UNIVERSITY OF DORTMUND

REIHE COMPUTATIONAL INTELLIGENCE

COLLABORATIVE RESEARCH CENTER 531

Design and Management of Complex Technical Processes and Systems by means of Computational Intelligence Methods

A Framework of Quantum-inspired Multi-Objective Evolutionary Algorithms and its Convergence Properties

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No. CI-228/07

Technical Report ISSN 1433-3325 April 2007

Secretary of the SFB 531 · University of Dortmund · Dept. of Computer Science/LS 2
44221 Dortmund · Germany

This work is a product of the Collaborative Research Center 531, "Computational Intelligence," at the University of Dortmund and was printed with financial support of the Deutsche Forschungsgemeinschaft.

A Framework of Quantum-inspired Multi-Objective Evolutionary Algorithms and its Convergence Properties

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Abstract. In this paper, a general framework of quantum-inspired multi-objective evolutionary algorithms is proposed based on the basic principles of quantum computing and general schemes of multi-objective evolutionary algorithms. One of the sufficient convergence conditions to Pareto optimal set is presented and it is proved under partially order set theory. Moreover, two algorithms are given as examples meeting this convergence condition, in which two improved Q-gates are used. Their convergence properties are discussed. Additionally, one counterexample is also given.

Keywords: Quantum computing, multi-objective evolutionary algorithms, Pareto optimal set, stochastic convergence.

1 Introduction

Many optimization problems in scientific and engineering fields involve simultaneously two or more objectives that are competing or in conflict with each other frequently. They are known as multi-objective optimization problems (MOP). Ordinary MOPs have a set of optimal solutions, which is called Pareto solutions set[1, 2]. The plot of the objective functions whose vectors of the decision variables are in the Pareto solutions set is called the Pareto front[1, 2].

As the capability of searching simultaneously whole of solution spaces using a population of feasible solutions based on stochastic mechanisms, evolutionary algorithms have more advantage in dealing with discontinuous and concave Pareto fronts than traditional mathematical programming techniques. A large number of multi-objective evolutionary algorithms (MOEA) that employ some innovative mechanisms have been proposed during the last two decades, such as , MOGA[3], NPGA[4], NSGA[5], SPGA[6], NSGA2[7], SPEA2[8] etc. Some important theoretical work related to MOEA has been done. Rudolph has investigated convergence properties of some MOEAs under partially ordered finite set theory[9, 10]. Hanne presented an evolutionary algorithm for approximating the efficient set of MOP [11].

Meanwhile, the quantum mechanical computational theory is attracting serious attention since their remarkable superiority was demonstrated by several quantum algorithms during the last 15 years, such as Shor's quantum factoring algorithm [12] and Grover's database search algorithm [13]. Integrating the quantum computing mechanisms and classical evolutionary algorithms, some quantum-inspired evolutionary algorithms (QEA) were proposed in[14-18], which are characterized by some quantum mechanics such as uncertainty, superposition, interference etc. In last two years, some specific algorithms combining MOEA with QEA, which are called quantum-inspired multi-objective evolutionary algorithms (QMOEA) in this paper, were proposed to solve the multi-objective knapsack problem (MOKP)[19, 20]. Those experiments results show better proximity performance as well as diversity maintenance. The theoretical analysis, such as the convergence properties, may be significative to design and assess QMOEAs. However, few theoretical results on the QMOEA have been done.

In this paper, we will propose a general framework of QMOEA, and discuss its sufficient convergence conditions to the Pareto optimal set and give several example algorithms. Therefore section 2 recalls some preliminary material on partially ordered set, section 3 presents a general framework of QMOEA, section 4 gives two convergent algorithms and a counterexample, their convergence properties are discussed in this section, section 5 presents a conclusion.

2 Preliminaries

MOP can be defined with a mathematical formulation as follows:

$$\text{optimum } f(\boldsymbol{x})=(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}),\ldots f_d(\boldsymbol{x}))^T \quad \text{s.t. } \boldsymbol{x} \in \boldsymbol{S} \qquad (2.1)$$

where f: $\boldsymbol{S} \rightarrow \boldsymbol{R}^d$ is a vector-valued objective function, $\boldsymbol{x} = [x_1,x_2,\ldots,x_k]^T$ is the vector of decision variables and the $\boldsymbol{S}$ is the feasible set, which is usually defined by some constraint functions,

$$S = \{x \in R^k : g_i(x) \le 0, h_j(x) = 0; i = 1,2...p, j = 1,2...q\}. \qquad (2.2)$$

Without loss of generality, let "optimum" mean "minimum" here. We say that a solution to a MOP is Pareto optimal if there exists no other feasible solution which would decrease some criteria without causing a simultaneous increase in at least one other criterion. The set comprising all of Pareto optimal solutions is just the *Pareto optimal set*, which is the goal of multi-objective optimization algorithms (MOA). Generally the image set of all feasible solutions in a MOP does not constitute a totally ordered set, instead a partially ordered set because of multi-criteria evaluation. The theoretical background on the partially ordered set in this paper roots in [9, 21]. Some of the basic definitions and theorem as are as follows.

Let F be a set., we can define a *partial order relation* "$\le$" which is a reflexive, antsymmetric and transitive relation on F, and a *strict partial order relation* "$<$" as an antireflexive, asymmetric and transitive relation which may be obtained by the former relation by setting $x < y := (x \le y) \wedge (x \ne y)$.

Definition 2.1 *Let F be some set. If the partial order relation "≤" is valid on F then the pair (F, ≤) is called a* **partially ordered set** *(or short:* **poset***). If x < y for some x,*

$y \in F$ then x is said to **dominate** *y. Distinct points x, $y \in F$ are said to be* **comparable** *when x < y, y < x or x = y. Otherwise, x and y are* **incomparable** *which is denoted by x || y. If each pair of distinct points of a poset (F, ≤) is comparable then (F, ≤) is called a* **totally ordered set** *or a* **chain**. *Dually, if each pair of distinct points of a poset (F, ≤) are incomparable then (F, ≤) is termed an* **antichain**. *An element $x^* \in F$ is called a* **minimal element** *of the poset (F, ≤) if there is no $x \in F$ such that $x < x^*$. The set of all minimal elements, denoted M (F, ≤), is said to be* **complete** [9, 21] *if for each $x \in F$ there is at least one $x^* \in M(F, \leq)$ such that $x^* \leq x$.*

If the poset $(F, \leq)$ is finite then the completeness of $M(F, \leq)$ is guaranteed [9]. Let $f: X \rightarrow F$ be a mapping from some set X to the poset $(F, \leq)$. For some $A \subseteq X$ the set $M_f(A, \leq) = \{a \in A: f(a) \in M(f(A), \leq)\}$ contains those elements from A whose images are minimal elements in the image space $f(A) = \{f(a): a \in A\}$. In order to clarify the notion of "stochastic convergence to the set of minimal elements" we need measures on the distances between finite point sets. Here the first measure is characterized as follows: If A and B are subset of a finite ground set X then $d(A,B) = |A \cup B| - |A \cap B|$ is a metric on the power set of X. the second measure uses quantity $\delta_B(A) = |A| - |A \cap B|$ counting the number of elements that are in set A but not in set B.

Definition 2.2 *Let A_t be a solutions set of a MOA at iteration $t \geq 0$ and $F_t = f(A_t)$ its associated image set, F^* denotes* the set of minimal elements*. The algorithm is said to* converge with probability 1 to the entire set of minimal elements *if*

$$d(F_t, F^*) \rightarrow 0 \text{ with probability 1 as } t \rightarrow \infty. \quad (2.3)$$

And the algorithm is said to converge with probability 1 to the set of minimal elements *if*

$$\delta_{F^*}(F_t) \rightarrow 0 \text{ with probability 1 as } t \rightarrow \infty. \quad (2.4)$$

Needless to say, $d(F_t, F^*) \rightarrow 0$ implies $\delta_{F^*}(F_t) \rightarrow 0$.

3 The Basic Principles and the General Framework of Quantum-inspired Multi-Objective Evolutionary Algorithms

A few researchers have proposed some QMOEAs that are mainly based on a particular MOEAs, such as Kim, Kim and Han's QMEA based on the NSGA2 in [20] and Meshoul, Mahdi and Batouche's algorithm based on SPEA2 in [19]. Here we present a new general QMOEA framework, which is based on the basic principles of QEA and the general schemes of MOEA.

3.1 The basic principles of quantum-inspired evolutionary algorithm

A. Q-bits' Chromosome Representation and Q-individual

The individuals' chromosomes in QEA utilize Q-bits representation which is a kind of probabilistic representation. *Q-bit* (or *qubit*) is abstraction of quantum bit. It is the smallest unit of information in QEA, which is defined with a pair of numbers (α,

β) [22]. Consequently, an individual's chromosome q can be defined as m Q-bits string

$$q = \begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_m \\ \beta_1 & \beta_2 & \ldots & \beta_m \end{pmatrix}$$

where $|\alpha_i|^2 + |\beta_i|^2 = 1, i = 1,2,\ldots m$. In this paper we call this kind of individual as *Q-individual*. This quantum representation mechanism has the advantage to represent a linear superposition of states. All possible combinations of decision variables values can be derived from a single Q-individual.

In fact, there are other similar probabilistic chromosome presentations, such as

$$q = (p_1 \quad p_2 \quad \ldots \quad p_m), 0 \le p_i \le 1, i = 1,2,\ldots m \,.$$

Here p_i denotes the probability that the ith Q-bit is in '0' state. However, here we use the former presentation because it accord with the Q-gates operators in better way. The Q-gates operators are developed from the Walsh-Hadamard transform and Hadamard gate used by the physical quantum computing theory [23].

B. Q-population and observing population

For more diversity, QEA maintains a population of Q-individuals, called *Q-population* in this paper, using $Q(t)=\{q_1^t, q_2^t, \ldots q_n^t\}$ at each generation t of the evolutionary iterative process where n is the size of population and q_j^t is a Q-individual defined as above text.

A quantum operator called *observing* is applied in order to obtain feasible solutions in QEA. This operator makes a population of binary solutions, $P(t)=\{x_1^t, x_2^t, \ldots x_n^t\}$, which is called *observing population* in the present paper. Each component x_j^t , $j=1,2,\ldots n$ is a length m binary string which is formed by selecting either 0 or 1 for each bit by using the probability either $|\alpha_i|^2$ or $|\beta_i|^2$, $k=1,2,\ldots m$ of q_j^t, respectively.

C.Uupdating Q-individual and Q-gate

In QEA process Q-individuals can be updated by applying a variety of Q-gate operators, by which the updated Q-bit with a new pair of number (α', β') should satisfy the normalization condition, $|\alpha'|^2 + |\beta'|^2 = 1$. The rotation gate acting on a single Q-bit is the basic Q-gate in QEA as follows:

$$R(\Delta\theta) = \begin{pmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) \\ \sin(\Delta\theta) & \cos(\Delta\theta) \end{pmatrix}$$

where $\Delta\theta$ is a rotation angle toward either 0 or 1 state depending on its objective sign. As the rotation gate is applied, a correlative binary individual to each Q-individual, which is called an *objective solution*, is often appointed in advance. The objective sign to each bit of a Q-individual is defined as the corresponding bit of the correlative objective binary individual, respectively. After the rotation gate $R(\Delta\theta)$ acting on a Q-bit (α, β), the updated Q-bit (α', β') satisfy $(\alpha', \beta')^T = R(\Delta\theta) \cdot (\alpha, \beta)^T$. Here $\Delta\theta$ should be designed in compliance with the application problem and each Q-bit possibly matches with different angles. Several rotation gate strategies can be referred in [15, 16, 24]. Here we give a simple $\Delta\theta$ strategy which roots in [24] as follows.

Let the observing individual of q_j^t be $x = \{x^1x^2\ldots x^m\}$ and the objective individual of q_j^t be $b = \{b^1b^2\ldots b^m\}$, where x and b are binary strings. If the $x^k = 0$, $b^k = 1$ and b dominates x then $\Delta\theta = \theta_0$. If the $x^k = 1$, $b^k = 0$ and b dominate x then $\Delta\theta = -\theta_0$. Otherwise, $\Delta\theta = 0$. Here $0<\Delta\theta<0.25\pi$ and the values from 0.001π to 0.05π are recommended for the magnitude of $\Delta\theta$.

Moreover NOT gate and H_ε gate are other two operator. The function of the former is to exchange the probabilities of '0' state and '1' state in the Q-bit. It can be defined as a transformation matrix $N = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The latter is extended from the rotation gate and was proposed by Han and Kim in [18]. If acted by H_ε gate, a Q-bit (α, β) would be updated as $(\alpha', \beta')^T = H_\varepsilon(\alpha, \beta, \Delta\theta)$, where for $(\alpha'', \beta'')^T = R(\Delta\theta)(\alpha, \beta)^T$

a) if $|\alpha''|^2 \le \varepsilon$ and $|\beta''|^2 \ge 1-\varepsilon$ then $(\alpha', \beta')^T = (\mathrm{sgn}(\alpha'')\cdot\sqrt{\varepsilon}, \mathrm{sgn}(\beta'')\cdot\sqrt{1-\varepsilon})^T$;

b) if $|\alpha''|^2 \ge 1-\varepsilon$ and $|\beta''|^2 \le \varepsilon$ then $(\alpha', \beta')^T = (\mathrm{sgn}(\alpha'')\cdot\sqrt{1-\varepsilon}, \mathrm{sgn}(\beta'')\cdot\sqrt{\varepsilon})^T$;

c) otherwise $(\alpha', \beta')^T = (\alpha'', \beta'')^T$.

Here $0 < \varepsilon << 1$.

3.2 The General Framework of Quantum-inspired Multi-objective Evolutionary Algorithms

The algorithms for MOP have two main goals in the iterative process: making current solutions as close as possible to the Pareto front and as diverse as possible. A number of good techniques have been used in order to improve MOEAs, some of them are so successful that they have become general schemes, such as nondominated rank sorting and selection, maintaining solutions diversity and reserving elitism solutions as an external population etc[25]. Integrating the basic principle of QEA and general schemes of MOEA, we propose a general framework of quantum-inspired multi-objective evolutionary algorithms as follows:

The Procedure of the QMOEAs' Basic Framework

```
    Begin
      t ← 0
i)    Initialize Q(t)
ii)   A(t) ={ }, C(t) = { }
iii)  While (not termination condition) do
          t ← t+1
iv)       Make P(t) by observing the state of Q(t-1)
v)        Evolve P(t)                 \\ Sometimes this step can be omitted.
vi)       Make C(t) = M_f( P(t) ∪ C(t-1), ≤ )    \\ Normally this step is eliminated.
vii)      Rebuild the archive set A(t);
          \\ Here A(t) ⊆ M_f( P(t) ∪ A(t-1), ≤ ) and maximize the diversity of
             those chosen elements in A(t).
viii)     Make Q(t) by updating Q(t-1) on Q-gates
    End
```

End

i) ~ ii) First the two external archive set $A(t)$, $C(t)$ and the Q-population $Q(t)$ are initialized. Set $A(0) = \phi$, $C(0) = \phi$. Make $Q(0) = \{q_j^0, j=1,2,\ldots n\}$, where each Q-bit in q_j^0 have the identical probability of '0' state and '1' state. In other word, each Q-bit of q_j^0 can be presented as $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$.It means that one Q-individual q_j^0 represents the linear superposition of all possible states with the same probability.

iii) Until the termination condition is satisfied, the QMOEA is running in the while loop.

iv) Binary solutions in $P(t)$ are formed by observing the state of $Q(t)$ as above subsection.

v) $P(t)$ can evolve by using some evolutionary operations, such as simple genetic algorithm, evolutionary strategy etc. In fact, this step is not indispensable; it may be omitted in some QMOEAs.

vi) According to the definition of $C(t) = M_f(P(t) \bigcup C(t\text{-}1))$, $C(t)$ consists of all nondominated solutions in $\bigcup_{t_1=1}^{t} P(t_1)$. Here some efficient techniques can be used, such as the fast nondominated sorting method which was proposed in NSGA2 [7]. The size of $C(t)$ will continually grow along with the iteration cycles. Since the size may be too huge, this step usually is not adopted in practice.

vii) As $A(\mathrm{t}) \subseteq M_f(P(t) \bigcup A(t\text{-}1), \leq)$, all elements of $A(t)$ are the nondominated solutions in $\bigcup_{t_1=1}^{t} P(t_1)$. Unlike $C(t)$, $A(t)$ is the archive set, its size is usually changeless. In order to maximize its diversity, some techniques can be used such as crowding-distance [7], entropy [17], clustering [26] etc.

viii) In this step, Q-individuals in $Q(t)$ are updated by applying Q-gates, such as the rotation gate, NOT gate and H_ε gate. When the rotation gate are applied, a correlative solution to each Q-individual, which is called a *objective solution*, is often selected from $A(t)$. Then the objective sign to each bit of a Q-individual is defined as the corresponding bit of the correlative objective solution, respectively.

4 On the Convergence Properties of QMOEA

4.1 One of the Sufficient Convergence Conditions to QMOEA

According to those definitions of MOP and partially ordered set in section 2, we look upon the image space of MOP, $(f(S), \leq)$, as a partially ordered set. The set $M(f(S),\leq)$, a subset of $(f(S),\leq)$, denotes the Pareto optimal set of the MOP. By the construction of the basic framework, $A(t)$ is the archives solutions set. Thus we can define the concept on convergence to the Pareto optimal set as follows.

Definition 4.1 *Let $F^* = M(f(S),\leq)$ and $A(t)$ be the archives solutions set of QMOEA. The QMOEA is said to* converge with probability 1 to the Pareto optimal set *if*

$$\delta_{F^*}(f(A(t))) \to 0 \text{ with probability 1 as } t \to \infty. \qquad (4.1)$$

Proposition 1 One of sufficient conditions by whose the QMOEA converges with probability 1 to its Pareto optimal set is that the set sequence $\{C(t)\}$ satisfy

$$d(f(C(t)), F^*) \to 0 \textit{ with probability } 1 \textit{ as } t \to \infty. \qquad (4.2)$$

where $F^* = M(f(S),\leq)$ is the minimal elements set of the image set $f(S)$.

Proof: Since $C(t) = M_f(P(t) \cup C(t-1)), t > 0$ and $C(0) = \phi$, $A(t) \subseteq M_f(P(t) \cup A(t-1)), t > 0$ and $A(0) = \phi$,

we can attain $A(t) \subseteq C(t)$ and further $f(A(t)) \subseteq f(C(t))$.

Let $S_0 = f(C(t)) - f(A(t))$. We can conclude that

$$\begin{aligned} d(f(C(t)), F^*) &= |f(C(t)) \cup F^*| - |f(C(t)) \cap F^*| \\ &= |f(A(t))| - |f(A(t)) \cap F^*| + |S_0 \cup F^*| - |f(C(t)) \cap F^*| \\ &= \delta_{F^*}(f(A(t))) + |S_0 \cup F^*| - |f(C(t)) \cap F^*| \\ &\geq \delta_{F^*}(f(A(t))). \end{aligned}$$

Since $\delta_{F^*}(f(A(t))) \leq d(f(C(t)), F^*)$, it is clear that if $d(f(C(t)), F^*) \to 0$ *with probability* 1 *as* $t \to \infty$ then $\delta_{F^*}(f(A(t))) \to 0$ with probability 1 as $t \to \infty$. Considering the definition (4.1), we have proved this proposition. ■

Proposition 2 Let S be a feasible solution set of MOP, $s \in S$ be an arbitrary from feasible solution. If the probability $P(s \in P(t))$ is independent each other for different t and there exists a real number $\varepsilon_0, 0<\varepsilon_0<1$, which satisfies $P(s \in P(t)) \geq \varepsilon_0$ for all $s \in S$, all $t > 0$, then $d(f(C(t)), F^*) \to 0$ *with probability* 1 *as* $t \to \infty$, where $F^* = M(f(S),\leq)$ is the minimal elements set of the image set $f(S)$.

Proof: In one 'while loop' of the basic framework the $P(t)$ maybe be changed in v) step. For avoiding the different understanding, the $P(t)$ always denotes its final result in v) step in following text.

First, we describe $d(f(C(t)), F^*) \to 0$ *with probability* 1 *as* $t \to \infty$ with a mathematical limit language as follows:

$\forall \varepsilon_1 \in R, 0 < \varepsilon_1 << 1, \exists N_0 \in N$ such that $P(d(f(C(t)), F^*) = 0) \geq 1 - \varepsilon_1$ for all t > N_0.

Second, we consider the preconditions that can guarantee $d(f(C(t)), F^*) = 0$. The poset $f(S)$ is complete since the feasible set S and its image poset $f(S)$ are finite. Let x be an arbitrary element of $f(S) - F^*$. There exists at least an element $y \in F^*$ and y dominate x. By the definitions of $C(t)$, It is guaranteed that if $y \in f(C(t_0))$ then $x \in f(C(t))$ is impossible for all $t \geq t_0$. Further, if $F^* \subseteq f(C(t_0))$ implies that any element of $f(S) - F^*$ will not stay in $f(C(t))$ for all $t \geq t_0$. In other words, $F^* \subseteq f(C(t_0))$ implies $(f(S) - F^*) \cap f(C(t)) = \phi$ for all $t \geq t_0$. Since $f(C(t))$ is a subset of $f(S)$, we can affirm that if $F^* \subseteq f(C(t_0))$ and $(f(S) - F^*) \cap f(C(t)) = \phi$ then $F^* = f(C(t))$. Hence, it is clear that if $F^* \subseteq f(C(t_0))$ then $d(f(C(t)),F^*) = |f(C(t)) \cup F^*| - |f(C(t)) \cap F^*| = 0$ for all $t \geq t_0$.

Third, we estimate the probability that all element of F^* enter into $f(C(t)$ in $K \times l$ iterations beginning from t_0, $P(F^* \subseteq f(C(t_0 + K \times l)))$, as follows. By construction of the basic QMOEA framework and those definitions of C(t), it is guaranteed the image set $f(C(t))$ is the minimal set of the union set $\bigcup_{t_1=1}^{t} f(P(t_1))$. As soon as an element of F^* has entered $f(P(t_0))$ then it will be saved in $f(C(t_0))$ and then it will stay in $f(C(t))$, $t \geq t_0$, forever.

Let $K = |F^*|$. Without loss of generality, we can assume that all elements of $|F^*|$ are label as $\{s_1, s_2, \ldots, s_K\}$. Taking into account that the probability $P(s \in P(t))$ is independent each other for different t, we can decompose these probability expressions in following inequations. Since there exists a real number ε_0,$0<\varepsilon_0<1$, which satisfies $P(s \in P(t)) \geq \varepsilon_0$ for all $s \in S$, all $t > 0$, we can estimate the probability that an element s_j, $j = 1,2,\ldots,K$, enter into $f(C(t))$ in l iterations beginning from t_0 as follows:

$$
\begin{aligned}
& P\left(s_j \in f(C(t_0 + l)) \middle| s_j \notin f(C(t_0))\right) \\
& = 1 - P\left(s_j \notin f(C(t_0 + l)) \middle| s_j \notin f(C(t_0))\right) \\
& = 1 - P((s_j \notin f(C(t_0))) and (s_j \notin f(P(t_0 + 1))) and (s_j \notin f(P(t_0 + 2))) \\
& \qquad and \ldots and (s_j \notin f(P(t_0 + l)))) \\
& = 1 - P(s_j \notin f(C(t_0))) \cdot P(s_j \notin f(P(t_0 + 1))) \cdot P(s_j \notin f(P(t_0 + 2))) \cdot \ \ldots \cdot P(s_j \notin f(P(t_0 + l))) \\
& \geq 1 - (1 - \varepsilon_0)^l ,
\end{aligned}
$$

where l and t_0 are arbitrary nature number.

Further, we can estimate the probability that all element of F^* enter into $f(C(t)$ in $K \times l$ iterations beginning from t_0 as follows:

$$
\begin{aligned}
& P\left(F^* \subseteq f(C(t_0 + K \times l))\right) \\
& \geq P((s_1 \in P(t_0 + l)) and (s_2 \in P(t_0 + 2l)) and \ \ldots and (s_K \in P(t_0 + K \times l))) \\
& \geq P\left(s_1 \in P(t_0 + l) \middle| s_1 \notin P(t_0)\right) \cdot P\left(s_2 \in P(t_0 + 2l) \middle| s_2 \notin P(t_0 + l)\right) \cdot \\
& \qquad \ldots \cdot P\left(s_K \in P(t_0 + K \times l) \middle| s_K \notin P(t_0 + (K-1) \times l)\right) \\
& \geq (1 - (1 - \varepsilon_0)^l)^K ,
\end{aligned}
$$

where l and t_0 are arbitrary nature number.

Finally, we can sum up the proof by a fit N_0 for arbitrary ε_1 as follows.

Let $N_1 = \log_{1-\varepsilon_0}^{1-(1-\varepsilon_1)^{\frac{1}{K}}}$. $\forall \varepsilon_1 \in R, 0 < \varepsilon_1 << 1$, we set $N_0 \geq K \cdot N_1 + 1$ and $N_0 \in N$. Let $t_0 = 1$ and t > N_0. With all the above conclusions we can conclude as follows:

$$
\begin{aligned}
& P\left(d(f(C(t),F^*) = 0\right) \\
& \geq P\left(F^* \subseteq f(C(t_0 + K \cdot N_1)\right) \\
& \geq (1 - (1 - \varepsilon_0)^{N_1})^K \\
& = 1 - \varepsilon_1 \cdot
\end{aligned}
$$

Summing up: $\forall \varepsilon_0, \varepsilon_1 \in R, 0 < \varepsilon_0, \varepsilon_1 << 1$, $\exists N_0 \in N$ such that $\left(1-(1-\varepsilon_0)^t\right)^{N_1} \geq 1 - \varepsilon_1$ for all $t > N_0$, i.e. it is true that $d(f(C(t)), F^*) \to 0$ *with probability* 1 *as* $t \to \infty$. ■

Considering the proposition 1 and proposition 2, we can immediately conclude the theorem 1 as follows, whose proof is omitted.

Theorem 1 (Sufficient Convergence Condition) Let S be a feasible solution set of MOP. One of the sufficient conditions by whose this QMOEA converges with probability 1 to its Pareto optimal set is that there exists a real number $\varepsilon_0, 0<\varepsilon_0<1$, which satisfies $P(s \in P(t)) \geq \varepsilon_0$ for all $s \in S$, $t > 0$ and $P(s \in P(t))$ is independent from each other for different t.

Remark 1 From theorem 1, we obtain a sufficient convergence condition of QMOEA to the Pareto optimal set. However, it is not indispensable. We can give an example which does not satisfy this sufficient condition but converge to its Pareto optimal set.

Remark 2 By the framework's construction, we know that $A(t) \subseteq C(t)$ would be true no matter what diversity preserving method. From the proof detail for proposition 1, theorem 1 would be true no matter what diversity preserving method to $A(t)$. In other words, the different diversity preserving methods do not change the convergence properties.

4.2 On the Convergence Property of QMOEA with H_ε Gate

The first example algorithm meeting the convergence condition is the MOEA with H_ε gate. The H_ε gate is firstly proposed by Kim and Han in [18] and we have simply described it in subsection (3.1). The procedure of this algorithm is similar with that basic framework but the step v) and vi) are eliminated and the H_ε gate is adopted in step viii) as follows.

The Procedure of the QMOEA with H_ε Gate

Begin
$t \leftarrow 0$
i) Initialize $Q(t)$
ii) $A(t) = \{\}$
iii) While (not termination condition) do
$t \leftarrow t+1$
iv) Make $P(t)$ by observing the state of $Q(t\text{-}1)$
v) Rebuild the archive set A(t);
\\ **Here $A(t) \subseteq M_f(P(t) \bigcup A(t\text{-}1), \leq)$ and maximize the diversity of those chosen elements in $A(t)$.**
vi) Make $Q(t)$ by updating Q(t-1) on H_ε gate
End
End

Theorem 2 The QMOEA with H_ε gate which is defined above converges with probability 1 to its Pareto optimal set.
Proof: Taking into account the definition of H_ε gate, we can conclude

$\exists 0 < \varepsilon < 1$, $\forall q_j^t = \begin{pmatrix} \alpha_{j1}^t & \alpha_{j2}^t & \cdots & \alpha_{jm}^t \\ \beta_{j1}^t & \beta_{j2}^t & \cdots & \beta_{jm}^t \end{pmatrix} \in Q(t)$, $\varepsilon \le |\alpha_{jk}^t|^2, |\beta_{jk}^t|^2 \le 1-\varepsilon$ where $j = 1,2,\ldots n$, $k = 1,2,\ldots,m$, $t > 0$, $0 < \varepsilon << 1$.

According to the observing operator, if we define the function *observing*(α, β) as one observing operator to Q-bit (α, β), it can only get either 1 or 0. We can estimate the probability of the observing result to a Q-bit $(\alpha_{jk}^t, \beta_{jk}^t)$ as follows:

$$P(observing(\alpha_{jk}^t, \beta_{jk}^t) = 0) = |\alpha_{jk}^t|^2 \ge \varepsilon,$$

$$P(observing(\alpha_{jk}^t, \beta_{jk}^t) = 1) = |\beta_{jk}^t|^2 \ge \varepsilon.$$

Let us now consider the probability $P(s \in P(t))$, $s \in S$, $t > 0$. On the assumption that s is an arbitrary element in S, s can be expressed as a binary string $\{s^1 s^2 \ldots s^m\}$, where s^k is either 0 or 1, $k = 1,2,\ldots,m$. Further we can conclude the probability of the observing result to a Q-individual q_j^t:

$P(observing(q_j^t)) = s) = \prod_{k=1}^{m} P(observing(q_{jk}^t) = s^k) \ge \varepsilon^m$, $j = 1,2,\ldots n$, $t > 0$. Thus we can conclude that $P(s \in P(t)) \ge P(observing(q_j^t)) = s) \ge \varepsilon^m$.

Moreover, Considering the construction of the algorithm, it is guaranteed that $P(s \in P(t))$ is independent each other for different t.

From the theorem 1, we can conclude the theorem 2. ■

4.3 On the Convergence Property of QMOEA Rotation Gate and N_ε Gate

The second example is the MOEA with the rotation gate and the N_ε gate. We have described the rotation gate and NOT gate in subsection (3.1). The N_ε gate is a modified NOT gate which is proposed in this paper. In fact its function is to exchange Q-bit's parameters with the probability ε. Its transformation matrix can be defined as follows:

N_ε: $N_\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with probability ε; $N_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with probability 1-ε, where 0<ε<<1.

The simplified procedure of this algorithm is similar with that basic framework but the step v) and vi) are eliminated and the rotation gate and the N_ε gate are adopted in step viii) as follows:

The Procedure of the QMOEA with Rotation Gate and NOT Gate with Probability

Begin

$t \leftarrow 0$

i) Initialize $Q(t)$

ii) $A(t) = \{\}$

iii) While (not termination condition) do

$t \leftarrow t+1$

iv) Make $P(t)$ by observing the state of $Q(t-1)$

v) Rebuild the archive set A(t);

\\ **Here $A(t) \subseteq M_f(P(t) \bigcup A(t\text{-}1), \leq)$ and maximize the diversity of those chosen elements in $A(t)$.**

vi) Make $Q(t)$ by updating Q(t-1) on rotation gate

vii) Update $Q(t)$ on the N_ε gate

End

End

Theorem 3 The QMOEA with the rotation gate and the N_ε gate which is defined above converges with probability 1 to its Pareto optimal set.

Proof: First we let the $Q(t)$ in vi) step of the algorithm above as follows:

$Q(t) = \{ q_j^t, j = 1,2,\ldots,n\}$ and $q_j^t = \begin{pmatrix} \alpha_{j1}^t & \alpha_{j2}^t & \ldots & \alpha_{jm}^t \\ \beta_{j1}^t & \beta_{j2}^t & \ldots & \beta_{jm}^t \end{pmatrix}$, $t > 0$.

Then let us consider the N_ε gate. After updated by N_ε gate, each Q-bit in q_j^t has been exchanged its parameters with the probability ε. According to the observing operator in iv) step of the algorithm, the probability of the observing result to kth Q-bit q_{jk}^t of q_j^t can be expressed as follows:

$$P(observing\,(q_{jk}^t) = 0) = (1-\varepsilon)|\alpha_{jk}^t|^2 + \varepsilon|\beta_{jk}^t|^2 = (1-2\varepsilon)|\alpha_{jk}^t|^2 + \varepsilon,$$

$$P(observing\,(q_{jk}^t) = 1) = \varepsilon|\alpha_{jk}^t|^2 + (1-\varepsilon)|\beta_{jk}^t|^2 = (1-2\varepsilon)|\beta_{jk}^t|^2 + \varepsilon.$$

Since 0<ε<<1, we can conclude

$P(observing(q_{jk}^t) = 0) \geq \varepsilon$ and $P(observing\,(q_{jk}^t) = 1) \geq \varepsilon$, $k = 1,2,\ldots m$.

Let us now consider the probability $P(s \in P(t))$, $s \in S$, $t > 0$. On the assumption that s is an arbitrary element in S, s can be expressed as a binary string $\{s^1 s^2 \ldots s^m\}$, where s^k is either 0 or 1, $k = 1,2,\ldots,m$. Further we can conclude the probability of the observing result to a Q-individual q_j^t:

$$P(observing(q_j^t)) = s) = \prod_{k=1}^{m} P(observing(q_{jk}^t) = s^k) \geq \varepsilon^m \quad ,j = 1,2,..\, n,\ t > 0.$$

Thus we can conclude $P(s \in P(t)) \geq P(observing(q_j^t)) = s) \ \geq \varepsilon^m$.

Moreover, Considering the construction of the algorithm, it is guaranteed that $P(s \in P(t))$ is independent each other for different t.

From the theorem 1, we can conclude the theorem 3. ■

4.4 On the Convergence Property of QMOEA Rotation Gate and Nε Gate

Now we give an algorithm accord with the basic framework, which does not satisfy the convergence condition in theorem 1. We show that it can not converge to the Pareto optimal set in a specific situation. Further, we can show that in another specific situation the algorithm converge to the Pareto optimal set whereas the $f(C(t))$ can not convergence to the entire Pareto optimal set.

The procedure of this algorithm can be defined as similar with the algorithm with rotation gate and Nε gate in subsection (4.3) but the vii) step is eliminated. In this algorithm $Q(t)$ is updated only by rotation gate, whose $\Delta\theta$ strategy is same as the description in subsection (3.1). By the $\Delta\theta$ strategy, the Q-bit will hold its state if the observing bit of the Q-bit equals to its objective sign. Let us assume that all solutions are presented by binary digit and the feasible space $S = \{0,1\}^4$, the Pareto optimal set $PS = \{1111,1110\}$ and 1111 dominate all elements of S - PS. In order to simplify the issue, we assume that $Q(t)$ and $P(t)$ are only one individual, i.e. $n = 1$. Then we give two specific situations as follows.

In the first case, we show a specific situation where the algorithm can not converge to the Pareto optimal set. Let $Q(t_0) = \{q^{t_0}\} = \left\{\begin{pmatrix} \alpha_1^{t_0} & \alpha_2^{t_0} & 1 & \alpha_4^{t_0} \\ \beta_1^{t_0} & \beta_2^{t_0} & 0 & \beta_4^{t_0} \end{pmatrix}\right\}$, $A(t_0) \cap PS = \phi$ and all elements in $A(t_0)$ accord with schema '**0*',where * denotes either 1 or 0. By the observing operator, it is clear that $P(t_0 + 1) = \{p^{t_0+1}\}$ and p^{t_0+1} accord with schema '**0*' because the third Q-bit of q^{t_0} is (1,0). Hence $A(t_0+1) \cap PS = \phi$ and all elements in $A(t_0+1)$ accord with schema '**0*' since $A(t_0+1) = M_f(A(t_0)) \cup P(t_0+1)$. As the objective solution $b(t_0+1) \in A(t_0+1)$, $b(t_0+1)$ accord with schema '**0*', too. By updating operator with $\Delta\theta$ strategy of the rotation gate, we can conclude that $Q(t_0+1) = \{q^{t_0+1}\} = \left\{\begin{pmatrix} \alpha_1^{t_0+1} & \alpha_2^{t_0+1} & 1 & \alpha_4^{t_0+1} \\ \beta_1^{t_0+1} & \beta_2^{t_0+1} & 0 & \beta_4^{t_0+1} \end{pmatrix}\right\}$. Now let $t = t_0 + 1$, we can repeat the deduction as above. Hence in this situation $P(1111 \in P(t)) = P(1110 \in P(t)) = 0$, $t > t_0$, so the convergence condition is not satisfied. Further, it can not converge to the Pareto optimal set forever because $A(t) \cap PS = \phi$, $t > t_0$. Here it is an example that a QMOEA does not satisfy the convergence condition.

Remark 4 From above text, we obtain an example which does not satisfy the convergence condition and does not converge to the Pareto optimal set.

In the second case, let $Q(t_0) = \{q^{t_0}\} = \left\{\begin{pmatrix} \alpha_1^{t_0} & \alpha_2^{t_0} & \alpha_3^{t_0} & 0 \\ \beta_1^{t_0} & \beta_2^{t_0} & \beta_3^{t_0} & 1 \end{pmatrix}\right\}$, $A(t_0) = \{1111\}$ and $C(t_0) = \{1111\}$. By the observing operator, it is clear that $P(t_0 + 1) = \{p^{t_0+1}\}$ and p^{t_0+1} accord with schema '***1'. Hence $A(t_0+1) = \{1111\}$ and $C(t_0+1) = \{1111\}$ since $A(t_0+1) = M_f(A(t_0) \cup P(t_0+1)$, $C(t_0+1) = M_f(C(t_0) \cup P(t_0+1)$ and 1111 dominate all elements of S - PS. By the updating operator with $\Delta\theta$ strategy of the rotation gate, $Q(t_0+1) = \{q^{t_0+1}\} = \left\{\begin{pmatrix} \alpha_1^{t_0+1} & \alpha_2^{t_0+1} & \alpha_3^{t_0+1} & 0 \\ \beta_1^{t_0+1} & \beta_2^{t_0+1} & \beta_3^{t_0+1} & 1 \end{pmatrix}\right\}$. Now let $t_0 = t_0 + 1$, we can repeat the deduction as above. Hence in this situation $P(1110 \in P(t)) = 0$, $t > t_0$, so the convergence condition is not satisfied. Further, $C(t)$ can not converge to the entire set PS forever because $1110 \notin C(t)$, $t > t_0$. But $A(t)$ converge to PS because $A(t) \subseteq PS$, $t > t_0$. So the convergence condition is sufficient but indispensable.

Remark 5 From above text, we obtain an counterexample in which $C(t)$ does not converge the entire Pareto optimal set but $A(t)$ converge to the Pareto optimal set.

5 Conclusions

In this article we have presented a general framework for quantum-inspired multiobjective evolutionary algorithms. Roughly speaking, this is an integration of the basic principles of quantum computing and general schemes of MOEA, such as Q-bit individual presentation, observing operator, Q-gate updating operator, external archive set, nondominated sorting, diversity preserving etc. We give one of sufficient convergence conditions for the basic framework and its proof bases on the partial set theory and probability theory. Then we present two algorithms those satisfy this convergence condition. One is with H_ε Gate and another is with the rotation gate and NOT gate with probability. These theoretical characters may be useful for designing QMOEAs.

Despite of these strong theoretical features on convergence, we need numerical results with these QMOEAs. Furthermore, efficiency and diversity are also significant to multiobjective optimization algorithms besides the convergence. These issues should be subject of future work.

6 Acknowledgments

The authors are thankful to the “China Scholarship Council (CSC)” for its financial support in form of a scholarship that made the present research possible, while one of the authors, Zhiyong Li, is studying at Universität Dortmund as a postdoctoral researcher in Germany.

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