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ESSAYS ON REVEALED PREFERENCE: CONTRIBUTIONS TO THE THEORY OF CONSUMER'S BEHAVIOR

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ABSTRACT

This thesis contributes to the theory of consumer's behavior. It is devided into three parts: The first part gives a brief history of the theoretical developments relevant to the two consecutive parts and provides a detailed outline. The second part contains theoretical contributions. Specifically, (i) it provides a unifying proof technique to show that preference cycles can be of arbitrary length for more than two but not for two commodities. An immediate corollary is that the Weak Axiom of Revealed Preference only implies the Strong Axiom for two commodities, (ii) it provides a simple graphical way to construct preference cycles, (iii) it shows that for two dimensional commodity spaces any homothetic utility function that rationalizes each pair of observations in a set of consumption data also rationalizes the entire set of observations, (iv) it explorers rationalizability issues for finite sets of observations of stochastic choice and gives two rationalizability theorems. The third part provides three practical contributions. Specfically, (i) it explorers some possible applications of a lemma used in the chapter on homothetic preferences in two dimensions, (ii) it suggests a procedure to decide whether or not to treat a consumer who violates the Generalized Axiom of Revealed Preference as "close enough" to utility maximization, (iii) it provides a new measure for the severity of a violation of utility maximization based on the extent to which the upper bound of the indifference map intersects the budget set.

PUBLICATIONS AND PRESENTATIONS

Some ideas and figures have appeared previously in the following publications:

"Revealed Preference and the Number of Commodities" (2007), Ruhr Economic Papers No. 36
"A Geometric Measure for the Violation of Utility Maximization" (2008), Ruhr Economic Papers No. 69
"Stochastic Revealed Preference and Rationalizability" (2008), Ruhr Economic Papers No. 70

Some ideas were presented in the following conferences and seminars:

Doctoral Meeting of Montpellier 1st edition (2008), Montpellier
Jahrestagung des Vereins für Socialpolitik (2008), Graz
Jahrestagung des Vereins für Socialpolitik (2009), Magdeburg
International Meeting of the Association for Public Economic Theory (2009), Galway
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ACRONYMS

AEI Afriat Efficiency Index

CD Cobb-Douglas

CES Constant Elasticity of Substitution

HARP	Homothetic Axiom of Revealed Preference
HE	Homothetic Efficiency
GARP	Generalized Axiom of Revealed Preference
PHARP	Pairwise Homothetic Axiom of Revealed Preference
RSO	Rationalizability in terms of Stochastic Orderings
RSOP	Rationalizability in terms of Stochastic Orderings on the normalized Price space
SARF	Strong Axiom of Revealed Favorability
SARP	Strong Axiom of Revealed Preference
SDF	Stochastic Demand Function
SS	Stochastic Substitutability
VEI	Volume Efficiency Index
VES	Variable Elasticity of Substitution
WARF	Weak Axiom of Revealed Favorability
WARP	Weak Axiom of Revealed Preference
WASRP	Weak Axiom of Stochastic Revealed Preference
WGARP	Weak Generalized Axiom of Stochastic Revealed Preference

Part I

INTRODUCTION

1 A BRIEF HISTORY OF REVEALED PREFERENCE

1.1 INTRODUCTION

Revealed preference has to be considered as one of the most influencial ideas in economics.[1] Paul Samuelson introduced his "Note on the Pure Theory of Consumer's Behavior" in 1938 – a paper in which he used the expression "selected over" instead of "revealed preferred". Varian (2006) recalls a search he conducted on Google scholar in 2005 with 3,600 results which contained the phrase "revealed preference". When I repeated the search in early 2009, it returned 17,100 results.[2]

A more formal introduction of revealed preference is given in the different chapters of this thesis. The main idea in Samuelson (1938) was to "[drop] off the last vestiges of the utility analysis" and to "start anew in direct attack upon the problem". He expressed the hope

> [...] that the orientation given here is more directly based upon those elements which must be taken as *data* by economic science [...]

In noticing that *utility* is unobservable, Samuelson's approach was to base the analysis of consumer behavior on the observable magnitutes *prices*, *expenditure* and *demand* at any price-wealth situation. If, then, a consumer demands a certain bundle x at a given price vector p and expenditure w, he *reveals* that he preferres this bundle over all other bundles which were affordable. Samuelson went on to postulate that

> [...] if an individual selects batch one over batch two, he does not at the same time select two over one.

This is what later became known as the Weak Axiom of Revealed Preference (WARP).

1 This chapter is roughly based on Mas-Colell (1982) and Varian (2006).
2 Clearly Google scholar has improved over time and now includes many more old papers as well as new unpublished working papers.

After Little's (1949) and Samuelson's (1948) graphical attempts to show how to construct indifference maps based on revealed preference relations and thereby showing that "a theory of consumer's demand can be based solely on consistent behaviour" (Little 1949) it became obvious that the two-dimensional graphical analysis was not sufficient.

Shortly afterwards, Houthakker (1950) provided a general proof for arbitrarily many goods. His idea was to extend the directly revealed preference relations to indirectly revealed preference relations. We might not observe that a consumer selects bundle one over bundle two, but we might nonetheless be able to establish a preference relation between one and two by observing that a consumer selects bundle one over bundle three, bundle three over bundle four, and so forth, until we arrive at bundle two.

Houthakker's idea extended the weak axiom and is now known as the Strong Axiom of Revelaed Preference (SARP). It can be related to the old "integrability problem" which can be traced back as far as Antonelli in 1886 (cf. Richter 1966): Given a demand function, how can we tell whether or not it could have been induced by a utility function? In Antonelli's version of integrability, restrictions are imposed on indirect demand functions (independent variables being commodity bundles) and direct preferences are recovered (cf. Mas-Colell 1982). Integrability conditions can also be imposed on direct demand functions to obtain indirect preferences; see Hurwicz and Uzawa (1971). The integrability conditions on direct demand functions – symmetry and negative definiteness of the Slutsky matrix – are well known.

While integrability theory considers infinitesimal changes and relies on a condition which does not admit an economic interpretation (symmetry), revealed preference theory relies on the SARP, which is a fairly intuitive condition. However, revealed preference and integrability theory are equivalent in the sense that their conditions "characterize the preference hypothesis by restrictions involving only the demand function" (Mas-Colell 1982).

A related question is whether one can find sufficient conditions on demand functions which not only guarantee that a consumer acts as a utility maximizer, but also that his preferences are completely "determined" by his choice behaviour, i.e. whether preferences are unique inside some class. This problem is treated in Mas-Colell (1978), where the author shows that a regularity condition (the "income Lipschitz property") on the demand together with the SARP is sufficient for uniqueness of an underlying continuous preference relation.

It has long been noticed that a notion of *stochastic preferences* or *random utility maximization* could be a fruitful extension of the revealed preference approach. Marschak (1959) was perhaps the first to connect the revealed preference approach to the psychometric literature (Thurstone 1927, Luce 1958). Simply put, stochastic preferences imply that while a consumer is a rational maximizer of utility, he might choose according to one preference relation in some situation and according to another preference relation in some other situation. Alternatively, one can argue that while we can observe choices, we might not be able to observe all aspects of the situation in which this choice is made. For example, the situation may consist of a given income, a price vector, and a weather condition, but we only observe the income and the price. In this case we can interpret the observed choice as the result of random utility maximization, where the random factor is the weather. Another interpretation is that we observe individual choices of a group of anonymous consumers which cannot be traced. In that case, we might ask whether the distribution of choices observed for that group in a variety of situations is consistent with utility maximization. Papers addressing these issues include among others Block and Marschak (1960), McFadden and Richter (1970), Falmagne (1978), Fishburn (1978), Cohen (1980), Barberá and Pattanaik (1986), Cohen and Falmagne (1990), Bandyopadhyay et al. (1999, 2002, 2004), and McFadden (2005).

1.2 AFRIAT'S APPROACH AND EMPIRICAL ANALYSIS

Afriat's (1967) approach to revealed preference proved to be quite successful for empirical work. The main idea was to not start with a demand function which specifies demand for all possible situations, but to start with a *finite* set of consumption data, i.e. observed prices, expenditures, and choices, and to ask *how* a researcher could *construct* a utility function consistent with the observations. A new and somewhat clearer exposition of Afriat's original paper was later provided by Diewert (1973).

Afriat's approach was constructive: He gave an algorithm which can be used to compute a utility function consistent with a finite set of observations, given that the data satisfied a condition he called "cyclic consistency". Cyclic consistency was later reformulated by Varian (1982) as the Generalized Axiom of Revealed Preference (GARP). The difference between SARP and GARP is that while SARP is built around single-valued demand, GARP also allows for multi-valued demand. This distinction is of importance because multi-valued demand leads to "flat" parts of indifference curves which violates SARP.

When economists became increasingly interested in estimating consumer demand functions – two of the most famous examples are Christensen et al. (1975) and Deaton and Muellbauer (1980) – Varian (1982) noted that one could start any analysis by testing nonparametrically whether a set of consumption data could have been generated by the maximization of a utility function. He also developed several other nonparametric tests (Varian 1983) for specific forms, like homotheticity.

This nonparametric approach to demand analysis became increasingly popular over the last decade with the ascent of experimental economics, specifically induced budgets experiments. Experimental data has a distinctive advantage over field data, as the experimenter can observe data in an ideal way, just as assumed in revealed preference theory.

Battalio et al. (1973) were perhaps the first to look at individual demand generated in a quasi-experimental setting. Their subjects were patients in a psychiatric hospital who were endowed with different amounts of wealth as reward for cooperative behavior and could choose from a set of different goods; the authors changed prices of these goods to obtain power against the alternative hypothesis of random demand. Sippel (1997) recruited subjects from a pool of university students and was perhaps the first to test the theory of consumer demand in a laboratory setting. The goods he offered ranged from videoclips to snacks. A similar experiment was conducted by Mattei (2000). Harbaugh et al. (2001) conducted a similar experiment with children as subjects. Andreoni and Miller (2002) (see also Andreoni and Vesterlund 2001) examined dictator games with changing transfer rates and treated donations and own payoff as two different goods; a very similar experiment was conducted by Fisman et al. (2007). Février and Visser (2004) conducted an experiment with different kinds of orange juices as goods. Choi et al. (2007*a*) looked at investments in risky assets. Besides experimental data, household level panel data has also been analyzed using revealed preference techniques, for example by Blundell et al. (2003) and Blundell et al. (2007).

The data generated in *induced budgets experiments* is, by all practical means, just as the theory of revealed preference assumes. An induced budget experiment consists of an income (or wealth) endowment of experimental subjects which they can spend on some goods offered by the experimenter who controls the prices of these goods. The income endowment usually is of no direct value to the subject, i.e. unspent income is lost for them, creating incentives to spend all income. In some experiments, such as the one by Andreoni and Miller (2002), subjects are even required to spend their entire income. Prices and income are varied, creating different budget sets, and subjects are asked to choose one of the affordable consumption bundles. For example, in Andreoni and Miller (2002), subjects were given a sheet of paper informing subjects about prices and income. Subjects were then asked to write down their preferred bundle. In Fisman et al. (2007) and Choi et al. (2007*a*) this was taken a step further: Subjects were presented budget lines on a computer screen and asked to click on their preferred point on the line with the computer mouse.

In Sippel's (1997) experiment the commodities subjects could choose from were commodities in the literal sense, e.g. snacks. Andreoni and Miller's (2002) treated "payment to self" and "payment to other" as two different commodities. This setup may require some leap of reasoning, as subjects could choose how much of their endowment they wanted to pass to an anonymous other subject, given different transfer rates (i.e. one unit of money passed would result in x units of money for the other subject). That is, "giving" was interpreted as a commodity, and the transfer rate was interpreted as the "price of giving". While this classical demand theory framework may not be the most natural environment to test preferences for altruism,[3] it did create a rich set of data to test whether (broadly defined) "altruistic" choices could be modelled as the result of utility maximization.

3 But note that tax deduction for charitable donations or the matching of donations of private persons by a company sponsoring a charity event are quite common.

The theory of revealed preference has therefore turned out to be not only a successful theoretical approach to consumer behavior without the need for introspective subjective utility, it has also generated a rich set of experimental and empirical examination which is likely to grow in the future.

2 CONTRIBUTIONS OF THIS THESIS

2.1 INTRODUCTION

This thesis contributes to the theory of revealed preference and its application. It is devided into three parts. Part i contains a brief history of revealed preference and this short introduction to the contributions. Part ii contains theoretical contributions. Part iii contains applications.

2.2 THEORY

Chapter 3 contains a geometric graph-based attack on a known and solved problem. It introduces a new and unifying approach to show that a preference cycle, by which a consumer reveals that he prefers a bundle to itself via a chain of other choices, can only be of irreducible length two in case of a two dimensionsal commodity space. In three or more dimensional commodity spaces a preference cycle can be of arbitrary length. Two corollaries are that WARP implies SARP in two dimensions but not in higher dimensions. The proof given here generalizes the existing proofs insofar as it provides necessary and sufficient conditions for preference cycles to exist. It is shown that the necessary conditions cannot hold in two dimensions, but the sufficient conditions can be fullfilled in more than two dimensions. It is also shown how one can use the employed methods to define demand for all positive prices such that in three dimensions preference cycles of arbitrary length can be constructed. The proof is intuitively appealing as is gives a geometric interpretation of preference cycles as paths on indifference surfaces.

Chapter 4 provides a simple graphical way to construct preference cycles in three dimensions by "taking a look behind the scenes" of intersecting budget hyperplanes. By looking at the positive orthant of the Euclidean coordinate system through the origin of the system, the search for preference cycles is substantially simplified given suitable budgets. An attempt is made to adapt the method to four dimensions.

Chapter 5 presents a proof to show that for two dimensional commodity spaces any homothetic utility function that rationalizes each pair of observations in a set of consumption data also rationalizes the entire set of observations. The result is stated as a pairwise version of Varian's Homothetic Axiom of Revealed Preference. The chapter provides an explicit way to compute scalar factors to construct homothetic preference relations. These scalar factors are later applied in Chapter 7. The chapter also provides insights into relations between different revealed preference axioms which can be tested using data from induced budgets experiments, and how to construct powerful tests for homotheticity.

Chapter 6 explorers rationalizability issues for finite sets of observations of stochastic choice in the framework introduced by Bandyopadhyay et al. (1999). Is is argued that a useful approach is to consider indirect preferences on budgets instead of direct preferences on commodity bundles. Stochastic choices are rationalizable in terms of stochastic orderings on the normalized price space if and only if there exists a solution to a linear feasibility problem. Together with the weak axiom of stochastic revealed preference the existence of a solution implies rationalizability in terms of stochastic orderings on the commodity space. Furthermore it is shown that the problem of finding sufficiency conditions for binary choice probabilities to be rationalizable bears similarities to the problem considered in this chapter. The chapter also provides a discussion about some difficulties with the notion of probability measures on the set of preferences.

2.3 APPLICATIONS

Chapter 7 continues the work started in Chapter 5. The result obtained in the theoretical part is used to provide a simplified nonparametric test. It is shown how the explicit scalar factors derived in Chapter 5 can be usefully applied to data sets which violate homotheticity. The new test and measures are applied to experimental data.

Chapter 8 suggests a procedure to decide whether or not to treat a consumer who violates GARP as "close enough" to utility maximization.

It is based on the reduction of the power the test has against random behavior. It can also be used to compare different efficiency indices.

If a consumer is inconsistent with GARP, we might need a measure for the severity of this inconsistency. In Chapter 9 a new measure based on the extent to which the indifference surfaces intersect the budget hyperplanes is proposed. The measure is intuively appealing and, as a cutoff-rule evaluated by Monte Carlo experiments, performs very well compared to the often used Afriat Efficiency Index. The results suggest that the new measure is better suited to capture small deviations from utility maximation.

Part II

THEORY

3 PREFERENCE CYCLES AND THE NUMBER OF COMMODITIES

3.1 INTRODUCTION

For quite some time it had been an open question in economic theory whether the Weak Axiom of Revealed Preference (WARP) as introduced by Samuelson (1938) was actually sufficient to guarantee that a demand function maximizes a utility function. Houthakker (1950) defined an apparently stronger condition, the Strong Axiom of Revealed Preference (SARP) and showed that this condition was indeed sufficient. Arrow (1959), however, remarked that there was still no proof "that the Weak Axiom is not sufficient to ensure the desired result. The question is still open." Uzawa (1959) showed that the Weak Axiom combined with certain regularity conditions implies the Strong Axiom.[1] Meanwhile, Rose (1958) showed that the Weak Axiom implies the Strong Axiom for two commodities, extending a limited geometrical argument by Hicks (1965 [1956], pp. 52–54).[2]

Finally, Gale (1960) constructed a counterexample for the case of three commodities: WARP was satisfied, SARP was violated. This, essentially, settled the question. Kihlstrom et al. (1976) provided a theoretical argument which yields an infinite number of demand functions that satisfy WARP but not SARP. Peters and Wakker (1994, 1996) showed how to embed Gale's example in higher dimensions without relying on isomorphic extensions, i.e. with strictly positive demand for every commodity for suitable budgets. John (1997) showed that there is a simpler proof of their results.

1 Samuelson is said to have expressed the view that these regularity conditions "are perhaps integrability conditions in disguise" (Gale, 1960), and Kihlstrom et al. (1976) commented that "it looks very much like the *strong axiom* itself".

2 In Chapter 5 is is shown that for similar reasons pairwise comparison of consumption decisions is sufficient to check for homothetic rationalization.

Samuelson (1953) raised the question whether the exclusion of cycles of a certain length would be sufficient to imply SARP. Even if WARP does not generally imply SARP, the exclusion of cycles of length k could exclude the possibility of cycles of length greater than k. This question was answered by Shafer (1977) who showed that there exists a demand function for three commodities which violates SARP, but has no revealed preference cycles of length less then or equal to any $k \geq 2$, which for $k = 2$ also proves that WARP does not imply SARP. Shafer's result was also extended into more than three dimensions by Peters and Wakker and John.

The results can be found in some advanced textbooks. However, the existing proofs are complicated and often long-winded. In this chapter a simple new approach is developed to show that with more than two commodities there can be preference cycles of arbitrary finite length whereas for two commodities cycles can only be of length 2. From this it immidiately follows (i) that WARP necessarily implies SARP for two commodities, (ii) that WARP does not imply SARP for more than two commodities. The approach here unifies the proofs of Rose, Gale, and Peters and Wakker insofar that it gives necessary and sufficient conditions for the existence of cycles of length greater than two. It is shown that in two dimensions the necessary conditions cannot hold, whereas in more than two dimensions the sufficient conditions can be satisfied. The employed methods might be useful for other researchers to find simpler proofs for their results. Section 3.5 shows a way to define demand for all positive prices such that in three dimensions preference cycles of arbitrary length can be constructed.

The proof technique is intuitively appealing as it admits an understanding of the reason by giving a geometric interpretion of preference cycles as paths on indifference surfaces.

3.2 PRELIMINARIES

We use the following notation: $\mathbb{R}^{\ell}_{+} = \{x \in \mathbb{R}^{\ell} : x \geqq 0\}$, $\mathbb{R}^{\ell}_{++} = \{x \in \mathbb{R}^{\ell} : x > 0\}$, where " $x \geqq y$ " means " $x_i \geq y_i$ for all i ", " $x \geq y$ " means " $x \geqq y$ and $x \neq y$ ", and " $x > y$ " means " $x_i > y_i$ for all i ". Note the convention to use subscripts to denote scalars or vector components and superscripts to index bundles.

Let $X = \mathbb{R}^{\ell}_{+}$ be the commodity space, where $\ell \geq 2$ denotes the number of different commodities. The price space is $P = \mathbb{R}^{\ell}_{++}$, and the space of price-income vectors is $P \times \mathbb{R}_{++}$. Consumers choose a single bundle $x^i = (x^i_1, \ldots, x^i_\ell)' \in X$ when facing a price vector $p^i = (p^i_1, \ldots, p^i_\ell) \in P$ and an income $w^i \in \mathbb{R}_{++}$. A budget set is then defined by $B^i = B(p^i, w^i) = \{x \in X : p^i x^i \leq w^i\}$. Demand is exhaustive, i.e. $p^i x^i = w^i$. Denote the boundary of the budget set B as $\bar{B} = \{x \in X : px = w\}$, so $x^i \in \bar{B}^i$. It is convenient to normalize prices by the level of expenditure at each observation, so that $p^i x^i = 1$ for all i. A set of n observations can then be denoted as $S = \{(x^i, p^i)\}_{i=1}^n$.

Let $R \subseteq X \times X$ be a binary relation on X. Instead of $(x^i, x^j) \in R$ we will write $x^i R x^j$. The following definitions are central in the theory of revealed preference:

Definition 3.1 (Revealed preference relations) An observation x^i is *directly revealed preferred* to x, written $x^i R x$, if $p^i x^i \geq p^i x$. It is *revealed preferred* to x, written $x^i R^* x$, if for some sequence of bundles $(x^j, x^k, \ldots, x^m)$ it is the case that $x^i R x^j$, $x^j R x^k$, …, $x^m R x$. In this case R^* is the *transitive closure* of the relation R.

Definition 3.2 (Weak Axiom of Revealed Preference) The data set S satisfies the Weak Axiom of Revealed Preference (WARP) if $x^i R x^j$, $x^i \neq x^j$, implies [not $x^j R x^i$].

Definition 3.3 (Strong Axiom of Revealed Preference) The data set S satisfies the Strong Axiom of Revealed Preference (SARP) if $x^i R^* x^j$, $x^i \neq x^j$, implies [not $x^j R x^i$].

Definition 3.4 (Revealed preferred set and convex monotonic hull) The set of bundles that are revealed preferred to a certain bundle x^0 (which does not have to be an observed choice) is given by the *convex monotonic hull* of all choices revealed preferred to x^0, i.e.

$$RP(x^0) = H_{\text{convex}}\Big(\big\{x \in X : x \geq x^i \text{ such that } x^i R^* x^0 \text{ for some } i = 1, \ldots, n\big\}\Big), \tag{3.1}$$

where the convex hull H_{convex} of a set of points $A = \{a^i\}_{i=1}^n$ is defined as

$$H_{\text{convex}}(A) = \left\{ \sum_{i=1}^{n} \lambda_i a^i : a^i \in A, \lambda_i \in \mathbb{R}_+, \sum_{i=1}^{n} \lambda_i = 1 \right\}. \tag{3.2}$$

See also Varian (1982) and Knoblauch (1992). The convex monotonic hull of a set of points $\{x^i\}$ will be denoted as[3]

$$\text{CMH}_{\text{convex}}(\{x^i\}) = H_{\text{convex}}\Big(\big\{x \in X : x \geq x^i \text{ for some } i = 1, \ldots, n\big\}\Big). \tag{3.3}$$

The set of observations S can be interpreted as an unweighted directed graph (digraph), i.e. a pair $G = (V, A)$ where V is the set of nodes or vertices (the observations) and A is the set of directed edges or arcs (the directly revealed preference relations). An arc $a_{ij} = \{x^i, x^j\}$ is directed from x^i to x^j and is an element of A if and only if $x^i R x^j$.[4] The graph can then be represented by a Boolean adjacency matrix $M = \{m_{ij}\}$ where $m_{ij} = 1$ if $x^i R x^j$ and $m_{ij} = 0$ otherwise.[5]

Definition 3.5 (Preference Cycles) An ordered set $\{(x^i, p^i)\}_{i=1}^k$ of k observations forms a *cycle of length* k if $p^i x^{i+1} \leq 1$ and $x^i \neq x^{i+1}$ for $i = 1, \ldots, k \mod k$, i.e. if x^i is indirectly revealed preferred to itself via the chain of observations $\{(x^i, p^i)\}_{i=1}^k$. A set $\{(x^i, p^i)\}_{i=1}^k$ forms a *cycle of irreducible length* k if it forms a cycle of length k and there is no shorter cycle (with a smaller k) by which x^i is indirectly revealed preferred to itself.

As an illustration, suppose we have a set of observations $\{(x^i, p^i)\}_{i=1}^5$ such that $x^1 R x^2$, …, $x^4 R x^5$. Suppose that also $x^5 R x^1$ and $x^2 R x^4$, but there are no other directly revealed preference relations. Then by $x^1 R^* x^5$

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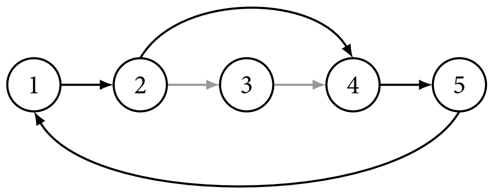
3 In a slight abuse of notation we will also write $\text{CMH}_{\text{convex}}(T)$ for a set $T = \{(x^i, p^i)\}$.

4 See also Wakker (1989), who takes a graph theoretic approach to revealed preference and defines alternatives as vertices and revealed preference relations as arcs. He shows that a choice function satisfies congruency (a condition "similiar" yet not equivalent to SARP, cf. Richter 1966) if and only if all dicircuits (directed paths that form a cycle) are reversible.

5 One can then use Warshall's (1962) algorithm to compute the transitive closure of the binary relation R. In the context of revealed preference theory this has first been pointed out and used by Varian (1982).



$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 3.1.: Left: The observations can be interpreted as nodes of a digraph. The shortest cycle includes nodes 1, 2, 4 and 5. Right: The Boolean adjacency matrix of the graph.

and $x^5 R x^1$ there is a preference cycle. The irreducible length of the shortest cycle in that data is four. See Figure 3.1.

Obviously, WARP implies the absence of cycles of irreducible length two, whereas SARP implies the absence of cycles of arbitrary irreducible length.

3.3 THEORY

Obviously any hyperplane that has an interior point of a convex polytope on one side will also have at least one vertex of the polytope on the same side.

This can be interpreted in the context of revealed preference: There can be observations that are strictly in a set $RP(x^0)$ and hence are redundant for the construction of $RP(x^0)$. If an observation x^i is directly preferred to such an interior point, the budget hyperplane $\bar{B}^i$ has to intersect the set $RP(x^0)$. Then $\bar{B}^i$ has at least one vertex of $RP(x^0)$ on its "left" side, so x^i is also directly revealed preferred to at least one other vertex of $RP(x^0)$. This leads to Proposition 3.1.

Proposition 3.1 (Necessary Conditions) Suppose $T = \{(x^i, p^i)\}_{i=1}^k$ is an ordered set of observations that forms a cycle of irreducible length k such that $x^1 R x^2$, $x^2 R x^3$, …, $x^{k-1} R x^k$, $x^k R x^1$. Then all of the observations in the cycle have to be distinct and non-redundant vertices of $\text{CMH}_{\text{convex}}(T)$, and the line segments connecting two observations of which one observation is directly revealed preferred to the other have to be edges on the boundary of $\text{CMH}_{\text{convex}}(T)$.

Proof Suppose $x^i \in T$ is not a vertex on the convex monotonic hull of T. Then x^{i-1} is directly revealed preferred to some x^j, $j \notin \{i-1, i\}$. If $j < i-1$ there exists a sequence $x^j R x^{j+1}$, $x^{j+1} R x^{j+2}$, …, $x^{i-2} R x^{i-1}$, $x^{i-1} R x^j$ of length $i - j < k$ which constitutes a preference cycle. If $j > i$ there exists a sequence $x^{i-1} R x^j$, $x^j R x^{j+1}$, …, $x^{k-1} R x^k$, $x^k R x^1$, …, $x^{j-1} R x^j$ of length $i - j + k < k$ which constitutes a preferences cycle.

Suppose that the line segment connecting x^{i-1} and x^i is not an edge of the convex monotonic hull. Then x^{i-1} also has to be directly revealed preferred to some x^j which is a vertex that causes the line to be strictly in the convex monotonic hull. Obviously this causes the cycle to be shorter than k by the same token as above.

Suppose $x^i \in T$ is a redundant vertex on the boundary so that $x^i = \lambda x^j + (1-\lambda)x^k$, $0 < \lambda < 1$, for some $x^j, x^k \in T$. Then x^i is on a line segment connecting x^j and x^k such that either (i) $x^j R x^k$ or $x^k R x^j$, or (ii) $x^j R x^i \wedge x^i R x^k$ or $x^k R x^i \wedge x^i R x^j$. Case (i) implies $x^j R x^i$ or $x^k R x^i$ respectively. Case (ii) implies $x^j R x^k$ or $x^k R x^j$ respectively. In either case, one bundle in $\{x^i, x^j, x^k\}$ is directly revealed preferred to two other bundles, which reduces the length of the preference cycle by the same token as above.

Suppose $x^i \in T$ is a redundant vertex on the boundary because it is a point on the monotonic extension of the convex hull of all bundles in T, so that $x^i \geq x^j$ for some x^j. Obviously any bundle directly revealed preferred to x^i will also be directly revealed preferred to x^j, which reduces the length of the preference cycle by the same token as above. ■

Proposition 3.2 For the case of commodity space $\mathbb{R}^2_+$ there cannot be cycles of irreducible length greater than two.

Proof Suppose there is a cycle $x^1 R x^2$, $x^2 R x^3$, …, $x^{k-1} R x^k$, $x^k R x^1$ of length k. By Proposition 3.1 the two edges connecting x^{k-1} with x^k and x^k with x^1 have to be on the boundary of the convex monotonic hull of all observations in the cycle. Because in a two-dimensional convex hull any vertex has only two edges, x^{k-1} and x^1 have to be either equal or on different sides of x^k. If $x^{k-1} = x^1$, there is a cycle of length two. If $x^{k-1} \neq x^1$, at one point an edge connecting some x^i with x^{i+1} needs to cut through the convex monotonic hull. Therefore there cannot be a cycle of irreducible length greater than two. See Figure 3.2. ■

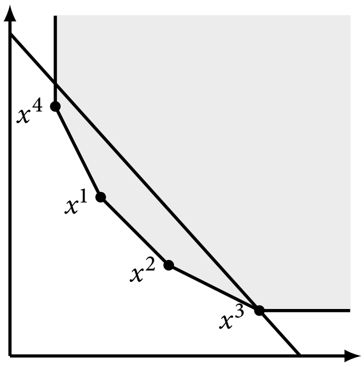


Figure 3.2.: Obviously x^3 cannot be directly revealed preferred to x^4 without also being directly revealed preferred to x^1 and x^2.

Corollary 3.1 For the case of commodity space $\mathbb{R}^2_+$ WARP implies SARP for any finite set S of data.

Proof Follows directly from Proposition 3.2. ■

Remark 3.1 In contrast to Rose's (1958) proof, Proposition 3.1 gives necessary conditions for the existence of preference cycles of length $k > 2$. Proposition 3.2 shows that these conditions cannot hold in the two commodity case.

Proposition 3.3 (Sufficient Conditions) Suppose $T' = \{x^i\}_{i=1}^k$ is a set of bundles such that all $x^i \in T'$ are distinct and non-redundant vertices on $\text{CMH}_{\text{convex}}(T')$. Then if there are non-intersecting line segments connecting all x^i with $x^{(i \mod k)+1}$ for all $x^i \in T'$ such that these line segments are edges of $\text{CMH}_{\text{convex}}(T')$, there exists a set of price vectors $\{p^i\}_{i=1}^k$, $p^i \in P$ for all $i \leq k$, such that $\{(x^i, p^i)\}_{i=1}^k$ forms a cycle of irreducible length k. [6]

Proof By the supporting hyperplane theorem there exists a hyperplane $H(p) = \{x \in X : px = 1\}$ such that $x^i, x^{i+1} \in H(p)$ and $x^j \notin H(p)$ for all $j \neq i, i+1$. Let p be the price vector at which x^{i+1} was chosen, so that $\bar{B}^{i+1} = H(p)$. Clearly, $x^{i+1}Rx^i$ and [not $x^{i+1}Rx^j$], i.e. $x^j \notin B^{i+1}$ for all $j \notin \{i, i+1\}$, $j \leq k$. ■

6 Note that $(i \mod k) + 1 = i + 1$ for all $i < k$ and $(i \mod k) + 1 = 1$ for $i = k$.

Proposition 3.4 For the case of commodity space $\mathbb{R}^{\ell}_{+}$, $\ell > 2$, there always exists a set of bundles $T' = \{x^i\}_{i=1}^{k}$ such that all $x^i \in T'$ are distinct and non-redundant vertices on $\text{CMH}_{\text{convex}}(T')$ and there are non-intersecting edges of $\text{CMH}_{\text{convex}}(T')$ that connect all x^i with $x^{(i \mod k)+1}$ for all $x^i \in T'$. For more than three commodities the shortest revealed preference cycle can therefore be of arbitrary irreducible length.

Proof A simple way to find a set of bundles T' that satisfies the conditions is to take a set of k distinct points from the intersection of an indifference surface of a concave utility function and a hyperplane $H(q) = \{x \in X : qx = 1\}$. The intersection of two convex sets is convex, so there are no interior or redundant points in the convex hull of the stereographic projection of all $x^i \in T'$ on a projective plane. Obviously the edges of that convex hull do not intersect and connect all x^i with $x^{(i \mod k)+1}$ for all projected points. (See also Figures 3.3, 3.4, and 3.5.) ■

Corollary 3.2 For the case of commodity space $\mathbb{R}^{\ell}_{+}$, $\ell > 2$, WARP does not imply SARP for any finite set S of data.

Proof Follows directly from Propositions 3.3 and 3.4. ■

Remark 3.2 The conditions in Proposition 3.3 extend the conditions given in Proposition 3.1. The combination is sufficient for the existence of preference cycles of arbitrary length. In the final step (Proposition 3.4) it was shown that the sufficiency conditions given in Proposition 3.3 can be met in three or more dimensions.

3.4 INTUITION

A graph $G(V, A)$ as defined in Section 3.2 that represents a preference cycle of irreducible length k is always a planar graph, i.e. a graph that can be drawn in the plane so that no edges intersect. Therefore $G(V, A)$ can always be embedded in an indifference surface of dimension three or more in the sense that every $v_i \in V$ is associated with a point on the surface, and every $a_{ij} \in A$ is associated with an edge. It cannot, however, be embedded in a two-dimensional indifference curve. That is to say, one cannot "extract" a preference cycle longer than two from a two-

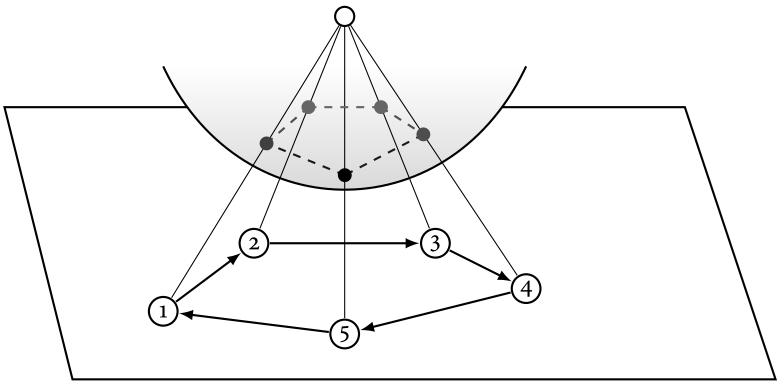


Figure 3.3.: A graph that represents a preference cycle can be "extracted" from a three-dimensional indifference surface.

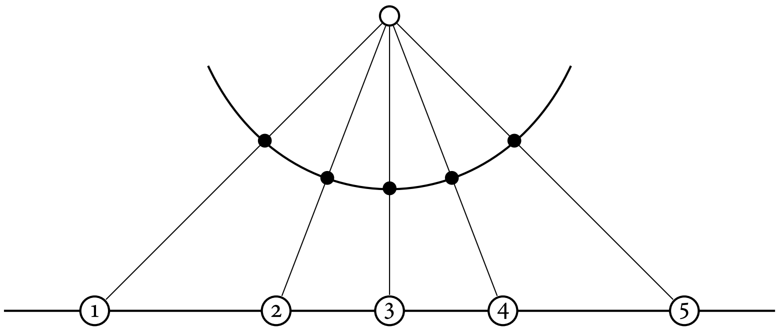


Figure 3.4.: No graph that represents a preference cycle can be "extracted" from a two-dimensional indifference curve.

dimensional commodity space, whereas higher dimensions allow this, as illustrated in Figures 3.3 and 3.4.

A similiar intuition yields the following: Just as there is only one distinct path on a circle (a closed curve) that connects a certain point on the circle with itself, there is no such path on an indifference curve (which is not closed). And just as there are infinitely many distinct paths on a sphere that connect a certain point on the sphere with itself, there are infinitely many paths on an indifference surface of a dimension greater than two.

Figure 3.5 shows how one can easily construct examples of preference cycles of arbitrary length in the three-dimensional commodity space. By Proposition 3.3, there exist price vectors such that each edge that connects two points is a line segment in the budget hyperplane at which one of the points was chosen, so that one of the points is directly revealed preferred to the other. This idea is also employed in Section 3.5 to define demand for all positve prices. Note that when one tries to use this method to construct a preference cycle in two dimensions, one obtains exactly two points – which is the maximal cycle length in two dimensions.

3.5 CONSTRUCTING DEMAND FOR ALL POSITIVE PRICES

In this section we will sketch a way to define demand for all positive prices in three dimensions such that only the weak but not the strong axiom is satisfied.

Define a line L which passes through the origin and the point $(1,1,1)$. The budget hyperplane $\bar{B}(p)$ defined by any price vector $p \in P$ will intersect L. Call this point of intersection $a(p)$. Let $u(x) = x_1^{1/3} x_2^{1/3} x_3^{1/3}$. The set $\{x \in X : u(x) = u(a(p))\}$ is an indifference surface of the preference represented by the Cobb-Douglas utility function $u(x)$ which is maximized by $a(p)$ if all prices in p are the same and $a(p) \in \bar{B}(p)$. For any other price vector for which $a(p) \in \bar{B}(p)$, the decision on $\bar{B}(p)$ which maximizes $u(x)$ will be different from $a(p)$ and hence be revealed preferred to $a(p)$ because $a(p) \in \bar{B}(p)$. The intersection of the sets $\{x \in X : u(x) = u(a(p))\}$ and $\bar{B}(p)$ defines a simple closed curve in the plane $\bar{B}(p)$.

Now take the minimal λ such that $\bar{B}(\lambda(1,1,1))$ is just tangent to the intersection of $\bar{B}(p)$ and $\{x \in X : u(x) = u(a(p))\}$. The set $\bar{B}((\lambda + \varepsilon)(1,1,1)) \cap \bar{B}(p) \cap \{x \in X : u(x) = u(a(p))\}$ consists of two elements for a sufficiently small $\varepsilon > 0$. Define the demand at p to be one of these elements, either always the one on "the left hand side" or "the right hand side", where the viewpoint is from the line L.

The set $\bar{B}((\lambda+\varepsilon)(1,1,1)) \cap \{x \in X : u(x) = u(a(p))\}$ defines a simple closed curve in the plane $\bar{B}((\lambda + \varepsilon)(1,1,1))$. Given demand as defined above, every bundle on this curve will be revealed preferred to itself via a chain of other bundles on the same curve. In fact, any bundle demanded at any price vector will be revealed preferred to itself via such a chain of bundles on the curve defined by $\bar{B}((\lambda + \varepsilon)(1,1,1)) \cap \{x \in X : u(x) = u(a(p))\}$. Note that the sufficiency conditions given in Proposition 3.3

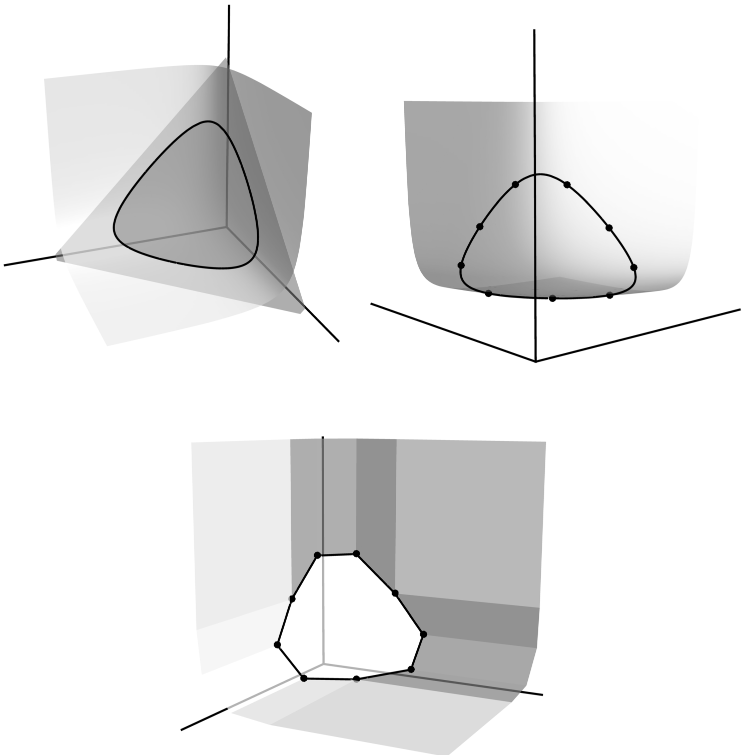


Figure 3.5.: A simple example for the construction of a cycle. Top Left: The indifference surface of the utility function $u(x_1, x_2, x_3) = x_1^{1/3} x_2^{1/3} x_3^{1/3}$ for $\bar{u} = 1$ and the plane $H = \{x \in \mathbb{R}^3_+ : 1/4(x_1 + x_2 + x_3) = 1\}$. Top Right: A set of points on the intersection of the indifference curve and the hyperplane. Bottom: The convex monotonic hull of the points.

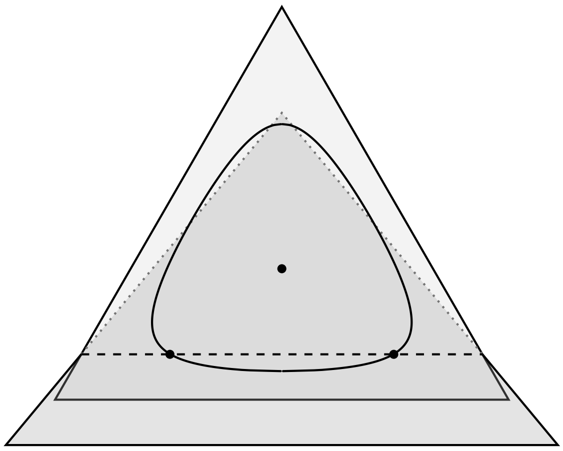


Figure 3.6.: Defining demand for all positive prices: The upper triangle (plane 1) depicts a budget plane with equal prices for each commodity; the point in the middle is it's center. The lower triangle is a budget plane which intersects the first plane at the dashed line (plane 2). The closed curve in the interior is the intersection of an indifference surface with plane 1. Demand at the price vector defining plane 2 is either the left or the right point; both points show the intersection of plane 2 with the indifference surface and plane 1.

are fullfilled. The demand therefore exhibits cycles of irreducible length $k > 2$.

Consider Figure 3.6. The upper triangle depicts a plane $\bar{B}(\lambda(1,1,1))$ and its center. The closed curve in the interior is the intersection of this plane and a set $\{x \in X : u(x) = u(a(p))\}$ for some $p \in P$. The darker lower triangle depicts the budget $\bar{B}(p)$; the dashed line shows the intersection of the two planes. The points on the left and right hand side show the intersection of the two planes with the set $\{x \in X : u(x) = u(a(p))\}$. If demand is defined to be the point on the right hand side, this bundle will be revealed preferred to all points on the segment of the curve between the two points. Changing p such that $\bar{B}(p)$ "rotates" around the center, we can construct cycles of arbirary length.

4

BEHIND THE SCENES: A GRAPHICAL WAY TO CONSTRUCT PREFERENCE CYCLES

4.1 INTRODUCTION

> *...you, who can actually* see *an angle, and contemplate the complete circumference of a Circle in the happy region of the Three Dimensions — how shall I make clear to you the extreme difficulty which we in Flatland experience in recognizing one another's configuration?*
>
> –Edwin A. Abbott, *Flatland*

The first proof that the Weak Axiom of Revealed Preference (WARP) does not imply the Strong Axiom of Revealed Preference (SARP) was provided by Gale (1960) by means of a counterexample. This chapter shows how one can easily construct preference cycles in three dimensions by "taking a look behind the scenes" of intersecting budget hyperplanes. By looking at the positive orthant of the Euclidean coordinate system through the origin of the system, the search for preference cycles is substantially simplified for suitable budgets.

Let $X = \mathbb{R}^\ell_+$ be the commodity space, where $\ell \geq 2$ denotes the number of different commodities. The price space is $P = \mathbb{R}^\ell_{++}$, and the space of price-income vectors is $P \times \mathbb{R}_{++}$. Consumers choose a single bundle $x^i = (x^i_1, \ldots, x^i_\ell)' \in X$ when facing a price vector $p^i = (p^i_1, \ldots, p^i_\ell) \in P$ and an income $w^i \in \mathbb{R}_{++}$. A budget set is then defined by $B^i = B(p^i, w^i) = \{x \in X : p^i x^i \leq w^i\}$. Demand is exhaustive, i.e. $p^i x^i = w^i$. Denote the boundary of the budget set B as $\bar{B} = \{x \in X : px = w\}$, so $x^i \in \bar{B}^i$. It is convenient to normalize prices by the level of expenditure at each observation, so that $p^i x^i = 1$ for all i. A set of n observations can then be denoted as $S = \{(x^i, p^i)\}^n_{i=1}$.

Let $R \subseteq X \times X$ be a binary relation on X. If $p^i x^i \geq p^i x$ then $\{x^i, x\} \in R$ and we say that the observation x^i is *directly revealed preferred* to x. For brevity, we write $x^i R x$. The observation x^i is *revealed preferred* to x, written $x^i R^* x$, if either $x^i R x$ or for some sequence of bundles $(x^j, x^k, \ldots, x^m)$ it is the case that $x^i R x^j$, $x^j R x^k$, …, $x^m R x$. In this case R^* is the *transitive closure* of the relation R.

The data set S satisfies the WARP if $x^i R x^j$, $x^i \neq x^j$, implies [not $x^j R x^i$]. The data set S satisfies the SARP if $x^i R^* x^j$, $x^i \neq x^j$, implies [not $x^j R x^i$].

An ordered set $\{(x^i, p^i)\}_{i=1}^k$ of k observations forms a *cycle of length* k if $p^i x^{i+1} \leq 1$ and $x^i \neq x^{i+1}$ for $i = 1, \ldots, k \mod k$, i.e. if x^i is indirectly revealed preferred to itself via the chain of observations $\{(x^i, p^i)\}_{i=1}^k$. A set $\{(x^i, p^i)\}_{i=1}^k$ forms a *cycle of irreducible length* k if it forms a cycle of length k and there is no shorter cycle (with a smaller k) by which x^i is indirectly revealed preferred to itself.

Obviously, WARP implies the absence of cycles of irreducible length two, whereas SARP implies the absence of cycles of arbitrary irreducible length.

In Section 4.2 it is shown how one can easily find a preference cycle of irreducible length three. Section 4.3 attempts to extend the approach to four dimensions.

4.2 THREE DIMENSIONS

For illustrative purposes consider the three budgets given by the price vectors $p_1 = (1,1,2)$, $p_2 = (2,1,1)$, $p_3 = (1,2,1)$. These three budgets are depicted in Figure 4.1.

Now take the coordinate system and rotate it until you look at the three budgets "from behind and below", i.e. through the origin (see Figure 4.2). Alternatively, project the three dimensional depiction including the axes on a suitable plane. Without the third dimensions it is now easy to see where we need to put the three choices associated with the three budgets in order to create a preference cycle of irreducible length three. As one can see in Figure 4.2, the three budgets are seperated into twelve segments.

The three outmost triangles are regions in which the subsets of the budgets associated with that triangle "are above" the remaining two

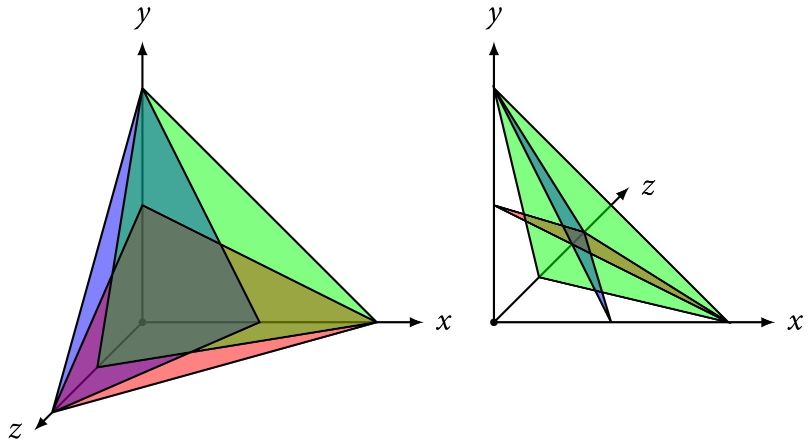


Figure 4.1.: Three intersecting budget hyperplanes, from two perspectives.

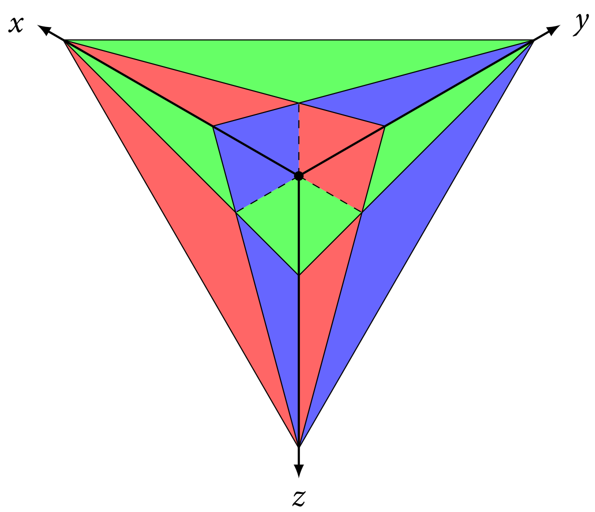


Figure 4.2.: A (stylized) look at the three budgets from Figure 4.1 through the origin. One can also think of it as Figure 4.1 after pressing it down on a flat surface.

budgets. The six inner triangles which extend from the intersection with the axes towards the origin in the center describe regions in which each of the budgets "is above" one budget and "below" another budget. Finally, the three quadrangles in the center decribe regions in which the budget is "below" both of the remaining two budgets.

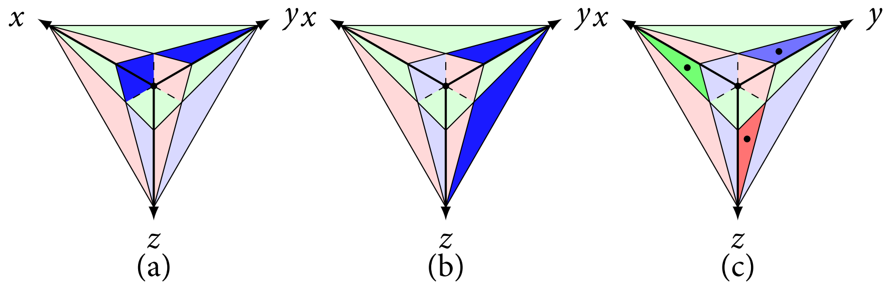


Figure 4.3.: We would like to construct a preference cycle of irreducible length three, such that x^1Rx^2, x^2Rx^3, x^3Rx^1. Consider the budget B^2, given by $p_2 = (2,1,1)$. First, x^1Rx^2 requires that x^2 be in the budget B^1, given by $p_1 = (1,1,2)$. This narrows the possible area down to the two dark segments in Figure 4.3(a). Next, [not x^3Rx^2] requires that x^2 is not in the budget B^3, given by $p_3 = (1,2,1)$. This narrows the possible area down to the two dark segments in Figure 4.3(b). Combining the two restrictions leaves only the one segment in the upper right corner. Apply the same reasoning to the remaining two choices. Figure 4.3(c) shows possible choices for a preference cycle.

Consider a preference cycle of irreducible length three: x^1Rx^2, x^2Rx^3, x^3Rx^1, such that [not x^2Rx^1], [not x^3Rx^2], [not x^1Rx^3]. Start with x^2: The condition x^1Rx^2 means that x^2 is in the budget set at which x^1 is chosen. This restricts the placement of x^2 to the two segments in which B^2 "is below" B^1. Furthermore, the condition [not x^3Rx^2] means that x^2 must not be in the budget set at which x^3 is chosen. This restricts the placement of x^2 to the two segments in which B^2 "is above" B^3. Combining the two restrictions narrows the placement of x^2 down to one segment. Applying the same reasoning to x^1 and x^3, we end up with one possible segment for each bundle. See Figure 4.3. Any placement in these segments will satisfy the conditions given above – the three bundles form a cycle of irreducible length three; WARP is satisfied, whereas SARP is violated.

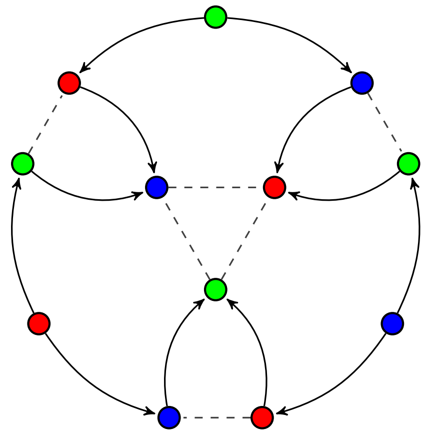


Figure 4.4.: A graph representing the segments of budget intersections. Each node represents a unique budget segment. An arrow from a node v_i to a node v_j indicates that any bundle in the budget of the segment represented by v_i is directly revealed preferred to a bundle in the segment represented by v_j. Dashed lines indicate that the represented segments share a border in Fig. 4.2 without yielding a revealed preference relation.

Figure 4.2 can also be represented by a directed graph: Define an ordered pair $G(V, A)$, where V is a set of nodes or vertices representing segments in Figure 4.2, and A is a set of ordered pairs of nodes, called arcs. Two nodes v_i and v_j are connected by an arc, i.e. $\{(v_i, v_j)\} \subseteq A$, if and only if any bundle *in the budget of the segment* represented by v_i is directly revealed preferred to any bundle *in the segment* represented by v_j. Consider the outer node north northeast of the center in Figure 4.4. This node represents a segment of bundle B^2 defined by $p_2 = (2, 1, 1)$. If the consumer choses a bundle x^2 in that segment when facing B^2, then any bundle in B^1 defined by $p_1 = (1, 1, 2)$ will be directly revealed preferred to x^2. This is because there is an arc from a node of B^1 directed to the node on which x^2 is chosen. Using a similar reasoning as for

Figure 4.2, one can easily find the nodes on which decisions have to be placed in order to construct a preference cycle.

4.3 FOUR DIMENSIONS

Suppose a person of the Fourth Dimension, condoscending to visit you, were to say 'Whenever you open you eyes, you see a Plane (which is of Two Dimensions) and you infer *a Solid (which is of Three); but in reality you also see (though you do not recognize) a Fourth Dimension, which is not colour nor brightness nor anything of the kind, but a true Dimension, although I cannot point out to you its direction, nor you can possibly measure it.'*
–Edwin A. Abbott, *Flatland*

In the previous section we represented three dimensional budgets as segments of a triangle. Budgets in four dimensions can be represented as segments in a tetrahedron. For a two-dimensional representation, the tetrahedron can be "unfolded". Consider Figure 4.5, which is an arrangement of four triangles. Each triangle is a 2-face of the original tetrahedron. Each of the four budgets depicted in Figure 4.5 appears explicitly in three of the four faces. Any bundle in any of these budgets is uniquely defined by three and only three points; one point in each face in which the budget appears.

To use graphs to represent the bugets, we now have to draw four seperate graphs, one for each face of the 3-simplex. This is done in Figure 4.6.

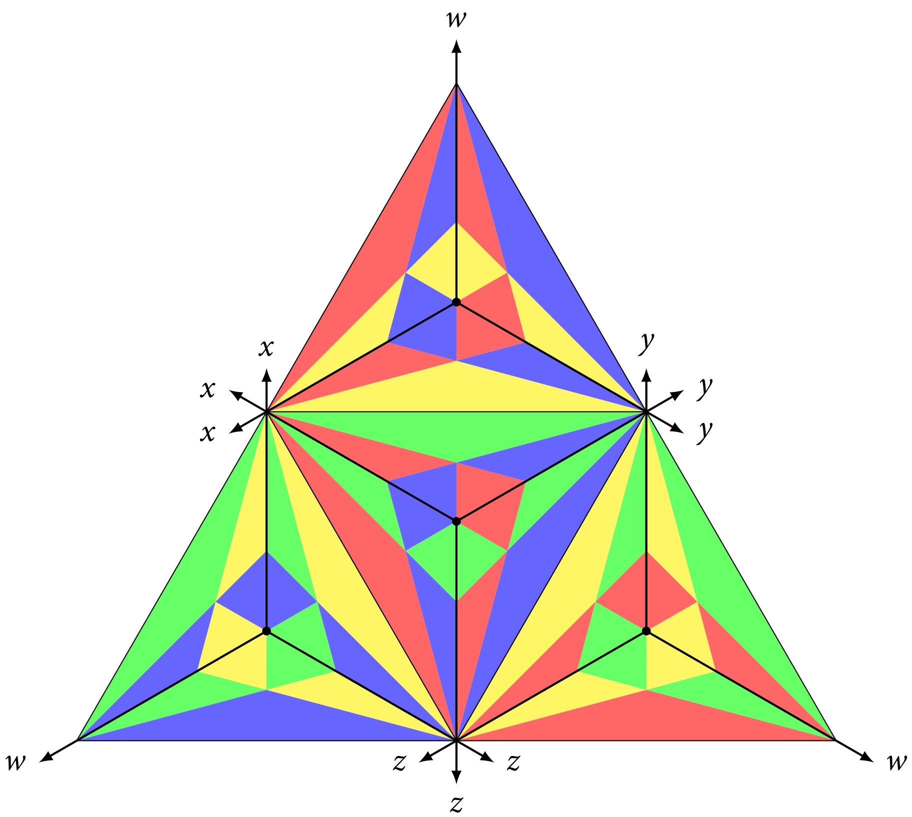


Figure 4.5.: Four budgets, four dimensions.

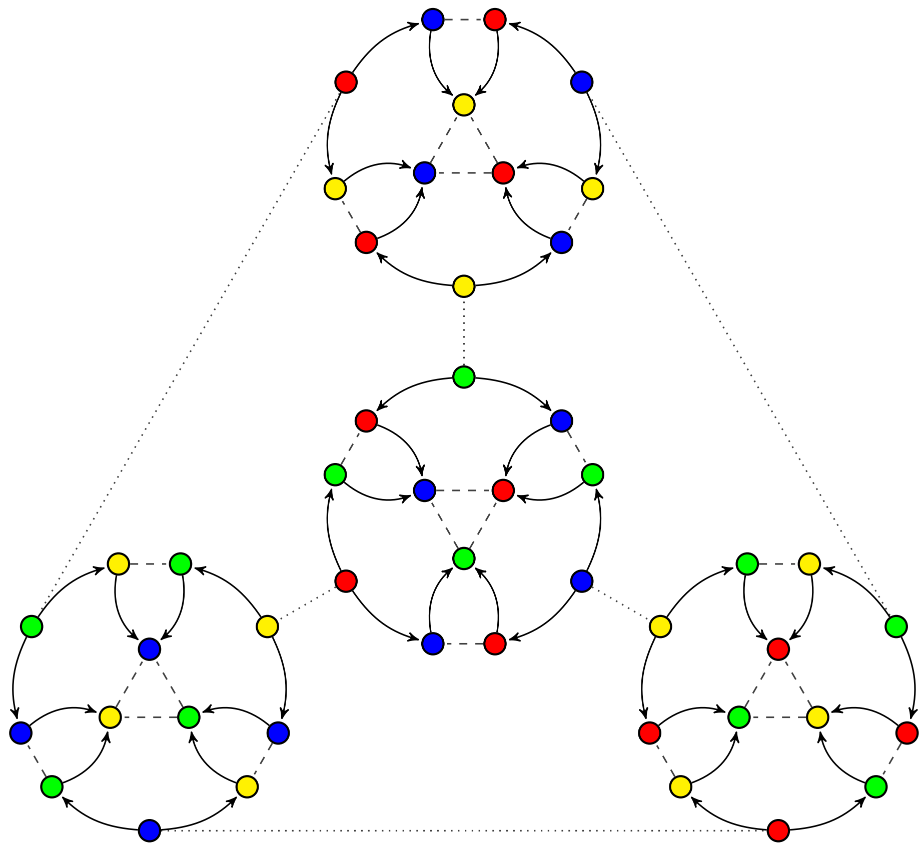


Figure 4.6.: The four graphs, representing four four-dimensional budgets. Dotted lines indicate that the two segments share an edge of the 3-simplex.

5 HOMOTHETIC PREFERENCES: THE TWO-COMMODITY CASE

5.1 INTRODUCTION

Homotheticity of consumer preferences is featured importantly in theory and applications. If preferences are homothetic, it is possible to deduce a consumer's entire preference relation from a single indifference set. Assuming homothetic preferences provides useful restrictions for the analysis of consumer demand. In applications researchers often focus on special types of homothetic preferences, like those defined by the CES utility function. A researcher who wishes to estimate homothetic demand functions using consumption data might wish to test if the data could have been generated by a homothetic utility function.

A common nonparametric test of the utility maximization hypothesis has been developed by Afriat (1967) and refined by Varian (1982, 1983). In applications, especially in controlled laboratory experiments, the commodity space is often only two dimensional (e.g. Harbaugh et al. 2001, Andreoni and Miller 2002, Choi et al. 2007*a*, and Fisman et al. 2007). For the two-commodity case Rose (1958) showed that satisfying the WARP as introduced by Samuelson (1938) is sufficient for utility maximization. In other words: Testing for WARP is equivalent to testing for SARP as introduced by Houthakker (1950).

Gale (1960) showed that Rose's finding does not hold for three dimensions, a result extended to arbitrary dimensions by Peters and Wakker (1994, 1996).[1] Banerjee and Murphy (2006) used the result for two dimensions to develop a simplified test for utility maximization.

In this chapter it is shown that Rose's result carries over to homothetic rationalization. That is, in the two-commodity case pairwise testing of observations is sufficient to test a set of observations on consumption choices for consistency with the maximization of a homothetic utility

1 An intuitive general proof can be found in Chapter 3.

function. The result is stated as a pairwise version of Varian's (1983) Homothetic Axiom of Revealed Preference (HARP).

Section 5.2 reviews the relevant part of revealed preference theory and introduces the Pairwise Homothetic Axiom of Revealed Preference (PHARP). In is shown that in two dimensions PHARP is equivalent to HARP. This section also provides an overview of the relationship between testable axioms of revealed preference. Chapter 7 contains applications.

5.2 THEORY

5.2.1 *Preliminaries*

For the particularities of the chapter it is convenient to slightly deviate from the notation introduced in Chapter 3. We use the following notation: $\mathbb{R}^\ell_+ = \{\mathbf{z} \in \mathbb{R}^\ell : \mathbf{z} \geq 0\}$, where $\mathbf{z} = (z_1, \ldots, z_l)$, $\mathbb{R}^\ell_{++} = \{\mathbf{z} \in \mathbb{R}^\ell : \mathbf{z} \gg 0\}$, where " $\mathbf{z} \geq 0$ " means " $z_i \geq 0$ for all i ", and " $\mathbf{z} \gg 0$ " means " $z_i > 0$ for all i ". Note the convention to use subscripts to denote scalars or vector components and superscripts to index bundles.

Let $\mathbb{R}^\ell_+$ be the consumption space, where $\ell \geq 2$ denotes the number of different commodities. The price space is $\mathbb{R}^\ell_{++}$, and the space of price-income vectors is $\mathbb{R}^\ell_{++} \times \mathbb{R}_{++}$. Consumers choose bundles $\mathbf{z}^i = (z^i_1, \ldots, z^i_l)' \in \mathbb{R}^\ell_+$ when facing a price vector $\mathbf{p}^i = (p^i_1, \ldots, p^i_l) \in \mathbb{R}^\ell_{++}$ and an income $w^i \in \mathbb{R}_{++}$. A budget set is then defined by $B^i = B(\mathbf{p}^i, w^i) = \{\mathbf{z} \in \mathbb{R}^\ell_+ : \mathbf{p}^i\mathbf{z}^i \leq w^i\}$. It is assumed that demand is exhaustive, i.e. $\mathbf{p}^i\mathbf{z}^i = w^i$.[2] An observation on a consumption choice is denoted by $(\mathbf{z}^i, B^i)$. Let S be a finite set of observations on a consumer, and let n denote the number of observations, so that $S = \{(\mathbf{z}^i, B^i)\}^n_{i=1}$.

An observation $\mathbf{z}^i$ is *directly revealed preferred* to a bundle $\mathbf{z}$, written $\mathbf{z}^i R \mathbf{z}$, if $\mathbf{p}^i\mathbf{z}^i > \mathbf{p}^i\mathbf{z}$. An observation $\mathbf{z}^i$ is *revealed preferred* to a bundle $\mathbf{z}$, written $\mathbf{z}^i R^* \mathbf{z}$, if for some sequence of bundles $\mathbf{z}^j, \mathbf{z}^k, \ldots, \mathbf{z}^m$ it is the case that $\mathbf{z}^i R \mathbf{z}^j$, $\mathbf{z}^j R \mathbf{z}^k$, …, $\mathbf{z}^m R \mathbf{z}$. In this case R^* is the *transitive closure* of the relation R. We also need a revealed preference relation which was not introduced before:

2 This condition is known as *budget balancedness* and is often violated in practice. Section 7.2.3 addresses the problem.

Definition 5.1 (Strict revealed preference relation) An observation $\mathbf{z}^i$ is *strictly directly revealed preferred* to a bundle $\mathbf{z}$, written $\mathbf{z}^i P \mathbf{z}$, if $\mathbf{p}^i \mathbf{z}^i > \mathbf{p}^i \mathbf{z}$.

5.2.2 *Revealed Preference Axioms*

The
preference are the WARP, SARP, and the Generalized Axiom of Revealed Preference (GARP). Definitions for WARP and SARP are given in Section 3.2 (Definition 3.2 and 3.3). GARP was introduced by Varian (1982) as a reformulation of Afriat's (1967) "cyclic consistency" condition:

Definition 5.2 (Generalized Axiom of Revealed Preference) The data set S satisfies the Generalized Axiom of Revealed Preference (GARP) if $\mathbf{z}^i R^* \mathbf{z}^j$ implies $[\text{not } \mathbf{z}^j P \mathbf{z}^i]$.

Definition 5.3 (Rationalizability) A utility function $u(x)$ *rationalizes* a data set S if $u(\mathbf{z}^i) \geq u(\mathbf{z})$ for all $\mathbf{z}$ such that $\mathbf{p}^i \mathbf{z}^i \geq \mathbf{p}^i \mathbf{z}$ for all $i = 1, \ldots, n$.

SARP is a necessary and sufficient condition for the existence of a strictly concave, strictly monotonic, continuous utility function that rationalizes the data (see Houthakker 1950 and Matzkin and Richter 1991). GARP is a necessary and sufficient condition for the existence of a concave, monotonic, continuous utility function that rationalizes the data (see Afriat 1967 and Varian 1982).

Definition 5.4 (Homothetic utility function) A utility function is *homothetic* if it is a positive monotonic transformation of a utility function that is homogeneous of degree 1.

HARP was introduced by Varian (1983):

Definition 5.5 (Homothetic Axiom of Revealed Preference) The data set S satisfies the Homothetic Axiom of Revealed Preference (HARP) if for all distinct choices of indices $(i, \ldots, m)$

$$(\mathbf{p}^i \mathbf{z}^j)(\mathbf{p}^j \mathbf{z}^k) \cdots (\mathbf{p}^m \mathbf{z}^i) \geq w^i w^j w^k \cdots w^m.$$

HARP is a necessary and sufficient condition for the existence of a concave, monotonic, continuous, homothetic utility function that rationalizes the data (see Varian 1983).

Rose (1958) showed that in two dimensions WARP is equivalent to SARP. Banerjee and Murphy (2006) introduced a weaker condition than GARP, WGARP, and showed that in two dimensions WGARP and GARP are equivalent.

Definition 5.6 (Weak Generalized Axiom of Revealed Preference) The data set S satisfies the Weak Generalized Axiom of Revealed Preference (WGARP) if $\mathbf{z}^i R \mathbf{z}^j$ implies [not $\mathbf{z}^j P \mathbf{z}^i$].

5.2.3 *Homotheticity and Two-Commodity Choice*

Following Varian (1983) and Knoblauch (1993), define a scalar $t^{i,j}$ for all i and j by

$$t^{i,j} = \min \left\{ \left(\frac{\mathbf{p}^i \mathbf{z}^k}{w^i} \right) \left(\frac{\mathbf{p}^k \mathbf{z}^l}{w^k} \right) \cdots \left(\frac{\mathbf{p}^m \mathbf{z}^j}{w^m} \right) \right\}, \tag{5.1}$$

where the minimum is over all distinct choices of indices $k, l, \ldots, m$.

Definition 5.7 (Homothetically revealed preference relation) Let $t^{i,i} = 1$. Then $t^{i,j}\mathbf{z}^i$ is *homothetically revealed preferred* to $\mathbf{z}^j$, written $t^{i,j}\mathbf{z}^i H \mathbf{z}^j$.

Note that $t = t^{i,j}$ is the smallest scalar for which $t\,\mathbf{z}^i H \mathbf{z}^j$, so that $t^{i,j}\mathbf{z}^i$ is a vertex on the set of bundles that are homothetically revealed preferred to $\mathbf{z}^j$ (see Knoblauch 1993). HARP is then equivalent to [not $\mathbf{z}^i P t^{j,i}\mathbf{z}^j$] for all i and j.

When the consumption space is two dimensional, the budgets can be ranked by the price ratio. Let $\mathbf{z}^i = (x^i, y^i)'$ and choose good x as the numeraire. Then $\mathbf{p}^i = (1, q^i)$, where q^i is the relative price of good y. Let the income w^i be redefined appropriately. Without loss of generality, let the data S be ordered by q such that $q^i \geq q^{i+1}$. If there are observations with the same q, let them be ordered such that $y^i/x^i \leq y^{i+1}/x^{i+1}$.

It is a well known fact that homotheticity implies that income expansion paths are straight lines through the origin.[3] It is easy to show that the slope of the expansion path, y/x, must increase as the relative price of y decreases: In the case of homotheticity, $(\mathbf{p}^i\mathbf{z}^j)(\mathbf{p}^j\mathbf{z}^i) \geq (\mathbf{p}^i\mathbf{z}^i)(\mathbf{p}^j\mathbf{z}^j)$. That is equivalent to $(q^i - q^j)(x^i y^j - y^i x^j) \geq 0$. If $i < j$, then $(q^i - q^j) \geq 0$, so it must be that $(x^i y^j - y^i x^j) \geq 0$. Thus $y^i/x^i \leq y^j/x^j$, and analogously for $i > j$. This is obviously a necessary condition for homotheticity, but it is not obvious that it is also sufficient.

In this chapter, we introduce a new axiom for homothetic choice, the PHARP, and show that in two dimensions, PHARP is already sufficient for the existence of a homothetic utility function that rationalizes the data.

Definition 5.8 (Pairwise Homothetic Axiom of Revealed Preference) The data satisfy the Pairwise Homothetic Axiom of Revealed Preference (PHARP) if for all distinct choices of indices i, j

$$(\mathbf{p}^i\mathbf{z}^j)(\mathbf{p}^j\mathbf{z}^i) \geq w^i w^j.$$

Theorem 5.1 For two-dimensional commodity spaces the following conditions are equivalent:

(1) there exists a concave, monotonic, continuous, non-satiated, homothetic utility function that rationalizes the data;

(2) the data satisfy HARP;

(3) the data satisfy PHARP.

The following Lemma will be helpful:

Lemma 5.1 Define a scalar $\theta^{i,j}$, where i and j are indices, by

$$\theta^{i,j} = \prod_{k=j}^{i-1} \frac{\mathbf{p}^{k+1}\mathbf{z}^k}{w^{k+1}} \text{ if } i > j \quad \text{and} \quad \theta^{i,j} = \prod_{k=i}^{j-1} \frac{\mathbf{p}^k\mathbf{z}^{k+1}}{w^k} \text{ if } i < j.$$

If the commodity space is two-dimensional, the data are ordered by q such that $q^i \geq q^{i+1}$, and the data satisfy PHARP, then $\theta^{i,j} = t^{i,j}$.

---

3 Note that the HARP also allows for expansion cones rather than expansion lines; see Section 5.2.5.

Proof of Lemma 5.1 Choose a $\mathbf{z}^0$ without loss of generality. It is first shown that $\theta^{1,0} = t^{1,0}$. Remember that the observations are ordered such that $q^i \geq q^{i+1}$.

$$\begin{aligned} \theta^{1,0} = \frac{\mathbf{p}^1\mathbf{z}^0}{\mathbf{p}^1\mathbf{z}^1} \leq \frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1}\frac{\mathbf{p}^i\mathbf{z}^0}{\mathbf{p}^i\mathbf{z}^i} &\Leftrightarrow (\mathbf{p}^1\mathbf{z}^0)(\mathbf{p}^i\mathbf{z}^i) - (\mathbf{p}^1\mathbf{z}^i)(\mathbf{p}^i\mathbf{z}^0) \leq 0 \\ &\Leftrightarrow (q^1 - q^i)(x^i y^0 - x^0 y^i) \leq 0. \end{aligned}$$

The last line is true because if $i > 1$, the first term is positive and the second term is negative, and vice versa if $i < 1$. Now suppose

$$\theta^{1,0} \leq \frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1} \cdots \frac{\mathbf{p}^k\mathbf{z}^0}{\mathbf{p}^k\mathbf{z}^k}$$

for a sequence $i, \ldots, k$ of length n. Then $\theta^{1,0}$ is also less than or equal to a sequence $i, \ldots, \ell$ of length $n+1$ because

$$\begin{aligned} &\frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1} \cdots \frac{\mathbf{p}^k\mathbf{z}^0}{\mathbf{p}^k\mathbf{z}^k} \leq \frac{\mathbf{p}^1\mathbf{z}^i}{\mathbf{p}^1\mathbf{z}^1} \cdots \frac{\mathbf{p}^k\mathbf{z}^\ell}{\mathbf{p}^k\mathbf{z}^k}\frac{\mathbf{p}^\ell\mathbf{z}^0}{\mathbf{p}^\ell\mathbf{z}^\ell} \\ \Leftrightarrow\ &(\mathbf{p}^k\mathbf{z}^0)(\mathbf{p}^\ell\mathbf{z}^\ell) \leq (\mathbf{p}^k\mathbf{z}^\ell)(\mathbf{p}^\ell\mathbf{z}^0), \end{aligned}$$

where the last line is true for similar reasons as above. So $\theta^{1,0} = t^{1,0}$.

It is now possible to show that $\theta^{n,0} = t^{n,0}$ implies $\theta^{n+1,0} = t^{n+1,0}$, which concludes the proof by induction: Write $\theta^{n+1,0}$ as $\theta^{n+1,n}\theta^{n,0}$ and note that $\theta^{n+1,n} = (\mathbf{p}^{n+1}\mathbf{z}^n)/(\mathbf{p}^{n+1}\mathbf{z}^{n+1})$. Then it is to be shown that

$$\frac{(\mathbf{p}^{n+1}\mathbf{z}^n)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})}\theta^{n,0} \leq \frac{(\mathbf{p}^{n+1}\mathbf{z}^i)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})}\frac{(\mathbf{p}^i\mathbf{z}^j)}{(\mathbf{p}^i\mathbf{z}^i)} \cdots \frac{(\mathbf{p}^k\mathbf{z}^0)}{(\mathbf{p}^k\mathbf{z}^k)},$$

for sequences of $i, \ldots, k$ of arbitrary length. By assumption,

$$\theta^{n,0} \leq \frac{(\mathbf{p}^n\mathbf{z}^i)}{(\mathbf{p}^n\mathbf{z}^n)}\frac{(\mathbf{p}^i\mathbf{z}^j)}{(\mathbf{p}^i\mathbf{z}^i)} \cdots \frac{(\mathbf{p}^k\mathbf{z}^0)}{(\mathbf{p}^k\mathbf{z}^k)}.$$

It is then sufficient that

$$\frac{(\mathbf{p}^{n+1}\mathbf{z}^n)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})} \leq \frac{(\mathbf{p}^{n+1}\mathbf{z}^i)}{(\mathbf{p}^{n+1}\mathbf{z}^{n+1})}\frac{(\mathbf{p}^n\mathbf{z}^n)}{(\mathbf{p}^n\mathbf{z}^i)}$$

holds, which is true if $n > 0$. The proof works analogously for $n < 0$. ■

Proof of Theorem 5.1 For (1) ⇔ (2), see Varian (1983). It is obvious that (2) ⇒ (3). We will show that (3) ⇒ (2).

Choose a $\mathbf{z}^0$ without loss of generality, i.e. assign indices such that q^0 is the highest, the lowest, or somewhere between the highest and the lowest relative price. Then [not $\mathbf{z}^0 P\theta^{1,0}\mathbf{z}^1$] if PHARP is satisfied. We need to show that this implies [not $\mathbf{z}^0 P\theta^{n,0}\mathbf{z}^n$] for all $n > 0$.

Suppose [not $\mathbf{z}^0 P\theta^{n,0}\mathbf{z}^n$]. Then

$$\begin{aligned}
& \mathbf{p}^0\mathbf{z}^0 \leq (\mathbf{p}^0\mathbf{z}^n)\theta^{n,0} \leq (\mathbf{p}^0\mathbf{z}^{n+1})\theta^{n+1,0} = (\mathbf{p}^0\mathbf{z}^{n+1})\theta^{n+1,n}\theta^{n,0} \\
\Leftrightarrow\ & \mathbf{p}^0\mathbf{z}^n \leq (\mathbf{p}^0\mathbf{z}^{n+1})\theta^{n+1,n} \\
\Leftrightarrow\ & (\mathbf{p}^0\mathbf{z}^n)(\mathbf{p}^{n+1}\mathbf{z}^{n+1}) \leq (\mathbf{p}^0\mathbf{z}^{n+1})(\mathbf{p}^{n+1}\mathbf{z}^n) \\
\Leftrightarrow\ & (q^0 - q^{n+1})(x^{n+1}y^n - x^n y^{n+1}) \leq 0.
\end{aligned}$$

It is easy to see that the last line is true because if $n > 0$ and PHARP is satisfied the first term on the left hand side is positive while the second term is negative. A similar argument applies when $n < 0$. This proves that [not $\mathbf{z}^0 P\theta^{1,0}\mathbf{z}^1$] implies [not $\mathbf{z}^0 P\theta^{n,0}\mathbf{z}^n$] for arbitrary $\mathbf{z}^0$. So PHARP implies HARP. ■

Figure 5.1 illustrates the necessary and sufficient conditions for homotheticity and the construction of an indifference map from a set of observations on a consumer. Figure 5.1.(a) shows five observations on a consumer. The dashed lines are rays through the origin and the observations. Figure 5.1.(b) illustrates the necessary condition for HARP: Four of the five observations are projected on the budget line of the remaining observation. Here the dashed lines indicate the slope of the budget lines the decisions were made on. As we move up the budget line of the observation in the middle, the shifted budget lines turn clockwise. This is a necessary condition for homotheticity. A piecewise linear indifference curve rationalizes the observations; note that the linear segments parallel the dashed lines. Figure 5.1.(c) shows how we can construct an indifference map using the scalar factors defined above.[4] The indifference curve to the upper right is the tightest bound on the indifference

4 The figure is similar to Figure 1 in Knoblauch (1993).

curve through the observation in the middle; the other two indifference curves are scaled down using factors 3/4 and 1/2.

5.2.4 *Relationships between the Axioms*

If a set of data fails to satisfy HARP or PHARP, what conclusions can be drawn? A consumer who violates GARP will necessarily also violate HARP. On the other hand, a consumer who satisfies GARP will not necessarily satisfy HARP. Absent a measure for the extent of any violation, testing data for homotheticity will therefore only yield additional information if the data already satisfy GARP. To the best of the author's knowledge this relationship between HARP and GARP has not been formally stated in the literature. It is therefore formally proven in Proposition 5.1.

Proposition 5.1 HARP implies GARP. GARP does not imply HARP.

Proof Varian (1983, Theorem 2) shows that HARP is equivalent to the existence of a concave, monotonic, continuous, non-satiated, homothetic utility function. He also shows (1983, Theorem 1) that the existence of a concave, monotonic, continuous, non-satiated utility function is equivalent to GARP. Homotheticity is an additional restriction on preferences, thus HARP implies GARP. To make the point clearer, consider the Afriat inequalities

$$U^i \leq U^j + \lambda^j \mathbf{p}^j (\mathbf{z}^i - \mathbf{z}^j), \tag{5.2}$$

for $i, j = 1, \ldots, N$, with N being the number of observations. GARP is equivalent to the existence of numbers $U^i, \lambda^i > 0$, $i = 1, \ldots, N$ such that the Afriat inequalities are satisfied. For convenience, normalize the prices by the level of expenditure so that $w^i = \mathbf{p}^i \mathbf{z}^i = 1$ for $i = 1, \ldots, N$. Let $\lambda^i = U^i$. Then the Afriat inequalities become

$$U^i \leq U^j \mathbf{p}^j \mathbf{z}^i. \tag{5.3}$$

Obviously, if there exist numbers U^i such that the inequalities 5.3 are satisfied, there also exist numbers $U^i, \lambda^i > 0$ such that the inequalities 5.2 are satisfied. The existence of numbers $U^i > 0$ such that the inequalities 5.3 are satisfied is equivalent to HARP (Varian 1983, Theorem 2).

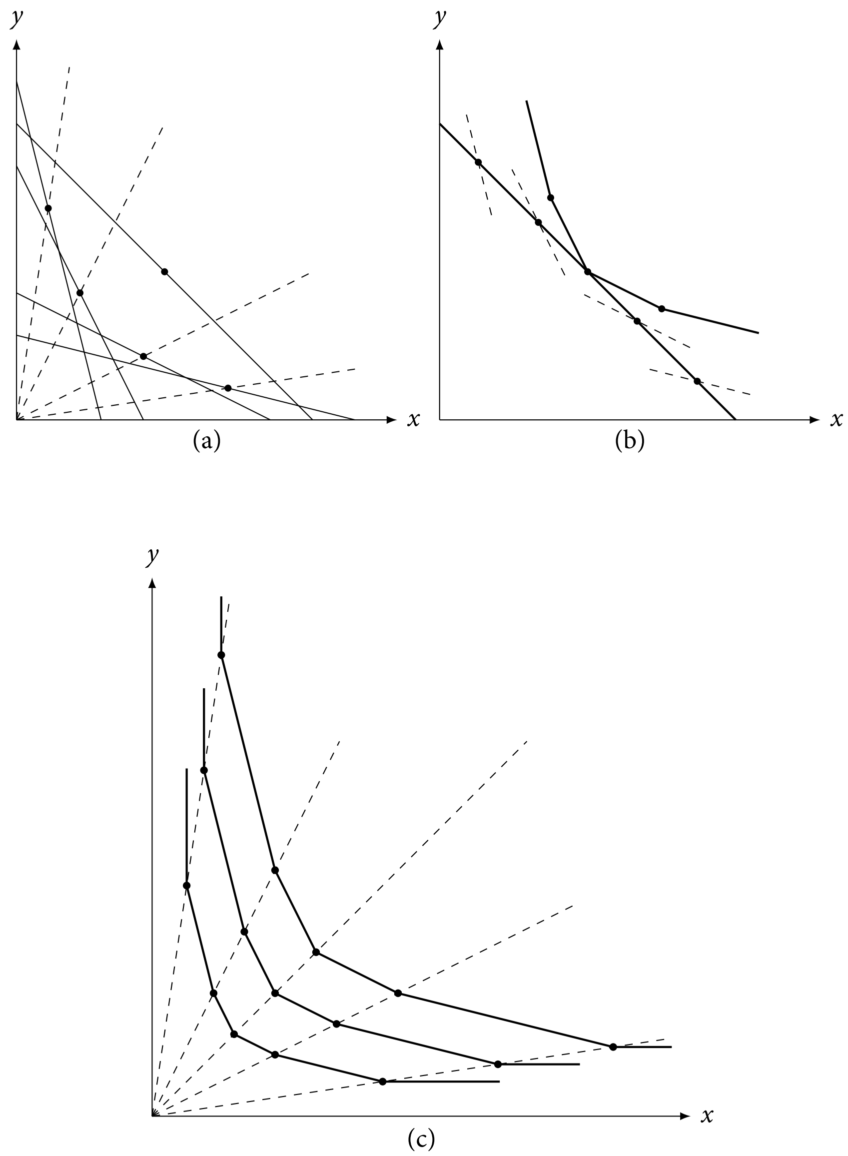


Figure 5.1.: Top left (a): Five observations on a consumer and rays through the origin and the observations. Top right (b): All observations are projected on one of the budget lines. The dashed lines show the slope of the projected observations' budgets. Bottom (c): The construction of piecewise linear indifference curves.

A simple counterexample demonstrates that GARP does not imply HARP. Suppose $\mathbf{p}^1 = (1,1)$ and $\mathbf{p}^2 = (2,4)$, $w^1 = w^2 = 1$. The budgets do not intersect, so GARP cannot be violated. But with $\mathbf{z}^1 = (1,0)$ and $\mathbf{z}^2 = (0,1/4)$, HARP is violated. ■

Varian (1983) implicitly intended HARP to be a restriction on preferences in addition to GARP. It is also interesting to explore the relationship between HARP and WARP:

Proposition 5.2 HARP is neither necessary nor sufficient for WARP or SARP.

Proof In the example shown in Figure 5.2.(a), WARP and SARP are satisfied: $\mathbf{z}^2 R^* \mathbf{z}^1$, [not $\mathbf{z}^1 R^* \mathbf{z}^2$]. HARP is violated because $y_1/x_1 < y_2/x_2$ and the relative price of good y is lower for budget B^1.

In the example shown in Figure 5.2.(b), WARP and SARP are violated: $\mathbf{z}^2 R^* \mathbf{z}^1$, $\mathbf{z}^1 R^* \mathbf{z}^2$. HARP is satisfied because the price vector is the same for both budgets, which implies $t^{1,2} = t^{2,1} = 1$. ■

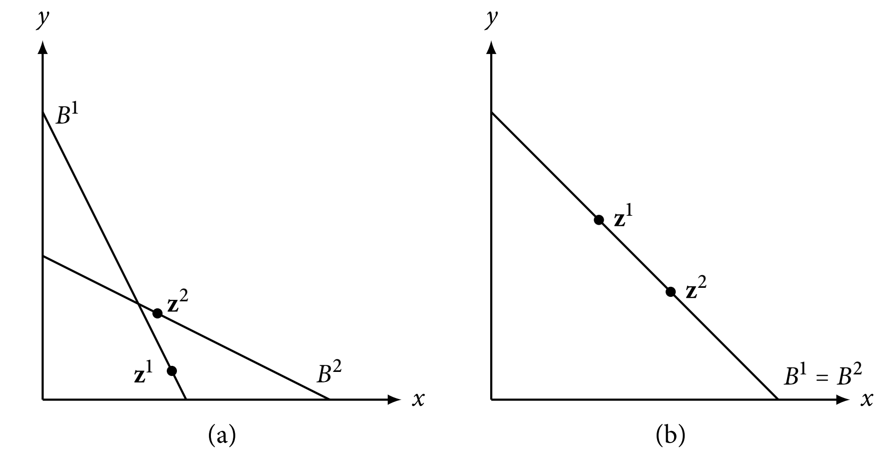


Figure 5.2.: Left: HARP is not necessary for WARP. Right: HARP is not sufficient for WARP.

Figure 5.3 summarizes the relationships between the axioms in the general case and in the case of two commodities, where "→" means "implies" and "↔" means "is equivalent to". See the appendix for a list of references to existing proofs and new proofs.

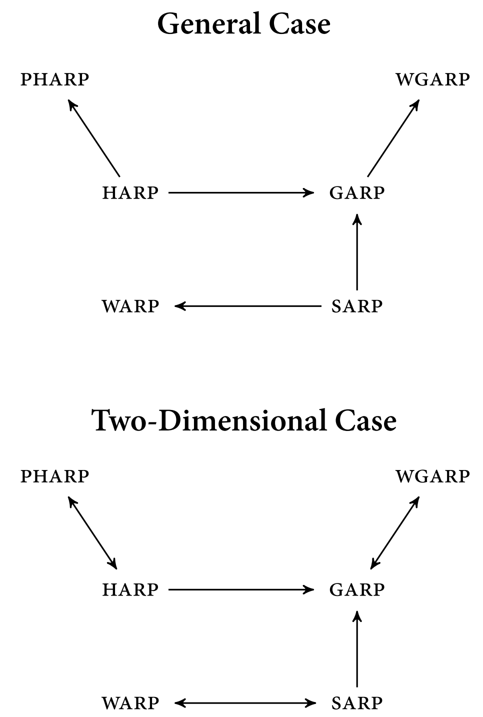


Figure 5.3.: Relationship between the axioms.

5.2.5 *Expansion Lines and Expansion Cones*

GARP does not exclude demand correspondences, i.e. multi-valued demand. Does this mean that HARP-consistent decisions of a consumer can imply expansion cones rather than expansion lines, or does HARP impose greater restrictions on GARP-consistent than WARP-consistent choices?[5] The answer is given in Proposition 5.3.

Proposition 5.3 A homothetic utility function that rationalizes a data set that satisfies HARP but not WARP implies expansion cones rather than expansion lines. If a homothetic utility function that implies expansion cones rationalizes a finite set of observations, the data set does not necessarily violate WARP. The demand function derived from such a utility function cannot, however, satisfy WARP.

Proof A data set that satisfies HARP also satisfies GARP. A data set can only satisfy GARP and simultaneously violate WARP if at least two different choices are made on the same budget set, i.e. if demand is multi-valued. Consider Figure 5.4. Suppose that $\mathbf{z}^1$ and $\mathbf{z}^2$ are chosen from budgets $B^1 = B^2$. Then GARP implies that the consumer is indifferent between $\mathbf{z}^1$, $\mathbf{z}^2$, and $\alpha\,\mathbf{z}^1 + (1-\alpha)\,\mathbf{z}^2$ for all $\alpha \in [0,1]$. HARP implies that the consumer is also indifferent between $\tau\,\mathbf{z}^1$, $\tau\,\mathbf{z}^2$, and $\alpha\,\tau\,\mathbf{z}^1 + (1-\alpha)\,\tau\,\mathbf{z}^2$ for any scalar $\tau > 0$. Thus any homothetic utility function which rationalizes the observations implies expansion cones.

Consider Figure 5.5. Both indifference curves rationalize the single observation. The solid indifference curve implies that the observation is the unique utility maximizing element in the budget. The dashed indifference curve implies that all points on the budget which lie between the two intersections of the budget with the rays through the origin yield the same utility. Obviously, no axiom is violated given the single observation. But while the solid indifference curve is consistent with WARP, the dashed indifference curve is not. This obviously holds in general: A utility function which implies expansion cones implies that more than one element of a single budget is utility maximizing, which violates WARP. But the observations which are rationalized do not violate WARP if they can also be rationalized by a strictly concave utility function. The reason is that the revealed preference relation is empirically defined and generally not complete, whereas a continuous utility function defined on $\mathbb{R}^\ell$ represents a complete preference relation. ■

5 I owe this interesting question to an anonymous referee.

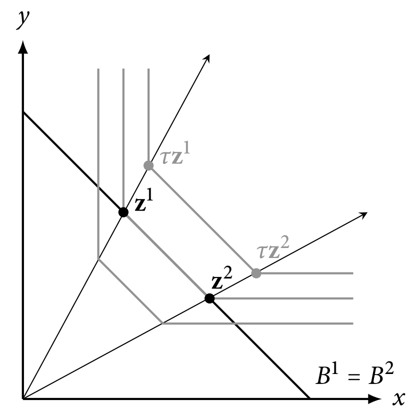


Figure 5.4.: Expansion cones.

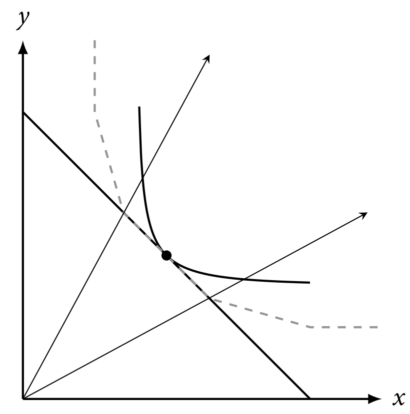


Figure 5.5.: Different indifference curves rationalize the observation.

6

STOCHASTIC REVEALED PREFERENCE AND RATIONALIZABILITY

6.1 INTRODUCTION

The idea that if there is any consistency in the behavior of individuals this consistency is of a stochastic nature is not new. It has already been considered in the economic literature by Georgescu-Roegen (1936), Quandt (1956), Luce (1958), Davidson and Marschak (1959), Debreu (1958), Block and Marschak (1960), Marschak (1959), Hildenbrand (1971), Barberá and Pattanaik (1986), and others. Bandyopadhyay, Dasgupta, and Pattanaik (1999) (henceforth BDP) initiated a line of investigation in which they explored choice behavior of a consumer who chooses in a stochastic fashion from different budget sets. In BDP (2002) this approach was extended by an interpretation of tuples of deterministic demand functions of different consumers as a stochastic demand function. They define a weak axiom of stochastic revealed preference which is implied by but does not imply rationalizability in terms of stochastic orderings on the commodity space.[1] In BDP (2004), the authors note that

> it is not at all obvious what would be a natural stochastic translation of the familiar strong axiom of revealed preference and what would be the implications of such a 'strong axiom of stochastic revealed preference'.

It is the purpose of this paper to explore rationalizability issues, provide a necessary and sufficient condition for rationalizability in terms of stochastic orderings, and to discuss related problems.

Suppose a consumer specifies a probability for each subset of a given budget such that the probability assignments add up to unity. Suppose further that we observe these probability assignments on a finite set of budgets. Can we find conditions on the probability assignments such

1 Formal definitions are given in Section 6.3.

that, if these conditions are satisfied, we cannot reject the hypothesis that the consumer has random preference orderings and, given the budget set, optimizes on the basis of his realized preference ordering?

Alternatively, suppose we observe single choices of many anonymous consumers on a finite set of budgets, such that we observe each individual decision but do not know by which consumer the decision was made. Can we find conditions on the choices such that, if these conditions are satisfied, we cannot reject the hypothesis that the choices were made by a set of maximizing consumers?

The problem is complicated by at least two factors. Firstly, in the context of stochastic revealed preference, budget sets are infinite sets of alternatives. The stochastic choice literature is usually confined to choices from finite sets.[2] Secondly, even in the deterministic case we are not in general able to recover the entire ranking of a consumer with only a finite set of observations. This is simply because a consumer might choose a in a situation where b is not available, and chooses b in a situation where a is not available. If there are no further observations which can be used to deduce a relation between a and b via a chain of other choices, we do not know if the consumer prefers a over b. In the stochastic case we are therefore only able to deduce minimal choice probabilities; for example, we might be able to deduce that the consumer prefers a over b in at least 30% of all cases and b over a in at least 20% of all cases.

It will be argued that a useful way to understand and analyze stochastic choices on standard budget sets is in terms of indirect preferences on the price-income space or the normalized price space. To this end Sakai's (1977) conditions for indirect preferences from which a utility function can be deduced are used. That is, the problem of finding a probability measure on orderings over the available commodity bundles is transformed into the problem of finding a probability measure on orderings over the budget sets from which choices are observed.

2 Falmagne (1978), who was the first to find conditions for rationalizability of stochastic choices by a probability distribution over linear preference orderings, explicitly confines himself to choices from finite sets of alternatives. Cohen (1980) extends Falmagne's approach to the case of an infinite overall set of alternatives, but again, all choice sets are finite subsets of the set of alternatives.

It is also shown that the rationalizability problem bears similarities to the problem of finding necessary and sufficient conditions for rationalizability of binary choice probabilities; this is specifically true for stochastic revealed preference conditions based on partial relations between alternatives. That is, a set of conditions sufficient for rationalizability is likely to also be applicable to the strand of literature concerned with binary choice. No finite set of necessary and sufficient conditions for each number of alternatives is known, and Fishburn (1990) showed that the set of conditions on the choice probabilities that are sufficient for rationalizability regardless of the number of alternatives must be infinite. This poses some problems for the framework considered here.

The remainder of the paper is organized as follows. Section 6.3 introduces the notation, and recalls the relevant work by BDP and Sakai. Section 6.4 introduces a linear feasibility problem which is solvable if and only if the choices are rationalizable in terms of stochastic orderings on the normalized price space. Combined with the weak axiom of stochastic revealed preference it implies the existence of probability distribution of orderings on the commodity space. Problems, in particular in connection with binary choices, are discussed. Section 6.5 concludes.

6.2 SOME TECHNICAL REMARKS

Before we begin the analysis in the framework of BDP, some important technical points have to be addressed. In particular, the question is how to define a stochastic preference.

In the framework of BDP (1999), the intuition of their notion of rationalizability is that the probability that a consumer chooses an element in a subset of a budget equals the probability that this consumer has a preference (ordering) which is maximized by an element in that subset, given the budget contraint.

BDP (1999) define a probability measure over "the class of all subsets" of a budget set, specified by a stochastic demand function. Because the commodity space X is the positive orthant of the ℓ-dimensional euclidean space and budgets are subsets of X, there is a natural algebra – the Borel σ-algebra generated by the open sets, denoted $\mathcal{X}$ – and hence it is unproblematic to define a budget B as a non-empty set in $\mathcal{X}$. One can then define a σ-additive probability measure on B. Alternatively, as in BDP (2004), this probability measure can be assumed to be only finitely additive, defined over the class of all subsets of a budget. This alternative is discussed further below.

While a σ-additive probability measure on a budget is not a problem, the other side of BDP's notion of rationalizability – a probability measure on the set of preferences or orderings – is more problematic. One could define the set of orderings $\mathbf{R}$ as a metric space with a Borel σ-algebra of subsets of $\mathbf{R}$, as for instance McFadden (2005) does for a set of utility functions. However, it is no simple matter to endow a set of preferences with a metric, at least if the metric should have the desirable property of being economically meaningful, i.e. that "similar preferences" with respect to the metric lead to similar decisions in similar situations. Both BDP (1999) and McFadden (2005) assume a σ-algebra on the set of preferences without showing that it exists.

If, however, all measures considered in the chapter are only finitely additive, then the technical problems do not occur. But then one could argue that the framework of BDP loses some of its possible interpretations and applications. For example, a uniform distribution on the budget would not be possible. The alternative interpretation that we observe choices of a (finite) number of anonymous consumers with deterministic preferences (BDP 2002) would still be valid, but the current framework only accounts for a number of consumers who each have the same income. This is a fairly unrealistic assumption.

Future research should therefore focus on the improvements of two aspects: First, it needs to be shown that the set of preferences considered in this framework can become a metric space, and a stochastic preference needs to be defined as a probability on the Borel subsets of $\mathbf{R}$. The work of Debreu (1969), Kannai (1970), Hildenbrand (1970, 1971, 1974), and Grodal (1974) are probably valuable starting points for this venture. Second, the framework of BDP (2002) needs to be extended to allow for income heterogeneity among different consumers.

In the chapter, we proceed in the following on the basis of BDP's framework of finitely additive measures. We do so because the main contribution is to point to an alternative way of framing the problem, which should ultimately also be of direct use for the more general case.

6.3 PRELIMINARIES

6.3.1 *Notation and Basic Concepts*

Let ℓ be the number of commodities, and let $X = \mathbb{R}^{\ell}_{+}$ be the *commodity space*.[3] The *normalized price space* $\mathbf{P}$ is defined by

$$\begin{aligned}\mathbf{P} = \big\{p : p = (p_1, p_2, \dots, p_\ell) \text{ and } p_i = \rho_i / w \quad (i = 1, 2, \dots, \ell) \\ \text{for some } (\rho_1, \rho_2, \dots, \rho_\ell, w) \in \mathbb{R}^{\ell}_{++} \times \mathbb{R}_{++}\big\},\end{aligned} \tag{6.1}$$

where ρ_i denotes the price of commodity i and w denotes the consumer's income; for most of the paper we shall assume that we observe consumption decisions on a finite set of n budgets. A *budget set* can then be defined by $\{x \in X : px \leq 1\}$. We will denote the budget sets as $B^i = B(p^i)$ and the upper bound of budget sets as $\bar{B}^i = \{x \in X : p^i x^i = 1\}$, where superscripts index the observation. Furthermore $\mathbf{B} \subseteq 2^X$ denotes the family of all budget sets, i.e. $\mathbf{B} = \bigcup \{B(p) : p \in P\}$.

Let h be a nonempty *demand correspondence (function)* on $\mathbf{B}$ which assigns to each B a nonempty subset $h(B)$. For most of the paper, we shall assume that h is a singleton, and denote $x^i = (x^i_1, x^i_2, \dots, x^i_\ell) = h(B^i)$. Furthermore we shall assume that the entire income is spent, such that $h(B^i) = h(\bar{B}^i)$.

Let $R \subseteq X \times X$ be a binary relation on X. If $p^i x^i \geq p^i x$ then $\{x^i, x\} \in R$ and we say that the observation x^i is *directly revealed preferred* to x. For brevity, we write $x^i R x$. The observation x^i is *revealed preferred* to x, written $x^i R^* x$, if either $x^i R x$ or for some sequence of bundles $(x^j, x^k, \dots, x^m)$ it is the case that $x^i R x^j$, $x^j R x^k$, ..., $x^m R x$. In this case R^* is the *transitive closure* of the relation R, i.e. $R^* = \bigcup_i R^i$. Let $\mathbf{R}$ be the set of all orderings over X.[4]

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3 Notation: $\mathbb{R}^{\ell}_{+} = \{x \in \mathbb{R}^{\ell} : x \geqq 0\}$, $\mathbb{R}^{\ell}_{++} = \{x \in \mathbb{R}^{\ell} : x > 0\}$, where " $x \geqq y$ " means " $x_i \geq y_i$ for all i ", and $x \neq y$ ", and " $x > y$ " means " $x_i > y_i$ for all i ". Note the convention to use subscripts to denote scalars or vector components and superscripts to index bundles.

4 We use the term "ordering" in the same sense as BDP. An ordering over $\mathbb{R}^{\ell}_{+}$ is binary relation R over $\mathbb{R}^{\ell}_{+}$ satisfying: (i) reflexivity: for all $x \in \mathbb{R}^{\ell}_{+}$, xRx; (ii) connectedness: for all distinct $x, y \in \mathbb{R}^{\ell}_{+}$, xRy or yRx; and (iii) transitivity: for all $x, y, z \in \mathbb{R}^{\ell}_{+}$, $[xRy$ and $yRz]$ implies xRz.

The *weak axiom of revealed preference* (WARP) asserts that R is asymmetric: For all $x, x' \in X$, $x \neq x'$, xRx' implies [not $x'Rx$]. The *strong axiom of revealed preference* (SARP) asserts that the transitive closure of R, R^*, is asymmetric: xR^*x' implies [not $x'R^*x$].

6.3.2 *Indirect Revealed Preference and Revealed Favorability*

There is a notion of *indirect revealed preference* due to Sakai (1977), Little (1979), and Richter (1979).[5] We will rely on Sakai's definitions and use the concept of *revealed favorability* in the following sense: Let $F \subseteq \mathbf{B} \times \mathbf{B}$ be a binary relation on $\mathbf{B}$.

Definition 6.1 (Revealed favorability relation) If $x^j \in B^i$ then there has to be an element $x \in B^i$ which is at least as good as x^j and we say that budget B^i is *revealed more favorable* than budget B^j. Given a set of observations on a consumer, we define the relation F^1 as $B^i\, F^1\, B^j$ if $x^j \in B^i$ and $B^i \neq B^j$. Let F be the *transitive closure* of the relation F^1. Let $\mathbf{F}$ be the set of all orderings on $\mathbf{B}$.

Definition 6.2 (Weak Axiom of Revealed Favorability) The *weak axiom of revealed favorability* (WARF) asserts that F^1 is asymmetric: For all $B, B' \in \mathbf{B}$, $B\, F^1\, B'$ implies [not $B'\, F^1\, B$].

Definition 6.3 (Strong Axiom of Revealed Favorability) The *strong axiom of revealed favorability* (SARF) asserts that the transitive closure of F^1, F, is asymmetric: $B\, F\, B'$ implies [not $B'\, F\, B$].

6.3.3 *Stochastic Revealed Preference and its Weak Axiom*

Next we recall the relevant part of the concepts used by BDP (1999, 2004).

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5 Sakai (1977) calls the relations on the price-income space *revealed favorability relations* and defines weak and strong axioms of revealed favorability by analogy with WARP and SARP. Little (1979) calls his relations *indirect preference relations* and employs the Congruence Axiom due to Richter (1966). See also Varian (1982), who explores the possibilities of ordinal comparisons between budgets in empirical analysis.

Definition 6.4 (Stochastic demand function) A *stochastic demand function* (SDF) is a rule g, which, for every normalized price vector $p \in P$ specifies exactly one (finitely additive) probability measure q over the class of all subsets of the budget set $B = B(p)$.

Let $q = g(p)$, where g is an SDF, and let A be a subset of a budget set $B(p)$. Then $q(A)$ is the probability that the bundle chosen by the consumer from the budget set $B(p)$ will be in the set A.

Definition 6.5 (Degenerate stochastic demand function) A stochastic demand function g is *degenerate* if, for every normalized price vector $p \in P$, there exists $x \in B(p)$ such that, for every subset A of $B(p)$, $x \in A$ implies $q(A) = 1$ and $x \notin A$ implies $q(A) = 0$, where $q = g(p)$.

Definition 6.6 (Weak Axiom of Stochastic Revealed Preference) A stochastic demand function g satisfies the *weak axiom of stochastic revealed preference* (WASRP) if, for all pairs of normalized price vectors p and p', and for all $A \subseteq B \cap B'$

$$q\,(B - B') \geq q'(A) - q(A), \tag{6.2}$$

where $q = g(p)$, $q' = g(p')$, $B = B(p)$ and $B' = B(p')$.

To interpret the WASRP, consider Figure 6.1. Suppose for simplicity that $q(A) = 0$. The WASRP then requires that the probability that the consumer chooses a bundle in $B - B'$ when facing budget B is at least as great as the the probability that the consumer chooses a bundle in A when facing budget B'. When switching from B to B', we "throw out" $B - B'$ and add $B' - B$. Because the addition of $B' - B$ should not increase the probability of choosing a bundle in A, any increase in the probability of choosing bundle A should be to due the bundles which were available under B but are no longer available under B'. Then the increase in the probability of choosing a bundle in A should not exceed the maximal probability that can be diverted from $B - B'$ to A. The figure also illustrates that WASRP implies the WARP in the case of a degenerate SDF (which can be interpreted as the demand function of a consumer with deterministic preferences): If the consumer chooses a bundle in A when facing budget B' (i.e. $q'(A) = 1$), then he chooses a bundle in $B - B'$ when facing budget B (i.e. $q(B - B') = 1$).

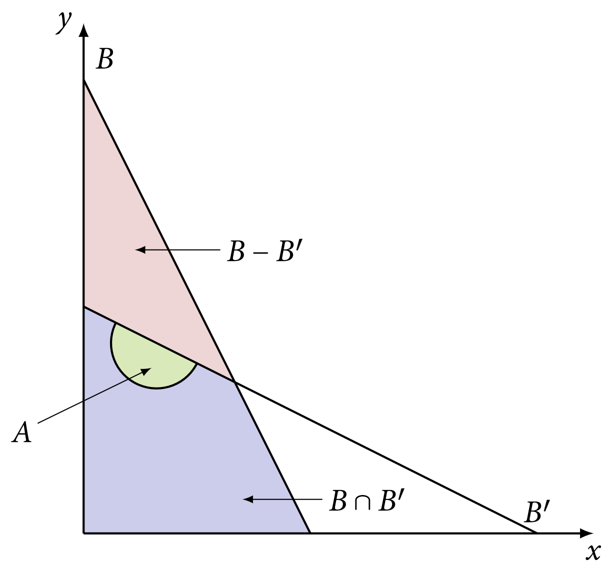

Figure 6.1.: The Weak Axiom of Stochastic Revealed Preference

A stochastic demand function which satisfies $q(\bar{B}) = 1$ is called *tight* (BDP 2004). The analysis here is confined to tight demand.

Definition 6.7 (Rationalizability in terms of stochastic orderings) A stochastic demand function g satisfies *rationalizability in terms of stochastic orderings* (RSO) if there exists a (finitely additive) probability measure r defined on $\mathbf{R}$ such that, for every normalized price vector p and every subset A of $B = B(p)$

$$q(A) = r[\{R \in \mathbf{R} : \text{there is a unique } R - \text{greatest element in } B \text{ and that element is in } A\}] \tag{6.3}$$

or $q(A) = r[\{R \in \mathbf{R} : \arg\max_B \in A\}]$ for short, where $q = g(p)$.

BDP (1999) show that RSO implies but is not implied by WASRP; the result holds for the framework of BDP (2004) which we consider here as well.

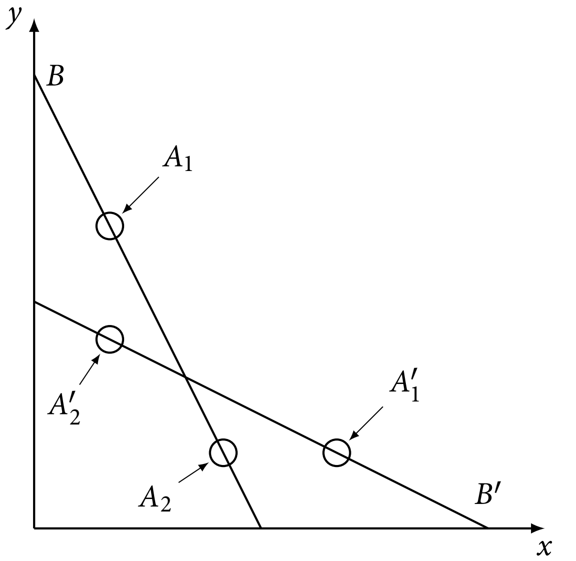

Figure 6.2.: An example

6.3.4 *Indirect Preferences and Stochastic Choice*

To extend WASRP to a stronger condition analogous to SARP it seems necessary to be able to utilize transitive closures of preference relations. But when we observe probability measures over all subsets of given budgets it is difficult to interpret these measures in terms of preference relations between elements of X. It is more obvious how to interpret the observations in terms of indirect preference relations or revealed favorability relations between elements of **B**: We can interpret $q^j(B^j \cap B^i)$ as the minimal share of the consumer's indirect preference relations which rank budget B^i over budget B^j.

Consider Figure 6.2. Suppose on budget B the consumer assigns the probabilities $q(A_1) = 6/8$ and $q(A_2) = 2/8$ to the sets A_1 and A_2 respectively. On budget B' he assigns the probabilities $q'(A'_1) = 4/8$ and $q'(A'_2) = 4/8$. Clearly he reveals that at least $2/8$ of his preference orderings rank B' over B, and at least $4/8$ of his preference orderings rank B oder B'.

Now suppose the indicated subsets of the budgets are singletons. The observed probability assignments are consistent with a consumer who has three different preferences R_a, R_b, R_c, such that A_1 is the R_a-greatest element of B and A'_2 is the R_a-greatest element of B' and the preference R_a is realized with probability $4/8$; A_1 is the R_b-greatest element of B and A'_1 is the R_b-greatest element of B' and the preference R_b is realized with probability $2/8$; A_2 is the R_c-greatest element of B and A'_1 is the R_c-greatest element of B' and the preference R_c is realized with probability $2/8$.

When considering indirect preferences, the only conditions imposed by the observed probability assignments are that the consumer has an indirect preference which ranks budget B over B' and is realized with a probability of at least $4/8$, and an indirect preference which ranks budget B' over B and is realized with a probability of at least $2/8$. For example the consumer could have two different indirect preferences F_a and F_b, such that B is the F_a-greatest element of $\{B, B'\}$ and the preference F_a is realized with probability $5/8$; B' is the F_b-greatest element of $\{B, B'\}$ and the preference F_b is realized with probability $3/8$.

6.4 RATIONALIZABILITY

6.4.1 *Rationalizability in Terms of Stochastic Orderings on the Normalized Price Space*

The idea of the notion of rationalizability considered in this section is that there exists a (finitely additive) probability measure on the set of indirect preferences which generates the observed choices. More formally:

Definition 6.8 (Rationalizability in terms of stochastic orderings on the normalized price space) Let $N = \{1, 2, \ldots, n\}$ be the set of indices of the observed budgets, and let $M \subset N$ with some index $k \in M$ and $k \notin N$. We say that a stochastic demand function g satisfies *rationalizability in terms of stochastic orderings on the normalized price space* (RSOP) if there exists a (finitely additive) probability measure f defined on $\mathbf{F}$ such that we can use f to generate the observed stochastic demand:

$$f\left[\left\{F \in \mathbf{F} : (\forall i \in M)\left[B^i F B^k\right]\right\}\right] \geq q^k\left(B^k \cap \bigcap_{i \in M} B^i\right) \tag{6.4a}$$

i.e. the sum over all indirect preferences which rank all budgets in $\{B^i\}_{i\in M}$ higher than B^k is greater than or equal to the choice probability assigned to the part of B^k that intersects with all $\{B^i\}_{i\in M}$. Furthermore,

$$f\left[\left\{F \in \mathbf{F} : (\forall i \in M)\left[B^k F B^i\right]\right\}\right] \leq q^k\left(B^k - \bigcup_{i \in M} B^i\right), \tag{6.4b}$$

i.e. the sum over all indirect preferences which rank all budgets in $\{B^i\}_{i\in M}$ lower than B^k is less than or equal to the choice probability assigned to the part of B^k that does not intersects with any B^i in $\{B^i\}_{i\in M}$.

Because the number of different indirect preferences is finite if the number of observations is finite, it is straightforward to test, at least in principle, for RSOP. Let $S(N)$ be the set of all ordered n-tuples of indices in N, i.e. the set of the $n!$ permutations of N. The elements of $S(N)$ will be indicated by σ, and more explicitly as $\sigma_i = \langle a, b, \ldots, e\rangle$ and $\sigma_i(1) = a$, $\sigma_i(2) = b$, etc. Let $\pi_i = \pi(\sigma_i)$ be the probability assigned to the ordering σ_i.

We now define the following linear feasibility problem:

$$\text{find } \Pi = (\pi_1, \pi_2, \ldots, \pi_{n!}) \tag{FP.1}$$

$$\text{satisfying } \pi_i \geq 0 \quad \text{for all } i = 1, 2, \ldots, n! \tag{FP.2}$$

$$\sum_{i=1}^{n!} \pi_i = 1 \tag{FP.3}$$

$$\sum_{\{i\in\{1,\ldots,n!\}:\sigma_i(j)<\sigma_i(k)\,\forall j\in M\}} \pi_i \geq q^k\left(B^k \cap \bigcap_{j\in M} B^j\right) \tag{FP.4}$$

$$\sum_{\{i\in\{1,\ldots,n!\}:\sigma_i(j)>\sigma_i(k)\,\forall j\in M\}} \pi_i \leq q^k\left(B^k - \bigcup_{j\in M} B^j\right) \tag{FP.5}$$

$$\text{for all nonempty } M \subset N \text{ and all } k \in N, k \notin M$$

Note that $\sum_{\{i\in\{1,\ldots,n!\}:\sigma_i(j)<\sigma_i(k)\forall j\in M\}} \pi_i$ denotes the sum over all probability assignments over preferences which rank *all* $j \in M$ higher than k, excluding preferences which rank one or more $j \in M$ lower than k.

Theorem 6.1 The following conditions are equivalent:
(1) there exists a (finitely additive) probability measure f over the set of all orderings on $\mathbf{B}$ that rationalizes the stochastic choices $\{q(B^i)\}_{i=1}^n$, i.e. RSOP is satisfied;
(2) the linear feasibility problem (FP) has a solution.

Proof Follows immediately from the definition of RSOP. ■

6.4.2 *Rationalizability in Terms of Stochastic Orderings on the Commodity Space*

Sakai (1977,Theorem 6) shows that if the entire income is spent, the (deterministic) demand at every normalized price vector is a singleton, and the demand function satisfies SARF, then a (direct) utility function can be deduced from the favorability relation. In analogy to Sakai's result, we arrive at the following interesting theorem.

Theorem 6.2 If RSOP is satisfied, then the stochastic demand function g satisfies rationalizability in terms of stochastic orderings (RSO).

Proof Note that an SDF g specifies exactly one finitely additive probability measure q over the class of all subsets of the budget. Thus the demand according to each preference the consumer has can be thought of as single valued.

Define a set of functions $g_R : \mathbf{B} \to X$ such that $g_R(B) = \arg\max_B R$, for all $R \in \mathbf{R}$. Under SARF and single valued demand, for every indirect preference $F \in \mathbf{F}$ there are direct preferences R such that $x\, R\, x'$ if and only if $B\, F\, B'$, where $x = g_R(B)$ and $x' = g_R(B')$. Let $\mathbf{Q}$ be the set of all $R \in \mathbf{R}$ consistent with a preference $F \in \mathbf{F}$. Note that $\{R \in \mathbf{Q}\} \subseteq \{R \in \mathbf{R}\}$. Then RSO is implied by the existence of a probability measure r defined on $\mathbf{Q}$ such that, for every normalized price vector p and every subset A of $B = B(p)$, we have $q(B) = r[\{R \in \mathbf{Q} : g_R(B) \in A\}]$. If [not RSO], then there does not exist r on $\mathbf{Q}$ such that $q^j(A) = r[\{R \in \mathbf{Q} : g_R(B) \in A\}]$ for all $A \subset B^j$ and all $j \in N$. Let

$$\begin{aligned} r\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B)\,R\,g_R(B^i)]\Big\}\Big] \\ := f\Big[\Big\{F \in \mathbf{F} : (\forall i \in M)\,[B\,F\,B^i]\Big\}\Big], \quad (6.5a) \\ r\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^i)\,R\,g_R(B)]\Big\}\Big] \\ := f\Big[\Big\{F \in \mathbf{F} : (\forall i \in M)\,[B^i\,F\,B]\Big\}\Big]. \quad (6.5b) \end{aligned}$$

Under the assumptions made above, we have

$$\begin{aligned} \Big\{R \in \mathbf{Q} : g_R(B^k) \in B^k \cap \bigcap_{i \in M} B^i\Big\} \\ \subseteq \Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^i)\,R\,g_R(B^k)]\Big\}, \\ (6.6a) \\ \Big\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i \in M} B^i\Big\} \\ \supseteq \Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^k)\,R\,g_R(B^i)]\Big\}, \\ (6.6b) \end{aligned}$$

and therefore

$$\begin{aligned} r\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in B^k \cap \bigcap_{i \in M} B^i\Big\}\Big] \\ \leq r\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^i)\,R\,g_R(B^k)]\Big\}\Big], \\ r\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i \in M} B^i\Big\}\Big] \\ \geq r\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^k)\,R\,g_R(B^i)]\Big\}\Big]. \end{aligned}$$

To see (6.6a), note that from

$$R_a \in \{R \in \mathbf{Q} : g_R(B^k) \in B^k \cap \bigcap_{i \in M} B^i\}$$

it follows that

$$(\forall i \in M)\,[g_{R_a}(B^i)\,R_a\,g_{R_a}(B^k)].$$

To see (6.6b), let

$$R_b \in \{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^k)\,R\,g_R(B^i)]\}.$$

Suppose $g_{R_b}(B^k) \notin B^k - \bigcup_{i\in M} B^i$, which implies $g_{R_b}(B^k) \in B^k \cap B^{i_0}$ for at least one $i_0 \in M$. Then we have $g_{R_b}(B^{i_0})\, R_b\, g_{R_b}(B^k)$. If also $g_{R_b}(B^k)\, R_b\, g_{R_b}(B^{i_0})$, then $B^{i_0}\, F\, B^k$ and $B^k\, F\, B^{i_0}$. This contradicts $R_b \in \mathbf{Q}$ because F is asymmetric.

The rest of the proof proceeds as follows: It will be shown that if RSOP is satisfied, it is always possible to assign probability in a way such that RSO is satisfied. In step 1.(a) it is shown that if there exists a probability measure r such that $q(B^k \cap \bigcap_{i\in M} B^i) = r[\{R \in \mathbf{Q} : g_R(B) \in B^k \cap \bigcap_{i\in M} B^i\}]$ holds, then there also exists a probability measure r such that $q(B^k \cap \bigcap_{i\in M} B^i) = r[\{R \in \mathbf{Q} : g_R(B) \in B^k \cap \bigcap_{i\in M} B^i\}]$ *and* $q(A) = r[\{R \in \mathbf{Q} : g_R(B) \in A\}]$ hold for any $A \subset B^k \cap \bigcap_{i\in M} B^i$, and similarly for $B^k - \bigcup_{i\in M} B^i$, which is shown in step 2.(a). That is, in identifying a violation of RSO, we can restrict the search to sets of the form $B^k \cap \bigcap_{i\in M} B^i$ or $B^k - \bigcap_{i\in M} B^i$. Steps 1.(b), 1.(c), 2.(b), and 2.(c) start with a probability measure which is inconsistent with RSO. It is then shown that there exists another probability measure obtained by appropriate addition or substraction that is consistent with RSO, given that RSOP is satisfied.

1.(a) Suppose that $A \subseteq B^k \cap \bigcap_{i\in M} B^i$ for some $M \subset N$ and $k \in N - M$. We have

$$q^k\Big(\big(B^k \cap \bigcap_{i\in M} B^i\big) - A\Big) + q^k(A) = q^k\big(B^k \cap \bigcap_{i\in M} B^i\big).$$

If

$$q^k\big(B^k \cap \bigcap_{i\in M} B^i\big) = r\Big[\big\{R \in \mathbf{Q} : g_R(B^k) \in B^k \cap \bigcap_{i\in M} B^i\big\}\Big],$$

then Eq. (6.6a) and the fact that

$$\begin{aligned}\big\{R \in \mathbf{Q} : g_R(B^k) \in B^k \cap \bigcap_{i\in M} B^i - A\big\} \cup \big\{R \in \mathbf{Q} : g_R(B^k) \in A\big\}\\ = \big\{R \in \mathbf{Q} : g_R(B^k) \in B^k \cap \bigcap_{i\in M} B^i\big\}\end{aligned}$$

tell us that letting $r[\{R \in \mathbf{Q} : g_R(B^k) \in A\}] := q^k(A)$ is consistent with RSOP.

1.(b) Now suppose that $q^k(A) > \tilde{r}[\{R \in \mathbf{R} : g_R(B^k) \in A\}]$ for $A = B^k \cap \bigcap_{i\in M} B^i$ for some probability measure $\tilde{r}$ on $\mathbf{Q}$ which satisfies

(6.5a) and (6.5b). Then $q^k(A) = \tilde{r}[\{R \in \mathbf{R} : g_R(B^k) \in A\}] + \delta$ for some $\delta > 0$. Suppose

$$\begin{aligned}&\tilde{r}\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in A\Big\}\Big] + \delta \\ &\qquad\qquad \le \tilde{r}\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^i)\,R\,g_R(B^k)]\Big\}\Big].\end{aligned}$$

Then by Eq. (6.6a) we can let $r[\{R \in \mathbf{Q} : g_R(B^k) \in A\}] := \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in A\}] + \delta$ and $r[\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i\in M} B^i\}] := \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i\in M} B^i\}] - \delta$. To see that this is consistent with RSOP, note that

$$\begin{aligned}q^k\Big(B^k - \bigcup_{i\in M} B^i\Big) &= 1 - \tilde{r}\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in B^k \cap \bigcap_{i\in M} B^i\Big\}\Big] - \delta \\ &= 1 - \Big(1 - \tilde{r}\Big[\Big\{R \in \mathbf{Q} : g_r(B^k) \in B^k - \bigcup_{i\in M} B^i\Big\}\Big]\Big) \\ &\quad - \delta \\ &= \tilde{r}\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i\in M} B^i\Big\}\Big] - \delta\end{aligned}$$

and with RSOP we have that

$$q^k\Big(B^k - \bigcup_{i\in M} B^i\Big) \ge \tilde{r}\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^k)\,R\,g_R(B^i)]\Big\}\Big],$$

such that

$$\begin{aligned}&\tilde{r}\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i\in M} B^i\Big\}\Big] - \delta \\ &\qquad\qquad \ge \tilde{r}\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^k)\,R\,g_R(B^i)]\Big\}\Big],\end{aligned}$$

so if $\tilde{r}$ is consistent with RSOP, so is r. If instead

$$\begin{aligned}&\tilde{r}\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in A\Big\}\Big] + \delta \\ &\qquad\qquad > \tilde{r}\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^i)\,R\,g_R(B^k)]\Big\}\Big] \\ &\qquad\qquad = f\Big[\Big\{F \in \mathbf{F} : (\forall i \in M)\,[B^i\,F\,B^k]\Big\}\Big],\end{aligned}$$

we have that $q^k(A) > f[\{F \in \mathbf{F} : (\forall i \in M)\,[B^i\,F\,B^k]\}]$, which contradicts RSOP.

1.(c) Now suppose that $q^k(A) < \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in A\}]$ for $A = B^k \cap \bigcap_{i \in M} B^i$. Then $q^k(A) = \tilde{r}[\{R \in \mathbf{Q} : r_R(B^k) \in A\}] - \delta$. We have that

$$\tilde{r}\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in A\Big\}\Big] - \delta$$
$$< \tilde{r}\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^i)\,R\,g_R(B^k)]\Big\}\Big]$$

and letting $r[\{R \in \mathbf{Q} : g_R(B^k) \in A\}] := \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in A\}] - \delta$ and $r[\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i \in M} B^i\}] := \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i \in M} B^i\}] + \delta$ is consistent with RSOP for similiar reasons as in step 1.(b).

2.(a) Suppose that $A \subseteq B^k - \bigcup_{i \in M} B^i$. We have

$$q^k\Big(B^k - \bigcup_{i \in M} B^i - A\Big) + q^k(A) = q^k\Big(B^k - \bigcup_{i \in M} B^i\Big).$$

If

$$q^k\Big(B^k - \bigcup_{i \in M} B^i\Big) = r\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i \in M} B^i\Big\}\Big],$$

then Eq. (6.6b) and the fact that

$$\Big\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i \in M} B^i - A\Big\} \cup \Big\{R \in \mathbf{Q} : g_R(B^k) \in A\Big\}$$
$$= \Big\{R \in \mathbf{Q} : g_R(B^k) \in B^k - \bigcup_{i \in M} B^i\Big\}$$

tell us that letting $r[\{R \in \mathbf{Q} : g_R(B^k) \in A\}] := q^k(A)$ is consistent with RSOP.

2.(b) Now suppose that $q^k(A) < \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in A\}]$ for $A = B^k - \bigcup_{i \in M} B^i$. Then $q^k(A) = \tilde{r}[\{R \in \mathbf{Q} : r_R(B^k) \in A\}] - \delta$. Suppose

$$\tilde{r}\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in A\Big\}\Big] - \delta$$
$$\geq \tilde{r}\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^k)\,R\,g_R(B^i)]\Big\}\Big].$$

Then by Eq. (6.6b) we can let $r[\{R \in \mathbf{Q} : g_R(B^k) \in A\}] := \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in A\}] + \delta$. This is consistent with RSOP for similiar reasons as in step 1.(b). If instead

$$\tilde{r}\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in A\Big\}\Big] - \delta$$
$$< \tilde{r}\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\,[g_R(B^k)\,R\,g_R(B^i)]\Big\}\Big]$$
$$= f\Big[\Big\{F \in \mathbf{F} : (\forall i \in M)\,[B^k\,F\,B^i]\Big\}\Big],$$

we have that $q^k(A) < f[\{F \in \mathbf{F} : (\forall i \in M)[B^k F B^i]\}]$, which contradicts RSOP.

2.(c) Now suppose that $q^k(A) > \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in A\}]$ for $A = B^k - \bigcup_{i \in M} B^i$. Then $q^k(A) = \tilde{r}[\{R \in \mathbf{Q} : r_R(B^k) \in A\}] + \delta$. We have that

$$\tilde{r}\Big[\Big\{R \in \mathbf{Q} : g_R(B^k) \in A\Big\}\Big] + \delta \\ > \tilde{r}\Big[\Big\{R \in \mathbf{Q} : (\forall i \in M)\left[g_R(B^k)\, R\, g_R(B^i)\right]\Big\}\Big],$$

and letting $r[\{R \in \mathbf{Q} : g_R(B^k) \in A\}] := \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in A\}] + \delta$ and $r[\{R \in \mathbf{Q} : g_R(B^k) \in B^k \cap \bigcap_{i \in M} B^i\}] := \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in B^k \cap \bigcap_{i \in M} B^i\}] - \delta$ is consistent with RSOP for similiar reasons as in step 1.(b).

Finally, note that any $A \subset B^k$ can always be written as the union of finitly many disjoint sets Z^j, i.e. $A = \bigcup_j Z^j$, where for every j, either $Z^j \subseteq B - \bigcup_j B^j$ or $Z^j \subseteq B \cap \bigcap_j B^j$. We then have that $q^k(A) = \sum_j q(Z^j)$. Then $q^k(A) \neq \tilde{r}[\{R \in \mathbf{Q} : g_R(B^k) \in A\}]$ implies that for at least one j we have that $q(Z^j) \neq \tilde{r}[\{R \in \mathbf{Q} : g_R(B) \in Z^j\}]$, and the same arguments as in steps 1 and 2 apply.

To see that RSOP is not necessary, suppose the SDF is degenerate and let $B \neq B'$ and $g(B) = g(B')$. This is obviously consistent with RSO, but (FP) has no solution. ∎

6.4.3 *Problems and Open Questions*

Consider the following construction: A budget B^i is *revealed more favorable by degree* $\varphi(i,j)$ than B^j if

$$\varphi(i,j) = \max\Big\{q^j\left(B^j \cap B^i\right), q^j\left(B^j \cap B^{M(1)}\right) \\ + \sum_{k=1}^{m-1} q^{M(k)}\left(B^{M(k)} \cap B^{M(k+1)}\right) \\ + q^{M(m)}\left(B^{M(m)} \cap B^i\right) - m\Big\}, \tag{6.7}$$

where the maximum is over all sets of indices $M \subseteq N - \{i,j\}$. Then obviously

$$\varphi(i,j) + \varphi(j,i) \leq 1 \tag{6.8}$$

is a necessary condition for RSOP. It may seem to be a reasonable conjecture that the condition is also sufficient, but unfortunately it is not, as will be shown below. But first note the following:

Claim 6.1 Identify a deterministic demand function with a degenerate stochastic demand function. For that demand function, condition (6.8) is equivalent to the strong axiom of revealed favorability.

Proof In the deterministic case, $B^i \, F \, B^j$ is equivalent to $\varphi(i,j) = 1$. To see this, note that (i) $\varphi(i,j) \in \{0,1\}$, (ii) $B^i \, F^1 \, B^j$ is equivalent to $q^j\left(B^j \cap B^i\right) = 1$, and (iii) $B^i \, F \, B^j$ is equivalent to $q^j\left(B^j \cap B^{M(1)}\right) = 1$ and

$$q^{M(1)}\left(B^{M(1)} \cap B^{M(2)}\right) = 1, \ldots, q^{M(m)}\left(B^{M(m)} \cap B^{M(i)}\right) = 1$$

for some $M \subset N$. So condition (6.8) is equivalent to asymmetry of F. ■

A "system of binary probabilities"

$$\left[\alpha_{ij} : i,j \in \{1,2,\ldots,n\}, i \neq j, \alpha_{ij} + \alpha_{ji} = 1\right]$$

is said to be "induced by rankings" (rationalizable) if there is a probability distribution on the set of $n!$ orderings of $\{1,2,\ldots,n\}$ such that, for all distinct i and j, α_{ij} is the sum of all probabilities attached to orderings which rank i over j (cf. Fishburn (1990)). The so-called *triangular condition*

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} \leq 2 \tag{6.9}$$

and its generalization

$$\alpha_{M(1)\,M(2)} + \alpha_{M(2)\,M(3)} + \ldots + \alpha_{M(m)\,M(1)} \leq m - 1 \tag{6.10}$$

for all sets of indices $M \subseteq N$ of length m is a necessary condition for rationalizability.[6] It was also conjectured to be a sufficient condition for rationalizability by Marschak (1959). In an unpublished paper, McFadden and Richter (1970) provided a counterexample for $n = 6$.[7] Later

6 For the generalized form, see for example Cohen and Falmagne (1990). In the case of binary probabilities, the generalized form can be deduced from the triangular condition.

7 A revision of the paper was later published as McFadden (2005).

on, Fishburn (1990) observed that the set of conditions on the choice probabilities that are sufficient for rationalizability regardless of n must be infinite.

This poses some problems for the framework considered here. Consider the counterexample of McFadden and Richter (1970) applied to the framework of stochastic revealed preference: For $n = 6$, let

$$\begin{aligned} \alpha_{12} = \alpha_{14} = \alpha_{34} = \alpha_{36} = \alpha_{56} = \alpha_{52} &= 1 \\ \alpha_{21} = \alpha_{41} = \alpha_{43} = \alpha_{63} = \alpha_{65} = \alpha_{25} &= 0 \\ \alpha_{ij} &= 1/2 \text{ for all other } i, j \end{aligned} \tag{6.11}$$

where $q^j\left(B^j \cap B^i\right) = \alpha_{ij}$. Then the triangular condition and its generalization are satisfied, and so is condition (6.8); but (FP) has no solution. Indeed, with $q^j\left(B^j \cap B^i\right) = \alpha_{ij}$, conditions (6.10) imply (6.8) because

$$\begin{aligned} \varphi(i,j) + \varphi(j,i) = {} & \alpha_{j\,M^i(1)} + \alpha_{M^i(1)\,M^i(2)} + \ldots + \alpha_{M^i(m)\,i} \\ & + \alpha_{i\,M^j(1)} + \alpha_{M^j(1)\,M^j(2)} + \ldots + \alpha_{M^j(m)\,j} \\ & - m^i - m^j, \end{aligned} \tag{6.12}$$

where M^i and M^j, $|M^i| = m^i$ and $|M^j| = m^j$, are the sets of indices used to construct $\varphi(i,j)$ and $\varphi(j,i)$, and with (6.12) and condition (6.10) we obtain

$$\begin{aligned} & \varphi(i,j) + \varphi(j,i) + m^i + m^j \leq (m^i + 1) + (m^j + 1) - 1 \\ \Leftrightarrow {} & \varphi(i,j) + \varphi(j,i) \leq 1. \end{aligned}$$

While it might also be possible that exploitation of the particularities of the framework of BDP, e.g. linearities of budgets, helps to find finite sets of necessary and sufficient conditions for stochastic revealed preference without applicability to the binary probability problem[8], the results of this section suggest that conditions for RSOP based on definitions for a partial revealed favorability relation between budgets suffer from similar problems as the conditions for rationalizability of binary probabilities. Therefore a "strong axiom of stochastic revealed favorability"

8 Suppose the commodity space is restricted to the positive orthant of the two-dimensional Euclidean space. Then, in analogy to deterministic revealed preference (see Rose (1958) and Chapters 3 and 5 of this thesis), WASRP might imply RSO.

could possibly also solve the problem of finding a finite set of necessary and sufficient conditions for systems of binary probabilities for each particular n.

6.5 CONCLUSIONS

The weak axiom of stochastic revealed preference, as introduced by Bandyopadhyay et al. (1999) and used in Bandyopadhyay et al. (2004), is a necessary but not sufficient condition for stochastic demand behavior to be rationalizable in terms of stochastic orderings on the commodity space. It was the purpose of this paper to explore rationalizability issues and to show how one can, in principle, test whether or not a finite set of observations of stochastic choice is rationalizable by stochastic orderings.

To this end the problem of finding a probability measure over orderings on the commodity space was transformed into a problem of finding a probability measure over orderings on the normalized price space. The advantage of this indirect approach is that it avoids the problems resulting from the infinity of the set of alternatives a consumer chooses from when facing a budget set defined in the usual way. Furthermore, it is interesting to note that rationalizability in terms of stochastic orderings on the normalized price space and the weak axiom of stochastic revealed preference together imply rationalizability in terms of stochastic orderings on the commodity space.

In Section 6.4.3 similarities with binary probability systems were pointed out. In particular it was shown that conditions based on partial revealed favorability relations are likely to suffer from similar problems as the conditions for rationalizability of binary probabilities.

Part III

APPLICATIONS

7

HOMOTHETIC PREFERENCES: APPLICATIONS IN TWO DIMENSIONS

7.1 INTRODUCTION

The proof that pairwise comparison of choices is sufficient for homothetic rationalization in two dimensions provides (see Chapter 5) a direct way to compute scalar factors needed to construct piecewise linear homothetic indifference curves through the choices. These scalars usually cannot be computed efficiently because a violation of homotheticity implies negative weight cycles in a graph representing the data. The direct way given in Chapter 5 for the case of two dimensions can still be used efficiently even if homotheticity is violated. It is shown how this feature can be usefully applied to get an idea about how severe a deviation from homotheticity is. We apply the test and measure developed in this chapter to data sets from two-person dictator experiments (Andreoni and Miller 2002, Fisman et al. 2007) and a two-asset risk experiment (Choi et al. 2007*a*).

The remainder is organized as follows. Section 7.2 introduces several application of the theoretical findings. It provides a simplified test for homotheticity, ways to reveal or measure the extent of deviation from homotheticity, and a test for discrete budget sets. It is also shown how one can construct powerful budget combinations to increase the probability of observing a violation of homotheticity under random choice. In Section 7.3 some of the ideas of Section 7.2 are applied to existing data sets. Section 7.4 concludes.

7.2 POSSIBLE APPLICATIONS

The notation here follows Chapter 5.

7.2.1 *Testing* HARP

Testing HARP requires shortest path algorithms such as the Warshall or Floyd-Warshall algorithm (Warshall 1962, Floyd 1962). Such algorithm can detect negative weight cycles in the data: Varian (1983) interprets the set of n observations as a weighted graph with n nodes and an associated n^2 cost matrix $\mathbf{C} = \{c_{ij}\}$, where $c_{ij} = \log \mathbf{p}^i \mathbf{z}^j / w^i$ is interpreted as the cost of moving from node i to node j. HARP then requires that moving from a node i to itself is never "cheaper" than zero.

The Floyd-Warshall algorithm takes $\{c_{ij}\}$ as input and provides as output a matrix $\{\bar{c}_{ij}\}$ which is the minimum cost of moving from node i to node j. In the case of two dimensions, however, we know from Theorem 2 that pairwise comparison is sufficient to detect violations of homotheticity. Hence a quicker way to test if a set of consumption data satisfies homotheticity is to compute the matrix $\mathbf{M} = \{m_{ij}\}$, where

$$m_{ij} = \frac{\mathbf{p}^i \mathbf{z}^j}{w^i} \frac{\mathbf{p}^j \mathbf{z}^i}{w^j}; \tag{7.1}$$

the PHARP is violated if and only if any element of $\mathbf{M}$ is less than 1. If there is a unique ordering of the relative prices, it is sufficient to only compute and check the subdiagonal for the ordered data; an algorithm to detect negative weight cycles is not needed.

7.2.2 *Revealing the Extent of Deviation*

If the data does not satisfy HARP, the scalar factors in Equation (5.1) cannot be computed using the Floyd-Warshall or any other efficient algorithm because shortest path problems are not well defined in the presence of negative weight cycles. But note that Lemma 5.1 provides an explicit way to compute the scalar factors in the case of two dimensions. The factors $\theta^{i,j}$ can then be used to recover homothetic preferences implied by the choices of a consumer, even if homotheticity is actually violated. While this may not seem reasonable, note that estimating parameters of homothetic functions with data that violate homotheticity is not uncommon.

For example, one can use this approach to graphically reveal the extent to which consumption choices deviate from homotheticity. Consider

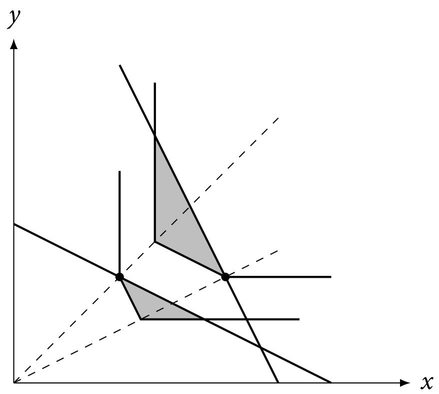

Figure 7.1.: Using the scalar factors $\theta^{i,j}$ from Lemma 1 we can construct curves through the choices. The size of the gray areas gives an idea about how severe the deviation from homotheticity is.

Figure 7.1: The data clearly violate homotheticity. Consequently, the implied homothetic indifference curves intersect the budget lines. The area between the curves and the budget lines gives us an idea about how severe the violation of homotheticity is. In Chapter 9 this approach is used to define an efficiency index that gives a measure of how severe a violation of utility maximization is. In general, neither this nor some variant of the common Afriat Efficiency Index (AEI, see Varian 1990) can be used to capture the extent of violations of homotheticity because the indifference map is not well defined. In two dimensions however, Lemma 5.1 allows us to do so.

7.2.3 *Budget Balancedness and Homothetic Efficiency*

All nonparametric tests based on revealed preference theory lose some of their simplicity and unambiguity if budget balancedness is violated, i.e. if consumers do not spend their entire income. For example, most subjects in Fisman et al. (2007) and Choi et. at. (2007*a*) did not spent their entire endowment.[1] GARP can still be usefully applied using, for example, the AEI, which roughly speaking is based on shifting budgets

1 This important aspect was pointed out by an anonymous referee.

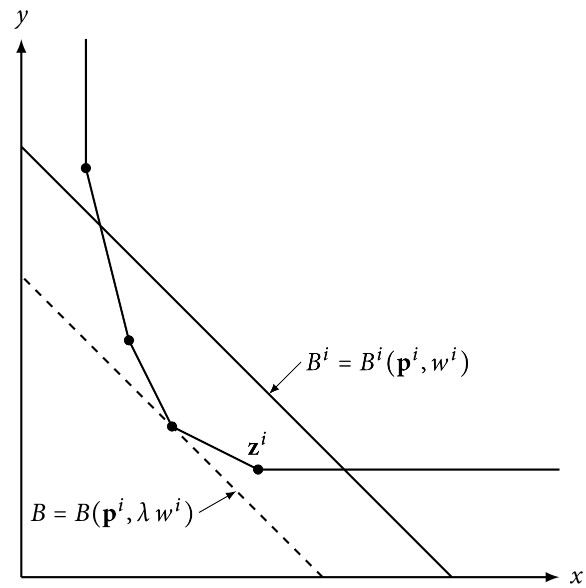

Figure 7.2.: The extent to which a budget has to be shifted towards the origin in order to be tangent to the implied homothetic indifference curve given an idea about how severe the deviation from homotheticity is. Note that in this example budget balancedness is violated.

towards the origin until the data "fits" GARP. This is not feasible when dealing with HARP (see also Proposition 7.1).

There are two straightforward solutions. One can redefine budgets such that endowment equals actual expenditure, so that budget balancedness is satisfied. Alternatively, one can compute efficiency indices based on the homothetic preference relations using the scalar factors from Lemma 5.1, which is only possible in two dimensions. After having constructed upper bounds of homothetic indifference curves using the scalars $\theta^{i,j}$, one can shift budgets towards the origin until all budgets are just tangent to the indifference curves. To do this, one needs to multiply income for each budget with a factor $\lambda \in [0,1]$. The minimum of all factors then gives an idea about how severe the deviation from homotheticity is, and subjects can be compared based on this measure of "homothetic efficiency". Figure 7.2 illustrates this idea.

More formally, the measure for homothetic efficiency which will be employed in Section 7.3 is defined as

$$\text{HE} = \min_{i,j\in\{1,\ldots,N\}} \left\{ \frac{(\theta^{i,j}\mathbf{z}^i)\mathbf{p}^j}{w^j} \right\}, \tag{7.2}$$

where $\theta^{i,j}$ is defined as in Lemma 5.1.

7.2.4 *Discrete Budget Sets*

One can also use the result of Lemma 5.1 for a nonparametric test of homotheticity for discrete budget sets. In economic experiments, budgets often are discrete (see, for example, Harbaugh et al. 2001, Andreoni and Miller 2002, or Chen et al. 2006). Figure 7.3 shows budgets for which commodities can only be demanded in integers. For certain homothetic preferences the consumer is constrained to choices that appear to violate homotheticity if the observer assumes that the budgets are given by the lines through the available bundles.

As noted above, by Lemma 5.1 it is possible to use Knoblauch's (1993) method of recovering homothetic preferences even if homotheticity is violated. One can then test if there have been bundles available on or below a budget line which are within the homothetically revealed preferred set. If not, the hypothesis of homotheticity cannot be rejected.

7.2.5 *The Power of a Test*

Bronars (1987) suggested a Monte Carlo approach to determine the power the test has against random behavior. The approximate power of the test is the percentage of random choices which violated GARP. For the simple case of just two observation, Bronars showed analytically that the probability of a violation of GARP, given random choice, is highest when two intersecting budgets are nearly parallel such that the length of a budget lying inside the other budget is large relative to the rest of the budget. Knowing the corresponding conditions for HARP would make it easier for researchers to design powerful budget combinations. The following proposition will be helpful towards this end:

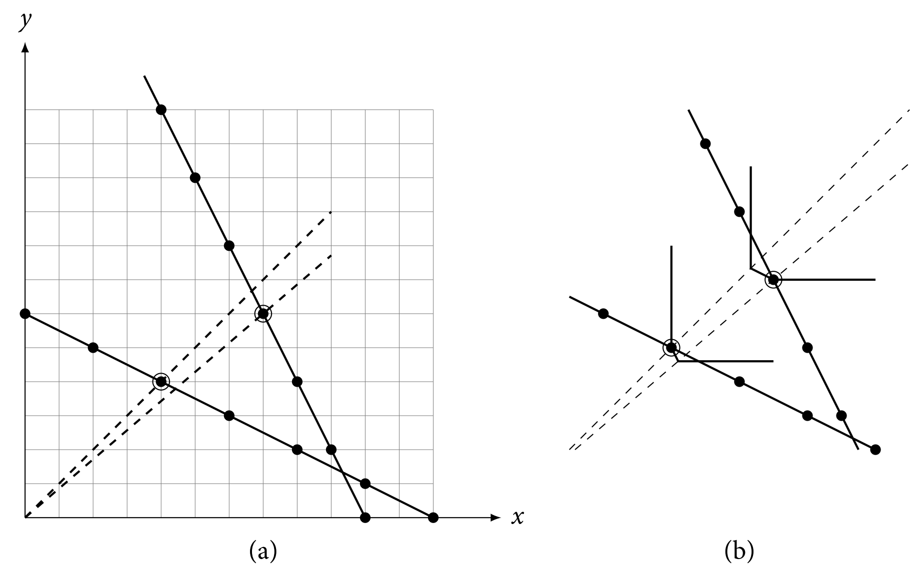

Figure 7.3.: A problem with discrete budget sets. Left: Varian's test will reject the hypothesis of homotheticity for the two choices. Note that not all bundles on the lines are available; the dots indicate available bundles, the circled dots indicate the choices made. Right: Using the scalar factors $\theta^{i,j}$ from Lemma 5.1 we can construct curves through the choices. Note that no alternative choices to the upper right of the curves are available on the budgets.

Proposition 7.1 Generate arbitrarily many new observations by shifting a budget B^i towards or away from the origin and using the intersection of the new budget with a ray through $\mathbf{z}^i$ as a new observation. Testing all augmented sets of observations obtained in this way for GARP is equivalent to testing the original set for HARP.

Proof Normalize the prices by the level of expenditure so that $\mathbf{p}^i\mathbf{z}^i = 1$ for $i = 1, \ldots, N$. Generate a new observation $(\mathbf{z}^{i'}, \mathbf{p}^{i'})$ by multiplying some $\mathbf{z}^i$ and $\mathbf{p}^i$ with a scalar factor $\tau > 0$.

If $\tau \geq t^{i,j}$, then $\tau\,\mathbf{z}^i H\mathbf{z}^j$. Using Eq. (5.1), this implies that $t^{i,j}\,\mathbf{z}^i$ and therefore $\tau\,\mathbf{z}^i$ is revealed preferred to $\mathbf{z}^j$ via a chain of bundles: $t^{i,k}\,\mathbf{z}^i R\mathbf{z}^k$, $t^{k,l}\,\mathbf{z}^k R\mathbf{z}^l, \ldots, t^{m,j}\,\mathbf{z}^m R\mathbf{z}^j$ for some sequence of indices $k, l, \ldots, m$. Thus [not $\mathbf{z}^j P t^{i,j}\,\mathbf{z}^i$] (HARP) is equivalent to [not $\mathbf{z}^j P\mathbf{z}^{i'}$] (GARP) for all $\mathbf{z}^{i'} = \tau\,\mathbf{z}^i$ with $\tau > t^{i,j}$.

If $\tau < t^{i,j}$, then [not $\tau\,\mathbf{z}^i H\mathbf{z}^j$] and [not $\mathbf{z}^{i'} R^* \mathbf{z}^j$]. Thus neither HARP nor GARP impose a condition on the relation between $\mathbf{z}^{i'}$ and $\mathbf{z}^j$. ■

Proposition 7.1 implies that a set of budgets which is powerful for a GARP test will also be powerful for a test for homotheticity. Additionally, the budgets for a HARP test do not even have to intersect; it is sufficient for the budgets to have similar slopes. Figure 7.4 illustrates this: Conditional on the observation on the budget with the steepest slope, the probability of a violation of HARP, given random choice, will be higher when this budget is paired with another budget with a similar slope. This is because homotheticity implies that the choice on the other budgets is made to the lower right side of the indicated expansion line, and the ratio $||\overline{L_i\,Y}||/||\overline{L_i\,X_i}||$ decreases as the slope decreases.

7.3 APPLICATIONS TO EXPERIMENTAL DATA

7.3.1 *Dictator Games*

The experiment of Andreoni and Miller (2002) was designed to test the rationalizability of altruistic choices. It is a generalized dictator game in which one subject (the dictator) allocates token endowments between himself and an anonymous other subject (the beneficiary) with different transfer rates. The payoffs to the dictator and the beneficiary are inter-

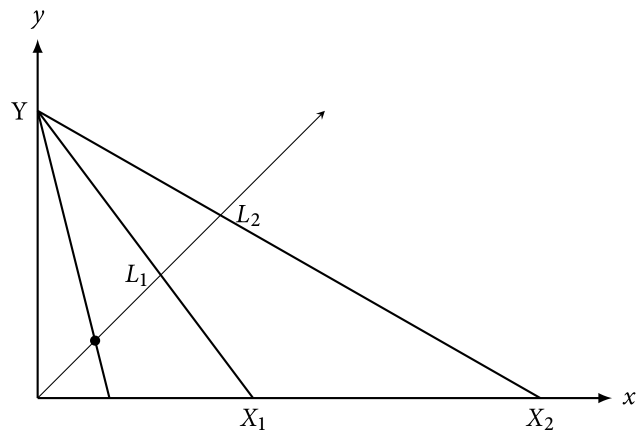

Figure 7.4.: The probability of observing a violation of HARP decreases as the difference in the slope increases.

preted as two distinct goods, and the transfer rates as the price ratio of these two goods.

Fisman et al. (2007) employed the same basic idea as Andreoni and Miller, but they presented subjects budgets graphically. Subjects could then make a decision by using a computer mouse to click on their desired bundle. This method allowed to collect a large set of observations per subject.[2]

Andreoni and Miller

The budgets presented to subjects were discrete: Subjects were only allowed to demand each of the two goods in integers. Therefore we employ the procedure described in Section 7.2.4 to test if any of the deviations for homotheticity could be due to the discreteness of budget. However, the tests for HARP and PHARP and the test that accounts for the

2 Note that none of the experiments considered here tested the hypothesis of income homogeneity of degree zero, which is implied by WARP. However, the data was tested for consistency with GARP, which allows multi-valued demand. Any apparent violation of income homogeneity could therefore also be interpretated as indifference between the different bundles.

discreteness of the budgets yield the same. We can therefore conclude that the setup is not sensitive with respect to discreteness.

The same data were also used to analyze differences between male and female subjects (see Andreoni and Vesterlund 2001); therefore data on the sex of a subject are available. Table 7.1 reports the number of subjects who satisfy PHARP and GARP; remember that in this case PHARP implies but is not implied by GARP.

		PHARP		GARP		ALL
		SATISFIED	VIOLATED	SATISFIED	VIOLATED	
SEX	MALE	67	28	89	6	95
	FEMALE	24	23	40	7	47
ALL		91	51	129	13	142

Table 7.1.: Subjects who satisfy PHARP and GARP in the dictator game of Andreoni and Vesterlund (2001) and Andreoni and Miller (2002).

A detailed analysis of the data is beyond the scope of this chapter, but it is interesting to note that male subjects are significantly less likely than female subjects to violate PHARP ($\chi^2 = 5.175$, $p = 0.0229$).[3]

Fisman et al.

As opposed to the Andreoni and Miller experiment, subjects were not required to make decisions on the upper bound of the budget but were free to click on any bundle on or below the budget line. Some of the subjects did not satisfy budget balancedness, and none of the subject satisfies GARP. Fisman et al. therefore had to resort to computing the

3 Andreoni and Vesterlund's finding that male subjects are more price sensitive than women might explain this result to a large extent.

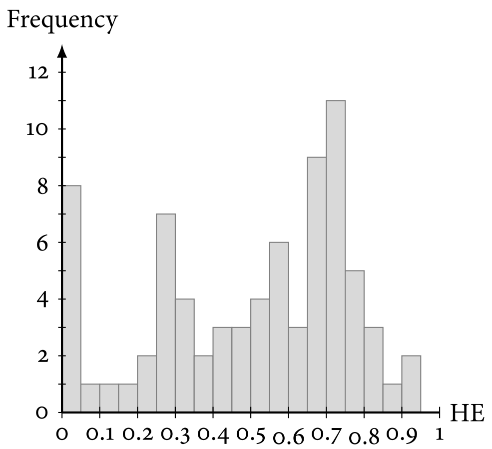

Figure 7.5.: Distribution of the homothetic efficiency index in the data of Fisman et. al.

Afriat Efficiency Index (or Critical Cost Efficiency Index) and concluded that most subjects were "close enough" to utility maximizing.

Using the observed expenditure instead of the actual endowment as income, 12 out of a total 76 subjects satisfy GARP. Out of these 12 subjects, 5 also satisfy PHARP.

To analyze the extent of deviation from homothetic rationalizability, we compute the HE index described in Section 7.2.3, Eq. (7.2). We also compute the AEI; the correlation coefficient between the HE and the AEI is $\rho = 0.791$, the Spearman rank correlation coefficient is $\rho^{\text{rank}} = 0.856$. Figure 7.5 shows the distribution of the HE. The mean HE is 0.501.[4]

Subject 14 occasionally passed substantial amounts of money when it was cheap to do so and had a rather high HE of 0.923. Figure 7.6 shows one of the decisions and the implied upper bound of the indifference curve through that point (gray curve). When we "impose" homotheticity on that subject's preferences and compute the convex monotonic hull of all points homothetically revealed preferred to the indicated choice (see Varian 1982 and Knoblauch 1993), we obtain a less steeper curve. The interpretation is that if we are willing to accept this subject's choices

4 It is not straightforward how to interpret the HE. One approach would be an extensive examination of the distribution of the HE under random behavior similar to the approach used in Chapters 8 and 9 which is beyond the scope of this chapter.

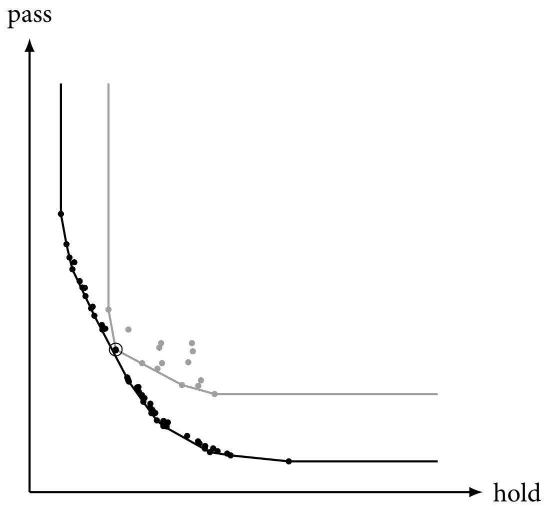

Figure 7.6.: Indifference curve for decision 35 of subject ID 14 as implied by choices, with and without imposing homotheticity.

as "close enough" to homothetic choices, we can recover a more precise indifference map. Without homotheticity, the subject's choices appear to be generated by preferences close to the Leontief type, whereas after imposing homotheticity, the preferences appear to be closer to utilitarian preferences.

7.3.2 *Two-Asset Risk Experiment*

Choi et al. (2007*a*) used graphical representations of simple portfolio choice problems to study behavior under uncertainty at the level of the individual subject. There was one symmetric treatment in which the two different assets on which subjects could allocate their endowment payed off with the same probability, and two asymmetric treatments with different probabilities.

The correlation coefficient between the HE and the AEI is $\rho = 0.792$, the Spearman rank correlation coefficient is $\rho^{\text{rank}} = 0.705$. Figure 7.7 shows the distribution of the HE. The mean HE is slightly lower in the symmetric treatment (0.731 and 0.759, respectively).

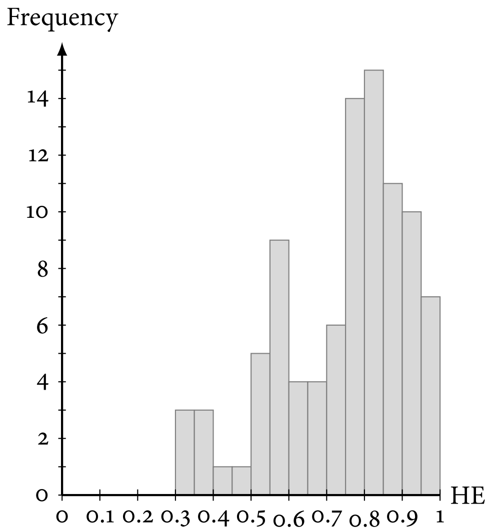

Figure 7.7.: Distribution of the homothetic efficiency index in the data of Choi et. al.

7.4 CONCLUSIONS

In Chapter 5 it was shown that for two dimensional commodity spaces any homothetic utility function that rationalizes each pair of observations in a set of consumption data also rationalizes the entire set of observations. The result exploits the possibility of ranking budgets by their slope, which is only possible when the consumption space is two-dimensional.

A straightforward application, given in this chapter, is to simplify the nonparametric test for homotheticity, so that the use of Warshall's algorithm can be avoided. Other possible applications are based on Lemma 5.1: It can be used to compute implied indifference curves that intersect the budget lines. This is useful to measure the extent of deviation from homotheticity, and to provide a nonparametric test of homotheticity for discrete budget sets. This was demonstrated by applying the described methods to experimental data.

Besides the explored features of the two-commoditiy case, the theoretical Chapter 5 also provided an overview of the relationship between different axioms of revealed preference which one can test for. We also showed how to design budgets in order to create a powerful test for homotheticity.

8 TESTING FOR UTILITY MAXIMIZATION WITH ERROR

8.1 INTRODUCTION

Revealed Preference methods offer an unambiguous way of testing whether a set of observations on consumption could have been generated by a single utility maximizing consumer. The test was originally developed by Afriat (1967). Varian (1982) showed that his Generalized Axiom of Revealed Preference (GARP) is equivalent to Afriat's condition of cyclic consistency. Consistency with GARP can easily be tested.

If a consumer's decisions are inconsistent with GARP one would like to have a test for "almost optimizing" behavior, or one might want to have an idea of how severe this violation of utility maximization is. One such measure is the Afriat efficiency index (AEI, Afriat 1972) or Critical Cost Efficiency Index, which is widely used.

Bronars (1987) suggests a Monte Carlo approach to determine the power the test has against random behavior. The approximate power of the test is the percentage of random choices which violated GARP.

Surprisingly it has rarely been noted that accepting certain consumers who exhibit less than 100% efficiency as "close enough" to GARP decreases the power of the test. Sippel (1996) attributes the first notice of the problem to Famulari (1995). Sippel shows that for data from three experiments (Battalio et al. 1973,Mattei 1994, Sippel 1997) the test for GARP loses most of its power when accepting most subjects as close enough to GARP. Fisman et al. (2007), Choi et al. (2007*a*) and Choi et al. (2007*b*) compute and compare the distribution of the AEI of their experimental data and of random choices based on Bronars' procedure and arrive at more optimistic results.

The aim of this paper is to establish a procedure for testing almost optimizing behavior based on the loss of power if we accept some consumers with GARP violations as close enough to GARP. This reveals the tradeoff between the level of false positives and false negatives. The researcher can then decide what level of power is still acceptable given the number of consumers he can then treat as utility maximizers. A small loss in power may be appropriate if that allows to treat many more consumers as utility maximizers – assuming that this is an objective – whereas a researcher should refrain from losing power if that loss would lead to only a small number of additional consumers accepted as close enough to GARP.[1]

The remainder of this chapter is organized as follows. Section 8.2 gives a short introduction to revealed preference theory and describes the suggested procedure. Section 8.3 applies the procedure to simulated utility maximizing data with stochastic error and to data from experimental dictator games. Section 8.4 concludes.

8.2 THEORY

8.2.1 *Preliminaries*

A set of observed consumption choices consists of a set of chosen bundles of commodities and the prices and incomes at which these bundles were chosen. Let $X = \mathbb{R}^\ell_+$ be the commodity space, where $\ell \geq 2$ denotes the number of different commodities.[2] The price space is $P = \mathbb{R}^\ell_{++}$, and the space of price-income vectors is $P \times \mathbb{R}_{++}$. Consumers choose bundles $x^i = (x^i_1, \dots, x^i_\ell)' \in X$ when facing a price vector $p^i = (p^i_1, \dots, p^i_\ell) \in P$ and an income $w^i \in \mathbb{R}_{++}$. A budget set is then defined by $B^i = B(p^i, w^i) = \{x \in X : p^i x^i \leq w^i\}$. The entire set of n observations on a consumer is denoted as $S = \{(x^i, B^i)\}^n_{i=1}$.

A utility function $u(x)$ *rationalizes* a set of observations S if $u(x^i) \geq u(x)$ for all x such that $p^i x^i \geq p^i x$ for all $i = 1, \dots, n$.

The following definitions are needed to recover consumer preferences that are implicit in a set of consumption choices: An observation x^i is *directly revealed preferred* to x, written $x^i R x$, if $p^i x^i \geq p^i x$; *revealed preferred* to x, written $x^i R^* x$, if for some sequence of bundles $(x^j, x^k, \dots,$

1 In Chapter 9 we use this approach to compare the AEI with a new efficiency index.

2 The following notation is used: For all $x, y \in \mathbb{R}^\ell$ we write $x \geq y$ for $x_i \geq y_i$ for all i, $x > y$ for $x_i \geq y_i$ and $x \neq y$ for all i, and $x \gg y$ for $x_i > y_i$ for all i. We denote $\mathbb{R}^\ell_+ = \{x \in \mathbb{R}^\ell : x \geq 0\}$ and $\mathbb{R}^\ell_{++} = \{x \in \mathbb{R}^\ell : x \gg 0\}$.

x^m) such that x^iRx^j, x^jRx^k, ..., x^mRx. In this case R^* is the *transitive closure* of the relation R; *strictly directly revealed preferred* to x, written x^iPx, if $p^ix^i > p^ix$.

For consistency with the maximization of a piecewise linear utility function, Varian (1982) introduced the following condition: The set of observations S satisfies the *Generalized Axiom of Revealed Preference* (GARP) if $x^iR^*x^j$ implies $\left[\text{not } x^jPx^i\right]$. It can then be shown (Afriat 1967, Varian 1982) that if the data satisfy GARP then there exists a concave, monotonic, continuous, non-satiated utility function that rationalizes the data.

Several goodness-of-fit measures have been proposed. Arguably the most popular measure for the severity of a violation is the Afriat efficiency index (AEI) due to Afriat (1972). Reporting the AEI has become a standard for experimental studies.[3] To obtain the AEI, budgets are shifted towards the origin until a set of observations is consistent with GARP. Let e be a number between 0 and 1. Define the relation $R(e)$ to be $x^iR(e)x^j$ if $ep^ix^i \geq p^ix$, and let $R^*(e)$ be the transitive closure of $R(e)$.

Definition 8.1 (Generalized Axiom of Revealed Preference at efficiency level e) A set of data satisfies the Generalized Axiom of Revealed Preference at efficiency level e (GARP(e)) if $x^iR^*(e)x^j$, $x^i \neq x^j$, implies $\left[\text{not } ep^jx^j > p^jx^i\right]$.

Then the AEI is the largest number such that GARP(e) is satisfied. See Gross (1995) for a survey of other measures.

8.2.2 *Power against Random Behavior*

Depending on the characteristics of the budget sets, the chance of violating GARP can differ substantially. A completely rational consumer will always be consistent. However, even a consumer who makes purely random decisions has a chance to satisfy GARP. Bronars (1987) suggests a Monte Carlo approach to determine the power the test has against random behavior. The approximate power of the test is the percentage of random choices which violated GARP. Bronars' first algorithm follows

3 See, for example, Sippel (1997), Mattei (2000), Harbaugh et al. (2001), Andreoni and Miller (2002), Février and Visser (2004), Choi et al. (2007*b*), Fisman et al. (2007).

Becker's (1962) example by inducing a uniform distribution across the budget hyperplane. For Bronars' second algorithm, the random choices are generated by drawing ℓ i.i.d uniform random variables, $z_1, \ldots, z_\ell$, for each price vector, and calculate budget shares $Sh_i = z_i / \sum_{j=1}^{\ell} z_j$. The random demand for commodity x_i is then calculated as $x_i = (Sh_i\, w)/p_i$.

8.2.3 *The Procedure*

Varian (1990) suggests a 95% AEI as the critical value for acceptance of GARP-violating sets of observations as utility maximizing, "for sentimental reasons". There is, however, no natural critical value. We therefore suggest to generate random choices on the budget sets and to recompute Bronars' power for all observed efficiency levels between 0 and 1. This will give us an idea of how much power the test loses if we accept GARP-violating observations as close enough to GARP. This procedure also allows one to compare different efficiency indices.

To approximate the power of the GARP test if we allow deviations from utility maximization, we need to generate random choices on the budget sets:

A1 Generate random choices on the budget sets of the observed data, following Bronars' first and second algorithm: (1) Draw a random point SP from the $(\ell - 1)$-simplex using a simplex point picking algorithm. The random demand for commodity x_i is then calculated as $x_i = (SP_i\, w)/p_i$. (2) Draw ℓ i.i.d uniform random variables, $z_1, \ldots, z_\ell$, for each price vector, and calculate budget shares $Sh_i = z_i / \sum_{i=1}^{\ell} z_j$. The random demand for commodity x_i is then calculated as $x_i = (Sh_i\, w)/p_i$.

A2 Repeat steps A1 many times for each set of observed budgets.

The second step is to compute the loss of power of the test for all possible values of AEI:

B1 Compute the AEI for each consumer in the observed data sets.

B2 Generate sets of observations following procedure A. Again, compute the AEI for each set.

B3 Sort the sets from B1 by their AEI values. For each set from B1, compute the percentage of sets from step B2 that have a higher AEI.

8.3 APPLICATION

To illustrate the procedure, we take data from a known generating function and add stochastic error to simulate measurement error. Ideally, we would like to accept all of the sets obtained in this way for reasonably low error terms as utility maximizing without thereby reducing the power of the applied test.

We use a similar procedure as applied in Fleissig and Whitney (2003, 2005). First, we generate data from a five commodity Cobb-Douglas utility function given by

$$U(x^*) = \prod_{i=1}^{5} x_i^{*\alpha i}, \qquad \text{with} \sum_{i=1}^{5} \alpha_i = 1 \tag{8.1}$$

We use random parameters each time by drawing each α_i from a uniform distribution $\mathcal{U}[.05, .95]$ and then normalizing it such that $\sum_{i=1}^{5} \alpha_i = 1$.

For the Monte-Carlo experiment we assume that we observe the demand according to the given utility function with some measurement error that fluctuates by κ% around the true demand; we use $\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\} = \{.05, .1, .2, .25\}$.

The datasets have $n = 20$ observations each, with expenditure w drawn from a uniform distribution $\mathcal{W} \sim \mathcal{U}[10000, 12000]$. Price vectors are drawn from a uniform distribution $\mathcal{P} \sim \mathcal{U}[95, 100]$. These expenditures and prices lead to many intersections of budget sets which can lead to many violations of GARP.

The data are generated by the following steps:

A1' Randomly draw 20 expenditure observations from the uniform distribution $\mathcal{W}$ and 20 price vectors for which each element is drawn from the uniform distribution $\mathcal{P}$.

A2' Generate utility maximizing demand for each budget. Denote the maximizing decisions x_i^* for $i = 1, \ldots, 5$.

A3' Generate demands with measurement error by multiplying $x_1^*, \ldots, x_5^*$ by a uniform random number, so that $x_i = x_i^*(1 + \varepsilon_i)$ for $i = 1, \ldots, 5$, where $\varepsilon_i \sim \mathcal{U}[-\kappa, \kappa]$ and $\kappa \in \{.05, .1, .2, .25\}$. To keep expenditure constant, normalize the x_i by multiplying each with $\lambda = w/(p \cdot x)$.

A4' Repeat steps A1' – A3' many times.

For illustrative purposes we proceed to execute procedures A, A', and B. For the latter we use the data generated in A' as the observed data sets. A and A' are repeated 10000 times. Bronars' power is based on his first algorithm.[4] We then plot the result measuring the power of the test on the horizontal axis and the fraction of observations accepted as close enough to utility maximization on the vertical axis; see Figure 8.1.

To further illustrate the procedure, we use laboratory data from two experimental dictator games of Andreoni and Miller (2002) and Fisman et al. (2007). In the former experiment, the same set of budgets was used for each subject. Bronars' power is based on the first algorithm with 20000 repetitions of procedure A. In the latter experiment, budgets were drawn randomly for each subject. Bronars' power is based on the first algorithm with 200 repetitions of procedure A for each subject's budget sets, resulting in 15200 repetitions.[5] The results are reported in Figure 8.2.

Finally, to find the optimal tradeoff, note that the curves obtained by the above procedure can be interpreted as a (non-linear) budget set. A researcher could specify a "utility function" for test power and the fraction of consumers accepted as close enough to GARP and find the optimal cutoff point. For example, a Cobb-Douglas type utility function $v\,(\text{Power}, \text{GARP}) = \text{Power}^{2/3}\text{GARP}^{1/3}$ is maximized by choosing a power

4 The results for Bronars' second algorithm are very similar.

5 In Fisman et al. (2007), 50 budgets were drawn from identical distributions, hence all subjects faced very similar budgets.

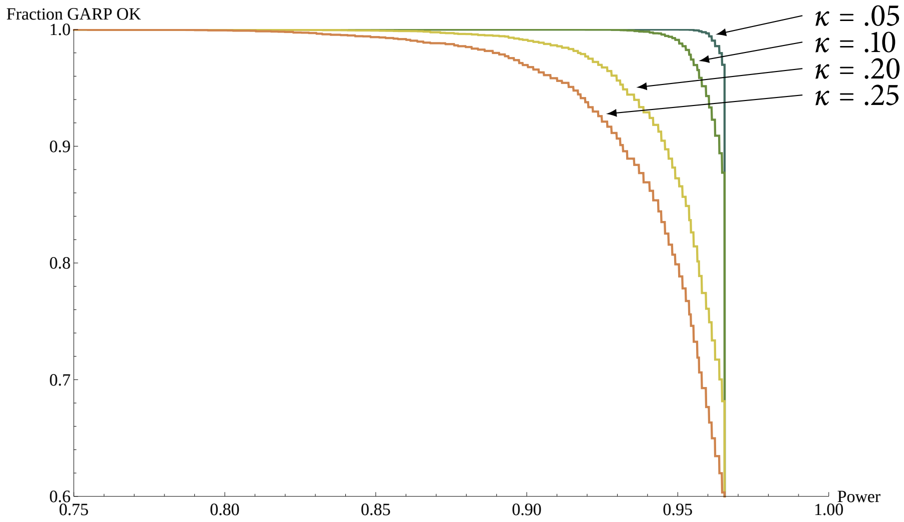

Figure 8.1.: Simulated Data. The plots show the fraction of simulated consumers who are accepted as close enough to GARP depending on the maintained power.

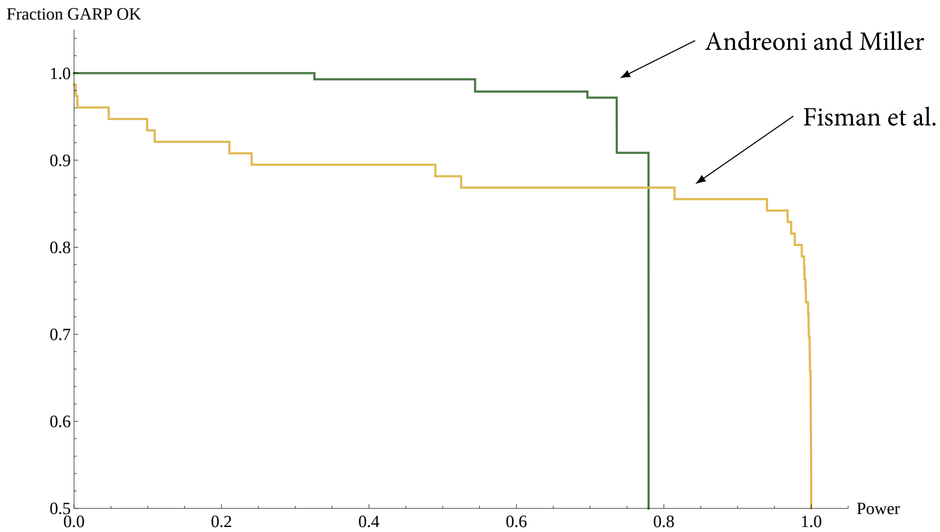

Figure 8.2.: Data from Andreoni and Miller (2002) and Fisman et al. (2007). The plots show the fraction of subjects who are accepted as close enough to GARP depending on the maintained power.

of 0.9679 with a fraction of 0.8421 subjects accepted as close enough to GARP for the data from Fisman et al. (2007).

8.4 DISCUSSION AND CONCLUSION

The application to simulated data shows that for reasonably small error terms one can accept almost all of the choices with stochastic error as close enough to utility maximizing without losing much power. For larger error terms the loss of power is substantial. The application to experimental data shows that the experiments from Andreoni and Miller (2002) and Fisman et al. (2007) suffer from lack of power once we allow deviation from 100% efficiency for most subjects, although it is noteworthy that for the data from Fisman et al. (2007) all subjects violated budget balancedness (did not spend all of their income) and yet we can accept many of the subjects as utility maximizers without much loss of power. With the introduction of the graphical presentation of budgets to subjects (Choi et al. 2007*b*) which allows to collect more data in experiments, we can expect to see more experiments of the kind of Choi et al. (2007*a*) and Fisman et al. (2007). Because in these experiments no subject was perfectly consistent with GARP, trading off power against accepting subjects as utility maximizers becomes a delicate business. The present paper may contribute to the analysis of such data.

The procedure described in this note can also be used to compare different methods of measuring the extent of violations of utility maximization. For this it is necessary to compute the different efficiency indices for simulated utility maximizing choices with stochastic errors and repeat this for Bronars' procedure. One can then compare the loss of power when basing the decision which observations to accept as close enough to utility maximizing on the different measures.

9

A GEOMETRIC MEASURE FOR THE VIOLATION OF UTILITY MAXIMIZATION

9.1 INTRODUCTION

If a consumer's decisions are inconsistent with GARP we might want to have an idea of how severe this violation of utility maximization is. Alternatively, we would like to have a test for "almost optimizing" behavior. One such measure is the Afriat efficiency index (AEI, Afriat 1972), which is widely used.

We propose a new measure based on the extent to which the upper bound of the indifference surface of a decision intersects the budget on which the decision was made. The idea is to use preference relations that are implicit in a set of observations to construct the set of bundles which are revealed preferred to a consumption choice. The boundary of this set can be interpreted as an upper bound for the indifference surface. If the data violate GARP, some of these sets will intersect the budget hyperplane on which the choice was made. We then compute the area (or volume in higher dimensions) of the intersection of the revealed preferred set and the budget.

We use the procedure described in Chapter 8 to decide whether or not to treat a consumer who violates GARP as "close enough" to utility maximization. It is based on the reduction of the power the test has against random behavior. When testing this procedure with a set of utility maximizing decisions with added stochastic error, our new geometric measure performs very well compared to the AEI.

The remainder is organized as follows: Section 9.2 first briefly summarizes the revealed preference approach and the AEI. The new geometric measure is introduced and it is shown how the procedure described in Chapter 8 can be applied to compare the measure with the AEI. Section 9.3 compares the new measure and its performance with the AEI. Section

9.4 discusses the advantages and disadvantages of the new measure and concludes.

9.2 THEORY

9.2.1 *Preliminaries*

A set of observed consumption choices consists of a set of chosen bundles of commodities and the prices and incomes at which these bundles were chosen. Let $X = \mathbb{R}^{\ell}_{+}$ be the commodity space, where $\ell \geq 2$ denotes the number of different commodities. The price space is $P = \mathbb{R}^{\ell}_{++}$, and the space of price-income vectors is $P \times \mathbb{R}_{++}$. Consumers choose bundles $x^i = (x^i_1, \ldots, x^i_\ell)' \in X$ when facing a price vector $p^i = (p^i_1, \ldots, p^i_\ell) \in P$ and an income $w^i \in \mathbb{R}_{++}$. A budget set is then defined by $B^i = B(p^i, w^i) = \{x \in X : p^i x^i \leq w^i\}$. The entire set of n observations on a consumer is denoted as $S = \{(x^i, B^i)\}^n_{i=1}$.

A utility function $u(x)$ *rationalizes* a set of observations S if $u(x^i) \geq u(x)$ for all x such that $p^i x^i \geq p^i x$ for all $i = 1, \ldots, n$.

The following definitions are needed to recover consumer preferences that are implicit in a set of consumption choices:

An observation x^i is
(1) *directly revealed preferred* to x, written $x^i R x$, if $p^i x^i \geq p^i x$;
(2) *revealed preferred* to x, written $x^i R^* x$, if either $x^i R x$ or for some sequence of bundles $(x^j, x^k, \ldots, x^m)$ such that $x^i R x^j, x^j R x^k, \ldots, x^m R x$. In this case R^* is the *transitive closure* of the relation R.
(3) *strictly directly revealed preferred* to x, written $x^i P x$, if $p^i x^i > p^i x$.

For consistency with the maximization of a piecewise linear utility function, Varian (1982) introduced the following condition: The set of observations S satisfies the *Generalized Axiom of Revealed Preference* (GARP) if $x^i R^* x^j$ implies [not $x^j P x^i$].

It can then be shown (Afriat 1967, Varian 1982) that if the data satisfy GARP then there exists a concave, monotonic, continuous, non-satiated utility function that rationalizes the data.

The set of bundles that are revealed preferred to a certain bundle x^0 (which does not have to be an observed choice) is given by the convex monotonic hull of all choices revealed preferred to x^0. The interior of the convex monotonic hull is used to compute an approximate overcompensation function by Varian (1982). Knoblauch (1992) shows that the set of bundles revealed preferred to x^0, denoted $RP(x^0)$, is just the convex monotonic hull of all bundles in S that are revealed preferred to x^0:

$$RP(x^0) = H_{\text{convex}}\Big(\big\{x \in X : x \geq x^i \text{ such that } x^i R^* x^0 \text{ for some } i = 1, \ldots, \big\}\Big). \tag{9.1}$$

where H_{convex} denotes the convex hull. See also Definition 3.4.

9.2.2 *Prior Measures*

Several goodness-of-fit measures have been proposed. Arguably the most popular measure for the severity of a violation is the Afriat efficiency index (AEI) due to Afriat (1972). Reporting the AEI has become a standard at least for experimental studies.[1] To obtain the AEI, budgets are shifted towards the origin until a set of observations is consistent with GARP. Let e be a number between 0 and 1. Define the relation $R(e)$ to be $x^i R(e) x^j$ if $ep^i x^i \geq p^i x$, and let $R^*(e)$ be the transitive closure of $R(e)$. Define GARP(e) as

$$\text{GARP}(e) \Leftrightarrow \text{ If } x^i R^*(e) x^j \text{ implies } [\text{not } ep^j x^j > p^j x^i]. \tag{9.2}$$

Then the AEI is the largest number e such that GARP(e) is satisfied.

Other measures include Varian's (1985) minimum perturbation test, based on the minimal movements of the data needed to accept the null hypothesis of utility maximization; Famulari's (1995) violation rate, which is the proportion of combinations that form violations among observations for which violations can be expected; and comparison of the observed number of violations with the maximum number of violations

1 See, for example, Sippel (1997), Mattei (2000), Harbaugh et al. (2001), Andreoni and Miller (2002), Février and Visser (2004), Fisman et al. (2007), Choi et al. (2007*a*), Choi et al. (2007*b*).

possible, as applied by Swofford and Whitney (1987) and McMillan and Amoako-Tuffou (1988). See Gross (1995) for a survey of measures and his own test, which is based on an estimate of wasted expenditure.

9.2.3 *The New Measure*

Obviously, if a consumer makes decisions that are incompatible with GARP, then at least for one choice the upper bound of the indifference curve through that point, as implied by the other choices, intersects the budget line he made the choice on.[2] The idea of our measure is to ask, "how much of a given budget did a consumer reveal to prefer to the actual choice he made on the budget?". To answer the question, we take the upper bound of the indifference curve through a choice x^i and compute the area between that curve and the budget line. That is to say, we compute the area of the intersection of the two sets B^i and $RP(x^i)$. This basic idea is illustrated in Figure 9.1 and 9.2.

The "size" of an intersection of B^i and $RP(x^i)$ is an area in two dimensions, and a volume in three dimensions. For simplicity, the generalization to arbitrary dimensions (the "hypervolume") will be also be called volume[3] and denoted by $vol(\text{Polytope})$. For example, the volume of an ℓ-dimensional hypercube h with edge length a is $vol(h) = a^\ell$.

Denote the volume of the intersection of a budget B^i and all bundles revealed preferred to x^i by

$$V(x^i) = vol\left(RP(x^i) \cap B^i\right). \tag{9.3}$$

Obviously, if S satisfies GARP, $V(x^i) = 0$ for all $i = 1, \ldots, n$.

To compare the extent of violation of GARP between many consumers who all made decisions on the same budgets, $V(x^i)$ does not have to be adjusted. However, if consumers made decisions on different budgets, the magnitude of $V(x^i)$ can be misleading. We therefore normalize the volume in the following way:

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2 Note that for illustrative purposes, we occasionally use terms only applicable to the two dimensional case.

3 The generalization of an area or volume to higher dimensions is also known as the *content*. See Weisstein (2008).

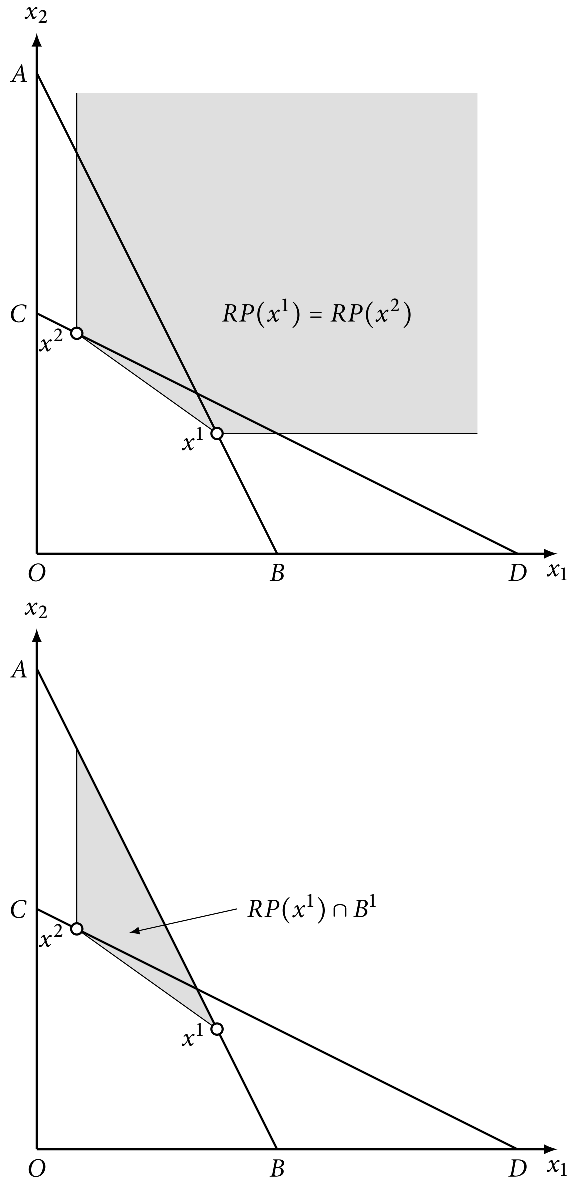

Figure 9.1.: Top: Two observations which violate GARP. The shaded area gives the set of all bundles revealed preferred to x^1 and x^2. Since x^1 and x^2 form a preference cycle the sets are necessarily identical. Bottom: The intersection of $RP(x^1)$ with the budget line AB on which x^1 was chosen.

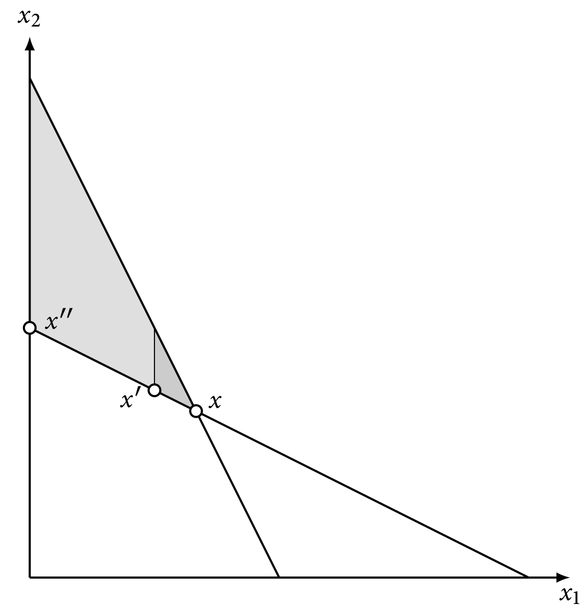

Figure 9.2.: Both $\{x', x\}$ and $\{x'', x\}$ lead to the same Afriat efficiency index of 1 - ε, but have different volume violation indices.

Denote the ratio of $V(x^i)$ to the entire volume of the budget B^i as

$$V^B(x^i) = \frac{V(x^i)}{vol(B^i)} \tag{9.4}$$

where

$$vol(B^i) = \left(\prod_{j=1}^{\ell} \frac{w^i}{p^i_j}\right)\frac{1}{\ell!}. \tag{9.5}$$

Given the $V(x^i)$, we would like an index that aggregates the different intersections. One obvious way to define the index is the mean of all $V(x^i)$.[4] Denote by

$$\mathrm{VI}^{\mathrm{mean}} = \frac{1}{n}\sum_{i=1}^{n} V^B(x^i) \tag{9.6}$$

an index using the mean of all $V^B(x^i)$. Because V^B is bounded between 0 and 1, we can define the *volume efficiency index* (VEI) as

$$\mathrm{VEI} = 1 - \mathrm{VI}^{\mathrm{mean}} \tag{9.7}$$

In two dimensions, computation is fairly simply. For higher dimensions, we use the program qhull, which implements the quick hull algorithm for convex hulls (see Barber et al. 1996).

9.2.4 *Power against Random Behavior*

Depending on the characteristic of the budget sets, the chance of violating GARP can differ substantially. A utility maximizing consumer will always be consistent and is not in "danger" of violating GARP. However, even a consumer who makes purely random decision has a chance to satisfy GARP. Bronars (1987) suggests a Monte Carlo approach to determine the power the test has against random behavior. The approximate power of the test is the percentage of random choices which violated GARP.

4 Another option is to take the maximal of all $V(x^i)$, or the median. The results are robust with respect to the aggregation method.

Bronars' first algorithm follows Becker's (1962) example by inducing a uniform distribution across the budget hyperplane. Using Bronars' second algorithm, the random choices are generated by drawing ℓ i.i.d uniform random variables, $z_1, \ldots, z_\ell$, for each price vector, and calculate budget shares $Sh_i = z_i / \sum_{j=1}^{\ell} z_j$. The random demand for commodity x_i is then calculated as $x_i = (Sh_i\, w)/p_i$.

One way to utilize Bronars' power is to compute it based on a range of values for an index that measures the severity of the violations, i.e. the power of the test is the percentage of random choices which do "worse" than the value of the index. For example, for an VEI or AEI of .9, compute the percentage of random choices which have a VEI or AEI lower than .9. This will give an idea of how much power the test loses if we allow consumers to deviate from optimizing behavior. See Section 9.3 for details.

9.2.5 *Theoretical Considerations*

In Section 9.3 we will evaluate the new measure based on Monte-Carlo experiments. However, it should already be pointed out that while the new measure is quite intuitive, it has a theoretical shortcoming. The AEI can be related to *wasted absolute income*, which is a real magnitude. The volume efficiency index is related to the fraction of the budget which is preferred to the actual decision, and puts equal weight on fractions of the same volume. Neither does it tell us anything about wasted income, nor does it say much about wasted utility.

Consider Figure 9.2. If we move x just a bit upwards on the steeper of the two budget lines we can find a utility function that rationalizes $\{x, x'\}$ and $\{x, x''\}$. Suppose that the data was collected with a small measurement error and that the consumer's actual decision was indeed a bit to the upper left of the observation x. While the AEI is 1 (or $1 - \varepsilon$) and raises little concern about the rationality of the consumer, the VEI suggests a small but substantial deviation from utility maximization if the data is $\{x, x'\}$ and a relatively large deviation if the data is $\{x, x''\}$.

To understand this unrobust behavior of the volume efficiency index, note that the set of observations $\{x, x'\}$ or $\{x, x''\}$ only implies that the shaded area is revealed preferred to x, but nothing can be said about *how much* it is preferred. Consider Figure 9.3, and suppose x^1 and x^2 are the actual decisions instead of the observed ones in Figure 9.2. Suppose the two indifference curves represent utility levels which, in absolute terms, are only marginally different. Then no bundle in the part of the budget which is preferred to x in Figure 9.2 adds much utility to x.

Based on these considerations we can expect the volume efficiency index to be more robust if the underlying preferences are homothetic or "almost" homothetic. In the diagram on the bottom of Figure 9.3 the bundle x^2 is projected on budget 1. Homotheticity implies that $t\,x^2$ is preferred over all bundles to the right of $t\,x^2$ on budget 1, so the actual decision made on budget 1 would have to be on the left of $t\,x^2$. This implies that if we measure decisions of a homothetic consumer with some measurement error or slight failures in the maximization process it is unlikely that we observe a decision pattern as the one depicted in Figure 9.2.

Notwithstanding these theoretical concerns, the results of the Monte-Carlo experiments in the next section imply that the volume efficiency index can be usefully applied.

9.3 COMPARISON: POWER AGAINST RANDOM BEHAVIOR

9.3.1 *Procedure*

The procedure suggested in Chapter 8 is useful to compare two different efficiency indices. To evaluate the two indices, we take data from known generating functions and add stochastic error to simulate measurement error. Ideally, we would like to accept all of the thusly obtained sets as utility maximizing without thereby reducing the power of the applied test.

We use a similar procedure as applied in Fleissig and Whitney (2003, 2005). First, we generate data from a five commodity utility function. The first function is a Cobb-Douglas type utility function given by

$$U^{\mathrm{CD}}(x^*) = \prod_{i=1}^{5} x_i^{*\alpha i}, \qquad \text{with } \sum_{i=1}^{5} \alpha_i = 1 \tag{9.8}$$

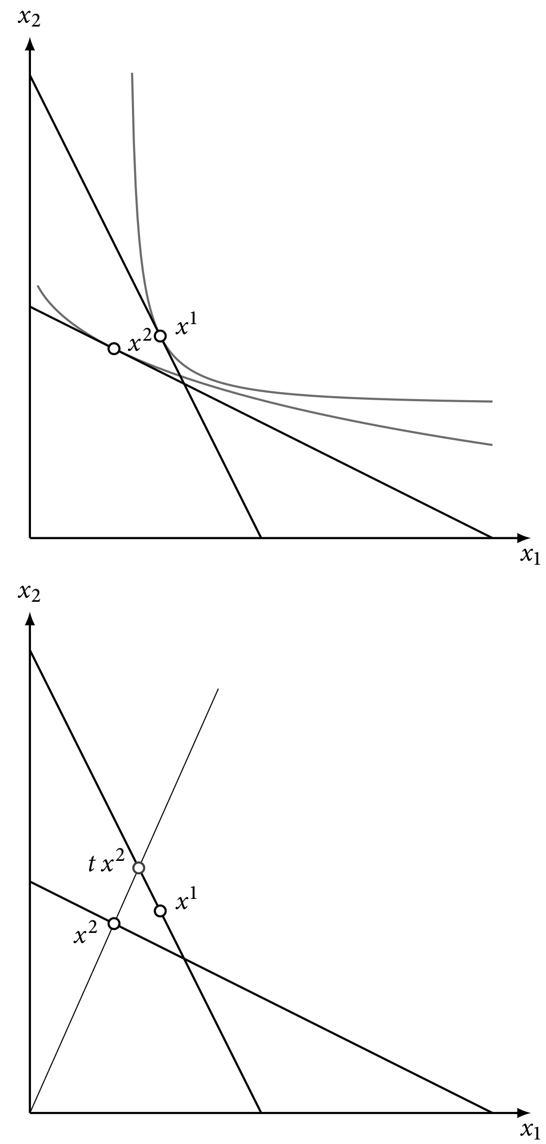

Figure 9.3.: Top: Wasting a large fraction of the budget does not necessarily mean that the achieved utility level could have been a lot higher. Bottom: The set $\{x^1, x^2\}$ does not satisfy homotheticity, because $t\,x^2$ would then be strictly preferred to x^1. Observations with measurement error are less likely to cause a low volume efficiency index if the underlying preferences are homothetic.

The second utility function is a non-homothetic utility function with variable elasticity of substitution:

$$U^{\text{VES}}(x^*) = \sum_{i=1}^{5} \alpha_i \left(x_i^* - \gamma_i\right)^{\beta_i}, \qquad \text{with} \sum_{i=1}^{5} \alpha_i = \sum_{i=1}^{5} \beta_i = 1 \tag{9.9}$$

We run the computations for both functions with a different set of random parameters each time by drawing each α_i and each β_i from a uniform distribution $\mathcal{U}[.05, .95]$ and then normalizing it such that $\sum_{i=1}^{5} \alpha_i = \sum_{i=1}^{5} \beta_i = 1$. We draw each γ_i from a uniform distribution $\mathcal{U}[-5, 5]$.[5]

For the Monte-Carlo experiment we assume that we observe the demand according to the given utility function with some measurement error that fluctuates by κ% around the true demand; we use $\{\kappa_1, \kappa_2, \kappa_3\} = \{.05, .1, .2\}$.

The datasets have $n = 20$ observations each with expenditure w drawn from a uniform distribution $\mathcal{W} \sim \mathcal{U}[10000, 12000]$. Price vectors are drawn from a uniform distribution $\mathcal{P}_1 \sim \mathcal{U}[95, 100]$. The same steps are repeated with a distribution $\mathcal{P}_2 \sim \mathcal{U}[90, 100]$. These expenditures and prices lead to many intersections of budget sets which can lead to many violations of GARP.

To summarize, we use 12 different settings, each one being an element of $\{\text{CD}, \text{VES}\} \times \{\kappa_1, \kappa_2, \kappa_3\} \times \{\mathcal{P}_1, \mathcal{P}_2\}$.

The data are generated by the following steps:

A1 Randomly draw n expenditure observations from a uniform distribution $\mathcal{W}$ and n price vectors for which each element is drawn from a uniform distribution $\mathcal{P} \in \{\mathcal{P}_1, \mathcal{P}_2\}$.

A2 Draw parameters α, β, γ from $\mathcal{U}[.05, .95]$ and $\mathcal{U}[-5, 5]$, respectively. Generate utility maximizing demand for each budget, using the respective functional form and set of parameters. Denote the maximizing decisions x_i^* for $i = 1, \dots, 5$.

A3 Generate demands with measurement error by multiplying $x_1^*, \dots, x_5^*$ by a uniform random number, so that $x_i = x_i^*(1 + \varepsilon_i)$ for $i =$

5 The simulations were also conducted using a fixed set of parameters. The results are very similar.

$1,\dots,5$, where $\varepsilon_i \sim \mathcal{U}[-\kappa,\kappa]$ and $\kappa \in \{.05,.1,.2\}$. To keep expenditure constant, normalize the x_i by multiplying each with $\lambda = w/(p \cdot x)$.

A4 Repeat steps A1 – A4 many times, say 10,000.

To approximate the power the GARP test has if we allow deviations from utility maximization, we need to generate random choices on the budget sets:

B1 Generate budgets as in step A1.

B2 Generate random choices on the budget sets of step B1, following Bronars' first and second algorithm: (1) Draw a random point *SP* from the 4-simplex using a simplex point picking algorithm. The random demand for commodity x_i is then calculated as $x_i = (SP_i\, w)/p_i$. (2) Draw five i.i.d uniform random variables, $z_1,\dots,z_5$, for each price vector, and calculate budget shares $Sh_i = z_i/\sum_{i=1}^{5} z_j$. The random demand for commodity x_i is then calculated as $x_i = (Sh_i\, w)/p_i$.

B3 Repeat steps B1 and B2 many times, at least as often as with A1 – A4.

The final step is to compute the loss of power of the test for all AEI and VEI values:

C1 Generate utility maximizing sets of observations with stochastic error, following procedure A. Then for each set of n budgets, compute the AEI and the VEI.

C2 Generate sets of observations following procedure B. Again, compute the AEI and the VEI for each set.

C3 Sort the sets from C1 by their AEI and VEI values, respectively. For each set from C1, compute the percentage of sets from step C2 that have a higher AEI and VEI value, respectively.

	ALGORITHM	
	FIRST	SECOND
$\mathcal{P}_1$	96.52%	90.80%
$\mathcal{P}_2$	99.58%	99.22%

Table 9.1.: Bronars' Power of the used data sets.

9.3.2 *Results*

Descriptive Statistics

Table 9.1 reports the Bronars' Power of the data sets generated by procedure B.

Table 9.2 reports the fraction of choice sets generated by procedure A which are inconsistent with GARP.

Loss of Power

The main result is that for all of the different data sets we generated, the loss of power is mostly smaller and never greater if the cutoff point is based on the VEI rather than the AEI. Perhaps a bit surprisingly, the result is robust with respect to the functional form. It suggests that the VEI is better suited than the AEI to capture small deviations from utility maximization and distinguish between a set of decisions that are close to utility maximizing on the one hand and purely random behavior on the other hand.

Figures 9.4 and 9.5 report the proportion of utility maximizing observations with stochastic error that are accepted as "consistent enough" with GARP, depending on the desired power of the test, for the Cobb-Douglas and VES utility function, respectively. Both figures report the results for $\mathcal{P}_1$ and Bronar's first algorithm. The results are very similar for the remaining configurations. In all cases, we lose less test power when basing decisions on the VEI.

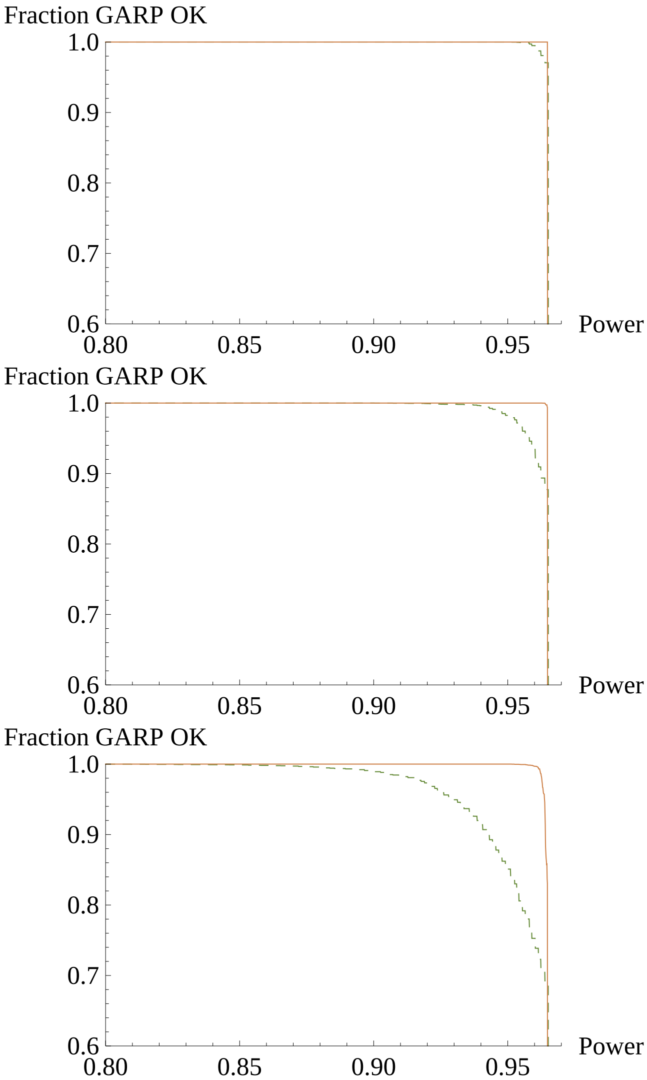

Figure 9.4.: Results for the Cobb-Douglas function, using $\mathcal{P}_1$ and Bronars' first algorithm. Top: $\kappa = .05$, middle: $\kappa = .1$, bottom: $\kappa = .2$. The figure reports the proportion of utility maximizing observations that are accepted as consistent with GARP, depending on the desired power of the test. The dashed line gives the proportion of accepted observations according to the AEI, and the solid line gives the proportion according to the VEI.

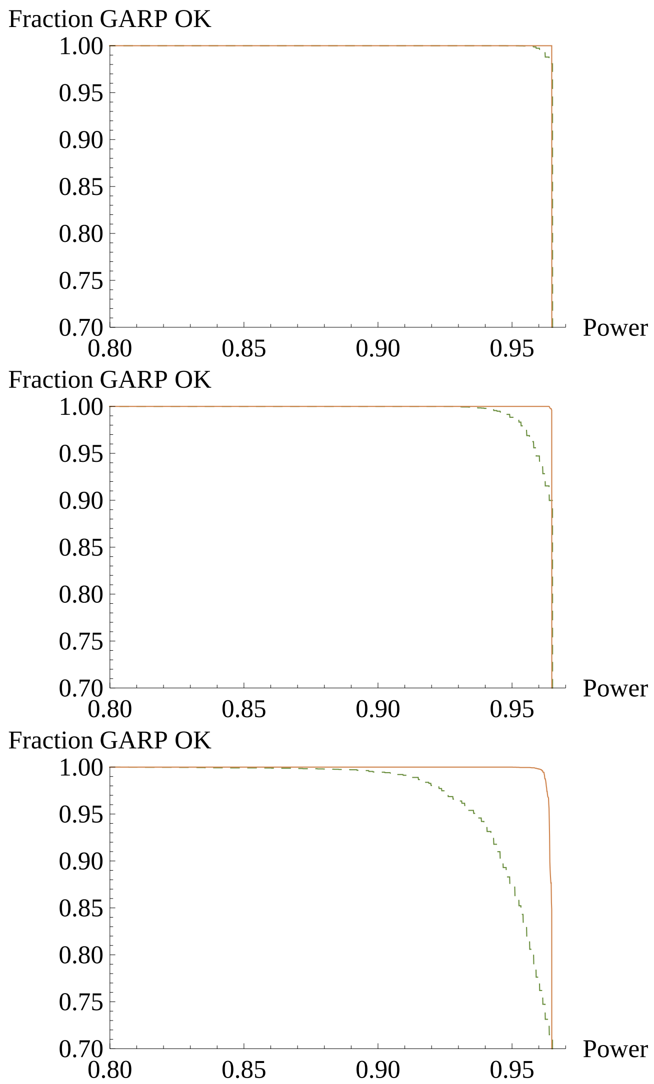

Figure 9.5.: Same as for Figure 9.4, but for the VES utility function.

	COBB-DOUGLAS FUNCTION		
	$\kappa = .05$	$\kappa = .1$	$\kappa = .2$
$\mathcal{P}_1$	2.94%	12.28%	31.09%
$\mathcal{P}_2$	1.06%	10.62%	38.39%
	VES FUNCTION		
	$\kappa = .05$	$\kappa = .1$	$\kappa = .2$
$\mathcal{P}_1$	1.92%	10.02%	28.51%
$\mathcal{P}_2$	0.40%	6.72%	32.00%

Table 9.2.: Fractions of GARP-inconsistent choice sets.

Table 9.3 reports the retained test power when all utility maximizing choices with stochastic error are accepted as utility maximizing, using Bronars' first algorithm.

9.4 DISCUSSION AND CONCLUSION

In this paper a new measure for the severity of a violation of utility maximization, the volume efficiency index, was suggested. The measure is based on the extent to which the upper bound of the indifference surface of a decision intersects the budget on which the decision was made. This measure has several advantages.

The measure is intuitively appealing as it can be easily illustrated with graphical tools covered in any intermediate course in microeconomic theory. In two dimensions the measure is easy to compute. It performs very well as a cutoff rule for determining whether or not observations on a single consumer can still be considered "close enough" to maximizing behavior (see also Chapter 8).

COBB-DOUGLAS FUNCTION						
	AEI			VEI		
	$\kappa = .05$	$\kappa = .1$	$\kappa = .2$	$\kappa = .05$	$\kappa = .1$	$\kappa = .2$
$\mathcal{P}_1$	94.92%	90.03%	78.51%	96.49%	96.30%	95.15%
$\mathcal{P}_2$	99.42%	98.73%	92.04%	99.58%	99.57%	99.36%

VES FUNCTION						
	AEI			VEI		
	$\kappa = .05$	$\kappa = .1$	$\kappa = .2$	$\kappa = .05$	$\kappa = .1$	$\kappa = .2$
$\mathcal{P}_1$	94.79%	92.80%	80.07%	96.49%	96.37%	95.03%
$\mathcal{P}_2$	99.48%	99.16%	96.03%	99.58%	99.57%	99.41%

Table 9.3.: The retained test power when all choices are accepted as utility maximizing, using Bronars' first algorithm.

A disadvantage is the computational effort needed to compute the measure in high dimensions.[6] However, note that the dimension of most data obtained by laboratory experiments is naturally bounded.

6 From experimentation with simulated data it seems that even Monte Carlo experiments are still quite feasible in six dimensions and 40 observation points per consumer.

Part IV

APPENDIX

APPENDIX A

Proof of the relations between the axioms as shown in Figure 5.3

The general case:

SARP ⇒ WARP: SARP is obviously a stronger condition than WARP.

not (WARP ⇒ SARP): See Gale (1960) for a counterexample.

SARP ⇒ GARP: SARP is a stronger condition than GARP. To see this, we can use Varian's (1982) list of equivalent definitions for SARP, specifically version 3: The data satisfy SARP if $x^i R^* x^j$ and $x^i \neq x^j$ implies not $x^j R x^i$. The data satisfy GARP if $x^i R^* x^j$ implies not $x^j P x^i$; note that $x^i P x^j$ implies $x^i R x^j$.

not (GARP ⇒ SARP): GARP allows multi-valued demand, SARP does not.

GARP ⇒ WGARP: GARP is obviously a stronger condition than WGARP.

not (WGARP ⇒ GARP): Suppose $\mathbf{p}^1 = (2,1,1)$, $\mathbf{p}^2 = (1,2,1)$, $\mathbf{p}^3 = (1,1,2)$, $\mathbf{z}^1 = \left(\frac{1}{10}, 0, \frac{8}{10}\right)$, $\mathbf{z}^2 = \left(\frac{8}{10}, \frac{1}{10}, 0\right)$, $\mathbf{z}^3 = \left(0, \frac{8}{10}, \frac{1}{10}\right)$, and $w^1 = w^2 = w^3 = 1$. Then WGARP is satisfied and GARP is not.

HARP ⇒ PHARP: HARP is obviously a stronger condition than PHARP.

not (HARP ⇒ PHARP): See the example for not (WGARP ⇒ GARP).

HARP ⇒ GARP: See proof of Proposition 5.1.

not (GARP ⇒ HARP): See proof of Proposition 5.1.

not (HARP ⇒ WARP): See proof of Proposition 5.2.

not (WARP ⇒ HARP): See proof of Proposition 5.2.

The two-dimensional case:

WARP $\Rightarrow$ SARP: See Rose (1958).

WGARP $\Rightarrow$ GARP: See Banerjee and Murphy (2006).

PHARP $\Rightarrow$ HARP: See Theorem 5.1 in Section 5.2.

■

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COLOPHON

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The custom size of the textblock was calculated using the directions given by Bringhurst: 12 pt Minion needs 144 pt for the string "abcdefghijklmnopqrstuvwxyz". This yields a good line length between 26–28 pc (312–336 pt) according to the copyfitting table in Bringhurst (2005, p. 29). Using a "*golden section textblock*" with a 1:1.62 ratio this results in a textblock of 336:544 pt.

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ERKLÄRUNG

Ich versichere, dass ich diese Dissertation selbständig verfasst habe. Bei der Erstellung der Arbeit habe ich mich ausschließlich der angegebenen Hilfsmittel bedient. Die Dissertation ist nicht bereits Gegenstand eines erfolgreich abgeschlossenen Promotions- oder sonstigen Prüfungsverfahrens gewesen.

Dortmund,

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