chi_\delta}{\partial \eta^\alpha} \frac{\overrightarrow{\partial}^L}{\partial \rho_\alpha} \frac{\partial^L \chi_\delta}{\partial \rho_\alpha} \frac{\overrightarrow{\partial}^L}{\partial \eta^\alpha} \right) \Bigg] G. \end{aligned} \tag{2.19}$$

Working this out and using the relation between the left and right derivatives of functions of Grassmann variables

$$\frac{\partial^R F}{\partial \theta^\alpha} = -(-1)^{\epsilon(F)} \frac{\partial^L F}{\partial \theta^\alpha}, \tag{2.20}$$

one obtains

$$\{F, G\}_{FDB} = F \left(\frac{1}{2} \frac{\overleftarrow{\partial}}{\partial \eta^\alpha} \frac{\overrightarrow{\partial}}{\partial \rho_\alpha} + \frac{1}{2} \frac{\overleftarrow{\partial}}{\partial \rho_\alpha} \frac{\overrightarrow{\partial}}{\partial \eta^\alpha} + \mathrm{i} \frac{\overleftarrow{\partial}}{\partial \eta^\alpha} \frac{\overrightarrow{\partial}}{\partial \eta^\alpha} - \frac{\mathrm{i}}{4} \frac{\overleftarrow{\partial}}{\partial \rho_\alpha} \frac{\overrightarrow{\partial}}{\partial \rho_\alpha} \right) G. \tag{2.21}$$

Note that the notion of right and left derivatives is now included in the vector notation. Also the sum over all α is understood. From now on the Dirac brackets are used instead of the Poisson brackets, and the constraints are implemented as strong equations, according to Dirac's method [29]. The only independent variables are then the η^α, so that in (2.21) only the third term remains.

Then one can construct a fermionic Moyal product from (2.21) which gives

$$F *_M G = F \exp\left(\frac{\hbar}{2} \frac{\overleftarrow{\partial}}{\partial \eta^\alpha} \frac{\overrightarrow{\partial}}{\partial \eta^\alpha} \right) G. \tag{2.22}$$

This shows that the deformation quantization of grassmannian mechanics leads to a star product that was postulated in [113]. For the star anticommutators one obtains

$$\{\eta^\alpha, \eta^\beta\}_{*_M} = \eta^\alpha *_M \eta^\beta + \eta^\beta *_M \eta^\alpha = \hbar\delta^{\alpha\beta}. \tag{2.23}$$

This means that the fermionic star product leads to a cliffordization of the Grassmann algebra of the fermionic coordinates.

With the fermionic star product (2.22) one can now calculate the star exponential of the Hamilton function $H = -\mathrm{i}\omega\eta^1\eta^2$ of the fermionic oscillator. Using the fact that $(-\mathrm{i}\omega\eta^1\eta^2) *_M (-\mathrm{i}\omega\eta^1\eta^2) = \frac{\hbar^2\omega^2}{4}$ one gets

$$\begin{aligned}
\mathrm{Exp}_M(Ht) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{-\mathrm{i}t}{\hbar}\right)^n H^{n*_M} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{\omega t}{2\mathrm{i}}\right)^{2n} + (-\mathrm{i}\omega\eta^1\eta^2)\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{t}{\mathrm{i}\hbar}\right)^{2n+1}\left(\frac{\hbar^2\omega^2}{4}\right)^n \\
&= \cos\left(\frac{\omega t}{2}\right) - \frac{2}{\hbar}\eta^1\eta^2\sin\left(\frac{\omega t}{2}\right) \qquad (2.24)\\
&= \pi^{(M)}_{1/2} e^{-\mathrm{i}\frac{\omega t}{2}} + \pi^{(M)}_{-1/2} e^{\mathrm{i}\frac{\omega t}{2}}, \qquad (2.25)
\end{aligned}$$

with the projectors

$$\pi^{(M)}_{1/2} = \frac{1}{2} - \frac{\mathrm{i}}{\hbar}\eta^1\eta^2 \qquad \text{and} \qquad \pi^{(M)}_{-1/2} = \frac{1}{2} + \frac{\mathrm{i}}{\hbar}\eta^1\eta^2. \tag{2.26}$$

Using expression (2.24) it is easy to see that the star exponential fulfills the defining differential equation

$$\mathrm{i}\hbar\frac{d}{dt}\mathrm{Exp}_M(Ht) = -\mathrm{i}\hbar\frac{\omega}{2}\sin\left(\frac{\omega t}{2}\right) - \mathrm{i}\omega\eta^1\eta^2\cos\left(\frac{\omega t}{2}\right) = H *_M \mathrm{Exp}_M(Ht). \tag{2.27}$$

The projectors or fermionic Wigner functions $\pi^{(M)}_{\pm 1/2}$ are idempotent and complete, i.e. $\pi^{(M)}_\alpha *_M \pi^{(M)}_\beta = \delta_{\alpha\beta}\pi^{(M)}_\alpha$ and $\pi^{(M)}_{1/2} + \pi^{(M)}_{-1/2} = 1$, so that the time development of the projectors can be calculated easily with (2.25) as

$$\mathrm{Exp}_M(-Ht) *_M \pi^{(M)}_{\pm 1/2} *_M \mathrm{Exp}_M(Ht) = \pi^{(M)}_{\pm 1/2} e^{\pm \mathrm{i}\omega t}. \tag{2.28}$$

The $*$-eigenvalue equations are

$$H *_M \pi^{(M)}_{1/2} = \frac{\hbar\omega}{2}\,\pi^{(M)}_{1/2}, \qquad H *_M \pi^{(M)}_{-1/2} = -\frac{\hbar\omega}{2}\,\pi^{(M)}_{-1/2}, \tag{2.29}$$

so the energy eigenvalues are

$$E_{\pm\frac{1}{2}} = \pm\frac{\hbar\omega}{2}\ , \tag{2.30}$$

and the spectral decomposition of H is

$$H = \sum_{n=\pm\frac{1}{2}} E_n \pi^{(M)}_n = -\mathrm{i}\omega\eta^1\eta^2. \tag{2.31}$$

Just as in the bosonic case it is also here possible to formulate the whole problem in holomorphic coordinates

$$f = \frac{1}{\sqrt{2}}\left(\eta^2 + \mathrm{i}\eta^1\right) \qquad \text{and} \qquad \bar{f} = \frac{1}{\sqrt{2}}\left(\eta^2 - \mathrm{i}\eta^1\right). \tag{2.32}$$

The fermionic Moyal star product (2.22) in these variables is

$$F *_M G = F \; e^{\frac{\hbar}{2}\left(\overleftarrow{\partial}_f \overrightarrow{\partial}_{\bar{f}} + \overleftarrow{\partial}_{\bar{f}} \overrightarrow{\partial}_f\right)} \; G, \tag{2.33}$$

which delines a fermionic star product that was suggested for example in [13]. With this star product one finds

$$f *_M \bar{f} = f\bar{f} + \frac{\hbar}{2}, \qquad \bar{f} *_M f = \bar{f}f + \frac{\hbar}{2}, \tag{2.34}$$

and

$$\{\bar{f}, \bar{f}\}_{*_M} = \{f, f\}_{*_M} = 0, \qquad \{\bar{f}, f\}_{*_M} = \hbar. \tag{2.35}$$

The Hamilton function in holomorphic coordinates is

$$H = \omega \bar{f} f, \tag{2.36}$$

and the time-evolution function is

$$\mathrm{Exp}_M(Ht) = \cos\left(\frac{\omega t}{2}\right) - \frac{2\mathrm{i}}{\hbar}\bar{f}f \sin\left(\frac{\omega t}{2}\right) = \pi^{(M)}_{-1/2} e^{\mathrm{i}\frac{\omega t}{2}} + \pi^{(M)}_{1/2} e^{-\mathrm{i}\frac{\omega t}{2}}, \tag{2.37}$$

with

$$\pi^{(M)}_{-1/2} = \frac{1}{2} - \frac{1}{\hbar}\bar{f}f \qquad \text{and} \qquad \pi^{(M)}_{1/2} = \frac{1}{2} + \frac{1}{\hbar}\bar{f}f. \tag{2.38}$$

It is obvious that these projectors satisfy the required orthonormality and completeness conditions, as well as the corresponding $*$-eigenvalue equations for the energy levels $E_{\pm 1/2} = \pm\frac{\hbar\omega}{2}$, so that the time development of (2.38) is also given by (2.28), while the time development of the holomorphic coordinates is given by:

$$\mathrm{Exp}_M(-Ht) *_M f *_M \mathrm{Exp}_M(Ht) = f e^{-\mathrm{i}\omega t} \qquad \text{and} \qquad \mathrm{Exp}_M(-Ht) *_M \bar{f} *_M \mathrm{Exp}_M(Ht) = \bar{f} e^{\mathrm{i}\omega t}. \tag{2.39}$$

Furthermore one can show that f and $\bar{f}$ act as annihilation and creation functions:

$$f *_M \pi^{(M)}_{-1/2} = \bar{f} *_M \pi^{(M)}_{1/2} = 0, \tag{2.40}$$

and

$$\bar{f} *_M \pi^{(M)}_{-1/2} *_M f = \hbar \pi^{(M)}_{1/2}, \qquad f *_M \pi^{(M)}_{1/2} *_M \bar{f} = \hbar \pi^{(M)}_{-1/2}. \tag{2.41}$$

The expression $\tau = \frac{2}{\hbar}\bar{f}f$ is an involution, i.e. $\tau *_M \tau = 1$. It thus has the two eigenvalues ± 1, and the projectors onto the even and odd eigenspaces are $\pi^{(M)}_{\pm} = \frac{1}{2}(1 \pm \tau)$ in agreement with (2.38). In the conventional operator approach to supersymmetric quantum mechanics the above quantities are represented as 2×2 matrices, and the star product corresponds to ordinary matrix multiplication. The matrices one uses are

$$\hat{f} = \sqrt{\hbar}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \hat{f}^\dagger = \sqrt{\hbar}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\tau} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.42}$$

The matrix representation $\hat{f}$ and $\hat{f}^\dagger$ for f and $\bar{f}$ reproduce the anticommutator relations (2.35) if one replaces the star product by the matrix multiplication. Using (2.38) and the $\hat{\tau}$-matrix, the matrix representation of the projectors becomes:

$$\hat{\pi}^{(M)}_{-1/2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{\pi}^{(M)}_{1/2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.43}$$

Relations such as (2.41) are then simple exercises in matrix multiplication. One sees here also that the projectors are normalized according to

$$\mathrm{Tr}\left(\hat{\pi}^{(M)}_{\pm 1/2}\right) = 1. \tag{2.44}$$

As in the bosonic case it is also possible to construct a normal star product for fermionic functions:

$$F *_N G = F e^{\hbar \overleftarrow{\partial}_f \overrightarrow{\partial}_{\bar{f}}} G. \tag{2.45}$$

We then have

$$f *_N \bar{f} = f\bar{f} + \hbar, \qquad \bar{f} *_N f = f\bar{f}, \tag{2.46}$$

and

$$\{\bar{f}, \bar{f}\}_{*_N} = \{f, f\}_{*_N} = 0, \qquad \{\bar{f}, f\}_{*_N} = \hbar, \tag{2.47}$$

which is consistent with the previous results for the Moyal product, as it must be. One easily calculates the time-evolution function

$$\mathrm{Exp}_N(Ht) = \pi_0^{(N)} + \pi_1^{(N)} e^{-\mathrm{i}\omega t}, \tag{2.48}$$

with

$$\pi_0^{(N)} = 1 - \frac{1}{\hbar}\bar{f}f, \qquad \pi_1^{(N)} = \frac{1}{\hbar}\bar{f}f. \tag{2.49}$$

These projectors satisfy the required properties, including the $*$-eigenvalue equations:

$$H *_N \pi_0^{(N)} = 0, \qquad H *_N \pi_1^{(N)} = \hbar\omega\, \pi_1^{(N)}, \tag{2.50}$$

so that the energy levels are 0 and $\hbar\omega$. As in the bosonic case there is a shift of $\frac{1}{2}\hbar\omega$ in the oscillator energy levels between the Moyal and the normal product, but now the spectrum is shifted upwards whereas the bosonic spectrum was shifted downwards.

Just as in the Moyal case $\bar{f}$ and f act as creation and annihilation functions:

$$f *_N \pi_0^{(N)} = \bar{f} *_N \pi_1^{(N)} = 0, \tag{2.51}$$

and

$$\bar{f} *_N \pi_0^{(N)} *_N f = \hbar\pi_1^{(N)}, \qquad f *_N \pi_1^{(N)} *_N \bar{f} = \hbar\pi_0^{(N)}. \tag{2.52}$$

2.3 The Supersymmetric Star Product

It is now straightforward to combine the bosonic and the fermionic star product into a supersymmetric star product. The supersymmetric star product of the Moyal type is

$$F *_{SU} G = F \exp\left[\frac{\hbar}{2}\left(\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a + \overleftarrow{\partial}_f \overrightarrow{\partial}_{\bar{f}} + \overleftarrow{\partial}_{\bar{f}} \overrightarrow{\partial}_f\right)\right] G. \tag{2.53}$$

The supersymmetric star product factorizes in a bosonic and a fermionic Moyal product, so that the supersymmetric star product of two purely bosonic functions reduces to the bosonic Moyal product of these functions and analogously for purely fermionic functions. The Hamilton function of the supersymmetric oscillator is just the sum of the Hamilton functions for the fermionic and the bosonic oscillators. So it can be written (with the supersymmetric star product) as

$$H_{SU} = \omega\left(\bar{f} *_{SU} f + \bar{a} *_{SU} a\right) = \omega\left(\bar{f}f + \bar{a}a\right) = H_F + H_B. \tag{2.54}$$

Another possibility to write H_{SU} that clarifies the relation of the bosonic and fermionic sectors and allows an easy generalization to nonlinear supersymmetry is

$$H_{SU} = \omega\left[(\bar{a} *_M a)\pi_{-1/2}^{(M)} + (a *_M \bar{a})\pi_{1/2}^{(M)}\right] = \omega\left[(\bar{a} *_M a)\pi_{-1/2}^{(M)} + (\bar{a} *_M a + \hbar)\pi_{1/2}^{(M)}\right]. \tag{2.55}$$

So one sees that there are two bosonic oscillators shifted by $\hbar\omega$ and that they live in sectors separated by the projectors $\pi^{(M)}_{\pm 1/2}$.

One can then define functions that relate the energy levels of these two sectors:

$$Q_+ = \sqrt{\frac{1}{\hbar}}\,(a *_{SU} \bar{f}) = \sqrt{\frac{1}{\hbar}}\,a\bar{f} \qquad \text{and} \qquad Q_- = \sqrt{\frac{1}{\hbar}}\,(\bar{a} *_{SU} f) = \sqrt{\frac{1}{\hbar}}\,\bar{a}f. \tag{2.56}$$

These functions are nilpotent,

$$Q_\pm *_{SU} Q_\pm = Q_\pm^2 = 0, \tag{2.57}$$

and the Hamiltonian of the supersymmetric oscillator may be written as

$$H_{SU} = \omega\{Q_+, Q_-\}_{*_{SU}}. \tag{2.58}$$

With (2.57) one sees immediately that H_{SU} is supersymmetric:

$$[Q_+, H_{SU}]_{*_{SU}} = [Q_-, H_{SU}]_{*_{SU}} = 0. \tag{2.59}$$

One may also use the hermitian functions

$$Q_1 = Q_+ + Q_- \qquad \text{and} \qquad Q_2 = -i(Q_+ - Q_-), \tag{2.60}$$

so that the supersymmetric Hamiltonian becomes

$$H_{SU} = \omega Q_1^{2*_{SU}} = \omega Q_2^{2*_{SU}}. \tag{2.61}$$

Since the supersymmetric star product factorizes into a bosonic and a fermionic part one can also choose a factor ansatz for the star exponential of the supersymmetric oscillator. The product of (1.120) and (2.25) leads to:

$$\begin{aligned}
\mathrm{Exp}_{SU}(H_{SU}t) &= \left[\cos\left(\frac{\omega t}{2}\right) - \frac{2\mathrm{i}}{\hbar}\bar{f}f\sin\left(\frac{\omega t}{2}\right)\right]\frac{1}{\cos\left(\frac{\omega t}{2}\right)}\exp\left[\left(\frac{2H_B}{\mathrm{i}\hbar\omega}\right)\tan\left(\frac{\omega t}{2}\right)\right] \\
&= \left[1 + \left(\frac{2H_F}{\mathrm{i}\hbar\omega}\right)\tan\left(\frac{\omega t}{2}\right)\right]\exp\left[\left(\frac{2H_B}{\mathrm{i}\hbar\omega}\right)\tan\left(\frac{\omega t}{2}\right)\right] \\
&= \exp\left[\left(\frac{2H_{SU}}{\mathrm{i}\hbar\omega}\right)\tan\left(\frac{\omega t}{2}\right)\right].
\end{aligned} \tag{2.62}$$

This ansatz fulfills the differential equation for the time evolution with the supersymmetric star product:

$$\begin{aligned}
\mathrm{i}\hbar\frac{d}{dt}\mathrm{Exp}_{SU}(H_{SU}t) &= (H_F *_M \mathrm{Exp}_M(H_F t))\,\mathrm{Exp}_M(H_B t) + (H_B *_M \mathrm{Exp}_M(H_B t))\,\mathrm{Exp}_M(H_F t) \\
&= H_{SU} *_{SU} \mathrm{Exp}_{SU}(H_{SU}t).
\end{aligned} \tag{2.63}$$

The Fourier-Dirichlet expansion is

$$\mathrm{Exp}_{SU}(H_{SU}t) = \sum_{n_F=-1/2}^{1/2}\sum_{n_B=0}^{\infty}\pi^{(M)}_{n_F}\pi^{(M)}_{n_B}e^{-\mathrm{i}\left(E_{n_F}+E_{n_B}\right)t/\hbar}. \tag{2.64}$$

This means that the supersymmetric projectors $\pi^{(SU)}_{n_F n_B}$ are just the product of the fermionic and the bosonic projectors. They also fulfill the $*$-eigenvalue equation

$$H_{SU} *_{SU} \pi^{(SU)}_{n_F n_B} = (E_{n_B} + E_{n_F})\pi^{(SU)}_{n_F n_B}, \tag{2.65}$$

and are idempotent:

$$\pi^{(SU)}_{n_F n_B} *_{SU} \pi^{(SU)}_{n'_F n'_B} = \delta_{n_F n'_F}\delta_{n_B n'_B}\pi^{(SU)}_{n_F n_B}. \tag{2.66}$$

The spectral resolution of the Hamilton function is

$$H_{SU} = \sum_{n_F=-1/2}^{1/2}\sum_{n_B=0}^{\infty}(E_{n_B}+E_{n_F})\,\pi^{(SU)}_{n_F n_B}. \tag{2.67}$$

The functions $Q_\pm$ now act on the supersymmetric projectors as:

$$Q_+ *_{SU} \pi^{(SU)}_{n_F n_B} *_{SU} Q_- = \hbar\pi^{(SU)}_{n_F+1,n_B-1} \qquad \text{and} \qquad Q_- *_{SU} \pi^{(SU)}_{n_F n_B} *_{SU} Q_+ = \hbar\pi^{(SU)}_{n_F-1,n_B+1} \tag{2.68}$$

and one sees that the $Q_\pm$ relate energy levels of the two sectors of the supersymmetric oscillator that lie on the same footing.

The functions $\pi^{(M)}_{\pm 1/2}$, Q_+ and Q_- fulfill the relations:

$$\pi^{(M)}_{\pm 1/2} *_{SU} \pi^{(M)}_{\pm 1/2} = \pi^{(M)}_{\pm 1/2}, \qquad Q_\pm *_{SU} \pi^{(M)}_{\mp 1/2} = Q_\pm \qquad \text{and} \qquad \pi^{(M)}_{\pm 1/2} *_{SU} Q_\pm = Q_\pm, \tag{2.69}$$

so that these functions form a Fredholm quadruple Ξ. With this quadruple one can then define an index [51]:

$$\begin{aligned} \mathrm{ind}\Xi &= \mathrm{tr}\left[\pi^{(M)}_{-1/2} - \frac{1}{\hbar}Q_+ *_{SU} Q_-\right] - \mathrm{tr}\left[\pi^{(M)}_{+1/2} - \frac{1}{\hbar}Q_- *_{SU} Q_+\right] \\ &= \mathrm{tr}\left[\pi^{(M)}_{-1/2}\left(\frac{1}{2} - \frac{a\bar{a}}{\hbar}\right)\right] - \mathrm{tr}\left[\pi^{(M)}_{+1/2}\left(\frac{1}{2} - \frac{a\bar{a}}{\hbar}\right)\right]. \end{aligned} \tag{2.70}$$

The trace tr is here the sum over all bosonic and fermionic states:

$$\mathrm{tr}[F] = \sum_{n_B=0}^{\infty}\sum_{n_F=\pm 1/2}\int d^2a\,\mathrm{Tr}(\pi^{(M)}_{n_B}\pi^{(M)}_{n_F} *_{SU} F), \tag{2.71}$$

where Tr is defined as in (2.44). The terms in the round brackets of (2.70) give the sum of the number of the bosonic states. Since all states with $E > 0$ are paired as they appear in the bosonic and the fermionic sector, these two terms cancel each other. The first term in the round brackets counts the number of states, so that the index is the difference of the number of states in the bosonic and the fermionic sector. Because of the pairing of states with $E > 0$ the index is zero if there is a state with $E = 0$ in the bosonic and the fermionic sector and it is one if only one of the sectors has a $E = 0$ state. This index is called the Witten index and describes if the supersymmetry is exact or broken [83]. Since the second terms in the round brackets of (2.70) cancel anyway, the same result can be obtained just with the elliptic pair $\pi^{(M)}_{\pm 1/2}$:

$$\mathrm{ind}\Xi = \mathrm{tr}\left[\pi^{(M)}_{-1/2} - \pi^{(M)}_{+1/2}\right]. \tag{2.72}$$

Just as one can construct a supersymmetric star product of the Moyal type it is also possible to construct a supersymmetric star product of the normal type:

$$F *_{SUN} G = F e^{\hbar\left(\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}} + \overleftarrow{\partial}_f\overrightarrow{\partial}_{\bar{f}}\right)} G. \tag{2.73}$$

All calculations that were done for the Moyal type star product can be done analogously for the normal type star product. For example the star exponential is now the product of (1.95) and (2.48) which leads to

$$\mathrm{Exp}_{SUN}(H_{SU}t) = e^{-H_{SU}/\hbar\omega}\exp\left[\frac{1}{\hbar}\left(\bar{f}f\,e^{-\mathrm{i}E_F t/\hbar} + \bar{a}a\,e^{-\mathrm{i}E_B t/\hbar}\right)\right]. \tag{2.74}$$

2.4 Supersymmetric Quantum Mechanics with Star Products

Crucial for the vanishing of the star commutator (2.59) is that the $Q_\pm$ are nilpotent (2.57). But this is already assured by the fermionic part of the $Q_\pm$, so that one can use a bosonic part that is more general than a or $\bar{a}$, as in (2.56). In supersymmetric quantum mechanics one usually goes over to the new coordinates

$$B = \frac{1}{\sqrt{2}}\left(W(q) + \mathrm{i}\frac{p}{\sqrt{m}}\right) \qquad \text{and} \qquad \bar{B} = \frac{1}{\sqrt{2}}\left(W(q) - \mathrm{i}\frac{p}{\sqrt{m}}\right), \tag{2.75}$$

where $W(q)$ is the superpotential and m an additional mass parameter. The $Q_\pm$ can then be generalized to $Q_+ = B\bar{f}$ and $Q_- = \bar{B}f$, which results in a system with interaction between the fermionic and the bosonic sector [83].

In the star product formalism one can proceed in an analogous way. Therefore one first transforms the Moyal product into the new coordinates (2.75), which gives

$$F *_M G = F e^{\frac{\hbar}{2}\frac{\partial W}{\partial q}\frac{1}{\sqrt{m}}\left(\overleftarrow{\partial}_B\overrightarrow{\partial}_{\bar{B}} - \overleftarrow{\partial}_{\bar{B}}\overrightarrow{\partial}_B\right)} G. \tag{2.76}$$

The star commutator and anticommutator are then

$$\{B, \bar{B}\}_{*_M} = W^2(q) + \frac{p^2}{m^2} \qquad \text{and} \qquad [B, \bar{B}]_{*_M} = \frac{\hbar}{\sqrt{m}}\frac{\partial W}{\partial q}. \tag{2.77}$$

To implement the new coordinates one uses the fact that the supersymmetric Hamilton function of an oscillator can be written as in (2.55). By analogy, the Hamilton function for the general supersymmetric system is then:

$$\begin{aligned}
H_{SU} &= \left(\bar{B} *_M B\right)\pi^{(M)}_{-1/2} + \left(B *_M \bar{B}\right)\pi^{(M)}_{1/2} && (2.78)\\
&= \left(\frac{1}{2}\{B,\bar{B}\}_{*_M} - \frac{1}{2}[B,\bar{B}]_{*_M}\right)\pi^{(M)}_{-1/2} + \left(\frac{1}{2}\{B,\bar{B}\}_{*_M} + \frac{1}{2}[B,\bar{B}]_{*_M}\right)\pi^{(M)}_{1/2} && (2.79)\\
&= \frac{1}{2}\left(\frac{p^2}{m} + W^2 - \frac{\hbar}{\sqrt{m}}\frac{\partial W}{\partial q}\right)\pi^{(M)}_{-1/2} + \frac{1}{2}\left(\frac{p^2}{m} + W^2 + \frac{\hbar}{\sqrt{m}}\frac{\partial W}{\partial q}\right)\pi^{(M)}_{1/2} && (2.80)\\
&= H_{(1)}\pi^{(M)}_{-1/2} + H_{(2)}\pi^{(M)}_{1/2}. && (2.81)
\end{aligned}$$

In the conventional operator approach to supersymmetric quantum mechanics the projectors are represented as 2×2 matrices, see Eq. (2.43), and the two systems $H_{(1)}$ and $H_{(2)}$ are represented as blocks of a matrix.

Using the orthogonality and idempotence of the projectors one can simplify the $*$-eigenvalue equation for H_{SU}:

$$\begin{aligned}
H_{SU} *_{SU} \pi^{(SU)}_{-1/2, n_B^{(1)}} &= H_{SU} *_{SU}\left(\pi^{(M)}_{n_B^{(1)}}\pi^{(M)}_{-1/2}\right) = \left(H_{(1)} *_M \pi^{(M)}_{n_B^{(1)}}\right)\pi^{(M)}_{-1/2} = E_{(1)}\pi^{(M)}_{n_B^{(1)}}\pi^{(M)}_{-1/2},\\
H_{SU} *_{SU} \pi^{(SU)}_{-1/2, n_B^{(2)}} &= H_{SU} *_{SU}\left(\pi^{(M)}_{n_B^{(2)}}\pi^{(M)}_{1/2}\right) = \left(H_{(2)} *_M \pi^{(M)}_{n_B^{(2)}}\right)\pi^{(M)}_{1/2} = E_{(2)}\pi^{(M)}_{n_B^{(2)}}\pi^{(M)}_{1/2}. \qquad (2.82)
\end{aligned}$$

This means that the problem is reduced to two bosonic $*$-eigenvalue equations:

$$\begin{aligned}
& H_{(1)} *_M \pi^{(M)}_{n_B^{(1)}} = \bar{B} *_M B *_M \pi^{(M)}_{n_B^{(1)}} = E_{(1)}\pi^{(M)}_{n_B^{(1)}} && (2.83)\\
\text{and} \quad & H_{(2)} *_M \pi^{(M)}_{n_B^{(2)}} = B *_M \bar{B} *_M \pi^{(M)}_{n_B^{(2)}} = E_{(2)}\pi^{(M)}_{n_B^{(2)}}. && (2.84)
\end{aligned}$$

The connection between the two systems can be found immediately with the help of the associativity of the star product:

$$H_{(1)} *_M \left(\bar{B} *_M \pi^{(M)}_{n_B^{(2)}} *_M B\right) = \bar{B} *_M \left(B *_M \bar{B} *_M \pi^{(M)}_{n_B^{(2)}} *_M B\right) = E_{(2)} \left(\bar{B} *_M \pi^{(M)}_{n_B^{(2)}} *_M B\right),$$
$$H_{(2)} *_M \left(B *_M \pi^{(M)}_{n_B^{(1)}} *_M \bar{B}\right) = B *_M \left(\bar{B} *_M B *_M \pi^{(M)}_{n_B^{(1)}} *_M \bar{B}\right) = E_{(1)} \left(B *_M \pi^{(M)}_{n_B^{(1)}} *_M \bar{B}\right). \tag{2.85}$$

$E_{(2)}$ is then also an eigenvalue of $H_{(1)}$ and $E_{(1)}$ is an eigenvalue of $H_{(2)}$, just as $\bar{B} *_M \pi^{(M)}_{n_B^{(2)}} *_M B$ is an eigenfunction of $H_{(1)}$ and $B *_M \pi^{(M)}_{n_B^{(1)}} *_M \bar{B}$ is an eigenfunction of $H_{(2)}$. One sees then that B and $\bar{B}$ relate two systems with supersymmetric partner potentials $V_{(1)} = \frac{1}{2}\left(W^2 - \frac{\hbar}{\sqrt{m}}\frac{\partial W}{\partial q}\right)$ and $V_{(2)} = \frac{1}{2}\left(W^2 + \frac{\hbar}{\sqrt{m}}\frac{\partial W}{\partial q}\right)$.

To show how the star product formalism works one considers the superpotential $W(q) = A\tanh(\alpha q)$. The two partner potentials are in this case

$$V_{(1)} = \frac{1}{2}\left(A^2 - A\left(A + \frac{\hbar\alpha}{\sqrt{m}}\right)\frac{1}{\cosh^2(\alpha q)}\right) \quad \text{and} \quad V_{(2)} = \frac{1}{2}\left(A^2 - A\left(A - \frac{\hbar\alpha}{\sqrt{m}}\right)\frac{1}{\cosh^2(\alpha q)}\right). \tag{2.86}$$

For $A = \hbar\alpha/\sqrt{m}$ these expressions become

$$V_{(1)} = \frac{\hbar^2\alpha^2}{2m}\left(1 - \frac{2}{\cosh^2(\alpha q)}\right) \qquad \text{and} \qquad V_{(2)} = \frac{\hbar^2\alpha^2}{2m}, \tag{2.87}$$

so the first system is the Pöschel-Teller potential and the second system is the free particle.

One can then first consider the free particle with the Hamiltonian $H_{(2)} = \frac{p^2}{2m} + \frac{\hbar^2\alpha^2}{2m}$. Using $\left(H_{(2)}\right)^{n *_M} = \left(H_{(2)}\right)^n$ one gets for the star exponential

$$\mathrm{Exp}_M\left(H_{(2)}t\right) = \exp\left(\frac{H_{(2)}t}{\mathrm{i}\hbar}\right). \tag{2.88}$$

The projectors can be obtained from the $*$-eigenvalue equation

$$H_{(2)}(p) *_M \pi^{(M)}_{k^{(2)}} = E_k\, \pi^{(M)}_{k^{(2)}} \tag{2.89}$$

as

$$\pi^{(M)}_{k^{(2)}} = \delta(p-k), \tag{2.90}$$

where the energy eigenvalues are $E_k = \frac{k^2}{2m} + \frac{\hbar^2\alpha^2}{2m}$. The Fourier-Dirichlet expansion is given by

$$\mathrm{Exp}_M\left(H_{(2)}t\right) = \int dk\, \pi^{(M)}_{k^{(2)}} \exp\left(\frac{E_k t}{\mathrm{i}\hbar}\right) = \exp\left(\frac{H_{(2)}t}{\mathrm{i}\hbar}\right). \tag{2.91}$$

The Pöschel-Teller potential as the supersymmetric partner potential has one bound state with energy zero and a continuum of reflectionless states. The projector $\pi^{(M)}_{0^{(1)}}$ for the bound state can be calculated in terms of the eigenfunctions ψ_n of the Hamilton operator according to (1.55). With the ground state wavefunction $\psi_{0^{(1)}} = \sqrt{\frac{\alpha}{2}}\frac{1}{\cosh(\alpha q)}$ of the Pöschel-Teller potential this gives the projector [23]:

$$\pi^{(M)}_{0^{(1)}} = \frac{\sin(2pq/\hbar)}{\sinh(2\alpha q)\sinh(p\pi/\alpha\hbar)}. \tag{2.92}$$

The projectors $\pi^{(M)}_{k^{(1)}}$ for the reflectionless states can now be obtained with the help of the functions B and $\bar{B}$ from the projectors $\pi^{(M)}_{k^{(2)}} = \delta(p-k)$:

$$\begin{aligned}
\pi^{(M)}_{k^{(1)}} &= \frac{1}{\hbar}\left[\bar{B} *_M \pi^{(M)}_{(2)k} *_M B\right] \\
&= \frac{1}{4\pi}\int dy \left(\frac{p^2}{\hbar m} + \mathrm{i}\frac{p\alpha}{m}\tanh\left(\alpha q + \alpha\frac{y}{2}\right) - \mathrm{i}\frac{p\alpha}{m}\tanh\left(\alpha q - \alpha\frac{y}{2}\right)\right. \\
&\qquad\qquad \left. + \frac{\hbar\alpha^2}{m}\tanh\left(\alpha q + \alpha\frac{y}{2}\right)\tanh\left(\alpha q - \alpha\frac{y}{2}\right)\right) e^{\mathrm{i}(p-k)y} \qquad (2.93) \\
&= \frac{p^2}{\hbar m}\delta(p-k) + \mathrm{i}\frac{4\pi p}{\alpha\hbar}\frac{\sin\left(2(p-k)q/\hbar\right)}{\sinh\left((p-k)\pi/\alpha\hbar\right)}. \qquad (2.94)
\end{aligned}$$

Equation (2.93) is the same result that one gets by calculating the projector with Eq. (1.55), using the wavefunction $\psi_k = \hat{B}e^{\mathrm{i}kq}$. $\hat{B}$ is the operator form of Eq. (2.75). For the case $W(q) = n\frac{\hbar\alpha}{\sqrt{m}}\tanh(\alpha q)$ with n bound states one can proceed in a similar way [23].

2.5 Spin and Star Products

In conventional quantum mechanics spin 1/2 fermions are described by using 2×2 Pauli matrices. As seen above, such matrices can be described in the star product formalism with products of Grassmann variables. This suggests the use of appropriate Grassmann variables in order to describe spin 1/2 particles in the framework of deformation quantization. With this motivation one introduces the Grassmann variables θ_i, $(i = 1, 2, 3)$, and in analogy to (2.22) the Clifford star product

$$F *_C G = F \exp\left(\frac{\hbar}{2}\sum_{n=1}^{d}\overleftarrow{\partial}_{\theta_n}\overrightarrow{\partial}_{\theta_n}\right) G. \qquad (2.95)$$

The variables θ_i form a Clifford algebra with respect to this product: $\{\theta_i, \theta_j\}_{*_C} = \hbar\delta_{ij}$. With the θ_i variables one can construct the quantities

$$\sigma^i = \varepsilon^{ijk}\frac{1}{2\mathrm{i}\hbar}[\theta_j, \theta_k]_{*_C} = \frac{1}{\mathrm{i}\hbar}\varepsilon^{ijk}\theta_j\theta_k, \qquad (2.96)$$

which fulfill the relations

$$[\sigma^i, \sigma^j]_{*_C} = 2\mathrm{i}\varepsilon^{ijk}\sigma^k \qquad \text{and} \qquad \{\sigma^i, \sigma^j\}_{*_C} = 2\delta^{ij}. \qquad (2.97)$$

The σ^i obviously correspond to the Pauli matrices $\hat{\sigma}_i$ in the operator formalism and will therefore be called Pauli functions. The Pauli functions are real, i.e. $\overline{\sigma_i} = \sigma_i$, where the complex conjugation $F \mapsto \overline{F}$ for a superfunction F is defined as in [11] and fulfills:

$$\overline{\overline{F}} = F, \qquad \overline{F_1 F_2} = \overline{F_2}\,\overline{F_1} \qquad \text{and} \qquad \overline{cF} = \overline{c}\,\overline{F}, \qquad (2.98)$$

where c is a complex number. The realness of the Pauli functions corresponds to the fact that the Pauli matrices are hermitian: $(\hat{\sigma}_i)^\dagger = \hat{\sigma}_i$.

In order to construct the analogy of the trace for the σ^i one must define the Hodge dual for Grassmann monomials, which maps a Grassmann monomial with grade r into a Grassmann monomial with grade $d-r$; d is the number of Grassmann basis elements, which is here three:

$$\star\left(\theta_{i_1}\theta_{i_2}\cdots\theta_{i_r}\right) = \frac{1}{(d-r)!}\varepsilon^{i_{r+1}\ldots i_d}_{i_1 i_2\ldots i_r}\theta_{i_{r+1}}\cdots\theta_{i_d}. \qquad (2.99)$$

The trace of a function $F(\theta_1, \theta_2, \theta_3)$ is then defined as:

$$\mathrm{Tr}(F) = \frac{2}{\hbar^3} \int d\theta_3 d\theta_2 d\theta_1 \, \star F, \tag{2.100}$$

where the integral is given by the Berezin integral. The Berezin integral has the property that $\int d\theta_i \, \theta_j = \delta_{ij} \hbar$. A $\hbar$ appears here, because the variables θ_i have in our case the unit $\sqrt{\hbar}$. This definition of the trace leads immediately to

$$\mathrm{Tr}(\sigma^i) = 0 \qquad \text{and} \qquad \mathrm{Tr}(\sigma^i *_C \sigma^j) = 2\delta^{ij}. \tag{2.101}$$

Moreover one can see the correspondence between spin and the fermionic quantum oscillator. If one considers the two dimensional fermionic oscillator as described above embedded in a three dimensional space with coordinates θ_i, $(i = 1, 2, 3)$, then the Clifford product corresponds to the fermionic Moyal product (2.23). The two canonical momenta are given according to (2.14) by $-\frac{\mathrm{i}}{2}\theta_1$ and $-\frac{\mathrm{i}}{2}\theta_2$, so that the angular momentum of the two dimensional fermionic oscillator is $\vec{S} = (0, 0, S_3)$ with

$$S_3 = -\frac{\mathrm{i}}{2}\left(\theta_1\theta_2 - \theta_2\theta_1\right) = -\mathrm{i}\theta_1\theta_2 = \frac{\hbar}{2}\sigma^3. \tag{2.102}$$

The Hamilton function of the fermionic oscillator (2.6) can then be written as $H = \omega S_3$ and the corresponding Wigner functions are in analogy to (2.26) given by

$$\pi^{(C)}_{\pm 1/2} = \frac{1}{2} \mp \frac{\mathrm{i}}{\hbar}\theta_1\theta_2 = \frac{1}{2}(1 \pm \sigma^3), \tag{2.103}$$

which shows that the $\pi^{(C)}_{\pm 1/2}$ are just nonrelativistic spin projectors. With the trace (2.100) the projectors (2.103) are normalized according to:

$$\mathrm{Tr}\left(\pi^{(C)}_{\pm 1/2}\right) = \mathrm{Tr}\left(\frac{1}{2} \pm \frac{1}{2}\sigma^3\right) = 1. \tag{2.104}$$

One sees that (2.100) is the Grassmann analogue of the matrix trace used in (2.44) and (2.71). Moreover (2.100) allows now also the calculation of expectation values, which makes clear that the projectors (2.103) correspond to spin up and spin down states, because the expectation values of the S_i are given by:

$$\begin{aligned}
\langle S_1 \rangle &= \mathrm{Tr}\left(\pi^{(C)}_{\pm 1/2} *_C \frac{\hbar}{2}\sigma^1\right) = 0 \quad , \quad \langle S_2 \rangle = \mathrm{Tr}\left(\pi^{(C)}_{\pm 1/2} *_C \frac{\hbar}{2}\sigma^2\right) = 0 \\
\langle S_3 \rangle &= \mathrm{Tr}\left(\pi^{(C)}_{\pm 1/2} *_C \frac{\hbar}{2}\sigma^3\right) = \pm\frac{\hbar}{2} \quad , \quad \langle \vec{S}^{\,2*_C} \rangle = \mathrm{Tr}\left(\pi^{(C)}_{\pm 1/2} *_C \frac{\hbar^2}{4}\vec{\sigma}^{\,2*_C}\right) = \frac{3}{4}\hbar^2.
\end{aligned} \tag{2.105}$$

The star exponential (2.25) allows the calculation of the time development of the σ^i:

$$\begin{aligned}
\sigma^1(t) &= \mathrm{Exp}_C(-Ht) *_C \sigma^1 *_C \mathrm{Exp}_C(Ht) = \sigma^1 \cos(\omega t) - \sigma^2 \sin(\omega t) \\
\sigma^2(t) &= \mathrm{Exp}_C(-Ht) *_C \sigma^2 *_C \mathrm{Exp}_C(Ht) = \sigma^1 \sin(\omega t) + \sigma^2 \cos(\omega t) \\
\sigma^3(t) &= \mathrm{Exp}_C(-Ht) *_C \sigma^3 *_C \mathrm{Exp}_C(Ht) = \sigma^3.
\end{aligned} \tag{2.106}$$

With these expressions it is easy to see that the $*$-Heisenberg equation $\mathrm{i}\hbar\frac{dF(t)}{dt} = [F(t), H(t)]_*$ for the spin is given by:

$$\begin{aligned}
&\frac{dS_1(t)}{dt} = \frac{1}{\mathrm{i}\hbar}\left[S_1(t), H\right]_{*_C} = -\omega S_2(t), \qquad \frac{dS_2(t)}{dt} = \frac{1}{\mathrm{i}\hbar}\left[S_2(t), H\right]_{*_C} = \omega S_1(t) \\
\text{and} \quad &\frac{dS_3(t)}{dt} = \frac{1}{\mathrm{i}\hbar}\left[S_3(t), H\right]_{*_C} = 0.
\end{aligned} \tag{2.107}$$

For $\omega = \frac{e}{m}B_3$, where B_3 is the third component of the magnetic field $\vec{B} = (0, 0, B_3)$, this leads to the equation of motion for the spin:

$$\frac{d\vec{S}}{dt} = \frac{e}{m}\vec{B} \times \vec{S}. \tag{2.108}$$

So here one has the case that the classical equations of motion follow from the star product time evolution.

Furthermore the spin $\vec{S}$ is in the fermionic θ-space the generator of rotations. A rotation $\vec{\varphi} = \varphi\vec{n}$ with angle φ and rotation axis $\vec{n}$ is described by the star exponential

$$\mathrm{Exp}_C(\vec{\varphi} \cdot \vec{S}) = e_{*_C}^{-\frac{1}{2}\mathrm{i}\vec{\varphi}\cdot\vec{\sigma}} = \cos\frac{\varphi}{2} - \mathrm{i}(\vec{\sigma} \cdot \vec{n}) \sin\frac{\varphi}{2}. \tag{2.109}$$

The vector $\vec{\theta} = (\theta_1, \theta_2, \theta_3)^T$ transforms then passively (in opposite to the active transformation of time development as in (2.106) which has the opposite sign structure) according to:

$$\mathrm{Exp}_C(\vec{\varphi} \cdot \vec{S}) *_C \vec{\theta} *_C \mathrm{Exp}_C(-\vec{\varphi} \cdot \vec{S}) = R(\vec{\varphi})\vec{\theta}, \tag{2.110}$$

where $R(\vec{\varphi})$ is the well known $SO(3)$-rotation matrix. The axial vector $\vec{\sigma}$ transforms in the same way. Note that the passive transformation (2.110) of the θ_i amounts to an active transformation of vectors $x = \sum_{i=1}^{3} x_i\theta_i$.

The spin Wigner function $\pi^{(C)}_{+1/2} = \pi^{(C)}_{++}$ corresponds to the density matrix $|+\rangle\langle+|$ and the spin Wigner function $\pi^{(C)}_{-1/2} = \pi^{(C)}_{--}$ corresponds to the density matrix $|-\rangle\langle-|$, so there should also exist off diagonal Wigner functions $\pi^{(C)}_{+-}$ and $\pi^{(C)}_{-+}$. With the ansatz $\pi^{(C)}_{+-} = a_0 + a_i\sigma^i$ and $\pi^{(C)}_{-+} = b_0 + b_i\sigma^i$ for these functions one can determine the coefficients by the functional analogues of the fundamental relations. The density matrices $|-\rangle\langle+|$ and $|+\rangle\langle-|$ fulfill:

$$\mathrm{Tr}\left(\pi^{(C)}_{+-}\right) = \mathrm{Tr}\left(\pi^{(C)}_{-+}\right) = 0, \tag{2.111}$$

$$\pi^{(C)}_{+-} *_C \pi^{(C)}_{+-} = \pi^{(C)}_{-+} *_C \pi^{(C)}_{-+} = 0, \tag{2.112}$$

$$\text{and} \quad \pi^{(C)}_{+-} *_C \pi^{(C)}_{-+} = \pi^{(C)}_{++}, \qquad \pi^{(C)}_{-+} *_C \pi^{(C)}_{+-} = \pi^{(C)}_{--}. \tag{2.113}$$

The results for $\pi^{(C)}_{+-}$ and $\pi^{(C)}_{-+}$ are

$$\pi^{(C)}_{+-} = \frac{1}{2}(\sigma^1 - \mathrm{i}\sigma^2) \qquad \text{and} \qquad \pi^{(C)}_{-+} = \frac{1}{2}(\sigma^1 + \mathrm{i}\sigma^2). \tag{2.114}$$

With these projectors the Pauli matrix $\hat{\sigma}^i$ can be written as

$$\hat{\sigma}^i = \begin{pmatrix} \mathrm{Tr}\left(\sigma^i *_C \pi^{(C)}_{++}\right) & \mathrm{Tr}\left(\sigma^i *_C \pi^{(C)}_{+-}\right) \\ \mathrm{Tr}\left(\sigma^i *_C \pi^{(C)}_{-+}\right) & \mathrm{Tr}\left(\sigma^i *_C \pi^{(C)}_{--}\right) \end{pmatrix} \tag{2.115}$$

and the angular momentum functions in the Schwinger representation (1.140) become

$$J_i = \sum_{s,s'=\pm} \bar{a}_s \mathrm{Tr}\left(\frac{1}{2}\sigma^i *_C \pi^{(C)}_{s,s'}\right) a_{s'}. \tag{2.116}$$

The $\mathfrak{su}(2)$-algebra structure (1.137) the angular momentum functions (1.140) fulfill with the star product (1.133) is then a reflection of the algebra (2.97) the sigma functions satisfy:

$$\begin{aligned} [J_i, J_j]_{*_M} &= \sum_{s,s'=\pm} \bar{a}_s \mathrm{Tr}\left(\left[\frac{1}{2}\sigma^i, \frac{1}{2}\sigma^j\right]_{*_C} *_C \pi^{(C)}_{s,s'}\right) a_{s'} \\ &= \sum_{s,s'=\pm} \bar{a}_s \mathrm{Tr}\left(\mathrm{i}\varepsilon^{ijk}\frac{1}{2}\sigma^k *_C \pi^{(C)}_{s,s'}\right) a_{s'} \\ &= \mathrm{i}\varepsilon_{ijk} J_k \end{aligned} \tag{2.117}$$

This makes clear how the angular momentum in the Schwinger representation is decomposed into spins and how the fermionic star product algebra translates into a bosonic one. What the Schwinger representation actually does is that it replaces the $\langle s|$-part of the Wigner function that annihilates a spin $s = \pm$ on the right by the bosonic annihilation function a_s. But with the annihilation and creation functions $\bar{a}_s$ and a_s one can construct arbitrary bosonic angular momentum Wigner functions. For example the simplest spin states ($j = \frac{1}{2}, m = \pm\frac{1}{2}$) are given by

$$\pi^{(M)}_{j=1/2,m=1/2} = \bar{a}_+ *_M \pi^{(M)}_{n_+=0,n_-=0} *_M a_+ \qquad \text{and} \qquad \pi^{(M)}_{j=1/2,m=-1/2} = \bar{a}_- *_M \pi^{(M)}_{n_+=0,n_-=0} *_M a_-, \tag{2.118}$$

where $\pi^{(M)}_{n_+,n_-}$ is the two dimensional harmonic oscillator Wigner function as discussed in the last chapter. The Wigner function for an arbitrary angular momentum is then

$$\pi^{(M)}_{j,m} = \frac{1}{(j+m)!(j-m)!} \bar{a}_+^{j+m} \bar{a}_-^{j-m} *_C \pi^{(M)}_{n_+=0,n_-=0} *_C a_-^{j-m} a_+^{j+m} \tag{2.119}$$

and fulfills (1.142).

As a further physical application one can show that the Pauli Hamilton function, which describes the 2-dimensional motion of a charged spin 1/2 particle in a constant magnetic field along the z-axis, may be described in the star product formalism in a supersymmetric framework. To this purpose introduce the Moyal-Clifford star product

$$F *_{MC} G = F \exp\left[\frac{\mathrm{i}\hbar}{2} \sum_{n=1}^{3} \left(\overleftarrow{\partial}_{q_n} \overrightarrow{\partial}_{p_n} - \overleftarrow{\partial}_{p_n} \overrightarrow{\partial}_{q_n} - \mathrm{i} \overleftarrow{\partial}_{\theta_n} \overrightarrow{\partial}_{\theta_n}\right)\right] G, \tag{2.120}$$

and the quantities

$$Q_1 = \frac{1}{\sqrt{2m}} \left[-\left(p_2 - eA_2\right)\sigma^1 + \left(p_1 - eA_1\right)\sigma^2\right], \tag{2.121}$$

$$Q_2 = \frac{1}{\sqrt{2m}} \left[\left(p_1 - eA_1\right)\sigma^1 + \left(p_2 - eA_2\right)\sigma^2\right], \tag{2.122}$$

where $\vec{A}(q_1, q_2)$ is the vector potential of the magnetic field. One finds

$$\{Q_1, Q_2\}_{*_{MC}} = \frac{1}{2m} \left[\left(p_1 - eA_1\right)^{2*_M} - \left(p_2 - eA_2\right)^{2*_M}\right] \{\sigma^1, \sigma^2\}_{*_C} = 0, \tag{2.123}$$

and

$$\begin{aligned} Q_1 *_{MC} Q_1 = Q_2 *_{MC} Q_2 &= \frac{1}{2m} \left[\left(p_1 - eA_1\right)^{2*_M} + \left(p_2 - eA_2\right)^{2*_M}\right] \\ &\quad + \frac{1}{2m} \left[\left(p_1 - eA_1\right), \left(p_2 - eA_2\right)\right]_{*_M} (\sigma^1 *_C \sigma^2). \end{aligned} \tag{2.124}$$

Calculating the star commutator

$$\left[\left(p_1 - eA_1\right), \left(p_2 - eA_2\right)\right]_{*_M} = -e\left[p_1, A_2\right]_{*_M} - e\left[A_1, p_2\right]_{*_M} = \mathrm{i}e\hbar B_3 \tag{2.125}$$

with $\vec{B} = \vec{\nabla} \times \vec{A}$ the Hamilton function is now

$$H_P = Q_1 *_{MC} Q_1 = Q_2 *_{MC} Q_2 = \frac{1}{2m} \left(\vec{p} - e\vec{A}\right)^{2*_M} - \frac{e\hbar}{2m} \vec{\sigma} \cdot \vec{B}. \tag{2.126}$$

This is the Pauli Hamilton function, with a gyromagnetic factor $g = 2$. Q_1, Q_2 and H_P form a supersymmetric algebra, as in Eq. (2.59).

One should note that (2.122) together with (2.126) is the Feynman trick (see page 79 of [98]) in the star product formalism, i.e. for commuting quantities $\vec{p}$ and $\vec{A}$ one has $\left[\left(\vec{p}-e\vec{A}\right)\cdot\hat{\vec{\sigma}}\right]^2=\left(\vec{p}-e\vec{A}\right)^2\hat{I}$, whereas in the non-commuting case an interaction term is induced: $\left[\left(\vec{p}-e\vec{A}\right)\cdot\vec{\sigma}\right]^{2*_{MC}}=\left(\vec{p}-e\vec{A}\right)^{2*_M}-\hbar e\vec{\sigma}\cdot\vec{B}$. Or more generally there exists a star product Gordon decomposition for two vector valued functions f_i and g_i of the variables q_i and p_i with $i=1,2,3$:

$$(\sigma^i f_i)*_{MC}(\sigma^j g_j)=\left(\varepsilon^{imn}\frac{1}{\mathrm{i}\hbar}\theta^m\theta^n f_i\right)*_{MC}\left(\varepsilon^{jrs}\frac{1}{\mathrm{i}\hbar}\theta^r\theta^s g_j\right)=f_i*_M g_i+\mathrm{i}\varepsilon^{ijm}(f_i*_M g_j)\sigma^m. \tag{2.127}$$

The next task is to find the Wigner function for (2.126). Since this problem separates in conventional quantum mechanics into a space and a spin part, one can here also consider the two terms of the Pauli Hamiltonian separately. First consider the bosonic part of this problem describing a charged particle in a magnetic field, which corresponds to the Landau problem. The magnetic field points in the direction of q_3 and can be described with the gauge potential $\vec{A}=\frac{B_3}{2}(-q_2,q_1,0)$. For this gauge potential the Moyal product in (2.126) reduces to a conventional product and the bosonic part is the Landau Hamiltonian

$$H_L=\frac{1}{2m}\left(\tilde{p}_1^2+\tilde{p}_2^2\right), \tag{2.128}$$

where one defines

$$\tilde{p}_1=p_1-eA_1=p_1+\frac{m\omega}{2}q_2 \qquad \text{and} \qquad \tilde{p}_2=p_2-eA_2=p_2-\frac{m\omega}{2}q_1 \tag{2.129}$$

with $\omega=\frac{eB}{m}$. In order to quantize this two dimensional system one transforms the Moyal product from canonical coordinates into $(q_i,\tilde{p}_i)$-coordinates, which leads to

$$f\,\tilde{*}_M\,g=f\exp\left[\frac{\mathrm{i}\hbar}{2}\left(\overleftarrow{\partial}_{q_1}\overrightarrow{\partial}_{\tilde{p}_1}-\overleftarrow{\partial}_{\tilde{p}_1}\overrightarrow{\partial}_{q_1}+\overleftarrow{\partial}_{q_2}\overrightarrow{\partial}_{\tilde{p}_2}-\overleftarrow{\partial}_{\tilde{p}_2}\overrightarrow{\partial}_{q_2}\right)+\frac{\mathrm{i}\hbar m\omega}{2}\left(\overleftarrow{\partial}_{\tilde{p}_1}\overrightarrow{\partial}_{\tilde{p}_2}-\overleftarrow{\partial}_{\tilde{p}_2}\overrightarrow{\partial}_{\tilde{p}_1}\right)\right]g. \tag{2.130}$$

The $*$-eigenvalue equation $\frac{1}{2m}\left(\tilde{p}_1^2+\tilde{p}_2^2\right)\ \tilde{*}_M\,\pi_n^{(\tilde{M})}=E_n\pi_n^{(\tilde{M})}$ can easily be calculated by comparison with the bosonic oscillator. As seen above the $*$-eigenfunctions of the bosonic oscillator depend only on the Hamiltonian. Therefore also the $\pi_n^{(\tilde{M})}$ should depend on $\tilde{p}_1$ and $\tilde{p}_2$ only. Taking this as an ansatz, only the second part of the star product (2.130), which can be written as

$$\exp\left[\frac{\mathrm{i}\hbar}{2}\left(\overleftarrow{\partial}_{\left(\frac{\tilde{p}_1}{m\omega}\right)}\overrightarrow{\partial}_{\tilde{p}_2}-\overleftarrow{\partial}_{\tilde{p}_2}\overrightarrow{\partial}_{\left(\frac{\tilde{p}_1}{m\omega}\right)}\right)\right], \tag{2.131}$$

has to be taken into account for the $*$-eigenvalue equation.

Setting $q=\frac{\tilde{p}_1}{m\omega}$ and $p=\tilde{p}_2$ the Landau Hamiltonian H_L reduces to the Hamiltonian of the bosonic harmonic oscillator and (2.131) becomes the Moyal product in canonical variables. Then it is clear that the $*$-eigenfunctions of the Landau Hamiltonian are in analogy to the one dimensional harmonic oscillator given by

$$\pi_n^{(\tilde{M})}(\tilde{p}_1,\tilde{p}_2)=\pi_n^{(\tilde{M})}(H_L)=2(-1)^n\exp\left(-\frac{2H_L}{\hbar\omega}\right)L_n\left(\frac{4H_L}{\hbar\omega}\right). \tag{2.132}$$

The energy levels are the Landau levels $E_n=\hbar\omega\left(n+\frac{1}{2}\right)$.

Since the system considered here is described in a four dimensional phase space one can expect that another observable which commutes with the Hamiltonian is needed to characterize all the energy $*$-eigenfunctions. To find such an observable it is useful to write the star product (2.130) in the two forms

$$\begin{aligned} f\,\tilde{*}_M\,g &= f\exp\left[\frac{\mathrm{i}\hbar}{2}\left(\overleftarrow{\partial}_{q_1}\overrightarrow{\partial}_{\tilde{p}_1}-\overleftarrow{\partial}_{\tilde{p}_1}(\overrightarrow{\partial}_{q_1}-m\omega\overrightarrow{\partial}_{\tilde{p}_2})+\overleftarrow{\partial}_{q_2}\overrightarrow{\partial}_{\tilde{p}_2}-\overleftarrow{\partial}_{\tilde{p}_2}(\overrightarrow{\partial}_{q_2}+m\omega\overrightarrow{\partial}_{\tilde{p}_1})\right)\right]g && (2.133\text{a}) \\ &= f\exp\left[\frac{\mathrm{i}\hbar}{2}\left((\overleftarrow{\partial}_{q_1}-m\omega\overleftarrow{\partial}_{\tilde{p}_2})\overrightarrow{\partial}_{\tilde{p}_1}-\overleftarrow{\partial}_{\tilde{p}_1}\overrightarrow{\partial}_{q_1}+(\overleftarrow{\partial}_{q_2}+m\omega\overleftarrow{\partial}_{\tilde{p}_1})\overrightarrow{\partial}_{\tilde{p}_2}-\overleftarrow{\partial}_{\tilde{p}_2}\overrightarrow{\partial}_{q_2}\right)\right]g && (2.133\text{b}) \end{aligned}$$

by simply rearranging the terms in the argument of the exponential function. By observing that the functions $\tilde{q}_1 = q_1 + \frac{1}{m\omega}\tilde{p}_2$ and $\tilde{q}_2 = q_2 - \frac{1}{m\omega}\tilde{p}_1$ fulfill the equations

$$(\partial_{q_1} - m\omega\partial_{\tilde{p}_2})\tilde{q}_i = 0 \qquad \text{and} \qquad (\partial_{q_2} + m\omega\partial_{\tilde{p}_1})\tilde{q}_i = 0 \tag{2.134}$$

it is obvious from equations (2.133) that every (analytical) function of $\tilde{q}_i$ commutes with every (analytical) function of $\tilde{p}_i$, e. g.

$$H_L \,\tilde{*}_M\, f(\tilde{q}_1, \tilde{q}_2) = H_L f(\tilde{q}_1, \tilde{q}_2) = f(\tilde{q}_1, \tilde{q}_2) H_L = f(\tilde{q}_1, \tilde{q}_2) \,\tilde{*}_M\, H_L. \tag{2.135}$$

On the one hand this means that $\tilde{q}_1$ and $\tilde{q}_2$ are two conserved phase space functions and on the other hand it follows that all functions of the form $f(\tilde{q}_1, \tilde{q}_2)\pi_n^{(\tilde{M})}(\tilde{p}_1, \tilde{p}_2)$ are $*$-eigenfunctions of the Hamiltonian as well. Obviously this function becomes a $*$-eigenfunction of the angular momentum

$$J_3 = q_1 p_2 - q_2 p_1 = -\frac{1}{\omega} H_L + \frac{m\omega}{2}(\tilde{q}_1^2 + \tilde{q}_2^2) \tag{2.136}$$

by choosing $f(\tilde{q}_1, \tilde{q}_2)$ to be a $*$-eigenfunction of $\frac{m\omega}{2}(\tilde{q}_1^2 + \tilde{q}_2^2)$. Using (2.134) only two terms in the argument of the exponential function contribute to the star product (2.133b) in this $*$-eigenvalue equation, so that

$$\begin{aligned} \frac{m\omega^2}{2}(\tilde{q}_1^2 + \tilde{q}_2^2) \;\tilde{*}_M\, f(\tilde{q}_1, \tilde{q}_2) &= \frac{m\omega^2}{2}(\tilde{q}_1^2 + \tilde{q}_2^2) \exp\left[\frac{\mathrm{i}\hbar}{2}\left(-\overleftarrow{\partial}_{\tilde{p}_1}\overrightarrow{\partial}_{q_1} - \overleftarrow{\partial}_{\tilde{p}_2}\overrightarrow{\partial}_{q_2}\right)\right] \\ &= \frac{m\omega^2}{2}(\tilde{q}_1^2 + \tilde{q}_2^2) \exp\left[\frac{\mathrm{i}\hbar}{2m\omega}\left(\overleftarrow{\partial}_{\tilde{q}_2}\overrightarrow{\partial}_{\tilde{q}_1} - \overleftarrow{\partial}_{\tilde{q}_1}\overrightarrow{\partial}_{\tilde{q}_2}\right)\right] f(\tilde{q}_1, \tilde{q}_2), \end{aligned} \tag{2.137}$$

where in the last step the definition of $\tilde{q}_i$ was used. Setting $q = \tilde{p}_2$ and $p = m\omega\tilde{q}_1$, the whole problem again reduces to the one dimensional harmonic oscillator, so that $f(\tilde{q}_1, \tilde{q}_2)$ becomes

$$\pi_l^{(\tilde{M})}(\tilde{q}_1, \tilde{q}_2) = 2(-1)^l \exp\left(-\frac{m\omega}{\hbar}(\tilde{q}_1^2 + \tilde{q}_2^2)\right) L_l\left(\frac{2m\omega}{\hbar}(\tilde{q}_1^2 + \tilde{q}_2^2)\right) \tag{2.138}$$

and the $*$-eigenvalues of $\frac{m\omega}{2}(\tilde{q}_1^2 + \tilde{q}_2^2)$ are $\hbar\left(l + \frac{1}{2}\right)$.

Thus, the Wigner functions of the Landau problem are $\pi_{nl}^{(\tilde{M})}(\tilde{q}_1, \tilde{q}_2, \tilde{p}_1, \tilde{p}_2) = \pi_l^{(\tilde{M})}(\tilde{q}_1, \tilde{q}_2)\pi_n^{(\tilde{M})}(\tilde{p}_1, \tilde{p}_2)$ and lead with the $*$-eigenvalue equation $H_L \,\tilde{*}_M\, \pi_{nl}^{(\tilde{M})} = E_n \pi_{nl}^{(\tilde{M})}$ to the Landau levels $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$, whereas the equation $J_3 \,\tilde{*}_M\, \pi_{nl}^{(\tilde{M})} = j_{nl}\pi_{nl}^{(\tilde{M})}$ gives rise to the angular momentum eigenvalues $j_{nl} = \hbar(l - n)$. The same results can be obtained with holomorphic coordinates, as it was done in [27].

It is now straightforward to include the spin, because the interaction term H_I in the Pauli Hamiltonian for $\vec{B} = (0, 0, B_3)$ can be written with (2.96) as

$$H_I = -\frac{e\hbar}{2m}\vec{\sigma} \cdot \vec{B} = -\mathrm{i}\omega\theta_1\theta_2 \tag{2.139}$$

which is nothing else than the Hamiltonian of the fermionic harmonic oscillator (2.6). Then one can combine the Clifford star product (2.95), which corresponds to the fermionic Moyal product in canonical coordinates for the harmonic oscillator, with the Moyal product (2.130) to a Moyal-Clifford product just like in (2.120). The $*$-eigenvalue equation $H_P *_{MC} \pi_{n,l,n_s}^{(MC)} = E_{n,n_s}\pi_{n,l,n_s}^{(MC)}$ then decomposes into a bosonic part that is equivalent to the Landau problem with the Wigner functions $\pi_{nl}^{(\tilde{M})}$ and energy levels $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$ and a fermionic part that is equivalent to the fermionic harmonic oscillator with Wigner functions (2.103) and energy levels $E_{n_s=\pm 1/2} = \pm\frac{\hbar\omega}{2}$. The full Wigner function for the Pauli Hamiltonian is then the product of these two, i.e. $\pi_{n,l,n_s}^{(MC)} = \pi_{n,l}^{(M)}\pi_{n_s}^{(C)}$ and the energy levels are $E_{n,n_s} = \hbar\omega\left(n + \frac{1}{2} \pm \frac{1}{2}\right)$.

2.6 The Dirac Equation

After having established the star product formalism in supersymmetric quantum mechanics and quantum mechanics with spin one can now extend the star product formalism to Dirac theory. In order to do that one uses the fact that the Dirac Hamiltonian is a supercharge with respect to the β-matrix. In the operator formalism a hermitian operator $\hat{Q} = \hat{Q}^\dagger$ is called supercharge with respect to the involution $\hat{\tau}$ if it consists just of an fermionic part [83], where the bosonic and the fermionic part of an hermitian operator can be projected out with the projection operator $\hat{\pi}_\pm = \frac{1}{2}(1 \pm \hat{\tau})$.

This can now be directly translated into the star product formalism, where the involution is $\tau = \frac{2}{\hbar}\bar{f}f$ and the projectors are $\pi_\pm = \pi_{\pm 1/2} = \frac{1}{2}(1 \pm \tau)$, see (2.43). With the projectors $\pi_\pm$ a phase space function F can then be decomposed into its even (bosonic) and odd (fermionic) part:

$$F = (\pi_+ *_M F *_M \pi_+ + \pi_- *_M F *_M \pi_-) + (\pi_+ *_M F *_M \pi_- + \pi_- *_M F *_M \pi_+) = F_B + F_F, \tag{2.140}$$

because with $\tau *_M \pi_\pm = \pm\pi_\pm$ and $[\tau, \pi_\pm]_{*_M} = 0$ follows ($*_M$ is here the fermionic Moyal product (2.33)):

$$[\pi_\pm *_M F *_M \pi_\pm, \tau]_{*_M} = 0 \qquad \text{and} \qquad \{\pi_\pm *_M F *_M \pi_\mp, \tau\}_{*_M} = 0. \tag{2.141}$$

In analogy to the operator formalism a phase space function $Q = \bar{Q}$ is then defined as a supercharge with respect to the involution τ if $Q = Q_F$, or equivalently $\{Q, \tau\}_{*_M} = 0$. An example for a supercharge are the functions (2.60). For example one easily calculates for $Q_1 = \sqrt{\frac{1}{\hbar}}(a\bar{f} + \bar{a}f)$ the star anticommutator with τ as $\{Q_1, \tau\}_{*_M} = 0$. The square of the supercharge then gives the supersymmetric Hamiltonian (see (2.61)).

The Dirac operator for a massless particle is the simplest example for a supercharge with respect to the β-matrix [83]:

$$\hat{H}_D = \hat{Q} = \begin{pmatrix} 0 & \hat{\vec{\sigma}} \cdot \hat{\vec{p}} \\ \hat{\vec{\sigma}} \cdot \hat{\vec{p}} & 0 \end{pmatrix} = \hat{D}\hat{f}^\dagger + \hat{D}^\dagger \hat{f}, \tag{2.142}$$

with $\hat{D} = \hat{D}^\dagger = \frac{1}{\sqrt{\hbar}}\hat{\vec{\sigma}} \cdot \hat{\vec{p}}$ and (2.42). The corresponding supersymmetric Hamilton operator is then defined as the square of $\hat{Q}$ divided by m in order to get the right units:

$$\hat{H}_{SU} = \frac{1}{m}\begin{pmatrix} \hbar\hat{D}^\dagger\hat{D} & 0 \\ 0 & \hbar\hat{D}\hat{D}^\dagger \end{pmatrix}, \tag{2.143}$$

which corresponds to the supersymmetric Hamilton operator if one substitutes $\hat{D}$ and $\hat{D}^\dagger$ by $\hat{B} = \mathcal{Q}(B)$ and $\hat{B}^\dagger = \mathcal{Q}(\bar{B})$. The difference is that (2.143) is a 4×4 matrix whereas the supersymmetric Hamilton operator is a 2×2 matrix.

The analogy of (2.142) in the star product formalism is then immediately given by:

$$H_D = Q = D\bar{f} + \bar{D}f, \tag{2.144}$$

with $D = \bar{D} = \frac{1}{\sqrt{\hbar}}\hat{\vec{\sigma}} \cdot \vec{p}$. It follows then, that $\{H_D, \tau\}_{*_{SU}} = 0$ for $\tau = \frac{2}{\hbar}\bar{f}f$, so that H_D is a supercharge. The supersymmetric Hamiltonian is:

$$\begin{aligned} H_{SU} = \frac{1}{m} Q *_{SU} Q &= \frac{1}{m}\left[(D *_M \bar{D})(\bar{f} *_M f) + (\bar{D} *_M D)(f *_M \bar{f})\right] \\ &= \frac{1}{m}\left[(D *_M \bar{D})\,\hbar\pi_{1/2} + (\bar{D} *_M D)\,\hbar\pi_{-1/2}\right], \end{aligned} \tag{2.145}$$

where the star product between D and $\bar{D}$ is a bosonic Moyal product and the star product between f and $\bar{f}$ is a fermionic Moyal product.

D and $\bar{D}$ are still 2×2 matrices, but one can now use the formalism of the last section to turn these matrices with the Clifford star product into Pauli functions. The σ^i in D and $\bar{D}$ are then no longer Pauli matrices but the corresponding expressions (2.96) and furthermore to the supersymmetric star product with its bosonic and fermionic Moyal part the Clifford part has to be added:

$$F *_{M\tilde{C}} G = F \exp\left[\frac{\mathrm{i}\hbar}{2}\sum_{n=1}^{3}\left(\overleftarrow{\partial}_{q_n}\overrightarrow{\partial}_{p_n} - \overleftarrow{\partial}_{p_n}\overrightarrow{\partial}_{q_n}\right) + \frac{\hbar}{2}\left(\overleftarrow{\partial}_f\overrightarrow{\partial}_{\bar{f}} + \overleftarrow{\partial}_{\bar{f}}\overrightarrow{\partial}_f\right) + \frac{\hbar}{2}\left(\sum_{n=1}^{3}\overleftarrow{\partial}_{\theta_n}\overrightarrow{\partial}_{\theta_n}\right)\right] G. \tag{2.146}$$

With this product one finds:

$$H_D *_{M\tilde{C}} H_D = (\vec{\sigma}\cdot\vec{p}) *_{M\tilde{C}} (\vec{\sigma}\cdot\vec{p}) = \vec{p}^{\,2}, \tag{2.147}$$

which corresponds to the relativistic relation between energy and momentum for massless particles: $E = |\vec{p}|$.

To describe the Hamilton function for massive particles and antiparticles one needs a generalized supercharge $H_D = Q + M *_{M\tilde{C}} \tau$, where M is a bosonic function that commutes with Q and τ: $[M, Q]_{*_{M\tilde{C}}} = [M, \tau]_{*_{M\tilde{C}}} = 0$. Take M to be of the form

$$M = M_+ \pi^{(M)}_{1/2} + M_- \pi^{(M)}_{-1/2}, \tag{2.148}$$

where $M_\pm$ are purely bosonic functions. Then

$$M *_{M\tilde{C}} \tau = M_+ \pi^{(M)}_{1/2} - M_- \pi^{(M)}_{-1/2}. \tag{2.149}$$

One sees here explicitly that the $M_\pm$ correspond to the rest mass of the particle and the antiparticle, respectively. One finds

$$\begin{aligned} H_D *_{M\tilde{C}} H_D &= (Q + M *_{M\tilde{C}} \tau) *_{M\tilde{C}} (Q + M *_{M\tilde{C}} \tau) \\ &= (\bar{D} *_{M\tilde{C}} D + M_+ *_{M\tilde{C}} M_+)\pi^{(M)}_{1/2} + (D *_{M\tilde{C}} \bar{D} + M_- *_{M\tilde{C}} M_-)\pi^{(M)}_{-1/2} \\ &= \left[p^i p^j (\sigma^i *_C \sigma^j) + (M_+ *_M M_+)\right]\pi^{(M)}_{1/2} + \left[p^i p^j (\sigma^i *_C \sigma^j) + (M_- *_M M_-)\right]\pi^{(M)}_{-1/2}. \end{aligned} \tag{2.150}$$

For the choice $M_\pm = m$ just the relativistic relation between energy and momentum for massive particles is obtained: $E^2 = |\vec{p}|^2 + m^2$.

In order to calculate the nonrelativistic limit for this Hamilton function one can use the resolvent method [106], with the resolvent $(H_D - m - z)^{-1*_{M\tilde{C}}}$, where the notation $(\ )^{-1*}$ denotes the formal inverse with respect to the star product. First define the expressions

$$A_\pm = H_D \pm m \pm z = Q \pm 2m\pi^{(M)}_{\pm 1/2} \pm z. \tag{2.151}$$

With $\pi^{(M)}_{1/2} *_{M\tilde{C}} \tilde{Q} = \tilde{Q} *_{M\tilde{C}} \pi^{(M)}_{-1/2}$ one has then

$$A_+ *_{M\tilde{C}} A_- = Q *_{M\tilde{C}} Q - 2mz - z^2 = A_- *_{M\tilde{C}} A_+. \tag{2.152}$$

This can be written as

$$\begin{aligned} A_-^{-1*_{M\tilde{C}}} &= A_+ *_{M\tilde{C}} (A_- *_{M\tilde{C}} A_+)^{-1*_{M\tilde{C}}} \\ &= \frac{1}{2m} A_+ *_{M\tilde{C}} \left(H_\infty - z - \frac{z^2}{2m}\right)^{-1*_{M\tilde{C}}}, \end{aligned} \tag{2.153}$$

with $H_\infty = \frac{1}{2m} Q *_{M\tilde{C}} Q$. Using the associativity of the star product it is easy to establish the identity

$$(A + B)^{-1*} = (1 + A^{-1*} * B)^{-1*} * A^{-1*}. \tag{2.154}$$

Using this relation with $A = H_\infty - z$ and $B = -\frac{z^2}{2m}$ equation (2.153) can be written as:

$$A_-^{-1*_{M\tilde{C}}} = \frac{1}{2m} A_+ *_{M\tilde{C}} \left(1 - \frac{z^2}{2m}(H_\infty - z)^{-1*_{M\tilde{C}}}\right)^{-1*_{M\tilde{C}}} *_{M\tilde{C}} (H_\infty - z)^{-1*_{M\tilde{C}}} \tag{2.155}$$

Substituting $A_\pm$ according to (2.151), this reads

$$(H_D - m - z)^{-1*_{M\tilde{C}}} = \left(\pi_{1/2}^{(M)} + \frac{Q+z}{2m}\right) *_{M\tilde{C}} \left(1 - \frac{z^2}{2m}(H_\infty - z)^{-1*_{M\tilde{C}}}\right)^{-1*_{M\tilde{C}}} *_{M\tilde{C}} (H_\infty - z)^{-1*_{M\tilde{C}}}. \tag{2.156}$$

The limit $m \to \infty$ is then:

$$\lim_{m\to\infty} (H_D - m - z)^{-1*_{M\tilde{C}}} = \pi_{1/2}^{(M)} *_{M\tilde{C}} (H_\infty - z)^{-1*_{M\tilde{C}}}. \tag{2.157}$$

With the idempotency of $\pi_{1/2}^{(M)}$, $[H_\infty, \pi_{1/2}^{(M)}]_{*_{M\tilde{C}}} = 0$ and $(1-x)^{-1*} = \sum_{i=0}^{\infty} x^{i*}$ one can then calculate:

$$\begin{aligned}
\pi_{1/2}^{(M)} *_{M\tilde{C}} (H_\infty - z)^{-1*_{M\tilde{C}}} &= \frac{1}{-z}\pi_{1/2}^{(M)} *_{M\tilde{C}} \left[1 + \frac{H_\infty}{z} + \left(\frac{H_\infty}{z}\right)^{2*_{M\tilde{C}}} + \cdots\right] \\
&= \frac{1}{-z}\left[\pi_{1/2}^{(M)} + \pi_{1/2}^{(M)} *_{M\tilde{C}} \pi_{1/2}^{(M)} *_{M\tilde{C}} \frac{H_\infty}{z} + \left(\pi_{1/2}^{(M)}\right)^{3*_{M\tilde{C}}} *_{M\tilde{C}} \left(\frac{H_\infty}{z}\right)^{2*_{M\tilde{C}}} + \cdots\right] \\
&= \frac{1}{-z}\pi_{1/2}^{(M)} *_{M\tilde{C}} \left[1 + \pi_{1/2}^{(M)} *_{M\tilde{C}} \frac{H_\infty}{z} + \left(\pi_{1/2}^{(M)} *_{M\tilde{C}} \frac{H_\infty}{z}\right)^{2*_{M\tilde{C}}} + \cdots\right] \\
&= \pi_{1/2}^{(M)} *_{M\tilde{C}} \left(\pi_{1/2}^{(M)} *_{M\tilde{C}} H_\infty - z\right)^{-1*_{M\tilde{C}}}
\end{aligned} \tag{2.158}$$

Putting this into equation (2.157) one can read off from the resolvent that the non-relativistic limit is the Pauli Hamiltonian:

$$H_P = \pi_{1/2}^{(M)} *_{M\tilde{C}} H_\infty = \frac{1}{2m}\pi_{1/2}^{(M)} *_{M\tilde{C}} Q *_{M\tilde{C}} Q = \frac{1}{2m}\left(\bar{D} *_{M\tilde{C}} D\right)\pi_{1/2}^{(M)}. \tag{2.159}$$

The interpretation of this equation is as follows. The projectors $\pi_{\pm 1/2}^{(M)}$ effectively project onto the subspaces describing particles and antiparticles. The projector $\pi_{1/2}^{(M)}$ indicates that the expression one is concerned with is in the positive energy sector: in the non-relativistic limit the contribution of the antiparticles vanishes. The coefficient of the projector determines the dynamics in the positive energy sector: it is just the non-relativistic Pauli Hamilton function for spin 1/2 particles: $H_P = \frac{1}{2m}\bar{D} *_{M\tilde{C}} D$.

One can now also include by hand a magnetic field by setting: $D = \bar{D} = \vec{\sigma} \cdot \left(\vec{p} - e\vec{A}\right)$, so that with (2.127) the Pauli Hamiltonian reads:

$$H_P = \frac{1}{2m}(\bar{D} *_{M\tilde{C}} D) = \frac{1}{2m}\left(\vec{p} - e\vec{A}\right)^{2*_M} - \frac{e\hbar}{2m}\vec{\sigma}\cdot\vec{B}, \tag{2.160}$$

in agreement with equation (2.126).

With the grassmannian representation of the Pauli matrices (2.96) it is also possible to give a more direct approach to Dirac theory. One uses therefore the fact that in the matrix representation the alpha and beta matrices have a tensor structure: $\hat{\alpha}^k = \hat{\sigma}^1 \otimes \hat{\sigma}^k$ and $\hat{\beta} = \hat{\sigma}^3 \otimes \hat{I}$. This can now be imitated in the grassmannian representation by starting with six grassmannian variables $\theta_1, \ldots, \theta_6$ and constructing two triples of $\sigma^k = \frac{2}{\mathrm{i}\hbar}\varepsilon^{klm}\theta_l\theta_m$, one for $k,l,m \in \{1,2,3\}$ and one for $k,l,m \in \{4,5,6\}$. The four functions defined as

$$\alpha^k = \sigma^k\sigma^4 \quad \text{for} \quad k = 1,2,3 \qquad \text{and} \qquad \beta = \sigma^6, \tag{2.161}$$

fulfill then the equations

$$\{\alpha^k, \alpha^l\}_{*_C} = 2\delta^{kl}, \qquad \{\alpha^k, \beta\}_{*_C} = 0 \qquad \text{and} \qquad \beta *_C \beta = 1, \tag{2.162}$$

where one used the Clifford star product (2.95) for $d = 6$. Conceptually Dirac's ansatz was turned around. While Dirac tried to find quantities α and β that fulfill the Dirac algebra, one can also look for a product such that the relations of the Dirac algebra are fulfilled, which leads to the Clifford star product.

In this approach to the Dirac theory one combined two copies of the three dimensional fermionic spaces which in the last section appeared to be suitable to describe spin. Thereby just the Grassmann subalgebra of even grade was used. But from the algebraic point of view one can ask whether it is necessary to use a Grassmann algebra with six generators to reproduce the Dirac algebra (2.162). Indeed, the functions

$$\alpha^k = \sqrt{\frac{2}{\hbar}}\sigma^k\theta_5 \qquad \text{and} \qquad \beta = \frac{2\mathrm{i}}{\hbar}\theta_4\theta_5, \tag{2.163}$$

also fulfill the Dirac algebra (2.162) by using five Grassmann variables and the star product (2.95) for $d = 5$. This representation corresponds to the one obtained by constructing the Dirac Hamiltonian as a supercharge from supersymmetric quantum mechanics as done above.

Since the Clifford algebra of the Dirac matrices is four dimensional it should also be possible to start with a four dimensional Grassmann algebra $\theta_1, \ldots, \theta_4$, that is turned into a Clifford algebra with the four dimensional Clifford star product. Indeed the dimensionless variables

$$\alpha^k = \sqrt{\frac{2}{\hbar}}\theta_k \qquad \text{and} \qquad \beta = \sqrt{\frac{2}{\hbar}}\theta_4 \tag{2.164}$$

obey the relations (2.162) and form another representation of the Dirac algebra. With respect to the Clifford star product the generators of the Grassmann algebra become here generators of the Clifford algebra.

This four dimensional representation of the Dirac algebra can be motivated by consideration of symmetry transformations [105]. With the definition of σ^k in equation (2.96) one could reproduce the commutation relations of the corresponding Pauli matrices and in equation (2.110) it was shown that the $S_k = \frac{\hbar}{2}\sigma^k$ generate rotations of the Grassmann algebra. So far only the even part of the Grassmann algebra was involved, so that the question arises what kind of transformation the θ_k are related to. The definition $K_k = \mathrm{i}\sqrt{\hbar/2}\,\theta_k = \mathrm{i}\frac{\hbar}{2}\,\alpha^k$ leads to the commutation relations

$$[S_k, S_l]_{*_C} = \mathrm{i}\hbar\varepsilon^{klm}S_m, \qquad [S_k, K_l]_{*_C} = \mathrm{i}\hbar\varepsilon^{klm}K_m \qquad \text{and} \qquad [K_k, K_l]_{*_C} = -\mathrm{i}\hbar\varepsilon^{klm}S_m, \tag{2.165}$$

so that $\vec{K}$ can be identified as the generator of the Lorentz boost. The star exponential $\mathrm{Exp}_C(\vec{\omega}\cdot\vec{K})$ transforms $\theta^\mu = (1, \vec{\theta})$ like a four vector:

$$\mathrm{Exp}_C\left(\vec{\omega}\cdot\vec{K}\right) *_C \theta^\mu *_C \overline{\mathrm{Exp}_C\left(\vec{\omega}\cdot\vec{K}\right)} = \mathrm{Exp}_C\left(\vec{\omega}\cdot\vec{K}\right) *_C \theta^\mu *_C \mathrm{Exp}_C\left(\vec{\omega}\cdot\vec{K}\right) = \Lambda^\mu{}_\nu(\vec{\omega})\,\theta^\nu. \tag{2.166}$$

Note that $\overline{\vec{K}} = -\vec{K}$ in contrast to $\overline{\vec{S}} = \vec{S}$, so that the sign structure is here different to that in (2.110).

Besides the continuous Lorentz transformations (2.110) and (2.166) there is also the discrete parity transformation $\mathcal{P}$ in the fermionic space, which acts as $\mathcal{P}(\vec{\theta}) = -\vec{\theta}$. This transformation cannot be represented without extending the algebra. By introducing an additional generator θ_4 to the Grassmann algebra and by extending the star exponential (2.95) to $d = 4$ a representation of the parity transformation can be given by

$$\mathcal{P}(F) = \beta *_C F *_C \beta \tag{2.167}$$

with the definition $\beta = \sqrt{2/\hbar}\,\theta_4$. As it should be the scalar 1 and the axial vector $\vec{\sigma}$ are unchanged under this transformation.

Since in the four dimensional representation (2.164) the alpha functions are just proportional to the θ_i, also $\alpha^\mu = (1, \vec{\alpha})$ transforms like a four vector, i.e. one has the transformation equations:

$$\mathrm{Exp}_C\left(\vec{\omega}\cdot\vec{K}\right) *_C \alpha^\mu *_C \mathrm{Exp}_C\left(\vec{\omega}\cdot\vec{K}\right) = \Lambda^\mu{}_\nu(\vec{\omega})\,\alpha^\nu \tag{2.168}$$

and

$$\mathrm{Exp}_C\left(\vec{\varphi}\cdot\vec{S}\right) *_C \vec{\alpha} *_C \mathrm{Exp}_C\left(-\vec{\varphi}\cdot\vec{S}\right) = R(\vec{\varphi})\,\vec{\alpha}. \tag{2.169}$$

Theses equations are independent of the representation of the alpha functions, because $K_k = \mathrm{i}\frac{\hbar}{2}\alpha^k$ and $S_k = \frac{1}{2\mathrm{i}}\varepsilon^{klm}\theta_l\theta_m = \frac{\hbar}{4\mathrm{i}}\varepsilon^{klm}\alpha^l *_C \alpha^m$ can be represented just with alpha functions so that the transformation equations (2.168) and (2.169) depend just on the algebraic behavior of the alpha functions, which is the same for all representations (2.164), (2.163) and (2.161).

The boosts (2.168) and the rotations (2.169) can be cast into one equation by going over to the functional analogue of the gamma matrices:

$$\gamma^0 = \beta \qquad \text{and} \qquad \gamma^k = \beta *_C \alpha^k \qquad \Rightarrow \qquad \{\gamma^\mu, \gamma^\nu\}_{*_C} = 2g^{\mu\nu}. \tag{2.170}$$

Star-multiplying (2.168) with β from the left and using that β anticommutes with $K_i \propto \alpha^i$, leads to

$$\mathrm{Exp}_C\left(-\vec{\omega}\cdot\vec{K}\right) *_C \gamma^\mu *_C \mathrm{Exp}_C\left(\vec{\omega}\cdot\vec{K}\right) = \Lambda^\mu{}_\nu(\vec{\omega})\,\gamma^\nu. \tag{2.171}$$

With the definition $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]_{*_C}$ the six generators of the Lorentz transformation can be written as

$$K_k = \mathrm{i}\frac{\hbar}{2}\alpha^k = \mathrm{i}\frac{\hbar}{2}\gamma^0 *_C \gamma^k = \frac{\hbar}{2}\sigma^{0k} \tag{2.172a}$$

$$S_k = -\mathrm{i}\frac{\hbar}{4}\varepsilon^{klm}\alpha^l *_C \alpha^m = \mathrm{i}\frac{\hbar}{4}\varepsilon^{klm}\gamma^k *_C \gamma^m = \frac{\hbar}{2}\sum_{l<m}\varepsilon^{klm}\sigma^{lm}. \tag{2.172b}$$

Therefore all Lorentz transformations are generated by $\frac{\hbar}{2}\sigma^{\mu\nu}$ with $\mu < \nu$. Because β commutes with $S_k \propto \varepsilon^{klm}\alpha^l *_C \alpha^m$, one can replace $\vec{\alpha}$ by $\vec{\gamma}$ in equation (2.169) and the resulting equation can finally be unified with (2.171) to

$$\mathrm{Exp}_C\left(-\frac{\hbar}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\right) *_C \gamma^\mu *_C \mathrm{Exp}_C\left(+\frac{\hbar}{4}\sigma^{\mu\nu}\omega_{\mu\nu}\right) = \Lambda^\mu{}_\nu(\omega_{\mu\nu})\,\gamma^\nu. \tag{2.173}$$

This is the usual form of Lorentz transformation known from Dirac theory.

The Clifford algebra (2.170) of the γ-functions can be constructed with each representation of the α- and β-functions (2.161), (2.163) and (2.164). For all these representations with $d = 4$, 5 or 6 generators θ_i a trace can be defined in the same way as in equation (2.100):

$$\mathrm{Tr}(F) = \frac{4}{\hbar^d}\int d\theta_d d\theta_{d-1}\dots d\theta_2 d\theta_1 \star F. \tag{2.174}$$

and with (2.170) all the well-known trace rules for the gamma matrices are reproduced. Also note that the trace $\mathrm{Tr}(F)$ projects out the scalar part of F, which is the fermionic equivalent of taking the vacuum expectation value.

With α_i and β the Dirac Hamiltonian is given by

$$H_D = \vec{\alpha}\cdot\vec{p} + \beta m \tag{2.175}$$

and by using $H_D *_{MC} H_D = \vec{p}^2 + m^2$ one can calculate the star exponential as

$$\mathrm{Exp}_{MC}(H_D t) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{\mathrm{i}\hbar}\right)^n H_D^{n *_{MC}} = \pi_{-E}^{(MC)}(\vec{p})\,e^{+\mathrm{i}tE/\hbar} + \pi_{+E}^{(MC)}(\vec{p})\,e^{-\mathrm{i}tE/\hbar}, \tag{2.176}$$

with the Wigner functions

$$\pi^{(MC)}_{\pm E}(\vec{p}) = \frac{1}{2}\left(1 \pm \frac{H_D}{E}\right) \tag{2.177}$$

and $E = \sqrt{\vec{p}^2 + m^2}$. The energy projectors $\pi^{(MC)}_{\pm E}(\vec{p})$ are idempotent, complete and fulfill the $*$-eigenvalue equations

$$H_D *_{MC} \pi^{(MC)}_{\pm E}(\vec{p}) = \pm E\, \pi^{(MC)}_{\pm E}(\vec{p}). \tag{2.178}$$

One can then find projectors that are also $*$-eigenfunctions of the spin, which is defined as $S_{\vec{s}} = \frac{\hbar}{2}\gamma^5 *_C (\vec{\gamma}\cdot\vec{s})$ with $\vec{s}$ being a unit vector orthogonal to $\vec{p}$. One has then $S_{\vec{s}} *_C S_{\vec{s}} = \left(\frac{\hbar}{2}\right)^2$, so that the star exponential for $S_{\vec{s}}$ is given by

$$\mathrm{Exp}_C(S_{\vec{s}}\,\varphi) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\varphi}{\mathrm{i}\hbar}\right)^n S_{\vec{s}}^{n*_C} = \pi^{(C)}_{-\frac{1}{2}}(\vec{s})\, e^{+\mathrm{i}\varphi/2} + \pi^{(C)}_{+\frac{1}{2}}(\vec{s})\, e^{-\mathrm{i}\varphi/2} \tag{2.179}$$

with the Wigner functions

$$\pi^{(C)}_{\pm\frac{1}{2}}(\vec{s}) = \frac{1}{2} \pm \frac{1}{\hbar} S_{\vec{s}}. \tag{2.180}$$

These are the star product analogues of the Dirac spin projectors and they obey the $*$-eigenvalue equation

$$S_{\vec{s}} *_C \pi^{(C)}_{\pm\frac{1}{2}}(\vec{s}) = \pm\frac{\hbar}{2}\pi^{(C)}_{\pm\frac{1}{2}}(\vec{s}). \tag{2.181}$$

Since one has for $\vec{p}\cdot\vec{s} = 0$:

$$\left[\beta, \gamma^5 *_C (\vec{\gamma}\cdot\vec{p})\right]_{*_C} = 0 \qquad \text{and} \qquad \left[\vec{p}\cdot\vec{\alpha}, \gamma^5 *_C (\vec{\gamma}\cdot\vec{s})\right]_{*_C} = 0, \tag{2.182}$$

so that $[H_D, S_{\vec{s}}]_{*_C} = 0$, the Wigner functions $\pi^{(MC)}_{\pm E}(\vec{p})$ and $\pi^{(C)}_{\pm\frac{1}{2}}(\vec{s})$ and the observables H_D and $S_{\vec{s}}$ commute under the star product. The Wigner function for the Dirac problem is therefore given by

$$\pi^{(MC)}_{\pm E,\pm\frac{1}{2}}(\vec{p},\vec{s}) = \pi^{(MC)}_{\pm E}(\vec{p}) *_{MC} \pi^{(C)}_{\pm\frac{1}{2}}(\vec{s}) \tag{2.183}$$

and the $*$-eigenvalue equations are

$$H_D *_{MC} \pi^{(MC)}_{\pm E,\pm\frac{1}{2}}(\vec{p},\vec{s}) = \pm E \pi^{(MC)}_{\pm E,\pm\frac{1}{2}}(\vec{p},\vec{s}) \qquad \text{and} \qquad S_{\vec{s}} *_{MC} \pi^{(MC)}_{\pm E,\pm\frac{1}{2}}(\vec{p},\vec{s}) = \pm\frac{\hbar}{2}\pi^{(MC)}_{\pm E,\pm\frac{1}{2}}(\vec{p},\vec{s}). \tag{2.184}$$

The Dirac Wigner functions are idempotent: $\pi^{(MC)}_{\pm E,\pm\frac{1}{2}}(\vec{p},\vec{s}) *_{MC} \pi^{(MC)}_{\pm E,\pm\frac{1}{2}}(\vec{p},\vec{s}) = \pi^{(MC)}_{\pm E,\pm\frac{1}{2}}(\vec{p},\vec{s})$ and with the trace (2.174) the Dirac Wigner functions (2.183) are normalized to 1.

It is now also possible to calculate the time development of the position according to

$$q_i(t) = \mathrm{Exp}_{MC}(-Ht) *_{MC} q_i *_{MC} \mathrm{Exp}_{MC}(Ht). \tag{2.185}$$

This expression can be calculated by shuffling all the powers of H_D that appear in the starexponential on the left side of q_i to the right side by using the relations

$$[H_D, q_i]_{*_{MC}} = -\mathrm{i}\hbar\alpha_i \qquad \text{and} \qquad \{H_D, \alpha_i\}_{*_{MC}} = 2p_i. \tag{2.186}$$

One obtains

$$\begin{aligned} H_D^{n*_{MC}} *_{MC} q_i &= H_D^{(n-k)*_{MC}} *_{MC} \Big[q_i *_{MC} H_D^{k*_{MC}} - \mathrm{i}\hbar\alpha_i *_{MC} H_D^{(k-1)*_{MC}} \\ &\qquad -(k-1)\mathrm{i}\hbar p_i *_{MC} H_D^{(k-2)*_{MC}}\Big] \qquad \text{for odd } k, \quad (2.187) \\ H_D^{n*_{MC}} *_{MC} q_i &= H_D^{(n-k)*_{MC}} *_{MC} \Big[q_i *_{MC} H_D^{k*_{MC}} - k\mathrm{i}\hbar p_i *_{MC} H_D^{(k-2)*_{MC}}\Big] \qquad \text{for even } k, \quad (2.188) \end{aligned}$$

so that it is possible to write

$$\begin{aligned}
&\mathrm{Exp}_{MC}(-Ht) *_{MC} q_i \\
&\quad = q_i + \sum_{k \text{ odd}} \frac{1}{k!}\left(\frac{\mathrm{i}t}{\hbar}\right)^k \left[q_i *_{MC} H_D^{k*_{MC}} - \mathrm{i}\hbar\alpha_i *_{MC} H_D^{(k-1)*_{MC}} - (k-1)\mathrm{i}\hbar p_i *_{MC} H_D^{(k-2)*_{MC}}\right] \\
&\quad\quad + \sum_{k \text{ even}} \frac{1}{k!}\left(\frac{\mathrm{i}t}{\hbar}\right)^k \left[q_i *_{MC} H_D^{k*_{MC}} - k\mathrm{i}\hbar p_i *_{MC} H_D^{(k-2)*_{MC}}\right]. \qquad (2.189)
\end{aligned}$$

Adding then

$$0 = \sum_{k \text{ odd}} \left[\frac{1}{k!}\left(\frac{\mathrm{i}t}{\hbar}\right)^k \mathrm{i}\hbar p_i *_{MC} H_D^{(k-2)*_{MC}} - \frac{1}{k!}\left(\frac{\mathrm{i}t}{\hbar}\right)^k \mathrm{i}\hbar p_i *_{MC} H_D^{(k-2)*_{MC}}\right] \qquad (2.190)$$

Eq. (2.189) turns into

$$\begin{aligned}
\mathrm{Exp}_{MC}(-Ht) *_{MC} q_i = {}& q_i *_{MC} \mathrm{Exp}_{MC}(-Ht) + p_i t *_{MC} H_D^{-1*_{MC}} *_{MC} \mathrm{Exp}_{MC}(-Ht) \\
& - \frac{\mathrm{i}\hbar}{2}\left(\alpha_i *_{MC} H_D^{-1*_{MC}} - p_i *_{MC} H_D^{-2*_{MC}}\right) *_{MC} \left(\mathrm{Exp}_{MC}(-Ht) - \mathrm{Exp}_{MC}(Ht)\right), \qquad (2.191)
\end{aligned}$$

where $H_D^{-1*_{MC}} = \frac{H_D}{\vec{p}^2 + m^2}$ is the inverse of H_D with respect to the Moyal-Clifford star product. This gives eventually for the time development

$$\begin{aligned}
q_i(t) &= \mathrm{Exp}_{MC}(-Ht) *_{MC} q_i *_{MC} \mathrm{Exp}_{MC}(Ht) \\
&= q_i + p_i t *_{MC} H_D^{-1*_{MC}} \\
&\quad + \frac{\mathrm{i}\hbar}{2}\left(\alpha_i - p_i *_{MC} H_D^{-1*_{MC}}\right) *_{MC} H^{-1*_{MC}} *_{MC} \left(\mathrm{Exp}_{MC}(2H_D t) - 1\right). \qquad (2.192)
\end{aligned}$$

The first two terms correspond to the classical movement while the last term is the well-known term that represents the Zitterbewegung.

In order to calculate the non-relativistic limit it is straightforward to translate the Foldy-Wouthuysen transformation [53] into the star product formalism. The time development of the Wigner function is given by [113]

$$\mathrm{i}\hbar \frac{\partial \pi(t)}{\partial t} = [H(t), \pi(t)]_{*_{MC}}. \qquad (2.193)$$

This can be translated into an equation for the unitary transformed Wigner function $\pi'(t) = U(t) *_{MC} \pi(t) *_{MC} U(t)^{-1}$, which leads to $\mathrm{i}\hbar\partial_t \pi'(t) = [H'(t), \pi'(t)]_{*_{MC}}$ with

$$H'(t) = U(t) *_{MC} (H(t) - \mathrm{i}\hbar\partial_t) *_{MC} U(t)^{-1}. \qquad (2.194)$$

The Hamiltonian can be written as

$$H_D = \beta m + \mathcal{E} + \mathcal{O} \qquad (2.195)$$

with

$$\beta m + \mathcal{E} = \frac{1}{2}\left(H_D + \beta *_C H_D *_C \beta\right) \quad \text{and} \quad \mathcal{O} = \frac{1}{2}\left(H_D - \beta *_C H_D *_C \beta\right).$$

The function $\mathcal{E}$ has positive parity and $\mathcal{O}$ is a function with negative parity.

Following the conventional Foldy-Wouthuysen transformation choose

$$U(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\beta}{2m} *_{MC} \mathcal{O}\right)^{*_{MC}}, \qquad (2.196)$$

so that (2.194) gives

$$
\begin{aligned}
H' &= \beta m *_{MC}\left(1+\frac{1}{2m}\mathcal{O}^{2*_{MC}}-\frac{1}{8m^3}\mathcal{O}^{4*_{MC}}\right)+\mathcal{E}-\frac{1}{8m^2}\left[\mathcal{O},\left([\mathcal{O},\mathcal{E}]_{*_{MC}}+\mathrm{i}\hbar\dot{\mathcal{O}}\right)\right]_{*_{MC}} \\
&\quad +\frac{1}{2m}\beta *_{MC}[\mathcal{O},\mathcal{E}]_{*_{MC}}-\frac{1}{3m^2}\mathcal{O}^{3*_{MC}}+\frac{\mathrm{i}\hbar}{2m}\beta *_{MC}\dot{\mathcal{O}}+\ldots, \qquad (2.197)
\end{aligned}
$$

where the first row just contains even functions only whereas the second row just consists of odd functions. This shows that (2.197) can be written as $H'=\beta m+\mathcal{E}'+\mathcal{O}'$. Repeating this transformation leads to $H''=\beta+\mathcal{E}'$, where all odd terms of the order $(\frac{1}{m^2})^2$ or higher are neglected.

For the Hamiltonian $H=\vec{\alpha}\cdot\left(\vec{p}-e\vec{A}\right)+\beta m+e\Phi$ one has

$$
\mathcal{E}=e\Phi \qquad \text{and} \qquad \mathcal{O}=\vec{\alpha}\cdot(\vec{p}-e\vec{A}). \qquad (2.198)
$$

Up to terms of order $(\frac{1}{m})^3$ in H'' the transformed Hamiltonian H'' is therefore given by

$$
\begin{aligned}
H'' &= m\beta *_{MC}\left(1+\frac{1}{2}\mathcal{O}^{2*_{MC}}-\frac{1}{8}\mathcal{O}^{4*_{MC}}\right)+m\mathcal{E}-\frac{m}{8}\left[\mathcal{O},\left([\mathcal{O},\mathcal{E}]_{*_{MC}}+\frac{\mathrm{i}\hbar}{m}\dot{\mathcal{O}}\right)\right]_{*_{MC}} \\
&= \beta\left(m+\frac{\left(\vec{p}-e\vec{A}\right)^{2*_{MC}}}{2m}-\frac{\vec{p}^4}{8m^3}\right)-\frac{e\hbar}{2m}\beta *_{MC}\vec{\sigma}\cdot\vec{B}+e\Phi \\
&\quad -\frac{e\hbar}{4m^2}\vec{\sigma}\cdot\left(\vec{E}\times\vec{p}\right)-\frac{e\hbar^2}{8m^2}\mathrm{div}\vec{E}. \qquad (2.199)
\end{aligned}
$$

In order to compare this result with the conventional operator expression one has to apply a Weyl transformation Θ_W, which transforms a product of phase space variables into the totally symmetrized product of the corresponding operators and the σ^i, α_i and β into the corresponding matrices. The Hamilton operator corresponding to (2.199) is then

$$
\begin{aligned}
\hat{H}'' &= \beta\left(m+\frac{1}{2m}\left(\hat{\vec{p}}-e\hat{\vec{A}}\right)^2-\frac{\hat{\vec{p}}^4}{8m^3}\right)-\frac{e\hbar}{2m}\beta\hat{\vec{\sigma}}\cdot\hat{\vec{B}}+e\hat{\Phi} \\
&\quad -\frac{e\hbar}{4m^2}\hat{\vec{\sigma}}\cdot\left(\hat{\vec{E}}\times\hat{\vec{p}}\right)-\frac{\mathrm{i}e\hbar^2}{8m^2}\hat{\vec{\sigma}}\cdot\mathrm{rot}\hat{\vec{E}}-\frac{e\hbar^2}{8m^2}\mathrm{div}\hat{\vec{E}}, \qquad (2.200)
\end{aligned}
$$

which is the conventional result. It was used that

$$
\mathcal{Q}_W\left(\vec{E}\times\vec{p}\right)=\frac{1}{2}\left(\hat{\vec{E}}\times\hat{\vec{p}}-\hat{\vec{p}}\times\hat{\vec{E}}\right)=\hat{\vec{E}}\times\hat{\vec{p}}+\frac{\mathrm{i}\hbar}{2}\mathrm{rot}\hat{\vec{E}}. \qquad (2.201)
$$

It is also possible to derive the Dirac equation in the star product formalism by using the fact that in the rest frame it should coincide with the $*$-eigenvalue equation (2.178). By setting $\vec{p}=0$ this equation becomes

$$
\left(\gamma^0 m\mp m\right)*_C\pi^{(MC)}_{\pm E}(0)=0 \qquad \text{with} \qquad \pi^{(MC)}_{\pm E}(0)=\frac{1}{2}\left(1\pm\gamma^0\right). \qquad (2.202)
$$

The solution $\pi^{(MC)}_{\pm E}(0)$ can also be directly obtained from (2.177). According to (2.171) the equations in (2.202) can be boosted into a moving frame by $S=\mathrm{Exp}_C(\vec{\omega}\cdot\vec{K})$, where the parameter $\vec{\omega}$ depends on the momentum $\vec{p}$ of the particle in the moving frame.

$$
S^{-1}*_C\left(\gamma^0 m\mp m\right)*_C\pi^{(MC)}_{\pm E}(0)*_C S=\left(S^{-1}*_C\gamma^0*_C S\,m\mp m\right)*_C S^{-1}\;*_C\pi^{(MC)}_{\pm E}(0)*_C S=0
$$

Equation (2.171) leads to $S^{-1} *_C \gamma^0 *_C S = \frac{\not p}{m}$ so that with the definition $\pi^{(MC)}_{\pm m}(p) = S^{-1} *_C \pi^{(MC)}_{\pm E}(0) *_C S$ the equation above turns into

$$(\not p \mp m) *_{MC} \pi^{(MC)}_{\pm m}(p) = 0 \qquad \text{with} \qquad \pi^{(MC)}_{\pm m}(p) = \frac{\pm \not p + m}{2m}, \tag{2.203}$$

which corresponds to the Dirac equation and the well-known energy projector.

The same discussion as for the Lorentz boost of the energy $*$-eigenvalue equation (2.178) can be repeated for the spin $*$-eigenvalue equation (2.181) with its solution (2.180). By assuming that $S_{\vec{s}} = \frac{\hbar}{2}\gamma_5 *_C (\vec{\gamma} \cdot \vec{s})$ is a valid spin observable in the rest frame it takes on the form $S_s = S^{-1} *_C S_{\vec{s}} *_C S = -\frac{\hbar}{2}\gamma_5 *_C \not s$ in the moving frame by applying a formal boost with $S = \mathrm{Exp}_C(\vec{\omega} \cdot \vec{K})$. The condition $\vec{s}^2 = 1$ and $\vec{p} \cdot \vec{s} = 0$ have to be translated into $s^\mu s_\mu = -1$ and $p^\mu s_\mu = 0$ respectively to ensure that $S_s *_C S_s = \left(\frac{\hbar}{2}\right)^2$ and $[S_s, H_D]_{*_C} = 0$ hold true in every frame. Finally the relativistic version of the spin $*$-eigenvalue equation and its solution become

$$S_s *_C \pi^{(C)}_{\pm\frac{1}{2}}(s) = -\frac{\hbar}{2}\gamma_5 *_C \not s *_C \pi^{(C)}_{\pm\frac{1}{2}}(s) = \pm\frac{\hbar}{2}\pi^{(C)}_{\pm\frac{1}{2}}(s) \quad \text{with} \quad \pi^{(C)}_{\pm\frac{1}{2}}(s) = \frac{1}{2} \pm \frac{1}{\hbar}S_s = \frac{1 \mp \gamma_5 *_C \not s}{2} \tag{2.204}$$

by replacing $S_{\vec{s}}$ with S_s in both (2.181) and (2.180). One can see that the spin projector $\pi^{(C)}_{\pm\frac{1}{2}}$ takes on the form which is know from Dirac theory. As in equation (2.183) the two projectors in equations (2.203) and (2.204) can be combined to the functional analogue of the Dirac projectors:

$$\pi^{(MC)}_{\pm m, \pm\frac{1}{2}}(p, s) = \pi^{(MC)}_{\pm m}(p) *_{MC} \pi^{(C)}_{\pm\frac{1}{2}}(s) = \pi^{(C)}_{\pm\frac{1}{2}}(s) *_{MC} \pi^{(MC)}_{\pm m}(p). \tag{2.205}$$

They fulfill both $*$-eigenvalue equations in (2.203) and (2.204) and they are idempotent and normalized with respect to the trace (2.174).

2.7 Fermionic Star Products and Chevalley Cliffordization

One important feature of the fermionic star product is that the Grassmann algebra of the fermionic phase space variables is transformed into a Clifford algebra. In mathematics this concept is known as Chevalley Cliffordization [20]. In this section it will be shown, that the physically motivated deformation with fermionic star products is equivalent to Chevalley Cliffordization. Starting point is a Grassmann algebra

$$\bigwedge V = \mathbb{R} \oplus V \oplus (V \wedge V) \oplus \ldots \oplus (\wedge_n V) \oplus \ldots, \tag{2.206}$$

where V is a vector space with Grassmann basis $\{\theta_1, \ldots, \theta_n\}$. In Chevalleys approach a Clifford algebra is constructed as the endomorphism algebra on the linear space of the Grassmann algebra. In order to achieve this one defines an element $i_{\theta_i} \in V^*$ by

$$i_{\theta_i}(\theta_j) = \theta_i \mathbin{\lrcorner_B} \theta_j = B(\theta_i, \theta_j) = g(\theta_i, \theta_j) + A(\theta_i, \theta_j), \tag{2.207}$$

where $B(\theta_i, \theta_j)$ is a bilinear form that consists of a symmetric part $g(\theta_i, \theta_j)$ and an antisymmetric part $A(\theta_i, \theta_j)$. If $i_{\theta_i}(\theta_j) = \delta_{ij}$ then i_{θ_i} is called the euclidian dual isomorphism. The so defined action of V^* on V can be extended to monomials of the θ_i if one requires for homogenous $u, v, w \in \bigwedge V$ the following rules:

$$\theta_i \mathbin{\lrcorner_B} \theta_j = B(\theta_i, \theta_j) \tag{2.208a}$$

$$\theta_i \mathbin{\lrcorner_B} (uv) = (\theta_i \mathbin{\lrcorner_B} u)v + (-1)^{\epsilon(u)} u(\theta_i \mathbin{\lrcorner_B} v) \tag{2.208b}$$

$$(uv) \mathbin{\lrcorner_B} w = u \mathbin{\lrcorner_B} (v \mathbin{\lrcorner_B} w), \tag{2.208c}$$

where $\epsilon(u)$ is the Grassmann grade of u. The extension to arbitrary $u, v, w \in \bigwedge V$ is then straightforward. From (2.208b) with $u = v = 1$ it is clear that $\theta_i \underset{B}{\lrcorner} 1 = 0$ and from (2.208c) with $u = v = 1$ it follows that $1 \underset{B}{\lrcorner} u = u$. For any two homogeneous u and v the equation

$$\epsilon(u \underset{B}{\lrcorner} v) = \epsilon(v) - \epsilon(u). \tag{2.209}$$

holds true as follows from equations (2.208a) and (2.208b).

For $u \in \bigwedge V$ one can define the operator

$$\gamma^B_{\theta_i} := \begin{cases} \bigwedge V & \rightarrow & \bigwedge V \\ u & \mapsto & \gamma^B_{\theta_i} u \ := \ \theta_i u + \theta_i \underset{B}{\lrcorner} u \end{cases} . \tag{2.210}$$

The map

$$\gamma^B := \begin{cases} V & \rightarrow & \mathcal{C}\ell(B, V) \\ \theta_i & \mapsto & \gamma^B_{\theta_i} := \theta_i \cdot + \theta_i \underset{B}{\lrcorner} \end{cases} , \tag{2.211}$$

where $\mathcal{C}\ell(B, V)$ is a Clifford algebra on V, is called Clifford map and it follows $\mathcal{C}\ell(B, V) \subset \mathrm{End}\,(\bigwedge V)$. With (2.208) it is easy to calculate how two $\gamma^B_{\theta_i}$ act on an arbitrary $u \in \bigwedge V$:

$$\gamma^B_{\theta_i}\gamma^B_{\theta_j} u = B(\theta_i, \theta_j)u + (\theta_i\theta_j)\underset{B}{\lrcorner} u + \theta_i\theta_j u + \big[\theta_i(\theta_j \underset{B}{\lrcorner} u) - \theta_j(\theta_i \underset{B}{\lrcorner} u)\big]. \tag{2.212}$$

The Clifford structure as endomorphisms on $\bigwedge V$ becomes obvious by taking the part that is symmetric in θ_i and θ_j:

$$\{\gamma^B_{\theta_i}, \gamma^B_{\theta_j}\}u = (\gamma^B_{\theta_i}\gamma^B_{\theta_j} + \gamma^B_{\theta_j}\gamma^B_{\theta_i})u = 2g(\theta_i, \theta_j)u. \tag{2.213}$$

Note that the Clifford algebra is constructed with the bilinear form B, but that for the Clifford structure only the symmetric part g is important. This situation is similar to the one encountered in the star product formalism, where different star products have the same antisymmetric part so that the anticommutator leads to the Poisson bracket.

The last point suggests that there is a connection between the Chevalley Cliffordization described so far and the fermionic version of the twisted product:

$$u \circ_B v = u \exp\left(\sum_{i,j} B(\theta_i, \theta_j)\overleftarrow{\partial}_{\theta_i}\overrightarrow{\partial}_{\theta_j}\right) v. \tag{2.214}$$

Considering without loss of generality two monomials u and v the n-th term in the expansion of $u \circ_B v$ is of grade $\epsilon(u) + \epsilon(v) - 2n$ one can compare the $\epsilon(u)$-th term with $u \underset{B}{\lrcorner} v$, which is of the same grade $\epsilon(v) - \epsilon(u)$ as was stated in equation (2.209). In fact, both turn out to be identical, i.e.

$$\begin{aligned} u \underset{B}{\lrcorner} v &= \frac{1}{\epsilon(u)!}\, u \left(\sum_{i,j} B(\theta_i, \theta_j)\overleftarrow{\partial}_{\theta_i}\overrightarrow{\partial}_{\theta_j}\right)^{\epsilon(u)} v \\ &= u \left(\sum_{\sum_{i,j=1}^{n} k_{ij} = \epsilon(u)} \prod_{i,j=1}^{n} \left(B(\theta_i, \theta_j)\overleftarrow{\partial}_{\theta_i}\overrightarrow{\partial}_{\theta_j}\right)^{k_{ij}}\right) v, \end{aligned} \tag{2.215}$$

where one uses the fact that the k_{ij} are either 1 or 0. To prove this equality one has to show that the three axioms of (2.208) are fulfilled. Using (2.215) the first axiom is trivial, the second one reduces to the Leibniz rule

$$\begin{aligned}
\theta_i \underset{B}{\lrcorner}(uv) &= \sum_j B(\theta_i,\theta_j)\vec{\partial}_{\theta_j}(uv) \\
&= \sum_j B(\theta_i,\theta_j)\left[(\partial_{\theta_j}u)v + (-1)^{\epsilon(u)}u(\partial_{\theta_j}v)\right] \\
&= (\theta_i \underset{B}{\lrcorner} u)v + (-1)^{\epsilon(u)}u(\theta_i \underset{B}{\lrcorner} v)
\end{aligned} \tag{2.216}$$

and a proof of (2.208c) can be found in appendix A. Therefore $u \underset{B}{\lrcorner} v$ is equal to the term of the expansion of $u \circ_B v$ in which all basis elements of θ_i in u are cancelled by corresponding derivatives $\overleftarrow{\partial}_{\theta_i}$. Such a term will only exist if $\epsilon(u) \leq \epsilon(v)$ and if the necessary derivatives appear, i.e. the corresponding $B(\theta_i,\theta_j)$ have to be non-zero.

One can now formulate the Clifford map with the help of a circle product as

$$\gamma^B_{\theta_i} u = \left(\theta_i + \theta_i \underset{B}{\lrcorner}\right) u = \theta_i \circ_B u. \tag{2.217}$$

With this notation (2.212) reads

$$\begin{aligned}
\gamma^B_{\theta_i}\gamma^B_{\theta_j} u &= \theta_i \circ_B \theta_j \circ_B u \\
&= \theta_i\theta_j u + \sum_{k,l} B(\theta_j,\theta_k)B(\theta_i,\theta_l)\vec{\partial}_{\theta_l}\vec{\partial}_{\theta_k} u + B(\theta_i,\theta_j)u \\
&\quad + \theta_i \sum_k B(\theta_j,\theta_k)\vec{\partial}_{\theta_k} u - \theta_j \sum_l B(\theta_i,\theta_l)\vec{\partial}_{\theta_l} u.
\end{aligned} \tag{2.218}$$

Therefore the anticommutator (2.213) can be written as

$$\{\gamma^B_{\theta_i},\gamma^B_{\theta_j}\} = \{\theta_i,\theta_j\}_{\circ_B} = 2g(\theta_i,\theta_j). \tag{2.219}$$

The same constructions that were described here can also be applied to the bosonic case in a similar manner.

The important point is now that having established $\gamma^B_{\theta_i} = \theta_i \circ_B$ one can use this as a definition for the Chevalley cliffordization map instead of definition (2.211). Defining the Chevalley cliffordization with the circle product allows us to generalize the definition (2.211) to monomials u of arbitrary Grassmann grade:

$$\tilde{\gamma}^B_u v = u \circ_B v. \tag{2.220}$$

With this generalized definition of the cliffordization map one immediately sees that the cliffordization is a homomorphism, because one trivially has:

$$\tilde{\gamma}^B_u \tilde{\gamma}^B_v = \tilde{\gamma}^B_{u\circ_B v}, \tag{2.221}$$

which is not true for the Clifford map (2.211).

The choice of the bilinear form $B(\theta_i,\theta_j)$ specifies the Clifford map γ^B defined in equation (2.211). Starting from a Clifford map specified by a symmetric bilinear form, i.e. $B(\theta_i,\theta_j) = g(\theta_i,\theta_j)$ Fauser used in [47] the concept of the so-called Wick isomorphism $e^{-F}\mathcal{C}\ell(g,V)e^{+F}$ to induce an additional antisymmetric scalar part determined by $F = F^{ij}\theta_i\theta_j$. It is here important to note that $e^{-F}\mathcal{C}\ell(g,V)e^{+F} \neq \mathcal{C}\ell(B,V)$. In order to see what the Wick isomorphism does and what it does not do, first consider the connection of $\underset{B}{\lrcorner}$ and $\underset{g}{\lrcorner}$, for which one has

$$\theta_i \underset{B}{\lrcorner} u = \sum_k \left(g(\theta_i,\theta_k) + A(\theta_i,\theta_k)\right)\vec{\partial}_{\theta_k} u = \theta_i \underset{g}{\lrcorner} u + \theta_i \underset{A}{\lrcorner} v, \tag{2.222}$$

but this does not generalize to $\gamma^B_{\theta_i}$, i.e. $\gamma^B_{\theta_i} u \neq \left(\gamma^g_{\theta_i} + \gamma^A_{\theta_i}\right) u$. So the connection of the two Clifford maps $\gamma^B_{\theta_i}$ and $\gamma^g_{\theta_i}$ cannot to be established by a simple linear combination. But also the Wick isomorphism cannot transform $\gamma^g_{\theta_i}$ into $\gamma^B_{\theta_i}$. To see this and to see how the Wick isomorphism transforms $\gamma^g_{\theta_i}$ one has to calculate $e^{-F}\gamma^g_{\theta_i} e^F u$. Therefore one first calculates

$$\theta_i \underset{B}{\lrcorner} F = \sum_j B(\theta_i, \theta_j)\vec{\partial}_{\theta_j} F^{kl}\theta_k\theta_l = \sum_j 2B(\theta_i,\theta_j)F^{jk}\theta_k, \tag{2.223}$$

where (2.215) was used. With equation (2.216) this leads to

$$\theta_i \underset{B}{\lrcorner} F^n = n(\theta_i \underset{B}{\lrcorner} F)F^{n-1} \qquad \Rightarrow \qquad \theta_i \underset{B}{\lrcorner} e^F = (\theta_i \underset{B}{\lrcorner} F)e^F, \tag{2.224}$$

so that one gets

$$e^{-F}\left[\theta_i \underset{B}{\lrcorner} (e^F u)\right] = \theta_i \underset{B}{\lrcorner} u + (\theta_i \underset{B}{\lrcorner} F)u. \tag{2.225}$$

With these equations one can eventually calculate

$$e^{-F}\gamma^g_{\theta_i} e^F u = \theta_i u + \theta_i \underset{g}{\lrcorner} u + (\theta_i \underset{g}{\lrcorner} F)u, \tag{2.226}$$

which is different from $\gamma^B_{\theta_i} u$! So the Wick isomorphism does not transform the $\gamma^g_{\theta_i}$ into $\gamma^B_{\theta_i}$, but it changes the term that amounts to a multiplication of u with a scalar. Such a term does not exist in the case of just one Clifford map as considered in (2.226). Terms where just a scalar is multiplied appear first in the case of two Clifford maps:

$$\begin{aligned} e^{-F}\left(\gamma^g_{\theta_i}\gamma^g_{\theta_j} e^F u\right) \quad = \quad & \theta_i\theta_j u + g(\theta_i,\theta_j)u + \theta_i(\theta_j \underset{g}{\lrcorner} F)u - \theta_j(\theta_i \underset{g}{\lrcorner} F)u \\ & +\theta_i(\theta_j \underset{g}{\lrcorner} u) - \theta_j(\theta_i \underset{g}{\lrcorner} u) + (\theta_i \underset{g}{\lrcorner} F)(\theta_j \underset{g}{\lrcorner} u) - (\theta_j \underset{g}{\lrcorner} F)(\theta_i \underset{g}{\lrcorner} v) + \theta_i \underset{g}{\lrcorner}(\theta_j \underset{g}{\lrcorner} u) - (\theta_j \underset{g}{\lrcorner} F)(\theta_i \underset{g}{\lrcorner} F)u \\ & +(\theta_i \underset{g}{\lrcorner}(\theta_j \underset{g}{\lrcorner} F))u \end{aligned} \tag{2.227}$$

namely the terms $g(\theta_i,\theta_j)u$ and $(\theta_i \underset{g}{\lrcorner}(\theta_j \underset{g}{\lrcorner} F))u$. It is always possible to choose F, such that

$$\theta_i \underset{g}{\lrcorner}(\theta_j \underset{g}{\lrcorner} F) = (\theta_i\theta_j) \underset{g}{\lrcorner} F = A(\theta_i,\theta_j), \tag{2.228}$$

which is shown explicitly in the following calculation, where (2.215) is used:

$$\begin{aligned} \theta_i \underset{g}{\lrcorner}(\theta_j \underset{g}{\lrcorner} F) \quad &= \quad \theta_i \underset{g}{\lrcorner}\left(\sum_k g(\theta_j,\theta_k)\vec{\partial}_{\theta_k} F^{rs}\theta_r\theta_s\right) \\ &= \quad \theta_i \underset{g}{\lrcorner}\left(2\sum_r g(\theta_j,\theta_r)F^{rs}\theta_s\right) \\ &= \quad 2\sum_{r,s} F^{rs} g(\theta_i,\theta_s)g(\theta_j,\theta_r) = A(\theta_i,\theta_j). \end{aligned} \tag{2.229}$$

So the Wick isomorphism has induced an antisymmetric scalar term $A(\theta_i,\theta_j)$ that combines with the symmetric term $g(\theta_i,\theta_j)$ to $B(\theta_i,\theta_j)$. If one then forms the anticommutator

$$\{e^{-F}\gamma^g_{\theta_i} e^F, e^{-F}\gamma^g_{\theta_j} e^F\} = 2g(\theta_i,\theta_j) \tag{2.230}$$

one sees that the anticommutator is not changed, because the antisymmetric part induced by the Wick isomorphism is cancelled out.

Just as here two different Clifford maps were compared one can also compare two different circle products. In the circle product formalism different circle-products are c-equivalent if they are related by the T-transformation

$$u \circ' v = T^{-1}(Tu \circ Tv), \tag{2.231}$$

with $T = \exp\left(T^{ij}\vec{\partial}_{\theta_i}\vec{\partial}_{\theta_j}\right)$. Transforming now the Clifford maps into the circle product notation as in (2.217) one notices that the Wick isomorphism is not a transformation that transforms $\circ_g$ into $\circ_B$, like a T-transformation would do. This can be seen from the simple fact that for $u = 1$ equation (2.227) leads to

$$\begin{aligned} e^{-F}\left(\gamma^g_{\theta_i}\gamma^g_{\theta_j} e^F\right) &= e^{-F}\left(\theta_i \circ_g \theta_j \circ_g e^F\right) \\ &= \theta_i\theta_j + g(\theta_i,\theta_j) + (\theta_i \underset{g}{\lrcorner} F)\theta_j + \theta_i(\theta_j \underset{g}{\lrcorner} F) + (\theta_i \underset{g}{\lrcorner} F)(\theta_j \underset{g}{\lrcorner} F) + \theta_i \underset{g}{\lrcorner}(\theta_j \underset{g}{\lrcorner} F), \end{aligned} \tag{2.232}$$

where more than one term of order two appears, while in $\theta_i \circ_B \theta_j$ there is just the term $\theta_i\theta_j$ of order two. The Wick isomorphism does not lead to a T-transformation of the corresponding circle product but it induces an antisymmetric scalar part and this scalar part is just the scalar part of the T-transformed circle product. So if $\langle F\rangle_0$ projects on the scalar part of F there is the following

Theorem 1

$$\langle \theta_{i_1} \circ_B \cdots \circ_B \theta_{i_n}\rangle_0 = \langle e^{-F}\left(\theta_{i_1} \circ_g \cdots \circ_g \theta_{i_n} \circ_g e^F\right)\rangle_0. \tag{2.233}$$

First consider the case of n being odd. Since the circle product always contracts an even number of basis elements, it is clear that $\langle \theta_{i_1} \circ_B \cdots \circ_B \theta_{i_{2m+1}}\rangle_0 = 0$. The same argument shows that on the right hand side $e^{-F}(\theta_{i_1} \circ_g \cdots \theta_{i_{2m+1}} \circ_g e^F)$ reduces to terms of the form $e^{-F}((\theta_{i_1} \cdots \theta_{i_{2m'+1}}) \circ_g e^F)$ with $m' = 0, \ldots, m$. But expanding the circle product leads to terms which all have grade higher than zero. This is because in order to reduce $\theta_{i_1} \cdots \theta_{i_{2m'+1}}$ to a constant one needs $2m'+1$ derivatives from $\circ_g$, but then there are also $2m'+1$ derivatives acting on e^F. An odd number of derivatives of e^F cannot create an constant term in the inner derivatives, since F has an even grade. So for odd n both sides of the theorem are zero.
For an even n the left hand side gives

$$\begin{aligned} \langle \theta_{i_1} \circ_B \cdots \circ_B \theta_{i_{2m}}\rangle_0 &= \sum_{\sigma\in S_{2m}} (-1)^\sigma B(\theta_{\sigma(i_1)},\theta_{\sigma(i_2)}) \cdots B(\theta_{\sigma(i_{2m-1})},\theta_{\sigma(i_{2m})}) \\ &= \sum_{\sigma\in S_{2m}} (-1)^\sigma (g(\theta_{\sigma(i_1)},\theta_{\sigma(i_2)}) + A(\theta_{\sigma(i_1)},\theta_{\sigma(i_2)})) \\ &\qquad \cdots (g(\theta_{\sigma(i_{2m-1})},\theta_{\sigma(i_{2m})}) + A(\theta_{\sigma(i_{2m-1})},\theta_{\sigma(i_{2m})})) \\ &= \sum_{\sigma\in S_{2m}} (-1)^\sigma \sum_{X=g,A} X(\theta_{\sigma(i_1)},\theta_{\sigma(i_2)}) \cdots X(\theta_{\sigma(i_{2m-1})},\theta_{\sigma(i_{2m})}). \end{aligned}$$

In order to calculate the right hand side one first notices that

$$\begin{aligned} \theta_{i_1} \circ_g \cdots \circ_g \theta_{i_{2m}} &= \theta_{i_1} \cdots \theta_{i_{2m}} + \sum_{\sigma\in S_{2m}} (-1)^\sigma \left[g(\theta_{\sigma(i_1)},\theta_{\sigma(i_2)})\theta_{\sigma(i_3)} \cdots \theta_{\sigma(i_{2m})}\right. \\ &\qquad + g(\theta_{\sigma(i_1)},\theta_{\sigma(i_2)}) g(\theta_{\sigma(i_3)},\theta_{\sigma(i_4)})\theta_{\sigma(i_5)} \cdots \theta_{\sigma(i_{2m})} \\ &\qquad \left. + \cdots + g(\theta_{\sigma(i_1)},\theta_{\sigma(i_2)}) \cdots g(\theta_{\sigma(i_{2m-1})},\theta_{\sigma(i_{2m})})\right], \end{aligned} \tag{2.234}$$

which corresponds to the Wick theorem. In each term of (2.234) one has the product of an even number of θ_i, which has to be circle multiplied with e^F. Using (2.229) it is easy to see that

$$\langle(\theta_{i_1} \cdots \theta_{i_{2r}}) \circ_g e^F\rangle_0 = \sum_{\sigma\in S_{2r}} (-1)^\sigma A(\theta_{\sigma(i_1)},\theta_{\sigma(i_2)}) \cdots A(\theta_{\sigma(i_{2r-1})},\theta_{\sigma(i_{2r})}). \tag{2.235}$$

Combining the results (2.234) and (2.235) one gets:

$$\langle e^{-F}(\theta_{i_1} \circ_g \cdots \circ_g \theta_{i_{2m}} \circ_g e^F)\rangle_0 = \sum_{\sigma \in S_{2m}} (-1)^\sigma \sum_{X=g,A} X(\theta_{\sigma(i_1)}, \theta_{\sigma(i_2)}) \cdots X(\theta_{\sigma(i_{2m-1})}, \theta_{\sigma(i_{2m})}), \tag{2.236}$$

which finishes the proof.□

This theorem tells us that the application of the Wick isomorphism leads to the same change in the scalar component as a c-equivalence transformation. The projection onto the scalar component corresponds physically to taking the vacuum expectation value. As shown above a T-transformation results just in a shift of the spectrum, i.e. it changes the vacuum, which is also the result of a Wick isomorphism [45]. Moreover, the situation is analogous to the bosonic case. In the bosonic case all star products in $\mathbb{R}^{2n}$ are c-equivalent and have the same antisymmetric scalar part that constitutes the Poisson bracket. In the fermionic case one uses antisymmetric variables, so all Clifford maps equivalent under the Wick isomorphism lead to the same symmetric scalar part. So while in the bosonic case the antisymmetric part is important, in the fermionic case the symmetric part is important, both physically and mathematically. Physically because it constitutes the fermionic Moyal bracket and mathematically because in mathematics one uses a symmetric bilinear form in order to construct a Clifford algebra.

Having shown that the star product formalism leads to a cliffordization clarifies on the one hand the mathematical nature of a fermionic deformation, but on the other hand it also allows to subsume various attempts to describe physics with the help of Chevalley cliffordization under the realm of the star product formalism which makes the underlying structures much clearer. Chevalley cliffordization was for example used to describe the Dirac equation in [44] and to describe the Wick theorem in [48]. This approach was the starting point for many investigations of the algebraic structures appearing [50]. In one of the next sections it will be shown how these structures appear in the star product formalism in the most natural way.

Chapter 3

Star Products in Quantum Field Theory

After having established the star product formalism in nonrelativistic quantum mechanics and Dirac theory one can now proceed to apply the star products in quantum field theory. For this purpose the approach of Curtright and Zachos can be used, who generalized the derivatives of the Moyal product to functional derivatives. In contrast to their work here the normal product will be used, which allows one to connect the star product formalism with the work of Brouder and Oeckl, who investigated the algebraic structure of quantum field theory.

3.1 The Forced Harmonic Oscillator

Before coming to quantum field theory one can first consider the harmonic oscillator in interaction with a time-dependent external source $J(t)$. The classical Hamilton function is

$$H = \omega a\bar{a} - J(t)\bar{a} - \bar{J}(t)a. \tag{3.1}$$

In the case of the normal star product the star exponential or time evolution function U_J is then characterized by the differential equation

$$\mathrm{i}\hbar \frac{d}{dt} U_J(t, t_i) = \left[H + \hbar(\omega\bar{a} - \bar{J}(t))\partial_{\bar{a}}\right] U_J(t, t_i), \tag{3.2}$$

which has the solution

$$\begin{aligned} U_J(t_f, t_i) \quad = \quad & e^{-a\bar{a}/\hbar} \exp\left[\frac{1}{\hbar} a\bar{a} e^{\mathrm{i}\omega(t_f - t_i)} + \frac{\mathrm{i}}{\hbar} a e^{\mathrm{i}\omega t_f} \int_{t_i}^{t_f} ds\, e^{-\mathrm{i}\omega s} \bar{J}(s)\right. \\ & \left. + \frac{\mathrm{i}}{\hbar} \bar{a} e^{-\mathrm{i}\omega t_f} \int_{t_i}^{t_f} ds\, e^{\mathrm{i}\omega s} J(s) - \frac{1}{\hbar} \int_{t_i}^{t_f} ds \int_{s}^{t_f} du\, e^{\mathrm{i}\omega(u-s)} J(s)\bar{J}(u)\right]. \end{aligned} \tag{3.3}$$

In the scattering situation one requires that the source term becomes negligible as $|t| \to \infty$. The asymptotic dynamics is then governed by the classical Hamilton function for the free system: $H_0 = H|_{J=0}$. The scattering function relates the asymptotic in-states to the asymptotic out-states, where the source term is effective only in an at first limited time interval $-T < t < T$:

$$S[J] = \lim_{T\to\infty} U(0, T) *_N U_J(T, -T) *_N U(-T, 0). \tag{3.4}$$

The phase space variables a and $\bar{a}$ develop in time under the influence of the free time evolution function $U(t,0) = \mathrm{Exp}_N(Ht)$ as solutions of the free equations of motion. One finds then with (1.95) for a general function $f(a,\bar{a})$

$$\begin{aligned}
& \mathrm{Exp}_N(-Ht_2) *_N f(a,\bar{a}) *_N \mathrm{Exp}_N(Ht_1) \\
= \; & \mathrm{Exp}_N(-Ht_2) *_N \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(\partial_a^n f(a,\bar{a})\right) \left(-a + a e^{-\mathrm{i}\omega t_1}\right)^n \mathrm{Exp}_N(Ht_1) \right] \\
= \; & \mathrm{Exp}_N(-Ht_2) *_N \left[f(a e^{-\mathrm{i}\omega t_1}, \bar{a}) \mathrm{Exp}_N(Ht_1) \right] \\
= \; & \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\bar{a} + \bar{a} e^{\mathrm{i}\omega t_2}\right)^n \mathrm{Exp}_N(-Ht_2) \sum_{m=0}^{\infty} \frac{1}{\hbar^m} \binom{n}{m} \left(\partial_{\bar{a}}^{n-m} f(a e^{-\mathrm{i}\omega t_1}, \bar{a})\right) \left(-a + a e^{-\mathrm{i}\omega t_1}\right)^m \mathrm{Exp}_N(Ht_1) \\
= \; & \mathrm{Exp}_N(Ht_1) \mathrm{Exp}_N(-Ht_2) \sum_{n,m=0}^{\infty} \frac{1}{\hbar^m} \frac{1}{m!n!} \left(-a + a e^{-\mathrm{i}\omega t_1}\right)^m \left(-\bar{a} + \bar{a} e^{\mathrm{i}\omega t_2}\right)^{m+n} \partial_{\bar{a}}^n f(a e^{-\mathrm{i}\omega t_1}, \bar{a}) \\
= \; & f(a e^{-\mathrm{i}\omega t_1}, \bar{a}^{\mathrm{i}\omega t_2}),
\end{aligned} \tag{3.5}$$

so that one has

$$U(0,T) *_N f(a,\bar{a}) *_N U(-T,0) = f\left(a e^{-\mathrm{i}\omega T}, \bar{a} e^{\mathrm{i}\omega T}\right). \tag{3.6}$$

For the harmonic oscillator with a time-dependent source this yields, from (3.3):

$$S[J] = \exp\left[\frac{\mathrm{i}}{\hbar} a \bar{j}(\omega) + \frac{\mathrm{i}}{\hbar} \bar{a} j(\omega) - \frac{1}{2\hbar} \iint ds\, du\, e^{-\mathrm{i}\omega|s-u|} J(s) \bar{J}(u) \right], \tag{3.7}$$

where $j(\omega) = \int ds J(s) e^{\mathrm{i}\omega s}$ is the Fourier transform of $J(s)$. Let $\phi(t) = a e^{-\mathrm{i}\omega t} + \bar{a} e^{\mathrm{i}\omega t}$, and let $J(t)$ be real. Then (3.7) may be written as

$$S[J] = e^{\frac{\mathrm{i}}{\hbar} \int dt J(t)\phi(t)} \exp\left[-\frac{1}{2\hbar^2} \iint dt dt' J(t) D_F(t-t') J(t') \right], \tag{3.8}$$

with

$$D_F(t) = \hbar \left[\theta(t) e^{-\mathrm{i}\omega t} + \theta(-t) e^{\mathrm{i}\omega t} \right] \tag{3.9}$$

As shown below the scattering function $S[J]$ corresponds to the scattering operator of quantum field theory, and $D_F(t)$ corresponds to the Feynman propagator (this correspondence is the reason for the factor $\hbar$ in the above equation). The generating functional is the vacuum expectation value of the scattering operator. In the phase space formalism this quantity can be calculated with $\pi_0^{(N)} = e^{-a\bar{a}/\hbar}$ (see (1.97)) as

$$\begin{aligned}
Z[J] &= \frac{1}{2\pi\hbar} \int d^2a\, S[J] *_N \pi_0^{(N)} \\
&= \frac{1}{2\pi\hbar} \int d^2a\, S[J] e^{-\overleftarrow{\partial}_a a} \pi_0^{(N)} \\
&= \frac{1}{2\pi\hbar} \int d^2a\, \exp\left[\frac{\mathrm{i}}{\hbar} \bar{a} j(\omega) - \frac{1}{2\hbar^2} \iint dt dt' J(t) D_F(t-t') J(t') \right] e^{-a\bar{a}/\hbar} \\
&= \exp\left[-\frac{1}{2\hbar^2} \iint dt dt' J(t) D_F(t-t') J(t') \right] \frac{1}{2\pi\hbar} \int d^2a\, \exp\left[-\frac{1}{\hbar} a\bar{a} + \frac{\mathrm{i}}{\hbar} \bar{a} j(\omega) \right] \\
&= \exp\left[-\frac{1}{2\hbar^2} \iint dt dt' J(t) D_F(t-t') J(t') \right],
\end{aligned} \tag{3.10}$$

where in the last step the normalization of the Wigner function was used. Decomposing $D_F(t-t')$ into the real and imaginary parts according to

$$D_F(t) = \hbar \left[\cos(\omega t) - \mathrm{i}\epsilon(t) \sin(\omega t) \right], \tag{3.11}$$

one can write (3.10) with $\epsilon(t) = 2\theta(t) - 1$ as

$$\begin{aligned}
&\exp\left[-\frac{1}{2\hbar^2}\iint dt dt' J(t) D_F(t-t') J(t')\right] \\
=\ &\exp\left[-\frac{1}{2\hbar}\iint dt dt' \left[J(t)\left(\frac{1}{2}\left(e^{\mathrm{i}\omega(t-t')} + e^{-\mathrm{i}\omega(t-t')}\right) + \left(\theta(t)-\frac{1}{2}\right)\left(e^{-\mathrm{i}\omega(t-t')} - e^{\mathrm{i}\omega(t-t')}\right)\right) J(t')\right]\right] \\
=\ &e^{-|j(\omega)|^2/2\hbar}\exp\left[-\frac{1}{2\hbar^2}\iint dt dt' J(t) D_R(t-t') J(t')\right], \qquad (3.12)
\end{aligned}$$

where

$$D_R(t) = \hbar\theta(t)(e^{-\mathrm{i}\omega t} - e^{\mathrm{i}\omega t}) = -2\mathrm{i}\hbar\theta(t)\sin(\omega t) \qquad (3.13)$$

is the retarded propagator. So up to a phase given by the retarded propagator the expression (3.10) is equal to $\exp\left[-|j(\omega)|^2/2\hbar\right]$.

One can also calculate off-diagonal matrix elements of the scattering operator by making use of the Wigner functions

$$\pi^{(N)}_{m,n} = \frac{1}{\sqrt{\hbar^m \hbar^n m! n!}}\pi^{(N)}_0 \bar{a}^m a^n, \qquad (3.14)$$

which are obviously straightforward generalizations of the projectors: $\pi^{(N)}_n = \pi^{(N)}_{n,n}$. The transition amplitude for the system to go from the ground state to the state with energy E_n under the influence of the source is then given by

$$\begin{aligned}
\mathrm{Amp}(0 \to n) &= \frac{1}{2\pi\hbar}\int d^2a\, \pi^{(N)}_{0,n} *_N S[J] *_N \pi^{(N)}_0 \\
&= \frac{1}{2\pi\hbar}\int d^2a\, \pi^{(N)}_{0,n} *_N \left(\exp\left[\frac{\mathrm{i}}{\hbar}\bar{a}j(\omega) - \frac{1}{2\hbar^2}\iint dt dt' J(t) D_F(t-t') J(t')\right] e^{-a\bar{a}/\hbar}\right) \\
&= \exp\left[-\frac{1}{2\hbar^2}\iint dt dt' J(t) D_F(t-t') J(t')\right] \frac{1}{2\pi\hbar}\int d^2a\, \frac{a^n}{\sqrt{\hbar^n n!}} e^{-a\bar{a}/\hbar} e^{\overleftarrow{\partial}_a(-a+\mathrm{i}j)} e^{-a\bar{a}/\hbar + \mathrm{i}\bar{a}j/\hbar} \\
&= \exp\left[-\frac{1}{2\hbar^2}\iint dt dt' J(t) D_F(t-t') J(t')\right] \frac{1}{2\pi\hbar}\int d^2a\, \frac{1}{\sqrt{\hbar^n n!}} \left(\mathrm{i}j(\omega)\right)^n e^{-a\bar{a}/\hbar} \\
&= \exp\left[-\frac{1}{2\hbar^2}\iint dt dt' J(t) D_F(t-t') J(t')\right] \frac{1}{\sqrt{\hbar^n n!}} \left(\mathrm{i}j(\omega)\right)^n. \qquad (3.15)
\end{aligned}$$

Using the factorization (3.12) for the exponential function, the probability for the above transition is

$$P_n = |\mathrm{Amp}(0 \to n)|^2 = \frac{|j(\omega)|^{2n}}{\hbar^n n!} e^{-|j(\omega)|^2/\hbar}. \qquad (3.16)$$

This corresponds to the well-known Poisson distribution for the number of emitted quanta in the field theoretical context:

$$P_n = e^{-\bar{n}}\frac{\bar{n}^n}{n!}, \qquad (3.17)$$

where $\bar{n}$ is the average number of emitted quanta:

$$\bar{n} = \sum_{n=0}^{\infty} n P_n = |j(\omega)|^2/\hbar. \qquad (3.18)$$

The aim in this section was to demonstrate that one can calculate quantities of physical interest working exclusively at the level of phase space, that is, within the framework of deformation quantization. These quantum mechanical results may be taken over to the field theoretical context by a formal extension to the case where the system considered has an infinite number of degrees of freedom.

3.2 The Wick Theorem

In order to see how the structures of perturbative quantum field theory arise in the star product formalism, one first notes that the Moyal product can be written as

$$(f *_{M} g)(z) = e^{M_{12}} f(z_1) g(z_2)\Big|_{z_1=z_2=z}, \tag{3.19}$$

where

$$M_{12} = \left(\frac{\mathrm{i}\hbar}{2}\right) \sum_{i,j=1}^{2d} \alpha^{ij} \frac{\partial}{\partial z_1^i} \frac{\partial}{\partial z_2^j}\ , \tag{3.20}$$

and z_α^i, $i = 1, \ldots, 2d$, is the i-th component of phase space point z_α. In canonical coordinates, $z = (q_1, \ldots, q_d, p_1, \ldots, p_d)$, M_{12} is proportional to the Poisson bracket operator:

$$M_{12} = \left(\frac{i\hbar}{2}\right) \sum_{i=1}^{d} \left(\frac{\partial}{\partial q_1^i} \frac{\partial}{\partial p_2^i} - \frac{\partial}{\partial p_1^i} \frac{\partial}{\partial q_2^i} \right). \tag{3.21}$$

Analogously, for two holomorphic coordinates:

$$(f *_{M} g)(a, \bar{a}) = e^{M_{12}} f(a_1, \bar{a}_1) g(a_2, \bar{a}_2)\Big|_{\substack{a_1=a_2=a \\ \bar{a}_1=\bar{a}_2=\bar{a}}} \tag{3.22}$$

with

$$M_{12} = \frac{\hbar}{2} (\partial_{a_1} \partial_{\bar{a}_2} - \partial_{\bar{a}_1} \partial_{a_2}), \tag{3.23}$$

and for the normal product one gets

$$(f *_{N} g)(a, \bar{a}) = e^{N_{12}} f(a_1, \bar{a}_1) g(a_2, \bar{a}_2)\Big|_{\substack{a_1=a_2=a \\ \bar{a}_1=\bar{a}_2=\bar{a}}}, \tag{3.24}$$

with $N_{12} = \hbar \partial_{a_1} \partial_{\bar{a}_2}$.

The Moyal product of r functions can then be written as

$$f_1 *_{M} f_2 *_{M} \cdots *_{M} f_r = \exp\left(\sum_{i<j} M_{ij} \right) \prod_{m=1}^{r} f_m(a_m, \bar{a}_m)\Big|_{\substack{a_m=a \\ \bar{a}_m=\bar{a}}}. \tag{3.25}$$

There is a similar formula for the normal product. For functions f_m which are linear in a and $\bar{a}$,

$$f_m(a, \bar{a}) = A_m a + B_m \bar{a}, \tag{3.26}$$

the star product may be written in the form of a Wick theorem. For example, the star product of four linear functions can be written by expanding the exponential:

$$\begin{aligned} f_1 *_{M} f_2 *_{M} f_3 *_{M} f_4 \quad = \quad & f_1 f_2 f_3 f_4 + G_{12}\ (f_3 f_4) + G_{13}\ (f_2 f_4) + G_{14}\ (f_3 f_3) \\ & + G_{23}\ (f_1 f_4) + G_{24}\ (f_1 f_3) + G_{34}\ (f_1 f_2) \\ & + G_{12} G_{34} + G_{13} G_{24} + G_{14} G_{23}, \end{aligned} \tag{3.27}$$

where the contractions

$$G_{ij} = M_{ij} f_i f_j = \frac{\hbar}{2} (A_i B_j - A_j B_i) \tag{3.28}$$

are constants. One has then the relation

$$M_{ij} = G_{ij} \frac{\partial}{\partial f_i} \frac{\partial}{\partial f_j}, \tag{3.29}$$

and (3.25) may be written as

$$f_1 *_M f_2 *_M \cdots *_M f_r = \exp\left[\sum_{i<j} G_{ij} \frac{\partial}{\partial f_i} \frac{\partial}{\partial f_j}\right] \prod_{m=1}^{r} f_m(a_m, \bar{a}_m). \tag{3.30}$$

It should be clear from the above that not only the original form, but also the various generalized Wick theorems which have been discussed in the literature [1],[88], are direct consequences of the structure of the relevant star products. The operator form of the Wick theorem can be obtained if one identifies the functions f_i with the fields and then applies the quantization operator $\mathcal{Q}$. In the Weyl quantization scheme one has

$$\hat{f}_1 \cdots \hat{f}_r = \mathcal{Q}_W \left\{ (f_1 *_M \cdots *_M f_n)(a, \bar{a}) \right\} = \mathcal{Q}_W \left\{ \exp\left[\sum_{i<j} M_{ij}\right] \prod_{m=1}^{r} f_m(a_m, \bar{a}_m) \Big|_{\substack{a_m = a \\ \bar{a}_m = \bar{a}}} \right\}. \tag{3.31}$$

For a quantization scheme which is c-equivalent to the Moyal scheme one uses the corresponding contractions X_{ij} instead of the Moyal contractions M_{ij}. One may write $X_{ij} = X_{\{ij\}} + M_{ij}$, where $X_{\{ij\}} = \frac{1}{2}(X_{ij} + X_{ji})$ is the symmetric part of X_{ij}, since the antisymmetric part is fixed for all c-equivalent star products, by property (iii) of the definition of the star product.

One additional important ingredient is the time ordering. The time-ordered product of r time-dependent operators is given by the prescription

$$\mathcal{T}\left\{\hat{f}_1(t_1) \cdots \hat{f}_r(t_r)\right\} = \mathcal{Q}_X \left\{ \exp\left[\sum_{i<j} (X_{\{ij\}} + \epsilon(t_i - t_j) M_{ij})\right] \prod_{m=1}^{r} f_m(a_m, \bar{a}_m, t_m) \Big|_{\substack{a_m = a \\ \bar{a}_m = \bar{a}}} \right\}, \tag{3.32}$$

since the transposition of two operators leaves $X_{\{ij\}}$ invariant, while the signs of $\epsilon(t_i - t_j)$ and of M_{ij} reverse. For the case of normal ordering the exponent in (3.32) may be written as

$$\begin{aligned} T_{ij} = N_{\{ij\}} + \epsilon(t_i - t_j) M_{ij} &= \frac{\hbar}{2} \left[(\partial_{a_i} \partial_{\bar{a}_j} + \partial_{\bar{a}_j} \partial_{a_i}) + \epsilon(t_i - t_j)(\partial_{a_i} \partial_{\bar{a}_j} - \partial_{\bar{a}_j} \partial_{a_i}) \right] \\ &= \frac{\hbar}{2} \left[(1 + \epsilon(t_i - t_j)) \partial_{a_i} \partial_{\bar{a}_j} + (1 - \epsilon(t_i - t_j)) \partial_{\bar{a}_j} \partial_{a_i} \right] \\ &= \hbar \left[\theta(t_1 - t_2) \partial_{a_i} \partial_{\bar{a}_j} + \theta(t_2 - t_1) \partial_{\bar{a}_j} \partial_{a_i} \right]. \end{aligned} \tag{3.33}$$

Suppose now that the functions f_m are linear in a and $\bar{a}$, and have a periodic time dependence:

$$f_m(t) = A_m a e^{-\mathrm{i}\omega t} + B_m \bar{a} e^{\mathrm{i}\omega t}. \tag{3.34}$$

By (3.33) the relevant contractions are

$$D_{ij}(t_i - t_j) = \hbar \left[A_i B_j \theta(t_i - t_j) e^{-\mathrm{i}\omega t} + A_j B_i \theta(t_j - t_i) e^{\mathrm{i}\omega t} \right], \tag{3.35}$$

which is a generalization of the expression in (3.9). In analogy to (3.29) one can write

$$T_{ij} = \iint dt dt' \frac{\delta}{\delta f_i(t)} D_{ij}(t - t') \frac{\delta}{\delta f_j(t')}, \tag{3.36}$$

where the $\delta/\delta f(t)$ are functional derivatives. For the operators

$$\hat{f}_m(t) = A_m \hat{a} e^{-\mathrm{i}\omega t} + B_m \hat{a}^\dagger e^{\mathrm{i}\omega t} \tag{3.37}$$

one gets the quantum mechanical form of Wick's theorem by inserting these expressions into (3.32):

$$\mathcal{T}\{\hat{f}_1(t_1)\cdots\hat{f}_r(t_r)\} = \mathcal{Q}_N \left\{ \exp\left[\sum_{i<j} \iint dt dt' \frac{\delta}{\delta f_i(t)} D_{ij}(t-t') \frac{\delta}{\delta f_j(t')}\right] f_1(t_1)\cdots f_r(t_r) \Bigg|_{\substack{a_m=a \\ \bar{a}_m=\bar{a}}} \right\}. \tag{3.38}$$

Since one has modified the star product contractions in (3.32) by the insertion of the $\epsilon(t_i - t_j)$ factor, the time-ordered product is not the Weyl transform of a star product. This can be seen form the fact that the time-ordered product is symmetric in its arguments, whereas the star products have an antisymmetric part.

The generalization of the foregoing results to the field theoretical context is formally straightforward: the free real scalar field is equivalent to an infinite collection of harmonic oscillators. One at first considers the system to be confined to a box of finite volume V. The Fourier representation of the free field is of the variables $a(\vec{k}),\ \bar{a}(\vec{k})$ is

$$\phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\hbar\omega_{\vec{k}}}} \left[a(\vec{k})e^{-\mathrm{i}kx} + \bar{a}(\vec{k})e^{\mathrm{i}kx}\right], \tag{3.39}$$

where $\hbar k^0 = \hbar\omega_{\vec{k}} = \sqrt{\hbar^2|\vec{k}|^2 + m^2}$ is the energy of a single quantum of the field. The normalization of the field is fixed by the equal-time commutator

$$\left[\phi(x), \dot{\phi}(y)\right]_* \Big|_{x^0=y^0} = \mathrm{i}\hbar\delta^{(3)}(\vec{x}-\vec{y}). \tag{3.40}$$

The Hamilton function in the normal product scheme is the generalization of $H = \omega\bar{a}a$:

$$H = \sum_{\vec{k}} \omega_{\vec{k}} \bar{a}(\vec{k}) a(\vec{k}). \tag{3.41}$$

The vacuum state in the normal product scheme is

$$\pi_0^{(N)} = \prod_{\vec{k}} \left(e^{a(\vec{k})\bar{a}(\vec{k})/\hbar}\right) = e^{-\sum_{\vec{k}} a(\vec{k})\bar{a}(\vec{k})/\hbar} \tag{3.42}$$

and is normalized in the usual manner:

$$\int \prod_{\vec{k}} \left(\frac{d^2 a_{\vec{k}}}{2\pi\hbar}\right) \pi_0^{(N)} = 1. \tag{3.43}$$

The vacuum expectation value of H vanishes:

$$\int \prod_{\vec{k}} \left(\frac{d^2 a_{\vec{k}}}{2\pi\hbar}\right) H *_N \pi_0^{(N)} = 0. \tag{3.44}$$

In the case of the Moyal product quantization scheme one would have found an infinite vacuum energy, arising from the zero-point energy in the spectrum. This fact has been used to argue that the normal product is the only admissible star product in the context of free field theory. From now on one shall go over to the continuum normalization of the fields:

$$\phi(x) = \int \frac{d^3 k}{(2\pi)^{\frac{3}{2}}} \frac{1}{\sqrt{2\hbar\omega_{\vec{k}}}} \left[a(\vec{k})e^{-\mathrm{i}kx} + \bar{a}(\vec{k})e^{\mathrm{i}kx}\right]. \tag{3.45}$$

To form the Moyal product of fields one first calculates the relevant contractions by generalizing (3.23) to a system with an infinite number of degrees of freedom,

$$\begin{aligned}\frac{1}{2}D(x_1-x_2) &= \iiint d^3k \frac{d^3k_1}{(2\pi)^{\frac{3}{2}}}\frac{d^3k_2}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega_{\vec{k}_1}}}\frac{1}{\sqrt{2\omega_{\vec{k}_2}}}\frac{\hbar}{2}\left[\frac{\delta}{\delta a_1(\vec{k})}\frac{\delta}{\delta \bar{a}_2(\vec{k})}-\frac{\delta}{\delta \bar{a}_1(\vec{k})}\frac{\delta}{\delta a_2(\vec{k})}\right]\\ &\quad\times\left(a_1(\vec{k}_1)e^{-\mathrm{i}k_1x_1}+\bar{a}_1(\vec{k}_1)e^{\mathrm{i}k_1x_1}\right)\left(a_2(\vec{k}_2)e^{-\mathrm{i}k_2x_2}+\bar{a}_2(\vec{k}_2)e^{\mathrm{i}k_2x_2}\right)\\ &= \frac{1}{2}\left[D^+(x_1-x_2)+D^-(x_1-x_2)\right],\end{aligned} \tag{3.46}$$

where

$$D^{\pm}(x)=\pm\int\frac{d^3k}{(2\pi)^3}\frac{\hbar}{2\omega_{\vec{k}}}e^{\mp\mathrm{i}kx}, \tag{3.47}$$

are the propagators for the components of positive and negative frequencies, and $D(x)$ is the Schwinger function. The Moyal product of the fields is then, in analogy to (3.30),

$$\phi(x_1)*_M\cdots *_M\phi(x_r)=\exp\left[\frac{1}{2}\sum_{i<j}\iint d^4x\,d^4y\,\frac{\delta}{\delta\phi_i(x)}D(x-y)\frac{\delta}{\delta\phi_j(y)}\right]\prod_{m=1}^{r}\phi_m(x_m)\Big|_{\phi_m=\phi}. \tag{3.48}$$

For the quantum field operators

$$\hat{\phi}(x)=\int\frac{d^3k}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\hbar\omega_{\boldsymbol{k}}}}\left[\hat{a}(\vec{k})e^{-\mathrm{i}kx}+\hat{a}^{\dagger}(\vec{k})e^{\mathrm{i}kx}\right] \tag{3.49}$$

one obtains

$$\hat{\phi}(x_1)\cdots\hat{\phi}(x_r)=\mathcal{Q}_W\left\{\exp\left[\frac{1}{2}\sum_{i<j}\iint d^4x\,d^4y\,\frac{\delta}{\delta\phi_i(x)}D(x-y)\frac{\delta}{\delta\phi_j(y)}\right]\prod_{m=1}^{r}\phi_m(x_m)\Big|_{\phi_m=\phi}\right\}. \tag{3.50}$$

However, the Moyal product is not appropriate in the field theory context. To treat local interactions in perturbation theory causality requires the use of the Feynman propagator, which propagates the positive frequencies forward in time, and the negative frequencies backwards in time. For this one needs the analogy of (3.38):

$$\mathcal{T}\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_r)\}=\mathcal{Q}_N\left\{\exp\left[\sum_{i<j}\iint d^4x\,d^4y\frac{\delta}{\delta\phi_i(x)}D_F(x-y)\frac{\delta}{\delta\phi_j(y)}\right]\prod_{m=1}^{r}\phi_m(x_m)\Big|_{\phi_m=\phi}\right\}. \tag{3.51}$$

Here D_F, the Feynman propagator, is given by the infinite dimensional generalization of (3.35):

$$\begin{aligned}D_F(x_1-x_2) &= \iiint d^3k\frac{d^3k_1}{(2\pi)^{\frac{3}{2}}}\frac{d^3k_2}{(2\pi)^{\frac{3}{2}}}\frac{1}{\sqrt{2\omega_{\vec{k}_1}}}\frac{1}{\sqrt{2\omega_{\vec{k}_2}}}\\ &\quad\times\hbar\left[\theta(t_1-t_2)\frac{\delta}{\delta a_1(\vec{k})}\frac{\delta}{\delta\bar{a}_2(\vec{k})}+\theta(t_2-t_1)\frac{\delta}{\delta\bar{a}_1(\vec{k})}\frac{\delta}{\delta a_2(\vec{k})}\right]\\ &\quad\times\left(a_1(\vec{k}_1)e^{-\mathrm{i}k_1x_1}+\bar{a}_1(\vec{k}_1)e^{\mathrm{i}k_1x_1}\right)\left(a_2(\vec{k}_2)e^{-\mathrm{i}k_2x_2}+\bar{a}_2(\vec{k}_2)e^{\mathrm{i}k_2x_2}\right)\\ &= \theta(t_1-t_2)D^+(x_1-x_2)-\theta(t_2-t_1)D^-(x_1-x_2).\end{aligned} \tag{3.52}$$

One may simplify (3.51) by using the symmetry of the Feynman propagator, $D_F(x_1 - x_2) = D_F(x_2 - x_1)$; it becomes:

$$\mathcal{T}\{\hat{\phi}(x_1)\cdots\hat{\phi}(x_r)\} = \mathcal{Q}_N \left\{ \exp\left[\frac{1}{2}\iint d^4x\, d^4y \frac{\delta}{\delta\phi(x)} D_F(x-y) \frac{\delta}{\delta\phi(y)}\right] \phi(x_1)\cdots\phi(x_2) \right\}. \tag{3.53}$$

Note that in this case it is no longer necessary to use different fields which are set equal only after the differentation; because of the symmetry the correct combinatorics are guaranteed by the Leibnitz rule for differentiation. Eq. (3.53) is the field-theoretic version of Wick's theorem.

The propagator for positive frequencies $D^+(x)$ is c-equivalent to $\frac{1}{2}D(x)$ by use of the transition operator

$$T = \exp\left[-\frac{1}{2}\iint d^4x\, d^4y \frac{\delta}{\delta\phi(x)} \frac{1}{2}\left[D^+(x-y) - D^-(x-y)\right] \frac{\delta}{\delta\phi(y)}\right]. \tag{3.54}$$

The time-ordered product for the field operators, Eq. (3.51), is the Weyl transform of the expression which results from the Moyal star product, Eq. (3.48), by replacing $\frac{1}{2}D$ by D^+, restricting the integration to positive times $x^0 > y^0$, and symmetrizing.

For $r = 2$ Wick's theorem is

$$\mathcal{T}\{\hat{\phi}(x_1)\hat{\phi}(x_2)\} = \mathcal{Q}_N\{\phi(x_1)\phi(x_2)\} + D_F(x_1 - x_2). \tag{3.55}$$

Since the vacuum expectation value of the normal product vanishes, this yields the familiar relation

$$D_F(x_1 - x_2) = \langle 0|\mathcal{T}\{\hat{\phi}(x_1)\hat{\phi}(x_2)\}|0\rangle. \tag{3.56}$$

Wick's theorem may also be written in the form of a generating function:

$$\mathcal{T}\left\{e^{\frac{\mathrm{i}}{\hbar}\int d^4x J(x)\hat{\phi}(x)}\right\} = \mathcal{Q}_N\left\{e^{\frac{\mathrm{i}}{\hbar}\int d^4x J(x)\phi(x)}\right\} \exp\left[-\frac{1}{2\hbar^2}\iint d^4x\, d^4y J(x) D_F(x-y) J(y)\right], \tag{3.57}$$

where $J(x)$ is an external source, and Eq. (3.51) results by expanding both sides of Eq. (3.57) in powers of J and comparing coefficients. Note that

$$\hat{S}[J] = \mathcal{T}\left\{e^{-\mathrm{i}\int d^4x J(x)\hat{\phi}(x)}\right\} = \mathcal{T}\left\{e^{-\frac{\mathrm{i}}{\hbar}\int d^4x \hat{H}_{\mathrm{int}}(x)}\right\} \tag{3.58}$$

is the scattering operator of quantum field theory, so that Eq. (3.57) is the perturbation expansion of the scattering operator for this interaction. This is just the operator form of our previous result, Eq. (3.10), which was derived completely within the phase space formalism of deformation quantization theory. The generating functional for the perturbation series is, by Eq. (3.57),

$$Z_0[J] = \langle 0|\hat{S}[J]|0\rangle = \exp\left[-\frac{1}{2\hbar^2}\iint d^4x\, d^4y\, J(x) D_F(x-y) J(y)\right], \tag{3.59}$$

in agreement with (3.10). When a self-interaction term is included in the interaction Hamiltonian, $\hat{H}_{\mathrm{int}} = -J\phi + V(\phi)$, the generating functional for the interacting theory becomes

$$Z[J] = \frac{1}{N} e^{-\frac{\mathrm{i}}{\hbar}\int d^4x V\left(\frac{\hbar}{\mathrm{i}}\frac{\delta}{\delta J(x)}\right)} Z_0[J], \tag{3.60}$$

where the normalization constant is $N = Z[J=0]$.

3.3 Quantum Groups and Twisted Products

As shown above the time ordering cannot be described with a star product, because the time ordered product is symmetric. In the first chapter a symmetric product of the star product type was called circle product. Circle products were originally introduced by Rota and Stein in the context of Hopf algebra theory. Here it will be shown that the circle product of Rota and Stein has just the form of a symmetric star product. It is then possible to unify the star product formalism with Hopf algebra theory and apply both aspects in quantum field theory. In order to establish the connection between star product and Hopf algebra theory first the necessary notation will be briefly reviewed.

$\mathbb{K}$ is the field of real or complex numbers. An algebra H is a vector space $\mathbb{K}$ with two linear maps $H \otimes H \to H$ (the product) and $\eta : \mathbb{K} \to H$ (the identity) such that the product is associative and the unit mapping is $\eta(1) = 1 \in H$. A coalgebra H is a vector space over $\mathbb{K}$ with two linear maps $\Delta : H \to H \otimes H$ (the coproduct) and $\varepsilon : H \to \mathbb{K}$ (the counit). In Sweedler notation for the coproduct: $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$. The coproduct must be coassociative: $\sum(\Delta u_{(1)}) \otimes u_{(2)} = \sum u_{(1)} \otimes (\Delta u_{(2)})$. The counit satisfies $\sum \varepsilon(u_{(1)})u_{(2)} = \sum u_{(1)}\varepsilon(u_{(2)}) = u$. A bialgebra is a vector space over $\mathbb{K}$ which is an algebra and a bialgebra, with the compatibility condition that Δ and ε are algebra homomorphisms: $\Delta(uv) = \Delta(u)\Delta(v)$ and $\varepsilon(uv) = \varepsilon(u)\varepsilon(v)$.

A bialgebra H is a Hopf algebra if there is a linear mapping $S : H \to H$ (called the antipode) such that $\sum S(u_{(1)})u_{(2)} = \sum u_{(1)}S(u_{(2)}) = \varepsilon(u)1$. A Hopf algebra is commutative if the algebra product is commutative, and cocommutative if $\sum u_{(1)} \otimes u_{(2)} = \sum u_{(2)} \otimes u_{(1)}$.

A quantum group is a Hopf algebra with a coquasitriangular structure: this is an invertible bilinear map $\mathcal{R} : H \times H \to \mathbb{K}$ such that

$$\mathcal{R}(uv, w) = \sum \mathcal{R}(u, w_{(1)})\mathcal{R}(v, w_{(2)}), \quad \mathcal{R}(u, vw) = \sum \mathcal{R}(u_{(1)}, w)\mathcal{R}(u_{(2)}, v). \tag{3.61}$$

For a commutative and cocommutative Hopf algebra the coquasitriangular structure can be explicitly given by the following rule. Let $u = u_1 \cdots u_n,\ v = v_1 \cdots v_m$ with $u_i, v_j \in H$. For $m \neq n$ $\mathcal{R}(u, v) = 0$. For $m = n$

$$\mathcal{R}(u, v) = \text{perm } \mathcal{R}(u_i, v_j) = \sum_{\sigma \in P} \mathcal{R}(u_1, v_{\sigma(1)}) \cdots \mathcal{R}(u_n, v_{\sigma(n)}), \tag{3.62}$$

where the sum is over all the permutations σ of the indices $1, \ldots, n$. This function is called the permanent of the square matrix $\mathcal{R}(u_i, v_j)$.

One can use $\mathcal{R}$ to define a twisted product, denoted by $\circ$. When the Hopf algebra is cocommutative this product may be written as

$$u \circ v = \sum \mathcal{R}(u_{(1)}, v_{(1)})u_{(2)}v_{(2)} = \sum u_{(1)}v_{(1)}\mathcal{R}(u_{(2)}, v_{(2)}). \tag{3.63}$$

It can be shown that the twisted product is associative.

Now let V be a vector space and $T(V)$ the tensor algebra over V. There is then a unique Hopf algebra structure on $T(V)$ such that $\Delta(v) = v \otimes 1 + 1 \otimes v,\ \ \varepsilon(v) = 0$ and $S(v) = -v$ for $v \in V$. For $v_1, v_2, \cdots v_n \in V$ the coproduct is

$$\Delta(v_1 \cdots v_n) = \sum_{k=0}^{n} \sum_{\sigma \in P_{nk}} v_{\sigma(1)} \cdots v_{\sigma(k)} \otimes v_{\sigma(k+1)} \cdots v_{\sigma(n)}, \tag{3.64}$$

where P_{nk} denotes the set of all permutations of $(1, ..., n)$ such that

$$\sigma(1) < \sigma(2) < ... < \sigma(k) \quad \text{and} \quad \sigma(k+1) < ... < \sigma(n). \tag{3.65}$$

We see from this that the coproduct is cocommutative. The antipode is

$$S(v_1 v_2 \cdots v_n) = (-1)^n v_n \cdots v_2 v_1. \tag{3.66}$$

For the special case of the symmetric algebra $S(V)$ over V the formula for the coproduct simplifies. To begin with one considers V to be two-dimensional, with basis elements $a, \bar{a}$. The coproduct of the monomial $u = a^m \bar{a}^n$ is then

$$\Delta u = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} a^i \bar{a}^j \otimes a^{m-i} \bar{a}^{n-j}. \tag{3.67}$$

In this case the twisted product can be written as an exponential:

$$u \circ v = u\, e^{(\mathcal{R}(a,a)\overleftarrow{\partial}_a \overrightarrow{\partial}_a + \mathcal{R}(\bar{a},\bar{a})\overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_{\bar{a}} + \mathcal{R}(a,\bar{a})\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} + \mathcal{R}(\bar{a},a)\overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a)}\, v, \tag{3.68}$$

which makes immediately obvious in which sense the circle product is a generalization of the star product. That the circle product (3.63) can be written in an exponential form as in (3.68) has been demonstrated for vector spaces of higher dimension in the literature [14]. In Appendix B a combinatoric proof is given that is more direct than these proofs. It should also be emphasized that no other assumptions are necessary for the result except the basic structures of the present section.

The connection of the above twisted product and the star products will now be discussed in more detail. In order to make contact to physics one first identifies the variables a and $\bar{a}$ with the holomorphic coordinates of a flat two dimensional phase space (the generalization to higher dimensions is straightforward). Since a and $\bar{a}$ then have dimension $\hbar^{-1/2}$ the coquasitriangular structure $\mathcal{R}$ must be proportional to $\hbar$, so that the twisted product fulfills

$$\lim_{\hbar \to 0} u \circ v = uv. \tag{3.69}$$

Since the circle product is an associative product it satisfies nearly all the requirements of a star product. To get a star product one must impose the additional requirement

$$\lim_{\hbar \to 0} \frac{1}{\hbar}(a \circ \bar{a} - \bar{a} \circ a) = \{a, \bar{a}\}_{PB} = 1, \tag{3.70}$$

If the circle product can be written in the exponential representation (3.68), condition (3.70) reduces to

$$\mathcal{R}(a, \bar{a}) - \mathcal{R}(\bar{a}, a) = \hbar, \tag{3.71}$$

so that the basic coquasitriangular structures are no longer independent. Condition (3.70) can be fulfilled in various ways, where a particularly convenient choice is $\mathcal{R} = \mathcal{R}_N$, with

$$\mathcal{R}_N(a, \bar{a}) = \hbar \quad \text{and} \quad \mathcal{R}_N(a, a) = \mathcal{R}_N(\bar{a}, \bar{a}) = \mathcal{R}_N(\bar{a}, a) = 0. \tag{3.72}$$

One can then directly prove for this case that the twisted product reproduces the well-known formula for the normal star product. For the choice (3.72) the only terms which survive in the sum (3.63) are those for which $u_{(2)}$ contains only a-factors and $v_{(2)}$ only $\bar{a}$-factors. Taking $u = a^r \bar{a}^s$ and $v = a^m \bar{a}^n$ one finds that the relevant terms in the coproduct formulae for u and v are $\binom{r}{i} a^i \bar{a}^s \otimes a^{r-i}$ and $\binom{n}{l} a^m \bar{a}^l \otimes \bar{a}^{n-l}$ with $r - i = n - l$. These terms yield a contribution to the sum in (3.63) which is

$$\binom{r}{i}\binom{n}{l} a^i \bar{a}^s a^m \bar{a}^l \mathcal{R}_N(a^{r-i}, \bar{a}^{n-l}) \quad = \quad \frac{r!}{i!} \frac{n!}{l!} \frac{\hbar^{n-l}}{(n-l)!} a^{r+m-(n-l)} \bar{a}^{s+l} \tag{3.73}$$

where (3.62) was used, which for the present case is $\mathcal{R}_N(a^n, \bar{a}^n) = n! \mathcal{R}(a, \bar{a})^n$. However, this is just the $(n-l)$-term in the expansion of the exponential

$$u \circ v = a^s \bar{a}^l\, e^{\mathcal{R}_N(a,\bar{a})\overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}}}\, a^m \bar{a}^n = u\, e^{\hbar \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}}}\, v, \tag{3.74}$$

which is the expression for the normal star product $u *_N v$. In the $2d$-dimensional case the only non-vanishing basic coquasitriangular structure is

$$\mathcal{R}_N(a_i, \bar{a}_j) = \hbar \delta_{ij}, \tag{3.75}$$

which leads to the corresponding d-dimensional normal product. The other possible generalization is to choose another scheme, for example $\mathcal{R} = \mathcal{R}_M$, with

$$\mathcal{R}_M(a, \bar{a}) = \frac{\hbar}{2} \quad , \quad \mathcal{R}_M(\bar{a}, a) = -\frac{\hbar}{2} \quad \text{and} \quad \mathcal{R}_M(a, a) = \mathcal{R}_M(\bar{a}, \bar{a}) = 0, \tag{3.76}$$

this leads to the Moyal product.

The general case of an antisymmetric circle product can be discussed with the filter functions that describe the different quantization schemes. The operator that corresponds to a function $f(a, \bar{a})$ can be written as:

$$\begin{aligned} \mathcal{Q}_\phi(f)(\hat{a}, \hat{a}^\dagger) &= \frac{1}{(2\pi)^2} \iint d^2\alpha\, d^2a\, e^{(\mu\alpha^2 + \nu\bar{\alpha}^2 + \lambda\alpha\bar{\alpha})} e^{-(\alpha\bar{a} - \bar{\alpha}a)} e^{(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a})} f(a, \bar{a}) \\ &= \frac{1}{(2\pi)^2} \iint d^2\alpha\, d^2a\, e^{(\mu\partial_{\bar{a}}^2 + \nu\partial_a^2 - \lambda\partial_a\partial_{\bar{a}})} e^{-(\alpha\bar{a} - \bar{\alpha}a)} e^{(\alpha\hat{a}^\dagger - \bar{\alpha}\hat{a})} f(a, \bar{a}) \end{aligned} \tag{3.77}$$

where $\phi_{\mu,\nu,\lambda}(\alpha, \bar{\alpha}) = e^{(\mu\alpha^2 + \nu\bar{\alpha}^2 + \lambda\alpha\bar{\alpha})}$ is the filter function that parametrises the ordering scheme. (Note that this is the parametrization where the holomorphic coordinates are considered as primary, see footnote on page 7.) For $\mu = \nu = \lambda = 0$ one gets Weyl ordering, for $\mu = \nu = 0$, $\lambda = \frac{\hbar}{2}$ normal ordering and for $\mu = \nu = 0$, $\lambda = -\frac{\hbar}{2}$ antinormal ordering. One can also easily see that the filter function, written as $e^{(\mu\partial_{\bar{a}}^2 + \nu\partial_a^2 - \lambda\partial_a\partial_{\bar{a}})}$, is the T-operator that relates the Moyal product to the star products that correspond to the ordering schemes given by the parameters μ, ν and λ. For example, the T-operator that relates the Moyal product to the most general twisted product satisfying the antisymmetry condition (3.71),

$$f \circ_A g = f\, e^{(\mathcal{R}(a,a)\overleftarrow{\partial}_a\overrightarrow{\partial}_a + \mathcal{R}(\bar{a},\bar{a})\overleftarrow{\partial}_{\bar{a}}\overrightarrow{\partial}_{\bar{a}} + \mathcal{R}(a,\bar{a})\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}} + (\mathcal{R}(a,\bar{a}) - \hbar)\overleftarrow{\partial}_{\bar{a}}\overrightarrow{\partial}_a)}\, g \tag{3.78}$$

according to $f \circ_A g = T^{-1}(Tf *_M Tg)$, is given by

$$T = e^{(-\frac{1}{2}\mathcal{R}(a,a)\partial_a^2 - \frac{1}{2}\mathcal{R}(\bar{a},\bar{a})\partial_{\bar{a}}^2 - (\mathcal{R}(a,\bar{a}) - \frac{\hbar}{2})\partial_a\partial_{\bar{a}})}. \tag{3.79}$$

Comparing the filter function and the T-operator one sees that there is a direct correspondence between the three independent basic coquasitriangular structures and the parameters μ, ν and λ.

The coquasitriangular structure can be recovered from the twisted product by the formula $\mathcal{R}(u, v) = \varepsilon(u \circ v)$ [15]. This just means that $\mathcal{R}(u, v)$ is given by the constant term in $u \circ v$. For example, for the normal star product of (3.74) one just gets the relations (3.72). Furthermore for $u, v \in S(V)$ monomials in a and $\bar{a}$ one finds that $\mathcal{R}_N(u, v) = 0$ unless $u = a^n$ and $v = \bar{a}^n$, in which case $\mathcal{R}_N(u, v) = n!\hbar^n$. This is consistent with the general rule (3.62) that the coquasitriangular structure of two monomials decomposes into the permanent of the basic coquasitriangular structures (3.62).

3.4 The Fermionic Case

In the fermionic case the basis of the vector space V consists of Grassmann variables $f_1, \ldots, f_n$ and one considers the antisymmetric algebra $\Lambda(V)$. A monomial in this algebra is at most linear in one of the f_i and has the form $u = f_{i_1} \cdots f_{i_r}$ with $r \leq n$. The use of fermionic variables leads in the definition of the coproduct (3.64) to a factor $(-1)^\sigma$ under the sum and the coquasitriangular structure given by

$$\mathcal{R}(u, v) = \det \mathcal{R}(u_i, v_i) = \sum_{\sigma \in P} (-1)^\sigma \mathcal{R}(u_1, v_{\sigma(1)}) \cdots \mathcal{R}(u_n, v_{\sigma(n)}). \tag{3.80}$$

If one now tries to construct a circle product like in (3.63) the two possible definitions of the circle product are no longer the same. For example the first term of p-th order in $\sum \mathcal{R}(u_{(1)}, v_{(1)}) u_{(2)} v_{(2)}$ is formed by the coproduct terms

$$u_1 \cdots u_p \otimes u_{p+1} \cdots u_n \quad \text{and} \quad v_1 \cdots v_p \otimes v_{p+1} \cdots v_m \tag{3.81}$$

whereas the same term in the definition $\sum u_{(1)}v_{(1)}\mathcal{R}(u_{(2)},v_{(2)})$ is formed by the coproduct terms

$$u_{p+1}\cdots u_n\otimes u_1\cdots u_p \quad\text{and}\quad v_{p+1}\cdots v_m\otimes v_1\cdots v_p. \tag{3.82}$$

But these terms differ in the coproduct by a sign factor of $(-1)^{p(n-p)}$ and $(-1)^{p(m-p)}$ respectively.

This ambiguity corresponds to the fact that one has to distinguish between left and right derivatives for fermionic variables. So the exponential representiation of the circle product can be written either with left or with right derivatives. For the definition of the fermionic circle product one chooses:

$$\begin{aligned} u\circ v &= \sum \mathcal{R}(u_{(1)},v_{(1)})u_{(2)}v_{(2)} && (3.83)\\ &= u\,\exp\left[\sum_{i,j=0}^{n}\mathcal{R}(f_i,f_j)\overleftarrow{\partial}^L_{f_i}\overrightarrow{\partial}^L_{f_j}\right]v = u\,\exp\left[-(-1)^{\pi(u)}\sum_{i,j=0}^{n}\mathcal{R}(f_i,f_j)\overleftarrow{\partial}_{f_i}\overrightarrow{\partial}_{f_j}\right]v, && (3.84)\end{aligned}$$

where one uses the relation between left and right derivative

$$\frac{\partial^R F}{\partial f_i} = -(-1)^{\pi(F)}\frac{\partial^L F}{\partial f_i} \tag{3.85}$$

and encoded the left and right derivatives in the vector arrows. The proof that (3.83) can be written in the exponential form (3.84) can be found in Appendix B.

Just as in the bosonic case one can also define a normal product for fermionic variables:

$$u *_N v = u\,\exp\left(-(-1)^{\pi(u)}\hbar\sum_{i=0}^{n}\overleftarrow{\partial}_{f_i}\overrightarrow{\partial}_{\bar{f}_i}\right)v \tag{3.86}$$

so that the basic coquasitriangular structures are:

$$\mathcal{R}_N(f_i,\bar{f}_j)=\hbar\delta_{ij}, \quad\text{and}\quad \mathcal{R}_N(f_i,f_j)=\mathcal{R}_N(\bar{f}_i,\bar{f}_j)=\mathcal{R}_N(\bar{f}_i,f_j)=0. \tag{3.87}$$

In order to describe the field theoretic case for fermionic fields

$$\begin{aligned} \psi(x) &= \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}\sum_s\left(f_{s,\vec{k}}u_s(k)e^{-\mathrm{i}kx}+h_{s,\vec{k}}v_s(k)e^{\mathrm{i}kx}\right), && (3.88)\\ \bar{\psi}(x) &= \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{k}}}}\sum_s\left(\bar{f}_{s,\vec{k}}\bar{u}_s(k)e^{\mathrm{i}kx}+\bar{h}_{s,\vec{k}}\bar{v}_s(k)e^{-\mathrm{i}kx}\right) && (3.89)\end{aligned}$$

we can take into account the time ordering in the normal star product (3.86). The time ordering destroys the antisymmetry of the normal star product and one gets a circle product for linear functions

$$\begin{aligned} f(t_1)\circ_{TN} g(t_2) = f(t_1)\exp\Bigg(\int d^3k\sum_s\Bigg[&\hbar\theta(t_1-t_2)\left(\frac{\overleftarrow{\delta}}{\delta f_{s,\vec{k}}}\frac{\overrightarrow{\delta}}{\delta\bar{f}_{s,\vec{k}}}+\frac{\overleftarrow{\delta}}{\delta h_{s,\vec{k}}}\frac{\overrightarrow{\delta}}{\delta\bar{h}_{s,\vec{k}}}\right)\\ &-\hbar\theta(t_2-t_1)\left(\frac{\overleftarrow{\delta}}{\delta\bar{f}_{s,\vec{k}}}\frac{\overrightarrow{\delta}}{\delta f_{s,\vec{k}}}+\frac{\overleftarrow{\delta}}{\delta\bar{h}_{s,\vec{k}}}\frac{\overrightarrow{\delta}}{\delta h_{s,\vec{k}}}\right)\Bigg]\Bigg)g(t_2). \qquad (3.90)\end{aligned}$$

The coquasitriangular structures for two fields are

$$\begin{aligned} \mathcal{R}_{TN}(\psi(x_1),\bar{\psi}(x_2)) &= \int\frac{d^3k}{(2\pi)^3 2E_{\vec{k}}}\Bigg[\hbar\theta(t_1-t_2)\sum_s u_s(k)\bar{u}_s(k)e^{-\mathrm{i}k(x_1-x_2)}\\ &\qquad -\hbar\theta(t_2-t_1)\sum_s v_s(k)\bar{v}_s(k)e^{\mathrm{i}k(x_1-x_2)}\Bigg]\\ &= \hbar\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{i}(\not{k}+m)}{k^2-m^2+\mathrm{i}\epsilon}e^{-\mathrm{i}k(x_1-x_2)} = S_F(x_1-x_2) && (3.91)\end{aligned}$$

and

$$\mathcal{R}_T(\psi(x_1),\psi(x_2)) = \mathcal{R}_T(\bar\psi(x_1),\bar\psi(x_2)) = 0, \quad \mathcal{R}_T(\bar\psi(x_1),\psi(x_2)) = -S_F(x_2-x_1). \tag{3.92}$$

Here the completeness relations for the spinors

$$\sum_s u_s(k)\bar u_s(k) = \not{k} + m, \quad \sum_s v_s(k)\bar v_s(k) = \not{k} - m \tag{3.93}$$

have been used. $S_F(x)$ is the Dirac propagator [97].

As in the bosonic case it is also possible to write multiple fermionic $\circ_T$-products. In quantum electrodynamics, for example, the vacuum expectation value of the scattering operator is

$$S_0 = \langle 0|T\exp\left[-\mathrm{i}e\int d^4x\,\hat{\bar\psi}(x)\gamma^\mu\hat\psi(x)A(x)\right]|0\rangle. \tag{3.94}$$

In the above notation this can be written as:

$$\begin{aligned}
S_0 &= \sum_{n=0}^{\infty}\frac{(-\mathrm{i}e)^n}{n!}\int dx_1\cdots dx_n \epsilon\left[\left(\bar\psi(x_1)\not{A}(x_1)\psi(x_1)\right)\circ_T\cdots\circ_T\left(\bar\psi(x_n)\not{A}(x_n)\psi(x_n)\right)\right]\\
&= \sum_{n=0}^{\infty}\frac{(-\mathrm{i}e)^n}{n!}\int dx_1\cdots dx_n\\
&\quad \epsilon\left[\exp\left[\sum_{i<j}\iint d^4x d^4y\left(S_F^{\alpha\beta}(x-y)\frac{\delta}{\delta\psi_i^\alpha(x)}\frac{\delta}{\delta\bar\psi_j^\beta(y)} - S_F^{\alpha\beta}(y-x)\frac{\delta}{\delta\bar\psi_j^\beta(y)}\frac{\delta}{\delta\psi_j^\beta(x)}\right)\right]\right.\\
&\quad \left.\times\psi_1^{\alpha_1}(x_1)\not{A}^{\alpha_1\beta_1}(x_1)\psi_1^{\beta_1}(x_1)\cdots\bar\psi_n^{\alpha_n}(x_n)\not{A}^{\alpha_n\beta_n}(x_n)\psi_n^{\beta_n}(x_n)\right]\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int dx_1\cdots dx_n\\
&\quad \sum_P\epsilon_P\sum_{\alpha_1,\ldots,\alpha_n}(-\mathrm{i}e)\left(\not{A}_{\alpha_1\beta_1}(x_1)S_F^{\beta_1\alpha_{P_1}}(x_1-x_{P_1})\right)\cdots(-\mathrm{i}e)\left(\not{A}_{\alpha_1\beta_1}(x_1)S_F^{\beta_1\alpha_{P_1}}(x_1-x_{P_1})\right)\\
&= \mathrm{Det}\left[I - e\not{A}(x)\frac{1}{\not{p}-m+\mathrm{i}\varepsilon}\right].
\end{aligned} \tag{3.95}$$

Here the Cayley-Hamilton formula for the expansion of the the determinant of a matrix Γ was used, i.e.

$$\mathrm{Det}(I-\Gamma) = \exp(\mathrm{Tr}\ln(I-\Gamma)) = \sum_n\frac{(-1)^n}{n!}\sum_{\alpha_1\cdots\alpha_n}\sum_P\epsilon_P\Gamma_{\alpha_1\alpha_{P_1}}\cdots\Gamma_{\alpha_n\alpha_{P_n}}, \tag{3.96}$$

for the matrix

$$\Gamma_{\alpha_1x,\alpha_2y} = -\mathrm{i}e\sum_\alpha\not{A}_{\alpha_1\alpha}[S_F(x-y)]_{\alpha\alpha_2}. \tag{3.97}$$

Here the notations Det and Tr indicate a sum over discrete variables and an integration over continuous ones. The result (3.95) can be written as

$$\begin{aligned}
S_0(A) &= \mathrm{Det}\left[I - e\not{A}\frac{1}{\not{p}-m+\mathrm{i}\epsilon}\right]\\
&= \mathrm{Det}\left[(\not{p}-e\not{A}-m+\mathrm{i}\epsilon)\frac{1}{\not{p}-m+\mathrm{i}\epsilon}\right] = \frac{\mathrm{Det}\left[\not{p}-e\not{A}-m+\mathrm{i}\epsilon\right]}{\mathrm{Det}\left[\not{p}-m+\mathrm{i}\epsilon\right]},
\end{aligned} \tag{3.98}$$

a well-known result usually derived using path integral methods [97].

Chapter 4

Star Products and Geometric Algebra

The approach to spin in the last chapter was analogous to the procedure in the conventional formalism. One starts with quantum mechanics without spin and then adds the spin structures that were described above with a fermionic star product. In this and the next chapter it will be shown that it is actually the other way round. The fermionic structures introduced in the last chapter appear already in classical mechanics in the context of geometric algebra. With the fermionic Clifford star product it is then possible to reformulate geometric algebra as deformed superanalysis. This allows to describe vector analysis, spin and differential geometry in a superanalytic language.

4.1 Geometric Algebra and the Clifford Star Product

As described in the last chapter one has the situation that grassmannian mechanics itself is not a physical theory, but can nevertheless be used to describe a physical phenomenon like spin. So one might wonder what the physical status of grassmannian mechanics actually is. The problem is not the theory itself but the interpretation, because in grassmannian mechanics one interprets the Grassmann numbers as dynamical variables. Such fermionic variables do not physically exist as dynamical variables but they serve as the basis vector structure of space and space-time. This can be seen if one compares the formalism of the last chapter with geometric algebra.

Geometric algebra goes back to early ideas of Hamilton, Grassmann and Clifford. But it was first developed into a full formalism by Hestenes in [73] and [74]. The formalism of geometric algebra is based on the definition of the geometric or Clifford product that is the sum of the scalar and the wedge product of vectors. This product equips the space with the algebraic structure of a Clifford algebra. The geometric product then appeared to be a very powerful tool, that allows to describe and generalize the structures of vector analysis, of complex analysis and of the theory of spin in a unified and clear formalism. The algebraic power of this concept is due to the fact that the geometric product is in contrast to the scalar product associative. This formalism can then be used to describe classical mechanics in the realm of geometric algebra instead of linear algebra [74]. The formalism can also be generalized from the algebra of space to the algebra of space-time in order to describe electrodynamics and special relativity [73, 34].

In [35] it was shown that geometric algebra can be expressed with the help of Grassmann variables. Comparing this grassmannian formulation of geometric algebra with the formalism of the last chapter leads immediately to the conclusion that the geometric product is actually the Clifford star-product and that the Grassmann variables are actually the basis vectors of space. The Clifford algebra that appeared as the deformation of the Grassmann algebra is then the Clifford algebra of the basis vectors. In order to make this explicit the formalism of geometric algebra will be shortly sketched in the following and it will then be shown how geometric algebra can be reformulated with the Clifford star product.

Starting point for geometric algebra [73] is an d-dimensional vector space over the real numbers with vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$. A multiplication, called geometric product, of vectors can then be denoted by juxtaposition of an indeterminate number of vectors so that one gets monomials $A, B, C, \ldots$. These monomials can be added in a commutative and associative manner: $A+B=B+A$ and $(A+B)+C=A+(B+C)$, so that they form polynomials also denoted by capital letters. The so obtained polynomials can be multiplied associatively, i.e. $A(BC)=(AB)C$ and they fulfill the distributive laws $(A+B)C=AC+BC$ and $C(A+B)=CA+CB$. Furthermore there exists a null vector $\boldsymbol{a}\mathbf{0}=\mathbf{0}$ and the multiplication with a scalar $\lambda\boldsymbol{a}=\boldsymbol{a}\lambda$, with $\lambda \in \mathbb{R}$. The connection between scalars and vectors can be given if one assumes that the product $\boldsymbol{ab}$ is a scalar iff $\boldsymbol{a}$ and $\boldsymbol{b}$ are collinear, so that $\sqrt{\boldsymbol{a}^2}$ is the length of the vector $\boldsymbol{a}$. These axioms define now the Clifford algebra $\mathcal{C}\ell(V)$ and the elements $A, B, C, \ldots$ of $\mathcal{C}\ell(V)$ are called Clifford or c-numbers.

Since the geometric product of two collinear vectors is a scalar, the symmetric part of the geometric product $\frac{1}{2}(\boldsymbol{ab}+\boldsymbol{ba}) = \frac{1}{2}((\boldsymbol{a}+\boldsymbol{b})^2-\boldsymbol{a}^2-\boldsymbol{b}^2)$ is a scalar denoted $\boldsymbol{a}\cdot\boldsymbol{b}=\frac{1}{2}(\boldsymbol{ab}+\boldsymbol{ba})$. The product $\boldsymbol{a}\cdot\boldsymbol{b}$ is the inner or scalar product. One can then decompose the geometric product into its symmetric and antisymmetric part:

$$\boldsymbol{ab}=\frac{1}{2}(\boldsymbol{ab}+\boldsymbol{ba})+\frac{1}{2}(\boldsymbol{ab}-\boldsymbol{ba})=\boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{a}\wedge\boldsymbol{b}, \tag{4.1}$$

where the antisymmetric part $\boldsymbol{a}\wedge\boldsymbol{b}=\frac{1}{2}(\boldsymbol{ab}-\boldsymbol{ba})$ is formed with the outer product. For the outer product one has obviously $\boldsymbol{a}\wedge\boldsymbol{b}=-\boldsymbol{b}\wedge\boldsymbol{a}$ and $\boldsymbol{a}\wedge\boldsymbol{a}=0$, so that $\boldsymbol{a}\wedge\boldsymbol{b}$ can be interpreted geometrically as an oriented area. The geometric product is constructed in such a way that it gives information about the relative directions of $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e. $\boldsymbol{ab}=\boldsymbol{ba}=\boldsymbol{a}\cdot\boldsymbol{b}\Rightarrow\boldsymbol{a}\wedge\boldsymbol{b}=0$ means that $\boldsymbol{a}$ and $\boldsymbol{b}$ are collinear whereas $\boldsymbol{ab}=-\boldsymbol{ba}=\boldsymbol{a}\wedge\boldsymbol{b}\Rightarrow\boldsymbol{a}\cdot\boldsymbol{b}=0$ means that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular. The important point is that the geometric product (4.1) is associative, in contrast to the scalar and the cross product of conventional vector analysis. This allows the construction of a much more powerful multivector formalism that includes complex analysis and the theory of spins. As will be shown below it is in so far much better suited for doing physics.

The first step in the construction of the multivector formalism is to define with the outer product simple r-vectors or r-blades

$$A_{(r)}=\boldsymbol{a}_1\wedge\boldsymbol{a}_2\wedge\ldots\wedge\boldsymbol{a}_r, \tag{4.2}$$

which can be interpreted as r-dimensional volume forms. The geometric product can then be generalized to the case of a vector and a r-blade:

$$\boldsymbol{a}A_{(r)}=\boldsymbol{a}\cdot A_{(r)}+\boldsymbol{a}\wedge A_{(r)}, \tag{4.3}$$

which is the sum of a $(r-1)$-blade $\boldsymbol{a}\cdot A_{(r)}=\frac{1}{2}(\boldsymbol{a}A_{(r)}-(-1)^r A_{(r)}\boldsymbol{a})$ and a $(r+1)$-blade $\boldsymbol{a}\wedge A_{(r)}=\frac{1}{2}(\boldsymbol{a}A_{(r)}+(-1)^r A_{(r)}\boldsymbol{a})$. Applying this recursively one sees that each c-number can be written as a polynomial of r-blades, and using a set of basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_r$ a c-number reads:

$$A=A^0+A^i\boldsymbol{e}_i+\frac{1}{2!}A^{i_1i_2}\boldsymbol{e}_{i_1}\wedge\boldsymbol{e}_{i_2}+\ldots+\frac{1}{n!}A^{i_1\ldots i_r}\boldsymbol{e}_{i_1}\wedge\boldsymbol{e}_{i_2}\wedge\ldots\boldsymbol{e}_{i_r}. \tag{4.4}$$

A is called multivector or r-vector if the highest appearing grade is r. It decomposes into several blades:

$$A=\langle A\rangle_0+\langle A\rangle_1+\ldots+\langle A\rangle_r=\sum_{n=0}^{r}\langle A\rangle_n, \tag{4.5}$$

where $\langle\ \rangle_n$ projects onto the term of grade n. A multivector $A_{(r)}$ is called homogeneous if all appearing blades have the same grade, i.e. $A_{(r)}=\langle A_{(r)}\rangle_r$. The geometric product of two homogeneous multivectors $A_{(r)}$ and $B_{(s)}$ can be written as

$$A_{(r)}B_{(s)}=\langle A_{(r)}B_{(s)}\rangle_{r+s}+\langle A_{(r)}B_{(s)}\rangle_{r+s-2}+\cdots+\langle A_{(r)}B_{(s)}\rangle_{|r-s|}. \tag{4.6}$$

The inner and the outer product stand now for the terms with the lowest and the highest grade:

$$A_{(r)}\cdot B_{(s)}=\langle A_{(r)}B_{(s)}\rangle_{|r-s|} \qquad \text{and} \qquad A_{(r)}\wedge B_{(s)}=\langle A_{(r)}B_{(s)}\rangle_{r+s}. \tag{4.7}$$

In the case $r=0$, i.e. one of the multivectors is a scalar function f, one defines $f\cdot B_{(s)}=0$ and $f\wedge B_{(s)}=fB_{(s)}$. One should note that the inner and outer product here in the general case do not correspond anymore to the symmetric and the antisymmetric part of the geometric product. For example in the case of two bivectors one has $A_{(2)}\wedge B_{(2)}=B_{(2)}\wedge A_{(2)}$, so that the outer product is symmetric. Actually one finds for the symmetric and the antisymmetric parts of $A_{(2)}B_{(2)}$:

$$\frac{1}{2}(A_{(2)}B_{(2)}+B_{(2)}A_{(2)})=A_{(2)}\cdot B_{(2)}+A_{(2)}\wedge B_{(2)} \qquad \text{and} \qquad \frac{1}{2}(A_{(2)}B_{(2)}-B_{(2)}A_{(2)})=\langle A_{(2)}B_{(2)}\rangle_2. \tag{4.8}$$

In general the commutativity of the outer and the inner product is given by:

$$A_{(r)}\wedge B_{(s)}=(-1)^{rs}B_{(s)}\wedge A_{(r)} \qquad \text{and} \qquad A_{(r)}\cdot B_{(s)}=(-1)^{r(s+1)}B_{(s)}\cdot A_{(r)} \tag{4.9}$$

and both products are always distributive:

$$A\wedge(B+C)=A\wedge B+A\wedge C \qquad \text{and} \qquad A\cdot(B+C)=A\cdot B+A\cdot C. \tag{4.10}$$

Just as the outer product of r-vectors is in general associative, i.e. $A\wedge(B\wedge C)=(A\wedge B)\wedge C$, for the inner product one gets:

$$A_{(r)}\cdot(B_{(s)}\cdot C_{(t)})=(A_{(r)}\cdot B_{(s)})\cdot C_{(t)} \qquad \text{for} \quad r+t\leq s. \tag{4.11}$$

If one has to calculate several products of different type, one uses the convention that the inner and the outer product always have to be calculated first, i.e.

$$A\wedge BC=(A\wedge B)C\neq A\wedge(BC) \qquad \text{and} \qquad A\cdot BC=(A\cdot B)C\neq A\cdot(BC). \tag{4.12}$$

The formalism of geometric algebra briefly sketched so far can now be described with Grassmann variables and the Clifford star product that turns the Grassmann algebra into a Clifford algebra. In order to make the equivalence even more obvious one goes over to the dimensionless Grassmann variables

$$\boldsymbol{\sigma}_i=\sqrt{\frac{2}{\hbar}}\theta_i. \tag{4.13}$$

These variables play here the role of dimensionless basis vectors and will therefore be written in bold face, whereas the θ_i played in the discussion of the last chapter the role of dynamical variables with dimension $\sqrt{\hbar}$. In the $\boldsymbol{\sigma}_i$-variables the Clifford star product (2.95) has the form

$$F*_C G=F\,\exp\left[\sum_{i=1}^{d}\frac{\overleftarrow{\partial}}{\partial\boldsymbol{\sigma}_i}\frac{\overrightarrow{\partial}}{\partial\boldsymbol{\sigma}_i}\right]G. \tag{4.14}$$

As a star product the Clifford star product is associative and distributive.

In order to show what the geometric algebra in terms of Grassmann variables and with the Clifford star product looks like one first considers the two dimensional euclidian case. There are two Grassmann basis elements $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, so that a general element of the Clifford algebra is a supernumber $A=A^0+A^1\boldsymbol{\sigma}_1+A^2\boldsymbol{\sigma}_2+A^{12}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2=\langle A\rangle_0+\langle A\rangle_1+\langle A\rangle_2$ and a vector corresponds to a supernumber with Grassmann grade one: $\boldsymbol{a}=a^1\boldsymbol{\sigma}_1+a^2\boldsymbol{\sigma}_2$. The Clifford star product of two of these supernumbers is

$$\boldsymbol{a}*_C\boldsymbol{b}=\boldsymbol{a}\boldsymbol{b}+\boldsymbol{a}\left[\sum_{i=1}^{2}\frac{\overleftarrow{\partial}}{\partial\boldsymbol{\sigma}_i}\frac{\overrightarrow{\partial}}{\partial\boldsymbol{\sigma}_i}\right]\boldsymbol{b}=(a^1b^2-a^2b^1)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2+a^1b^1+a^2b^2\equiv\boldsymbol{a}\wedge\boldsymbol{b}+\boldsymbol{a}\cdot\boldsymbol{b}, \tag{4.15}$$

where the symmetric and the antisymmetric part of the Clifford star product is given by:

$$\frac{1}{2}(\boldsymbol{a}*_C\boldsymbol{b}+\boldsymbol{b}*_C\boldsymbol{a})=a^1b^1+a^2b^2\equiv\boldsymbol{a}\cdot\boldsymbol{b} \tag{4.16}$$

$$\text{and} \qquad \frac{1}{2}(\boldsymbol{a}*_C\boldsymbol{b}-\boldsymbol{b}*_C\boldsymbol{a})=(a^1b^2-a^2b^1)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2=\boldsymbol{a}\boldsymbol{b}\equiv\boldsymbol{a}\wedge\boldsymbol{b}, \tag{4.17}$$

which are terms with Grassmann grade 0 and 2 respectively. Note that now a juxtaposition like $\boldsymbol{ab}$ is just as in the notation of superanalysis the product of supernumbers and not the Clifford product, which should be described explicitly with the star product (4.14). Note further that the $\boldsymbol{\sigma}_i$ form an orthonormal basis under the scalar product: $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \frac{1}{2}(\boldsymbol{\sigma}_i *_C \boldsymbol{\sigma}_j + \boldsymbol{\sigma}_j *_C \boldsymbol{\sigma}_i) = \delta_{ij}$.

The unit 2-blade $\mathtt{i} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ can be interpreted as the generator of $\frac{\pi}{2}$-rotations because by multiplying from the right one gets

$$\boldsymbol{\sigma}_1 *_C \mathtt{i} = \boldsymbol{\sigma}_1 \cdot \mathtt{i} = \boldsymbol{\sigma}_2 \quad , \quad \boldsymbol{\sigma}_2 *_C \mathtt{i} = \boldsymbol{\sigma}_2 \cdot \mathtt{i} = -\boldsymbol{\sigma}_1 \quad \text{and} \quad \boldsymbol{\sigma}_1 *_C \mathtt{i} *_C \mathtt{i} = -\boldsymbol{\sigma}_1, \tag{4.18}$$

so that a vector $\boldsymbol{x} = x^1\boldsymbol{\sigma}_1 + x^2\boldsymbol{\sigma}_2$ is transformed into $\boldsymbol{x}' = \boldsymbol{x} *_C \mathtt{i} = \boldsymbol{x} \cdot \mathtt{i} = x^1\boldsymbol{\sigma}_2 - x^2\boldsymbol{\sigma}_1$. The relation $\mathtt{i}^{2*_C} = -1$ describes then a reflection and furthermore one has with (2.98): $\overline{\mathtt{i}} = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = -\mathtt{i}$, so that $\mathtt{i}$ corresponds to the imaginary unit. The connection between the two dimensional vector space with vectors $\boldsymbol{x}$ and the Gauss plane with complex numbers z is established just by star multiplying $\boldsymbol{x}$ with $\boldsymbol{\sigma}_1$:

$$z = \boldsymbol{\sigma}_1 *_C \boldsymbol{x} = x^1 + \mathtt{i}x^2. \tag{4.19}$$

Such a bivector that results from star multiplying two vectors is also called spinor. While the bivector $\mathtt{i}$ generates a rotation of $\frac{\pi}{2}$ when acting from the right, the spinor z generates a general combination of a rotation and dilation when acting from the right. One can see this by writing $z = x^1 + \mathtt{i}x^2 = |z|e_{*_C}^{\mathtt{i}\varphi}$ with $|z|^2 = z *_C \overline{z} = (x^1)^2 + (x^2)^2$. Acting from the right with z causes then a dilation by $|z|$ and a rotation by φ, one has for example: $\boldsymbol{\sigma}_1 *_C z = \boldsymbol{x}$, which is the inversion of (4.19). Here one can see that the formalism of geometric algebra reproduces complex analysis and gives it a geometric meaning.

After having described the geometric algebra of the euclidian 2-space one can now turn to the euclidian 3-space with basis vectors $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$ and $\boldsymbol{\sigma}_3$ and with the Clifford star product (4.14) for $d = 3$. The basis vectors are orthogonal: $\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \delta_{ij}$ and a general c-number written as a supernumber has the form

$$A = A^0 + A^1\boldsymbol{\sigma}_1 + A^2\boldsymbol{\sigma}_2 + A^3\boldsymbol{\sigma}_3 + A^{12}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 + A^{31}\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 + A^{23}\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + A^{123}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3. \tag{4.20}$$

This multivector has now four different simple multivector parts. Besides the scalar part A^0 there is the pseudoscalar part corresponding to $I_{(3)} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$, which can be interpreted as a right handed volume form, because a parity operation gives $(-\boldsymbol{\sigma}_1)(-\boldsymbol{\sigma}_2)(-\boldsymbol{\sigma}_3) = -I_{(3)}$. Moreover $I_{(3)}$ has also the properties of an imaginary unit: $\overline{I_{(3)}} = -I_{(3)}$ and $I_{(3)} *_C I_{(3)} = I_{(3)} \cdot I_{(3)} = -1$. While the pseudoscalar $I_{(3)}$ is an oriented volume element the bivector part with the basic 2-blades

$$\mathtt{B}_1 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 = I_3 *_C \boldsymbol{\sigma}_1 \quad , \quad \mathtt{B}_2 = \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 = I_3 *_C \boldsymbol{\sigma}_2 \quad \text{and} \quad \mathtt{B}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = I_3 *_C \boldsymbol{\sigma}_3 \tag{4.21}$$

describes oriented area elements. Each of the $\mathtt{B}_i$ plays in the plane it defines the same role as the $\mathtt{i}$ of the two dimensional euclidian plane defined above. Star-multiplying with the pseudoscalar $I_{(3)}$ is equivalent to taking the Hodge dual, for example to each bivector $\mathtt{B} = b^1\mathtt{B}_1 + b^2\mathtt{B}_2 + b^3\mathtt{B}_3$ corresponds a vector $\boldsymbol{b} = b^1\boldsymbol{\sigma}_1 + b^2\boldsymbol{\sigma}_2 + b^3\boldsymbol{\sigma}_3$, which can be expressed by the equation $\mathtt{B} = I_{(3)} *_C \boldsymbol{b}$. This duality can for example be used to write the geometric product of two vectors $\boldsymbol{a} = a^1\boldsymbol{\sigma}_1 + a^2\boldsymbol{\sigma}_2 + a^3\boldsymbol{\sigma}_3$ and $\boldsymbol{b} = b^1\boldsymbol{\sigma}_1 + b^2\boldsymbol{\sigma}_2 + b^3\boldsymbol{\sigma}_3$ as:

$$\boldsymbol{a} *_C \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} + I_{(3)} *_C (\boldsymbol{a} \times \boldsymbol{b}), \tag{4.22}$$

where $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^3 a^i b^i$ and $\boldsymbol{a} \times \boldsymbol{b} = \varepsilon_{ij}{}^k a^i b^j \boldsymbol{\sigma}_k$. Furthermore one finds:

$$\boldsymbol{\sigma}_1 \times \boldsymbol{\sigma}_2 = -I_{(3)} *_C \boldsymbol{\sigma}_1 *_C \boldsymbol{\sigma}_2 = -I_{(3)} *_C \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_3 \tag{4.23}$$

and cyclic permutations.

The multivector part of (4.20) with even Grassmann grade can be described in the basis 1, $\mathtt{Q}_1 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$, $\mathtt{Q}_2 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_3$, $\mathtt{Q}_3 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$, note that $\mathtt{Q}_1 = \mathtt{B}_1$, $\mathtt{Q}_3 = \mathtt{B}_3$, but $\mathtt{Q}_2 = -\mathtt{B}_2$. The bivectors $\mathtt{Q}_i$ fulfill

$$\mathtt{Q}_1^{2*_C} = \mathtt{Q}_2^{2*_C} = \mathtt{Q}_3^{2*_C} = \mathtt{Q}_1 *_C \mathtt{Q}_2 *_C \mathtt{Q}_3 = -1, \tag{4.24}$$

so that the even multivectors $Q = q^0 + q^i \mathtt{Q}_i$ form a closed subalgebra under the Clifford star product, namely the quaternion algebra. The multivector part of (4.20) with odd grade does not close under the Clifford star product, but nevertheless one can reinvestigate the definition of the Pauli functions in (2.96). Replacing in (2.96) the scalar i by the pseudoscalar $I_{(3)}$ one sees that the basis vectors $\boldsymbol{\sigma}_i$ fulfill

$$[\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j]_{*_C} = 2\varepsilon_{ijk} I_{(3)} *_C \boldsymbol{\sigma}_k \qquad \text{and} \qquad \{\boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j\}_{*_C} = 2\delta_{ij}, \tag{4.25}$$

which justifies denoting them $\boldsymbol{\sigma}_i$. With the pseudoscalar $I_{(3)}$ the trace (2.100) can be written as

$$\mathrm{Tr}(F) = 2 \int d\boldsymbol{\sigma}_3 d\boldsymbol{\sigma}_2 d\boldsymbol{\sigma}_1 \star F = 2 \int d\boldsymbol{\sigma}_3 d\boldsymbol{\sigma}_2 d\boldsymbol{\sigma}_1 \, I_{(3)} *_C F. \tag{4.26}$$

So one has here achieved with the Clifford star product a cliffordization of the three dimensional Grassmann algebra of the $\boldsymbol{\sigma}_i$.

Just as in the two dimensional case one can also investigate in three dimensions the role of spinors and rotations. To this purpose one first considers a vector transformation of the form

$$\boldsymbol{x} \to \boldsymbol{x}' = -\boldsymbol{u} *_C \boldsymbol{x} *_C \boldsymbol{u}, \tag{4.27}$$

where $\boldsymbol{u}$ is a three dimensional unit vector: $\boldsymbol{u} = u^1\boldsymbol{\sigma}_1 + u^2\boldsymbol{\sigma}_2 + u^3\boldsymbol{\sigma}_3$ with $|\boldsymbol{u}| = \sqrt{(u^1)^2 + (u^2)^2 + (u^3)^2} = 1$. This transformation can be identified as a reflection if one decomposes $\boldsymbol{x}$ into a part collinear to $\boldsymbol{u}$ and a part orthogonal to $\boldsymbol{u}$:

$$\boldsymbol{x} = \boldsymbol{x}_\parallel + \boldsymbol{x}_\perp = (\boldsymbol{x} \cdot \boldsymbol{u} + \boldsymbol{x}\boldsymbol{u}) *_C \boldsymbol{u}, \tag{4.28}$$

with $\boldsymbol{x}_\parallel = (\boldsymbol{x} \cdot \boldsymbol{u})\boldsymbol{u}$ and $\boldsymbol{x}_\perp = (\boldsymbol{x}\boldsymbol{u}) *_C \boldsymbol{u} = (\boldsymbol{x}\boldsymbol{u}) \cdot \boldsymbol{u}$. One can check that

$$\boldsymbol{x}_\parallel *_C \boldsymbol{u} = \boldsymbol{u} *_C \boldsymbol{x}_\parallel \Rightarrow \boldsymbol{x}_\parallel \| \boldsymbol{u} \qquad \text{and} \qquad \boldsymbol{x}_\perp *_C \boldsymbol{u} = -\boldsymbol{u} *_C \boldsymbol{x}_\perp \Rightarrow \boldsymbol{x}_\perp \perp \boldsymbol{u}. \tag{4.29}$$

This decomposition of $\boldsymbol{x}$ can most easily be obtained if one just star-divides $\boldsymbol{x} *_C \boldsymbol{u} = \boldsymbol{x} \cdot \boldsymbol{u} + \boldsymbol{x} \wedge \boldsymbol{u}$ by $\boldsymbol{u}$, which gives with $\boldsymbol{u}^{-1*_C} = \boldsymbol{u}$:

$$\boldsymbol{x} = (\boldsymbol{x} \cdot \boldsymbol{u}) *_C \boldsymbol{u}^{-1*_C} + (\boldsymbol{x}\boldsymbol{u}) *_C \boldsymbol{u}^{-1*_C} = (\boldsymbol{x} \cdot \boldsymbol{u})\boldsymbol{u} + (\boldsymbol{x}\boldsymbol{u}) *_C \boldsymbol{u} = \boldsymbol{x}_\parallel + \boldsymbol{x}_\perp. \tag{4.30}$$

Using (4.29) one sees that the transformation (4.27) turns $\boldsymbol{x}$ into $\boldsymbol{x}' = -\boldsymbol{u} *_C \boldsymbol{x} *_C \boldsymbol{u} = -\boldsymbol{x}_\parallel + \boldsymbol{x}_\perp$, so that only the component collinear to $\boldsymbol{u}$ is inverted, which amounts to a reflection in the plane where $\boldsymbol{u}$ is the normal vector. Two successive transformations (4.27) lead to:

$$\boldsymbol{x} \to \boldsymbol{x}'' = -\boldsymbol{v} *_C \boldsymbol{x}' *_C \boldsymbol{v} = \boldsymbol{v} *_C \boldsymbol{u} *_C \boldsymbol{x} *_C \boldsymbol{u} *_C \boldsymbol{v} = U *_C \boldsymbol{x} *_C \overline{U}, \tag{4.31}$$

where U can be written as:

$$U = \boldsymbol{v} *_C \boldsymbol{u} = \boldsymbol{v} \cdot \boldsymbol{u} + \boldsymbol{v} \wedge \boldsymbol{u} = \cos\left(\frac{1}{2}|\mathtt{A}|\right) + \mathtt{A}_0 \sin\left(\frac{1}{2}|\mathtt{A}|\right) = e_{*_C}^{\frac{1}{2}\mathtt{A}}. \tag{4.32}$$

The angle between the unit vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is described by an bivector $\mathtt{A} = \boldsymbol{v} \wedge \boldsymbol{u} = \boldsymbol{v}\boldsymbol{u} = |\boldsymbol{v}\boldsymbol{u}| A_0$. Hereby the unit bivector $\mathtt{A}_0 = \boldsymbol{v}\boldsymbol{u}/|\boldsymbol{v}\boldsymbol{u}|$ defines the plane in which the angle lies, while the magnitude $|\boldsymbol{v}\boldsymbol{u}|$ gives the angle in radians, furthermore it fulfills $\mathtt{A}_0 *_C \mathtt{A}_0 = -1$. If one chooses for example the basis vectors $\boldsymbol{\sigma}_k$ for $\boldsymbol{u}$ and $\boldsymbol{v}$, $\mathtt{A}_0$ is given by one of the bivectors in (4.21). The additional factor 1/2 in (4.32) becomes clear if one investigates the action of the transformation (4.31). Therefore one proceeds analogously to the discussion of the reflection (4.27). One first decomposes the vector $\boldsymbol{x}$ into a part $\boldsymbol{x}_\parallel$ in the plane defined by $\mathtt{A}$ and a part $\boldsymbol{x}_\perp$ perpendicular to that plane. This is done analogously to (4.30) by star-dividing $\boldsymbol{x} *_C \mathtt{A} = \boldsymbol{x} \cdot \mathtt{A} + \boldsymbol{x} \wedge \mathtt{A}$ by $\mathtt{A}$ which leads to

$$\boldsymbol{x} = (\boldsymbol{x} \cdot \mathtt{A}) *_C \mathtt{A}^{-1*_C} + (\boldsymbol{x}\mathtt{A}) *_C \mathtt{A}^{-1*_C} = \boldsymbol{x}_\parallel + \boldsymbol{x}_\perp, \tag{4.33}$$

with $\boldsymbol{x}_\parallel *_C \mathtt{A} = -\mathtt{A} *_C \boldsymbol{x}_\parallel$ and $\boldsymbol{x}_\perp *_C \mathtt{A} = \mathtt{A} *_C \boldsymbol{x}_\perp$. One then has for the transformation (4.31):

$$U *_C \boldsymbol{x} *_C \overline{U} = e_{*_C}^{-\mathtt{A}/2} *_C \boldsymbol{x} *_C e_{*_C}^{\mathtt{A}/2} = \boldsymbol{x}_\perp + \boldsymbol{x}_\parallel *_C e_{*_C}^{\mathtt{A}}. \tag{4.34}$$

So the component perpendicular to the plane defined by $\mathtt{A}$ is not changed while the component inside this plane is rotated in that plane with the help of the spinor $e_{*_C}^{\mathtt{A}}$ by an angle of magnitude $|\mathtt{A}|$, just as described in the two dimensional case above. One sees here why the rotation in the two dimensional case could be written just by acting with a spinor from the right. This is due to the fact that when the vector lies in the plane of rotation one has

$$e_{*_C}^{-\mathtt{A}/2} *_C \boldsymbol{x}_\parallel *_C e_{*_C}^{\mathtt{A}/2} = \boldsymbol{x}_\parallel *_C e_{*_C}^{\mathtt{A}}. \tag{4.35}$$

A rotation can be described with the bivector $\mathtt{A}$, but also with the dual vector $\boldsymbol{a}$ defined by $\mathtt{A} = I_{(3)} *_C \boldsymbol{a}$, where the direction of $\boldsymbol{a}$ defines the axis of rotation, while the magnitude gives the angle in radians $|\boldsymbol{a}| = |\mathtt{A}|$. So U can also be written as:

$$U = e_{*_C}^{-\frac{1}{2} I_{(3)} *_C \boldsymbol{a}}, \tag{4.36}$$

which corresponds to the star exponential (2.109).

It is now straightforward to generalize the formalism to the case of d dimensions and an arbitrary metric $\eta_{ij} = \mathrm{diag}(\underbrace{1,\dots,1}_{p},\underbrace{-1,\dots,-1}_{q})$, with $p+q=d$. One has then d basis vectors $\boldsymbol{\sigma}_i$ which together with the star product

$$F *_C G = F \exp\left[\sum_{i,j=1}^{d} \eta_{ij} \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\sigma}_i} \frac{\overrightarrow{\partial}}{\partial \boldsymbol{\sigma}_j}\right] G \tag{4.37}$$

form the Clifford algebra $\mathcal{C}\ell(\eta) = \mathcal{C}\ell_{p,q}$. In the d-dimensional case an additional operation becomes important, namely the commutator product that is defined for two multivectors A and B as

$$A \times B = \frac{1}{2}\left(A *_C B - B *_C A\right) = \frac{1}{2}\left[A, B\right]_{*_C}, \tag{4.38}$$

which should not be confused with the vector cross product as used in (4.22). The cross product of two three-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and the commutator product of the corresponding bivectors $\mathtt{A} = I_{(3)} *_C \boldsymbol{a}$ and $\mathtt{B} = I_{(3)} *_C \boldsymbol{b}$ are connected according to

$$-I_{(3)} *_C (\boldsymbol{a} \times \boldsymbol{b}) = \frac{1}{2}\left[I_{(3)} *_C \boldsymbol{a}, I_{(3)} *_C \boldsymbol{b}\right]_{*_C} = \mathtt{A} \times \mathtt{B}. \tag{4.39}$$

The special feature of the commutator product is, that the commutator product $A_{(r)} \times B_{(2)}$ of an r-blade $A_{(r)} = A^{i_1\dots i_r}\boldsymbol{\sigma}_{i_1}\dots\boldsymbol{\sigma}_{i_r}$ and a two blade $B_{(2)} = B^{jk}\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k$ gives again an r-blade:

$$\begin{aligned}
A_{(r)} \times B_{(2)} = \frac{1}{2} A^{i_1\dots i_r} B^{jk} \Big[& \boldsymbol{\sigma}_{i_1}\dots\boldsymbol{\sigma}_{i_r}\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k - \boldsymbol{\sigma}_j\boldsymbol{\sigma}_k\boldsymbol{\sigma}_{i_1}\dots\boldsymbol{\sigma}_{i_r} \\
& + \sum_s \Big((-1)^{r-s}\eta_{ji_s}\boldsymbol{\sigma}_{i_1}\dots\check{\boldsymbol{\sigma}}_{i_s}\dots\boldsymbol{\sigma}_{i_r}\boldsymbol{\sigma}_k - (-1)^{r-s}\eta_{ki_s}\boldsymbol{\sigma}_{i_1}\dots\check{\boldsymbol{\sigma}}_{i_s}\dots\boldsymbol{\sigma}_{i_r}\boldsymbol{\sigma}_j \Big) \\
& - \sum_s \Big((-1)^{s-1}\eta_{ki_s}\boldsymbol{\sigma}_j\boldsymbol{\sigma}_{i_1}\dots\check{\boldsymbol{\sigma}}_{i_s}\dots\boldsymbol{\sigma}_{i_r} - (-1)^{s-1}\eta_{ji_s}\boldsymbol{\sigma}_k\boldsymbol{\sigma}_{i_1}\dots\check{\boldsymbol{\sigma}}_{i_s}\dots\boldsymbol{\sigma}_{i_r} \Big) \\
& + \frac{1}{2!}\sum_{s<t}(-1)^{s+t}\left(\eta_{ji_s}\eta_{ki_t} - \eta_{ki_t}\eta_{ji_s} - \eta_{ki_s}\eta_{ji_t} + \eta_{ji_t}\eta_{ki_s}\right)\boldsymbol{\sigma}_{i_1}\dots\check{\boldsymbol{\sigma}}_{i_s}\dots\check{\boldsymbol{\sigma}}_{i_t}\dots\boldsymbol{\sigma}_{i_r}\Big] \\
= A^{i_r\dots i_r} B^{jk} \sum_s & (-1)^{s-1}\left(\eta_{ki_s}\boldsymbol{\sigma}_j\boldsymbol{\sigma}_{i_1}\dots\check{\boldsymbol{\sigma}}_{i_s}\dots\boldsymbol{\sigma}_{i_r} - \eta_{ji_s}\boldsymbol{\sigma}_k\boldsymbol{\sigma}_{i_1}\dots\check{\boldsymbol{\sigma}}_{i_s}\dots\boldsymbol{\sigma}_{i_r}\right).
\end{aligned} \tag{4.40}$$

It is also easy to see, that $A\times$ acts as a derivative, i.e.

$$A \times (B *_C C) = (A \times B) *_C C + B *_C (A \times B) \tag{4.41}$$

and that the commutator product fulfills a Jacobi-identity

$$A \times (B \times C) = (A \times B) \times C + B \times (A \times C). \tag{4.42}$$

In general one can not only work in the basis of the $\boldsymbol{\sigma}_i$ but also in an arbitrary basis of basis vectors $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_d$, which do not have to be orthonormal. Then one can calculate the reciprocal base with vectors $\boldsymbol{b}^1, \boldsymbol{b}^2, \ldots, \boldsymbol{b}^d$, which is defined by the relation $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta_i^j$. The $\boldsymbol{b}^j$-vectors can be constructed with the help of the pseudoscalar $B_{(d)} = \boldsymbol{b}^1 \boldsymbol{b}^2 \ldots \boldsymbol{b}^d = \lambda I_{(d)}$, where λ is a real number. The space on which the basis vector $\boldsymbol{b}_j$ is normal is given for a d-dimensional euclidian space by the $(d-1)$-blade $(-1)^{j-1}\boldsymbol{b}_1\boldsymbol{b}_2 \ldots \check{\boldsymbol{b}}_j \ldots \boldsymbol{b}_d$, where $\check{\boldsymbol{b}}_j$ means that this basis vector is missing. The corresponding reciprocal basis vector is then given by

$$\boldsymbol{b}^j = (-1)^{j-1}\boldsymbol{b}_1\boldsymbol{b}_2 \ldots \check{\boldsymbol{b}}_j \ldots \boldsymbol{b}_d *_C B_{(d)}^{-1 *_C} / |B_{(d)}|^2, \tag{4.43}$$

where $B_{(d)}^{-1 *_C} = \frac{1}{\lambda} I_{(d)}^{-1 *_C} = \frac{1}{\lambda}\boldsymbol{\sigma}_d \boldsymbol{\sigma}_{d-1} \ldots \boldsymbol{\sigma}_1$ is the inverse of $B_{(d)}$ with respect to the Clifford star product. The absolute value of the d-blade $B_{(d)}$ is given by $|B_{(d)}| = \sqrt{B_{(d)} *_C \overline{B_{(d)}}}$. With the above definitions one has then $\boldsymbol{b}_i \cdot \boldsymbol{b}^j = \delta_i^j$, which in the case of a orthonormal euclidian basis reduces to $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}^i$.

The reciprocal basis allows the definition of the nabla operator

$$\boldsymbol{\nabla} = \boldsymbol{\sigma}^i \partial_i. \tag{4.44}$$

The nabla operator can act on multivector fields in the following ways. First the generalized gradient is given by

$$\text{grad} A = \boldsymbol{\nabla} *_C A, \tag{4.45}$$

which reduces to the conventional gradient if A is a scalar field, and acting on a vector field $\boldsymbol{a} = a^i(x_1, x_2, x_3)\boldsymbol{\sigma}_i$ one obtains

$$\boldsymbol{\nabla} *_C \boldsymbol{a} = \boldsymbol{\nabla} \cdot \boldsymbol{a} + \boldsymbol{\nabla} \wedge \boldsymbol{a} = \operatorname{div} \boldsymbol{a} + I_{(3)} *_C \operatorname{rot} \boldsymbol{a}. \tag{4.46}$$

Furthermore there is a generalized divergence

$$\text{div} A = \boldsymbol{\nabla} \cdot A \tag{4.47}$$

and a generalized rotation

$$\text{rot} A = \boldsymbol{\nabla} A. \tag{4.48}$$

Note that the product rule for the nabla operator acting on a product of multivectors is given by

$$\boldsymbol{\nabla} *_C (F *_C G) = (\boldsymbol{\nabla} *_C F) *_C G + \boldsymbol{\sigma}^i *_C F *_C \partial_i G, \tag{4.49}$$

because in general the $\boldsymbol{\sigma}_i$ do not commute with the multivector F. This can also be written in a coordinate-free manner with accents that indicate which functions is being differentiated:

$$\boldsymbol{\nabla} *_C (F *_C G) = \acute{\boldsymbol{\nabla}} *_C \acute{F} *_C G + \acute{\boldsymbol{\nabla}} *_C F *_C \acute{G}. \tag{4.50}$$

The star product formalism in the context of geometric calculus as described so far has the advantage that it gives an explicit expression for the geometric product. In contrast to the star product formalism geometric algebra as Hestenes constructed it, it is formulated with respect to the scalar and the wedge product, which represent the lowest and the highest order terms of the geometric product. All other terms of the geometric product are then formulated with the help of these two products. This approach is very practical, especially if one has only terms that are at most bivectors. But in the general case the highest and the lowest terms of an expansion have on a formal level the same status as all other terms. The star product gives now all these terms of different grade as terms of an expansion, that can be calculated in a straightforward fashion.

4.2 Vector Manifolds

In geometric algebra the points of a manifold are treated as vectors, so that a manifold can be seen as a surface in a flat background space. The at least $(d+1)$-dimensional flat background space is spanned by the rectangular basis vectors $\boldsymbol{\sigma}_a$ and it is equipped with the constant metric η_{ab}. The corresponding Clifford star product is then given by

$$F *_C G = F \exp\left[\sum_{a,b=1}^{d+1} \eta_{ab} \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\sigma}_a} \frac{\vec{\partial}}{\partial \boldsymbol{\sigma}_b}\right] G, \tag{4.51}$$

from which it follows that

$$\boldsymbol{\sigma}_a \cdot \boldsymbol{\sigma}_b = \eta_{ab}. \tag{4.52}$$

With (4.43) it is possible to calculate the reciprocal basis vectors $\boldsymbol{\sigma}^a$. The denominator in (4.43) is the determinant and the nominator leads to the cofactor of η_{ab}, so that the reciprocal basis vectors are

$$\boldsymbol{\sigma}^a = \eta^{ab} \boldsymbol{\sigma}_b, \tag{4.53}$$

where η^{ab} is the inverse of η_{ab}, i.e. $\eta_{ab}\eta^{bc} = \delta_a^c$. One has then

$$\boldsymbol{\sigma}_a \cdot \boldsymbol{\sigma}^b = \delta_a^b \qquad \text{and} \qquad \boldsymbol{\sigma}^a \cdot \boldsymbol{\sigma}^b = \eta^{ab}, \tag{4.54}$$

or more generally

$$\boldsymbol{\sigma}_a *_C \boldsymbol{\sigma}_b = \eta_{ab} + \boldsymbol{\sigma}_a \boldsymbol{\sigma}_b, \qquad \boldsymbol{\sigma}_a *_C \boldsymbol{\sigma}^b = \delta_a^b + \boldsymbol{\sigma}_a \boldsymbol{\sigma}^b, \qquad \text{and} \qquad \boldsymbol{\sigma}^a *_C \boldsymbol{\sigma}^b = \eta^{ab} + \boldsymbol{\sigma}^a \boldsymbol{\sigma}^b. \tag{4.55}$$

It is then straightforward to calculate coordinate transformations. In a three dimensional euclidian space, i.e. $\eta_{ab} = \delta_{ab}$, a vector in spherical coordinates is given by

$$\boldsymbol{x} = x^a(r,\theta,\varphi)\boldsymbol{\sigma}_a = r\sin\theta\cos\varphi\boldsymbol{\sigma}_1 + r\sin\theta\sin\varphi\boldsymbol{\sigma}_2 + r\cos\theta\boldsymbol{\sigma}_3, \tag{4.56}$$

so that the corresponding spherical basis vectors are $\boldsymbol{\tau}_r = \partial_r \boldsymbol{x}$, $\boldsymbol{\tau}_\theta = \partial_\theta \boldsymbol{x}$ and $\boldsymbol{\tau}_\varphi = \partial_\varphi \boldsymbol{x}$. The spherical metric can then be calculated as $g_{ab} = \boldsymbol{\tau}_a \cdot \boldsymbol{\tau}_b = \mathrm{diag}(1, r^2, r^2\sin^2\theta)$ and the three dimensional Clifford star product in spherical coordinates is

$$F *_C G = F \exp\left[\sum_{a,b=1}^{3} g_{ab} \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\tau}_a} \frac{\vec{\partial}}{\partial \boldsymbol{\tau}_b}\right] G. \tag{4.57}$$

Embedded in a $(d+1)$-dimensional background space with basis vectors $\boldsymbol{\sigma}_a$ one can then imagine a d-dimensional vector manifold M. The vector manifold is parametrized by smooth functions $f^a(x^1,\dots,x^d)$ as $\boldsymbol{x}(x^1,\dots,x^d) = f^a(x^1,\dots,x^d)\boldsymbol{\sigma}_a$, more common is the following notation, where the coordinates and the functions have the same name: $\boldsymbol{x}(x^1,\dots,x^d) = x^a(x^1,\dots,x^d)\boldsymbol{\sigma}_a = x^a(x^i)\boldsymbol{\sigma}_a$. The vectors

$$\boldsymbol{\xi}_i(\boldsymbol{x}) = \frac{\partial \boldsymbol{x}}{\partial x^i} \tag{4.58}$$

are the frame vectors of the manifold, which in the ambient space can be expanded as

$$\boldsymbol{\xi}_i(\boldsymbol{x}) = \xi_i^a(\boldsymbol{x})\boldsymbol{\sigma}_a. \tag{4.59}$$

The $\boldsymbol{\xi}_i(\boldsymbol{x})$ span the tangent space $T_{\boldsymbol{x}}M$, on which the Clifford star product acts as

$$F *_C G = F \exp\left[\sum_{i,j=1}^{d} g_{ij}(\boldsymbol{x}) \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\xi}_i} \frac{\vec{\partial}}{\partial \boldsymbol{\xi}_j}\right] G, \tag{4.60}$$

so that the scalar product of two basis vectors is given in the tangent space by

$$\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j = g_{ij}. \tag{4.61}$$

In the ambient space the scalar product $\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_j = (\xi_i^a \boldsymbol{\sigma}_a) \cdot (\xi_j^b \boldsymbol{\sigma}_b)$ also has to be g_{ij}, so that the metric of the vector manifold and of the ambient space are connected according to

$$\xi_i^a \xi_j^b \eta_{ab} = g_{ij}. \tag{4.62}$$

The $\xi_i^a(\boldsymbol{x})$ are here tetrad fields and the condition (4.62) assures that (4.61) is valid intrinsically, i.e. calculated with the Clifford star product (4.60) and externally, i.e. calculated with the Clifford star product (4.51) and the expansion (4.59).

For an orientable manifold there exists a global unit-pseudoscalar $I_d(\boldsymbol{x}) = \boldsymbol{\xi}_1 \boldsymbol{\xi}_2 \dots \boldsymbol{\xi}_d / |\boldsymbol{\xi}_1 \boldsymbol{\xi}_2 \dots \boldsymbol{\xi}_d|$, which allows with (4.43) the calculation of the reciprocal frame vectors $\boldsymbol{\xi}^i$ of $T_{\boldsymbol{x}}M$. In the tangent space (4.43) gives for the reciprocal base vectors $\boldsymbol{\xi}^i = g^{ij}\boldsymbol{\xi}_j$, where g^{ij} is the inverse of g_{ij}, i.e. $g_{ij}g^{jk} = \delta_i^k$, so that

$$\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}^j = \delta_i^j, \tag{4.63}$$

where the scalar product is calculated intrinsically with (4.60). In the ambient space the reciprocal frame vectors can be expanded as $\boldsymbol{\xi}^i = \xi_a^i \boldsymbol{\sigma}^a$, and to make sure that (4.63) is also valid in the ambient space with the Clifford star product (4.51) the expansion coefficients have to fulfill

$$\xi_i^a \xi_a^j = \delta_i^j. \tag{4.64}$$

Finally it is easy to show that one has as well intrinsically in the tangent space as extrinsically in the ambient space the relation

$$\boldsymbol{\xi}^i \cdot \boldsymbol{\xi}^j = g^{ij}. \tag{4.65}$$

In general one has for both, the extrinsic Clifford star product (4.51) and the intrinsic Clifford star product (4.60):

$$\boldsymbol{\xi}_i *_C \boldsymbol{\xi}_j = g_{ij} + \boldsymbol{\xi}_i \boldsymbol{\xi}_j, \qquad \boldsymbol{\xi}_i *_C \boldsymbol{\xi}^j = \delta_i^j + \boldsymbol{\xi}_i \boldsymbol{\xi}^j, \qquad \text{and} \qquad \boldsymbol{\xi}^i *_C \boldsymbol{\xi}^j = g^{ij} + \boldsymbol{\xi}^i \boldsymbol{\xi}^j. \tag{4.66}$$

With the unit pseudoscalar one can furthermore define a projector P on the vector manifold, which projects an arbitrary multivector $A(\boldsymbol{x})$ in the ambient space onto the vector manifold:

$$P(A(\boldsymbol{x}), \boldsymbol{x}) = (A(\boldsymbol{x}) \cdot I_{(d)}(\boldsymbol{x})) *_C I_{(d)}^{-1*_C}(\boldsymbol{x}). \tag{4.67}$$

A vector $\boldsymbol{v} = v^a \boldsymbol{\sigma}_a$ in the ambient space can then be decomposed into an intrinsic part

$$P(\boldsymbol{v}) = (\boldsymbol{\xi}_i \cdot \boldsymbol{v})\boldsymbol{\xi}^i = (v_a \xi_i^a)\boldsymbol{\xi}^i \tag{4.68}$$

which is tangent to the manifold and an extrinsic part $P_\perp(\boldsymbol{v}) = \boldsymbol{v} - P(\boldsymbol{v})$. Especially for a tangent vector one has $P(\boldsymbol{\xi}_i) = \boldsymbol{\xi}_i$. Applying the projector to the nabla operator of the ambient space gives a vector derivative intrinsic to the manifold:

$$\boldsymbol{\partial} = P(\boldsymbol{\nabla}) = \boldsymbol{\xi}^i(\boldsymbol{\xi}_i \cdot \boldsymbol{\nabla}) = \boldsymbol{\xi}^i(\xi_i^a \partial_a) = \boldsymbol{\xi}^i \partial_i \tag{4.69}$$

and for a tangent vector $\boldsymbol{a}$ the directional derivative in the $\boldsymbol{a}$-direction is $\boldsymbol{a} \cdot \boldsymbol{\partial} = a^i \partial_i = a^i \xi_i^a \partial_a = \boldsymbol{a} \cdot \boldsymbol{\nabla}$. With the intrinsic vector derivative (4.69) the cotangent frame vectors $\boldsymbol{\xi}^i$ can also be obtained as the gradient of the coordinate functions $x^i(\boldsymbol{x})$ that arise from the inversion of the vector manifold parametrization $\boldsymbol{x} = \boldsymbol{x}(x^1, \dots, x^d)$:

$$\boldsymbol{\xi}^i = \boldsymbol{\partial} x^i. \tag{4.70}$$

If one now applies the directional derivative $\boldsymbol{a} \cdot \boldsymbol{\partial}$ on a tangent multi-vector field $A(\boldsymbol{x})$ the result does not in general lie completely inside the manifold. So if one wants to have a purely intrinsic result one has to use the projection operator P again. This leads to the definition of a new type of derivative that acts on tangent multi-vector fields and returns tangent multi-vector fields. This new derivative is the covariant derivative and is defined by:

$$(\boldsymbol{a}\cdot\boldsymbol{D})A(\boldsymbol{x}) = P\big((\boldsymbol{a}\cdot\boldsymbol{\partial})A(\boldsymbol{x})\big). \tag{4.71}$$

In the case of a scalar field $f(\boldsymbol{x})$ on the manifold the covariant and the intrinsic derivative are the same:

$$(\boldsymbol{a}\cdot\boldsymbol{\partial})f = (\boldsymbol{a}\cdot\boldsymbol{D})f, \tag{4.72}$$

while for tangent vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$ one has

$$(\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{b} = P\big((\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{b}\big) + P_\perp\big((\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{b}\big) = (\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b} + \boldsymbol{b}\cdot\mathtt{S}(\boldsymbol{a}), \tag{4.73}$$

where $\mathtt{S}(\boldsymbol{a})$ is the so called shape tensor, which is a bivector that describes both intrinsic and extrinsic properties of the vector manifold. The covariant derivative can be seen as a map that maps two tangent vectors into a third tangent vector, fulfilling the defining relations of an affine connection:

$$(\boldsymbol{a}\cdot\boldsymbol{D})(\boldsymbol{b}+\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b} + (\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{c}, \tag{4.74a}$$

$$\big((\boldsymbol{a}+\boldsymbol{b})\cdot\boldsymbol{D}\big)\boldsymbol{c} = \big((\boldsymbol{a}\cdot\boldsymbol{D}) + (\boldsymbol{b}\cdot\boldsymbol{D})\big)\boldsymbol{c}, \tag{4.74b}$$

$$(f\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b} = f(\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b}, \tag{4.74c}$$

$$(\boldsymbol{a}\cdot\boldsymbol{D})f\boldsymbol{b} = \big((\boldsymbol{a}\cdot\boldsymbol{D})f\big)\boldsymbol{b} + f(\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b}. \tag{4.74d}$$

As a tangent vector $(\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b}$ can be expanded in the $\boldsymbol{\xi}_i$ base:

$$(\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b} = a^j\big((D_j b^i)\boldsymbol{\xi}_i + b^i(D_j\boldsymbol{\xi}_i)^k\boldsymbol{\xi}_k\big) = a^j\big(\partial_j b^i + b^k\Gamma^i_{jk}\big)\boldsymbol{\xi}_i, \tag{4.75}$$

where

$$\Gamma^i_{jk} = (D_j\boldsymbol{\xi}_k)\cdot\boldsymbol{\xi}^i = \big(D_j\boldsymbol{\xi}_k\big)^i \tag{4.76}$$

is the i-th component of $D_j\boldsymbol{\xi}_k$, which extrinsically can be written as

$$\Gamma^i_{jk} = \big(D_j\xi^a_k\boldsymbol{\sigma}_a\big)\cdot\xi^i_b\boldsymbol{\sigma}^b = (\partial_j\xi^a_k)\xi^i_a. \tag{4.77}$$

One of the properties the Γ^k_{ij} fulfill is the metric compatibility which can be found if one applies D_k on both sides of (4.61):

$$\partial_k g_{ij} - \Gamma^l_{ki}g_{lj} - \Gamma^l_{kj}g_{li} = 0, \tag{4.78}$$

which means that the Γ^i_{jk} are the Christoffel symbols and $(\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b}$ is the Levi-Civita connection. The symmetry in the lower indices of Γ^i_{jk} can be seen from the holonomy condition that is fulfilled because the frame vectors (4.58) form a coordinate basis:

$$\partial_i\boldsymbol{\xi}_j - \partial_j\boldsymbol{\xi}_i = (\partial_i\partial_j - \partial_j\partial_i)\boldsymbol{x} = 0. \tag{4.79}$$

Projecting into the manifold gives

$$D_i\boldsymbol{\xi}_j - D_j\boldsymbol{\xi}_i = 0, \tag{4.80}$$

so that the symmetry of the Γ^i_{jk} in the lower indices follows. From (4.79) follows further, that

$$(\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{b} - (\boldsymbol{b}\cdot\boldsymbol{\partial})\boldsymbol{a} = \big(a^j(\partial_j b^i) - b^j(\partial_j a^i)\big)\boldsymbol{\xi}_i \tag{4.81}$$

is an intrinsic quantity that corresponds to the Lie-derivative or the Jacobi-Lie-bracket

$$\mathscr{L}_{\boldsymbol{a}}\boldsymbol{b} = [\boldsymbol{a},\boldsymbol{b}]_{JLB} \equiv (\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{b} - (\boldsymbol{b}\cdot\boldsymbol{\partial})\boldsymbol{a} = (\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b} - (\boldsymbol{b}\cdot\boldsymbol{D})\boldsymbol{a}. \tag{4.82}$$

The holonomy condition (4.79) can then be written with $\boldsymbol{\xi}_i \cdot \boldsymbol{\partial} = \partial_i$ as $\mathscr{L}_{\boldsymbol{\xi}_i}\boldsymbol{\xi}_j = \left[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j\right]_{JLB} = 0$. It is easy to see that the Lie-derivative fulfills the relations

$$
\begin{aligned}
[\mathscr{L}_{\boldsymbol{a}}, \mathscr{L}_{\boldsymbol{b}}]\, \boldsymbol{c} &= (\mathscr{L}_{\boldsymbol{a}}\mathscr{L}_{\boldsymbol{b}} - \mathscr{L}_{\boldsymbol{b}}\mathscr{L}_{\boldsymbol{a}})\boldsymbol{c} = \mathscr{L}_{[\boldsymbol{a},\boldsymbol{b}]_{JLB}}\boldsymbol{c} && (4.83)\\
\mathscr{L}_{\boldsymbol{a}}(f\boldsymbol{b}) &= \big((\boldsymbol{a}\cdot\boldsymbol{D})f\big)\boldsymbol{b} + f(\mathscr{L}_{\boldsymbol{a}}\boldsymbol{b}). && (4.84)
\end{aligned}
$$

One can also conclude with (4.73) that since $[\boldsymbol{a}, \boldsymbol{b}]_{JLB}$ is an intrinsic quantity, the extrinsic parts in the Jacobi-Lie-bracket have to cancel, i.e.

$$\boldsymbol{a}\cdot \mathtt{S}(\boldsymbol{b}) = \boldsymbol{b}\cdot \mathtt{S}(\boldsymbol{b}). \tag{4.85}$$

Equation (4.75) shows that the intrinsic change of a vector field $\boldsymbol{b}$ in direction $\boldsymbol{a}$ consists of two parts, on the one hand the active change of the coefficients of the vector field and on the other hand a correction which corresponds to an passive change of the basis vectors due to the curvature of the manifold. If these two contributions cancel each other as one moves along a curve $\boldsymbol{c}(t)$ in the manifold the vector $\boldsymbol{b}(\boldsymbol{c}(t))$ does not move in the local frame of the $\boldsymbol{\xi}_i(\boldsymbol{c}(t))$. One says then that the vector $\boldsymbol{b}$ is parallel transported along $\boldsymbol{c}(t)$ and the condition for the parallel transport is

$$(\dot{\boldsymbol{c}}(t)\cdot\boldsymbol{D})\boldsymbol{b} = 0. \tag{4.86}$$

If the tangent vector $\dot{\boldsymbol{c}}(t)$ is parallel transported in its own direction the resulting curve is a geodesic and fulfills

$$(\dot{\boldsymbol{c}}(t)\cdot\boldsymbol{D})\dot{\boldsymbol{c}}(t) = \left(\ddot{c}^i + \Gamma^i_{jk}\dot{c}^k\dot{c}^j\right)\boldsymbol{\xi}_i = 0, \tag{4.87}$$

where the $\dot{c}^i$ are the components of $\dot{\boldsymbol{c}}$ in the $\boldsymbol{\xi}_i$ frame.

The covariant derivative of tangent vector fields can then be generalized to the covariant derivative of multivector fields by applying (4.73) to the Clifford star product $\boldsymbol{b} *_C \boldsymbol{c}$ of two tangent vector fields $\boldsymbol{b}$ and $\boldsymbol{c}$:

$$
\begin{aligned}
(\boldsymbol{a}\cdot\boldsymbol{D})(\boldsymbol{b} *_C \boldsymbol{c}) &= \big((\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{b}\big) *_C \boldsymbol{c} + (\mathtt{S}(\boldsymbol{a})\cdot\boldsymbol{b}) *_C \boldsymbol{c} + \boldsymbol{b} *_C \big((\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{c}\big) + \boldsymbol{b} *_C (\mathtt{S}(\boldsymbol{a})\cdot\boldsymbol{c})\\
&= (\boldsymbol{a}\cdot\boldsymbol{\partial})(\boldsymbol{b} *_C \boldsymbol{c}) + \mathtt{S}(\boldsymbol{a}) \times (\boldsymbol{b} *_C \boldsymbol{c}), && (4.88)
\end{aligned}
$$

where one uses the associativity of the Clifford star product and $\mathtt{S}(\boldsymbol{a})\cdot\boldsymbol{b} = \frac{1}{2}(\mathtt{S}(\boldsymbol{a}) *_C \boldsymbol{b} - \boldsymbol{b} *_C \mathtt{S}(\boldsymbol{a}))$. In general one has then

$$(\boldsymbol{a}\cdot\boldsymbol{\partial})A = (\boldsymbol{a}\cdot\boldsymbol{D})A + A\times\mathtt{S}(\boldsymbol{a}), \tag{4.89}$$

where $A\times B = \frac{1}{2}(A *_C B - B *_C A) = \frac{1}{2}\left[A, B\right]_{*_C}$ is the commutator product (not to be confused with the vector cross product used in (4.22); the cross product of two three-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and the commutator product of the corresponding bivectors $\mathtt{A} = I_{(3)} *_C \boldsymbol{a}$ and $\mathtt{B} = I_{(3)} *_C \boldsymbol{b}$ are connected according to $-I_{(3)} *_C (\boldsymbol{a}\times\boldsymbol{b}) = \frac{1}{2}\left[I_{(3)} *_C \boldsymbol{a}, I_{(3)} *_C \boldsymbol{b}\right]_{*_C} = \mathtt{A}\times\mathtt{B}$). The commutator product of an r-vector and a bivector gives again an r-vector so that all terms in (4.89) are r-vectors. Furthermore it is clear that (4.89) reduces to (4.73) if A is a vector field and to (4.72) if A is a scalar field.

A natural generalization of the Lie-derivative to multivectors is given by the Schouten-Nijenhuis bracket

$$
\begin{aligned}
\mathscr{L}_{A_{(r)}}B_{(s)} = \left[A_{(r)}, B_{(s)}\right]_{SNB} &= (-1)^{r-1}(A_{(r)}\cdot\boldsymbol{D})B_{(s)} + (-1)^{rs}(-1)^{s-1}(B_{(s)}\cdot\boldsymbol{D})A_{(r)}\\
&= (\acute{\boldsymbol{D}}\cdot A_{(r)})\acute{B}_{(s)} + (-1)^{rs}(\acute{\boldsymbol{D}}\cdot B_{(s)})\acute{A}_{(r)}. && (4.90)
\end{aligned}
$$

That the Schouten-Nijenhuis bracket can be written in this way can be seen from fact that (4.90) has the grade $r+s-1$, fulfills

$$
\begin{aligned}
\left[A_{(r)}, B_{(s)}\right]_{SNB} &= (-1)^{rs}\left[B_{(s)}, A_{(r)}\right]_{SNB} && (4.91)\\
\text{and}\quad \left[A_{(r)}, B_{(s)}C_{(t)}\right]_{SNB} &= \left[A_{(r)}, B_{(s)}\right]_{SNB}C_{(t)} + (-1)^{rs+s}B_{(s)}\left[A_{(r)}, C_{(t)}\right]_{SNB} && (4.92)
\end{aligned}
$$

and reduces for scalar functions f, g and vector fields $\boldsymbol{a}$ and $\boldsymbol{b}$ to

$$[f,g]_{SNB}=0, \qquad [\boldsymbol{a},f]_{SNB}=(\boldsymbol{a}\cdot\boldsymbol{D})f \qquad \text{and} \qquad [\boldsymbol{a},\boldsymbol{b}]_{SNB}=\mathscr{L}_{\boldsymbol{a}}\boldsymbol{b}. \tag{4.93}$$

Furthermore one has the Jacobi-identity

$$\begin{aligned}(-1)^{rt}\left[\left[A_{(r)},B_{(s)}\right]_{SNB},C_{(t)}\right]_{SNB}+(-1)^{rs}\left[\left[B_{(s)},C_{(t)}\right]_{SNB},A_{(r)}\right]_{SNB}&\\ +(-1)^{st}\left[\left[C_{(t)},A_{(r)}\right]_{SNB},B_{(s)}\right]_{SNB}&=0.\end{aligned} \tag{4.94}$$

4.3 Exterior Calculus

The exterior calculus can be constructed by noting that the cotangent frame vector or 1-form (4.70) can be written with (4.72) as [56]

$$\boldsymbol{\xi}^k=\boldsymbol{D}x^k=\boldsymbol{\partial}x^k\equiv\boldsymbol{d}x^k. \tag{4.95}$$

In order to see how the directional covariant derivative acts on a general 1-form $\boldsymbol{\omega}=\omega_i\boldsymbol{\xi}^i$ one first applies D_j on both sides of $\boldsymbol{\xi}_i\cdot\boldsymbol{\xi}^j=\delta_i^j$ which gives with (4.76)

$$(D_j\boldsymbol{\xi}^i)\cdot\boldsymbol{\xi}_k=(D_j\boldsymbol{\xi}^i)_k=-\Gamma^i_{jk}, \tag{4.96}$$

so that the covariant derivative of $\boldsymbol{\omega}$ reads

$$(\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{\omega}=a^j\big((D_j\omega_i)\boldsymbol{\xi}^i+\omega_i(D_j\boldsymbol{\xi}^i)_k\boldsymbol{\xi}^k\big)=a^j\big(\partial_j\omega_i-\omega_k\Gamma^k_{ji}\big)\boldsymbol{\xi}^i. \tag{4.97}$$

That the exterior product with the covariant derivative $\boldsymbol{D}$ corresponds to the exterior derivative can be seen if one applies the exterior derivative on $\boldsymbol{\xi}^i$:

$$\begin{aligned}\boldsymbol{dd}x^i=\boldsymbol{D}\boldsymbol{\xi}^i &= \boldsymbol{\xi}^jD_j(\boldsymbol{\xi}^kD_kx^i)\\ &= \boldsymbol{\xi}^j\left[(D_j\boldsymbol{\xi}^k)(D_kx^i)+\boldsymbol{\xi}^kD_jD_kx^i\right]\\ &= -\boldsymbol{\xi}^j\boldsymbol{\xi}^l\Gamma^k_{jl}\partial_kx^i+\boldsymbol{\xi}^j\boldsymbol{\xi}^k\partial_j\partial_kx^i=0,\end{aligned} \tag{4.98}$$

due to the antisymmetry in the upper indices and the symmetry in the lower indices. The closedness of $\boldsymbol{\xi}^i$ can for example be used to calculate the relation of the Γ^i_{jk} and the metric:

$$\Gamma^i_{jk}=(D_j\boldsymbol{\xi}_k)\cdot\boldsymbol{\xi}^i = \frac{1}{2}\left[(D_j\boldsymbol{\xi}_k)+(D_k\boldsymbol{\xi}_j)\right]\cdot\boldsymbol{\xi}^i \tag{4.99}$$

$$= \frac{1}{2}\left[\boldsymbol{\xi}_j\cdot(\boldsymbol{D}\boldsymbol{\xi}_k)+\Gamma^l_{mk}g_{jl}\boldsymbol{\xi}^m+\boldsymbol{\xi}_k\cdot(\boldsymbol{D}\boldsymbol{\xi}_j)+\Gamma^l_{mj}g_{kl}\boldsymbol{\xi}^m\right]\cdot\boldsymbol{\xi}^i \tag{4.100}$$

$$= \frac{1}{2}\left[\boldsymbol{\xi}_j\cdot(\boldsymbol{D}g_{km}\boldsymbol{\xi}^m)+\boldsymbol{\xi}_k\cdot(\boldsymbol{D}g_{jm}\boldsymbol{\xi}^m)+(\partial_mg_{jk})\boldsymbol{\xi}^m\right]\cdot\boldsymbol{\xi}^i \tag{4.101}$$

$$= \frac{1}{2}\left[(\partial_ng_{km})\boldsymbol{\xi}_j\cdot\boldsymbol{\xi}^n\boldsymbol{\xi}^m+(\partial_ng_{jm})\boldsymbol{\xi}_k\cdot\boldsymbol{\xi}^n\boldsymbol{\xi}^m+(\partial_mg_{jk})\boldsymbol{\xi}^m\right]\cdot\boldsymbol{\xi}^i \tag{4.102}$$

$$= \frac{1}{2}\left[(\partial_ng_{km})(\delta^n_j\boldsymbol{\xi}^m-\delta^m_j\boldsymbol{\xi}^n)+(\partial_ng_{jm})(\delta^n_k\boldsymbol{\xi}^m-\delta^m_k\boldsymbol{\xi}^n)+(\partial_mg_{jk})\boldsymbol{\xi}^m\right]\cdot\boldsymbol{\xi}^i \tag{4.103}$$

$$= \frac{1}{2}g^{il}\left[\partial_jg_{kl}+\partial_kg_{jl}-\partial_lg_{jk}\right], \tag{4.104}$$

where one uses in (4.99)

$$\boldsymbol{\xi}_j\cdot(\boldsymbol{D}\boldsymbol{\xi}_k)=\boldsymbol{\xi}_j\cdot(\boldsymbol{\xi}^iD_i\boldsymbol{\xi}_k)=\boldsymbol{\xi}_j\cdot(\boldsymbol{\xi}^i\Gamma^l_{ik}\boldsymbol{\xi}_l)=\Gamma^l_{ik}(\delta^i_j\boldsymbol{\xi}_l-g_{jl}\boldsymbol{\xi}^i)=D_j\boldsymbol{\xi}_k-\Gamma^l_{ik}g_{jl}\boldsymbol{\xi}^i, \tag{4.105}$$

in (4.100) the metric compatibility (4.78) and in (4.101) $\boldsymbol{D}\boldsymbol{\xi}^i = 0$.

The expression (4.104) can be used to show that the shape bivector can be written as

$$\mathtt{S}(\boldsymbol{a}) = \frac{1}{2}\big(\boldsymbol{\xi}^i \boldsymbol{\partial} a_i - \boldsymbol{\xi}_i \boldsymbol{\partial} a^i + \boldsymbol{\xi}^i (\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{\xi}_i\big), \tag{4.106}$$

or

$$\mathtt{S}_i = \mathtt{S}(\boldsymbol{\xi}_i) = \frac{1}{2}\boldsymbol{\xi}^j\boldsymbol{\xi}^k \partial_k g_{ij} + \frac{1}{2}\boldsymbol{\xi}^j \partial_i \boldsymbol{\xi}_j. \tag{4.107}$$

This can be proved by calculating

$$\begin{aligned}
\boldsymbol{b}\cdot\mathtt{S}(\boldsymbol{a}) &= \frac{1}{2}\big(b^i(\partial_j a_i)\boldsymbol{\xi}^j - b^j(\partial_j a_i)\boldsymbol{\xi}^i - b_i(\partial_j a^i)\boldsymbol{\xi}^j + b^j(\partial_j a^i)\boldsymbol{\xi}_i + a^j b^i(\partial_j \boldsymbol{\xi}_i) - a^j b^k \boldsymbol{\xi}^i(\boldsymbol{\xi}_k\cdot\partial_j\boldsymbol{\xi}_i)\big) && (4.108)\\
&= \frac{1}{2}\big(a^k b^i(\partial_j g_{ik})\boldsymbol{\xi}^j - a^k b^i(\partial_i g_{jk})\boldsymbol{\xi}^j + a^i b^j(\partial_i \boldsymbol{\xi}_j) - a^j b^k \boldsymbol{\xi}^i(\boldsymbol{\xi}_k\cdot\partial_j\boldsymbol{\xi}_i)\big) && (4.109)\\
&= -a^k b^i \Gamma^l_{ki}\boldsymbol{\xi}_l + \frac{1}{2}a^j b^k \boldsymbol{\xi}^i(\partial g_{ki}) + \frac{1}{2}a^i b^j(\partial_i\boldsymbol{\xi}_j) - \frac{1}{2}a^j b^k\boldsymbol{\xi}^i(\boldsymbol{\xi}_k\cdot\partial_j\boldsymbol{\xi}_i) && (4.110)\\
&= -a^k b^i \Gamma^l_{ki}\boldsymbol{\xi}_l + \frac{1}{2}a^i b^j \boldsymbol{\xi}^k\left[(\partial_i\boldsymbol{\xi}_j)\cdot\boldsymbol{\xi}_k\right] + \frac{1}{2}a^i b^j(\partial_i\boldsymbol{\xi}_j) && (4.111)\\
&= a^i b^j(\partial_i\boldsymbol{\xi}_j) - a^i b^j\Gamma^k_{ij}\boldsymbol{\xi}_k && (4.112)\\
&= (\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{D})\boldsymbol{b} && (4.113)\\
&= P_\perp\big((\boldsymbol{a}\cdot\boldsymbol{\partial})\boldsymbol{b}\big), && (4.114)
\end{aligned}$$

which corresponds to the definition (4.73). In (4.109) relation (4.104) was used, in (4.110) relation (4.61) and in (4.111) one uses

$$\boldsymbol{\xi}^k\left[(\partial_i\boldsymbol{\xi}_j)\cdot\boldsymbol{\xi}_k\right] = \xi^k_c\boldsymbol{\sigma}^c\left[(\partial_i\xi^a_j\boldsymbol{\sigma}_a)\cdot\xi^b_k\boldsymbol{\sigma}_b\right] = (\partial_i\xi^a_j)\boldsymbol{\sigma}_a = \partial_i\boldsymbol{\xi}_j. \tag{4.115}$$

While the exterior derivative of the reciprocal basis vectors is zero, the exterior derivative of a general 1-form $\boldsymbol{\omega} = \omega_i\boldsymbol{\xi}^i$ is a two-form

$$\boldsymbol{d\omega} = (\boldsymbol{D}\omega_j)\boldsymbol{\xi}^j + \omega_j\boldsymbol{D}\boldsymbol{\xi}^j = (\partial_i\omega_j)\boldsymbol{\xi}^i\boldsymbol{\xi}^j. \tag{4.116}$$

A general r-form is then a covariant r-blade $A^{(r)}$ and can be written as [56]

$$A^{(r)} = \frac{1}{r!}A_{i_1 i_2\ldots i_r}\boldsymbol{d}x^{i_1}\boldsymbol{d}x^{i_2}\ldots\boldsymbol{d}x^{i_r} = \frac{1}{r!}A_{i_1 i_2\ldots i_r}\boldsymbol{\xi}^{i_1}\boldsymbol{\xi}^{i_2}\ldots\boldsymbol{\xi}^{i_r}. \tag{4.117}$$

Applying the exterior differential, to $A^{(r)}$ gives with (4.98)

$$\boldsymbol{d}A^{(r)} = \frac{1}{r!}\left(\frac{\partial A_{i_1 i_2\ldots i_r}}{\partial x^j}\right)\boldsymbol{d}x^j\boldsymbol{d}x^{i_1}\boldsymbol{d}x^{i_2}\ldots\boldsymbol{d}x^{i_r} = \frac{1}{r!}\left(\frac{\partial A_{i_1 i_2\ldots i_r}}{\partial x^j}\right)\boldsymbol{\xi}^j\boldsymbol{\xi}^{i_1}\boldsymbol{\xi}^{i_2}\ldots\boldsymbol{\xi}^{i_r}, \tag{4.118}$$

which is a $(r+1)$-form or a covariant $(r+1)$-blade. Furthermore it is easy to see that (4.98) generalizes to

$$\boldsymbol{dd}A^{(r)} = 0. \tag{4.119}$$

In a three dimensional manifold the vector operations like grad, div and rot can be represented with the exterior derivative as

$$\begin{aligned}
\boldsymbol{d}f &= (\partial_1 f)\,\boldsymbol{d}x^1 + (\partial_2 f)\,\boldsymbol{d}x^2 + (\partial_3 f)\,\boldsymbol{d}x^3, && (4.120)\\
\boldsymbol{d}A^{(1)} &= (\partial_2 A_3 - \partial_3 A_2)\boldsymbol{d}x^2\boldsymbol{d}x^3 + (\partial_3 A_1 - \partial_1 A_3)\boldsymbol{d}x^3\boldsymbol{d}x^1 + (\partial_1 A_2 - \partial_2 A_1)\boldsymbol{d}x^1\boldsymbol{d}x^2, && (4.121)\\
\boldsymbol{d}A^{(2)} &= (\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3)\boldsymbol{d}x^1\boldsymbol{d}x^2\boldsymbol{d}x^3, && (4.122)
\end{aligned}$$

which gives the foundations for the integral theorems. Consider for example the integral of a scalar function f over a curve $\boldsymbol{c}(t) = c^1(t)\boldsymbol{\sigma}_1 + c^2(t)\boldsymbol{\sigma}_2$. The curve is a vector manifold with one coordinate $x^1 = t$ and tangent vector field $\boldsymbol{\xi}_t$ which is equal to the reciprocal frame field $\boldsymbol{\xi}^t = \boldsymbol{d}t$. The integral from $t = 0$ to $t = T$ of f over the curve is then

$$\begin{aligned}
\int_{\boldsymbol{c}} \boldsymbol{d}f &= \int_{\boldsymbol{c}} \boldsymbol{\xi}^t \partial_t f \\
&= \int_{\boldsymbol{c}} \boldsymbol{d}t \left(\frac{\partial f}{\partial c^1}\frac{dc^1}{dt} + \frac{\partial f}{\partial c^2}\frac{dc^2}{dt} \right) \\
&= \lim_{\substack{n\to\infty \\ \Delta\boldsymbol{c}\to 0}} \sum_{r=1}^{n} \left|\Delta\boldsymbol{c}(\boldsymbol{c}(t_r)) \cdot \boldsymbol{d}t(\boldsymbol{c}(t_r))\right| \left(\frac{\partial f}{\partial c^1}\frac{dc^1}{dt} + \frac{\partial f}{\partial c^2}\frac{dc^2}{dt} \right) \\
&= \lim_{n\to\infty} \sum_{r=1}^{n} \left|dt\, \boldsymbol{\xi}_t(\boldsymbol{c}(t_r)) \cdot \boldsymbol{\xi}^t(\boldsymbol{c}(t_r))\right| \left(\frac{\partial f}{\partial c^1}\frac{dc^1}{dt} + \frac{\partial f}{\partial c^2}\frac{dc^2}{dt} \right) \\
&= \int_0^T dt \left(\frac{\partial f}{\partial c^1}\frac{dc^1}{dt} + \frac{\partial f}{\partial c^2}\frac{dc^2}{dt} \right) \\
&= f\Big|_0^T, \qquad (4.123)
\end{aligned}$$

which is the easiest version of the Stokes theorem

$$\int_S \boldsymbol{d}A^{(r)} = \int_{\partial S} A^{(r)}. \qquad (4.124)$$

The important point is here that in the geometric algebra formalism the duality of the infinitesimal volume element $dx^1 dx^2 \dots dx^d \boldsymbol{\xi}_1 \boldsymbol{\xi}_2 \dots \boldsymbol{\xi}_d$ and the differential forms $\boldsymbol{\xi}^1 \boldsymbol{\xi}^2 \dots \boldsymbol{\xi}^d$ can be expressed with the scalar product, so that the ordinary scalar integral remains.

It is then also straightforward to translate other structures of exterior calculus into the language of superanalytic geometric algebra, for example the Hodge dual is given by

$$\star\left(\boldsymbol{\xi}^{i_1}\boldsymbol{\xi}^{i_2}\dots\boldsymbol{\xi}^{i_r}\right) = \frac{\sqrt{|g|}}{(d-r)!}\varepsilon^{i_1\dots i_r}{}_{i_{r+1}\dots i_d}\boldsymbol{\xi}^{i_{r+1}}\dots\boldsymbol{\xi}^{i_d}, \qquad (4.125)$$

with $\varepsilon^{i_1\dots i_r}{}_{i_{r+1}\dots i_d} = g^{i_1 j_1}\dots g^{i_r j_r}\varepsilon_{j_1\dots j_r i_{r+1}\dots i_d}$ and $\varepsilon_{i_1\dots i_d} = 1$ for even permutations. In the euclidian or Minkowski case the Hodge dual can be written as

$$\star A^{(r)} = (-1)^{(d-r)r + r(r-1)/2} I^{(d)} *_C A^{(r)}. \qquad (4.126)$$

For example in a d-dimensional Minkowski space with reciprocal basis vectors $\boldsymbol{\gamma}^\mu$ one has

$$\begin{aligned}
I^{(d)} *_C \boldsymbol{\gamma}^{\mu_1}\dots\boldsymbol{\gamma}^{\mu_r} &= \frac{1}{(d-r)!}\varepsilon_{\mu_{r+1}\dots\mu_d\mu_r\dots\mu_1}\boldsymbol{\gamma}^{\mu_{r+1}}\dots\boldsymbol{\gamma}^{\mu_d}\boldsymbol{\gamma}^{\mu_r}\dots\boldsymbol{\gamma}^{\mu_1} *_C \boldsymbol{\gamma}^{\mu_1}\dots\boldsymbol{\gamma}^{\mu_r} \qquad (4.127) \\
&= (-1)^{(d-r)r + r(r-1)/2}\frac{1}{(d-r)!}\varepsilon_{\mu_1\dots\mu_d}\boldsymbol{\gamma}^{\mu_{r+1}}\dots\boldsymbol{\gamma}^{\mu_d}\boldsymbol{\gamma}^{\mu_r}\dots\boldsymbol{\gamma}^{\mu_1} *_C \boldsymbol{\gamma}^{\mu_1}\dots\boldsymbol{\gamma}^{\mu_r} \qquad (4.128) \\
&= (-1)^{(d-r)r + r(r-1)/2} g_{\mu_1\nu_1}g^{\mu_1\mu_1}\dots g_{\mu_r\nu_r}g^{\mu_r\mu_r}\frac{1}{(d-r)!}\varepsilon^{\nu_1\dots\nu_r}{}_{\mu_{r+1}\dots\mu_d}\boldsymbol{\gamma}^{\mu_{r+1}}\dots\boldsymbol{\gamma}^{\mu_d} \qquad (4.129) \\
&= (-1)^{(d-r)r + r(r-1)/2} \star \boldsymbol{\gamma}^{\mu_1}\dots\boldsymbol{\gamma}^{\mu_r}, \qquad (4.130)
\end{aligned}$$

where there is no summation over $\mu_1, \dots, \mu_r$. Applying the Hodge star operator twice gives

$$\begin{aligned}
\star\star\boldsymbol{\gamma}^{\mu_1}\dots\boldsymbol{\gamma}^{\mu_r} &= \star(-1)^{(d-r)r+(r-1)r/2} I^{(d)} *_C \boldsymbol{\gamma}^{\mu_1}\dots\boldsymbol{\gamma}^{\mu_r} && (4.131)\\
&= (-1)^{(d-r)r+(r-1)r/2}(-1)^{r(d-r)+((d-r)-1)(d-r)/2} I^{(d)} *_C I^{(d)} *_C \boldsymbol{\gamma}^{\mu_1}\dots\boldsymbol{\gamma}^{\mu_r} && (4.132)\\
&= (-1)^{(r-1)r/2}(-1)^{(d^2-2rd+r^2-d+r)/2}(-1)^{-d(d-1)/2}\overline{I^{(d)}} *_C I^{(d)} *_C \boldsymbol{\gamma}^{\mu_1}\dots\boldsymbol{\gamma}^{\mu_r} && (4.133)\\
&= (-1)^{r(d-r)} g^{11}\dots g^{dd}\boldsymbol{\gamma}^{\mu_1}\dots\boldsymbol{\gamma}^{\mu_r}, && (4.134)
\end{aligned}$$

so that in the euclidian case one has for the inverse Hodge star operator

$$\star^{-1} = (-1)^{r(d-r)}\star = (-1)^{(r-1)r/2} I^{(d)} *_C, \tag{4.135}$$

while in the four dimensional Minkowski case one has an additional minus sign, i.e. $\star^{-1} = (-1)^{r(d-r)+1}\star$.

With the Hodge star operator as defined in (4.125) the coderivative $\boldsymbol{d}^\dagger$ is given in the Riemannian case as

$$\boldsymbol{d}^\dagger A^{(r)} = (-1)^{dr+d+1} \star \boldsymbol{d} \star A^{(r)} \tag{4.136}$$

and in the Minkowski case as

$$\boldsymbol{d}^\dagger A^{(r)} = (-1)^{dr+d} \star \boldsymbol{d} \star A^{(r)}. \tag{4.137}$$

Writing this down in components one directly sees that the coderivative as an operator that maps an r-form into an $(r-1)$-form can be written as

$$\boldsymbol{d}^\dagger A^{(r)} = -\boldsymbol{d} \cdot A^{(r)}. \tag{4.138}$$

The interior product that maps an r-blade $A^{(r)}$ into an $(r-1)$-blade is just the scalar product with a vector $\boldsymbol{a} = a^i \boldsymbol{\xi}_i$:

$$i_{\boldsymbol{a}} A^{(r)} = \boldsymbol{a} \cdot A^{(r)} = \frac{1}{(r-1)!} a^j A_{ji_2\dots i_r} \boldsymbol{\xi}^{i_2}\dots\boldsymbol{\xi}^{i_r}, \tag{4.139}$$

so that one has for example

$$i_{\boldsymbol{\xi}_1}\boldsymbol{\xi}^1\boldsymbol{\xi}^2 = \boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}^1\boldsymbol{\xi}^2 = \boldsymbol{\xi}^2, \qquad i_{\boldsymbol{\xi}_1}\boldsymbol{\xi}^2\boldsymbol{\xi}^3 = \boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}^2\boldsymbol{\xi}^3 = 0 \qquad \text{and} \qquad i_{\boldsymbol{\xi}_1}\boldsymbol{\xi}^3\boldsymbol{\xi}^1 = \boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}^3\boldsymbol{\xi}^1 = -\boldsymbol{\xi}^3. \tag{4.140}$$

The interior product can be generalized to the case of two multivectors $A_{(r)}$ and $B^{(s)}$:

$$i_{A_{(r)}} B^{(s)} = \overline{A_{(r)}} \cdot B^{(s)}. \tag{4.141}$$

With this generalized interior product one can for example write the contracted exterior derivative of a 1-form as

$$i_{\boldsymbol{ab}}\boldsymbol{d\omega} = (\boldsymbol{a}\cdot\boldsymbol{\partial})(\boldsymbol{b}\cdot\boldsymbol{\omega}) - (\boldsymbol{b}\cdot\boldsymbol{\partial})(\boldsymbol{a}\cdot\boldsymbol{\omega}) - \boldsymbol{\omega}\cdot[\boldsymbol{a},\boldsymbol{b}]_{JLB} = (\partial_i\omega_j)(a^i b^j - a^j b^i) \tag{4.142}$$

or in general

$$\begin{aligned}
i_{\boldsymbol{a}_1\boldsymbol{a}_2\dots\boldsymbol{a}_{r+1}}\boldsymbol{d}A^{(r)} &= \sum_{n=1}^{r+1}(-1)^{n+1}(\boldsymbol{a}_n\cdot\boldsymbol{\partial})\left(\overline{\boldsymbol{a}_1\dots\check{\boldsymbol{a}}_n\dots\boldsymbol{a}_{r+1}}\right)\cdot A^{(r)}\\
&\quad + \sum_{m<n}(-1)^{m+n}\left(\overline{[\boldsymbol{a}_m,\boldsymbol{a}_n]_{JLB}\,\boldsymbol{a}_1\dots\check{\boldsymbol{a}}_m\dots\check{\boldsymbol{a}}_n\dots\boldsymbol{a}_{r+1}}\right)\cdot A^{(r)}. && (4.143)
\end{aligned}$$

The Lie derivative of a 1-form is defined with the interior product as

$$i_{\boldsymbol{b}}\mathscr{L}_{\boldsymbol{a}}\boldsymbol{\omega} = \mathscr{L}_{\boldsymbol{a}}(i_{\boldsymbol{b}}\boldsymbol{\omega}) - i_{\mathscr{L}_{\boldsymbol{a}}\boldsymbol{b}}\boldsymbol{\omega}, \tag{4.144}$$

so that Cartan's magic formula follows

$$\mathscr{L}_{\boldsymbol{a}}\boldsymbol{\omega} = (\boldsymbol{d}i_{\boldsymbol{a}} + i_{\boldsymbol{a}}\boldsymbol{d})\,\boldsymbol{\omega} = \left(a^i(\partial_i\omega_j) + (\partial_j a^i)\omega_i\right)\boldsymbol{\xi}^j. \tag{4.145}$$

The exterior derivative can then be written with the Lie derivative as

$$\boldsymbol{d\omega} = \boldsymbol{\xi}^i \mathscr{L}_{\boldsymbol{\xi}_i} \boldsymbol{\omega}, \tag{4.146}$$

which is the generalization of $\boldsymbol{d}f = \boldsymbol{\xi}^i \partial_i f$, and for the coderivative one has similarly

$$\delta\boldsymbol{\omega} = -\boldsymbol{\xi}_i \cdot \mathscr{L}_{\boldsymbol{\xi}_i} \boldsymbol{\omega}. \tag{4.147}$$

It is easy to see that the Lie derivative of a 1-form fulfills the following relations

$$\begin{aligned}
[\mathscr{L}_{\boldsymbol{a}}, \mathscr{L}_{\boldsymbol{b}}]\,\boldsymbol{\omega} &= (\mathscr{L}_{\boldsymbol{a}}\mathscr{L}_{\boldsymbol{b}} - \mathscr{L}_{\boldsymbol{b}}\mathscr{L}_{\boldsymbol{a}})\boldsymbol{\omega} = \mathscr{L}_{[\boldsymbol{a},\boldsymbol{b}]_{JLB}}\boldsymbol{\omega}, && (4.148)\\
[\mathscr{L}_{\boldsymbol{a}}, i_{\boldsymbol{b}}]\,\boldsymbol{\omega} &= (\mathscr{L}_{\boldsymbol{a}} i_{\boldsymbol{b}} - i_{\boldsymbol{b}}\mathscr{L}_{\boldsymbol{a}})\boldsymbol{\omega} = i_{[\boldsymbol{a},\boldsymbol{b}]_{JLB}}\boldsymbol{\omega}, && (4.149)\\
\boldsymbol{d}\mathscr{L}_{\boldsymbol{a}}\boldsymbol{\omega} &= \mathscr{L}_{\boldsymbol{a}}\boldsymbol{d\omega}. && (4.150)
\end{aligned}$$

The Lie derivative of an r-form is

$$\mathscr{L}_{\boldsymbol{a}} A^{(r)} = (\boldsymbol{d} i_{\boldsymbol{a}} + i_{\boldsymbol{a}} \boldsymbol{d})\, A^{(r)} = \boldsymbol{D}(\boldsymbol{a} \cdot A^{(r)}) + \boldsymbol{a} \cdot (\boldsymbol{D} A^{(r)}). \tag{4.151}$$

Up to now only the coordinate basis of the $\boldsymbol{\xi}_i$ was used, in general it is also possible to use a non-coordinate basis given by

$$\boldsymbol{\vartheta}_r = \vartheta^i_r \boldsymbol{\xi}_i \qquad \text{and} \qquad \boldsymbol{\xi}_i = \vartheta^r_i \boldsymbol{\vartheta}_r, \tag{4.152}$$

where ϑ^i_r are functions of the x^k, with $\vartheta^r_i \vartheta^j_r = \delta^j_i$ and $g_{ij} = \vartheta^r_i \vartheta^s_j g_{rs}$. Analogously the reciprocal non-coordinate basis $\boldsymbol{\vartheta}^r$ can be expanded with the ϑ^r_i in the reciprocal coordinate basis of the $\boldsymbol{\xi}^i$. A special choice for the non-coordinate frame fields is obtained by the conditions $\boldsymbol{\vartheta}_r \cdot \boldsymbol{\vartheta}_s = \eta_{rs}$ and $\partial_i \boldsymbol{\vartheta}_r = 0$. This means the $\boldsymbol{\vartheta}_r$ span a (pseudo)-euclidian base and they move on the vector-manifold so that

$$D_i \boldsymbol{\vartheta}_r = -\boldsymbol{\vartheta}_r \cdot \mathtt{S}_i. \tag{4.153}$$

This shows that the shape tensor that in the $\boldsymbol{\vartheta}_r$-frame has the form $\mathtt{S}_r = \mathtt{S}(\boldsymbol{\vartheta}_r) = \vartheta^i_r \mathtt{S}_i$ is proportional to the Fock-Ivanenko bivector Γ_i [102], i.e. $\mathtt{S}_i = -2\Gamma_i$.

For non-coordinate basis vectors the Jacobi-Lie bracket is no longer zero, one rather has

$$\begin{aligned}
[\boldsymbol{\vartheta}_r, \boldsymbol{\vartheta}_s]_{JLB} &= \vartheta^i_r (\boldsymbol{\xi}_i \cdot \boldsymbol{D})(\vartheta^j_s \boldsymbol{\xi}_j) - \vartheta^i_s (\boldsymbol{\xi}_i \cdot \boldsymbol{D})(\vartheta^j_r \boldsymbol{\xi}_j) && (4.154)\\
&= \vartheta^i_r \left[(D_i \vartheta^j_s)\boldsymbol{\xi}_j + \vartheta^j_s (D_i \boldsymbol{\xi}_j)\right] - \vartheta^i_s \left[(D_i \vartheta^j_r)\boldsymbol{\xi}_j + \vartheta^j_r (D_i \boldsymbol{\xi}_j)\right] && (4.155)\\
&= \left[\vartheta^i_r D_i \vartheta^j_s - \vartheta^i_s D_i \vartheta^j_r\right] \boldsymbol{\xi}_j && (4.156)\\
&= \left[\partial_r \vartheta^j_s - \partial_s \vartheta^j_r\right] \vartheta^t_j \boldsymbol{\vartheta}_t && (4.157)\\
&= C^t_{rs} \boldsymbol{\vartheta}_t, && (4.158)
\end{aligned}$$

with

$$C^t_{rs} = [\boldsymbol{\vartheta}_r, \boldsymbol{\vartheta}_s]_{JLB} \cdot \boldsymbol{\vartheta}^t = \left[\partial_r \vartheta^j_s - \partial_s \vartheta^j_r\right] \vartheta^t_j. \tag{4.159}$$

For tangent vector fields $\boldsymbol{a} = a^r \boldsymbol{\vartheta}_r$ and $\boldsymbol{b} = b^s \boldsymbol{\vartheta}_s$ it follows then that

$$\mathscr{L}_{\boldsymbol{a}} \boldsymbol{b} = [\boldsymbol{a}, \boldsymbol{b}]_{JLB} = \left(a^r (\partial_r b^s) - b^r (\partial_r a^s)\right)\boldsymbol{\vartheta}_s + a^r b^s \left[\boldsymbol{\vartheta}_r, \boldsymbol{\vartheta}_s\right]_{JLB}, \tag{4.160}$$

which reduces in a coordinate basis to (4.81).

In the non-coordinate basis the Γ^t_{rs} are given by

$$\begin{aligned}
\Gamma^t_{rs} = -\left[(\boldsymbol{\vartheta}_r \cdot \boldsymbol{D})\boldsymbol{\vartheta}^t\right] \cdot \boldsymbol{\vartheta}_s &= -\left[(\vartheta^i_r(\boldsymbol{\xi}_i \cdot \boldsymbol{D})\vartheta^t_k \boldsymbol{\xi}^k\right] \cdot \vartheta^j_s \boldsymbol{\xi}_j && (4.161)\\
&= \vartheta^t_j \vartheta^i_r \partial_i \vartheta^j_s + \Gamma^k_{ij} \vartheta^i_r \vartheta^t_k \vartheta^j_s && (4.162)\\
&= \vartheta^t_j \vartheta^i_r \partial_i \vartheta^j_s + \frac{1}{2} g^{kl} \left[\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right] \vartheta^i_r \vartheta^t_k \vartheta^j_s && (4.163)\\
&= \vartheta^t_j \vartheta^i_r \partial_i \vartheta^j_s + \frac{1}{2} g^{tu} \left[\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right] \vartheta^i_r \vartheta^l_u \vartheta^j_s && (4.164)\\
&= \vartheta^t_j \partial_r \vartheta^j_s + \frac{1}{2} g^{tu} \left[\partial_i (\vartheta^v_j \vartheta^w_l g_{vw}) + \partial_j (\vartheta^v_i \vartheta^w_l g_{vw}) - \partial_l (\vartheta^v_i \vartheta^w_j g_{vw})\right] \vartheta^i_r \vartheta^l_u \vartheta^j_s && (4.165)\\
&= \frac{1}{2} g^{tu} \left[\partial_r g_{su} + \partial_s g_{ru} - \partial_u g_{rs}\right] + \frac{1}{2} g^{tu} (C_{urs} + C_{usr} - C_{sru}). && (4.166)
\end{aligned}$$

In (4.162) one uses

$$\vartheta^r_i \partial_t \vartheta^i_s = -\vartheta^i_s \partial_t \vartheta^r_i, \tag{4.167}$$

which results from acting with ∂_t on $\vartheta^r_i \vartheta^i_s = \delta^r_s$, in (4.164) one uses

$$\vartheta^t_i g^{il} = \vartheta^t_i (\vartheta^i_r \vartheta^l_s g^{rs}) = \delta^t_r \vartheta^l_s g^{rs} = \vartheta^l_s g^{ts} \tag{4.168}$$

and in (4.165) one uses the definition

$$C_{rsu} = g_{tu} C^t_{rs}. \tag{4.169}$$

While in the coordinate base $\left[\boldsymbol{\xi}_i, \boldsymbol{\xi}_j\right]_{JLB} = 0$ insured that the Γ^k_{ij} are symmetric in the lower indices, one has with (4.162) and (4.159) in the non-coordinate basis the relation

$$\Gamma^t_{rs} - \Gamma^t_{sr} = C^t_{rs}, \tag{4.170}$$

which implies that the non-coordinate 1-forms $\boldsymbol{\vartheta}^r$ are not closed:

$$\begin{aligned}
\boldsymbol{d}\boldsymbol{\vartheta}^r = \boldsymbol{\xi}^j D_j (\vartheta^r_i \boldsymbol{\xi}^i) &= \frac{1}{2} (\partial_i \vartheta^r_j - \partial_j \vartheta^r_i) \boldsymbol{\xi}^i \boldsymbol{\xi}^j && (4.171)\\
&= \frac{1}{2} \left(\vartheta^s_i (\boldsymbol{\vartheta}_s \cdot \boldsymbol{\partial}) \vartheta^r_j - \vartheta^s_j (\boldsymbol{\vartheta}_s \cdot \boldsymbol{\partial}) \vartheta^r_i\right) \vartheta^i_t \vartheta^j_u \boldsymbol{\vartheta}^t \boldsymbol{\vartheta}^u && (4.172)\\
&= \frac{1}{2} \left(\vartheta^i_u (\boldsymbol{\vartheta}_t \cdot \boldsymbol{\partial}) \vartheta^r_i - \vartheta^j_t (\boldsymbol{\vartheta}_u \cdot \boldsymbol{\partial}) \vartheta^r_j\right) \boldsymbol{\vartheta}^t \boldsymbol{\vartheta}^u && (4.173)\\
&= -\frac{1}{2} \left(\vartheta^r_i (\boldsymbol{\vartheta}_t \cdot \boldsymbol{\partial}) \vartheta^i_u - \vartheta^r_j (\boldsymbol{\vartheta}_u \cdot \boldsymbol{\partial}) \vartheta^j_t\right) \boldsymbol{\vartheta}^t \boldsymbol{\vartheta}^u && (4.174)\\
&= -\frac{1}{2} C^r_{tu} \boldsymbol{\vartheta}^t \boldsymbol{\vartheta}^u, && (4.175)
\end{aligned}$$

which is the Maurer-Cartan equation, that with (4.170) can also be written as $\boldsymbol{d}\boldsymbol{\vartheta}^r = -\Gamma^r_{st} \boldsymbol{\vartheta}^s \boldsymbol{\vartheta}^t$. The exterior derivative of a general non-coordinate 1-form $\boldsymbol{\alpha} = \alpha_r \boldsymbol{\vartheta}^r$ is

$$\boldsymbol{d}\boldsymbol{\alpha} = (\boldsymbol{D}\alpha_r) \boldsymbol{\vartheta}^r + \alpha_r \boldsymbol{D}\boldsymbol{\vartheta}^r = (\partial_r \alpha_s - \alpha_t \Gamma^t_{rs}) \boldsymbol{\vartheta}^r \boldsymbol{\vartheta}^s, \tag{4.176}$$

which should be compared with (4.116), and for the exterior derivative of a general r-form in the non-coordinate basis $A^{(r)} = \frac{1}{r!} A_{s_1 \ldots s_r} \boldsymbol{\vartheta}^{s_1} \ldots \boldsymbol{\vartheta}^{s_r}$ one obtains

$$\boldsymbol{d}A^{(r)} = \frac{(-1)^r}{(r+1)!} \left(\partial_{[s_{r+1}} A_{s_1 \ldots s_r]} - \Gamma^t_{[s_{r+1} s_k} A_{s_1 \ldots s_{k-1} t s_{k+1} \ldots s_r]}\right) \boldsymbol{\vartheta}^{s_1} \boldsymbol{\vartheta}^{s_2} \ldots \boldsymbol{\vartheta}^{s_{r+1}}, \tag{4.177}$$

where the square brackets antisymmetrize the lower indices.

4.4 Tensor Calculus

The formalism developed so far can then in a straightforward fashion be extended to tensor calculus. A tensor is a multilinear map of r vectors and s one-forms into the real numbers, and can be written as

$$\mathsf{T} = T^{i_1\cdots i_s}_{j_1\cdots j_r}\boldsymbol{\xi}_{i_1}\otimes\cdots\otimes\boldsymbol{\xi}_{i_s}\otimes\boldsymbol{\xi}^{j_1}\otimes\cdots\otimes\boldsymbol{\xi}^{j_r}. \tag{4.178}$$

The components of the tensor can be obtained as

$$T^{i_1\cdots i_s}_{j_1\cdots j_r} = \mathsf{T}(\boldsymbol{\xi}^{i_1},\ldots,\boldsymbol{\xi}^{i_s},\boldsymbol{\xi}_{j_1},\ldots,\boldsymbol{\xi}_{j_r}) = i_{\boldsymbol{\xi}^{i_1}\otimes\cdots\otimes\boldsymbol{\xi}_{j_r}}\mathsf{T} = (\boldsymbol{\xi}^{i_1}\otimes\cdots\otimes\boldsymbol{\xi}_{j_r})\cdot\mathsf{T}. \tag{4.179}$$

A change of the base according to

$$\boldsymbol{\xi}_{i'} = \xi^i_{i'}\boldsymbol{\xi}_i \qquad \text{and} \qquad \boldsymbol{\xi}^{j'} = \xi^{j'}_j\boldsymbol{\xi}^j, \tag{4.180}$$

where $\xi^i_{i'}$ and $\xi^{j'}_j$ are functions with $\xi^i_{i'}\xi^{j'}_i = \delta^{j'}_{i'}$ and

$$\xi^i_{i'} = \frac{\partial\boldsymbol{\xi}_{i'}}{\partial\boldsymbol{\xi}_i} = \frac{\partial\boldsymbol{\xi}^i}{\partial\boldsymbol{\xi}^{i'}} \qquad \text{and} \qquad \xi^{j'}_j = \frac{\partial\boldsymbol{\xi}^{j'}}{\partial\boldsymbol{\xi}^j} = \frac{\partial\boldsymbol{\xi}_j}{\partial\boldsymbol{\xi}_{j'}} \tag{4.181}$$

leads to a change of the tensor components according to

$$\begin{aligned}\mathsf{T} &= T^{i'_1\cdots i'_s}_{j'_1\cdots j'_r}\boldsymbol{\xi}_{i'_1}\otimes\cdots\otimes\boldsymbol{\xi}_{i'_s}\otimes\boldsymbol{\xi}^{j'_1}\otimes\cdots\otimes\boldsymbol{\xi}^{j'_r}\\ &= T^{i'_1\cdots i'_s}_{j'_1\cdots j'_r}\frac{\partial\boldsymbol{\xi}^{i_1}}{\partial\boldsymbol{\xi}^{i'_1}}\cdots\frac{\partial\boldsymbol{\xi}^{i_s}}{\partial\boldsymbol{\xi}^{i'_s}}\frac{\partial\boldsymbol{\xi}^{j'_1}}{\partial\boldsymbol{\xi}^{j_1}}\cdots\frac{\partial\boldsymbol{\xi}^{j'_r}}{\partial\boldsymbol{\xi}^{j_r}}\boldsymbol{\xi}_{i_1}\otimes\cdots\otimes\boldsymbol{\xi}_{i_s}\otimes\boldsymbol{\xi}^{j_1}\otimes\cdots\otimes\boldsymbol{\xi}^{j_r}.\end{aligned} \tag{4.182}$$

It is clear that the special case of a totally anti-symmetric tensor that maps r vectors into a scalar is a r-form, i.e. it has the form

$$\mathsf{A} = A_{i_1,\ldots,i_r}\boldsymbol{\xi}^{i_1}\wedge\ldots\wedge\boldsymbol{\xi}^{i_r} = A_{i_1,\ldots,i_r}\boldsymbol{\xi}^{i_1}\ldots\boldsymbol{\xi}^{i_r}. \tag{4.183}$$

The covariant derivative and the Lie derivative of a tensor can be obtained by acting on the components and the basis vectors of the tensor. As an example one can consider the metric tensor

$$\mathsf{g} = g_{ij}\boldsymbol{\xi}^i\otimes\boldsymbol{\xi}^j = g_{ij}\boldsymbol{d}x^i\otimes\boldsymbol{d}x^j, \tag{4.184}$$

that maps two vectors $\boldsymbol{a} = a^i\boldsymbol{\xi}_i$ and $\boldsymbol{b} = b^i\boldsymbol{\xi}_i$ into a scalar according to

$$\mathsf{g}(\boldsymbol{a},\boldsymbol{b}) = i_{\boldsymbol{a}\otimes\boldsymbol{b}}\mathsf{g} = (a^k\boldsymbol{\xi}_k\otimes b^l\boldsymbol{\xi}_l)\cdot(g_{ij}\boldsymbol{\xi}^i\otimes\boldsymbol{\xi}^j) = g_{ij}a^ib^j. \tag{4.185}$$

The covariant derivative of g is then

$$D_k\mathsf{g} = (D_kg_{ij})\boldsymbol{\xi}^i\otimes\boldsymbol{\xi}^j + g_{ij}(D_k\boldsymbol{\xi}^i)\otimes\boldsymbol{\xi}^j + g_{ij}\boldsymbol{\xi}^i\otimes(D_k\boldsymbol{\xi}^j) \tag{4.186}$$

$$= (\partial_kg_{ij} - \Gamma^l_{ki}g_{lj} - \Gamma^l_{kj}g_{il})\boldsymbol{\xi}^i\otimes\boldsymbol{\xi}^j, \tag{4.187}$$

so that the condition (4.78) can be written as $(D_k\mathsf{g})_{ij} = 0$. The Lie derivative of the metric tensor is

$$\mathscr{L}_{\boldsymbol{a}}\mathsf{g} = (\mathscr{L}_{\boldsymbol{a}}g_{ij})\boldsymbol{\xi}^i\otimes\boldsymbol{\xi}^j + g_{ij}(\mathscr{L}_{\boldsymbol{a}}\boldsymbol{\xi}^i)\otimes\boldsymbol{\xi}^j + g_{ij}\boldsymbol{\xi}^i\otimes(\mathscr{L}_{\boldsymbol{a}}\boldsymbol{\xi}^j) \tag{4.188}$$

$$= \big(a^k\partial_kg_{ij} + g_{kj}(\partial_ia^k) + g_{ik}(\partial_ja^k)\big)\boldsymbol{\xi}^i\otimes\boldsymbol{\xi}^j \tag{4.189}$$

$$= \big(a^k(\partial_kg_{ij} - \partial_ig_{jk} - \partial_jg_{ik}) + \partial_ia_j + \partial_ja_i\big)\boldsymbol{\xi}^i\otimes\boldsymbol{\xi}^j \tag{4.190}$$

$$= (\partial_ia_j - \Gamma^k_{ij}a_k + \partial_ja_i - \Gamma^k_{ji}a_k)\boldsymbol{\xi}^i\otimes\boldsymbol{\xi}^j. \tag{4.191}$$

If the metric tensor does not change in the direction of the vector field $\boldsymbol{a}$, i.e.

$$(\mathscr{L}_{\boldsymbol{a}}\mathsf{g})_{ij} = a_{i;j} - a_{j;i} = 0, \tag{4.192}$$

with $a_{i;j} = \partial_j a_i - \Gamma^k_{ji} a_k$, then $\boldsymbol{a}$ generates an isometry and is a killing vector.

The above tensor concept can be generalized in several ways. For example one can consider a function that maps r contravariant and s covariant blades of arbitrary grade into a scalar, i.e. tensors of the form

$$\mathsf{T} = T^{i_1 \dots i_r}_{j_1 \dots j_s} A^{(r_1)}_{i_1} \otimes \dots \otimes A^{(r_r)}_{i_r} \otimes B^{j_1}_{(s_1)} \otimes \dots \otimes B^{j_s}_{(s_s)}. \tag{4.193}$$

The other possibility is to consider multivector valued tensors. In this case a tensor maps a number of (multi)vectors into a multivector. The above notation runs into difficulties if one generalizes tensors in this way. So the tensor concept is founded in geometric algebra on a linear map $\mathsf{F}(\boldsymbol{a})$ that maps a vector $\boldsymbol{a}$ into another vector, that in general does not have to lie in the same space as $\boldsymbol{a}$. The linear map F is then generalized to multivectors by the rule

$$\mathsf{F}(\boldsymbol{ab}) = \mathsf{F}(\boldsymbol{a})\mathsf{F}(\boldsymbol{b}), \tag{4.194}$$

so that it is grade preserving, i.e. $\mathsf{F}(A_{(r)}) = \langle \mathsf{F}(A_{(r)}) \rangle_r$. The adjoint map $\mathsf{F}^\dagger$ is defined by

$$\boldsymbol{a} \cdot \mathsf{F}^\dagger(\boldsymbol{b}) = \mathsf{F}(\boldsymbol{a}) \cdot \boldsymbol{b}, \tag{4.195}$$

with $(\mathsf{F}^\dagger)^\dagger(\boldsymbol{a}) = \mathsf{F}(\boldsymbol{a})$ and $(\mathsf{FG})^\dagger(\boldsymbol{a}) = \mathsf{G}^\dagger\mathsf{F}^\dagger(\boldsymbol{a})$. It is easy to see that the definition (4.195) of the adjoint map generalizes to bivectors as $\mathtt{B}_1 \cdot \mathsf{F}(\mathtt{B}_2) = \mathsf{F}^\dagger(\mathtt{B}_1) \cdot \mathtt{B}_2$ and to multivectors as

$$\langle A *_C \mathsf{F}^\dagger(B) \rangle_0 = \langle \mathsf{F}(A) *_C \mathtt{B} \rangle_0. \tag{4.196}$$

The determinant of F is defined as

$$\mathsf{F}(I_{(d)}) = \det(\mathsf{F}) I_{(d)}, \tag{4.197}$$

where a short calculation shows that $\det(\mathsf{FG}) I_{(d)} = \det(\mathsf{G}) \det(\mathsf{F}) I_{(d)}$ and $\det(\mathsf{F}^\dagger) = \det(\mathsf{F})$. Having defined the determinant it is then possible to calculate the inverse F^{-1} of F. To this purpose one notices that from (4.197) it follows that

$$\det(\mathsf{F}) I_{(d)} *_C B = \mathsf{F}(I_{(d)}) *_C B = \mathsf{F}(I_{(d)} *_C \mathsf{F}^\dagger(B)), \tag{4.198}$$

where one uses in the last step

$$\mathsf{F}(A_{(r)} \cdot \mathsf{F}^\dagger(B_{(s)})) = \mathsf{F}(A_{(r)}) \cdot B_{(s)} \qquad \text{for } r \geq s. \tag{4.199}$$

Setting now $A = I_{(d)} *_C B$ one obtains

$$\det(\mathsf{F}) A = \mathsf{F}\big(I_{(d)} *_C \mathsf{F}^\dagger(I_{(d)}^{-1 *_C} *_C A)\big) \tag{4.200}$$

$$\Rightarrow \quad \mathsf{F}^{-1}(A) = \frac{1}{\det(\mathsf{F})} I_{(d)} *_C \mathsf{F}^\dagger(I_{(d)}^{-1 *_C} *_C A). \tag{4.201}$$

The components of the tensor F in a $\boldsymbol{\xi}_i$-base are obtained as

$$\mathsf{F}_{ij} = \boldsymbol{\xi}_i \cdot \mathsf{F}(\boldsymbol{\xi}_j) \qquad \text{and} \qquad \mathsf{F}^{ij} = \boldsymbol{\xi}^i \cdot \mathsf{F}(\boldsymbol{\xi}^j) \tag{4.202}$$

and the components of the adjoint tensor are found by transposition:

$$\mathsf{F}^\dagger_{ij} = \mathsf{F}^\dagger(\boldsymbol{\xi}_j) \cdot \boldsymbol{\xi}_i = \boldsymbol{\xi}_j \cdot \mathsf{F}(\boldsymbol{\xi}_i) = \mathsf{F}_{ji}. \tag{4.203}$$

The coefficients of $\mathsf{F}(\boldsymbol{a})$ are $\boldsymbol{\xi}_i \cdot \mathsf{F}(\boldsymbol{a}) = \mathsf{F}_{ij} a^j$, which is the product of a matrix and a vector. Similarly one obtains the product of two matrices with (4.195) as

$$(\mathsf{FG})_{ij} = \mathsf{FG}(\boldsymbol{\xi}_j) \cdot \boldsymbol{\xi}_i = \mathsf{G}(\boldsymbol{\xi}_j) \cdot \mathsf{F}^\dagger(\boldsymbol{\xi}_i) = \mathsf{G}(\boldsymbol{\xi}_j) \cdot \big(\boldsymbol{\xi}_k(\boldsymbol{\xi}^k \cdot \mathsf{F}^\dagger(\boldsymbol{\xi}_i))\big) = \mathsf{G}(\boldsymbol{\xi}_j) \cdot \boldsymbol{\xi}_k \mathsf{F}^{\dagger\,k}{}_i = \mathsf{F}_i{}^k \mathsf{G}_{kj}. \tag{4.204}$$

Furthermore if one changes the basis according to $\boldsymbol{\vartheta}_r = \vartheta^i_r \boldsymbol{\xi}_i$ the coefficients transform according to

$$\mathsf{F}_{rs} = \boldsymbol{\vartheta}_r \cdot \mathsf{F}(\boldsymbol{\vartheta}_s) = \vartheta^i_r \boldsymbol{\xi}_i \cdot \mathsf{F}(\vartheta^j_s \boldsymbol{\xi}_j) = \vartheta^i_r \vartheta^j_s \mathsf{F}_{ij}, \tag{4.205}$$

which is just the transformation property in (4.182).

4.5 Curvature and Torsion

Curvature can be described if one transports a vector around a closed path and measures the difference of the initial and the transported vector. The path can be thought of as spanned by two tangent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and closes by $[\boldsymbol{a},\boldsymbol{b}]_{JLB}$. One can then act with a curvature operator on a tangent vector $\boldsymbol{c} = c^r\boldsymbol{\vartheta}_r$:

$$\begin{aligned}&[(\boldsymbol{a}\cdot\boldsymbol{D})(\boldsymbol{b}\cdot\boldsymbol{D}) - (\boldsymbol{b}\cdot\boldsymbol{D})(\boldsymbol{a}\cdot\boldsymbol{D}) - [\boldsymbol{a},\boldsymbol{b}]_{JLB}\cdot\boldsymbol{D}]\,\boldsymbol{c} \\ &\qquad = a^r b^s c^t\left(D_r D_s - D_s D_r - C^u_{rs} D_u\right)\boldsymbol{\vartheta}_t = a^r b^s c^t R^u_{rst}\boldsymbol{\vartheta}_u,\end{aligned} \tag{4.206}$$

with

$$R^u_{rst} = \left[\big(D_r D_s - D_s D_r - [\boldsymbol{\vartheta}_r,\boldsymbol{\vartheta}_s]_{JLB}\cdot\boldsymbol{D}\big)\boldsymbol{\vartheta}_t\right]\cdot\boldsymbol{\vartheta}^u \tag{4.207}$$

$$= [D_r(\Gamma^w_{st}\boldsymbol{\vartheta}_w) - D_s(\Gamma^w_{rt}\boldsymbol{\vartheta}_w) - C^w_{rs}(D_w\boldsymbol{\vartheta}_t)]\cdot\boldsymbol{\vartheta}^u \tag{4.208}$$

$$= \partial_r\Gamma^u_{st} - \partial_s\Gamma^u_{rt} + \Gamma^u_{rw}\Gamma^w_{st} - \Gamma^u_{sw}\Gamma^w_{rt} + \Gamma^w_{rs}\Gamma^u_{wt} - \Gamma^w_{sr}\Gamma^u_{wt}, \tag{4.209}$$

which in the case of a coordinate basis reduces to

$$R^l_{ijk} = \left[\big(D_i D_j - D_j D_i\big)\boldsymbol{\xi}_k\right]\cdot\boldsymbol{\xi}^l = \partial_i\Gamma^l_{jk} - \partial_j\Gamma^l_{ik} + \Gamma^l_{im}\Gamma^m_{jk} - \Gamma^l_{jm}\Gamma^m_{ik}. \tag{4.210}$$

Since the curvature operator maps three vectors into a fourth one it can also be written as a tensor

$$\mathsf{R} = R^u_{rst}\boldsymbol{\vartheta}_u\otimes\boldsymbol{\vartheta}^r\otimes\boldsymbol{\vartheta}^s\otimes\boldsymbol{\vartheta}^t. \tag{4.211}$$

In general the curvature operator can act on a multivector A which with (4.89) can be written as

$$\begin{aligned}&\big[(\boldsymbol{a}\cdot\boldsymbol{D})(\boldsymbol{b}\cdot\boldsymbol{D}) - (\boldsymbol{b}\cdot\boldsymbol{D})(\boldsymbol{a}\cdot\boldsymbol{D}) - \big[\boldsymbol{a},\boldsymbol{b}\big]_{JLB}\cdot\boldsymbol{D}\big]A \\ &\qquad = \big[(\boldsymbol{a}\cdot\boldsymbol{\partial})\mathtt{S}(\boldsymbol{b}) - (\boldsymbol{b}\cdot\boldsymbol{\partial})\mathtt{S}(\boldsymbol{a}) + \mathtt{S}(\boldsymbol{a})\times\mathtt{S}(\boldsymbol{b}) - \mathtt{S}([\boldsymbol{a},\boldsymbol{b}]_{JLB})\big]\times A = \mathtt{R}(\boldsymbol{ab})\times A,\end{aligned} \tag{4.212}$$

which reduces to

$$[(\boldsymbol{a}\cdot\boldsymbol{D})(\boldsymbol{b}\cdot\boldsymbol{D}) - (\boldsymbol{b}\cdot\boldsymbol{D})(\boldsymbol{a}\cdot\boldsymbol{D}) - [\boldsymbol{a},\boldsymbol{b}]_{JLB}\cdot\boldsymbol{D}]\,\boldsymbol{c} = \mathtt{R}(\boldsymbol{ab})\cdot\boldsymbol{c} \tag{4.213}$$

acting on a vector. The bivector-valued function of a bivector

$$\mathtt{R}(\boldsymbol{ab}) = (\boldsymbol{a}\cdot\boldsymbol{\partial})\mathtt{S}(\boldsymbol{b}) - (\boldsymbol{b}\cdot\boldsymbol{\partial})\mathtt{S}(\boldsymbol{a}) + \mathtt{S}(\boldsymbol{a})\times\mathtt{S}(\boldsymbol{b}) - \mathtt{S}([\boldsymbol{a},\boldsymbol{b}]_{JLB}) \tag{4.214}$$

fulfills the Ricci and Bianchi identities

$$\boldsymbol{a}\cdot\mathtt{R}(\boldsymbol{bc}) + \boldsymbol{b}\cdot\mathtt{R}(\boldsymbol{ca}) + \boldsymbol{c}\cdot\mathtt{R}(\boldsymbol{ab}) = 0 \tag{4.215}$$

$$\text{and}\qquad (\boldsymbol{a}\cdot\boldsymbol{D})\mathtt{R}(\boldsymbol{bc}) + (\boldsymbol{b}\cdot\boldsymbol{D})\mathtt{R}(\boldsymbol{ca}) + (\boldsymbol{c}\cdot\boldsymbol{D})\mathtt{R}(\boldsymbol{ab}) = 0. \tag{4.216}$$

Comparing (4.206) with (4.213) shows that the curvature may be described by a bivector-valued function of a bivector according to

$$a^r b^s c^t R^u_{rst}\boldsymbol{\vartheta}_u = \mathtt{R}(\boldsymbol{ab})\cdot\boldsymbol{c}. \tag{4.217}$$

But it is also possible to describe it by a scalar-valued function of a bivector, i.e. a 2-form $R^u_t(\boldsymbol{ab}) = i_{\boldsymbol{ab}}R^u_t$ according to

$$a^r b^s c^t R^u_{rst}\boldsymbol{\vartheta}_u = c^t R^u_t(\boldsymbol{ab})\boldsymbol{\vartheta}_u. \tag{4.218}$$

It is now easy to see from this definition and (4.209) that the curvature 2-form R^u_t is

$$R^u_t = \big(\partial_v\Gamma^u_{wt} + \Gamma^u_{rt}\Gamma^r_{wv} + \Gamma^u_{vr}\Gamma^r_{wt}\big)\boldsymbol{\vartheta}^v\boldsymbol{\vartheta}^w, \tag{4.219}$$

which also can be expressed in another way. To this purpose one notices that the exterior derivative of $\boldsymbol{\vartheta}_r$ is a vector-valued 1-form:

$$\boldsymbol{d}\boldsymbol{\vartheta}_r = \boldsymbol{\vartheta}^s D_s \boldsymbol{\vartheta}_r = \Gamma^t_{sr} \boldsymbol{\vartheta}^s \boldsymbol{\vartheta}_t = \boldsymbol{\omega}_r^t \boldsymbol{\vartheta}_t, \tag{4.220}$$

where $\boldsymbol{\omega}_r^t = \Gamma^t_{sr} \boldsymbol{\vartheta}^s$. With $\boldsymbol{\omega}_r^t$ the curvature 2-form (4.219) can also be written as

$$R_t^u = \boldsymbol{d}\boldsymbol{\omega}_t^u + \boldsymbol{\omega}_r^u \boldsymbol{\omega}_t^r, \tag{4.221}$$

which is the first Cartan structure equation. Exterior differentiation of (4.221) gives the Bianchi identity for the curvature 2-form:

$$\boldsymbol{d}R^r_s + \boldsymbol{\omega}_t^r R_s^t - R_t^r \boldsymbol{\omega}_s^t = 0. \tag{4.222}$$

It is possible that the path spanned by two tangent vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is not closed by $[\boldsymbol{a}, \boldsymbol{b}]_{JLB}$. This is measured by the torsion

$$(\boldsymbol{a} \cdot \boldsymbol{D})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{D})\boldsymbol{a} - [\boldsymbol{a}, \boldsymbol{b}]_{JLB} = a^r b^s T^t_{rs} \boldsymbol{\vartheta}_t \tag{4.223}$$

with

$$T^t_{rs} = \left[D_r \boldsymbol{\vartheta}_s - D_s \boldsymbol{\vartheta}_r - [\boldsymbol{\vartheta}_r, \boldsymbol{\vartheta}_s]_{JLB} \right] \cdot \boldsymbol{\vartheta}^t = \Gamma^t_{rs} - \Gamma^t_{sr} - C^t_{rs}, \tag{4.224}$$

which reduces in a coordinate basis to $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$. This means that for non-vanishing torsion the Γ^k_{ij} are no longer symmetric in the lower indices so that $\boldsymbol{dd}x^i$ is no longer zero and the exterior differential of an r-form is given by

$$\begin{aligned} \boldsymbol{d}A^{(r)} = \boldsymbol{D}A^{(r)} \quad = \quad & \frac{1}{r!} \left(\frac{\partial A_{i_1 i_2 \ldots i_r}}{\partial x^j} \right) \boldsymbol{D}x^j \boldsymbol{D}x^{i_1} \boldsymbol{D}x^{i_2} \ldots \boldsymbol{D}x^{i_r} \\ & + \frac{1}{r!} A_{i_1 i_2 \ldots i_r} \left[\boldsymbol{D}\boldsymbol{D}x^{i_1} \boldsymbol{D}x^{i_2} \ldots \boldsymbol{D}x^{i_r} - \boldsymbol{D}x^{i_1} \boldsymbol{D}\boldsymbol{D}x^{i_2} \boldsymbol{D}x^{i_3} \ldots \boldsymbol{D}x^{i_r} \right. \\ & \qquad \left. + \ldots + (-1)^{r-1} \boldsymbol{D}x^{i_1} \boldsymbol{D}x^{i_2} \ldots \boldsymbol{D}\boldsymbol{D}x^{i_r} \right]. \end{aligned} \tag{4.225}$$

The torsion maps two vectors into a third one and so can also be written as a tensor

$$\mathsf{T} = T^t_{rs} \boldsymbol{\vartheta}_t \otimes \boldsymbol{\vartheta}^r \otimes \boldsymbol{\vartheta}^s. \tag{4.226}$$

The other possibility is to describe the torsion with a scalar-valued function of a bivector, i.e. a 2-form $T^t(\boldsymbol{ab}) = i_{\boldsymbol{ab}} T^t$ according to

$$a^r b^s T^t_{rs} \boldsymbol{\vartheta}_t = T^t(\boldsymbol{ab}) \boldsymbol{\vartheta}_t. \tag{4.227}$$

It is then easy to see with (4.224) that the torsion 2-form can be written as

$$T^t = \left(\Gamma^t_{rs} - \frac{1}{2} C^t_{sr} \right) \boldsymbol{\vartheta}^r \boldsymbol{\vartheta}^s. \tag{4.228}$$

With the Cartan 1-form $\boldsymbol{\omega}_r^t$ this can also be written as

$$T^t = \boldsymbol{d}\boldsymbol{\vartheta}^t + \boldsymbol{\omega}_r^t \boldsymbol{\vartheta}^r, \tag{4.229}$$

which is the second Cartan structure equation, and applying the exterior differentiation on both sides of (4.229) gives the second Bianchi-identity

$$\boldsymbol{d}T^t + \boldsymbol{\omega}_r^t T^r = R_r^t \boldsymbol{\vartheta}^r. \tag{4.230}$$

4.6 Rotor Groups and Bivector Algebras

Following [34] it is now straightforward to translate the theory of Lie groups and Lie algebras into a superanalytic language. Starting point is the fact that an orthogonal transformation can be decomposed into several reflections. A reflection of a vector $\boldsymbol{x}$ at a plane with normal unit-vector $\boldsymbol{u}$ can be written as $-\boldsymbol{u} *_C \boldsymbol{x} *_C \boldsymbol{u}^{-1 *_C}$. The $\boldsymbol{u}$ form under the Clifford star product the group $Pin(p,q)$ if one has a metric with (p,q)-signature. An element $U \in Pin(p,q)$ is a multivector with $U *_C \overline{U} = \pm 1$. The multivectors of even Grassmann grade are closed under the Clifford star product and form the subgroup $Spin(p,q)$, which is a double covering of $SO(p,q)$. An element $S \in Spin(p,q)$ fulfills $S *_C \overline{S} = \pm 1$ and a transformation $S *_C \boldsymbol{x} *_C S^{-1 *_C}$ gives again a vector-valued result. The elements $R \in Spin(p,q)$ with $R *_C \overline{R} = +1$ are called rotors and form the rotor group $Spin^+(p,q)$, which in the euclidian case is equal to the spin-group. For a rotor one has $R^{-1 *_C} = \overline{R}$, so that a multivector A transforms as $R *_C A *_C \overline{R}$.

A rotor can be written as a starexponential of a bivector. This can be seen if one considers a path $R(t)$ in the rotor group manifold. Differentiating $\boldsymbol{x}(t) = R(t) *_C \boldsymbol{x}_0 *_C \overline{R}(t)$ one obtains

$$\begin{aligned} \frac{d}{dt}\boldsymbol{x}(t) &= \dot{R}(t) *_C \boldsymbol{x}_0 *_C \overline{R}(t) + R(t) *_C \boldsymbol{x}_0 *_C \dot{\overline{R}}(t) && (4.231)\\ &= \dot{R}(t) *_C \overline{R}(t) *_C \boldsymbol{x}(t) - \boldsymbol{x}(t) *_C \dot{R}(t) *_C \overline{R}(t), && (4.232) \end{aligned}$$

where one uses

$$\dot{R}(t) *_C \overline{R}(t) + R(t) *_C \dot{\overline{R}}(t) = 0, \tag{4.233}$$

which follows from differentiating $R(t) *_C \overline{R}(t) = 1$. Since the left hand side of (4.231) is vector-valued the right hand side $\dot{R}(t) *_C \overline{R}(t) *_C \boldsymbol{x}(t) - \boldsymbol{x}(t) *_C \dot{R}(t) *_C \overline{R}(t)$ has to be vector-valued too, so that $\dot{R}(t) *_C \overline{R}(t)$ has to be a bivector $\frac{1}{2}\mathtt{B}(t)$. With this bivector (4.233) can be written as a defining equation for the rotor, i.e.

$$\dot{R}(t) = -R(t) *_C \dot{\overline{R}}(t) *_C R(t) = \frac{1}{2}\mathtt{B}(t) *_C R(t). \tag{4.234}$$

It is now easy to see that the bivector $\mathtt{B}$ is independent of t. This is because one has on the one hand

$$\frac{d}{dt}R(t+u) = \frac{1}{2}\mathtt{B}(t+u) *_C R(t+u) \tag{4.235}$$

and on the other hand

$$\frac{d}{dt}\Big(R(t) *_C R(u)\Big) = \frac{1}{2}\mathtt{B}(t) *_C R(t) *_C R(u) = \frac{1}{2}\mathtt{B}(t) *_C R(t+u), \tag{4.236}$$

so that $\mathtt{B}(t+u) = \mathtt{B}(t)$ is actually independent of t and (4.234) can be integrated to

$$R(t) = e_{*_C}^{\frac{t}{2}\mathtt{B}}. \tag{4.237}$$

This result is only true in the euclidian case where the rotor group manifold is connected. In general a rotor can be written as

$$R(t) = \pm e_{*_C}^{\frac{t}{2}\mathtt{B}}. \tag{4.238}$$

On the other hand it is also easy to show that the transformation with the above constructed rotor conserves the vector-grade, i.e.

$$\boldsymbol{x}(t) = e_{*_C}^{\frac{t}{2}\mathtt{B}} *_C \boldsymbol{x}_0 *_C e_{*_C}^{-\frac{t}{2}\mathtt{B}} \tag{4.239}$$

is vector-valued. If one considers the derivatives of $\boldsymbol{x}(t)$:

$$\begin{aligned} \frac{d}{dt}\boldsymbol{x}(t) &= e_{*_C}^{\frac{t}{2}\mathtt{B}} *_C (\mathtt{B} \cdot \boldsymbol{x}_0) *_C e_{*_C}^{-\frac{t}{2}\mathtt{B}}, && (4.240)\\ \frac{d^2}{dt^2}\boldsymbol{x}(t) &= e_{*_C}^{\frac{t}{2}\mathtt{B}} *_C \big(\mathtt{B} \cdot (\mathtt{B} \cdot \boldsymbol{x}_0)\big) *_C e_{*_C}^{-\frac{t}{2}\mathtt{B}} && (4.241) \end{aligned}$$

and so on, $\boldsymbol{x}(t)$ can be written as

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + t(\mathtt{B}\cdot\boldsymbol{x}_0) + \frac{1}{2!}t^2\big(\mathtt{B}\cdot(\mathtt{B}\cdot\boldsymbol{x}_0)\big) + \cdots\,, \tag{4.242}$$

which is vector-valued.

As a simple example one can consider the rotation in a two dimensional vector space with vectors $\boldsymbol{x} = x^1\boldsymbol{\sigma}_1 + x^2\boldsymbol{\sigma}_2$. The bivector $\mathtt{B} = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ generates the rotation given by

$$\boldsymbol{x}' = R(t) *_C \boldsymbol{x} *_C \overline{R(t)} = e_{*_C}^{\frac{t}{2}\mathtt{B}} *_C \boldsymbol{x} *_C e_{*_C}^{-\frac{t}{2}\mathtt{B}} = (x^1\cos t + x^2\sin t)\boldsymbol{\sigma}_1 + (x^2\cos t - x^1\sin t)\boldsymbol{\sigma}_2. \tag{4.243}$$

This means that the bivector $\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ generates a right rotation, i.e. a rotation in the mathematically negative direction, whereas the bivector $\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ generates a left rotation, i.e. a rotation in the mathematically positive direction.

The bivector basis $\mathtt{B}_i$ of a rotor constitutes an algebra under the commutator product

$$\mathtt{B}_i \times \mathtt{B}_j = C_{ij}^k\mathtt{B}_k, \tag{4.244}$$

where the C_{ij}^k are the structure constants (note that one has here an additional factor $\frac{1}{2}$ due to the definition of the commutator product). Furthermore one can directly calculate

$$\kappa_{ij} = \mathtt{B}_i\cdot\mathtt{B}_j, \tag{4.245}$$

which is (proportional to) the Killing metric. Note that here no detour over the adjoint matrix representation of the algebra elements has to be made to define the Killing metric. As an example one can consider the group $SO(3)$. Given a three dimensional euclidian space with basis vectors $\boldsymbol{\sigma}_i$ the rotor is given by

$$R = R_0 + R_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + R_2\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 + R_3\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2, \tag{4.246}$$

with $R *_C \overline{R} = R_0^2 + R_1^2 + R_2^2 + R_3^2 = 1$, so that the rotor can also be parametrized with three parameters α, θ and φ as[1]:

$$R(\alpha,\theta,\varphi) = \cos\alpha\cos\theta + \sin\alpha\cos\varphi\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + \sin\alpha\sin\varphi\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 + \cos\alpha\sin\theta\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2. \tag{4.247}$$

The three basis bivectors $\mathtt{B}_1 = \boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$, $\mathtt{B}_2 = \boldsymbol{\sigma}_3\boldsymbol{\sigma}_1$ and $\mathtt{B}_3 = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ fulfill

$$\mathtt{B}_i \times \mathtt{B}_j = -\varepsilon_{ijk}\mathtt{B}_k \qquad \text{and} \qquad \kappa_{ij} = \mathtt{B}_i\cdot\mathtt{B}_j = -\delta_{ij}. \tag{4.248}$$

It is easy to see that the group vector manifold, which for $SO(3)$ is an S^3 embedded in a four dimensional euclidian space with basis vectors $\boldsymbol{\tau}_a$, can be read off from (4.247) as

$$\boldsymbol{r}_R(\alpha,\theta,\varphi) = \cos\alpha\cos\theta\,\boldsymbol{\tau}_1 + \sin\alpha\cos\varphi\,\boldsymbol{\tau}_2 + \sin\alpha\sin\varphi\,\boldsymbol{\tau}_3 + \cos\alpha\sin\theta\,\boldsymbol{\tau}_4. \tag{4.249}$$

On this group vector manifold one can apply the formalism described in the last section and calculate the coordinate basis $\boldsymbol{\xi}_1 = \partial_\alpha\boldsymbol{r}_R$, $\boldsymbol{\xi}_2 = \partial_\theta\boldsymbol{r}_R$, $\boldsymbol{\xi}_3 = \partial_\varphi\boldsymbol{r}_R$ and the coordinate metric $g_{ij} = \boldsymbol{\xi}_i\cdot\boldsymbol{\xi}_j$.

The rotors act on themselves by left- and right-translation. A left-translation with a rotor R' is given by $\ell_{R'}R = R' *_C R$ and on the group vector manifold by $\ell_{R'}\boldsymbol{r}_R = \boldsymbol{r}_{R' *_C R}$. The left-translation induces a map $T_R\ell_{R'}$ between the tangent spaces at $\boldsymbol{r}_R$ and $\boldsymbol{r}_{R' *_C R}$. A vector field $\boldsymbol{a}(\boldsymbol{r}_R)$ on the group vector manifold is

[1] In general $R *_C \overline{R}$ is not as in the above case a scalar but a multivector, so that $R *_C \overline{R} = 1$ gives not one but several conditions on the R_i. This insures that the degrees of freedom in the even multivector R correspond to the number of linear independent bivectors, which constitute the group algebra.
Note further that the parametrization in α, θ and φ is much easier than the parametrization in t_i that follows if one rewrites with the Baker-Campbell-Hausdorff formula the rotor $R = e_{*_C}^{\frac{1}{2}(t_1\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 + t_2\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 + t_3\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3)}$ in the from of (4.246).

left-invariant if $T_R \ell_{R'} \boldsymbol{a}(\boldsymbol{r}_R) = \boldsymbol{a}(\boldsymbol{r}_{R' *_C R})$. Left-invariant vector fields on the group vector manifold can be obtained if one considers the multivector fields on the rotors given by $B_i^{\text{left}}(R) = R *_C \mathtt{B}_i$. For two rotors R and R' one has

$$B_i^{\text{left}}(R' *_C R) = R' *_C B_i^{\text{left}}(R). \tag{4.250}$$

Just as to each rotor R in the $\boldsymbol{\sigma}_a$-space corresponds a vector $\boldsymbol{r}_R$ in the $\boldsymbol{\tau}_a$-space there is also for each multivector field $B_i^{\text{left}}(R)$ in the $\boldsymbol{\sigma}_a$-space a left invariant vector field $\boldsymbol{\vartheta}_{B_i^{\text{left}}(R)}(\boldsymbol{r}_R) \equiv \boldsymbol{\vartheta}_i$ in the $\boldsymbol{\tau}_a$-space. These vector fields are closed under the Jacobi-Lie-bracket, i.e. they form a Lie subalgebra of all vector fields on $\boldsymbol{r}_R$ and they form a non-coordinate basis on $\boldsymbol{r}_R$. For the $SO(3)$-case one has for example

$$\boldsymbol{\vartheta}_{B_i^{\text{left}}(R)} \cdot \boldsymbol{\vartheta}_{B_j^{\text{left}}(R)} = \delta_{ij} \qquad \text{and} \qquad \left[\boldsymbol{\vartheta}_{B_i^{\text{left}}(R)}, \boldsymbol{\vartheta}_{B_j^{\text{left}}(R)}\right]_{JLB} = \varepsilon_{ijk} \boldsymbol{\vartheta}_{B_k^{\text{left}}(R)}. \tag{4.251}$$

The multivector fields $B_i^{\text{left}}(R)$ are uniquely defined by the bivectors at $R = 1$ and the corresponding left-invariant vector fields are uniquely defined by their value in $\boldsymbol{r}_{R=1}$. In the $SO(3)$-example the tangent space at $\boldsymbol{r}_{R=1}(0,0,0) = \boldsymbol{\tau}_1$ is spanned by the vectors $\boldsymbol{\vartheta}_{\mathtt{B}_i} = \boldsymbol{\tau}_{i+1}$ and constitutes the $\mathfrak{so}(3)$ algebra in the $\boldsymbol{\tau}_a$-space, where the commutator product in the bivector algebra corresponds here in the $\mathfrak{so}(3)$-case to the vector cross product on the $\boldsymbol{\vartheta}_{\mathtt{B}_i}$-space, i.e. one has an algebra anti-homomorphism between the algebra of left-invariant vector fields and the bivector algebra[2]

$$\boldsymbol{\vartheta}_{\mathtt{B}_i \times \mathtt{B}_j} = -\boldsymbol{\vartheta}_{\mathtt{B}_i} \times \boldsymbol{\vartheta}_{\mathtt{B}_j}. \tag{4.252}$$

To each basis-bivector $\mathtt{B}_i$ of the bivector algebra a two form Θ^i can be found so that $i_{\mathtt{B}_i} \Theta^j = \overline{\mathtt{B}_i} \cdot \Theta^j = \delta_i^j$ and to the two-forms Θ^i correspond then in the $\boldsymbol{\tau}_a$-space one forms $\boldsymbol{\vartheta}^{\Theta^i} \equiv \boldsymbol{\vartheta}^i$ that generalize to reciprocal non-coordinate basis vector fields on $\boldsymbol{r}_R$, which clearly obey the Maurer-Cartan equation (4.175). For a r-form $\boldsymbol{A}^{(r)}$ on the group vector manifold that is vector-valued in the $\boldsymbol{\sigma}_a$-space one can then in analogy to (4.143) define the BRST-operator s as

$$\begin{aligned} \left(\overline{\boldsymbol{\vartheta}_1 \boldsymbol{\vartheta}_2 \ldots \boldsymbol{\vartheta}_{r+1}}\right) \cdot s\boldsymbol{A}^{(r)} \quad &= \quad \sum_{n=1}^{r+1} (-1)^{n+1} \mathtt{B}_n \cdot \left(\left(\overline{\boldsymbol{\vartheta}_1 \ldots \check{\boldsymbol{\vartheta}}_n \ldots \boldsymbol{\vartheta}_{r+1}}\right) \cdot \boldsymbol{A}^{(r)} \right) \\ &\quad + \sum_{m<n} (-1)^{m+n} \left(\overline{[\boldsymbol{\vartheta}_m, \boldsymbol{\vartheta}_n]_{JLB}\, \boldsymbol{\vartheta}_1 \ldots \check{\boldsymbol{\vartheta}}_m \ldots \check{\boldsymbol{\vartheta}}_n \ldots \boldsymbol{\vartheta}_{r+1}} \right) \cdot \boldsymbol{A}^{(r)}. \end{aligned} \tag{4.253}$$

Note that the first scalar product on the right hand side is the scalar product in the $\boldsymbol{\sigma}_a$-space and the second scalar product is the scalar product in the $\boldsymbol{\tau}_a$-space. The s-operator has then the form (see for example [5] and the references therein):

$$s = \boldsymbol{\vartheta}^i \mathtt{B}_i + \frac{1}{2} C_{ij}^k \boldsymbol{\vartheta}^j \boldsymbol{\vartheta}^i \frac{\partial}{\partial \boldsymbol{\vartheta}^k}. \tag{4.254}$$

Combining a left-translation with R' and a right-translation with $\overline{R'}$ gives $R' *_C R *_C \overline{R'}$, which is an inner automorphism on the rotors. The derivative at the identity is the adjoint representation, i.e. the adjoint action of the rotor group on the bivector algebra is given by [34]

$$\mathrm{Ad}_R \mathtt{B} = R *_C \mathtt{B} *_C \overline{R}, \tag{4.255}$$

where $\mathtt{B} = b^i \mathtt{B}_i$ is a general element of the bivector algebra, to which corresponds a vector $\boldsymbol{b} = b^i \boldsymbol{\vartheta}_{\mathtt{B}_i}$ in the $\boldsymbol{\vartheta}_{\mathtt{B}_i}$-space. Ad_R is a bivector algebra homomorphism, i.e.

$$\mathrm{Ad}_R(\mathtt{A} \times \mathtt{B}) = \mathrm{Ad}_R \mathtt{A} \times \mathrm{Ad}_R \mathtt{B} \tag{4.256}$$

[2]For an arbitrary algebra one can analogously define a generalized vector cross product with the structure constants C_{ij}^k instead of ε_{ij}^k.

and a left action, i.e.

$$\mathrm{Ad}_{R*_C R'} = \mathrm{Ad}_R \mathrm{Ad}_{R'}. \tag{4.257}$$

For all elements R of the rotor group the adjoint action (4.255) constitutes the adjoint bivector orbit of $\mathtt{B}$, to which in the $\boldsymbol{\vartheta}_{\mathtt{B}_i}$-space corresponds an orbit vector manifold. In the $SO(3)$-case the adjoint action (4.255) leaves $|\mathtt{B}|^2 = \sum_{i=1}^{3}(b^i)^2 = |\boldsymbol{b}|^2$ invariant, so that the adjoint orbit vector manifold is an S^2.

Let now $\mathtt{A}$ be an element of the bivector algebra and consider the rotor $R(t) = e_{*_C}^{\frac{t}{2}\mathtt{A}}$. The adjoint action of this one-parameter rotor subgroup gives a curve in the bivector orbit and the derivative at $t = 0$ is

$$\mathrm{ad}_{\mathtt{A}}\mathtt{B} = \left.\frac{d}{dt}\right|_{t=0} R(t) *_C \mathtt{B} *_C \overline{R(t)} = \mathtt{A} \times \mathtt{B}. \tag{4.258}$$

In the $\boldsymbol{\vartheta}_{\mathtt{B}_i}$-space the vector $\boldsymbol{\vartheta}_{\mathtt{A}\times\mathtt{B}}$ is the tangent vector in direction $\boldsymbol{\vartheta}_{\mathtt{A}}$ to the orbit vector manifold in the point $\boldsymbol{\vartheta}_{\mathtt{B}}$, i.e. $\boldsymbol{\vartheta}_{\mathtt{A}\times\mathtt{B}}$ generates the adjoint action corresponding to $\mathtt{A}$. It is also possible to define the coadjoint action Ad^*_R of the rotor group on a two form Θ by

$$\overline{\mathtt{B}} \cdot \mathrm{Ad}^*_R\Theta = \overline{\mathrm{Ad}_R\mathtt{B}} \cdot \Theta, \tag{4.259}$$

which is the right action $\mathrm{Ad}^*_R\Theta = \overline{R} *_C \Theta *_C R$. The coadjoint left action is given by $\mathrm{Ad}^*_{\overline{R}}\Theta$. Infinitesimally one has $\overline{\mathtt{B}} \cdot \mathrm{ad}^*_{\mathtt{A}}\Theta = \overline{\mathrm{ad}_{\mathtt{A}}\mathtt{B}} \cdot \Theta$, or $\mathrm{ad}^*_{\mathtt{A}}\Theta = \Theta \times \mathtt{A}$. In the $SO(3)$-case the rotor acts on an euclidian space where the basis vectors and the reciprocal basis vectors are actually the same, so that $\mathtt{B}_i = \Theta^i$ and there is no difference between the adjoint and the coadjoint action.

In the above discussion the rotor R acts intrinsically from the left on a vector space. But more generally a rotor in an ambient space can also act from the left on a vector manifold $\boldsymbol{x}(x^i)$ by $\boldsymbol{x}' = R *_C \boldsymbol{x} *_C \overline{R}$ if $\boldsymbol{x}'$ is again a point in the vector manifold. The left-action of the rotor $R(t) = e_{*_C}^{\frac{t}{2}\mathtt{B}}$ induces on the vector manifold $\boldsymbol{x}(x^i)$ the vector field[3]

$$\left.\frac{d}{dt}\right|_{t=0} R(t) *_C \boldsymbol{x} *_C \overline{R(t)} = \mathtt{B} \cdot \boldsymbol{x}. \tag{4.260}$$

If on the other hand a tangent vector field $\boldsymbol{a}(\boldsymbol{x})$ on the vector manifold is given that can be expressed as $\boldsymbol{a}(\boldsymbol{x}) = \mathtt{B} \cdot \boldsymbol{x}$, for a constant bivector $\mathtt{B}$ in the ambient space, then the flow $\boldsymbol{x}(t)$ generated by $\boldsymbol{a}(\boldsymbol{x})$ is due to a rotor action. Furthermore one has an algebra anti-homomorphism between the bivector algebra in the ambient space and the induced vector fields on the vector manifold, given by

$$[\mathtt{A} \cdot \boldsymbol{x}, \mathtt{B} \cdot \boldsymbol{x}]_{JLB} = -(\mathtt{A} \times \mathtt{B}) \cdot \boldsymbol{x}. \tag{4.261}$$

This relation can be proved by direct calculation. For bivectors $\mathtt{A} = A^{bc}\boldsymbol{\sigma}_b\boldsymbol{\sigma}_c$ and $\mathtt{B} = B^{ef}\boldsymbol{\sigma}_e\boldsymbol{\sigma}_f$ in the ambient space spanned by the basis vectors $\boldsymbol{\sigma}_a$ the right hand side of (4.261) gives:

$$\begin{aligned}(\mathtt{A} \times \mathtt{B}) \cdot \boldsymbol{x} &= x^a A^{bc} B^{ef} (\eta_{ce}\boldsymbol{\sigma}_b\boldsymbol{\sigma}_f + \eta_{bf}\boldsymbol{\sigma}_c\boldsymbol{\sigma}_e - \eta_{cf}\boldsymbol{\sigma}_b\boldsymbol{\sigma}_e - \eta_{be}\boldsymbol{\sigma}_c\boldsymbol{\sigma}_f) \cdot \boldsymbol{\sigma}_a \\ &= 4x^a A^{bc} B^{ef} \eta_{ae}\eta_{bf}\boldsymbol{\sigma}_c - 4x^a A^{bc} B^{ef} \eta_{ab}\eta_{ec}\boldsymbol{\sigma}_f. \end{aligned} \tag{4.262}$$

The vector field induced by the bivector $\mathtt{A}$ can be expanded in the coordinate basis on the vector manifold, i.e.:

$$\mathtt{A} \cdot \boldsymbol{x} = 2x_b A^{cb}\boldsymbol{\sigma}_c = a^i\boldsymbol{\xi}_i = a^i\xi^c_i\boldsymbol{\sigma}_c, \tag{4.263}$$

so that the corresponding coefficients are $a^i = 2x_b A^{cb}\xi^i_c$ and similarly for the vector field induced by the

[3] The corresponding right-action $\boldsymbol{x}' = \overline{R(t)} *_C \boldsymbol{x} *_C R(t)$ induces the vector field $-\mathtt{B} \cdot \boldsymbol{x} = \boldsymbol{x} \cdot \mathtt{B}$.

bivector $\mathtt{B}$ one has the coefficients $b^i = 2x_e B^{fe}\xi^i_f$. The left hand side of (4.261) gives then

$$\begin{aligned}
[\mathtt{A}\cdot\boldsymbol{x},\mathtt{B}\cdot\boldsymbol{x}]_{JLB} &= \left(2x_b A^{cb}\xi^i_c\partial_i(2x_e B^{fe}\xi^j_f) - 2x_e B^{fe}\xi^i_f\partial_i(2x_b A^{cb}\xi^j_c)\right)\xi^a_j\boldsymbol{\sigma}_a && (4.264)\\
&= 4A^{bc}B^{ef}\left(x_b\eta_{eh}\xi^i_c\partial_i(x^h\xi^j_f) - x_e\eta_{bh}\xi^i_f\partial_i(x^h\xi^j_c)\right)\xi^a_j\boldsymbol{\sigma}_a && (4.265)\\
&= 4A^{bc}B^{ef}\left(x_b\eta_{ec}\delta^a_f + x_b x_e\Gamma^a_{cf} - x_e\eta_{bf}\delta^a_c - x_b x_e\Gamma^a_{fc}\right)\boldsymbol{\sigma}_a && (4.266)\\
&= 4x^a A^{bc}B^{ef}\eta_{ab}\eta_{ef}\boldsymbol{\sigma}_f - 4x^a A^{bc}B^{ef}\eta_{ae}\eta_{bf}\boldsymbol{\sigma}_c && (4.267)
\end{aligned}$$

which is up to a sign the same result as the right hand side (4.262).

The rotor in the ambient space acts not only on the vectors $\boldsymbol{x}$ of the vector manifold, but in the same way also on tangent vectors $\boldsymbol{a}$ at the manifold which are vectors in the ambient space too. The transformation of $\boldsymbol{x}$ and $\boldsymbol{a}$ in the ambient space of the vector manifold induce a transformation in the tangent bundle. The tangent bundle manifold can be seen as a $2d$-dimensional vector manifold in a $(2d+2)$-dimensional ambient space with basis vectors $\boldsymbol{\sigma}_a$ and $\boldsymbol{\tau}_a$, i.e. as

$$(\boldsymbol{x}+\boldsymbol{a})(x^i,a^i) = x^a(x^i)\boldsymbol{\sigma}_a + a^j\xi^a_j(x^i)\boldsymbol{\tau}_a. \tag{4.268}$$

Analogously one can define multivector bundles, for example a bivector bundle manifold has the form

$$(\boldsymbol{x}+\mathtt{B})(x^i,B^{jk}) = x^a(x^i)\boldsymbol{\sigma}_a + B^{jk}\xi^a_j(x^i)\xi^b_k(x^i)\boldsymbol{\tau}_a\boldsymbol{\tau}_b. \tag{4.269}$$

The tangential lift of the rotor action is given by $R *_C \boldsymbol{x} *_C \overline{R} + R *_C \boldsymbol{a} *_C \overline{R}$, where the rotor acts on the $\boldsymbol{\tau}_a$-space in the same way as on the $\boldsymbol{\sigma}_a$-space. In the case of a flat vector manifold the tangent bundle is just a $2d$-dimensional vector space and the rotor acts separately and intrinsically on both subspaces. Instead of two rotors that act separately on the $\boldsymbol{\sigma}_a$ and $\boldsymbol{\tau}_a$ spaces one can consider also a lifted rotor with a bivector $\mathtt{B}_{\text{lifted}}$ that is the sum of the two single bivectors, so that one can write $R_{\text{lifted}} *_C (\boldsymbol{x}+\boldsymbol{a}) *_C \overline{R_{\text{lifted}}}$. If one describes the tangent vector in a reciprocal ambient space, i.e. as a one-form $\boldsymbol{\alpha}$ the cotangent bundle has the form

$$(\boldsymbol{x}+\boldsymbol{\alpha})(x^i,\alpha_i) = x^a(x^i)\boldsymbol{\sigma}_a + \alpha_i\xi^i_a(x^i)\boldsymbol{\tau}^a \tag{4.270}$$

and the corresponding cotangent lift is given by $\overline{R} *_C \boldsymbol{x} *_C R + \overline{R} *_C \boldsymbol{\alpha} *_C R$ or $\overline{R_{\text{lifted}}} *_C (\boldsymbol{x}+\boldsymbol{\alpha}) *_C R_{\text{lifted}}$.

In order to construct unitary transformations [36] one can consider a $2d$-dimensional space with basis vectors $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ for $i=1,\ldots,d$. The two subspaces spanned by $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ should have the same metric, i.e. $\boldsymbol{\alpha}_i\cdot\boldsymbol{\alpha}_j = \boldsymbol{\beta}_i\cdot\boldsymbol{\beta}_j$ and $\boldsymbol{\alpha}_i\cdot\boldsymbol{\beta}_j = 0$. On this space one can define the bivector

$$\mathtt{J} = \sum_{i=1}^d \boldsymbol{\alpha}_i\boldsymbol{\beta}_i = \sum_{i=1}^d \mathtt{J}_i, \tag{4.271}$$

which connects the two subspaces according to

$$\boldsymbol{\alpha}_i\cdot\mathtt{J} = \boldsymbol{\beta}_i \qquad \text{and} \qquad \boldsymbol{\beta}_i\cdot\mathtt{J} = -\boldsymbol{\alpha}_i, \tag{4.272}$$

so that one has

$$(\boldsymbol{\alpha}_i\cdot\mathtt{J})\cdot\mathtt{J} = -\boldsymbol{\alpha}_i \qquad \text{and} \qquad (\boldsymbol{\beta}_i\cdot\mathtt{J})\cdot\mathtt{J} = -\boldsymbol{\beta}_i \tag{4.273}$$

or in general for a vector $\boldsymbol{x} = a^i\boldsymbol{\alpha}_i + b^i\boldsymbol{\beta}_i$ one has $(\boldsymbol{x}\cdot\mathtt{J})\cdot\mathtt{J} = -\boldsymbol{x}$. The $2n$-dimensional vector $\boldsymbol{x}$ corresponds to an n-dimensional complex vector with components

$$x^k = \boldsymbol{x}\cdot\boldsymbol{\alpha}_k + \mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{\beta}_k = a^k + \mathrm{i}\,b^k. \tag{4.274}$$

The complex internal product can then be written as

$$\langle\boldsymbol{x}|\boldsymbol{y}\rangle = x^k\overline{y}_k = (\boldsymbol{x}\cdot\boldsymbol{\alpha}^k + \mathrm{i}\,\boldsymbol{x}\cdot\boldsymbol{\beta}^k)(\boldsymbol{y}\cdot\boldsymbol{\alpha}_k - \mathrm{i}\,\boldsymbol{y}\cdot\boldsymbol{\beta}_k) = \boldsymbol{x}\cdot\boldsymbol{y} + \mathrm{i}(\boldsymbol{x}\boldsymbol{y})\cdot\mathtt{J}. \tag{4.275}$$

A unitary transformation generated by the rotor R leaves the above complex product invariant, i.e. $\langle \boldsymbol{x}|\boldsymbol{y}\rangle = \langle R *_C \boldsymbol{x} *_C \overline{R} | R *_C \boldsymbol{y} *_C \overline{R}\rangle$. This means that on the one hand the scalar product $\boldsymbol{x}\cdot\boldsymbol{y}$ has to be invariant and on the other hand the more restrictive condition that

$$(\boldsymbol{x}\boldsymbol{y})\cdot \mathtt{J} = \big((R *_C \boldsymbol{x} *_C \overline{R})(R *_C \boldsymbol{y} *_C \overline{R})\big)\cdot \mathtt{J} = (\boldsymbol{x}\boldsymbol{y})\cdot(\overline{R} *_C \mathtt{J} *_C R), \tag{4.276}$$

which means that $\mathtt{J} = R *_C \mathtt{J} *_C \overline{R}$ is the defining relation for the unitary rotor and with the ansatz $R = e_{*_C}^{\mathtt{B}/2}$ one obtains the defining relation for the bivector $\mathtt{B}$

$$\mathtt{B}\times\mathtt{J} = 0. \tag{4.277}$$

A bivector that fulfills this equation can easily be found if one considers that

$$\big((\boldsymbol{x}\cdot\mathtt{J})(\boldsymbol{y}\cdot\mathtt{J})\big)\times\mathtt{J} = -(\boldsymbol{x}\boldsymbol{y})\times\mathtt{J}, \tag{4.278}$$

so that $\mathtt{B}$ has the form

$$\mathtt{B} = \boldsymbol{x}\boldsymbol{y} + (\boldsymbol{x}\cdot\mathtt{J})(\boldsymbol{y}\cdot\mathtt{J}). \tag{4.279}$$

Putting in this formula the basis vectors for $\boldsymbol{x}$ and $\boldsymbol{y}$ one obtains the d^2 basis bivectors of the $\mathfrak{u}(d)$-algebra:

$$\mathtt{E}_{ij} = \boldsymbol{\alpha}_i\boldsymbol{\alpha}_j + \boldsymbol{\beta}_i\boldsymbol{\beta}_j, \qquad \mathtt{F}_{ij} = \boldsymbol{\alpha}_i\boldsymbol{\beta}_j - \boldsymbol{\beta}_i\boldsymbol{\alpha}_j \qquad \text{and} \qquad \mathtt{J}_i = \boldsymbol{\alpha}_i\boldsymbol{\beta}_i \tag{4.280}$$

for $i < j = 1, \ldots, d$. It is easy to show that these basis bivectors form a closed algebra under the commutator product. The bivector $\mathtt{J}$ is part of the $\mathfrak{u}(d)$-algebra, if one excludes this generator of a global phase one obtains the $\mathfrak{su}(d)$-algebra. For example the bivector basis of $\mathfrak{su}(2)$ is given by

$$\mathtt{B}_1 = \boldsymbol{\alpha}_1\boldsymbol{\alpha}_2 + \boldsymbol{\beta}_1\boldsymbol{\beta}_2, \qquad \mathtt{B}_2 = \boldsymbol{\alpha}_1\boldsymbol{\beta}_2 - \boldsymbol{\beta}_1\boldsymbol{\alpha}_2 \qquad \text{and} \qquad \mathtt{B}_3 = \boldsymbol{\alpha}_1\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_2 \tag{4.281}$$

and it is easy to see that these bivectors fulfill the same commutator-product algebra as the $\mathfrak{so}(3)$ basis bivectors.

In order to describe $Gl(n)$ one proceeds similarly to the unitary case. One considers a $2n$-dimensional space spanned by the basis vectors $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ for $i = 1, \ldots, n$, but now the sign of the metric in the spaces spanned by $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_i$ is opposite, i.e. the Clifford star product is given by

$$*_C = \exp\left[\eta_{ij}\frac{\overleftarrow{\partial}}{\partial\boldsymbol{\alpha}_i}\frac{\vec{\partial}}{\partial\boldsymbol{\alpha}_j} - \eta_{ij}\frac{\overleftarrow{\partial}}{\partial\boldsymbol{\beta}_i}\frac{\vec{\partial}}{\partial\boldsymbol{\beta}_j}\right], \tag{4.282}$$

so that $\boldsymbol{\alpha}_i\cdot\boldsymbol{\alpha}_j = \eta_{ij}$, $\boldsymbol{\beta}_i\cdot\boldsymbol{\beta}_j = -\eta_{ij}$ and $\boldsymbol{\alpha}_i\cdot\boldsymbol{\beta}_j = 0$. On this space one defines

$$\mathtt{K} = \boldsymbol{\alpha}_i\boldsymbol{\beta}^i, \tag{4.283}$$

which relates the two subspaces according to

$$\boldsymbol{\alpha}_i\cdot\mathtt{K} = -\boldsymbol{\beta}_i \qquad \text{and} \qquad \boldsymbol{\beta}_i\cdot\mathtt{K} = -\boldsymbol{\alpha}_i, \tag{4.284}$$

or in general for a vector $\boldsymbol{x} = a^i\boldsymbol{\alpha}_i + b^i\boldsymbol{\beta}_i$ one has $(\boldsymbol{x}\cdot\mathtt{K})\cdot\mathtt{K} = \boldsymbol{x}$. While $\mathtt{J}$ generates a complex structure, $\mathtt{K}$ generates a 0-structure, i.e. one can decompose a vector $\boldsymbol{x}$ according to

$$\boldsymbol{x} = \frac{1}{2}\big(\boldsymbol{x} + \boldsymbol{x}\cdot\mathtt{K}\big) + \frac{1}{2}\big(\boldsymbol{x} - \boldsymbol{x}\cdot\mathtt{K}\big) = \boldsymbol{x}_+ + \boldsymbol{x}_-, \tag{4.285}$$

so that $\boldsymbol{x}_+\cdot\boldsymbol{x}_+ = \boldsymbol{x}_-\cdot\boldsymbol{x}_- = 0$. One has then two subspaces V_+ and V_- that are defined by

$$\boldsymbol{x}_+\cdot\mathtt{K} = \boldsymbol{x}_+ \qquad \text{and} \qquad \boldsymbol{x}_-\cdot\mathtt{K} = \boldsymbol{x}_-. \tag{4.286}$$

A $Gl(n)$-transformation now transforms a vector in V_+ into another vector in V_+, i.e.

$$(R *_C \boldsymbol{x}_+ *_C \overline{R}) \cdot \mathtt{K} = R *_C \boldsymbol{x}_+ *_C \overline{R}, \tag{4.287}$$

which can be written as

$$\boldsymbol{x}_+ = \boldsymbol{x}_+ \cdot (\overline{R} *_C \mathtt{K} *_C R) \tag{4.288}$$

or $\mathtt{K} = R *_C \mathtt{K} *_C \overline{R}$. With the same argumentation as above one can see that a bivector generator must have the form

$$\mathtt{B} = \boldsymbol{x}\boldsymbol{y} - (\boldsymbol{x} \cdot \mathtt{K})(\boldsymbol{y} \cdot \mathtt{K}), \tag{4.289}$$

note the different sign compared with (4.279). The n^2 basis bivectors of $\mathfrak{gl}(n)$ are then

$$\mathtt{E}_{ij} = \boldsymbol{\alpha}_i\boldsymbol{\alpha}_j - \boldsymbol{\beta}_i\boldsymbol{\beta}_j, \qquad \mathtt{F}_{ij} = \boldsymbol{\alpha}_i\boldsymbol{\beta}_j - \boldsymbol{\beta}_i\boldsymbol{\alpha}_j \qquad \text{and} \qquad \mathtt{K}_i = \boldsymbol{\alpha}_i\boldsymbol{\beta}_i \tag{4.290}$$

for $i < j = 1, \ldots, n$. It is here also easy to show that these bivectors form a closed algebra under the commutator product. Note that the doubling of the dimension is here necessary to have sufficient degrees of freedoms for the bivector algebra. In the case of an orthogonal group this is not necessary, so that the generating bivectors live directly in the space on which the transformation acts.

What is actually happening in the $Gl(n)$-case is that one performs a transformation of the variables of the vector $\boldsymbol{x} = a^i\boldsymbol{\alpha}_i + b^i\boldsymbol{\beta}_i$ into variables q^i, p^i, $\boldsymbol{\eta}_i$ and $\boldsymbol{\rho}_i$ according to

$$\boldsymbol{x}_+ = \frac{1}{2}\big(\boldsymbol{x} + \boldsymbol{x} \cdot \mathtt{K}\big) = \frac{1}{2}(a^i - b^i)(\boldsymbol{\alpha}_i - \boldsymbol{\beta}_i) \equiv q^i\boldsymbol{\eta}_i \tag{4.291}$$

$$\boldsymbol{x}_- = \frac{1}{2}\big(\boldsymbol{x} - \boldsymbol{x} \cdot \mathtt{K}\big) = \frac{1}{2}(a^i + b^i)(\boldsymbol{\alpha}_i + \boldsymbol{\beta}_i) \equiv p^i\boldsymbol{\rho}_i. \tag{4.292}$$

It is then straightforward to transform the star product (4.282) and the generators (4.290) into these new variables. For the star product one obtains

$$*_C = \exp\left[\frac{\eta_{ij}}{2}\left(\frac{\overleftarrow{\partial}}{\partial\boldsymbol{\eta}_i}\frac{\vec{\partial}}{\partial\boldsymbol{\rho}_j} + \frac{\overleftarrow{\partial}}{\partial\boldsymbol{\rho}_i}\frac{\vec{\partial}}{\partial\boldsymbol{\eta}_j}\right)\right], \tag{4.293}$$

which is a fermionic version of the Moyal product

$$*_M = \exp\left[\frac{\mathrm{i}\hbar}{2}\eta^{ij}\left(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\vec{\partial}}{\partial p^j} - \frac{\overleftarrow{\partial}}{\partial p^i}\frac{\vec{\partial}}{\partial q^j}\right)\right]. \tag{4.294}$$

This suggests that the vector $\boldsymbol{x} = q^i\boldsymbol{\eta}_i + p^i\boldsymbol{\rho}_i$ can not only be transformed with a fermionic star exponential as described above, but can also be transformed in the bosonic coefficients with a bosonic star exponential according to [3]

$$e_{*_M}^{\alpha_{ij}M^{ij}} *_M q^k *_M e_{*_M}^{-\alpha_{ij}M^{ij}} = q^k + \alpha_{ij}\left[M^{ij}, q^k\right]_{*_M} + \frac{1}{2!}\alpha_{ij}\alpha_{lm}\left[M^{lm}, \left[M^{ij}, q^k\right]_{*_M}\right]_{*_M} + \ldots, \tag{4.295}$$

where $[f, g]_{*_M} = f *_M g - g *_M f$ is the star-commutator. In analogy to the fermionic case one can now demand that for a $Gl(n)$ transformation the q^k have to be a linear combination of the q^i alone and no terms in p^i should appear. This means that $\left[M^{ij}, q^k\right]_{*_M}$ must be a function of the q^i alone. This is achieved if one chooses the bosonic generators

$$E^{ij} = q^ip^j + q^jp^i, \qquad F^{ij} = q^ip^j - q^jp^i, \qquad \text{and} \qquad K^i = q^ip^i, \tag{4.296}$$

which form a closed algebra under the Moyal star-commutator.

4.7 Spinors

It is now also possible to describe spinors in the language of geometric algebra [55]. To this purpose one notices that multivectors of even grade play a special role in the Clifford algebra $\mathcal{C}\ell_{p,q}$ because they form the subalgebra $\mathcal{C}\ell^+_{p,q}$, which is isomorphic to a geometric algebra of smaller dimension:

$$\mathcal{C}\ell^+_{p,q} \simeq \mathcal{C}\ell_{q,p-1} \simeq \mathcal{C}\ell_{p,q-1}. \tag{4.297}$$

As shown in the discussion of the complex numbers to each $d = (p+q)$-dimensional vector $\boldsymbol{x}$ corresponds an element of the even subalgebra, which is called a paravector and can be obtained according to

$$\underline{\boldsymbol{x}} = \boldsymbol{x} *_C \boldsymbol{u}, \tag{4.298}$$

where $\boldsymbol{u}$ is a unit vector. While $\boldsymbol{x}$ as a grade-one quantity is invariant under the involution $\overline{\boldsymbol{x}} = \boldsymbol{x}$, the paravector is invariant under the hermitian conjugate

$$\underline{\boldsymbol{x}}^\dagger \equiv \boldsymbol{u} *_C \overline{\underline{\boldsymbol{x}}} *_C \boldsymbol{u} = \boldsymbol{u} *_C \boldsymbol{u} *_C \boldsymbol{x} *_C \boldsymbol{u} = \boldsymbol{x} *_C \boldsymbol{u} = \underline{\boldsymbol{x}}. \tag{4.299}$$

The transformation under a passive rotation is given for a vector by

$$\boldsymbol{x}' = R *_C \boldsymbol{x} *_C \overline{R}, \tag{4.300}$$

where $R = e_{*_C}^{\mathrm{B}\varphi/2}$ is an element of the group $Spin_+(p,q)$. The paravector transforms oppsitely to (4.300) as

$$\underline{\boldsymbol{x}}' = R *_C \underline{\boldsymbol{x}} *_C R^\dagger = R *_C \boldsymbol{x} *_C \overline{R} *_C \boldsymbol{u} = \boldsymbol{x}' *_C \boldsymbol{u}. \tag{4.301}$$

Besides the vectors and paravectors that transform as in (4.300) and (4.301) there are also multivectors ψ called spinors that transform according to

$$\psi' = R *_C \psi. \tag{4.302}$$

In the vector case the product $\boldsymbol{x} *_C \boldsymbol{x} = |\boldsymbol{x}|^2$ is invariant under rotation, in the paravector case the product $\underline{\boldsymbol{x}} *_C \overline{\underline{\boldsymbol{x}}} = |\boldsymbol{x}|^2$ and in the spinor case the product

$$\overline{\psi'} *_C \psi' = \overline{\psi} *_C \overline{R} *_C R *_C \psi = \overline{\psi} *_C \psi. \tag{4.303}$$

The spinor ψ can obviously be represented by the even multivectors of $\mathcal{C}\ell^+_{p,q}$.

In two euclidian dimensions the relation of vectors, paravectors and spinors simplify due to the fact that in two dimensions a vector transforms as

$$\boldsymbol{x}' = R_{\varphi/2} *_C \boldsymbol{x} *_C \overline{R_{\varphi/2}} = R_\varphi *_C \boldsymbol{x}. \tag{4.304}$$

A paravector transforms then according to

$$\underline{\boldsymbol{x}}' = R_{\varphi/2} *_C \underline{\boldsymbol{x}} *_C R^\dagger_{\varphi/2} = R_{\varphi/2} *_C \boldsymbol{x} *_C \overline{R_{\varphi/2}} *_C \boldsymbol{u} = R_\varphi *_C \boldsymbol{x} *_C \boldsymbol{u} = R_\varphi *_C \underline{\boldsymbol{x}}\,, \tag{4.305}$$

i.e. in two dimensions the paravectors as even multivectors of maximal grade two and spinors as general even multivectors are the same and they both correspond to complex numbers as discussed above. Furthermore the hermitian conjugate and the involution are the same in two dimensions, for example if $\boldsymbol{u} = \boldsymbol{\sigma}_2$ one has for an even multivector U in two dimensions

$$U^\dagger = \boldsymbol{\sigma}_2 *_C \overline{U} *_C \boldsymbol{\sigma}_2 = \boldsymbol{\sigma}_2 *_C (U_1 + U_2\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) *_C \boldsymbol{\sigma}_2 = U_1 + U_2\boldsymbol{\sigma}_2\boldsymbol{\sigma}_1 = \overline{U}. \tag{4.306}$$

In three euclidian dimensions the relation (4.304) is no longer valid, so that here paravectors and spinors have a different transformation behavior and so cannot be identified. But in three dimensions the maximal even grade is two, so that here paravectors and spinors are both multivectors with maximal grade two and are related according to

$$\underline{\boldsymbol{x}} = \psi *_C \psi^\dagger. \tag{4.307}$$

This becomes obvious if one considers the transformation of the paravector:

$$\underline{\boldsymbol{x}}' = R *_C \psi *_C \psi^\dagger *_C R^\dagger = R *_C \psi *_C \boldsymbol{u} *_C \overline{\psi} *_C \overline{R} *_C \boldsymbol{u} = \psi' *_C \psi'^\dagger, \tag{4.308}$$

with $\psi'^\dagger = (R *_C \psi)^\dagger = \boldsymbol{u} *_C \overline{\psi} *_C \overline{R} *_C \boldsymbol{u}$. The Pauli spinor can therefore be represented in three dimensions as

$$\psi = \psi^0 + \psi^k \mathtt{B}_k = (\psi^0 + \psi^3 \mathtt{B}_3) + (\psi^2 + \psi^1 \mathtt{B}_3) *_C \mathtt{B}_2, \tag{4.309}$$

which is isomorphic to a quaternion. The representation (4.309) shows that $\mathtt{B}_3 = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$ plays the role of the unitary unit i and the even multivector ψ corresponds in the conventional formalism to the spinor

$$\hat{\psi} = \begin{pmatrix} \psi^0 + \mathrm{i}\psi^3 \\ -\psi^2 + \mathrm{i}\psi^1 \end{pmatrix}. \tag{4.310}$$

From (4.310) and (4.309) one has the following correspondences

$$\hat{\psi}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \leftrightarrow \psi_+ = 1 \quad \text{and} \quad \hat{\psi}_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \leftrightarrow \psi_- = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_3 \tag{4.311}$$

and the $*$-eigenvalue equation can be written as:

$$\hat{\sigma}_3 \hat{\psi}_\pm = \pm \hat{\psi}_\pm \leftrightarrow \boldsymbol{\sigma}_3 *_C \psi_\pm *_C \boldsymbol{\sigma}_3 = \pm \psi_\pm. \tag{4.312}$$

It is clear that a spinor in geometric algebra is a rotor that is not normalized, i.e. $\psi = |\psi| R$ with $|\psi|^2 = (\psi^0)^2 + (\psi^1)^2 + (\psi^2)^2 + (\psi^3)^2$. Furthermore in the star product formalism the hermitian product $\langle \hat{\psi}, \hat{\phi} \rangle_H = \hat{\psi}^\dagger \hat{\phi}$ can be written as

$$\langle \psi, \phi \rangle_H = \frac{1}{2} \left(\overline{\psi} *_C \phi - \mathtt{B}_3 *_C \overline{\psi} *_C \phi *_C \mathtt{B}_3 \right) = \overline{\psi} *_C \phi + \left[\left(\overline{\psi} *_C \phi \right) \times \mathtt{B}_3 \right] *_C \mathtt{B}_3 \tag{4.313}$$

and $\hat{\psi}^\dagger \hat{\sigma}^k \hat{\psi}$ can be written as $\langle \psi *_C \boldsymbol{\sigma}_3 *_C \overline{\psi} *_C \boldsymbol{\sigma}_k \rangle_0$.

In order to describe Dirac spinors in geometric algebra one has to go over to the four dimensional space-time. Geometric algebra in the Minkowski space is called space-time algebra $\mathcal{C}\ell_{1,3}$. The space-time basis vectors are $\boldsymbol{\gamma}_0$, $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$ and $\boldsymbol{\gamma}_3$ and the Clifford star product is

$$F *_C G = F \exp \left[\eta_{\mu\nu} \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\gamma}_\mu} \frac{\vec{\partial}}{\partial \boldsymbol{\gamma}_\nu} \right] G, \tag{4.314}$$

so that

$$\boldsymbol{\gamma}_\mu \cdot \boldsymbol{\gamma}_\nu = \frac{1}{2} \left(\boldsymbol{\gamma}_\mu *_C \boldsymbol{\gamma}_\nu + \boldsymbol{\gamma}_\nu *_C \boldsymbol{\gamma}_\mu \right) = \eta_{\mu\nu}, \tag{4.315}$$

or $\{\boldsymbol{\gamma}_\mu, \boldsymbol{\gamma}_\nu\}_{*_C} = 2\eta_{\mu\nu}$. A space-time vector is then given by $\boldsymbol{x} = x^\mu \boldsymbol{\gamma}_\mu$ and a general space-time multivector has the form

$$A = A_{(0)} + A^\mu_{(1)} \boldsymbol{\gamma}_\mu + A^{\mu\nu}_{(2)} \boldsymbol{\gamma}_\mu \boldsymbol{\gamma}_\nu + A^{\mu\nu\rho}_{(3)} \boldsymbol{\gamma}_\mu \boldsymbol{\gamma}_\nu \boldsymbol{\gamma}_\rho + A_{(4)} \boldsymbol{\gamma}_0 \boldsymbol{\gamma}_1 \boldsymbol{\gamma}_2 \boldsymbol{\gamma}_3. \tag{4.316}$$

Just as in the three dimensional case it is now possible to define with a timelike vector $\boldsymbol{u}$, i.e. $\boldsymbol{u}^{2*_C} = 1$, a paravector $\underline{\boldsymbol{x}} = \boldsymbol{x} *_C \boldsymbol{u}$ and a hermitian conjugation $A^\dagger = \boldsymbol{u} *_C \overline{A} *_C \boldsymbol{u}$. The choice of $\boldsymbol{u}$ defines a space-time split and the easiest choice is $\boldsymbol{u} = \boldsymbol{\gamma}_0$, so that the paravector is given by

$$\underline{\boldsymbol{x}} = \boldsymbol{x} *_C \boldsymbol{\gamma}_0 = \boldsymbol{x} \cdot \boldsymbol{\gamma}_0 + \boldsymbol{x} \boldsymbol{\gamma}_0 = x^0 + \mathtt{x} = t + x^i \boldsymbol{\gamma}_i \boldsymbol{\gamma}_0. \tag{4.317}$$

One should note that $\mathtt{x} = \boldsymbol{x}\boldsymbol{\gamma}_0 = x^i\boldsymbol{\gamma}_i\boldsymbol{\gamma}_0$ is a space-time bivector, but corresponds to a space vector. This is due to the fact that the two-blades $\boldsymbol{\gamma}_i\boldsymbol{\gamma}_0$ behave in space-time like the $\boldsymbol{\sigma}_i$ in space:

$$\boldsymbol{\sigma}_i *_C \boldsymbol{\sigma}_j = I_{(3)} *_C \sigma_k \quad \widehat{=} \quad \boldsymbol{\gamma}_j\boldsymbol{\gamma}_i = I_{(4)} *_C \boldsymbol{\gamma}_k\boldsymbol{\gamma}_0 \qquad \text{for cyclic } i,j,k, \tag{4.318}$$

$$\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j = \frac{1}{2}\left(\boldsymbol{\sigma}_i *_C \boldsymbol{\sigma}_j + \boldsymbol{\sigma}_j *_C \boldsymbol{\sigma}_i\right) \quad \widehat{=} \quad \frac{1}{2}\left(\boldsymbol{\gamma}_i\boldsymbol{\gamma}_0 *_C \boldsymbol{\gamma}_j\boldsymbol{\gamma}_0 + \boldsymbol{\gamma}_j\boldsymbol{\gamma}_0 *_C \boldsymbol{\gamma}_i\boldsymbol{\gamma}_0\right) = \delta_{ij}, \tag{4.319}$$

$$I_{(3)} = \boldsymbol{\sigma}_1 *_C \boldsymbol{\sigma}_2 *_C \boldsymbol{\sigma}_3 \quad \widehat{=} \quad \boldsymbol{\gamma}_1\boldsymbol{\gamma}_0 *_C \boldsymbol{\gamma}_2\boldsymbol{\gamma}_0 *_C \boldsymbol{\gamma}_3\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 = I_{(4)}, \tag{4.320}$$

where on the left hand side the three dimensional euclidian Clifford star product and on the right hand side the four dimensional Clifford star product (4.314) is used.

The Dirac spinor Ψ is now a general element of $\mathcal{C}\ell^+_{1,3}$, which can be written with two Pauli spinors ψ_I, ψ_{II} and using $\boldsymbol{\sigma}_i \widehat{=} \boldsymbol{\gamma}_i\boldsymbol{\gamma}_0$ as

$$\begin{aligned}
\Psi &= \psi_I + \psi_{II} *_C \boldsymbol{\gamma}_3\boldsymbol{\gamma}_0 && (4.321)\\
&= (\psi_I^0 + \psi_I^1\boldsymbol{\sigma}_2 *_C \boldsymbol{\sigma}_3 + \psi_I^2\boldsymbol{\sigma}_3 *_C \boldsymbol{\sigma}_1 + \psi_I^3\boldsymbol{\sigma}_1 *_C \boldsymbol{\sigma}_3) \\
&\quad +(\psi_{II}^0 + \psi_{II}^1\boldsymbol{\sigma}_2 *_C \boldsymbol{\sigma}_3 + \psi_{II}^2\boldsymbol{\sigma}_3 *_C \boldsymbol{\sigma}_1 + \psi_{II}^3\boldsymbol{\sigma}_1 *_C \boldsymbol{\sigma}_3) *_C \boldsymbol{\gamma}_3\boldsymbol{\gamma}_0 && (4.322)\\
&\widehat{=} \psi_I^0 - \psi_I^1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 - \psi_I^2\boldsymbol{\gamma}_3\boldsymbol{\gamma}_1 - \psi_I^3\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2 + \psi_{II}^0\boldsymbol{\gamma}_3\boldsymbol{\gamma}_0 + \psi_{II}^1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_0 - \psi_{II}^2\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0 + \psi_{II}^3\boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3 && (4.323)\\
&= \Psi_1 + \Psi_2^\dagger *_C \boldsymbol{\gamma}_1\boldsymbol{\gamma}_3 - (\Psi_3 + \Psi_4 *_C \boldsymbol{\gamma}_1\boldsymbol{\gamma}_3) *_C \boldsymbol{\gamma}_0\boldsymbol{\gamma}_3. && (4.324)
\end{aligned}$$

The Ψ_μ are the "complex" components of the four-spinor, where the role of the unitary unit i is played here by the two-blade $\boldsymbol{\gamma}_2\boldsymbol{\gamma}_1$, so that $\Psi_\mu = \Psi_{\mu,\mathrm{Re}} + \Psi_{\mu,\mathrm{Im}}\boldsymbol{\gamma}_2\boldsymbol{\gamma}_1$. The even multivector Ψ corresponds then to the four spinor

$$\hat{\Psi} = \begin{pmatrix} \Psi_{1,\mathrm{Re}} + \mathrm{i}\Psi_{1,\mathrm{Im}} \\ \Psi_{2,\mathrm{Re}} + \mathrm{i}\Psi_{2,\mathrm{Im}} \\ \Psi_{3,\mathrm{Re}} + \mathrm{i}\Psi_{3,\mathrm{Im}} \\ \Psi_{4,\mathrm{Re}} + \mathrm{i}\Psi_{4,\mathrm{Im}} \end{pmatrix}. \tag{4.325}$$

The hermitian product $\langle\hat{\Psi},\hat{\Phi}\rangle_H = \hat{\Psi}^\dagger\hat{\gamma}_0\hat{\Phi}$ can in the formalism of geometric algebra be written as

$$\langle\Psi,\Phi\rangle_H = \Psi_1^\dagger *_C \Phi_1 + \Psi_2^\dagger *_C \Phi_2 - \Psi_3^\dagger *_C \Phi - \Psi_3^\dagger *_C \Phi_3 - \Psi_4^\dagger *_C \Phi_4. \tag{4.326}$$

Furthermore all bilinear covariants can be translated in this language, as examples only the scalar $\langle\Psi *_C \overline{\Psi}\rangle_0$ and the vector observable $\boldsymbol{J} = \Psi *_C \boldsymbol{\gamma}_0 *_C \overline{\Psi}$ should be mentioned.

4.8 Symplectic Vector Manifolds

A symplectic vector space can be considered as a $2d$-dimensional euclidian space with vectors

$$\boldsymbol{z} = z^a\boldsymbol{\zeta}_a = q^m\boldsymbol{\eta}_m + p^m\boldsymbol{\rho}_m, \tag{4.327}$$

where $a = 1,\ldots,2d$ and $m = 1,\ldots,d$, and a closed two form

$$\Omega = \frac{1}{2}\Omega_{ab}\boldsymbol{\zeta}^a\boldsymbol{\zeta}^b = \sum_{m=1}^{d}\boldsymbol{\eta}^m\boldsymbol{\rho}^m = \sum_{m=1}^{d}\boldsymbol{d}q^m\boldsymbol{d}p^m, \tag{4.328}$$

where Ω_{ab} is a non-degenerate, antisymmetric matrix [91]. The euclidian metric on the vector space defines a scalar product and a relation between vectors and one forms. The two form Ω gives now an additional possibility to establish such structures, i.e. one can define the symplectic scalar product as

$$\boldsymbol{z} \cdot_{Sy} \boldsymbol{w} \equiv \boldsymbol{i}_{\boldsymbol{zw}}\Omega = (\boldsymbol{wz})\cdot\Omega = \boldsymbol{z}\cdot(\Omega\cdot\boldsymbol{w}) = z^a\Omega_{ab}w^b \tag{4.329}$$

and furthermore one can map with Ω a vector in a one form according to $\boldsymbol{z}^\flat = i_{\boldsymbol{z}}\Omega = \boldsymbol{z}\cdot\Omega$, so that $\boldsymbol{z}\cdot_{Sy}\boldsymbol{w} = -\boldsymbol{z}\cdot\boldsymbol{w}^\flat$ (the other possibility to define $\flat$ used for example in [73] is $\Omega\cdot\boldsymbol{x} = -\boldsymbol{x}\cdot\Omega$). The inverse map of a one form into a vector can be described with the bivector

$$\mathtt{J} = \frac{1}{2}J^{ab}\boldsymbol{\zeta}_a\boldsymbol{\zeta}_b = \frac{1}{2}\sum_{a,b=1}^{2d}\Omega_{ab}\boldsymbol{\zeta}_a\boldsymbol{\zeta}_b = \sum_{m=1}^{d}\boldsymbol{\eta}_m\boldsymbol{\rho}_m, \tag{4.330}$$

so that the vector corresponding to a one form $\boldsymbol{\omega}$ is given by $\boldsymbol{\omega}^\natural = \mathtt{J}\cdot\boldsymbol{\omega}$. The map $\natural$ should be inverse to $\flat$, from which follows that $J^{ab} = (\Omega_{ab}^{-1})^T = \Omega^{ba}$. Especially for the nabla operator $\boldsymbol{\nabla} = \boldsymbol{d} = \boldsymbol{\zeta}^a\frac{\partial}{\partial z_a}$ one has

$$\boldsymbol{d}^\natural = \sum_{m=1}^{d}\left(\boldsymbol{\eta}_m\frac{\partial}{\partial p^m} - \boldsymbol{\rho}_m\frac{\partial}{\partial q^m}\right), \tag{4.331}$$

so that for example the Hamilton equations can be written as

$$\dot{\boldsymbol{z}} = \boldsymbol{d}^\natural H, \tag{4.332}$$

or explicitly

$$(\dot{q}^m\boldsymbol{\eta}_m + \dot{p}^m\boldsymbol{\rho}_m) = \mathtt{J}\cdot\left(\boldsymbol{\eta}^m\frac{\partial H}{\partial q^m} + \boldsymbol{\rho}^m\frac{\partial H}{\partial p^m}\right) = \sum_{m=1}^{d}\left(-\boldsymbol{\rho}_m\frac{\partial H}{\partial q^m} + \boldsymbol{\eta}_m\frac{\partial H}{\partial p^m}\right). \tag{4.333}$$

Furthermore the Poisson bracket can be written as

$$\{F,G\}_{PB} = F\,\overleftarrow{\boldsymbol{d}}\cdot_{Sy}\overrightarrow{\boldsymbol{d}}\,G = J^{ab}\frac{\partial F}{\partial x^a}\frac{\partial G}{\partial x^b}. \tag{4.334}$$

The bivector $\mathtt{J}$ plays the role of the compatible complex structure to Ω [91], because one has

$$(\boldsymbol{z}\cdot\mathtt{J})\cdot_{Sy}(\boldsymbol{w}\cdot\mathtt{J}) = \boldsymbol{z}\cdot_{Sy}\boldsymbol{w} \qquad \text{and} \qquad \boldsymbol{z}\cdot_{Sy}(\boldsymbol{w}\cdot\mathtt{J}) > 0 \qquad \forall\boldsymbol{z}\neq 0. \tag{4.335}$$

Furthermore one has $\mathtt{J}\cdot\mathtt{J} = -1$, $(\boldsymbol{z}\cdot\mathtt{J})\cdot\mathtt{J} = -\boldsymbol{z}$ and the symplectic scalar product can be written as $\boldsymbol{z}\cdot_{Sy}\boldsymbol{w} = (\boldsymbol{z}\cdot\mathtt{J})\cdot\boldsymbol{w}$. A metric space with a two form Ω and a compatible complex structure is a Kähler space.

A symplectic vector manifold is an even-dimensional vector manifold $\boldsymbol{x}(x^i)$ with a closed two form $\Omega(\boldsymbol{x}) = \frac{1}{2}\Omega_{ij}\boldsymbol{\xi}^i\boldsymbol{\xi}^j$, i.e.

$$\partial_i\Omega_{jk} + \partial_j\Omega_{ki} + \partial_k\Omega_{ij} = 0. \tag{4.336}$$

The tangent spaces at the symplectic vector manifold are symplectic vector spaces. A vector field $\boldsymbol{z}(\boldsymbol{x})$ on a symplectic vector manifold is symplectic if $\boldsymbol{z}^\flat$ is closed, i.e. if $\boldsymbol{d}(\boldsymbol{z}\cdot\Omega) = 0$. Symplectic vector fields conserve the symplectic structure, i.e. $\mathscr{L}_{\boldsymbol{z}}\Omega = \boldsymbol{d}i_{\boldsymbol{z}}\Omega = 0$ and they form an algebra under the Jacobi-Lie bracket, i.e. for two symplectic vector fields $\boldsymbol{z}(\boldsymbol{x})$ and $\boldsymbol{w}(\boldsymbol{x})$ one has $\boldsymbol{d}\big([\boldsymbol{z},\boldsymbol{w}]_{JLB}\cdot\Omega\big) = 0$. If $\boldsymbol{z}^\flat$ is not only closed but also exact, the vector field is called hamiltonian. According to the Poincaré lemma every closed form is locally exact, so that a symplectic vector field is locally hamiltonian. This means for a local (global) hamiltonian vector field $\boldsymbol{h}_H$ exists locally (globally) a function H so that

$$\boldsymbol{h}_H\cdot\Omega = \boldsymbol{d}H, \tag{4.337}$$

which in the coordinate basis reads

$$\boldsymbol{h}_H = \boldsymbol{d}^\natural H = J^{ij}(\partial_j H)\boldsymbol{\xi}_i. \tag{4.338}$$

The Lie-bracket of two symplectic vector fields $\boldsymbol{z}(\boldsymbol{x})$ and $\boldsymbol{w}(\boldsymbol{x})$ is always a hamiltonian vector field with hamiltonian $(\boldsymbol{z}\boldsymbol{w})\cdot\Omega$, i.e. $\boldsymbol{h}_{(\boldsymbol{z}\boldsymbol{w})\cdot\Omega} = [\boldsymbol{z},\boldsymbol{w}]_{JLB}$ or

$$[\boldsymbol{z},\boldsymbol{w}]_{JLB}\cdot\Omega = \boldsymbol{d}((\boldsymbol{z}\boldsymbol{w})\cdot\Omega). \tag{4.339}$$

This follows easily using $\mathscr{L}_{\boldsymbol{z}}\Omega = \mathscr{L}_{\boldsymbol{w}}\Omega = 0$ and $\boldsymbol{d}\Omega = 0$:

$$\begin{aligned}
[\boldsymbol{z},\boldsymbol{w}]_{JLB}\cdot\Omega &= \mathscr{L}_{\boldsymbol{z}} i_{\boldsymbol{w}}\Omega - i_{\boldsymbol{w}}\mathscr{L}_{\boldsymbol{z}}\Omega && (4.340)\\
&= (\boldsymbol{d}i_{\boldsymbol{z}} + i_{\boldsymbol{z}}\boldsymbol{d})i_{\boldsymbol{w}}\Omega && (4.341)\\
&= \boldsymbol{d}\big((\boldsymbol{z}\boldsymbol{w})\cdot\Omega\big) + i_{\boldsymbol{z}}(\boldsymbol{d}i_{\boldsymbol{w}} + i_{\boldsymbol{w}}\boldsymbol{d})\Omega && (4.342)\\
&= \boldsymbol{d}\big((\boldsymbol{z}\boldsymbol{w})\cdot\Omega\big) + i_{\boldsymbol{z}}\mathscr{L}_{\boldsymbol{w}}\Omega && (4.343)\\
&= \boldsymbol{d}\big((\boldsymbol{z}\boldsymbol{w})\cdot\Omega\big). && (4.344)
\end{aligned}$$

With a hamiltonian vector field the Poisson bracket can then be written as

$$\mathscr{L}_{\boldsymbol{h}_H} F = \boldsymbol{h}_H\cdot\boldsymbol{d}F = \{F,H\}_{PB}, \tag{4.345}$$

or, using (4.337) in this equation, as

$$\{F,G\}_{PB} = i_{\boldsymbol{h}_F\boldsymbol{h}_G}\Omega = (\boldsymbol{h}_G\boldsymbol{h}_F)\cdot\Omega. \tag{4.346}$$

It is easy to see that the hamiltonian vector fields form a Lie subalgebra of the symplectic vector fields with

$$[\boldsymbol{h}_F,\boldsymbol{h}_G]_{JLB} = -\boldsymbol{h}_{\{F,G\}_{PB}}. \tag{4.347}$$

Given a symplectic vector field $\boldsymbol{z}$ that preserves the Hamilton function H, i.e. $\mathscr{L}_{\boldsymbol{z}}\Omega = \Omega$ and $\mathscr{L}_{\boldsymbol{z}}H = 0$, then this symplectic vector field $\boldsymbol{z}$ can be written locally as a hamiltonian vector field $\boldsymbol{h}_F$ with

$$\mathscr{L}_{\boldsymbol{h}_F} H = \boldsymbol{h}_F\cdot\boldsymbol{d}H = \{F,H\}_{PB} = 0, \tag{4.348}$$

which shows that F is a conserved quantity. This is Noethers theorem for the symplectic case.

The metric $g_{ij}(\boldsymbol{x})$ on the vector manifold is induced by the ambient space and so exists naturally on the vector manifold. It was used in the above discussion just to contract vector fields and forms with the scalar product. The metric can also be used to define a compatible almost complex structure. This is a bivector field $\mathtt{J}(\boldsymbol{x})$, that maps via the scalar product a tangent vector into another tangent vector. If the structures $g_{ij}(\boldsymbol{x})$, $\mathtt{J}(\boldsymbol{x})$ and $\Omega(\boldsymbol{x})$ are compatible the metric scalar product of two tangent vectors $\boldsymbol{z}$ and $\boldsymbol{w}$ in a point $\boldsymbol{x}$ can be written as

$$\boldsymbol{z}\cdot\boldsymbol{w} = \boldsymbol{z}\cdot_{Sy}(\boldsymbol{w}\cdot\mathtt{J}) \tag{4.349}$$

and the symplectic product can be written as

$$\boldsymbol{z}\cdot_{Sy}\boldsymbol{w} = (\boldsymbol{z}\cdot\mathtt{J})\cdot\boldsymbol{w}. \tag{4.350}$$

A vector manifold with these three compatible structures is a Kähler vector manifold.

Symplectic manifolds of special physical interest are cotangent bundles, for which the symplectic two-form is globally exact. The cotangent bundle of a d-dimensional euclidian vector space is a $2d$-dimensional euclidian vector space with elements $\boldsymbol{q}+\boldsymbol{\pi} = q^m\boldsymbol{\eta}_m + p_m\boldsymbol{\rho}^m$. On this vector space one can define with a vector $\boldsymbol{a}+\boldsymbol{\omega} = a^m\boldsymbol{\eta}_m + \omega_m\boldsymbol{\rho}^m$ a canonical one form $\boldsymbol{\theta}(\boldsymbol{q}+\boldsymbol{\pi})$ by

$$(\boldsymbol{a}+\boldsymbol{\omega})\cdot\boldsymbol{\theta}(\boldsymbol{q}+\boldsymbol{\pi}) = a^m p_m, \tag{4.351}$$

so that $\boldsymbol{\theta} = p_m\boldsymbol{\eta}^m = p_m\boldsymbol{d}q^m$, where the nabla operator is given by $\boldsymbol{\nabla} = \boldsymbol{d} = \boldsymbol{\eta}^m\frac{\partial}{\partial q^m} + \boldsymbol{\rho}_m\frac{\partial}{\partial p_m}$. The symplectic two form on the cotangent bundle can then be obtained as

$$\Omega = -\boldsymbol{d}\boldsymbol{\theta} = \boldsymbol{\eta}^m\boldsymbol{\rho}_m = \boldsymbol{d}q^m\boldsymbol{d}p_m. \tag{4.352}$$

The above definitions generalize readily to the case of a cotangent bundle of a d-dimensional vector manifold. In a $(2d+2)$-dimensional ambient vector space with basis vectors $\boldsymbol{\sigma}_a$ and $\boldsymbol{\tau}^a$ the cotangent bundle can be described as the $2d$-dimensional vector manifold $(\boldsymbol{q}+\boldsymbol{\pi})(q^i, p_j) = q^a(q^i)\boldsymbol{\sigma}_a + p_j \xi_a^j(q^i)\boldsymbol{\tau}^a$, with tangent vectors $\boldsymbol{a}+\boldsymbol{\omega} = a^i\xi_i^a\boldsymbol{\sigma}_a + \omega_i\xi_a^i\boldsymbol{\tau}^a$ at this bundle vector manifold. With a projection operator $T\pi_{\boldsymbol{q}}$ defined as

$$T\pi_{\boldsymbol{q}}(\boldsymbol{a}+\boldsymbol{\omega}) = T\pi_{\boldsymbol{q}}(\boldsymbol{a}) = a^i\xi_i^a\boldsymbol{\tau}_a, \tag{4.353}$$

which is the tangent function of the projection $\pi_{\boldsymbol{q}}(\boldsymbol{q}+\boldsymbol{\pi}) = \boldsymbol{q}$, one can write (4.351) as

$$(\boldsymbol{a}+\boldsymbol{\omega})\cdot\boldsymbol{\theta}(\boldsymbol{q}+\boldsymbol{\pi}) = T\pi_{\boldsymbol{q}}(\boldsymbol{a}+\boldsymbol{\omega})\cdot\boldsymbol{\pi}, \tag{4.354}$$

so that $\boldsymbol{\theta} = p_i\boldsymbol{\xi}^i = p_i\boldsymbol{d}q^i$.

The special feature of a cotangent bundle manifold, namely that the symplectic two form Ω is globally exact, i.e. $\Omega = -\boldsymbol{d\theta}$, allows to define globally the Liouville vector field $\boldsymbol{l}$ by

$$\boldsymbol{l}\cdot\Omega = -\boldsymbol{\theta}, \tag{4.355}$$

which in local coordinates is given by $\boldsymbol{l} = p_i\xi_a^i\boldsymbol{\tau}^a$, while the directional derivative is $\boldsymbol{l}\cdot\boldsymbol{d} = p_i\frac{\partial}{\partial p_i}$. The Liouville vector field fulfills

$$\mathscr{L}_{\boldsymbol{l}}\boldsymbol{\theta} = \boldsymbol{\theta} \qquad \text{and} \qquad \mathscr{L}_{\boldsymbol{l}}\Omega = \Omega. \tag{4.356}$$

This follows easily with the Cartan formula:

$$\mathscr{L}_{\boldsymbol{l}}\boldsymbol{\theta} = i_{\boldsymbol{l}}\boldsymbol{d\theta} + \boldsymbol{d}i_{\boldsymbol{l}}\boldsymbol{\theta} = -i_{\boldsymbol{l}}\Omega = \boldsymbol{\theta} \tag{4.357}$$

and applying $\boldsymbol{d}$ on both sides one obtains $\boldsymbol{d}(\mathscr{L}_{\boldsymbol{l}}\boldsymbol{\theta}) = \boldsymbol{d\theta}$, which is equivalent to $\mathscr{L}_{\boldsymbol{l}}\Omega = \Omega$. The Liouville vector field can then be used to measure the order of a scalar function on the cotangent bundle that is polynomial in the fibres. Such a scalar function of order k has the form

$$f(q^i, p_i) = \frac{1}{k!}f^{i_1\ldots i_k}(q^i)p_{i_1}\ldots p_{i_k} \tag{4.358}$$

so that acting with the directional derivative in direction of the Liouville vector field on $f(q^i, p_i)$ gives $\boldsymbol{l}\cdot\boldsymbol{d}f = \mathscr{L}_{\boldsymbol{l}}f = kf$. Scalar valued functions that are polynomial in the fibres can be obtained with an isomorphism $\mathcal{P}$ from tensors $\mathsf{T} = T^{i_1,\ldots,i_k}(q^i)\boldsymbol{\xi}_{i_1}\otimes\ldots\otimes\boldsymbol{\xi}_{i_k}$ on the base space according to

$$\mathcal{P}(\mathsf{T}) = \frac{1}{k!}T^{i_1,\ldots,i_k}(q^i)p_{i_1}\ldots p_{i_k}. \tag{4.359}$$

A tangent vector field $\boldsymbol{a} = a^j(q^i)\boldsymbol{\xi}_j$ on the configuration space $\boldsymbol{q}(q^i)$ is then mapped into a scalar function on the cotangent bundle that is linear in the fibres:

$$\mathcal{P}(\boldsymbol{a}) = \mathcal{P}(a^j(q^i)\boldsymbol{\xi}_j) = a^j(q^i)p_j. \tag{4.360}$$

The scalar function $\mathcal{P}(\boldsymbol{a})$ is the so-called momentum of $\boldsymbol{a}$ and $\mathcal{P}$ is the universal momentum map of the cotangent bundle T^*Q. Furthermore one has

$$\begin{aligned}
\mathcal{P}([\boldsymbol{a},\boldsymbol{b}]_{JLB}) &= \mathcal{P}\left(\left(a^i\frac{\partial}{\partial q^i}b^j - b^i\frac{\partial}{\partial q^i}a^j\right)\boldsymbol{\xi}_j\right) && (4.361)\\
&= \left(a^i\frac{\partial}{\partial q^i}b^j - b^i\frac{\partial}{\partial q^i}a^j\right)p_j && (4.362)\\
&= -\{a^jp_j, b^ip_i\}_{PB} && (4.363)\\
&= -\{\mathcal{P}(\boldsymbol{a}),\mathcal{P}(\boldsymbol{b})\}_{PB}. && (4.364)
\end{aligned}$$

In the discussion so far the symplectic structure was defined via a two form. The metric on the vector manifold that is induced from the ambient space was then used to contract vectors and forms. But this contraction is actually independent of the metric. So a metric structure is actually not necessary to define a symplectic structure. In the case of a cotangent bundle it suffices to use the natural duality on this space. This duality can also be described with a star product, for example on the cotangent bundle of a vector space one can define

$$F *_D G = F \exp\left[\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\eta}_a}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\rho}^a}\right] G, \tag{4.365}$$

so that (4.351) reads $i_{\boldsymbol{a}+\boldsymbol{\omega}}\boldsymbol{\theta}(\boldsymbol{q}+\boldsymbol{\pi}) = i_{\boldsymbol{a}}\boldsymbol{\pi} = \langle \boldsymbol{a} *_D \boldsymbol{\pi}\rangle_0 = \boldsymbol{a} \cdot_D \boldsymbol{\pi}$ and further $i_{(\boldsymbol{a}+\boldsymbol{\omega})(\boldsymbol{b}+\boldsymbol{\chi})}\Omega = \boldsymbol{a} \cdot_D \boldsymbol{\chi} - \boldsymbol{b} \cdot_D \boldsymbol{\omega}$, which can easily be generalized to manifolds [90]. The other possibility is to define a symplectic star product, by using Ω_{ij} instead of the metric η_{ij} in the fermionic star product. On a $2d$-dimensional vector space the symplectic star product in Darboux coordinates is given by

$$F *_{Sy} G = F \exp\left[\Omega_{ab}\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\zeta}_a}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\zeta}_b}\right] G = F \exp\left[\sum_{m=1}^{d}\left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\eta}_m}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\rho}_m} - \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\rho}_m}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\eta}_m}\right)\right] G. \tag{4.366}$$

On a $2d$-dimensional vector manifold the tangent space can also be spanned by Darboux basis vectors $\boldsymbol{\eta}_m = \eta^i_m \boldsymbol{\xi}_i$ and $\boldsymbol{\rho}_m = \rho^i_m \boldsymbol{\xi}_i$ so that one has analogously

$$F *_{Sy} G = F \exp\left[\Omega_{ij}\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\xi}_i}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\xi}_j}\right] G = F \exp\left[\sum_{m=1}^{d}\left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\eta}_m}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\rho}_m} - \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\rho}_m}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\eta}_m}\right)\right] G. \tag{4.367}$$

The indices are now lowered and raised with Ω_{ij}, i.e. for a tangent vector $\boldsymbol{a} = a^i \boldsymbol{\xi}_i$ one has $a_i = \Omega_{ij} a^j$ and $\boldsymbol{\xi}^i = \Omega^{ij}\boldsymbol{\xi}_j$, where $\Omega_{ij}\Omega^{jk} = \delta^k_i$. The relations $\flat$ and $\natural$ between vectors and one forms can then be written as

$$\begin{aligned} \boldsymbol{a}^\flat &= a^i \Omega_{ij}\boldsymbol{\xi}^j = (\Omega^T_{ji} a^i)\boldsymbol{\xi}^j, & (4.368)\\ \boldsymbol{\omega}^\natural &= \omega_i \Omega^{ij}\boldsymbol{\xi}_j = (\Omega^{ji\,T}\omega_i)\boldsymbol{\xi}_j = J^{ji}\omega_i \boldsymbol{\xi}_j. & (4.369)\end{aligned}$$

Furthermore it follows for the scalar products that

$$\boldsymbol{\xi}_i \cdot_{Sy} \boldsymbol{\xi}_j = \Omega_{ij}, \qquad \boldsymbol{\xi}^i \cdot_{Sy} \boldsymbol{\xi}_j = -\boldsymbol{\xi}_j \cdot_{Sy} \boldsymbol{\xi}^i = \delta^i_j \qquad \text{and} \qquad \boldsymbol{\xi}^i \cdot_{Sy} \boldsymbol{\xi}^j = -\Omega^{ij} = J^{ij}. \tag{4.370}$$

If one establishes the symplectic structure with the symplectic star product and not with a metric star product and a two form, the contraction of vectors and one forms has to be defined with the symplectic scalar product $\boldsymbol{\xi}_i \cdot_{Sy} \boldsymbol{\xi}^j = -\delta^j_i$. This leads to a different sign structure compared with the case of a metric star product, for example instead of (4.337) one has for a hamiltonian vector field on a vector space with a symplectic star product

$$\boldsymbol{h}_H \cdot_{Sy} \Omega = -\boldsymbol{d}H \tag{4.371}$$

and since $\boldsymbol{a} \cdot_{Sy} \boldsymbol{\partial} = -\boldsymbol{a} \cdot \boldsymbol{\partial}$ there is no minus sign on the right side of (4.347). So these two sign conventions correspond to the usage of an metric or a symplectic star product on the vector space.

4.9 Poisson Vector Manifolds

A vector manifold M with a bivector $\mathtt{J}(\boldsymbol{x}) = \frac{1}{2}J^{ij}\boldsymbol{\xi}_i\boldsymbol{\xi}_j$ and

$$J^{ij}\partial_i J^{kl} + J^{ik}\partial_i J^{lj} + J^{il}\partial_i J^{jk} = 0 \tag{4.372}$$

is a Poisson vector manifold, where (4.372) can also be expressed with (4.90) as

$$[\mathtt{J}, \mathtt{J}]_{SNB} = 0. \tag{4.373}$$

The bivector $\mathtt{J}$ defines as discussed above a map from $T^*_{\boldsymbol{x}}M$ to $T_{\boldsymbol{x}}M$ by $\boldsymbol{\alpha}^\natural = \mathtt{J}\cdot\boldsymbol{\alpha} = J^{ij}\alpha_j\boldsymbol{\xi}_i$, where $\boldsymbol{\alpha} = \alpha_i\boldsymbol{\xi}^i$ is an element of $T^*_{\boldsymbol{x}}M$. Especially the hamiltonian vector field (4.338) can be written as

$$\boldsymbol{h}_H = i_{\boldsymbol{d}H}\mathtt{J} = \mathtt{J}\cdot\boldsymbol{d}H. \tag{4.374}$$

The Poisson bracket is then given by

$$\{F,G\}_{PB} = i_{\boldsymbol{d}F\boldsymbol{d}G}\mathtt{J} = (\boldsymbol{d}G\boldsymbol{d}F)\cdot\mathtt{J}, \tag{4.375}$$

where (4.372) insures the Jacobi-identity of the Poisson bracket. With the Poisson bracket the hamiltonian vector field $\boldsymbol{h}_H$ can be defined for all scalar functions F as

$$\boldsymbol{h}_H\cdot\boldsymbol{d}F = \{F,H\}_{PB}. \tag{4.376}$$

Equating (4.346) and (4.375) shows how Ω and $\mathtt{J}$ determine each other:

$$(\boldsymbol{h}_G\boldsymbol{h}_F)\cdot\Omega = (\boldsymbol{d}G\boldsymbol{d}F)\cdot\mathtt{J}. \tag{4.377}$$

Since a Poisson manifold can be odd-dimensional the hamiltonian vector fields in general do not span the tangent space of the Poisson manifold. This suggests to define the range $\mathrm{ran}(\mathtt{J}(\boldsymbol{x}))$ of $\mathtt{J}(\boldsymbol{x})$ as the span of all tangent vectors that can be expressed as $\boldsymbol{\alpha}^\natural = \mathtt{J}\cdot\boldsymbol{\alpha}$ for a one form $\boldsymbol{\alpha}\in T^*_{\boldsymbol{x}}M$. The range of $\mathtt{J}(\boldsymbol{x})$ is also the span of all hamiltonian vector fields at $\boldsymbol{x}$. The dimension of $\mathrm{ran}(\mathtt{J}(\boldsymbol{x}))$ is the rank of the Poisson manifold in $\boldsymbol{x}$ and equal to the rank of the matrix J^{ij}, which is an even number because of the anti-symmetry of J^{ij}. The even-dimensional vector space $\mathrm{ran}(\mathtt{J}(\boldsymbol{x}))$ is then the tangent space of a symplectic leaf in the point $\boldsymbol{x}$. The symplectic leaf is a submanifold of the Poisson manifold, which follows from the Frobenius theorem, that states that a system of vector fields on a manifold is integrable iff it is in involution and rank-invariant. Equation (4.347) shows that the hamiltonian vector fields are in involution and they are rank-invariant because they conserve the Poisson bivector and so especially also the rank, i.e. for all functions H one has

$$\mathscr{L}_{\boldsymbol{h}_H}\mathtt{J} = 0. \tag{4.378}$$

The Poisson manifold is then foliated by symplectic leafs. Only when the rank of a Poisson manifold M is everywhere equal to $\dim(M)$ the Poisson manifold itself is a symplectic manifold.

The formalism developed so far can now directly be generalized to multivectors, which leads to Poisson calculus (see [107] and the references therein). The r-vector that corresponds to an r-form is given by

$$\left(A^{(r)}\right)^\natural = \frac{1}{r!}J^{k_1i_1}\dots J^{k_ri_i}A_{i_1\dots i_r}\boldsymbol{\xi}_{k_1}\dots\boldsymbol{\xi}_{k_r} \tag{4.379}$$

and in analogy to (4.141) one has $i_{A^{(r)}}B_{(s)} = \overline{A^{(r)}}\cdot B_{(s)}$, so that

$$\overline{\boldsymbol{\alpha}_1\dots\boldsymbol{\alpha}_r}\cdot\left(A^{(r)}\right)^\natural = (-1)^r\overline{\boldsymbol{\alpha}_1^\natural\dots\boldsymbol{\alpha}_r^\natural}\cdot A^{(r)}. \tag{4.380}$$

It is then also possible to define a Poisson bracket for one forms by

$$\{\boldsymbol{\alpha},\boldsymbol{\beta}\}_{PB} = \mathscr{L}_{\boldsymbol{\alpha}^\natural}\boldsymbol{\beta} - \mathscr{L}_{\boldsymbol{\beta}^\natural}\boldsymbol{\alpha} + \boldsymbol{d}\big((\boldsymbol{\beta}\boldsymbol{\alpha})\cdot\mathtt{J}\big), \tag{4.381}$$

so that $\{\boldsymbol{\alpha},\boldsymbol{\beta}\}^\natural_{PB} = \left[\boldsymbol{\alpha}^\natural,\boldsymbol{\beta}^\natural\right]_{JLB}$. With this Poisson bracket one can further define an exterior differential $\tilde{\boldsymbol{d}}$ in analogy to (4.143) as

$$\begin{aligned}(\overline{\boldsymbol{\alpha}_1\boldsymbol{\alpha}_2\dots\boldsymbol{\alpha}_{r+1}})\cdot\tilde{\boldsymbol{d}}A_{(r)} \quad = \quad & \sum_{n=1}^{r+1}(-1)^{n+1}(\boldsymbol{\alpha}_n^\natural\cdot\boldsymbol{\partial})\left(\overline{\boldsymbol{\alpha}_1\dots\check{\boldsymbol{\alpha}}_n\dots\boldsymbol{\alpha}_{r+1}}\right)\cdot A_{(r)} \\ & + \sum_{m<n}(-1)^{m+n}\left(\overline{\{\boldsymbol{\alpha}_m,\boldsymbol{\alpha}_n\}_{PB}\boldsymbol{\alpha}_1\dots\check{\boldsymbol{\alpha}}_m\dots\check{\boldsymbol{\alpha}}_n\dots\boldsymbol{\alpha}_{r+1}}\right)\cdot A_{(r)},\end{aligned} \tag{4.382}$$

which can also be written as $\tilde{\boldsymbol{d}}A_{(r)} = \left[\mathtt{J}, A_{(r)}\right]_{SNB}$.

The easiest nonconstant Poisson tensor fulfilling (4.372) is a linear tensor

$$J^{ij}(\boldsymbol{x}) = C^{ij}_k x^k, \tag{4.383}$$

where the antisymmetry of J^{ij} and (4.372) ensure that the C^{ij}_k are structure constants of a Lie algebra. The corresponding Poisson bracket is the so called Lie-Poisson bracket

$$\{F, G\}_{LPB} = C^{ij}_k x^k \partial_i F \partial_j G. \tag{4.384}$$

The fundamental example is the Lie-Poisson structure on $\mathfrak{g}^*$. To this purpose one considers the bivector space spanned by the basis bivectors $\mathtt{B}_i$ with bivector algebra (4.244) and its reciprocal basis with two forms Θ^i, i.e. $\overline{\mathtt{B}_i} \cdot \Theta^j = \delta^j_i$. For scalar-valued functions F and G of general two forms $\Theta = \theta_i \Theta^i$ a Lie-Poisson bracket is given by

$$\{F, G\}_{LPB}(\Theta) = C^k_{ij}\theta_k \frac{\partial F}{\partial \theta_i}\frac{\partial G}{\partial \theta_j} = \overline{(\mathtt{d}F \times \mathtt{d}G)} \cdot \Theta, \tag{4.385}$$

where $\mathtt{d}$ is the exterior differential in the bivector basis: $\mathtt{d} = \mathtt{B}_i \frac{\partial}{\partial \theta_i}$. In the $SO(3)$-case, where $\Theta^i = \mathtt{B}_i$ the Lie-Poisson bracket can be written as

$$\{F, G\}_{LPB}(\mathtt{B}) = \overline{\mathtt{B}} \cdot \big((I_{(3)} *_C \boldsymbol{d})F \times (I_{(3)} *_C \boldsymbol{d})G\big) = \overline{\mathtt{B}} \cdot (\mathtt{d}F \times \mathtt{d}G)\,. \tag{4.386}$$

The symplectic leaves induced by the symplectic foliation with the Lie-Poisson bracket on $\mathfrak{g}^*$ are the orbits of the coadjoint action of the corresponding group G on $\mathfrak{g}^*$. This can be seen if one considers a scalar linear function $H(\Theta) = \overline{\mathtt{B}} \cdot \Theta = b^i\theta_i$ on $\mathfrak{g}^*$ with $\mathtt{d}H = \mathtt{B}$. For the Lie-Poisson bracket one has then for any scalar function F on $\mathfrak{g}^*$:

$$\{F, H\}_{LPB}(\Theta) = \overline{(\mathtt{d}F \times \mathtt{d}H)} \cdot \Theta = -\overline{(\mathtt{B} \times \mathtt{d}F)} \cdot \Theta = -\overline{(\mathrm{ad}_{\mathtt{B}}\mathtt{d}F)} \cdot \Theta = -\overline{\mathtt{d}F} \cdot \mathrm{ad}^*_{\mathtt{B}}\Theta. \tag{4.387}$$

On the other hand one can define in analogy to (4.376) the hamiltonian bivector field $\mathtt{h}_H$ of the Hamilton function $H(\Theta)$ as

$$\overline{\mathtt{h}_H(\Theta)} \cdot \mathtt{d}F = \{F, H\}_{LPB}(\Theta) = \overline{(\mathtt{d}F \times \mathtt{d}H)} \cdot \Theta = -\overline{\mathrm{ad}^*_{\mathtt{B}}\Theta} \cdot \mathtt{d}F, \tag{4.388}$$

so that $\mathtt{h}_H(\Theta) = -\mathrm{ad}^*_{\mathtt{B}}\Theta = -\mathrm{ad}^*_{\mathtt{d}H}\Theta$. This means that the hamiltonian bivector fields $\mathtt{h}_H$ that span the tangent space of the symplectic leaf are, up to a sign, the generators of the coadjoint action determined by $\mathtt{B}$. If Θ varies now over the coadjoint orbit one can define a skew-symmetric bilinear form on the orbit by

$$\Omega_\Theta(\mathrm{ad}^*_{\mathtt{A}}\Theta, \mathrm{ad}^*_{\mathtt{B}}\Theta) = \overline{\mathtt{A} \times \mathtt{B}} \cdot \Theta, \tag{4.389}$$

which defines on the coadjoint orbit a symplectic structure, that is the restriction of the Lie-Poisson bracket to the orbit [90]. Ω_Θ can be seen as a generalized antisymmetric tensor of the form (4.193) that maps two bivectors into a scalar.

The next step is to investigate the hamiltonian action of a rotor group on a Poisson vector manifold. The scalar functions $P_1, \ldots, P_r$ on the Poisson manifold M generate a hamiltonian action of a Lie group G on M if their Poisson brackets satisfy

$$\{P_i, P_j\}_{PB} = -C^k_{ij}P_k, \tag{4.390}$$

where the C^k_{ij} are the structure constants of the Lie algebra $\mathfrak{g}$ of G. The corresponding hamiltonian vector field $\boldsymbol{h}_{P_i}$ satisfy then with (4.347)

$$\left[\boldsymbol{h}_{P_i}, \boldsymbol{h}_{P_j}\right]_{JLB} = C^k_{ij}\boldsymbol{h}_{P_k} \tag{4.391}$$

and therefore generate a local action of G on M. The quotient manifold M/G inherits a Poisson structure from M. Functions $\tilde{F}, \tilde{H}$ on M/G correspond to G-invariant functions F, H on M, i.e. functions with $\boldsymbol{h}_{P_i} \cdot \boldsymbol{d}F = \{F, P_i\}_{PB} = 0$ for $i = 1, \ldots, r$. The Poisson bracket $\{\tilde{F}, \tilde{H}\}^{M/G}_{PB}$ on M/G corresponds then to the G-invariant function $\{F, H\}_{PB}$ on M. The so defined Poisson bracket on M/G fulfills clearly the defining relations for a Poisson bracket, so that it remains to show that the Poisson bracket $\{F, H\}_{PB}$ of two G-invariant functions on M is again a G-invariant function. But this is just a consequence of the Jacobi-identity:

$$\{\{F, H\}_{PB}, P_i\}_{PB} = \{\{F, P_i\}_{PB}, H\}_{PB} + \{F, \{H, P_i\}_{PB}\}_{PB} = 0. \tag{4.392}$$

If there is a hamiltonian system on M where each of the P_i is a first integral, i.e. $\{P_i, H\}_{PB} = 0$ for $i = 1, \ldots, r$, one says that G is a hamiltonian symmetry group. The Hamiltonian H is then a G-invariant function and there exists a reduced hamiltonian system on M/G with Hamiltonian $\tilde{H}$ whose solutions are the projections of the solutions of the system on M.

If the hamiltonian action is given by a rotor group the aim is to find the Hamilton function $P_{\mathtt{B}}$ of the vector field $\mathtt{B} \cdot \boldsymbol{x}$, that is induced according to (4.260) by the rotor left-action with bivector $\mathtt{B}$, i.e.

$$\boldsymbol{h}_{P_{\mathtt{B}}} = \mathtt{B} \cdot \boldsymbol{x}. \tag{4.393}$$

Since $\boldsymbol{h}_{P_{\mathtt{B}}} \cdot \boldsymbol{d}H = \{H, P_{\mathtt{B}}\}_{PB}$, it is possible to write the defining relation for $P_{\mathtt{B}}$ as

$$\{H, P_{\mathtt{B}}\}_{PB} = (\mathtt{B} \cdot \boldsymbol{x}) \cdot \boldsymbol{d}H, \tag{4.394}$$

for all scalar functions H. $P_{\mathtt{B}}$ is defined by (4.393) only up to a function G with $\boldsymbol{h}_G = 0$, so that $\boldsymbol{h}_{P_{\mathtt{B}}+G} = \boldsymbol{h}_{P_{\mathtt{B}}}$. Furthermore one has for two bivectors $\mathtt{A}$ and $\mathtt{B}$ with (4.347) and (4.261)

$$\boldsymbol{h}_{\{P_{\mathtt{A}}, P_{\mathtt{B}}\}_{PB}} = \boldsymbol{h}_{P_{\mathtt{A}\times\mathtt{B}}}. \tag{4.395}$$

While in the symplectic case a symplectic vector field is always locally hamiltonian, in the Poisson case an infinitesimal Poisson automorphism is in general not locally hamiltonian. This means that if the rotor left-action is canonical, i.e. $\mathscr{L}_{\mathtt{B}\cdot\boldsymbol{x}}\mathtt{J} = 0$, there does not exist in general a function $P_{\mathtt{B}}$, so that (4.393) is fulfilled. The additional condition that $\mathtt{B} \cdot \boldsymbol{x}$ is also hamiltonian can be expressed with the momentum map. A momentum map is here a two form $\Pi(\boldsymbol{x})$ with

$$i_{\mathtt{B}}\Pi = \overline{\mathtt{B}} \cdot \Pi = P_{\mathtt{B}}. \tag{4.396}$$

So if the hamiltonian vector field $\boldsymbol{h}_{P_{\mathtt{B}}}$ corresponding to the function $P_{\mathtt{B}} = \overline{\mathtt{B}} \cdot \Pi$ is the same as the vector field $\mathtt{B} \cdot \boldsymbol{x}$ induced by the rotor left-action, i.e. if one has $\boldsymbol{h}_{\overline{\mathtt{B}}\cdot\Pi} = (\mathtt{J} \cdot \boldsymbol{d}) \cdot (\overline{\mathtt{B}} \cdot \Pi) = \mathtt{B} \cdot \boldsymbol{x}$, then Π is a momentum map. If a momentum map of a rotor action exists and H is a Hamilton function that is invariant under the rotor action, then equation (4.394) reduces to $\{H, P_{\mathtt{B}}\}_{PB} = 0$ and the momentum map is a constant of the motion described by H. This follows because $\{H, P_{\mathtt{B}}\}_{PB} = 0$ means that $P_{\mathtt{B}}$ is constant along the hamiltonian flow of H, which must then also be true for the left hand side of (4.396), i.e. for Π, because $\mathtt{B}$ is constant. This is the Noether theorem in the Poisson case.

If on the other hand a hamiltonian action of a rotor group with r bivector generators on a Poisson vector manifold is given, there are scalar functions $P_{\mathtt{B}_1}, \ldots, P_{\mathtt{B}_r}$ on the Poisson manifold that generate the hamiltonian action. The momentum map is then

$$\Pi(\boldsymbol{x}) = P_{\mathtt{B}_i}(\boldsymbol{x})\Theta^i. \tag{4.397}$$

A momentum map $\Pi(\boldsymbol{x})$ that is determined by a hamiltonian group action is equivariant, i.e. it respects the rotor left-action on the vector manifold:

$$\Pi(R *_C \boldsymbol{x} *_C \overline{R}) = R *_C \Pi(\boldsymbol{x}) *_C \overline{R}, \tag{4.398}$$

which can also be written more precisely as

$$\overline{\mathrm{Ad}_R\mathtt{B}} \cdot \Pi(R *_C \boldsymbol{x} *_C \overline{R}) \equiv P_{\mathrm{Ad}_R\mathtt{B}}(R *_C \boldsymbol{x} *_C \overline{R}) = P_{\mathtt{B}}(\boldsymbol{x}) \equiv \overline{\mathtt{B}} \cdot \Pi(\boldsymbol{x}) \tag{4.399}$$

To see that the momentum map (4.397) is equivariant, it suffices to show the infinitesimal version of (4.398)

$$\begin{aligned}\big(\boldsymbol{h}_{P_{\mathtt{B}_j}}\cdot\boldsymbol{d}\big)P_{\mathtt{B}_i}\Theta^i &= \mathtt{B}_j\times\Pi && (4.400)\\ \{P_{\mathtt{B}_i},P_{\mathtt{B}_j}\}_{PB}\Theta^i &= P_{\mathtt{B}_i\mathtt{B}_j\times\Theta^i} && (4.401)\\ -C^k_{ij}P_{\mathtt{B}_k}\Theta^i &= P_{\mathtt{B}_i}C^i_{jk}\Theta^k. && (4.402)\end{aligned}$$

Infinitesimal equivariance [90] implies that

$$P_{\mathtt{A}\times\mathtt{B}} = \{P_{\mathtt{A}},P_{\mathtt{B}}\}_{PB}. \tag{4.403}$$

In this case momentum maps are Poisson maps, i.e. for scalar-valued functions F and G on $\mathfrak{g}^*$ one has

$$\{F,G\}_{LPB}(\Pi(\boldsymbol{x})) = \{F(\Pi(\boldsymbol{x})),G(\Pi(\boldsymbol{x}))\}_{PB}. \tag{4.404}$$

To prove this one shows that the left hand side of (4.404) can be written as

$$\{F,G\}_{LPB}(\Pi(\boldsymbol{x})) = \overline{\mathtt{d}F\times\mathtt{d}G}\cdot\Pi(\boldsymbol{x}) = P_{\mathtt{d}F\times\mathtt{d}G} = \{P_{\mathtt{d}F},P_{\mathtt{d}G}\}_{PB}, \tag{4.405}$$

where one uses in the last step infinitesimal equivariance. The right hand side of (4.404) gives the same result:

$$\{F(\Pi(\boldsymbol{x})),G(\Pi(\boldsymbol{x}))\}_{PB} = J^{ij}\partial_iF(\Pi(\boldsymbol{x}))\partial_jG(\Pi(\boldsymbol{x})) = J^{ij}\partial_iP_{\mathtt{d}F}\partial_jP_{\mathtt{d}G} = \{P_{\mathtt{d}F},P_{\mathtt{d}G}\}_{PB}, \tag{4.406}$$

using

$$\partial_iF(\Pi(\boldsymbol{x})) = \overline{\mathtt{d}F}\cdot\partial_i\Pi(\boldsymbol{x}) = \partial_i\left(\overline{\mathtt{d}F}\cdot\Pi(\boldsymbol{x})\right) = \partial_iP_{\mathtt{d}F}. \tag{4.407}$$

A special case for a momentum map is the momentum map of the cotangent lift of a rotor action on a vector manifold $\boldsymbol{q}=q^a(q^i)\boldsymbol{\sigma}_a$. In order to find this momentum map one first states that it is possible to find for a tangent vector field $\boldsymbol{a}(\boldsymbol{q})=a^i\xi^a_i\boldsymbol{\sigma}_a$ a function $P_{\boldsymbol{a}}(q^i,p_i)=P_{\boldsymbol{a}}(\boldsymbol{q}+\boldsymbol{\pi})$ on the cotangent bundle, which is given with the projection operator (4.353) as:

$$P_{\boldsymbol{a}}(q^i,p_i) = T\pi_{\boldsymbol{q}}(\boldsymbol{a})\cdot(\boldsymbol{q}+\boldsymbol{\pi}) = a^j\xi^a_j\boldsymbol{\tau}_a\cdot(q^b\boldsymbol{\sigma}_b+p_k\xi^k_b\boldsymbol{\tau}^b) = a^j(q^i)p_j. \tag{4.408}$$

The other possibility to obtain $P_{\boldsymbol{a}}(q^i,p_i)$ is to use the universal momentum map (4.359):

$$\mathcal{P}(\boldsymbol{a}) = a^j(q^i)p_j = P_{\boldsymbol{a}}(q^i,p_i). \tag{4.409}$$

These functions form an algebra on the cotangent bundle, i.e.

$$\{P_{\boldsymbol{a}},P_{\boldsymbol{b}}\}_{PB} = \frac{\partial P_{\boldsymbol{a}}}{\partial q^i}\frac{\partial P_{\boldsymbol{b}}}{\partial p_i} - \frac{\partial P_{\boldsymbol{b}}}{\partial q^i}\frac{\partial P_{\boldsymbol{a}}}{\partial p_i} = \left(\frac{\partial a^i}{\partial q^j}b^j - \frac{\partial b^i}{\partial q^j}a^j\right)p_i = -P_{[\boldsymbol{a},\boldsymbol{b}]_{JLB}}. \tag{4.410}$$

The rotor action of a rotor $R(t)=e_{*_C}^{\frac{t}{2}\mathtt{B}}$ on the vector manifold $\boldsymbol{q}$ induces a flow $\boldsymbol{q}(t)=R(t)*_C\boldsymbol{q}*_C\overline{R(t)}$ and a tangential vector field $\boldsymbol{b}=\mathtt{B}\cdot\boldsymbol{q}$. The inverse cotangent lift of this rotor action is

$$(\boldsymbol{q}+\boldsymbol{\pi})(t) = \overline{R_{\text{lifted}}(-t)}*_C(\boldsymbol{q}+\boldsymbol{\pi})*_C R_{\text{lifted}}(-t) = R_{\text{lifted}}(t)*_C(\boldsymbol{q}+\boldsymbol{\pi})*_C\overline{R_{\text{lifted}}(t)}, \tag{4.411}$$

which induces on the cotangent bundle a tangent vector field $\boldsymbol{b}_{\text{lifted}}=\mathtt{B}_{\text{lifted}}\cdot(\boldsymbol{q}+\boldsymbol{\pi})$, where $\mathtt{B}_{\text{lifted}}$ can be written as $\mathtt{B}_{\text{lifted}}=\mathtt{B}+T\pi_{\boldsymbol{q}}(\mathtt{B})$. The vector field $\boldsymbol{b}_{\text{lifted}}$ is then the hamiltonian vector field of $P_{\boldsymbol{b}}$, i.e. $\boldsymbol{b}_{\text{lifted}}=\boldsymbol{h}_{P_{\boldsymbol{b}}}$. This can be proved very easily if one considers that the cotangent lift of a rotor action leaves the canonical one-form invariant, i.e.

$$\mathscr{L}_{\boldsymbol{b}_{\text{lifted}}}\boldsymbol{\theta} = 0. \tag{4.412}$$

Cartan's magic formula (4.145) gives then

$$\boldsymbol{b}_{\text{lifted}} \cdot \Omega = -i_{\boldsymbol{b}_{\text{lifted}}} \boldsymbol{d\theta} = \boldsymbol{d} i_{\boldsymbol{b}_{\text{lifted}}} \boldsymbol{\theta} = \boldsymbol{d}(\boldsymbol{b}_{\text{lifted}} \cdot \boldsymbol{\theta}). \tag{4.413}$$

On the other hand one has with (4.351) and (4.353)

$$\boldsymbol{b}_{\text{lifted}} \cdot \boldsymbol{\theta}(\boldsymbol{q}+\boldsymbol{\pi}) = T\pi_{\boldsymbol{q}}(\boldsymbol{b}_{\text{lifted}}) \cdot \boldsymbol{\pi} = T\pi_{\boldsymbol{q}}(\boldsymbol{b}) \cdot \boldsymbol{\pi} = P_{\boldsymbol{b}}(\boldsymbol{q}+\boldsymbol{\pi}). \tag{4.414}$$

Putting this into (4.413) gives

$$\boldsymbol{b}_{\text{lifted}} \cdot \Omega = \boldsymbol{d} P_{\boldsymbol{b}}, \tag{4.415}$$

which shows that $\boldsymbol{b}_{\text{lifted}}$ is the hamiltonian vector field of $P_{\boldsymbol{b}}$, so that for a scalar function $F(\boldsymbol{q}+\boldsymbol{\pi}) = F(q^i, p_i)$

$$\begin{aligned} \boldsymbol{b}_{\text{lifted}} \cdot \boldsymbol{d}F(\boldsymbol{q}+\boldsymbol{\pi}) &= \left.\frac{\partial}{\partial t}\right|_{t=0} F(R_{\text{lifted}} *_C (\boldsymbol{q}+\boldsymbol{\pi}) *_C \overline{R_{\text{lifted}}}) && (4.416) \\ &= \frac{\partial F}{\partial q^i}(\mathtt{B} \cdot \boldsymbol{q})^i + \frac{\partial F}{\partial p_i}(T\pi_{\boldsymbol{q}}(\mathtt{B}) \cdot \boldsymbol{\pi})_i = \{F, P_{\boldsymbol{b}}\}_{PB} \ , && (4.417) \end{aligned}$$

with $\frac{\partial P_{\boldsymbol{b}}}{\partial p_i} = b^i = (\mathtt{B} \cdot \boldsymbol{q})^i$ and

$$\begin{aligned} \frac{\partial P_{\boldsymbol{b}}}{\partial q^i} &= \frac{\partial}{\partial q^i} T\pi_{\boldsymbol{q}}(\mathtt{B} \cdot \boldsymbol{q}) \cdot \boldsymbol{\pi} = \frac{\partial}{\partial q^i} T\pi_{\boldsymbol{q}} \left(\left.\frac{\partial}{\partial t}\right|_{t=0} R_{\text{lifted}} *_C \boldsymbol{q} *_C \overline{R_{\text{lifted}}} \right) \cdot \boldsymbol{\pi} && (4.418) \\ &= T\pi_{\boldsymbol{q}} \left(\frac{\partial \boldsymbol{q}}{\partial q^i} \right) \cdot \left.\frac{\partial}{\partial t}\right|_{t=0} \overline{R_{\text{lifted}}} *_C \boldsymbol{\pi} *_C R_{\text{lifted}} = -(T\pi_{\boldsymbol{q}}(\mathtt{B}) \cdot \boldsymbol{\pi})_i. && (4.419) \end{aligned}$$

The momentum map of the cotangent lift of a rotor action on the vector manifold $\boldsymbol{q}$ is then given for $\boldsymbol{b} = \mathtt{B} \cdot \boldsymbol{q}$ by

$$\overline{\mathtt{B}} \cdot \Pi(\boldsymbol{q}+\boldsymbol{\pi}) = T\pi_{\boldsymbol{q}}(\mathtt{B} \cdot \boldsymbol{q}) \cdot (\boldsymbol{q}+\boldsymbol{\pi}) = \mathcal{P}(\mathtt{B} \cdot \boldsymbol{q}) = P_{\boldsymbol{b}}(\boldsymbol{q}+\boldsymbol{\pi}). \tag{4.420}$$

Moreover this momentum map is also equivariant:

$$\begin{aligned} \overline{\mathtt{B}} \cdot \Pi(R_{\text{lifted}} *_C (\boldsymbol{q}+\boldsymbol{\pi}) *_C \overline{R_{\text{lifted}}}) &= T\pi_{\boldsymbol{q}}\big(\mathtt{B} \cdot (R *_C \boldsymbol{q} *_C \overline{R})\big) \cdot (R_{\text{lifted}} *_C (\boldsymbol{q}+\boldsymbol{\pi}) *_C \overline{R_{\text{lifted}}}) && (4.421) \\ &= T\pi_{\boldsymbol{q}}(\mathrm{Ad}_{\overline{R}}\mathtt{B} \cdot \boldsymbol{q}) \cdot (\boldsymbol{q}+\boldsymbol{\pi}) && (4.422) \\ &= \overline{\mathrm{Ad}_{\overline{R}}\mathtt{B}} \cdot \Pi(\boldsymbol{q}+\boldsymbol{\pi}) && (4.423) \\ &= \overline{\mathtt{B}} \cdot \mathrm{Ad}^*_{\overline{R}}\Pi(\boldsymbol{q}+\boldsymbol{\pi}), && (4.424) \end{aligned}$$

using in the second step that

$$\begin{aligned} \overline{R} *_C \big(\mathtt{B} \cdot (R *_C \boldsymbol{q} *_C \overline{R})\big) *_C R &= \overline{R} *_C \frac{1}{2}\big(\mathtt{B} *_C R *_C \boldsymbol{q} *_C \overline{R} - R *_C \boldsymbol{q} *_C \overline{R} *_C \mathtt{B}\big) *_C R && (4.425) \\ &= (\overline{R} *_C \mathtt{B} *_C R) \cdot \boldsymbol{q} = \mathrm{Ad}_{\overline{R}}\mathtt{B} \cdot \boldsymbol{q}. && (4.426) \end{aligned}$$

A simple example is the action of the rotation group on a three dimensional euclidian vector space with vectors $\boldsymbol{q} = q^i \boldsymbol{\eta}_i$ for $i = 1, 2, 3$. The tangent bundle is then a six dimensional euclidian vector space with vectors $\boldsymbol{q}+\boldsymbol{\pi} = q^i \boldsymbol{\eta}_i + p_i \boldsymbol{\rho}^i$ and a canonical symplectic structure $\Omega = \boldsymbol{\eta}^i \boldsymbol{\rho}_i$. A rotation on the $\boldsymbol{q}$-space is generated by the bivectors

$$\mathtt{B}_i = \frac{1}{2}\varepsilon_{ijk} \boldsymbol{\eta}_j \boldsymbol{\eta}_k. \tag{4.427}$$

For example a rotation around the $\boldsymbol{\eta}_3$-axis is generated by $\mathtt{B}_3 = \boldsymbol{\eta}_1\boldsymbol{\eta}_2$ and the corresponding vector field is $\boldsymbol{b}_3 = \mathtt{B}_3 \cdot \boldsymbol{q} = q^2\boldsymbol{\eta}_1 - q^1\boldsymbol{\eta}_2$. The lifted rotation is a rotation that acts in the $\boldsymbol{\rho}_i$-space just the same way as in the $\boldsymbol{\eta}_i$-space, the lifted generator is then $\mathtt{B}_3^{\text{lifted}} = \boldsymbol{\eta}_1\boldsymbol{\eta}_2 + \boldsymbol{\rho}^1\boldsymbol{\rho}^2$ and the corresponding lifted vector field is given by

$$\boldsymbol{b}_3^{\text{lifted}} = \mathtt{B}_3^{\text{lifted}} \cdot (\boldsymbol{q}+\boldsymbol{\pi}) = q^2\boldsymbol{\eta}_1 - q^1\boldsymbol{\eta}_2 + p_2\boldsymbol{\rho}^1 - p_1\boldsymbol{\rho}^2. \tag{4.428}$$

The Hamilton function $P_{\mathtt{B}_3}$ that generates this vector field fulfills $\boldsymbol{b}_3^{\text{lifted}} \cdot \Omega = \boldsymbol{d}P_{\mathtt{B}_3}$ or

$$p^2\boldsymbol{\eta}^1 - p^1\boldsymbol{\eta}^2 - q^2\boldsymbol{\rho}^1 + q^1\boldsymbol{\rho}^2 = \boldsymbol{\eta}^1\frac{\partial P_{\mathtt{B}_3}}{\partial q^1} + \boldsymbol{\eta}^2\frac{\partial P_{\mathtt{B}_3}}{\partial q^2} + \boldsymbol{\rho}_1\frac{\partial P_{\mathtt{B}_3}}{\partial p_1} + \boldsymbol{\rho}_2\frac{\partial P_{\mathtt{B}_3}}{\partial p_2}, \tag{4.429}$$

which is solved by the angular momentum function. The angular momentum functions $P_{\mathtt{B}_i} = \varepsilon_{ij}^k q^j p_k$ are the generators of the active rotations, that rotate the q^i as well as the p_i coefficients. They form the algebra $\{P_{\mathtt{B}_i}, P_{\mathtt{B}_j}\}_{PB} = \varepsilon_{ijk}P_{\mathtt{B}_k}$, so that there is a hamiltonian action of the rotations on the six dimensional symplectic space. The momentum map $\Pi(q^i, p_i) = P_{\mathtt{B}_j}(q^i, p_i)\Theta^j$ is just the angular momentum bivector $\mathtt{L} = \boldsymbol{qp}$ and connects the generators of the active and passive rotations.

Another simple example is the circle action of S^1 on S^2 [91]. The two-dimensional sphere $\boldsymbol{x}(\theta,\varphi) = \sin\theta\cos\varphi\boldsymbol{\sigma}_1 + \sin\theta\sin\varphi\boldsymbol{\sigma}_2 + \cos\theta\boldsymbol{\sigma}_3$ is a symplectic vector manifold with the symplectic two form

$$\Omega = x^1\boldsymbol{\sigma}^2\boldsymbol{\sigma}^3 + x^2\boldsymbol{\sigma}^3\boldsymbol{\sigma}^1 + x^3\boldsymbol{\sigma}^1\boldsymbol{\sigma}^2\big|_{S^2} = \sin\theta\boldsymbol{\xi}^\theta\boldsymbol{\xi}^\varphi, \tag{4.430}$$

which is the volume form on the S^2. A left rotation around the $\boldsymbol{\sigma}_3$-axis is generated by $\mathtt{B} = -\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ and induces on S^2 the vector field

$$\mathtt{B}\cdot\boldsymbol{x} = \sin\theta\cos\varphi\boldsymbol{\sigma}_2 - \sin\theta\sin\varphi\boldsymbol{\sigma}_1 = \partial_\varphi\boldsymbol{x} = \boldsymbol{\xi}_\varphi. \tag{4.431}$$

The Hamilton function $P_{\mathtt{B}}$ that generates this vector field fulfills according to (4.337) the equation $\boldsymbol{\xi}_\varphi \cdot \Omega = \boldsymbol{d}P_{\mathtt{B}}$, or

$$-\sin\theta\boldsymbol{\xi}^\theta = \boldsymbol{\xi}^\varphi\partial_\varphi P_{\mathtt{B}} + \boldsymbol{\xi}^\theta\partial_\theta P_{\mathtt{B}}, \tag{4.432}$$

which is solved by $P_{\mathtt{B}} = \cos\theta = x^3$.

As a third example one can consider a four dimensional symplectic vector space with vectors $\boldsymbol{x} = a^1\boldsymbol{\alpha}_1 + a^2\boldsymbol{\alpha}_2 + b^1\boldsymbol{\beta}_1 + b^2\boldsymbol{\beta}_2$ and symplectic two form $\Omega = \boldsymbol{\alpha}^1\boldsymbol{\beta}^1 + \boldsymbol{\alpha}^2\boldsymbol{\beta}^2$. The Lie group that acts symplectically on the four dimensional vector space is the $SU(2)$ with bivector generators $\mathtt{B}_i$ given in (4.281). The action of this group has an equivariant momentum map defined by

$$P_{\mathtt{B}} = \overline{\mathtt{B}}\cdot\Pi(\boldsymbol{x}) = \frac{1}{2}\big(\boldsymbol{x}(\mathtt{B}\cdot\boldsymbol{x})\big)\cdot\Omega. \tag{4.433}$$

That this is an momentum map follows with $\big(\boldsymbol{x}(\mathtt{B}\cdot\boldsymbol{y})\big)\cdot\Omega = \big(\boldsymbol{y}(\mathtt{B}\cdot\boldsymbol{x})\big)\cdot\Omega$ from

$$\boldsymbol{y}\cdot\boldsymbol{d}P_{\mathtt{B}}(\boldsymbol{x}) = \frac{1}{2}\big(\boldsymbol{y}(\mathtt{B}\cdot\boldsymbol{x})\big)\cdot\Omega + \frac{1}{2}\big(\boldsymbol{x}(\mathtt{B}\cdot\boldsymbol{y})\big)\cdot\Omega = \big(\boldsymbol{y}(\mathtt{B}\cdot\boldsymbol{x})\big)\cdot\Omega, \tag{4.434}$$

so that $\boldsymbol{d}P_{\mathtt{B}} = \boldsymbol{h}_{P_{\mathtt{B}}}\cdot\Omega$. The momentum map $\Pi(x) = P_{\mathtt{B}_i}(\boldsymbol{x})\Theta^i$ can be calculated with

$$\overline{B}\cdot\Pi(\boldsymbol{x}) = B^i P_{\mathtt{B}_i} = \frac{1}{2}\big(\boldsymbol{x}(\mathtt{B}\cdot\boldsymbol{x})\big)\cdot\Omega. \tag{4.435}$$

One obtains

$$P_{\mathtt{B}_1}(\boldsymbol{x}) = a^2b^1 - a^1b^2, \tag{4.436}$$

$$P_{\mathtt{B}_2}(\boldsymbol{x}) = a^1a^2 + b^1b^2, \tag{4.437}$$

$$P_{\mathtt{B}_3}(\boldsymbol{x}) = (a^1)^2 - (a^2)^2 + (b^1)^2 - (b^2)^2. \tag{4.438}$$

Restricting $\boldsymbol{x}$ now on an S^3, i.e. $(a^1)^2 + (a^2)^2 + (b^1)^2 + (b^2)^2 = 1$, one has $|\Pi(\boldsymbol{x})|_{S^3}| = 1/2$. This means that the momentum map maps an S^3 in the $\boldsymbol{x}$-space onto a two dimensional sphere with radius $1/2$ in the $\mathtt{B}_i$-space. $\Pi(\boldsymbol{x})|_{S^3}$ is the Hopf fibration.

Applying now the concepts discussed so far to the cotangent space of a group manifold T^*G, which is a vector manifold with vectors $\boldsymbol{r} + \boldsymbol{\vartheta}$, one arrives at the Lie-Poisson reduction [90]. As seen above the rotors act on the group vector manifold with a left translation ℓ_R which induces the tangential maps $T\ell_R$ and $T^*\ell_R$. A scalar function $F(\boldsymbol{r}+\boldsymbol{\vartheta}) = F(R, \dot{R})$ on T^*G is left invariant if $F \circ T^*\ell_R = F$. Such left invariant functions can be identified with reduced functions on $\mathfrak{g}$, i.e. $F(\boldsymbol{r}+\boldsymbol{\vartheta}) = F(R, \dot{R}) = F(1, \overline{R} *_C \dot{R}) = f(\Theta)$, where $\overline{R} *_C \dot{R}$ is an element of the bivector algebra that can also be expressed in the dual basis. This reduction can now be described with the momentum map $\Pi : T^*G \to \mathfrak{g}^*$, i.e. $F(\boldsymbol{r}+\boldsymbol{\vartheta}) = f(\Pi(\boldsymbol{r}+\boldsymbol{\vartheta}))$. One has then a Poisson map between the Poisson bracket of left invariant functions on T^*G and the Lie-Poisson bracket of reduced functions on $\mathfrak{g}^*$. In this way a left invariant Hamilton function on T^*G induces a Lie-Poisson dynamic on $\mathfrak{g}^*$. This will be explained for the example of the rigid body in the next section.

Chapter 5

Physical Applications for Superanalytic Geometric Algebra

One can now apply the formalism described in the last chapter to physics. It is clear that all the applications in the literature of geometric algebra can immediately be translated into the superanalytic formalism. In addition here the Lie-Poisson and the Euler-Poincaré reduction may be discussed for the rigid rotor. Moreover while the fermionic Clifford star product gives in classical mechanics the geometric structure, it appears then natural to combine the Clifford star product with the bosonic Moyal product in order to obtain a non-commutative version of geometric algebra that describes the quantum case. The consequences of deforming geometric algebra on space, space-time and phase space will be described in the following.

5.1 The Rigid Body

The rigid body is an example where the formalism described above can be shown to work very effectively. If one considers a free rigid body $\mathcal{B}$ in a three-dimensional ambient space spanned by the basis vectors $\boldsymbol{\sigma}_a$ and a body-fixed coordinate system $\boldsymbol{\xi}_i(t)$, a point of the body in the ambient space is given by

$$\boldsymbol{x}(t) = R(t) *_C \boldsymbol{x}_{\mathcal{B}} *_C \overline{R}(t), \tag{5.1}$$

where $\boldsymbol{x}_{\mathcal{B}}$ is the vector in the body-fixed system. The velocity is then given by

$$\begin{aligned} \dot{\boldsymbol{x}} &= \dot{R} *_C \boldsymbol{x}_{\mathcal{B}} *_C \overline{R} + R *_C \boldsymbol{x}_{\mathcal{B}} *_C \dot{\overline{R}} && (5.2)\\ &= R *_C (\overline{R} *_C \dot{R} *_C \boldsymbol{x}_{\mathcal{B}} - \boldsymbol{x}_{\mathcal{B}} *_C \overline{R} *_C \dot{R}) *_C \overline{R} && (5.3)\\ &= \dot{R} *_C \overline{R} *_C \boldsymbol{x} - \boldsymbol{x} *_C \dot{R} *_C \overline{R} && (5.4)\\ &= 2(\dot{R} *_C \overline{R}) \cdot \boldsymbol{x}, && (5.5)\end{aligned}$$

where one uses $\overline{R} *_C R = 1 \Rightarrow \dot{\overline{R}} *_C R + \overline{R} *_C \dot{R} = 0$. And for the body-fixed velocity one obtains

$$\dot{\boldsymbol{x}}_{\mathcal{B}} = \overline{R} *_C \dot{\boldsymbol{x}} *_C R = 2(\overline{R} *_C \dot{R}) \cdot \boldsymbol{x}_{\mathcal{B}}. \tag{5.6}$$

On the other hand one has $\dot{\boldsymbol{x}} = \boldsymbol{\omega} \times \boldsymbol{x}$, where $\boldsymbol{\omega}$ is the axial vector of angular velocity. Using that the vector cross product can be written as $\boldsymbol{a} \times \boldsymbol{b} = -(I_{(3)} *_C \boldsymbol{a}) \cdot \boldsymbol{b}$ this leads to

$$\dot{\boldsymbol{x}} = -(I_{(3)} *_C \boldsymbol{\omega}) \cdot \boldsymbol{x} = -\mathtt{W} \cdot \boldsymbol{x}, \tag{5.7}$$

where

$$\mathtt{W} = -2\dot{R} *_C \overline{R} = I_{(3)} *_C \boldsymbol{\omega} \tag{5.8}$$

is the angular velocity bivector that generates the rotation. Equation (5.8) can be rewritten to obtain the rotor equation

$$\dot{R} = -\frac{1}{2}\mathtt{W} *_C R, \tag{5.9}$$

which integrates for constant angular velocity to $R = e_{*C}^{-\frac{t}{2}\mathtt{W}}$. With the angular velocity bivector (5.8) one can also write (5.7) as

$$\dot{\boldsymbol{x}} = (R *_C \boldsymbol{x}_{\mathcal{B}} *_C \overline{R}) \cdot \mathtt{W} = R *_C (\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{W}_{\mathcal{B}}) *_C \overline{R}, \tag{5.10}$$

where $\mathtt{W}_{\mathcal{B}} = \overline{R} *_C \mathtt{W} *_C R = -2\overline{R} *_C \dot{R}$, so that the rotor equation becomes $\dot{R} = -\frac{1}{2} R *_C \mathtt{W}_{\mathcal{B}}$.

The angular momentum bivector is given by

$$\begin{aligned} \mathtt{L} &= \int d^3x\, \rho(\boldsymbol{x})\, \boldsymbol{x}\dot{\boldsymbol{x}} = \int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, (R *_C \boldsymbol{x}_{\mathcal{B}} *_C \overline{R})(R *_C (\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{W}_{\mathcal{B}}) *_C \overline{R}) && (5.11)\\ &= R *_C \left(\int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, \boldsymbol{x}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{W}_{\mathcal{B}}) \right) *_C \overline{R} = R *_C \mathtt{I}(\mathtt{W}_{\mathcal{B}}) *_C \overline{R}, && (5.12) \end{aligned}$$

where the bivector-valued function of a bivector

$$\mathtt{I}(\mathtt{B}) = \int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, \boldsymbol{x}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{B}), \tag{5.13}$$

corresponds to the inertial tensor. The equation of motion of the free rigid body can be obtained from

$$\begin{aligned} 0 = \dot{\mathtt{L}} &= \dot{R} *_C \mathtt{I}(\mathtt{W}_{\mathcal{B}}) *_C \overline{R} + R *_C \mathtt{I}(\mathtt{W}_{\mathcal{B}}) *_C \dot{\overline{R}} + R *_C \mathtt{I}(\dot{\mathtt{W}}_{\mathcal{B}}) *_C \overline{R} && (5.14)\\ &= R *_C \left(\mathtt{I}(\dot{\mathtt{W}}_{\mathcal{B}}) - \mathtt{W}_{\mathcal{B}} \times \mathtt{I}(\mathtt{W}_{\mathcal{B}}) \right) *_C \overline{R} && (5.15) \end{aligned}$$

as $\mathtt{I}(\dot{\mathtt{W}}_{\mathcal{B}}) - \mathtt{W}_{\mathcal{B}} \times \mathtt{I}(\mathtt{W}_{\mathcal{B}}) = 0$, which are for $\mathtt{W}_{\mathcal{B}} = I_{(3)} *_C \boldsymbol{\omega}_{\mathcal{B}} = \sum_{j=1}^3 \omega_{\mathcal{B}j} I_{(3)} *_C \boldsymbol{\xi}_j$ and $\mathtt{I}(\mathtt{B}) = \sum_{j=1}^3 I_j B_j I_{(3)} *_C \boldsymbol{\xi}_j$ the Euler equations

$$\begin{aligned} I_1\dot{\omega}_{\mathcal{B}1} - \omega_{\mathcal{B}2}\omega_{\mathcal{B}3}(I_2 - I_3) &= 0, && (5.16)\\ I_2\dot{\omega}_{\mathcal{B}2} - \omega_{\mathcal{B}3}\omega_{\mathcal{B}1}(I_3 - I_1) &= 0, && (5.17)\\ I_3\dot{\omega}_{\mathcal{B}3} - \omega_{\mathcal{B}1}\omega_{\mathcal{B}2}(I_1 - I_2) &= 0, && (5.18) \end{aligned}$$

where the I_j are the principal moments of inertia. Alternatively one can also calculate

$$\begin{aligned} 0 = \dot{\mathtt{L}} &= R *_C \dot{\mathtt{L}}_{\mathcal{B}} *_C \overline{R} + \mathtt{L} \times \mathtt{W} && (5.19)\\ &= R *_C \left(\dot{\mathtt{L}}_{\mathcal{B}} + \mathtt{L}_{\mathcal{B}} \times \mathtt{W}_{\mathcal{B}} \right) *_C \overline{R} && (5.20)\\ &= R *_C \left(\mathtt{I}(\dot{\mathtt{W}}_{\mathcal{B}}) - \mathtt{W}_{\mathcal{B}} \times \mathtt{I}(\mathtt{W}_{\mathcal{B}}) \right) *_C \overline{R}, && (5.21) \end{aligned}$$

where (5.19) should be compared with the corresponding vector equation $\dot{\boldsymbol{L}} = R *_C \dot{\boldsymbol{L}}_{\mathcal{B}} *_C \overline{R} + \boldsymbol{\omega} \times \boldsymbol{L}$, where $\mathtt{L} = I_{(3)} *_C \boldsymbol{L}$.

The aim is now to derive the equations of motion in the Lagrange or Hamilton formalism. The kinetic energy of the free rigid body can be written with (5.6) as

$$\begin{aligned} T &= \frac{1}{2} \int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, |2(\overline{R} *_C \dot{R}) \cdot \boldsymbol{x}_{\mathcal{B}}|^2 && (5.22)\\ &= \frac{1}{2} \int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, |\mathtt{W}_{\mathcal{B}} \cdot \boldsymbol{x}_{\mathcal{B}}|^2 && (5.23)\\ &= \frac{1}{2}\overline{\mathtt{W}_{\mathcal{B}}} \cdot \mathtt{I}(\mathtt{W}_{\mathcal{B}}) && (5.24)\\ &= \frac{1}{2}\overline{\mathtt{W}} \cdot \mathtt{L}. && (5.25) \end{aligned}$$

Equation (5.22) is the left invariant Lagrangian $L(R,\dot{R})$ and (5.24) the reduced Lagrangian $l(\mathtt{W}_{\mathcal{B}})$ of the free rigid body. This means that the dynamics is transferred by (5.1) from the vectors $\boldsymbol{x}(t)$ to the rotors or the generating bivectors, i.e. one considers the dynamics on the rotor group or the bivector algebra respectively.

The question is now how to vary the corresponding Lagrangians. In analogy to the matrix representation [90] one has

$$\begin{aligned}\delta\mathtt{W}_{\mathcal{B}} = \delta(-2\overline{R} *_C \dot{R}) &= 2\overline{R} *_C \delta R *_C \overline{R} *_C \dot{R} - 2\overline{R} *_C \delta\dot{R} & (5.26)\\ &= -\overline{R} *_C \delta R *_C \mathtt{W}_{\mathcal{B}} - 2\overline{R} *_C \delta\dot{R} & (5.27)\end{aligned}$$

and defining the bivector $\mathtt{B} = 2\overline{R} *_C \delta R$ so that

$$\dot{\mathtt{B}} = \mathtt{W}_{\mathcal{B}} *_C \frac{1}{2}\mathtt{B} + 2\overline{R} *_C \delta\dot{R}, \tag{5.28}$$

one obtains

$$\delta\mathtt{W}_{\mathcal{B}} = -\dot{\mathtt{B}} + \mathtt{W}_{\mathcal{B}} \times \mathtt{B}. \tag{5.29}$$

The variation

$$\begin{aligned}0 = \delta l(\mathtt{W}_{\mathcal{B}}) = \delta \int dt\, \frac{1}{2}\overline{\mathtt{W}_{\mathcal{B}}} \cdot \mathtt{I}(\mathtt{W}_{\mathcal{B}}) &= \int dt \int d^3x_{\mathcal{B}}\, \rho(\boldsymbol{x}_{\mathcal{B}})\, \overline{\delta\mathtt{W}_{\mathcal{B}}} \cdot [\boldsymbol{x}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{W}_{\mathcal{B}})] & (5.30)\\ &= \int dt\, \overline{\mathtt{I}(\mathtt{W}_{\mathcal{B}})} \cdot \left(-\dot{\mathtt{B}} + \mathtt{W}_{\mathcal{B}} \times \mathtt{B}\right) & (5.31)\\ &= \int dt\, \left[\mathtt{I}(\dot{\mathtt{W}}_{\mathcal{B}}) + \mathtt{I}(\mathtt{W}_{\mathcal{B}}) \times \mathtt{W}_{\mathcal{B}}\right] \cdot \overline{\mathtt{B}}, & (5.32)\end{aligned}$$

leads then again to the Euler equations, where one uses in (5.30)

$$\overline{\mathtt{W}_{\mathcal{B}}} \cdot [\boldsymbol{x}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B}} \cdot \delta\mathtt{W}_{\mathcal{B}})] = \overline{\delta\mathtt{W}_{\mathcal{B}}} \cdot [\boldsymbol{x}_{\mathcal{B}}(\boldsymbol{x}_{\mathcal{B}} \cdot \mathtt{W}_{\mathcal{B}})] \tag{5.33}$$

and in (5.31) equation (5.29).

The procedure described above is the Euler-Poincaré reduction in the rotor-case. Given is a left invariant rotor Lagrangian $L(R,\dot{R})$ and its reduction to the bivector algebra $l(\mathtt{W}_{\mathcal{B}})$. The variation of $L(R,\dot{R})$ corresponds to the variation of $l(\mathtt{W}_{\mathcal{B}})$ for variations $\delta\mathtt{W}_{\mathcal{B}} = -\dot{\mathtt{B}} + \mathtt{W}_{\mathcal{B}} \times \mathtt{B}$, where $\mathtt{B}$ is a bivector that vanishes at the endpoints. The Euler-Lagrange equation for the rotor corresponds to the bivector equation

$$\frac{d}{dt}\frac{\delta l}{\delta\mathtt{W}_{\mathcal{B}}} = \mathtt{W}_{\mathcal{B}} \times \frac{\delta l}{\delta\mathtt{W}_{\mathcal{B}}}. \tag{5.34}$$

The Euler-Poincaré reconstruction of the rotor from the bivector $\mathtt{W}_{\mathcal{B}}$ can then be done with the rotor equation and in a last step the dynamics $\boldsymbol{x}(t)$ is reobtained by (5.1). The Clifford calculus can also be used to treat the case of a spinning top very elegantly without Euler angles, this is described in [102].

In the Hamilton formalism the analogous construction is called Lie-Poisson reduction and can also be done in the rotor case. The Hamiltonian (5.24) of the free rigid body can be written as

$$H = \frac{1}{2}\left(\frac{L_{\mathcal{B}1}^2}{I_1} + \frac{L_{\mathcal{B}2}^2}{I_2} + \frac{L_{\mathcal{B}3}^2}{I_3}\right). \tag{5.35}$$

With the Lie-Poisson bracket (4.386)

$$\{F,G\}_{LPB}(\mathtt{L}_{\mathcal{B}}) = \overline{\mathtt{L}_{\mathcal{B}}} \cdot \left((I_{(3)} *_C \boldsymbol{\nabla}F) \times (I_{(3)} *_C \boldsymbol{\nabla}G)\right) = \overline{\mathtt{L}_{\mathcal{B}}} \cdot (\mathtt{d}F \times \mathtt{d}G) \tag{5.36}$$

the Euler equations are obtained by $\dot{L}_{\mathcal{B}i} = \{L_{\mathcal{B}i}, H\}_{LPB}$. They preserve the coadjoint orbit, i.e. the Casimir function $|\mathtt{L}_{\mathcal{B}}|^2$ is a constant of motion: $\{(L_{\mathcal{B}1}^2 + L_{\mathcal{B}2}^2 + L_{\mathcal{B}3}^2), H\}_{LPB} = 0$. The conserved quantity that results from the left-invariance is the angular momentum, which follows from the calculation in (5.14) and (5.15).

The procedure described above is the bivector version of the Poincaré equation [72]. In order to derive the Poincaré equation one considers a vector manifold $\boldsymbol{x}(q^i)$ with coordinate basis vectors $\boldsymbol{\xi}_i = \partial_i \boldsymbol{x}$ and non-coordinate basis vectors $\boldsymbol{\vartheta}_r = \vartheta^i_r \boldsymbol{\xi}_i$. For a scalar-valued function $f(q^i(t))$ on a trajectory $\boldsymbol{q}(t) = \boldsymbol{x}(q^i(t))$ one has

$$\frac{d}{dt} f = \frac{\partial f}{\partial q^i} \frac{dq^i}{dt} = \dot{q}^i \partial_i f, \tag{5.37}$$

or $\frac{d}{dt} = \dot{q}^i \partial_i$. In the non-coordinate basis the coefficients are $s^r = \vartheta^r_i \dot{q}^i$, so that $\frac{d}{dt} = s^r \partial_r$. The variation of the trajectory $\boldsymbol{q}(t) = \boldsymbol{q}(t, u = 0)$ is given by

$$\delta q^i = \left. \frac{d}{du} \right|_{u=0} q^i(t, u) = w^i, \tag{5.38}$$

where the coefficients in the non-coordinate basis are $w^r = \vartheta^r_i w^i$. So there is a vector $\boldsymbol{s} = \dot{\boldsymbol{q}}$ that describes the variation along the trajectory and a vector $\boldsymbol{w}$ that describes the orthogonal variation of the trajectory. It is now important that these variations commute, i.e. the operators

$$\frac{d}{dt} = \boldsymbol{s} \cdot \boldsymbol{\partial} = s^r \partial_r \qquad \text{and} \qquad \frac{d}{du} = \boldsymbol{w} \cdot \boldsymbol{\partial} = w^r \partial_r \tag{5.39}$$

must commute:

$$\begin{aligned} s^r \partial_r (w^s \partial_s) &= w^s \partial_s (s^r \partial_r) && (5.40) \\ s^r w^s (\partial_r \partial_s - \partial_s \partial_r) &= \left(\frac{d}{du} s^t - \frac{d}{dt} w^t \right) \partial_t && (5.41) \\ s^r w^s C^t_{rs} \partial_t &= \left(\frac{d}{du} s^t - \frac{d}{dt} w^t \right) \partial_t && (5.42) \\ \frac{d}{du} s^t &= \frac{d}{dt} w^t + C^t_{rs} s^r w^s, && (5.43) \end{aligned}$$

where one uses (4.159) in (5.42). For the vectors $\boldsymbol{s}$ and $\boldsymbol{w}$ one has then

$$\frac{d}{du} \boldsymbol{s} = \frac{d}{dt} \boldsymbol{w} + [\boldsymbol{s}, \boldsymbol{w}]_{JLB} . \tag{5.44}$$

This equation can now be used for varying the Lagrange function $L(q^i(t, u), s^r(t, u))$:

$$\begin{aligned} 0 = \delta S &= \int_a^b dt\, \delta L && (5.45) \\ &= \int_a^b dt \left[\frac{\partial L}{\partial q^i} \frac{\partial q^i}{\partial u} + \frac{\partial L}{\partial s^r} \left(\frac{d}{dt} w^r + C^r_{st} s^s w^t \right) \right]_{u=0} && (5.46) \\ &= \int_a^b dt \left[\left(\partial_r L + \frac{\partial L}{\partial s^s} s^t C^s_{tr} - \frac{d}{dt} \frac{\partial L}{\partial s^r} \right) w^r + \frac{d}{dt} \left(\frac{\partial L}{\partial s^r} w^r \right) \right]_{u=0}, && (5.47) \end{aligned}$$

from which the Poincaré equation follows

$$\frac{d}{dt} \frac{\partial L}{\partial s^r} - \frac{\partial L}{\partial s^s} s^t C^s_{tr} = \partial_r L. \tag{5.48}$$

If the configuration space is a rotor group the Lagrange function is $L = L(R, \dot{R})$ and one has to vary $R(t, u)$. Instead of vectors $\boldsymbol{s}$ and $\boldsymbol{w}$ the variations are described by bivectors

$$\mathtt{s} = 2\overline{R} *_C \dot{R} \qquad \text{and} \qquad \mathtt{w} = 2\overline{R} *_C \delta R, \tag{5.49}$$

so that the operators (5.39) are now expressed as $\frac{d}{dt} = \overline{\mathtt{s}} \cdot \mathtt{d}$ and $\frac{d}{du} = \overline{\mathtt{w}} \cdot \mathtt{d}$. It follows further that

$$\begin{aligned} \frac{d\mathtt{s}}{du} &= -2\overline{R} *_C \delta R *_C \overline{R} *_C \dot{R} + 2\overline{R} *_C \delta\dot{R} = -\frac{1}{2}\mathtt{w} *_C \mathtt{s} + 2\overline{R} *_C \delta\dot{R}, && (5.50)\\ \frac{d\mathtt{w}}{dt} &= -2\overline{R} *_C \dot{R} *_C \overline{R} *_C \delta R + 2\overline{R} *_C \delta\dot{R} = -\frac{1}{2}\mathtt{s} *_C \mathtt{w} + 2\overline{R} *_C \delta\dot{R}. && (5.51)\end{aligned}$$

Equating the expressions for $2\overline{R} *_C \delta\dot{R}$ gives the bivector analog of (5.44):

$$\frac{d}{du}\mathtt{s} = \frac{d}{dt}\mathtt{w} + \mathtt{s} \times \mathtt{w}. \tag{5.52}$$

The variation of the action is now given by

$$\begin{aligned} 0 = \delta S &= \int_a^b dt\, \delta L && (5.53)\\ &= \int_a^b dt \left[\overline{\mathtt{w}} \cdot \mathtt{d}L + \overline{\frac{\delta L}{\delta \mathtt{s}}} \cdot \left(\frac{d}{dt}\mathtt{w} + \mathtt{s} \times \mathtt{w}\right)\right]_{u=0} && (5.54)\\ &= \int_a^b dt \left[\overline{\mathtt{w}} \cdot \left(\mathtt{d}L - \frac{d}{dt}\frac{\delta L}{\delta \mathtt{s}} + \frac{\delta L}{\delta \mathtt{s}} \times \mathtt{s}\right) + \frac{d}{dt}\left(\overline{\mathtt{w}} \cdot \frac{\delta L}{\delta \mathtt{s}}\right)\right]_{u=0}, && (5.55)\end{aligned}$$

so that the bivector version of the Poincaré equation follows

$$\frac{d}{dt}\frac{\delta L}{\delta \mathtt{s}} - \frac{\delta L}{\delta \mathtt{s}} \times \mathtt{s} = \mathtt{d}L. \tag{5.56}$$

In the Hamilton formalism the hamiltonian is given by $H(q^i, p_i) = p_j \dot{q}^j(q^i, p_i) - L(q^i, s^r(q^i, \dot{q}^i(q^i, p^i)))$ and the canonical momentum is $p_i = \frac{\partial L}{\partial \dot{q}^i} = \vartheta^r_i \frac{\partial L}{\partial s^r}$. The Poincaré equation follows then as

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial s^r} = \left\{\frac{\partial L}{\partial s^r}, H\right\}_{PB} &= \{\vartheta^i_r p_i, H\}_{PB} && (5.57)\\ &= \frac{\partial \vartheta^i_r}{\partial q^j} p_i \left(\dot{q}^j + p_k \frac{\partial \dot{q}^k}{\partial p_j} - \frac{\partial L}{\partial s^t}\frac{\partial s^t}{\partial \dot{q}^l}\frac{\partial \dot{q}^l}{\partial p_j}\right) \\ &\quad -\vartheta^i_r \left(p_j \frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial L}{\partial q^i} - \frac{\partial L}{\partial s^t}\frac{\partial s^t}{\partial q^i} - \frac{\partial L}{\partial s^t}\frac{\partial s^t}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial q^i}\right) && (5.58)\\ &= \frac{\partial \vartheta^i_r}{\partial q^j}\vartheta^t_i \vartheta^j_s \frac{\partial L}{\partial s^t} s^s + \vartheta^i_r \frac{\partial L}{\partial s^t}\dot{q}^j \frac{\partial \vartheta^t_j}{\partial q^i} + \vartheta^i_r \frac{\partial L}{\partial q^i} && (5.59)\\ &= \vartheta^t_i \left(\frac{\partial \vartheta^i_r}{\partial q^j}\vartheta^j_s - \frac{\partial \vartheta^i_s}{\partial q^j}\vartheta^j_r\right) s^s \frac{\partial L}{\partial s^t} + \vartheta^i_r \frac{\partial L}{\partial q^i} && (5.60)\\ &= C^t_{rs} s^s \frac{\partial L}{\partial s^t} + \partial_r L. && (5.61)\end{aligned}$$

The Hamilton equations $\dot{z}^i = \{z^i, H\}_{PB}$ in the bivector case, i.e. for a Hamilton function $H(\mathtt{z})$ with a bivector $\mathtt{z} = z^i \mathtt{B}_i$ are obtained by using the Lie-Poisson bracket instead of the Poisson bracket. In the $\mathfrak{so}(3)$-case the Hamilton equations read then

$$\dot{\mathtt{z}} = \mathtt{z} \times \mathtt{d}H = -\mathrm{ad}^*_{\mathtt{d}H}\mathtt{z}. \tag{5.62}$$

5.2 Geometric Algebra and the Kepler Problem

As the second example consider the solution of the Kepler problem by spinors [76, 7]. One uses here the fact that the radial position vector $\boldsymbol{r} = r_1\boldsymbol{\sigma}_1 + r_2\boldsymbol{\sigma}_2 + r_3\boldsymbol{\sigma}_3$ can be written as a rotated and dilated basis vector:

$$\boldsymbol{r} = U *_C \boldsymbol{\sigma}_1 *_C \overline{U}. \tag{5.63}$$

The components r_i of $\boldsymbol{r}$ can then be expressed in terms of the components u_i of $U = u_1 + u_2\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + u_3\boldsymbol{\sigma}_3\boldsymbol{\sigma}_1 + u_4\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$:

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & -u_3 & -u_4 \\ -u_4 & u_3 & u_2 & -u_1 \\ u_3 & u_4 & u_1 & u_2 \\ -u_2 & u_1 & -u_4 & u_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \tag{5.64}$$

which is the well known Kustaanheimo-Stiefel transformation [86]. Comparing (5.63) and (5.64) leads to the notational correspondence

$$\boldsymbol{r} = U *_C \boldsymbol{\sigma}_1 *_C \overline{U} \qquad \leftrightarrow \qquad \vec{r} = L_{\vec{u}}\,\vec{u}, \tag{5.65}$$

where $\vec{r}$ and $\vec{u}$ are four dimensional space vectors considered as tupels of numbers as in the conventional formalism. One should note here that the KS-transformation increases the degrees of freedom by one, which means that the bivector U in (5.63) is not unique [76]. This gauge freedom can be reduced by imposing an additional constraint on U as will be shown below. Squaring (5.65) leads to the relations

$$U *_C \overline{U} = |U|^2 = r \qquad \leftrightarrow \qquad L_{\vec{u}} L_{\vec{u}}^T = \vec{u}^{\,2} = r, \tag{5.66}$$

with $r = |\boldsymbol{r}| = |\vec{r}\,| = r_1^2 + r_2^2 + r_3^2$. Differentiating (5.65) with respect to t one obtains the KS-transformation for the velocities as

$$\dot{\boldsymbol{r}} = \dot{U} *_C \boldsymbol{\sigma}_1 *_C \overline{U} + U *_C \boldsymbol{\sigma}_1 *_C \dot{\overline{U}} \qquad \leftrightarrow \qquad \dot{\vec{r}} = 2L_{\vec{u}}\,\dot{\vec{u}}. \tag{5.67}$$

One can then choose for the constraint

$$\dot{U} *_C \boldsymbol{\sigma}_1 *_C \overline{U} = U *_C \boldsymbol{\sigma}_1 *_C \dot{\overline{U}} \qquad \leftrightarrow \qquad \dot{r}_4 = 0, \tag{5.68}$$

which means that the surplus fourth component r_4 stays zero for all times. With this constraint it is possible to invert the geometric algebra relation (5.67) for U. Implementing (5.68) in (5.67) gives $\dot{\boldsymbol{r}} = 2\dot{U} *_C \boldsymbol{\sigma}_1 *_C \overline{U}$, which can be solved for $\dot{U}$, so that the inverse relation to (5.67) is

$$\dot{U} = \frac{1}{2r}\dot{\boldsymbol{r}} *_C U *_C \boldsymbol{\sigma}_1 \qquad \leftrightarrow \qquad \dot{\vec{u}} = \frac{1}{2r} L_{\vec{u}}^T \dot{\vec{r}}. \tag{5.69}$$

By introducing a fictitious time s which is defined as

$$\frac{d}{ds} = r\frac{d}{dt}, \qquad \frac{dt}{ds} = r \tag{5.70}$$

it is then possible to regularize the divergent $1/r$-potential so that (5.69) reads $\frac{dU}{ds} = \frac{1}{2}\dot{\boldsymbol{r}} *_C U *_C \boldsymbol{\sigma}_1$ or

$$\frac{d^2U}{ds^2} = \frac{1}{2}\left(r\,\ddot{\boldsymbol{r}} *_C U *_C \boldsymbol{\sigma}_1 + \dot{\boldsymbol{r}} *_C \frac{dU}{ds} *_C \boldsymbol{\sigma}_1 \right) = \frac{1}{2}\left(\ddot{\boldsymbol{r}} *_C \boldsymbol{r} + \frac{1}{2}\dot{\boldsymbol{r}}^{2*_C} \right) *_C U. \tag{5.71}$$

Substituting now the inverse square force

$$m\ddot{\boldsymbol{r}} = -k\frac{\boldsymbol{r}}{r^3} \tag{5.72}$$

one obtains:

$$\frac{d^2U}{ds^2} = \frac{1}{2m}\left(\frac{1}{2}m\dot{\boldsymbol{r}}^{2*_C} - \frac{k}{r} \right) *_C U = \frac{E}{2m}U, \tag{5.73}$$

which is the equation of motion for a harmonic oscillator. This equation can be solved straightforwardly and much easier than the equation for $\boldsymbol{r}$. The orbit can then be calculated by (5.63).

The Kepler problem can also be treated in the canonical formalism. For this purpose one first needs the KS-transformation for the momentum. If $\boldsymbol{w} = \sum_{n=1}^{4} w_n \boldsymbol{\sigma}_n$ is the canonical momentum corresponding to $\boldsymbol{u} = \sum_{n=1}^{4} u_n \boldsymbol{\sigma}_n$ the KS-transformation is given by

$$\boldsymbol{p} = \frac{1}{4r}\left(W *_C \boldsymbol{\sigma}_1 *_C \overline{U} + U *_C \boldsymbol{\sigma}_1 *_C \overline{W}\right) \qquad \leftrightarrow \qquad \vec{p} = \frac{1}{2r} L_{\vec{u}} \vec{w}, \tag{5.74}$$

with $W = w_1 + w_2 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 + w_3 \boldsymbol{\sigma}_3 \boldsymbol{\sigma}_1 + w_4 \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2$. For $\boldsymbol{p}^{2 *_C} = p_1^2 + p_2^2 + p_3^2$ one gets with (5.74)

$$\boldsymbol{p}^{2 *_C} = \frac{1}{4r} |W|^2 - p_4^2, \tag{5.75}$$

where $|W|^2 = W *_C \overline{W} = w_1^2 + w_2^2 + w_3^2 + w_4^2$ and

$$p_4 = \frac{1}{2r}\left(u_1 w_2 - u_2 w_1 + u_3 w_4 - u_4 w_3\right). \tag{5.76}$$

Equation (5.75) allows to transform the Hamiltonian into u_i- and w_i-coordinates. This is done in several steps [103]. Starting from the Hamiltonian $H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) - \frac{k}{r}$ one first extends the phase space by a q_0- and a p_0-coordinate and forms the homogeneous Hamiltonian as $H_1 = H + p_0$. This leads for the zero component to two additional Hamilton equations

$$\frac{dq_0}{dt} = \frac{\partial H_1}{\partial p_0} = 1 \qquad \text{and} \qquad \frac{dp_0}{dt} = -\frac{\partial H_1}{\partial q_0} = -\frac{\partial H_1}{\partial t} = -\frac{\partial p_0}{\partial t}, \tag{5.77}$$

which shows that q_0 corresponds to the time t and p_0 is a constant and corresponds to the negative energy of the system, so that $H_1 = H + p_0 = 0$ for a conservative force. Since the time is now a coordinate the development of the system has to be described with a different parameter. This development parameter is the fictitious time s that is connected to the time by (5.70). The relation (5.70) can be implemented if one chooses $H_2 = rH_1$. The Hamilton equations that describe then the development according to s are differential equations with respect to s:

$$\frac{dq_i}{ds} = \frac{\partial H_2}{\partial p_i} \qquad \text{and} \qquad \frac{dp_i}{ds} = -\frac{\partial H_2}{\partial q_i} \qquad \text{for } \; i = 0, 1, 2, 3. \tag{5.78}$$

Especially for the zero component one gets $\frac{dq_0}{ds} = \frac{dt}{ds} = \frac{\partial H_2}{\partial p_0} = r$ which corresponds to (5.70). After having so far regularized the Hamiltonian one can then go over to KS-coordinates and obtains with (5.75)

$$H_3 = \frac{1}{8m}\left(w_1^2 + w_2^2 + w_3^2 + w_4^2\right) - \frac{1}{2m} r p_4^2 - k - Er. \tag{5.79}$$

Imposing now the constraint $p_4 = 0$, which for $w_i = m\dot{u}_i$ is just (5.68), and considering bound states with $E < 0$ the Hamiltonian is given by

$$H_4 = \frac{1}{8m}\left(w_1^2 + w_2^2 + w_3^2 + w_4^2\right) + |E|\left(u_1^2 + u_2^2 + u_3^2 + u_4^2\right) - k, \tag{5.80}$$

which describes a four dimensional harmonic oscillator with fixed energy and frequency $\omega = (|E|/2m)^{1/2}$.

The above discussed transformation of the Kepler problem can now be used to calculate the energy levels of the hydrogen atom. To this purpose one introduces holomorphic coordinates

$$a_n = \frac{1}{\sqrt{2}}\left(\sqrt{4m\omega}\, u_n + \mathrm{i} \frac{1}{\sqrt{4m\omega}} w_n\right) \tag{5.81}$$

so that the Hamiltonian H_4 in (5.80) can be written as:

$$H_4 = \omega \left(\sum_{n=1}^{4} a_n \bar{a}_n \right) - e^2, \tag{5.82}$$

where $k = e^2$. Introducing then holomorphic coordinates for left and right moving quanta

$$a_{R_{12}} = \frac{1}{\sqrt{2}} \left(a_1 - \mathrm{i} a_2 \right), \quad a_{L_{12}} = \frac{1}{\sqrt{2}} \left(a_1 + \mathrm{i} a_2 \right) \quad \text{and} \quad a_{R_{34}} = \frac{1}{\sqrt{2}} \left(a_3 - \mathrm{i} a_4 \right), \quad a_{L_{34}} = \frac{1}{\sqrt{2}} \left(a_3 + \mathrm{i} a_4 \right) \tag{5.83}$$

the Hamiltonian (5.82) turns into

$$H_4 = \omega \left(a_{R_{12}} \bar{a}_{R_{12}} + a_{L_{12}} \bar{a}_{L_{12}} + a_{R_{34}} \bar{a}_{R_{34}} + a_{L_{34}} \bar{a}_{L_{34}} \right) - e^2. \tag{5.84}$$

One can now quantize this system with the Moyal product. The Moyal star product transforms under the KS-transformation and the above transformations into

$$*_M = \exp \left[\sum_{n=1}^{4} \frac{\mathrm{i}\hbar}{2} \left(\overleftarrow{\partial}_{u_n} \vec{\partial}_{w_n} - \overleftarrow{\partial}_{w_n} \vec{\partial}_{u_n} \right) \right] = \exp \left[\frac{\hbar}{2} \sum_{X = R_{12}, L_{12}, R_{34}, L_{34}} \left(\overleftarrow{\partial}_{a_X} \vec{\partial}_{\bar{a}_X} - \overleftarrow{\partial}_{\bar{a}_X} \vec{\partial}_{a_X} \right) \right]. \tag{5.85}$$

The energy levels can then be obtained by the $*$-eigenvalue equation

$$H_4 *_M \pi^{(M)}_{n_1 n_2 n_3 n_4} = 0, \tag{5.86}$$

where $\pi^{(M)}_{n_1 n_2 n_3 n_4}$ is the product of four Wigner functions of the one dimensional harmonic oscillator given in (1.118). Eq. (5.86) gives then

$$e^2 = \hbar\omega \left(n_{R_{12}} + n_{L_{12}} + n_{R_{34}} + n_{L_{34}} + 2 \right). \tag{5.87}$$

To get the energy levels of the hydrogen atom one has to impose the constraint

$$p_4 = a_{R_{12}} \bar{a}_{R_{12}} - a_{L_{12}} \bar{a}_{L_{12}} + a_{R_{34}} \bar{a}_{R_{34}} - a_{L_{34}} \bar{a}_{L_{34}} = 0, \tag{5.88}$$

which for the energy levels corresponds to $n_{R_{12}} - n_{L_{12}} + n_{R_{34}} - n_{L_{34}} = 0$ or $n_{R_{12}} + n_{R_{34}} = n_{L_{12}} + n_{L_{34}} \equiv n - 1$. Putting this and $\omega = \sqrt{|E|/2m}$ into (5.87) one gets the well known energy levels of the hydrogen atom

$$E_n = -\frac{e^4 m}{2\hbar} \frac{1}{n^2}. \tag{5.89}$$

5.3 Active and Passive Rotations on Space and the Theoretical Prediction of Spin

In the classical case the fermionic Clifford star product gives the geometric structure, while the coefficients are bosonic, commuting scalars. It is then possible to go over to the noncommuting case by demanding that the coefficients have to be multiplied by a bosonic star product. The Moyal and the Clifford star product are then combined to the Moyal-Clifford star product that acts on functions on the phase space that are vector valued on the physical space, it has the form

$$F *_{MC} G = F \exp \left[\sum_{n=1}^{3} \left(\frac{\mathrm{i}\hbar}{2} \left(\overleftarrow{\partial}_{q_n} \vec{\partial}_{p_n} - \overleftarrow{\partial}_{p_n} \vec{\partial}_{q_n} \right) + \overleftarrow{\partial}_{\boldsymbol{\sigma}_n} \vec{\partial}_{\boldsymbol{\sigma}_n} \right) \right] G. \tag{5.90}$$

In section 4.6 it was demonstrated that an arbitrary transformation on a vector space can be represented in a star product formalism by doubling the dimensions, while passive rotations could be described intrinsically without such a doubling. Since here only rotations will be considered the doubled fermionic coodinates do not for reasons of simplicity appear in the above star product. The combination of bosonic and fermionic star products as above was also considered in a different context in [40, 54].

To see the consequences of a Moyal deformation in geometric algebra one can consider as a simple example the Moyal-Clifford product of two vectors in $d = 2$ dimensions. The deformed generalization of (4.15) can be written as

$$\boldsymbol{a} *_{MC} \boldsymbol{b} = (a_1 *_M b_2 - a_2 *_M b_1)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 + a_1 *_M b_1 + a_2 *_M b_2. \tag{5.91}$$

Under the Moyal product the coefficients in general do not commute if they are functions of q_n and p_n. This means that the Moyal-Clifford product of the same vectors $\boldsymbol{a} *_{MC} \boldsymbol{a}$ is in general not a scalar, but has also a bivector part. It is this additional bivector part, which appears only for $\hbar \neq 0$, that constitutes the spin as a physical observable. This can be seen if one considers the minimal substituted Hamiltonian which is in the formalism of deformed geometric algebra given by:

$$H = \frac{1}{2m}\Big[(p_1 + eA_1)\,\boldsymbol{\sigma}_1 + (p_2 + eA_2)\,\boldsymbol{\sigma}_2 + (p_3 + eA_3)\,\boldsymbol{\sigma}_3\Big]^{2*_{MC}} \tag{5.92}$$

$$\begin{aligned} &= \frac{1}{2m}\Big[(p_1 + eA_1)^{2*_M} + (p_2 + eA_2)^{2*_M} + (p_3 + eA_3)^{2*_M}\Big] \\ &\quad + \frac{1}{2m}\Big[(p_1 + eA_1)\,,(p_2 + eA_2)\Big]_{*_M}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 + \frac{1}{2m}\Big[(p_1 + eA_1)\,,(p_3 + eA_3)\Big]_{*_M}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 \\ &\quad + \frac{1}{2m}\Big[(p_2 + eA_2)\,,(p_3 + eA_3)\Big]_{*_M}\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3. \end{aligned} \tag{5.93}$$

The first three terms $H_0 = \frac{1}{2m}\sum_{n=1}^{3}(p_n + eA_n)^{2*_M}$ describe the Landau problem of a charged particle in a magnetic field which can be solved in the star product formalism as described in section 2.5 or [27]. The other three terms that describe the interaction of the spin and the magnetic field only appear because by introducing the bosonic Moyal product the phase space variables no longer commute. If the magnetic field points in $\boldsymbol{\sigma}_3$-direction the vector potential is given by $\boldsymbol{A} = -\frac{B_3}{2}q_2\boldsymbol{\sigma}_1 + \frac{B_3}{2}q_1\boldsymbol{\sigma}_2$ and only the first Moyal-commutator in (5.93) contributes:

$$\mathtt{H}_{\text{Spin}} = \frac{1}{2m}\Big[(p_1 + eA_1)\,,(p_2 + eA_2)\Big]_{*_M}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 = \frac{\hbar\omega}{2}\sigma^3 \tag{5.94}$$

where $\omega = \frac{eB_3}{m}$ and $\sigma^3 = -\mathrm{i}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2$ is a real quaternion, which is constructed according to (4.13) and (2.96). The difference between this calculation and the conventional approach is, that in the conventional formalism the Clifford structure is introduced in an ad hoc manner by inserting Pauli matrices by hand in (5.92). The Pauli matrices describe then the spin and lead analogously to the additional term in the Hamiltonian. This approach, which is also known as the Feynman trick, is actually wrong, because the Pauli matrices are tuple representations of the basis bivectors. But $(\boldsymbol{p} + e\boldsymbol{A})$ is neither a bivector nor an axial vector, it is a vector and so one has to insert the basis vectors and not the basis bivectors. That the Feynman trick leads nevertheless to a sensible result is due to the fact that there are in three dimensions as many basis vectors as basis bivectors and that they fulfill a similar algebra. In contrast to the tuple formalism in geometric algebra the Clifford structures do not have to be added by hand, they are just the basis vectors that already exist in classical mechanics, but become apparent as physical objects in the quantum case.

The $*$-eigenfunctions of $\mathtt{H}_{\text{Spin}}$ are $\pi^{(C)}_{\pm 1/2} = \frac{1}{2}(1 \mp \mathrm{i}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2)$, i.e.

$$\mathtt{H}_{\text{Spin}} *_C \pi^{(C)}_{\pm 1/2} = \pm\frac{\hbar\omega}{2}\pi^{(C)}_{\pm 1/2} \tag{5.95}$$

so that $\mathtt{H}_{\text{Spin}}$ can be decomposed as

$$\mathtt{H}_{\text{Spin}} = \frac{\hbar\omega}{2}\left(\pi^{(C)}_{+1/2} - \pi^{(C)}_{-1/2}\right). \tag{5.96}$$

The $\pi^{(C)}_{\pm 1/2}$ are the spin Wigner functions and as such they are projectors:

$$\pi^{(C)}_{\pm 1/2} *_C \pi^{(C)}_{\pm 1/2} = \pi^{(C)}_{\pm 1/2} \quad \text{and} \quad \pi^{(C)}_{+1/2} *_C \pi^{(C)}_{-1/2} = \pi^{(C)}_{-1/2} *_C \pi^{(C)}_{+1/2} = 0. \tag{5.97}$$

It is then clear that the $*$-eigenfunctions of the whole Hamiltonian (5.93) are products of the Moyal eigenfunctions of H_0 and the spin Wigner functions $\pi^{(C)}_{\pm}$

$$H *_{MC} \pi_n^{(M)} \pi^{(C)}_{\pm 1/2} = \left(H_0 + \frac{\hbar\omega}{2}\big(\pi^{(C)}_{+1/2} - \pi^{(C)}_{-1/2}\big) \right) *_{MC} \pi_n^{(M)} \pi^{(C)}_{\pm 1/2} = \left(E_n \pm \frac{\hbar\omega}{2} \right) \pi_n^{(M)} \pi^{(C)}_{\pm 1/2}. \tag{5.98}$$

It is evident that the time development and the expectation values of the spin are calculated just as in section 2.5. While there the construction of the spin term was based on pseudoclassical mechanics, one can see here that the spin does not need to be constructed and added a posteriori, but rather appears naturally by deforming geometric algebra.

The Moyal deformation of geometric algebra gives rise to multivector valued extra terms. Such a multivector is invariant under a combined transformation of the bosonic coefficients and a compensating transformation of the fermionic basis vectors. The bosonic transformation of the coefficients is an active transformations and the fermionic transformation of the basis vectors is a passive transformation. In a tuple formalism this difference can not be made and so active and passive transformations are mixed up with left and right transformation, whereas in a multivector formalism one rather has that an active right transformation corresponds to a passive left transformation and the other way round. To illustrate the concept of active and passive transformations in the star product formalism one can consider rotations in space. In the three dimensional euclidian space with vectors $\boldsymbol{x} = x^i \boldsymbol{\sigma}_i$ the active rotations [3] are generated by the angular momentum functions

$$L^i = \varepsilon^{ijk} x^j p^k, \tag{5.99}$$

which fulfill with the three dimensional Moyal product the active algebra

$$\left[L^i, L^j\right]_{*_M} = \mathrm{i}\hbar\varepsilon^{ijk} L^k. \tag{5.100}$$

An active left-rotation has then the form

$$\boldsymbol{x}' = \overline{U} *_M \boldsymbol{x} *_M U = e_{*_M}^{-\frac{\mathrm{i}}{\hbar}\alpha_k L^k} *_M \boldsymbol{x} *_M e_{*_M}^{\frac{\mathrm{i}}{\hbar}\alpha_k L^k} = \big(R^i_j x^j\big)\boldsymbol{\sigma}_i, \tag{5.101}$$

where the R^i_j is the well known rotation matrix. The corresponding passive rotation [4, 36] is generated by the bivectors

$$\mathtt{B}_i = \frac{1}{2}\varepsilon_{ijk}\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k \tag{5.102}$$

that fulfill as seen above the passive algebra

$$\mathtt{B}_i \times \mathtt{B}_j = -\varepsilon_{ijk}\mathtt{B}_k, \tag{5.103}$$

so that the passive left-rotation is given by

$$\boldsymbol{x}' = \overline{R} *_C \boldsymbol{x} *_C R = e_{*_C}^{-\frac{1}{2}\alpha^k \mathtt{B}_k} *_C \boldsymbol{x} *_C e_{*_C}^{\frac{1}{2}\alpha^k \mathtt{B}_k} = x^i \big(R^j_i \boldsymbol{\sigma}_j\big). \tag{5.104}$$

It is clear that the above transformations generalize to arbitrary multivectors $A(x^i)$ and that such a multivector is invariant under a composed active and a compensating passive transformation [82]. The generator of such a composed transformation is then the sum of the active and passive generators, so that one has infinitesimally

$$\left[L^i + \frac{1}{2}\mathtt{B}_i, A(x^n)\right]_{*_{MC}} = \left[L^i, A(x^n)\right]_{*_M} + \mathtt{B}_i \times A(x^n) = \left[\varepsilon^{ijk} x^j \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x^k} + \mathtt{B}_i \times\right] A(x^n). \tag{5.105}$$

In the conventional formalism one states that in quantum mechanics one has to go over from the angular momentum operator $\hat{L}_i$ to the operator $\hat{J}_i$ that includes also a Pauli matrix. In geometric algebra this follows from the invariance behavior of multivectors.

In order to establish the relation of the Wigner functions obtained above and the spinors described in section 4.7 one has to notice that the deformation of the three dimensional geometric algebra with the Moyal star product means that one works on the complex even subalgebra $\mathcal{C}\ell^+_{0,3}(\mathbb{C})$. But this subalgebra is isomorph to the full real Clifford algebra $\mathcal{C}\ell_{0,3}(\mathbb{R})$, i.e. $\mathcal{C}\ell^+_{0,3}(\mathbb{C}) \simeq \mathcal{C}\ell_{0,3}(\mathbb{R})$, with the substitutions

$$1 \leftrightarrow 1, \quad \mathrm{i} \leftrightarrow I_{(3)}, \quad \boldsymbol{\sigma}_i\boldsymbol{\sigma}_j \leftrightarrow \boldsymbol{\sigma}_i\boldsymbol{\sigma}_j \quad \text{and} \quad -\mathrm{i}\varepsilon_{ijk}\boldsymbol{\sigma}_j\boldsymbol{\sigma}_k \leftrightarrow \boldsymbol{\sigma}_i. \tag{5.106}$$

For the spin hamiltonian and the Wigner functions this means:

$$\frac{\hbar\omega}{2\mathrm{i}}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 \leftrightarrow \frac{\hbar\omega}{2}\boldsymbol{\sigma}_3 \quad \text{and} \quad \frac{1}{2}(1 \mp \mathrm{i}\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2) \leftrightarrow \frac{1}{2}(1 \pm \boldsymbol{\sigma}_3) \tag{5.107}$$

So for $\pi^{(C)}_{\pm 1/2} = \frac{1}{2}(1 \pm \boldsymbol{\sigma}_3)$ and $\psi_+ = 1$, $\psi_- = \boldsymbol{\sigma}_1\boldsymbol{\sigma}_3$ one has

$$\pi^{(C)}_{\pm 1/2} = \psi_\pm *_C \pi^{(C)}_{+1/2} *_C \overline{\psi_\pm}, \tag{5.108}$$

i.e. the spinors relate the two Wigner functions. Furthermore it is possible to relate the $*$-eigenvalue equations:

$$\begin{aligned}
& \boldsymbol{\sigma}_3 *_C \psi_\pm *_C \boldsymbol{\sigma}_3 = \pm\psi_\pm & (5.109)\\
\Leftrightarrow \quad & \boldsymbol{\sigma}_3 *_C \psi_\pm *_C \boldsymbol{\sigma}_3 *_C \frac{1}{2}(1+\boldsymbol{\sigma}_3) *_C \overline{\psi_\pm} = \pm\psi_\pm *_C \frac{1}{2}(1+\boldsymbol{\sigma}_3) *_C \overline{\psi_\pm} & (5.110)\\
\Leftrightarrow \quad & \boldsymbol{\sigma}_3 *_C \pi^{(C)}_{\pm 1/2} = \pm\pi^{(C)}_{\pm} & (5.111)
\end{aligned}$$

and

$$\begin{aligned}
& \boldsymbol{\sigma}_3 *_C \pi^{(C)}_{\pm 1/2} = \pm\pi^{(C)}_{\pm 1/2} & (5.112)\\
\Leftrightarrow \quad & \boldsymbol{\sigma}_3 *_C \psi_\pm *_C \pi^{(C)}_{+1/2} *_C \overline{\psi_\pm} = \pm\psi_\pm *_C \pi^{(C)}_{+1/2} *_C \overline{\psi_\pm} & (5.113)\\
\Rightarrow \quad & (\pm\psi_\pm - \boldsymbol{\sigma}_3 *_C \psi_\pm *_C \boldsymbol{\sigma}_3) *_C (1+\boldsymbol{\sigma}_3) = 0 & (5.114)\\
\Rightarrow \quad & \boldsymbol{\sigma}_3 *_C \psi_\pm *_C \boldsymbol{\sigma}_3 = \pm\psi_\pm. & (5.115)
\end{aligned}$$

5.4 Space-Time Algebra and Relativistic Quantum Mechanics

The formalism of space-time algebra can also be used to describe relativistic kinematics. If a particle is moving in the $\boldsymbol{\gamma}_0$-system along $\boldsymbol{x}(\tau)$, where τ is the proper time, the proper velocity is given by $\boldsymbol{u}(\tau) = \frac{d}{d\tau}\boldsymbol{x}(\tau)$, with $\boldsymbol{u}^{2*_C} = 1$. For the space-time split of the proper velocity one obtains:

$$\underline{\boldsymbol{u}} = \boldsymbol{u} *_C \boldsymbol{\gamma}_0 = \boldsymbol{u}\cdot\boldsymbol{\gamma}_0 + \boldsymbol{u}\boldsymbol{\gamma}_0 = u_0 + \mathtt{u} = \frac{d}{d\tau}\left(\boldsymbol{x}(\tau) *_C \boldsymbol{\gamma}_0\right) = \frac{d}{d\tau}\left(t + \mathtt{x}\right) = \frac{dt}{d\tau} + \frac{d\mathtt{x}}{dt}\frac{dt}{d\tau}. \tag{5.116}$$

Comparing the scalar and the bivector part leads to

$$u_0 = \boldsymbol{u}\cdot\gamma_0 = \frac{dt}{d\tau} \qquad \text{and} \qquad \mathtt{u} = \frac{d\mathtt{x}}{dt} = \frac{d\mathtt{x}}{d\tau}\frac{d\tau}{dt} = \frac{\boldsymbol{u}\gamma_0}{\boldsymbol{u}\cdot\gamma_0} \tag{5.117}$$

and with $1 = \boldsymbol{u}^{2*_C} = u_0^2(1 - \mathtt{u}^{2*_C})$ one has

$$u_0 = \boldsymbol{u}\cdot\boldsymbol{\gamma}_0 = \frac{1}{\sqrt{1-\mathtt{u}^{2*_C}}} = \gamma. \tag{5.118}$$

It is now also possible to specify a Lorentz transformation from a coordinate system $\boldsymbol{\gamma}_\mu$ to an in $\boldsymbol{\gamma}_1$-direction moving coordinate system $\boldsymbol{\gamma}'_\mu$. For the coefficients this transformation is given by $t = \gamma(t' + \beta x'^1)$, $x^1 = \gamma(x'^1 + \beta t')$, $x^2 = x'^2$, and $x^3 = x'^3$. The condition $\boldsymbol{x} = x^\mu \boldsymbol{\gamma}_\mu = x'^\mu \boldsymbol{\gamma}'_\mu$ leads then to

$$\boldsymbol{\gamma}'_0 = \gamma(\boldsymbol{\gamma}_0 + \beta\boldsymbol{\gamma}_1) \qquad \text{and} \qquad \boldsymbol{\gamma}'_1 = \gamma(\boldsymbol{\gamma}_1 + \beta\boldsymbol{\gamma}_0). \tag{5.119}$$

Introducing the angle α so that $\beta = \tanh(\alpha)$ this can be written as

$$\begin{aligned} \boldsymbol{\gamma}'_0 &= \cosh(\alpha)\boldsymbol{\gamma}_0 + \sinh(\alpha)\boldsymbol{\gamma}_1 = e_{*C}^{\alpha\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0} *_C \boldsymbol{\gamma}_0, &(5.120)\\ \boldsymbol{\gamma}'_1 &= \cosh(\alpha)\boldsymbol{\gamma}_1 + \sinh(\alpha)\boldsymbol{\gamma}_0 = e_{*C}^{\alpha\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0} *_C \boldsymbol{\gamma}_1, &(5.121)\end{aligned}$$

or with $L_1 = e_{*C}^{\alpha\boldsymbol{\gamma}_1\boldsymbol{\gamma}_0/2}$ as $\boldsymbol{\gamma}'_\mu = L_1 *_C \boldsymbol{\gamma}_\mu *_C \overline{L_1}$.

General passive Lorentz transformations as rotations in Minkowski space are generated by the space-time bivectors. In four dimensions one has six bivectors:

$$\begin{aligned} &\mathtt{B}_1 = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_0, \qquad \mathtt{B}_2 = \boldsymbol{\gamma}_2\boldsymbol{\gamma}_0, \qquad \mathtt{B}_3 = \boldsymbol{\gamma}_3\boldsymbol{\gamma}_0 &(5.122)\\ \text{and}\quad &\mathtt{B}_4 = I_{(4)} *_C \mathtt{B}_1 = \boldsymbol{\gamma}_3\boldsymbol{\gamma}_2, \quad \mathtt{B}_5 = I_{(4)} *_C \mathtt{B}_2 = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_3, \quad \mathtt{B}_6 = I_{(4)} *_C \mathtt{B}_3 = \boldsymbol{\gamma}_2\boldsymbol{\gamma}_1, &(5.123)\end{aligned}$$

with the pseudoscalar $I_{(4)} = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3$, For these bivectors one has $\mathtt{B}_{1,2,3}^{2*_C} = 1$, $\mathtt{B}_{4,5,6}^{2*_C} = -1$ and

$$\begin{aligned} &\mathtt{B}_i \times \mathtt{B}_j = -\varepsilon_{ijk}(I_{(4)} *_C \mathtt{B}_k), \qquad \mathtt{B}_i \times (I_{(4)} *_C \mathtt{B}_j) = \varepsilon_{ijk}\mathtt{B}_k, &(5.124)\\ &(I_{(4)} *_C \mathtt{B}_i) \times \mathtt{B}_j = \varepsilon_{ijk}\mathtt{B}_k, \qquad (I_{(4)} *_C \mathtt{B}_i) \times (I_{(4)} *_C \mathtt{B}_j) = \varepsilon_{ijk}\mathtt{B}_k. &(5.125)\end{aligned}$$

In analogy to the tuple formalism one can also write the bivectors as

$$\sigma_{\mu\nu} = \frac{I_{(4)}}{2} *_C \left[\boldsymbol{\gamma}_\mu, \boldsymbol{\gamma}_\nu\right]_{*C}. \tag{5.126}$$

The generators for the passive boosts and rotations are

$$\mathtt{K}_i = \frac{1}{2}\sigma_{0i} \qquad \text{and} \qquad \mathtt{L}_i = \frac{1}{2}\sum_{j<k}\varepsilon_{ijk}\sigma_{jk} \tag{5.127}$$

and they satisfy in the case of the nonstandard metric (for the standard metric one has to replace $I_{(4)}$ by $-I_{(4)}$):

$$[\mathtt{L}_i, \mathtt{L}_j]_{*C} = -I_{(4)} *_C \varepsilon_{ijk}\mathtt{L}_k, \qquad [\mathtt{L}_i, \mathtt{K}_j]_{*C} = -I_{(4)} *_C \varepsilon_{ijk}\mathtt{K}_k, \quad \text{and} \quad [\mathtt{K}_i, \mathtt{K}_j]_{*C} = I_{(4)} *_C \varepsilon_{ijk}\mathtt{L}_k. \tag{5.128}$$

The passive Lorentz transformation is then given by

$$\boldsymbol{x}' = e_{*C}^{\frac{1}{4}I_{(4)}*_C\alpha^{\mu\nu}\sigma_{\mu\nu}} *_C \boldsymbol{x} *_C e_{*C}^{-\frac{1}{4}I_{(4)}*_C\alpha^{\mu\nu}\sigma_{\mu\nu}} = x^\mu\left(\Lambda_\mu^\nu\boldsymbol{\gamma}_\nu\right) \tag{5.129}$$

where Λ_ν^μ is the well known Lorentz transformation matrix.

In the light of geometric algebra it now becomes clear that Dirac by factorizing the Klein-Gordon equation, found nothing else than the basis vectors of space-time. Taking these into account the Lorentz transformation in the coefficients of a four vector $\boldsymbol{p} = p^\mu\boldsymbol{\gamma}_\mu$ have to be compensated by a passive Lorentz transformation in the basis vectors if one demands invariance. This passive Lorentz transformation which in geometric algebra is naturally given had to be constructed by Dirac a posteriori to insure Lorentz invariance of his equation [73, 37]. It would now be straightforward to formulate Dirac theory in the star product formalism, which would reproduce the results of section 2.6. But the important point is here that by doing this one ignores two severe conceptual problems that lie in Dirac theory itself [41] and are again made obvious by the star product formalism.

The first problem is that one uses in the star product formalism for Dirac theory a four dimensional fermionic Clifford star product in order to generate the Clifford algebra of the $\boldsymbol{\gamma}_\mu$, but on the other hand a three dimensional bosonic Moyal product. Using the three dimensional Moyal product just reflects the special role the time plays in Dirac theory. But for the star product formalism this means that the algebraic structure is not supersymmetric and that one can not represent active Lorentz transformations. With a three dimensional Moyal product one can only represent active rotations, to include the boosts one would need a four dimensional star product, i.e.

$$f *_M g = f \exp\left[\frac{\mathrm{i}\hbar}{2}\eta^{\mu\nu}\left(\frac{\overleftarrow{\partial}}{\partial q^\mu}\frac{\vec{\partial}}{\partial p^\nu} - \frac{\overleftarrow{\partial}}{\partial p^\mu}\frac{\vec{\partial}}{\partial q^\nu}\right)\right] g. \tag{5.130}$$

Here again the nonstandard metric should be chosen, so that the three dimensional part reduces to the conventional Moyal product. The generators of an active Lorentz transformation are then

$$M^{\mu\nu} = q^\mu p^\nu - p^\mu q^\nu, \tag{5.131}$$

with

$$\left[M^{\mu\nu}, M^{\rho\sigma}\right]_{*_M} = \mathrm{i}\hbar\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\mu\sigma}M^{\rho\nu} - \eta^{\nu\sigma}M^{\rho\mu}\right). \tag{5.132}$$

The generators of boosts and rotations are

$$K^i = M^{01} \qquad \text{and} \qquad L^i = \sum_{j<k}\varepsilon^{ijk}M^{jk}. \tag{5.133}$$

They form the following active Moyal star-commutator algebra

$$\left[L^i, L^j\right]_{*_M} = \mathrm{i}\hbar\varepsilon^{ijk}L^k, \qquad \left[L^i, K^j\right]_{*_M} = \mathrm{i}\hbar\varepsilon^{ijk}K^k \qquad \text{and} \qquad \left[K^i, K^j\right]_{*_M} = -\mathrm{i}\hbar\varepsilon^{ijk}L^k, \tag{5.134}$$

so that an active Lorentz transformation of the four-vector $\boldsymbol{q} = q^\mu\boldsymbol{\gamma}_\mu$ is given by

$$\boldsymbol{q}' = e_{*_M}^{-\frac{\mathrm{i}}{\hbar}\alpha_{\mu\nu}M^{\mu\nu}} *_M \boldsymbol{q} *_M e_{*_M}^{\frac{\mathrm{i}}{\hbar}\alpha_{\mu\nu}M^{\mu\nu}} = \left(\Lambda^\mu_\nu q^\nu\right)\boldsymbol{\gamma}_\mu. \tag{5.135}$$

Taking the translations with the generators p_μ into account the Lorentz algebra is with $\left[p_\mu, p_\nu\right]_{*_M} = 0$ and

$$\left[M^{\mu\nu}, p_\rho\right]_{*_M} = \mathrm{i}\hbar\left(\eta^{\mu\rho}p_\nu - \eta^{\nu\rho}p_\mu\right) \tag{5.136}$$

extended to the Poincaré algebra.

The second severe conceptual problem that is related closely to the first one is that the Dirac equation has no classical limit. This contradicts the philosophy of deformation quantization, where quantum mechanics is a deformation of classical mechanics and where the limit $\hbar \to 0$ leads again to classical mechanics. So following the philosophy of deformation quantization means to deform a manifest covariant version of the canonical formalism to obtain relativistic quantum mechanics. Covariance of the canonical formalism means that the physical laws, expressed by Poisson bracket relations, have to be invariant under a transformation from one inertial system into another inertial system. Such transformations preserving the Poisson brackets are canonical transformations, so that a canonical system is relativistically invariant if one has a canonical realization of the relativity group. Manifest covariance means that in addition to the requirement of relativistic invariance of the physical laws the labeled trajectory of a particle in configuration space $\vec{q}(t)$ has to behave like a world line. This means that the relativity postulate leads only to the requirement of a Poisson bracket realization of the Poincare group, while manifest covariance requires that the dynamical quantities $(t, \vec{q}(t))$ constitute an space-time event [104]. There are now two approaches to a manifest covariant extension of the canonical formalism in classical mechanics. The first approach is that one describes the particles by their canonical coordinates and the time coordinate and then derives conditions that describe the fact that

$(t, \vec{q}(t))$ transforms like an event in space-time. These additional conditions lead then to the consequence that no interactions are allowed [104].

The alternative method is to use a parameter formalism. In this approach the events that constitute the world lines are labeled by an observer independent parameter s that increases monotonically as the world line is traversed. The four space-time coordinates of an event on the world line are then functions of this parameter and going from one inertial system to another one does not change the parameter. The four space-time coordinates are then regarded as the dynamical quantities, while the parameter s describes the evolution of the system. So the time has no longer the two roles of a dynamical variable and an evolution parameter. But this just fits perfectly to the solution of the first conceptual problem, because using the four dimensional Moyal product (5.130) for deformation quantization means that the one particle phase space is extended by the two variables q^0 and p^0, which means that the time development is not described by the time, that is now a phase space coordinate, but by an additional parameter. So what is actually deformed by the four dimensional Moyal product (5.130) is parametrized Hamiltonian dynamics. And in the limit $\hbar \to 0$ the star product reduces to the conventional product so that one reobtains the classical undeformed parametrized Hamiltonian dynamics, so that the conceptual problem of the missing classical limit is also solved. In the operator formalism of canonical quantization this would mean that time is no longer a scalar but an operator, for a discussion concerning the existence of such a time operator see [59].

It is now straightforward to develop a parametrized relativistic mechanics [43, 81]. One defines to this purpose a parameter-dependent action

$$S = \int_{s_1}^{s_2} ds\, L_s(q^\mu, \mathring{q}^\mu, s), \tag{5.137}$$

where $\mathring{q}^\mu$ is the derivation with respect to the parameter s:

$$\mathring{q}^\mu = \frac{dq^\mu}{ds}. \tag{5.138}$$

Requiring that the variation of the action vanishes: $\delta S = 0$ leads to the parametrized version of the Euler-Lagrange equation:

$$\frac{d}{ds}\frac{\partial L_s}{\partial \mathring{q}^\mu} - \frac{\partial L_s}{\partial q^\mu} = 0. \tag{5.139}$$

With the Legendre transformation

$$K(q^\mu, p_\mu, s) = \mathring{q}^\mu p_\mu - L_s(q^\mu, \mathring{q}^\mu, s) \tag{5.140}$$

one then obtains the parametrized Hamilton equations:

$$\mathring{q}^\mu = \frac{\partial K}{\partial p_\mu} \qquad \text{and} \qquad \mathring{p}_\mu = -\frac{\partial K}{\partial q^\mu}. \tag{5.141}$$

Using the Hamilton equations to calculate

$$\frac{d}{ds} f(q^\mu, p_\mu, s) = \{f, K\}_{PB} + \frac{\partial f}{\partial s} \tag{5.142}$$

one arrives at the four-space Poisson bracket

$$\{f, g\}_{PB} = \frac{\partial f}{\partial q^\mu}\frac{\partial g}{\partial p_\mu} - \frac{\partial g}{\partial q_\mu}\frac{\partial f}{\partial p^\mu}, \tag{5.143}$$

for which follows

$$\{q^\mu, p_\nu\}_{PB} = \delta^\mu_\nu \qquad \text{and} \qquad \{q^\mu, q^\nu\}_{PB} = \{p_\mu, p_\nu\}_{PB} = 0. \tag{5.144}$$

For example the covariant Hamiltonian of the free particle is

$$K = \frac{\eta^{\mu\nu}}{2m} p_\mu p_\nu \tag{5.145}$$

so that the Hamilton equations (5.141) lead to

$$\begin{aligned} & \mathring{p}^\mu = 0 && \Rightarrow && p_\mu = p_{0\mu} = \text{const} \\ \text{and} \quad & \mathring{q}^\mu = \frac{p^\mu}{m} && \Rightarrow && q^\mu = q_0^\mu + \frac{p_0^\mu}{m} s. \end{aligned} \tag{5.146}$$

Variation of q^μ gives then $\delta q^\mu \delta q_\mu = \frac{p_0^\mu p_{0\mu}}{m^2} (\delta s)^2 = (\delta s)^2$ with the initial condition $m^2 = p_0^\mu p_{0\mu}$, which shows that the parameter s is just the proper time.

In the case of a charged particle in an electromagnetic field the Hamiltonian (5.145) generalizes to

$$K = \frac{\eta^{\mu\nu}}{2m} \left[p_\mu - eA_\mu\right] \left[p_\nu - eA_\nu\right] = \frac{1}{2m} \pi^\mu \pi_\mu, \tag{5.147}$$

with the kinetic momentum $\pi_\mu = p_\mu - eA_\mu$. The Hamilton equations (5.141) lead to

$$\mathring{q}_\mu = \frac{\pi_\mu}{m} \qquad \text{and} \qquad \mathring{p}_\mu = \frac{e}{m} \pi^\nu \partial_\mu A_\nu. \tag{5.148}$$

Combining these two equations gives $\mathring{p}_\mu = e\mathring{q}^\nu \partial_\mu A_\nu$ and for the derivation of the kinetic momentum with respect to s one obtains $\mathring{\pi}_\mu = \mathring{p}_\mu - e\partial_\nu A_\mu \mathring{q}^\nu$. Equating then the expressions for $\mathring{p}_\mu$ gives the Lorentz force law

$$\mathring{\pi}_\mu = eF_{\mu\nu} \mathring{q}^\nu. \tag{5.149}$$

The classical mass is then a constant associated to the kinetic momentum which can be obtained as follows. With (5.149) and (5.148) one can calculate

$$\mathring{\pi}_\mu \pi^\mu = \frac{1}{2} \frac{d}{ds} \left(\pi_\mu \pi^\mu\right) = emF_{\mu\nu} \mathring{q}^\nu \mathring{q}^\mu = 0. \tag{5.150}$$

From $\frac{d}{ds} \left(\pi_\mu \pi^\mu\right) = 0$ follows then that $\pi_\mu \pi^\mu = \pi_{0\mu} \pi_0^\mu$ is an integration constant with respect to s. In order to be consistent with the case $A_\mu = 0$, where $p_\mu p^\mu = p_{0\mu} p_0^\mu = m^2$ one chooses the integration constant as $\pi_{0\mu} \pi_0^\mu = m^2$. This shows that the classical mass is a secondary concept in the proper time formalism, while energy and momentum are primary concepts.

Just as in the nonrelativistic case the connection of the four dimensional Poisson bracket (5.143) and the four dimensional star product (5.130) is given by

$$\lim_{\hbar \to 0} \frac{1}{\mathrm{i}\hbar} \left[f, g\right]_{*_M} = \{f, g\}_{PB}, \tag{5.151}$$

so that the star commutators of the canonical coordinates are

$$\left[q^\mu, p_\nu\right]_{*_M} = \mathrm{i}\hbar \delta_\nu^\mu \qquad \text{and} \qquad \left[q^\mu, q^\nu\right]_{*_M} = \left[p_\mu, p_\nu\right]_{*_M} = 0. \tag{5.152}$$

The structures of deformation quantization in the nonrelativistic case can then be generalized to the four dimensional case in a straightforward manner. The development of the system in s is generated by the four dimensional Hamiltonian. In the star product formalism this is described by the star exponential, which is in the four dimensional case given by

$$\mathrm{Exp}_M(Ks) = e_{*_M}^{-\mathrm{i}sK/\hbar} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-\mathrm{i}s}{\hbar}\right)^n K^{n*_M}, \tag{5.153}$$

where K^{n*_M} is the n-fold star product. The star exponential fulfills the proper time generalization of the time dependent Schrödinger equation:

$$\mathrm{i}\hbar\frac{d}{ds}\mathrm{Exp}_M(Ks) = K *_M \mathrm{Exp}_M(Ks). \tag{5.154}$$

The calculations to determine the spectrum and the Wigner eigenfunctions then parallels the calculations in the non-relativistic case.

But there is now also the additional effect due to noncommutativity known from the nonrelativistic case. Combining the Moyal product (5.130) and the Clifford product (4.314) into one supersymmetric formalism one obtains a noncommutative version of space-time algebra. In the commutative or classical case the generalized Hamiltonian (5.147) can be written as

$$K = \frac{1}{2m}\boldsymbol{\pi} *_C \boldsymbol{\pi} = \frac{1}{2m}\boldsymbol{\pi}\cdot\boldsymbol{\pi} \tag{5.155}$$

with $\boldsymbol{\pi} = \pi^\mu\boldsymbol{\gamma}_\mu$. But if one introduces noncommutativity via the Moyal product, the Moyal product of π_μ and π_ν is in general not symmetric in the indices, one rather has

$$[\pi_\mu, \pi_\nu]_{*_M} = \mathrm{i}\hbar e F_{\mu\nu}. \tag{5.156}$$

This leads then to the appearance of an additional term that describes the spin:

$$K = \frac{1}{2m}\boldsymbol{\pi} *_{MC} \boldsymbol{\pi} = \frac{1}{2m}\left(\pi^\mu *_M \pi^\nu\right)\left(\boldsymbol{\gamma}_\mu *_C \boldsymbol{\gamma}_\nu\right) = \frac{1}{2m}\pi^\mu\pi_\mu + \frac{1}{2m}\left(\pi^\mu *_M \pi^\nu\right)\boldsymbol{\gamma}_\mu\boldsymbol{\gamma}_\nu. \tag{5.157}$$

In the case of a stationary particle in a homogenous magnetic field (5.157) reduces to

$$K = -\frac{m}{2} + \mathrm{i}\frac{e\hbar}{2m}B_3\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2, \tag{5.158}$$

so that one has the spin eigenfunctions $\frac{1}{2} \pm \frac{\mathrm{i}}{2}\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2$, that fulfil

$$\mathrm{i}\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2 *_C \left(\frac{1}{2} \pm \frac{\mathrm{i}}{2}\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\right) = \pm\left(\frac{1}{2} \pm \frac{\mathrm{i}}{2}\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\right). \tag{5.159}$$

5.5 Deformed Geometric Algebra on the Phase Space and Supersymmetric Quantum Mechanics

The concept of deforming geometric algebra that on space and space-time produced extra spin terms can now also be used to deform geometric algebra on the phase space as described in section 4.7. It will be shown here that this deformation leads naturally to the appearance of supersymmetric quantum mechanics. While in section 2.4 supersymmetric quantum mechanics was constructed via pseudoclassical mechanics one will see here that it appears naturally and geometrically. One can restrict oneself to the simplest case of a flat two dimensional phase space, i.e. a point in the phase space is given by $\boldsymbol{z} = z^i\boldsymbol{\zeta}_i = q\boldsymbol{\eta} + p\boldsymbol{\rho}$ and the Clifford star product is given by

$$A *_C B = A\,\exp\left[\eta_{ij}\frac{\overleftarrow{\partial}}{\partial\boldsymbol{\zeta}_i}\frac{\overrightarrow{\partial}}{\partial\boldsymbol{\zeta}_j}\right]B, \tag{5.160}$$

where $\eta_{ij} = \mathrm{diag}(1,1)$ is here the euclidian metric on the vector space. Furthermore one has a closed two-form

$$\Omega = \frac{1}{2}\Omega_{ij}\boldsymbol{\zeta}^i\boldsymbol{\zeta}^j = \boldsymbol{\eta}\boldsymbol{\rho} = \boldsymbol{d}q\boldsymbol{d}p. \tag{5.161}$$

A Hamilton function can now be written as the square of the vector

$$\boldsymbol{w} = W(q)\boldsymbol{\eta} + p\boldsymbol{\rho}, \tag{5.162}$$

where $W(q)$ is the superpotential, one has then

$$H = \frac{1}{2}\boldsymbol{w} *_C \boldsymbol{w} = \frac{1}{2}\boldsymbol{w}\cdot\boldsymbol{w} = \frac{1}{2}\left[p^2 + W^2(q)\right] \tag{5.163}$$

and in holomorphic coordinates $B = \frac{1}{\sqrt{2}}(W(q)+\mathrm{i}p)$, $\bar{B} = \frac{1}{\sqrt{2}}(W(q)-\mathrm{i}p)$ and $\boldsymbol{f} = \frac{1}{\sqrt{2}}(\boldsymbol{\eta}+\mathrm{i}\boldsymbol{\rho})$, $\bar{\boldsymbol{f}} = \frac{1}{\sqrt{2}}(\boldsymbol{\eta}-\mathrm{i}\boldsymbol{\rho})$ one obtains

$$\boldsymbol{w} = B\bar{\boldsymbol{f}} + \bar{B}\boldsymbol{f} = \boldsymbol{Q}_+ + \boldsymbol{Q}_- \tag{5.164}$$

and $H = B\bar{B}$.

Up to now the coefficients were commuting quantities, but one can go over to the noncommutative or quantum case by demanding that the coefficients have to be multiplied by the Moyal product. In this case the square of $\boldsymbol{w}$ is no longer a scalar, but one has an bivector valued extra term

$$\begin{aligned} H_{\text{Susy}} = \frac{1}{2}\boldsymbol{w} *_{MC} \boldsymbol{w} &= \frac{1}{2}\big[(W(q) *_M W(q))(\boldsymbol{\eta} *_C \boldsymbol{\eta}) + (W(q) *_M p)(\boldsymbol{\eta} *_C \boldsymbol{\rho}) \\ &\quad +(p *_M W(q))(\boldsymbol{\rho} *_C \boldsymbol{\eta}) + (p *_M p)(\boldsymbol{\rho} *_C \boldsymbol{\rho})\big] \qquad (5.165) \\ &= \frac{1}{2}\left[p^2 + W^2(q)\right] + \frac{\hbar}{2}\frac{\partial W(q)}{\partial q}\frac{1}{\mathrm{i}}\boldsymbol{\eta}\boldsymbol{\rho}. \qquad (5.166) \end{aligned}$$

The next thing one has to notice is that $\boldsymbol{\eta}$, $\boldsymbol{\rho}$ and $-\mathrm{i}\boldsymbol{\eta}\boldsymbol{\rho}$ fulfill under the Clifford star product the Pauli algebra, i.e. one has for the star commutators and anticommutators of these real basis elements of the two dimensional Clifford algebra:

$$[\boldsymbol{\eta}, \boldsymbol{\rho}]_{*_C} = 2\boldsymbol{\eta}\boldsymbol{\rho}, \qquad [\boldsymbol{\eta}, -\mathrm{i}\boldsymbol{\eta}\boldsymbol{\rho}]_{*_C} = -2\mathrm{i}\boldsymbol{\rho}, \qquad [\boldsymbol{\rho}, -\mathrm{i}\boldsymbol{\eta}\boldsymbol{\rho}]_{*_C} = 2\mathrm{i}\boldsymbol{\eta} \tag{5.167}$$

$$\text{and} \quad \{\boldsymbol{\eta}, \boldsymbol{\eta}\}_{*_C} = \{\boldsymbol{\rho}, \boldsymbol{\rho}\}_{*_C} = \{-\mathrm{i}\boldsymbol{\eta}\boldsymbol{\rho}, -\mathrm{i}\boldsymbol{\eta}\boldsymbol{\rho}\}_{*_C} = 2, \tag{5.168}$$

while the other star commutators and star anticommutators vanish. This means that $\boldsymbol{\eta}$, $\boldsymbol{\rho}$ and $-\mathrm{i}\boldsymbol{\eta}\boldsymbol{\rho}$ would be represented in a tuple representation by the Pauli matrices, so that H_S is the supersymmetric Hamiltonian. Furthermore for the holomorphic basis vectors $\boldsymbol{f} = \frac{1}{\sqrt{2}}(\boldsymbol{\eta} + \mathrm{i}\boldsymbol{\rho})$ and $\bar{\boldsymbol{f}} = \frac{1}{\sqrt{2}}(\boldsymbol{\eta} - \mathrm{i}\boldsymbol{\rho})$ one has the tuple representation

$$\frac{1}{\sqrt{2}}\boldsymbol{f} \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{2}}\bar{\boldsymbol{f}} \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.169}$$

The two eigen-multivectors of $-\mathrm{i}\boldsymbol{\eta}\boldsymbol{\rho}$ are $\pi_\pm^{(C)} = \frac{1}{2}(1 \mp \mathrm{i}\boldsymbol{\eta}\boldsymbol{\rho})$, i.e. for these multivectors one has

$$-\mathrm{i}\boldsymbol{\eta}\boldsymbol{\rho} *_C \pi_\pm^{(C)} = \pm\pi_\pm^{(C)}. \tag{5.170}$$

In the star product formalism these multivectors are fermionic Wigner functions and as such they are projectors:

$$\pi_\pm^{(C)} *_C \pi_\pm^{(C)} = \pi_\pm^{(C)} \quad \text{and} \quad \pi_+^{(C)} *_C \pi_-^{(C)} = \pi_-^{(C)} *_C \pi_+^{(C)} = 0. \tag{5.171}$$

The holomorphic basis vectors $\frac{1}{\sqrt{2}}\boldsymbol{f}$ and $\frac{1}{\sqrt{2}}\bar{\boldsymbol{f}}$ serve here as lowering and raising operators, i.e. one has

$$\bar{\boldsymbol{f}} *_C \pi_+^{(C)} *_C \boldsymbol{f} = 2\pi_-^{(C)} \quad \text{and} \quad \boldsymbol{f} *_C \pi_-^{(C)} *_C \bar{\boldsymbol{f}} = 2\pi_+^{(C)}, \tag{5.172}$$

while the other combinations give zero.

With the multivectors $\pi_\pm^{(C)}$ the supersymmetric Hamilton function (5.166) can then be written as

$$\begin{aligned} H_{\mathrm{Susy}} &= \frac{1}{2}\left[p^2 + W^2(q) - \hbar\frac{\partial W(q)}{\partial q}\right]\left(\frac{1}{2} - \frac{\mathrm{i}}{2}\boldsymbol{\eta}\boldsymbol{\rho}\right) \\ &\quad + \frac{1}{2}\left[p^2 + W^2(q) + \hbar\frac{\partial W(q)}{\partial q}\right]\left(\frac{1}{2} + \frac{\mathrm{i}}{2}\boldsymbol{\eta}\boldsymbol{\rho}\right) \qquad (5.173) \\ &= H_1\pi_+^{(C)} + H_2\pi_-^{(C)}. \qquad (5.174) \end{aligned}$$

From (5.171) it is then clear that the Moyal-Clifford star eigenfunctions of H_{Susy} are a product of $\pi_+^{(C)}$ and Moyal star eigenfunctions of H_1 or products of $\pi_-^{(C)}$ and Moyal star eigenfunctions of H_2. The Moyal star eigenfunctions for supersymmetric partner potentials were for example discussed in [24].

The vectors $\boldsymbol{Q}_\pm$ defined in (5.164) fulfill

$$\boldsymbol{Q}_\pm *_{MC} \boldsymbol{Q}_\pm = 0, \quad \boldsymbol{Q}_- *_{MC} \boldsymbol{Q}_+ = H_1\pi_+^{(C)}, \quad \boldsymbol{Q}_+ *_{MC} \boldsymbol{Q}_- = H_2\pi_-^{(C)} \qquad (5.175)$$

so that H_{Susy} can be written as

$$H_{\mathrm{Susy}} = \frac{1}{2}\{\boldsymbol{Q}_+, \boldsymbol{Q}_-\}_{*_{MC}}, \qquad (5.176)$$

and with (5.175) one has $\left[\boldsymbol{Q}_+, H_{\mathrm{Susy}}\right]_{*_{MC}} = \left[\boldsymbol{Q}_-, H_{\mathrm{Susy}}\right]_{*_{MC}} = 0$. Defining finally

$$\boldsymbol{Q}_1 = \boldsymbol{Q}_+ + \boldsymbol{Q}_- \quad \text{and} \quad \boldsymbol{Q}_2 = -\mathrm{i}(\boldsymbol{Q}_+ - \boldsymbol{Q}_-) \qquad (5.177)$$

the supersymmetric Hamilton function factorizes as

$$H_{\mathrm{Susy}} = \frac{1}{2}\boldsymbol{Q}_1 *_{MC} \boldsymbol{Q}_1 = \frac{1}{2}\boldsymbol{Q}_2 *_{MC} \boldsymbol{Q}_2. \qquad (5.178)$$

In conclusion one sees here that supersymmetric quantum mechanics appears naturally if one deforms geometric algebra on the phase space. The deformation of geometric algebra on the three space induced an extra bivector valued term that splits the system in a version with spin up and one with spin down, i.e. the noncommutativity transforms the Schrödinger Hamilton into the Pauli Hamiltonian. The analogue procedure on the phase space leads similarly to a split into two supersymmetric partner systems. The appearance of these structures can also be stated in a different way: Just as the factorization of the Klein-Gordon equation exhibits in Dirac theory the Clifford structure of space-time, the factorization of a Hamilton function into supercharges exhibits the Clifford structure of the phase space.

5.6 Active and Passive Transformations on the Phase Space

A flat phase space can be considered as an $2d$-dimensional euclidian vector space with vectors (4.327) and a two-form (4.328). The time development is described by the hamiltonian vector field $\boldsymbol{h}_H = \dot{q}^n\boldsymbol{\eta}_n + \dot{p}^n\boldsymbol{\rho}_n = J^{ij}\partial_j H\boldsymbol{\zeta}_i$, so that one has for a scalar phase space function f

$$\dot{f} = \dot{\boldsymbol{z}}\cdot(\boldsymbol{d}f) = (\boldsymbol{h}_H\cdot\boldsymbol{d})f = \mathscr{L}_{\boldsymbol{h}_H} f = \{f, H\}_{PB}. \qquad (5.179)$$

where $\boldsymbol{h}_H\cdot\boldsymbol{d}$ is the Liouville operator. The above equation for the time development can immediately be generalized from 0-forms f to arbitrary r-forms. For example the time development of the symplectic two-form is given by $\dot{\Omega} = \mathscr{L}_{\boldsymbol{h}_H}\Omega = 0$, which means that the symplectic form is preserved by the time evolution.

The temporal development of a system can be described by an active time transformation of the coefficients, which corresponds to the Hamilton equations

$$\dot{z}^i = \mathscr{L}_{\boldsymbol{h}_H} z^i = J^{ij}\partial_j H. \tag{5.180}$$

In the formalism of geometric algebra it is also possible to write down a time transformation of the basis vectors

$$\dot{\boldsymbol{\zeta}}_i = \mathscr{L}_{\boldsymbol{h}_H} \boldsymbol{\zeta}_i = -J^{jk}\partial_k\partial_i H \boldsymbol{\zeta}_j, \tag{5.181}$$

which corresponds to the Jacobi equation that appeared in the path integral formulation of classical mechanics [65].

Active and passive time development can directly be discussed for the example of the harmonic oscillator. The Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$ generates via the star exponential $U(t) = e_{*_M}^{-\frac{\mathrm{i}}{\hbar}Ht}$ an active rotation of the state vector $\boldsymbol{z}_0 = q\boldsymbol{\eta} + p\boldsymbol{\rho}$ according to [114]

$$\boldsymbol{z}(t) = \overline{U(t)} *_M \boldsymbol{z}_0 *_M U(t) = (q\cos t + p \sin t)\boldsymbol{\eta} + (-q\sin t + p\cos t)\boldsymbol{\rho} = q(t)\boldsymbol{\eta} + p(t)\boldsymbol{\rho}. \tag{5.182}$$

The same transformation passively can be achieved with the rotor $R(t) = e_{*_C}^{\frac{1}{2}\mathtt{H}t}$ and the bivector $\mathtt{H} = \boldsymbol{\eta}\boldsymbol{\rho}$ as

$$\boldsymbol{z}(t) = R(t) *_C \boldsymbol{z}_0 *_C \overline{R(t)} = q(\cos t\, \boldsymbol{\eta} - \sin t\, \boldsymbol{\rho}) + p(\sin t\, \boldsymbol{\eta} + \cos t\, \boldsymbol{\rho}) = q\boldsymbol{\eta}(t) + p\boldsymbol{\rho}(t). \tag{5.183}$$

With the hamiltonian vector-field $\boldsymbol{h}_H = p\boldsymbol{\eta} - q\boldsymbol{\rho}$ and the relation $\{f, g\}_{PB} = \lim_{\hbar\to 0} \frac{1}{\mathrm{i}\hbar}\, [f, g]_{*_M}$ the active Hamilton equations $\dot{z}^i = \mathscr{L}_{\boldsymbol{h}_H} z^i$ can be written as

$$\dot{q} = \lim_{\hbar\to 0} \frac{1}{\mathrm{i}\hbar}\, [q, H]_{*_M} = p \qquad \text{and} \qquad \dot{p} = \lim_{\hbar\to 0} \frac{1}{\mathrm{i}\hbar}\, [p, H]_{*_M} = -q. \tag{5.184}$$

With (5.181) one can then calculate the corresponding time inverted passive Hamilton equations. Using the Clifford star commutator defined by

$$\left[A_{(r)}, B_{(s)}\right]_{*_C} = A_{(r)} *_C B_{(s)} - (-1)^{rs} B_{(s)} *_C A_{(r)} \tag{5.185}$$

these equations can be written as

$$\dot{\boldsymbol{\eta}} = \frac{1}{\mathrm{i}}\, [\boldsymbol{\eta}, \mathtt{H}]_{*_C} = \boldsymbol{\rho} \qquad \text{and} \qquad \dot{\boldsymbol{\rho}} = \frac{1}{\mathrm{i}}\, [\boldsymbol{\rho}, \mathtt{H}]_{*_C} = -\boldsymbol{\eta}, \tag{5.186}$$

where $\mathtt{H} = \frac{\mathrm{i}}{2}\boldsymbol{\eta}\boldsymbol{\rho}$ is the passive Hamiltonian. The passive Hamiltonian is connected with the active one through (5.185) and (5.181) by

$$\frac{1}{\mathrm{i}}\, [\boldsymbol{\zeta}_i, \mathtt{H}]_{*_C} = -J^{jk}\partial_k\partial_i H \boldsymbol{\zeta}_j. \tag{5.187}$$

The passive Hamiltonian $\mathtt{H}$ is here just the free Hamiltonian of pseudoclassical mechanics [11] (the additional factor $\frac{1}{2}$ is due to the definition of the Clifford product which is defined without a factor $\frac{1}{2}$).

A Lagrangian that takes into account both the time development according to (5.180) and the time development according to (5.181) should be called the extended Lagrangian and has the form

$$\begin{aligned} \widetilde{\mathcal{L}}_E &= y_i \left(\dot{z}^i - J^{ij}\partial_j H\right) + \mathrm{i}\boldsymbol{\zeta}_j \left(\partial_t \delta^j_l - J^{jk}\partial_l\partial_k H\right) \boldsymbol{\lambda}^l \\ &= y_i \dot{z}^i + \mathrm{i}\boldsymbol{\zeta}_j \dot{\boldsymbol{\lambda}}^j - \widetilde{\mathcal{H}}_E, \end{aligned} \tag{5.188}$$

where the extended Hamiltonian $\widetilde{\mathcal{H}}_E$ is given by

$$\widetilde{\mathcal{H}}_E = y_i J^{ij}\partial_j H + \mathrm{i}\boldsymbol{\zeta}_j J^{jk}\partial_l\partial_k H \boldsymbol{\lambda}^l. \tag{5.189}$$

The extended Lagrangian first appeared in the path integral approach to classical mechanics [65, 66], where the classical analogue of the quantum generating functional was considered:

$$Z_{CM}\left[J\right] = N \int Dz\, \delta\left[z(t) - z_{cl}(t)\right]\, \exp\left[\int dt\, J\phi\right]. \tag{5.190}$$

The delta function here constrains all possible trajectories to the classical trajectory obeying (5.180). It can be written as

$$\delta\left[z(t) - z_{cl}(t)\right] = \delta\left[\dot{z}^i - \Omega^{ij}\partial_j H\right] \det\left[\delta^i_j \partial_t - \Omega^{ik}\partial_k\partial_j H\right]. \tag{5.191}$$

The delta function on the right side can be expressed by a Fourier transform

$$\delta\left[\dot{z}^i - \Omega^{ij}\partial_j H\right] = \int Dy_i \exp\left[\mathrm{i}\int dt\, y_i\left(\dot{z}^i - \Omega^{ij}\partial_j H\right)\right] \tag{5.192}$$

and the determinant can be written in terms of Grassmann variables as

$$\det\left[\delta^i_j \partial_t - \Omega^{ik}\partial_k\partial_j H\right] = \int D\boldsymbol{\lambda}^i D\boldsymbol{\zeta}_i\, \exp\left[-\int dt\, \boldsymbol{\zeta}_i\left[\delta^i_j\partial_t - \Omega^{ik}\partial_k\partial_j H\right]\boldsymbol{\lambda}^j\right], \tag{5.193}$$

so that $Z_{CM}\left[0\right]$ becomes

$$Z_{CM}\left[0\right] = \int Dz^i Dy_i D\boldsymbol{\lambda}^j D\boldsymbol{\zeta}_j\, \exp\left[\mathrm{i}\int dt\, \widetilde{\mathcal{L}}_E\right]. \tag{5.194}$$

The important point is here that the path integral formalism of classical mechanics gives the fermionic basis vectors of geometric algebra the physical interpretation of ghosts. On the other hand the superanalytic formulation of geometric algebra has naturally the fermionic structures that in the conventional formalism have to be added ad hoc and per hand.

The z^i and $\boldsymbol{\zeta}_i$ form together with the newly introduced variables y_i and $\boldsymbol{\lambda}^i$ the extended phase space. On this extended phase space one can then introduce an extended canonical structure. This can easily be done in analogy to the Moyal and the Clifford star product structures of the phase space. Defining the extended Moyal-Clifford star product as

$$F *_{EMC} G = F\, \exp\left[\frac{\mathrm{i}}{2}\left(\frac{\overleftarrow{\partial}}{\partial z^k}\frac{\overrightarrow{\partial}}{\partial y_k} - \frac{\overleftarrow{\partial}}{\partial y_k}\frac{\overrightarrow{\partial}}{\partial z^k}\right) + \frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial \boldsymbol{\lambda}^k}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\zeta}_k} + \frac{\overleftarrow{\partial}}{\partial \boldsymbol{\zeta}_k}\frac{\overrightarrow{\partial}}{\partial \boldsymbol{\lambda}^k}\right)\right] G \tag{5.195}$$

the extended Poisson bracket has the form

$$\{F, G\}_{EPB} = \frac{1}{\mathrm{i}}\left[F *_{EMC} G - (-1)^{\epsilon(F)\epsilon(G)} G *_{EMC} F\right], \tag{5.196}$$

where $\epsilon(F)$ gives the Grassmann grade of F. In the bosonic part of the extended Clifford star product a factor $\hbar$ can be included like in the Moyal product, so that in the definition of the extended Poisson bracket (5.196) the limit $\hbar \to 0$ has to be taken. The extended canonical relations are then given by

$$\{z^i, y_j\}_{EPB} = \delta^i_j \qquad \text{and} \qquad \{\boldsymbol{\zeta}_i, \boldsymbol{\lambda}^j\}_{EPB} = -\mathrm{i}\delta^j_i, \tag{5.197}$$

while all other extended Poisson brackets vanish. Furthermore one can calculate the equations of motion as

$$\begin{aligned}
\dot{z}^i &= \{z^i, \widetilde{\mathcal{H}}_E\}_{EPB} = \Omega^{ij}\partial_j H, &(5.198)\\
\dot{\boldsymbol{\zeta}}_i &= \{\boldsymbol{\zeta}_i, \widetilde{\mathcal{H}}_E\}_{EPB} = -\Omega^{jk}\partial_k\partial_i H \boldsymbol{\zeta}_j, &(5.199)\\
\dot{y}_i &= \{y_i, \widetilde{\mathcal{H}}_E\}_{EPB} = -z_j\Omega^{jk}\partial_k\partial_i H - \mathrm{i}\boldsymbol{\zeta}_j\Omega^{jk}\partial_k\partial_l\partial_i H\boldsymbol{\lambda}^l, &(5.200)\\
\dot{\boldsymbol{\lambda}}^i &= \{\boldsymbol{\lambda}^i, \widetilde{\mathcal{H}}_E\}_{EPB} = \Omega^{ij}\partial_j\partial_k H\boldsymbol{\lambda}^k. &(5.201)
\end{aligned}$$

The extended Hamiltonian also generates the time development of r-vectors and r-forms according to [69]

$$\dot{X} = \mathscr{L}_{\boldsymbol{h}} X = \{X, \widetilde{\mathcal{H}}_E\}_{EPB}. \tag{5.202}$$

Having now a superanalytic formalism for classical mechanics that takes into account active and passive time development, one can ask if there is a supersymmetry in this formalism, i.e. a symmetry that relates the bosonic coefficients with the fermionic basis vectors. This supersymmetry was found by Gozzi et al. in [65]. There it was shown that $\widetilde{\mathcal{H}}_E$ is invariant under the following BRST-transformation

$$\delta z^k = \boldsymbol{\varepsilon}\boldsymbol{\lambda}^k, \qquad \delta\boldsymbol{\zeta}_k = \mathrm{i}\boldsymbol{\varepsilon} y_k, \qquad \delta\boldsymbol{\lambda}^k = \delta y_k = 0 \tag{5.203}$$

and the following anti-BRST-transformation

$$\delta z^k = -\boldsymbol{\varepsilon}\Omega^{kl}\boldsymbol{\zeta}_l, \qquad \delta\boldsymbol{\lambda}^k = \mathrm{i}\overline{\boldsymbol{\varepsilon}}\Omega^{kl} y_l, \qquad \delta\boldsymbol{\zeta}_k = \delta y_k = 0, \tag{5.204}$$

where $\boldsymbol{\varepsilon}$ and $\overline{\boldsymbol{\varepsilon}}$ are Grassmann variables. These symmetries are generated by

$$\boldsymbol{Q}_{BRST} = y_j \boldsymbol{\lambda}^j \qquad \text{and} \qquad \boldsymbol{Q}_{\overline{BRST}} = \boldsymbol{\zeta}_j \Omega^{jk} y_k \tag{5.205}$$

according to $\delta X = \{X, \boldsymbol{\varepsilon}\boldsymbol{Q}_{BRST} + \overline{\boldsymbol{\varepsilon}}\boldsymbol{Q}_{\overline{BRST}}\}_{EPB}$. The two charges $\boldsymbol{Q}_{BRST}$ and $\boldsymbol{Q}_{\overline{BRST}}$ are conserved, i.e.

$$\{\boldsymbol{Q}_{BRST}, \widetilde{\mathcal{H}}_E\}_{EPB} = \{\boldsymbol{Q}_{\overline{BRST}}, \widetilde{\mathcal{H}}_E\}_{EPB} = 0 \tag{5.206}$$

and fulfill

$$\{\boldsymbol{Q}_{BRST}, \boldsymbol{Q}_{BRST}\}_{EPB} = \{\boldsymbol{Q}_{\overline{BRST}}, \boldsymbol{Q}_{\overline{BRST}}\}_{EPB} = \{\boldsymbol{Q}_{BRST}, \boldsymbol{Q}_{\overline{BRST}}\}_{EPB} = 0. \tag{5.207}$$

Conclusions

After the discussion so far, one can now come back to the original question if a formally unified way to describe physics is possible. The most obvious formal break between classical and quantum physics is that classical physics is described on the phase space while quantum mechanics is described on a Hilbert space. The bosonic star product formalism gives here an alternative approach that allows to formulate also quantum mechanics on the phase space. Although the star product formalism is not always optimal for practical calculations and there are still many open questions, it nevertheless shows a way to overcome this first formal break between classical and quantum physics.

The second formal break between classical and quantum physics is that one uses in classical mechanics a vector formalism, while in quantum physics one actually is using a multivector formalism. This formal break is not so well noticed, because the multivectors in quantum mechanics are described as tuples in additional representation spaces. Closely related to this is the question which status fermionic degrees of freedom have. Conventionally fermionic degrees of freedom are connected with quantum mechanics, but they nevertheless appear also in classical physics for example in the classical BRST-formalism and in pseudoclassical mechanics, where they can be used to describe spin. Geometric algebra gives here a natural and confined picture that is interesting in several ways. Firstly the Clifford calculus of geometric algebra can be formulated as a Grassmann calculus that is deformed with a fermionic star product. So Grassmann variables that play in geometric algebra the role of basis vectors are no longer introduced a posteriori and ad hoc, but in a natural and geometric manner. Furthermore geometric algebra is a multivector formalism that can be used in classical and quantum mechanics. Secondly by using a fermionic star product to describe geometric algebra one obtains a fermionic counterpart of the bosonic star product formalism, which means that geometry and quantum mechanics are formulated in the same formalism. Thirdly one can combine the bosonic and the fermionic star product formalism into a noncommutative version of geometric algebra. By doing so one realizes that spin terms are naturally generated by the noncommutativity. The bosonic star product gives in this way a natural transition from a classical to a quantum geometry.

So the central question is which role Grassmann variables actually play. Conventionally Grassmann variables are introduced ad hoc and a posteriori, because they are needed or just because it is possible to introduce them. In this way they represent a sort of mathematical epicycle. But this is exactly the opposite of what Grassmann originally intended with his theory of extension. Grassmanns program, that is beautifully described in [78], was to constitute a unified geometric calculus. With the work of Clifford this program was completed for the flat space. (To be historically correct one has to state, that Grassmann himself found the Clifford product even before Clifford, but did not realize its fundamental importance.) But due to unfortunate historical circumstances the early form of geometric algebra was not applied in physics, instead the tuple formalism was established by Gibbs and Heavyside in the beginning of the 20th century. The tuple formalism does not include the algebraic structures Grassmann and Clifford already found. Unfortunately theses structures play a central role if one wants to describe curvature or noncommutativity and so they had to be reintroduced supplementary. In the historical process this did not happen systematically so that several lines of development emerged. The Grassmann algebra was for example reintroduced in geometry by Cartan through his calculus of differential forms, which is a homogenous multivector formalism. Later on Kähler generalized the calculus of differential forms into an inhomogenous multivector formalism by introducing a Clifford structure, which led to a formalism that is equivalent to geometric algebra [82]. The opposite direction of development was initiated by Dirac who discovered the Clifford structure in physics. Shortly afterwards Fock and Ivanenko generalized the Dirac calculus to curved spaces and constructed the Grassmann product as the antisymmetrized matrix product of Dirac matrices [52]. This formalism, that is also equivalent to geometric algebra, was later on very effectively used in general relativity [102]. Moreover the spinor calculus, that was introduced into physics by Pauli and later on generalized by Penrose to spinor and twistor geometry, can also be subsumed under geometric algebra [55].

The description of geometric algebra as a deformed Grassmann algebra gives now also the possibility to interpret the program of supersymmetry and BRST-symmetry from a Clifford calculus point of view. The fermionic basis vectors and their algebraic structure become here apparent if one factorizes functions that are defined on the space spanned by these basis vectors. For example the factorization of a Hamilton function into supercharges exhibits the basis vector structure of the phase space and the Clifford star-anticommutator is then nothing else than the scalar product on this space. Furthermore one can transform on the one hand actively the bosonic coefficients and on the other hand the passive basis vectors. Since these transformations are related one can also find transformations that relate bosonic and fermionic degrees of freedom. If the parallelism of bosonic and fermionic star product structures is founded in geometry one can turn the logic around and demand such supersymmetric star product structures for a physical theory. This implies in the case of relativistic quantum mechanics the postulate of a four dimensional Moyal product, which leads to the quantum proper time formalism. So the star product formalism overcomes here naturally a further formal break between classical and quantum mechanics, namely that the classical relativistic Hamilton formalism is manifest covariant, while in Dirac theory time plays a special role. So far the results can be summarized in the following table:

Classical Physics	Quantum Physics	Formal Synthesis
Phase Space	Hilbert Space	Bosonic Star Products
Vector Formalism	Multivector Formalism	Fermionic Star Products
Manifest Covariance	Covariance	Supersymmetric Star Products

In the prefrace of his book [70] Grassmann stated “I am aware that the form which I have given the science is imperfect.” and he went on to say “there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into a living communication with contemporary developments.” This prediction indeed proofed to be true when more than hundred years later Hestenes and Kaehler resumed Grassmanns program in the context of Dirac theory. Today geometric algebra is applied in a great variety of areas [34] and this work was a further step in this program. Future research will show in how far geometric algebra can be seen as the basis of supersymmetric structures.

Appendix A

In this appendix it will be shown that the representation (2.215) fulfills axiom (2.208c), i.e. $(uv)\underset{B}{\lrcorner}w = u\underset{B}{\lrcorner}(v\underset{B}{\lrcorner}w)$. Without restriction of generality one chooses $u = \theta_1 \ldots \theta_r$, $v = \theta_{r+1} \ldots \theta_s$ and $w = \theta_{i_1} \ldots \theta_{i_t}$ with $t \geq s$. Using the abbreviations $B(\theta_i, \theta_j) = B_{i,j}$ and $\partial_{\theta_i} = \partial_i$ it follows for the left hand side:

$$
\begin{aligned}
(uv)\underset{B}{\lrcorner}w &= uv\frac{1}{s!}\left(\sum_{i,j} B_{i,j}\overleftarrow{\partial}_i\vec{\partial}_j\right)^s w \\
&= \theta_i \ldots \theta_s \left(\sum_{\sigma \in S_{s,t}} B_{1,i_{\sigma(1)}} \cdots B_{s,i_{\sigma(s)}} (\overleftarrow{\partial}_1 \vec{\partial}_{i_{\sigma(1)}}) \cdots (\overleftarrow{\partial}_s \vec{\partial}_{i_{\sigma(s)}})\right) \theta_{i_1} \ldots \theta_{i_t} \\
&= \theta_i \cdots \theta_s \left(\sum_{\sigma \in S_{s,t}} B_{1,i_{\sigma(1)}} \cdots B_{s,i_{\sigma(s)}} \overleftarrow{\partial}_1 \cdots \overleftarrow{\partial}_s \vec{\partial}_{i_{\sigma(s)}} \cdots \vec{\partial}_{i_{\sigma(1)}}\right) \theta_{i_1} \ldots \theta_{i_t} \\
&= (-1)^{s(s-1)/2} \sum_{\sigma \in S_{s,t}} B_{1,i_{\sigma(1)}} \cdots B_{s,i_{\sigma(s)}} \vec{\partial}_{i_{\sigma(s)}} \cdots \vec{\partial}_{i_{\sigma(1)}} \theta_{i_1} \ldots \theta_{i_t},
\end{aligned} \tag{A.1}
$$

where $S_{s,t}$ is the set of all permutations of s elements out of t.

The right hand side of (2.208c) leads to:

$$
\begin{aligned}
u\underset{B}{\lrcorner}(v\underset{B}{\lrcorner}w) &= u\underset{B}{\lrcorner}\left[\theta_{r+1} \cdots \theta_s \left(\sum_{\sigma \in S_{s-r,t}} B_{r+1,i_{\sigma(r+1)}} \cdots B_{s,i_{\sigma(s)}} \overleftarrow{\partial}_{r+1} \cdots \overleftarrow{\partial}_s \vec{\partial}_{i_{\sigma(s)}} \cdots \vec{\partial}_{i_{\sigma(r+1)}}\right) \theta_{i_1} \cdots \theta_{i_t}\right] \\
&= u\underset{B}{\lrcorner}\left[(-1)^{(s-r)(s-r-1)/2} \sum_{\sigma \in S_{s-r,t}} B_{r+1,i_{\sigma(r+1)}} \cdots B_{s,i_{\sigma(s)}} \vec{\partial}_{i_{\sigma(s)}} \cdots \vec{\partial}_{i_{\sigma(r+1)}} \theta_{i_1} \ldots \theta_{i_t}\right] \\
&= (-1)^{(s-r)(s-r-1)/2} \sum_{\sigma \in S_{s-r,t}} B_{r+1,i_{\sigma(r+1)}} \cdots B_{s,i_{\sigma(s)}} \theta_1 \cdots \theta_r \\
&\quad \times \left[\sum_{\sigma' \in S_{r,t}} B_{1,i_{\sigma'(1)}} \cdots B_{r,i_{\sigma'(r)}} \overleftarrow{\partial}_1 \cdots \overleftarrow{\partial}_r \vec{\partial}_{i_{\sigma'(r)}} \cdots \vec{\partial}_{i_{\sigma'(1)}}\right] \vec{\partial}_{i_{\sigma(s)}} \cdots \vec{\partial}_{i_{\sigma(r+1)}} \theta_{i_1} \cdots \theta_{i_t} \\
&= (-1)^{[(s-r)(s-r-1)+r(r-1)]/2} \sum_{\substack{\sigma \in S_{s-r,t} \\ \sigma' \in S_{r,t}}} B_{1,i_{\sigma'(1)}} \cdots B_{r,i_{\sigma'(r)}} B_{r+1,i_{\sigma(r+1)}} \cdots B_{s,i_{\sigma(s)}} \\
&\qquad \times \vec{\partial}_{i_{\sigma'(r)}} \cdots \vec{\partial}_{i_{\sigma'(1)}} \vec{\partial}_{i_{\sigma(s)}} \cdots \vec{\partial}_{i_{\sigma(r+1)}} \theta_{i_1} \cdots \theta_{i_t} \\
&= (-1)^{s(s-1)/2} \sum_{\substack{\sigma \in S_{s-r,t} \\ \sigma' \in S_{r,t}}} B_{1,i_{\sigma'(1)}} \cdots B_{r,i_{\sigma'(r)}} B_{r+1,i_{\sigma(r+1)}} \cdots B_{s,i_{\sigma(s)}} \\
&\qquad \times \vec{\partial}_{i_{\sigma(s)}} \cdots \vec{\partial}_{i_{\sigma(r+1)}} \vec{\partial}_{i_{\sigma'(r)}} \cdots \vec{\partial}_{i_{\sigma'(1)}} \theta_{i_1} \cdots \theta_{i_t} \\
&= (-1)^{s(s-1)/2} \sum_{\sigma \in S_{s,t}} B_{1,i_{\sigma(1)}} \cdots B_{s,i_{\sigma(s)}} \vec{\partial}_{i_{\sigma(s)}} \cdots \vec{\partial}_{i_{\sigma(1)}} \theta_{i_1} \cdots \theta_{i_t},
\end{aligned} \tag{A.2}
$$

which is the same result as the left hand side. In the last step one uses that a term in the sum is zero if $\sigma(i) = \sigma'(j)$ because of the fermionic character of the derivatives.

Appendix B

To prove that $u \circ v$ can be written as an exponential function one decomposes the monomials $u = a^m \bar{a}^n$ and $v = a^r \bar{a}^s$ with the coproduct (3.67). Then the definition of the twisted product (3.63) leads to

$$u \circ v = \sum_{i,j,k,l=0}^{m,n,r,s} \binom{m}{i}\binom{n}{j}\binom{r}{k}\binom{s}{l} \mathcal{R}(a^i \bar{a}^j, a^k \bar{a}^l) a^{m+r-i-k} \bar{a}^{n+s-j-l}. \tag{B.1}$$

In this sum the $\mathcal{R}(a^i \bar{a}^j, a^k \bar{a}^l)$ are non-vanishing only if $i + j = k + l$. One includes this condition by setting

$$i = p_1 + p_3,\ j = p_2 + p_4,\ k = p_1 + p_4 \quad \text{and} \quad l = p_2 + p_3, \tag{B.2}$$

so that $i + j = k + l = p_1 + p_2 + p_3 + p_4$. Hence the sum over i, j, k, l is replaced by:

$$\sum_{i,j,k,l=0}^{m,n,r,s} \quad \rightarrow \quad \sum_{q=0}^{m+n=r+s} \sum_{p_1+p_2+p_3+p_4=q} . \tag{B.3}$$

This substitution alone would lead to multiple counting, for example the term $i = j = k = l = 1$ in (B.1) can be written with $p_1 = p_2 = 1$, $p_3 = p_4 = 0$ and with $p_1 = p_2 = 0$, $p_3 = p_4 = 1$. But the multiple counting is taken into account in the calculation of $\mathcal{R}(a^i \bar{a}^j, a^k \bar{a}^l)$. To see this one separates p_1 factors from a^i and a^k and p_2 factors from $\bar{a}^j$ and $\bar{a}^l$:

$$a^i \bar{a}^j = \overbrace{a \cdots a}^{p_1}\, \overbrace{a \cdots \cdot a}^{p_3}\, \overbrace{\bar{a} \cdots \bar{a}}^{p_2}\, \overbrace{\bar{a} \cdots \cdot \bar{a}}^{p_4} \tag{B.4}$$

$$a^k \bar{a}^l = \underbrace{a \cdots a}_{p_1}\, \underbrace{\bar{a} \cdots \cdot \bar{a}}_{p_3}\, \underbrace{\bar{a} \cdots \bar{a}}_{p_2}\, \underbrace{a \cdots \cdot a}_{p_4}. \tag{B.5}$$

Different values for p_1 and p_2, that fulfill the condition (B.2) lead to different separations of $\mathcal{R}(a^i \bar{a}^j, a^k \bar{a}^l)$ into the four basic coquasitriangular structures, but how often does such a separation appear? There are $\binom{p_1+p_3}{p_1}\binom{p_2+p_4}{p_4}$ ways to separate p_1 factors from a^i and p_2 factors from $\bar{a}^j$; secondly the permutation of the a's and $\bar{a}$'s leads to a factor $(p_1 + p_4)!(p_2 + p_3)!$. Then the coquasitriangular structure can be written as:

$$\mathcal{R}(a^i \bar{a}^j, a^k \bar{a}^l) = \sum_{\substack{p_1,p_2=0 \\ (B.2)}}^{i,j} \binom{p_1+p_3}{p_1}\binom{p_2+p_4}{p_2} (p_1+p_4)!(p_2+p_3)! \mathcal{R}(a,a)^{p_1} \mathcal{R}(\bar{a},\bar{a})^{p_2} \mathcal{R}(a,\bar{a})^{p_3} \mathcal{R}(\bar{a},a)^{p_4}. \tag{B.6}$$

The sum in (B.6) corresponds exactly to the multiple counting, so that (B.1) can be written as:

$$\begin{aligned}
u \circ v &= \sum_{q=0}^{m+n=r+s} \sum_{p_1+p_2+p_3+p_4=q} \frac{\mathcal{R}(a,a)^{p_1} \mathcal{R}(\bar{a},\bar{a})^{p_2} \mathcal{R}(a,\bar{a})^{p_3} \mathcal{R}(\bar{a},a)^{p_4}}{p_1! p_2! p_3! p_4!} \\
&\qquad \frac{m!\, a^{m-p_1-p_3}}{(m-p_1-p_3)!} \frac{n!\, \bar{a}^{n-p_2-p_4}}{(n-p_2-p_4)!} \frac{r!\, a^{r-p_1-p_4}}{(r-p_1-p_4)!} \frac{s!\, \bar{a}^{s-p_2-p_3}}{(s-p_2-p_3)!} \\
&= a^m \bar{a}^n \Bigg[\sum_{q=0}^{\infty} \frac{1}{q!} \sum_{p_1+p_2+p_3+p_4=q} \binom{q}{p_1, p_2, p_3, p_4} \\
&\qquad (\mathcal{R}(a,a) \overleftarrow{\partial}_a \overrightarrow{\partial}_a)^{p_1} (\mathcal{R}(\bar{a},\bar{a}) \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_{\bar{a}})^{p_2} (\mathcal{R}(a,\bar{a}) \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}})^{p_3} (\mathcal{R}(\bar{a},a) \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a)^{p_4} \Bigg] a^r \bar{a}^s \\
&= u\, e^{(\mathcal{R}(a,a) \overleftarrow{\partial}_a \overrightarrow{\partial}_a + \mathcal{R}(\bar{a},\bar{a}) \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_{\bar{a}} + \mathcal{R}(a,\bar{a}) \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} + \mathcal{R}(\bar{a},a) \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a)}\, v.
\end{aligned}$$

Here the symbols

$$\binom{q}{p_1, p_2, p_3, p_4} = \frac{q!}{p_1! p_2! p_3! p_4!} \tag{B.7}$$

are the multinomial coefficients which occur in the multinomial formula

$$(a_1 + \cdots + a_n)^q = \sum_{p_1 + \cdots + p_n = q} \binom{q}{p_1, \cdots, p_n} a_1^{p_1} \cdots a_n^{p_n}. \tag{B.8}$$

In the fermionic case one has to prove, analogously, that

$$u \circ v = \sum \mathcal{R}(u_{(1)}, v_{(1)}) u_{(2)} v_{(2)} = u \, \exp \left[\sum_{i,j} \mathcal{R}(f_i, f_j) \overleftarrow{\partial}^L_{f_i} \vec{\partial}^L_{f_j} \right] v \tag{B.9}$$

for $u = f_{i_1} \cdots f_{i_r}$ and $v = f_{j_1} \cdots f_{j_s}$. In the p-th order one combines the coproduct terms of the permutations σ for u and $\tilde{\sigma}$ for v that lead to

$$\begin{aligned}
& \sum_{\substack{\sigma \in P_{p,r} \\ \tilde{\sigma} \in P_{p,s}}} (-1)^{\sigma} (-1)^{\tilde{\sigma}} \mathcal{R}(f_{\sigma(i_1)} \cdots f_{\sigma(i_p)}, f_{\tilde{\sigma}(j_1)} \cdots f_{\tilde{\sigma}(j_p)}) f_{\sigma(i_{p+1})} \cdots f_{\sigma(i_r)} f_{\tilde{\sigma}(j_{p+1})} \cdots f_{\tilde{\sigma}(j_s)} \\
= & \sum_{\substack{\sigma \in P_{p,r}, \tilde{\sigma} \in P_{p,s} \\ \sigma' \in P}} (-1)^{\sigma + \tilde{\sigma} + \sigma'} \mathcal{R}(f_{\sigma(i_1)}, f_{\sigma'(\tilde{\sigma}(j_1))}) \cdots \mathcal{R}(f_{\sigma(i_p)}, f_{\sigma'(\tilde{\sigma}(j_p))}) \\
& \qquad f_{\sigma(i_{p+1})} \cdots f_{\sigma(i_r)} f_{\tilde{\sigma}(j_{p+1})} \cdots f_{\tilde{\sigma}(j_s)} \\
= & \sum_{\sigma, \tilde{\sigma}, \sigma'} (-1)^{\sigma + \tilde{\sigma} + \sigma'} \mathcal{R}(f_{\sigma(i_1)}, f_{\sigma'(\tilde{\sigma}(j_1))}) \cdots \mathcal{R}(f_{\sigma(i_p)}, f_{\sigma'(\tilde{\sigma}(j_p))}) \\
& \qquad f_{\sigma(i_1)} \cdots f_{\sigma(i_r)} \overleftarrow{\partial}^L_{f_{\sigma(i_1)}} \cdots \overleftarrow{\partial}^L_{f_{\sigma(i_p)}} \vec{\partial}^L_{f_{\tilde{\sigma}(j_p)}} \cdots \vec{\partial}^L_{f_{\tilde{\sigma}(j_1)}} f_{\tilde{\sigma}(j_1)} \cdots f_{\tilde{\sigma}(j_s)} \\
= & \sum_{\sigma, \tilde{\sigma}, \sigma'} \mathcal{R}(f_{\sigma(i_1)}, f_{\sigma'(\tilde{\sigma}(j_1))}) \cdots \mathcal{R}(f_{\sigma(i_p)}, f_{\sigma'(\tilde{\sigma}(j_p))}) \\
& \qquad f_{i_1} \cdots f_{i_n} \overleftarrow{\partial}^L_{f_{\sigma(i_1)}} \vec{\partial}^L_{f_{\sigma'(\tilde{\sigma}(j_1))}} \cdots \overleftarrow{\partial}^L_{f_{\sigma(i_p)}} \vec{\partial}^L_{f_{\sigma'(\tilde{\sigma}(j_p))}} f_{j_1} \cdots f_{j_s} \\
= & \frac{1}{p!} f_{i_1} \cdots f_{i_r} \left(\sum_{i,j} \mathcal{R}(f_i, f_j) \overleftarrow{\partial}^L_{f_i} \vec{\partial}^L_{f_j} \right)^p f_{j_1} \cdots f_{j_s},
\end{aligned} \tag{B.10}$$

where one uses the relation (3.80) for the fermionic coquasitriangular structure. The expression (B.10) is just the p-th order term in the exponential form.

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