On the analysis of unreplicated factorial designs

Thesis

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Ying Chen

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Referees: Professor Dr. Joachim Kunert

Professor Dr. Claus Weihs

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To

my motherland: *the People's Republic of China*

my parents: *Zuxie Chen* (father) and *Meizhen Lou* (mother)

and

my wife: *Qiwen Wu*

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CHAPTER 1

Introduction

1.1 Formulation of the problem in the analysis of unreplicated fractional factorial designs

Since the 1980's, loss of markets to Japan has caused attention of the U.S.A. and Western Europe to return to the enormous potential that experimental design possesses for the improvement of product design, for the improvement of manufacturing processes, and hence for improvement of overall product quality. In the initial stage of developing an industrial process and improving a product design or a manufacturing process, experimental studies based on factorial designs are often used to determine which factors among a number of possibilities can affect the process. As factorial designs require a number of runs that grows exponentially with the number of factors to be analyzed, the replicated fully factorial designs are not applicable when the experiment is expensive and the number of factors is large.

To decrease the number of runs, often unreplicated fractional factorial designs are used. These designs and other orthogonal arrays have proven useful in a screening to isolate preponderant factors. Because experimenters always consider as many factors as possible in a screening experiment, unreplicated fractional designs usually are saturated. That is the number m of factors considered is equal to $n - 1$, where n is the number of experimental runs.

Thus, a problem in the analysis of the designs arises: In full factorial designs, or in high-resolution designs, the higher order interactions can be supposed to be not active, and the squared mean of their estimates can be used to estimate the error variance. However, in saturated designs, although we can estimate all n effects (including the overall mean) with n observations, there are no degrees of freedom left to estimate the error variance. Consequently, we can no longer use standard ANOVA (F-tests or t-tests) to identify the active effects. Hence, the analysis of unreplicated fractional factorial designs presents a challenge.

As pointed out by Claudio Benski (Hamada & Balakrishnan (1998), comment): "Economical and technical reasons have contributed considerable appeal to the ever-increasing use of unreplicated experimental designs in industrial settings. It is therefore important to determine what kind of approach should an experimenter adopt to assess the statistical significance of the considered factors in the absence of an independent noise estimate. The consequences of a mistake on this decision-making process can be enormous and industry would certainly be very interested in a clear answer in this area. Obviously, this is still a very open and active research field."

1.2 Solutions for the analysis of unreplicated fractional factorial designs

Fortunately, in the screening stage of industrial experimentation it is frequently true that "Pareto Principle" applies; i.e., a large proportion of process variation is associated with a small proportion of the process variables (Box and Meyer (1986)). Under such an assumption of effect sparsity, some methods for the analysis of unreplicated fractional factorial designs were proposed.

The first acceptable solution for the analysis of unreplicated designs is the normal or half-normal probability plot proposed by Daniel (1959). His method consists of drawing in normal or half-normal probability paper the estimates of the effects: on the graph, the estimates corresponding to inactive columns (the majority) form an approximately straight line and the significant effects appear at a distance as outliers in a regression line. Figure 1.1 shows the half-normal plot of the example II given by Box and Meyer (1986), where the data are from Taguchi and Wu (1980). In this example, there are 15 effects, from which 2 effects are declared as active.

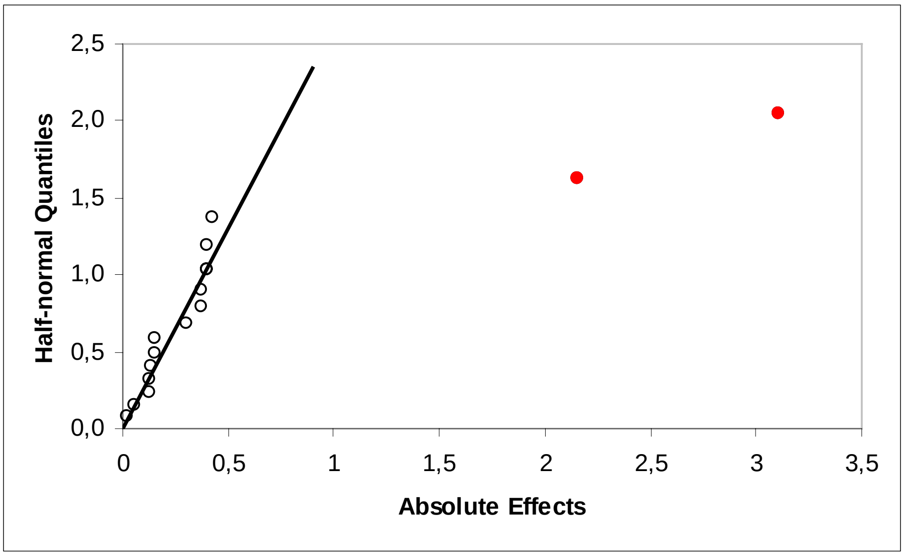


Figure 1.1: Half-normal plot (data from Taguchi and Wu (1980))

The main disadvantage of graphical methods is that their interpretation is subjective. Even when all effects are noise, the plotted points, due to randomness, will not lie perfectly on a straight line. An idea of the extent of non-linearity to be expected may be obtained by looking at the forty pages of plots of pure noise given in Daniel (1976, pp. 84-123). Frequently, only experienced analysts can judge whether an apparent deviation from the linearity is significant or not. Hence, there is a problem of non-uniqueness of interpretation for a half-normal plot.

For example, Figure 1.2 presents the half-normal plot of the example given by Box (1988). The data are from the work of Quinlan (1985), in which 15 factors are considered. From this half-normal plot, two different conclusions might be obtained. For the first one, the "error line" (the dashed line) would be determined from all points, thus only the largest two absolute contrasts would be regarded as falling off the line. For the second one, we can imagine drawing the straight line through the smallest 7 absolute contrasts only (the solid line). Then the largest 7 or 8 absolute contrasts would be considered as active. Figure 1.3 shows the half-normal plot of an artificial experiment given by Ye *et al*. (2001). In the artificial example, 7 out of 15 effects are active. Since there is no large difference between the absolute effects, using a half-normal plot might not detect any active effects.

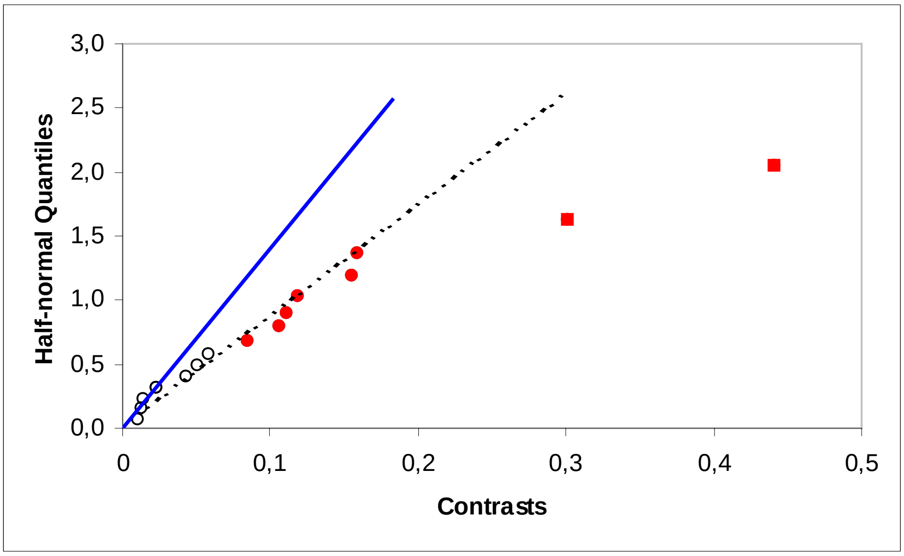


Figure 1.2: Half-normal plot (Data from Quinlan (1985), analysis based on the average of ln(y) (see Box (1988)).)

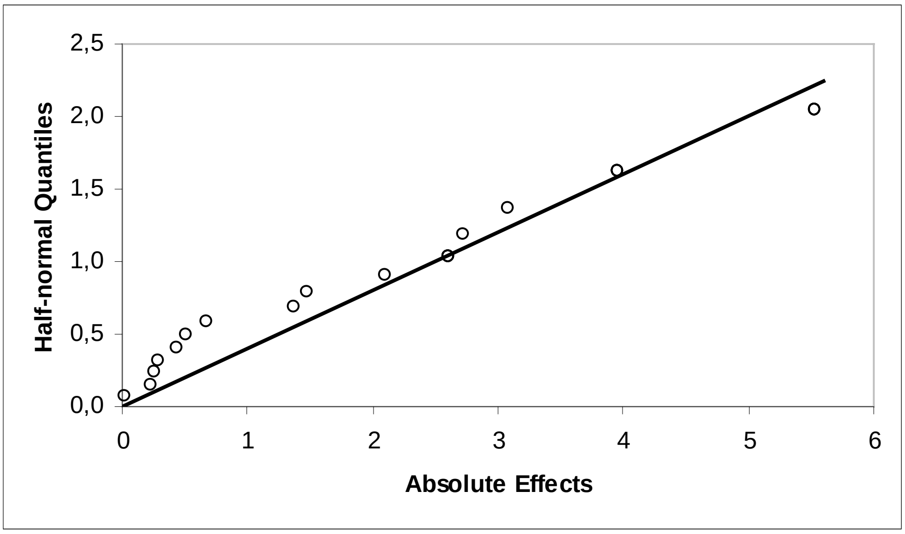


Figure 1.3: Half-normal plot, artificial example of Ye, Hamada and Wu (2001).

In order to eliminate the subjectivity of the graphical methods for detecting significant effects, a large number of formal procedures has been proposed, see e.g. Birnbaum (1959, 1961), Holms and Berrettoni (1969), Zahn (1975b), Seheult and Tukey (1982), Box and Meyer (1986), Johnson and Tukey (1987), Voss (1988), Benski (1989), Lenth (1989), Bissell (1989, 1992), Le and Zamar (1992), Juan and Pena (1992), Loh (1992), Dong (1993), Schneider *et al.* (1993), Venter and Steel (1996), Loughin and Noble (1997), Hamada and Balakrishnan (1998), Lawson *et al.* (1998), Al-Shiha and Yang (1999), Voss and Wang (1999), Schoen and Kaul (2000), Aboukalam and Al-Shiha (2001), Ye *et al.* (2001).

1.3 Statistical model for the analysis of unreplicated fractional factorial designs

For the analysis of unreplicated fractional factorial designs, generally the following statistical model is used:

$$\boldsymbol{y} = \sum_{i=0}^{m} \boldsymbol{x}_i \beta_i + \boldsymbol{\varepsilon}, \tag{1.1}$$

where $\boldsymbol{y} = (y_1, \ldots, y_n)'$ is the vector of observations and n is the number of runs of the experiment. $\beta_0, \beta_1, \ldots, \beta_m$ are unknown parameters, the vectors $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ are known and determined by the design of the experiment, $\boldsymbol{x}_0 = \mathbf{1}_n$ is the n dimensional vector of ones, and $m = n–1$. Finally, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)'$ is the vector of errors. We assume

a) ε_i, $i = 1, \ldots, n$ are independent random variables with expectation zero.

b) ε_i have common variance σ^2.

Further, we might assume

c) $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)' \sim N(\mathbf{0}, \sigma^2 I_n)$,

d) There are at most r ($1 \le r < m$) active effects, i.e. at most r of $\beta_1, \ldots, \beta_m$ differ from zero.

In the analysis, the observation vector $\boldsymbol{y} = (y_1, \ldots, y_n)'$ is used to decide which of the $\beta_1, \ldots, \beta_m$ are nonzero, if any. It is clear that this is a multiple decision problem. Hence, some authors propose a multiple testing procedure that tries to control the multiple level, or experiment-wise error rate.

It has been argued that this is not appropriate for factorial experiments. Missing an important factor can be just as bad as erroneously including an inactive factor. Since we have the opportunity of making a confirmation experiment to check our predictions, the standard for accepting a factor as active does not have to be as stringent as in e.g. medical experiments. However, from our experience it is also important to see whether there is any active factor at all. It may happen that an experiment went wrong and produced basically only random numbers. We hence propose a two stage procedure: in a first step we test the global hypothesis

$$H_0 : \beta_i = 0, \quad 1 \le i \le m, \qquad (1.2)$$

with a level α test. If this test rejects, then we use a less stringent method to decide which of the β_i are nonzero. That is, we decide for one of the hypotheses

$$H_{i_1 \ldots i_k} : \beta_{i_1} \ne 0, \ldots, \beta_{i_k} \ne 0, \text{ while } \beta_j = 0, \text{ for } j \ne i_1, \ldots, i_k,$$

where k, $1 \le k < m$, is an unknown integer and must be estimated.

It is well known that in the orthogonal case under conditions a) and b),

$$\hat{\beta}_i = \boldsymbol{x}_i' \boldsymbol{y} / (\boldsymbol{x}_i' \boldsymbol{x}_i), 0 \le i \le m, \qquad (1.3)$$

is the best linear unbiased estimate (BLUE) for β_i and under conditions a) – c) it is normally distributed with expectation β_i and variance

$$\tau^2 = \sigma^2 / (\boldsymbol{x}_i' \boldsymbol{x}_i), \qquad (1.4)$$

in which $\tau^2 = \sigma^2 / n$ whenever the design is a two-level fractional design. Additionally, $\hat{\beta}_0, \hat{\beta}_1, \ldots, \hat{\beta}_m$ are independent random variables. For all $i > 0$, the estimate $\hat{\beta}_i$ is a contrast.

1.4 What is studied in this thesis

This thesis mainly deals with the problem how to analyse unreplicated fractional factorial designs using quantitative methods.

The first two formal tests for the identification of active contrasts in unreplicated factorial designs were proposed by Birnbaum (1959, 1961), the former of which can identify only one active contrast and was expanded by Zahn (1975a); the later of which is based on optimal invariant multi-decision principle and has not been developed beyond a maximum of two active contrasts.

The idea of Birnbaum (1959) and Zahn (1975a) to solve this problem is to get an estimate of τ (or σ) in (1.4), and then use this estimate as the denominator of a test-statistic. As said before, the difficulty of using standard ANOVA (*F*-tests or *t*-tests) to test the significance of contrasts in unreplicated fractional factorial designs consists in getting an independent estimate of τ. After obtaining an estimate $\hat{\tau}$ of τ, we can use the following test statistic $t_\tau = |\hat{\beta}_i| / \hat{\tau}$ to test the significance of contrasts. The critical region of the test is then $t_\tau > t_{df;1-\alpha}$.

One of the approaches to the estimation of τ is to consider the smallest contrasts as errors to estimate τ (Wilk *et al.* (1963)) if we know that there is at least one inactive contrast. Under the assumption of effect sparsity, we might use more contrasts as errors to estimate τ to improve the efficiency of the estimates (Voss (1988), Berk and Picard (1991), Haaland and O’Connell (1995), etc.). Another approach to the estimation of τ is the robust estimates based on order statistics of the absolute contrasts. For example, Daniel (1959) and Birnbaum (1959) used the 0.683 quantile of the absolute contrasts to estimate τ; Zahn (1975a) used the slope of the regression line through the origin on Daniel's half-normal plot; Lenth (1989) used the pseudo standard error (PSE); Dong (1993) used the adaptive standard error (ASE); Juan and Pena (1992) used the iterative version of Lenth's (1989) PSE. A third approach to estimation of τ is the *M*-estimate that downweights the largest contrasts (Le and Zamar (1992), Aboukalam and Al-Shiha (2001)). Additionally, Wilk *et al.* (1963), Schneider *et al.* (1993) used MLE to estimate τ.

These direct methods based on $\hat{\tau}$ (or $\hat{\sigma}$) have become very important in practice, especially Lenth's (1989) estimate PSE is widely applied. In such a situation, however, the statistic t_τ^2 is no longer *F*-distributed because the estimates obtained this way are no longer independent estimates. The critical values can then be obtained as quantiles of the empirical distribution function or approximated by critical values from the usual *t*-distribution (see, e.g., Lenth (1989), Dong (1993)).

There are also other kinds of quantitative methods to identify active contrasts in unreplicated fractional factorial designs, e.g., outlier-detection techniques (Seheult and Tukey (1982), Benski (1989), Le and Zamar (1992), Loh (1992)), hybrid procedures (Benski (1989), Loh (1992), Lawson *et al*. (1998), Modified Loh's (Hamada and Balakrishnan (1998))), Bayesian methods (Box and Meyer (1986)) and non-parametric method (Loughin and Nobel (1997)).

In the screening stage, the assumption of effect sparsity is frequently true, but not always (Hurley (1995)). If there are too many active contrasts (more than 50%, say), most of such directed methods based on $\hat{\tau}$ will over-estimate τ. This will result in misidentifying some active contrasts as inactive. Most of the methods based on outlier-detection techniques and hybrid procedures consisting of outlier-detection procedures do not allow too many active contrasts, too. Since missing an important factor will almost certainly mean failure in optimizing the process, it is very important to find all important factors in the screening stage. Therefore, in the screening stage we should be careful when using methods that can work correctly only under the assumption of effect sparsity. However, most of the existing methods are based on the assumption of effect sparsity.

Haaland and O'Connell (1995) showed that the power of PSE-, ASE- and TSE (trimmed standard error)- based tests are sensitive to the choice of the two tuning constants, one of which is the quantile determining which of the contrasts is used to estimate s_0 in the initial step, and the other of which is the multiplier of s_0 used to trim large contrasts before calculating the final scale estimate. The tuning constants should be differently chosen for different numbers of really active contrasts to improve the power. Such an improvement of their power, similarly to Bayesian methods, is at the cost of the complexity of the computation. In practice, the methods which can be simply calculated are always popular, however.

Consequently, it is then natural to require a test method used in the screening stage to have the following properties: 1) The method can still work correctly without the assumption of effect sparsity; 2) Its power can be increased when the information about the number of active contrasts given is utilized; 3) It can be easily computed.

In this thesis, we propose a new quantitative method which satisfies the above three conditions. Firstly, the new method does not use an estimate of the error variance and does not need the assumption of effect sparsity; secondly, it can utilize the prior information about the range of the number of really active contrasts to increase its power; thirdly its computation is much simpler than that of Bayesian methods or permutation methods.

This new method, called $MaxU_r$, is described in chapter 3. Some of the statistical properties of the corresponding method are discussed with the help of several theorems in the same chapter. The exact distribution of $MaxU_r$ under H_0 is analytically given. The power function is also analytically given under the assumption of normality. Simulated critical values of $MaxU_r$ are presented. With the help of the software *Mathematica*, the exact quantiles and power of the test $MaxU_r$ are given in the special case when there are only three contrasts (e.g., in the 2^2 design). Unfortunately, more realistic sizes of experiments are numerically not tractable. Finally, four examples are given to illustrate the use of the new method $MaxU_r$.

In order to examine the whole performance of a new method, it is necessary to compare it with other existing methods. It is clear that we can not know much about its performance if we compare them only using a few data sets obtained from some experiments because of the randomness of samples. It is more suitable to compare their probabilities of detecting active contrasts when there are indeed active contrasts. To fairly compare them, the type I error of all compared test methods should be controlled at the same level. Since it is very difficult for most test methods to obtain their exact distribution and power functions, such comparisons are often done by means of computer simulation studies.

The usual procedure of a simulation study can be briefly described as follows. Firstly, the experiment-wise error rate (EER) or the individual error rate (IER) of each of the compared methods has to be set at the same level α when there are no active contrasts at all such that the comparison is fair. Then, their power of detecting individual active contrasts when there are some active contrasts has to be compared. In the power comparison, the normal distribution of experimental errors is usually used. The following parameters are some of those that should be considered in the comparison:

1. *The number of experimental runs*. For 2-level factorials, 8-, 16-, 32- and 64-run experiments are often used, especially 16-run experiment is mostly used.

2. *The level of EER or IER*. The level 0.03 of EER was used by Dong (1993). The level 0.05 of EER was used by Le and Zamar (1992), Benski and Cabau (1995), Haaland and O'Connell(1995), etc.; the level 0.10 of EER was used by Le and Zamar (1992), Al-Shiha and Yang (1999), Aboukalam and Al-Shiha (2001), Ye *et al.* (2001), etc.; the level 0.20 of EER was used by Loughin and Noble (1997). The level 0.05 of IER was used by Loughin and Noble (1997), 0.044 of IER by Hamada and Balakrishnan(1998), etc..

In substance, the motive of controlling IER, instead of EER, at about 0.05 is the same as that of controlling EER at 0.20. It is at the cost of increasing the probability of type I error to improve the ability of detecting active contrasts when some of the contrasts are active. In other words, we choose such a large EER (say 0.20) that the methods have a decent chance of detecting active contrasts when there are active contrasts. In fact, the EER of most methods will be greater than 0.20 when IER is set around 0.05 (see, e.g., Table 1 of Hamada and Balakrishnan (1998)).

3. *The distribution of experimental errors (or contrasts)*: Normal distribution of errors was most commonly used (Zahn (1975b), Voss (1988), Berk and Picard (1991), Loh (1992), Dong (1993), Benski and Caban (1995), Haaland and O'Connell(1995), Loughin and Noble (1997), Hamada and Balakrishnan (1998), Lawson *et al.* (1998), Al-Shiha and Yang (1999), Ye *et al.* (2001), etc.). Besides, student's *t*-distribution was used by Loughin and Noble (1997), Hamada and Balakrishnan (1998), Aboukalam and Al-Shiha (2001). Exponential distribution was used by Loughin and Noble (1997). Lenth (Hamada and Balakrishnan (1998), comment) recommended using a skewed distribution of errors. Aboukalam and Al-Shiha (2001) used the mixture distribution $0.9N(0,1)$ + $0.1N(2,1)$ of contrasts as a skewed model.

4. *The active effects*: There are two approaches in the treatment of active effects. The first approach is to consider active effects as nonzero constants: Some authors used the same magnitude for all active effects (Haaland and O'Connell (1995), Loughin and Noble (1997), Hamada and Balakrishnan (1998), Al-Shiha and Yang (1999), Aboukalam and Al-Shiha(2001), Ye *et al.* (2001), etc.) while some authors used different magnitudes for active effects (Zahn (1975b), Voss (1988), Berk and Picard (1991), Dong (1993), Haaland and O'Connell (1995), Ye *et al.* (2001), etc.). The second approach is to consider active effects as random variables: Benski and Cabau (1995) used scaled shifted distribution $N(0,k)$ for active effects. Le and Zamar (1992) considered active effects which were generated from an uniform distribution on the interval (4,10) with signs randomly assigned.

5. *The evaluation standards*: Power was the mostly used criterion (e.g., Le and Zamar (1992), Dong (1993), Haaland and O'Connell (1995), Loughin and Nobel (1997), Hamada and Balakrishnan (1998), Aboukalam and Al-Shiha (2001), Ye *et al*. (2001)), EER, IER were used by Loughin and Nobel (1997), Hamada and Balakrishnan (1998), and Ye *et al*. (2001), etc.. Lenth (Hamada and Balakrishnan (1998), comment) proposed "decent-chance detection capability" (DCDC(α)) as a criterion. Benski and Cabau (1995) considered *Merit Q* which contained the correct rate and the error rate of detecting active effects at the same time. In addition, Aboukalam and Al-Shiha (2001) used the probability that an effect is considered inactive given that it is really an inactive effect (pow_{II}) as a criterion. In fact, pow_{II} is nothing than the compliment (1 – IER), however.

We carry out a simulation study to compare the new method proposed in chapter 3 with 12 other existing methods to examine the whole performance of the new method. All of the thirteen compared methods have the potential of detecting at least up to six active contrasts out of fifteen. The procedure of our simulation is reported in chapter 4.

Like most authors did, we use 16-run 2-level-factorial experiments and standard normal distribution of errors for our simulation study. We tune EER of all compared methods at 0.05 when there are no active contrasts at all. We use EER instead of IER because, as said before, using large EER or using IER to improve the power of detecting active contrasts is reasonable only when there is no other alternative. In addition, the performance of a test method may vary depending on the level of EER — not only the power but also the range of the number of active contrasts which it can declare as active may be changed if its EER is controlled at 0.05 instead of 0.20. Moreover, it may be more suitable to control EER at a small level when there are no active contrasts at all than strictly to control IER when there are indeed some active effects. That means we would rather declare some inactive effects as active than misjudge any active effects as inactive when there are indeed some active effects. We think more active effects can be detected that way.

In the treatment of active effects, we also consider the active effects as nonzero constants. As said earlier, some authors used active effects with the same magnitude, and some authors used the active effects with different magnitudes. However, only a few authors used both cases for active effects in their comparison. In addition, although a few authors considered both cases for the active effects in their simulation studies, they did not report the changes of the power of the compared methods between the two different cases for the active effects when the number of active effects is fixed. The performance of the compared methods may depend on whether the active effects have the same magnitude or not. Hence, not only the cases where all active effects have the same magnitude but also the cases where active effects have different magnitudes are studied here, and the performances of the methods are compared with respect to the two different cases for active effects. Maybe the sizes of active effects of different magnitudes should be selected as a random sample from a distribution (see, e.g., Benski and Cabau (1995), Haaland and O'Connell (1995)).

Among reported simulation studies, the power was most often used as a criterion. The **power** is defined by the expected fraction of active effects that are declared active. Additional to the usual power, we also use the following four versions of power to evaluate the compared methods — **power I**, **II**, **III** and **IV**. **Power I** is defined to measure the ability of a method to reject the global null hypothesis (no active effects at all). **Power II** is defined to measure the ability of detecting all active effects. **Power III** is defined to measure the ability of exactly detecting all active effects. **Power IV** is defined to measure the ability of detecting all larger active effects except for the smallest active effects. For more details see chapter 4. It is worth noting that the **power III** is just the case 1 in Benski and Cabau (1995), in addition, their case 1 + case 2 is equivalent to the **power II**.

One of the disadvantages of the power criterion is that it only considers the ability of a test method to detect the active effects, but does not measure the error rate of a method to misjudge inactive effects as active. Hence, some authors used IER as an additional criterion. (see, Hamada and Balakrishnan (1998), Aboukalam and Al-Shiha (2001), Ye *et al*. (2001), etc.). But there are still difficulties when two criteria are separately used at the same time. It often happens that IER increases when the power increases; some methods have the highest power, but they also have the largest IER; some methods have smaller IER, though they have lower power. It is difficult to judge which methods are better if the power and IER are separately used at the same time.

The *Merit Q* proposed by Benski and Cabau (1995) seems to be the only known criterion that considers not only the correct rate of a method in identifying active effects but also at the same time the error rate of a method in misjudging inactive effects as active. However, *Merit Q* gives the same measurement when an active effect is declared as inactive as when an inactive effect is declared as active. That might be not suitable in the screening stage. The main objective of a screening experiment is to identify among many factors a few significant factors for further studies. Misidentification of inactive effects as being active may not be a serious mistake, as long as all important factors have been identified. But failure to find an important factor might be fatal for our aim to optimize the process (see also the comment by Haaland in Hamada & Balakrishnan (1998)). Hence, the above two cases should be weighted differently.

One of the possible approaches that satisfy the above conditions is to use the loss of decision (LD). We could choose a reasonable loss function to measure various cases. In our simulation study, we use three versions of the LD, namely **LD2L**, **LD1L9** and **LD1L0** which are derived from different weight functions.

CHAPTER 2

Overview over Existing Methods

Many quantitative methods have been proposed in recent years to test the significance of the effects in unreplicated fractional factorial designs: e.g., Birnbaum (1959, 1961), Holms and Berrettoni (1969), Zahn (1975a), Seheult and Tukey (1982), Box and Meyer (1986), Johnson and Tukey (1987), Voss (1988), Benski (1989), Bissell (1989), Lenth (1989), Stephenson *et al*. (1989), Berk and Picard (1991), Bissell (1992), Juan and Pena (1992), Le and Zamar (1992), Loh (1992), Box and Meyer (1993), Dong (1993), Schneider *et al*. (1993), Benski and Caban (1995), Haaland and O'Connell (1995), Venter and Steel (1996), Kunert (1997), Loughin and Nobel (1997), Hamada and Balakrishnan (1998), Lawson *et al*. (1998) , Al-Shiha and Yang (1999), Voss (1999), Kinateder *et al*. (2000), Schoen and Kaul (2000), Aboukalam and Al-Shiha (2001), Ye *et al*. (2001).

Among the above methods, most were proposed for orthogonal designs. Only three of them are for the non-orthogonal case: Box and Meyer (1993), Kunert (1997) and Kinateder *et al*. (2000). In this chapter, we will only review some of the existing methods for the orthogonal case.

2.1 Orthogonal case

For the orthogonal case, there have been many quantitative methods for analysing unreplicated fractional factorial designs since Birnbaum (1959), e.g., Birnbaum (1961), Holms and Berrettoni (1969), Zahn (1975a), Seheult and Tukey (1982), Box and Meyer (1986), Johnson and Tukey (1987), Voss (1988), Benski (1989), Bissell (1989), Lenth (1989), Stephenson *et al*. (1989), Berk and Picard (1991), Bissell (1992), Juan and Pena (1992), Le and Zamar (1992), Loh (1992), Dong (1993), Schneider *et al*. (1993), Benski and Caban (1995), Haaland and O'Connell (1995), Venter and Steel (1996), Loughin and Nobel (1997), Hamada and Balakrishnan (1998), Lawson *et al*. (1998), Al-Shiha and Yang (1999), Voss (1999), Voss and Wang (1999), Schoen and Kaul (2000), Aboukalam and Al-Shiha (2001), Ye *et al*. (2001). In addition, there are some graphical methods, e.g., Daniel (1959), which can also be used as quantitative method. Probably, there will be some other methods that we have not mentioned above.

In what follows, we will review the 12 quantitative methods for analyzing unreplicated designs in detail, which were given in Seheult and Tukey (1982), Box and Meyer (1986), Johnson and Tukey (1987), Benski (1989), Bissell (1989), Lenth (1989), Juan and Pena (1992), Le and Zamar (1992), Dong (1993), Loughin and Nobel (1997), Lawson, Grimshaw and Burt (1998) and Al-Shiha and Yang (1999). These methods will be studied in chapter 4 with the help of computer simulations to compare them with the new method $MaxU_r$.

Seheult and Tukey (1982)

Seheult and Tukey (1982) used an outlier procedure, which they called *threshold analysis*, based on the quartiles of a synthetic batch of contrasts, namely zero plus all the contrasts with both signs giving a total of $2m+1$ items. Because the synthetic batch is symmetric, the hinge spread is twice the size of the hinges that are the median of the unsigned contrasts plus zero. They recommended that the contrasts are considered active when their absolute value is more than one-and-half hinge spreads outside the corresponding hinge (Tukey (1977), Section 2D), i.e., more than $1\times \text{hinge} + 1.5\times \text{hinge spread}$ = $(1+1.5\times 2)\times \text{hinge}$ = $4\times$ hinge = $4\times \text{med}_0$, where med_0 is the median of the unsigned contrasts plus zero. Their testing procedure can be described as follows:

1) Calculate

$$\text{med}_0 = \underset{1\le i\le m+1-s}{\text{median}}\left(0, |\hat{\beta}_i|\right), \qquad (2.1.1)$$

beginning with s = 1.

2) If $\max\limits_{1\le i\le m+1-s}\left(|\hat{\beta}_i|\right) \Big/ \underset{1\le i\le m+1-s}{\text{median}}\left(0,|\hat{\beta}_i|\right) > C_{m+1-s} = 4$, remove the largest absolute contrast and re-index the remaining contrasts, then add s to one and repeat steps 1) and 2).

3) If $\max\limits_{1\le i\le m+1-s}\left(|\hat{\beta}_i|\right) \Big/ \underset{1\le i\le m+1-s}{\text{median}}\left(0,|\hat{\beta}_i|\right) \le C_{m+1-s} = 4$, stop and the contrasts removed in step 2) are considered active.

Box and Meyer (1986)

Box and Meyer (1986) suggested a Bayesian method based on the assumption of effect sparsity, i.e. there is a small proportion of active effects α_{active}. They assumed that the inactive effects β_i are zero and the active effects β_i have a $N\left(0, \tau^2_{active}\right)$ distribution. Thus, contrasts $\hat{\beta}_i$ = e_i corresponding to inactive effects have distribution $N\left(0, \tau^2\right)$; contrasts $\hat{\beta}_i = \beta_i + e_i$ corresponding to active effects have distribution $N\left(0, \delta^2\tau^2\right)$, where $\delta^2 = \left(\tau^2 + \tau^2_{active}\right)\big/\tau^2$; and $\hat{\beta}_1,\ldots,\hat{\beta}_m$ are i.i.d. from the scale-contaminated normal distribution $(1-\alpha_{active})N(0,\tau^2)$ + $\alpha_{active}N(0,\delta^2\tau^2)$.

They first computed the posterior probability of each of the 2^m possible events (i.e. an effect is active or not) $a_{j(r)}, j = 1,\ldots,2^m$, which is the event that a particular set of r (r=0,1,...,m) of the m effects is active:

$$p\left(a_{j(r)}\middle|\hat{\beta},\alpha_{active},\delta\right) \propto \left[\frac{\alpha_{active}\cdot\delta^{-1}}{1-\alpha_{active}}\right]^r \left[1-\varphi\cdot f_{(r)}\right]^{-m/2}, \qquad (2.1.2)$$

where $\varphi = 1 - 1\big/\delta^2 = \tau^2_{active}\big/(\tau^2+\tau^2_{active})$, $f_{(r)} = \hat{\boldsymbol{\beta}}'_{j(r)}\hat{\boldsymbol{\beta}}_{j(r)}\big/\hat{\boldsymbol{\beta}}'\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{\beta}}_{j(r)}$ is the vector of contrasts corresponding to active effects of $a_{j(r)}$ and $\hat{\boldsymbol{\beta}} = (\hat{\beta}_1,\ldots,\hat{\beta}_m)'$.

Then, they computed the marginal posterior probability p_i that an effect i is active given $\hat{\boldsymbol{\beta}}$, α_{active} and δ:

$$p_i = \sum_{a_{j(r)}:\text{effect } i \text{ active}} p\left(a_{j(r)} \middle| \hat{\boldsymbol{\beta}}, \alpha_{active}, \delta\right), \qquad (2.1.3)$$

which is the sum of the posterior probabilities $p\left(a_{j(r)} \middle| \hat{\boldsymbol{\beta}}, \alpha_{active}, \delta\right)$ of all the events in which effect i is active. They recommended that the effects whose marginal posterior probability p_i exceeds 0.5 are declared active.

They also recommended calculating the following integral

$$p_i = \int_0^\infty p_{i|\tau} \cdot p\left(\tau \mid \hat{\boldsymbol{\beta}}\right) \cdot d\tau \qquad (2.1.4)$$

through numerical integration to compute p_i for i =1,...,m rather than summing over 2^m combinations, in which

$$p_{i|\tau} = \frac{\frac{\alpha_{active}}{\delta} \exp\left(\frac{-\hat{\beta}_i^2}{2\delta^2\tau^2}\right)}{\frac{\alpha_{active}}{\delta} \exp\left(\frac{-\hat{\beta}_i^2}{2\delta^2\tau^2}\right) + (1-\alpha_{active}) \exp\left(\frac{-\hat{\beta}_i^2}{2\tau^2}\right)} \qquad (2.1.5)$$

and

$$p\left(\tau \mid \hat{\boldsymbol{\beta}}\right) \propto \tau^{-n} \prod_{j=1}^{m} \left[(1-\alpha_{active}) \exp\left(\frac{-\hat{\beta}_j^2}{2\tau^2}\right) + \frac{\alpha_{active}}{\delta} \exp\left(\frac{-\hat{\beta}_j^2}{2\delta^2\tau^2}\right) \right]. \qquad (2.1.6)$$

To determine the values of α_{active} and δ, Box and Meyer (1986) studied ten published data sets of unreplicated fractional factorials. They showed that the estimated values of α_{active} range between 0.13 and 0.27 with an average of about 0.20; and estimated values of δ vary widely between 2.7 and 18 with an average of about 9.6. Based on the results, they recommended 0.2 and 10 for α_{active} and δ, respectively.

Johnson and Tukey (1987)

Johnson and Tukey (1987) suggested a procedure based on display ratios which are the unsigned contrasts divided by their respective typical order statistics; i.e.

$$\textit{display ratio} = \frac{|\hat{\beta}|_{(i)}}{|z|_i}, \qquad (2.1.7)$$

where $|\hat{\beta}|_{(i)}$ is the i-th order statistic of the absolute contrasts, $|z|_i$ is the median of the i-th half-normal order statistic in a sample of size m. $|z|_i$ can be approximated by

$|z|_i = \Phi^{-1}\left(\frac{3i+3m}{6m+2}\right)$, in which $\Phi^{-1}(\cdot)$ is the inverse normal distribution function. Each of the *m* *display ratio*s provides an estimate of τ. Then, they defined the *ratio-to-scale* statistics, which are computed as:

$$\textit{ratio-to-scale} = \frac{\textit{display ratio}}{\textit{median display ratio}}. \quad (2.1.8)$$

Their test statistic is the ratio-to-scale of the largest size of contrasts, i.e.

$$R_{JT87} = \frac{\textit{display ratio}\text{ (for largest - size contrast)}}{\textit{median display ratio}\text{ (for all contrasts)}}. \quad (2.1.9)$$

The critical values $R_{\alpha,m}$ for R_{JT87} given in their Table 12 (Johnson and Tukey (1987), pp. 201) were obtained through 2048 simulations and some approximate formulas for the critical values were proposed as well. Their testing procedure can be written as follows:

1) Calculate *display ratio*s using (2.1.7), beginning with all contrasts.
2) Calculate the test statistic R_{JT87} using (2.1.9).
3) If $R_{JT87} > R_{\alpha,m}$, remove the largest absolute contrast and repeat steps 1)-3) with the remaining contrasts.
4) If $R_{JT87} \le R_{\alpha,m}$, stop, the contrasts removed in step 3) are considered active.

Benski (1989)

Benski (1989) suggested the use of the Olsson's version of the Shapiro-Wilk *W* test for normality to test the presence of active effects. An outlier test was used in parallel with the normality test to enhance the power of the normality testing procedure. To compound the significance levels of both tests, Fisher's procedure was used. His testing procedure is as follows:

1) Calculate Olsson's version W' of the Shapiro-Wilk statistic

$$W' = \frac{\left(\sum_{i=1}^{m} z_i \hat{\beta}_{(i)}\right)^2}{\sum_{i=1}^{m} z_i^2 \cdot \sum_{i=1}^{m} (\hat{\beta}_{(i)} - \bar{\hat{\beta}})^2}, \quad (2.1.10)$$

where $\bar{\hat{\beta}}$ is the average of the ordered contrasts $\hat{\beta}_{(i)}$ and z_i, $i = 1,\ldots,m$ are expected standard normal order statistics in a sample of size *m* and can be approximated using the inverse normal distribution, that is

$$z_i = \Phi^{-1}(p_i), \quad (2.1.11)$$

where $\Phi^{-1}(\cdot)$ is the inverse normal distribution, $p_i = (i-a)/(m-2a+1)$ and

$$a = \begin{cases} 0.275499 + 0.072884 \cdot (\ln(m))^{0.41148}, & 1 < i < m \\ 0.205146 + 0.1314965 \cdot (\ln(m))^{0.226701}, & i = 1 \text{ or } i = m \end{cases}. \quad (2.1.12)$$

2) Obtain the significance level P_1 of the W' test.

$$P_1 = \exp(C), \text{ for } P_1 > 0.005, \quad (2.1.13)$$

where

$$C = \frac{(W' - A)/B + 0.0486128}{0.02760309} - \ln(100), \quad (2.1.14)$$

$$A = 1.031918 - 0.183573 \cdot (0.1m)^{-0.5447402},$$

$$B = -0.5084706 + 2.076782 \cdot (0.1m)^{-0.4905993}.$$

3) If P_1 is small, calculate the significance level P_2 of the outlier test, which is the significance level of any data point outside of the interval $[-2d_F, +2d_F,]$, where $d_F = F_U - F_L$ is the inter-quartile range and F_L and F_U are the first and third quartiles of the contrasts $\hat{\beta}_i$, respectively. P_2 can be estimated using the approximate formula:

$$P_2 \approx 2\Phi\left[4\Phi^{-1}\left(\frac{1}{4}\right)\right] + \frac{0.4}{m} = 0.00698 + \frac{0.4}{m}, \quad (2.1.15)$$

which is obtained under normality and is only a function of sample size. If P_1 is not small, go to step 5).

4) Calculate the combined significance level P_c, which can be obtained using the fact that $-2 \cdot \ln(P_1 P_2)$ has approximately a chi-square distribution with 4 degrees of freedom. If the combined test is rejected, declare the contrast that is the largest in absolute value active; then, remove it and repeat steps 1) - 4) with the remaining contrasts.

5) Stop. The contrasts removed in step 4) are considered active.

Bissell (1989)

Bissell (1989) suggested using Bartlett's (1937) test for variance homogeneity to identify the presence of active effects. For 2-level designs, his test statistic can be written by

$$Bi = \ln\left(\frac{1}{k}\sum_{i=1}^{k} \hat{\beta}_i^2\right) - \frac{1}{k}\sum_{i=1}^{k} \ln(\hat{\beta}_i^2), \quad (2.1.17)$$

where k is the number of the contrasts considered.

Bissel (1989) suggested using Bi sequentially for which the critical value at the i-th stage is based on the remaining $k = m - i + 1$ effects being inactive; the critical value is based on an appropriate F distribution. His testing procedure can be written as follows:

1) Calculate *Bi*, beginning with all contrasts, i.e., $k = m$.
2) If $Bi > C_{\alpha,k}$, which is the critical value based on *k* effects being inactive, remove the largest absolute contrast and repeat steps 1) and 2) with the remaining contrasts until $Bi \leq C_{\alpha,k}$.
3) If $Bi \leq C_{\alpha,k}$, stop, the contrasts removed in step 2) are considered active.

Lenth (1989)

Lenth (1989) considered a robust estimator of the contrast standard error τ, which he called the pseudo standard error estimate or PSE:

$$\text{PSE} = 1.5 \cdot \underset{|\hat{\beta}_j|<2.5s_0}{\text{median}}\left(|\hat{\beta}_i|\right), \tag{2.1.18}$$

where

$$s_0 = 1.5 \cdot \underset{i}{\text{median}}\left(|\hat{\beta}_i|\right). \tag{2.1.19}$$

Then, he defined a margin of error for $\hat{\beta}_i$ with approximately 95% confidence

$$\text{ME} = t_{0.975;d} \times \text{PSE}, \tag{2.1.20}$$

where $t_{0.975;d}$ is the 0.975 quantile of a *t*-distribution on $d = m/3$ degrees of freedom; and defined also a simultaneous margin of error

$$\text{SME} = t_{\gamma;d} \times \text{PSE}, \tag{2.1.21}$$

where $\gamma = (1 + 0.95^{1/m})/2$. He recommended that the contrasts whose absolute values are greater than SME are clearly active, those whose absolute values are not greater than ME cannot be considered active, and those whose absolute values are between SME and ME are in a zone of uncertainty where a good argument can be made both for being active and for being happenstance of inactive contrasts.

Juan and Pena (1992)

Juan and Pena (1992) suggested a different estimator IMAD_0 for τ. It is similar to Lenth's (1989) PSE except that the calculation is iterative. Their study showed that the estimator based on the interquartile range d_F behaves poorly and IMAD_0 has better MSE than PSE when more than 25% of the effects are active. It also showed that using the trimmed median is generally better than the trimmed mean when more than 20% of the effects are active. Their testing procedure can be written as follows:

1) Compute IMAD_0 using the following iterative procedure:

   (1) Compute MAD_0, beginning with all contrasts

$$\text{MAD}_0 = \underset{1\le i\le m}{\text{median}}\left(|\hat{\beta}_i|\right). \tag{2.1.22}$$

(2) Take those values $\hat{\beta}_i$ that satisfy

$$|\hat{\beta}_i| \le w\,\text{MAD}_0, \tag{2.1.23}$$

where w is a previously determinated constant and must be greater than 2.

(3) With those values $\hat{\beta}_i$ recompute MAD_0. If the MAD_0 stops changing, the last MAD_0 is the IMAD_0; otherwise, repeat steps (2)-(3).

2) Identify active contrasts: If

$$|\hat{\beta}_i| \big/ \text{IMAD}_0 \ge w_c, \tag{2.1.24}$$

the contrast $\hat{\beta}_i$ is considered active. If w = 3.5 is chosen, they recommended w_c=4, 4.4 and 4.8 for the 8-, 16- and 32-run designs at level 0.05, respectively.

Le and Zamar (1992)

Le and Zamar (1992) suggested using an outlier test based on the ratio of two estimates of scale, a non-robust estimate divided by a robust one. Their test statistic is

$$R_{LZ92} = \frac{S_2}{S_1}, \tag{2.1.25}$$

where S_1 and S_2 are two M-estimates of τ that are the solutions of the equation

$$\frac{1}{m}\sum_{i=1}^{m}\rho\left(\frac{\hat{\beta}_i - T}{S}\right) = E\left[\rho(Z)\right], \tag{2.1.26}$$

for the even functions $\rho=\rho_1$ and $\rho=\rho_2$, respectively; T is a robust estimate of location and Z has a standard normal distribution.

They used the following ρ-functions

$$\rho_1(x) = \begin{cases} x^2 & \text{if } |x| < c \\ c^2 & \text{otherwise} \end{cases} \tag{2.1.27}$$

and

$$\rho_2(x) = \rho_1(x) + b(x^4 - 6x^2), \tag{2.1.28}$$

where c = 0.9, b= 0.11 and T is the median of the sample when the location parameter is not specified. Their testing procedure is as follows:

1) Calculate S_1
2) Calculate S_2
3) Calculate R_{LZ92}. If $R_{LZ92} > R_{\alpha,k}$, remove the largest absolute contrast and repeat steps 1)-3) with the remaining contrasts; otherwise, go to step 4).
4) Stop, the removed contrasts in step 3) are considered active.

The critical values $R_{\alpha,k}$ are given in their Table 1.

Dong (1993)

Similar to Lenth (1989), Dong (1993) considered an estimator for τ, the adaptive standard error (ASE) based on the trimmed mean of squared contrasts rather than the trimmed median of the unsigned contrasts:

$$\text{ASE} = \sqrt{\frac{1}{m_{inactive}} \sum_{|\hat{\beta}_i| \le 2.5 s_0} \hat{\beta}_i^2}\,, \tag{2.1.29}$$

where $m_{inactive}$ is the number of inactive contrasts declared by $|\hat{\beta}_i| \le 2.5 s_0$ and s_0 is defined earlier in (2.1.19). He used

$$|\hat{\beta}_i| > t_{\gamma, m_{inactive}} \cdot \text{ASE} \tag{2.1.30}$$

to test whether a contrast $\hat{\beta}_i$ is active or not, where $\gamma = (1 + 0.98^{1/m})/2$. Dong (1993) also suggested iteratively calculating ASE until it stops changing when there is a large number of active effects.

Loughin and Nobel (1997)

Loughin and Nobel (1997) suggested a non-parametric method, which is a permutation test based on the Birnbaum's (1961) test statistic. Their testing procedure is as follows:

1) Compute $\hat{\boldsymbol{\beta}}$ from $\boldsymbol{y}$, then order the contrasts $\hat{\beta}_1, \hat{\beta}_2, \ldots \hat{\beta}_m$ and the columns $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots \boldsymbol{x}_m$ of the design matrix $\boldsymbol{X}$ to correspond to the ordered absolute contrasts $|\hat{\beta}|_{(1)} \le |\hat{\beta}|_{(2)} \le \ldots \le |\hat{\beta}|_{(m)}$. Without loss of generality, we assume $|\hat{\beta}_i| = |\hat{\beta}|_{(i)}$, $i = 1,2,\ldots,m$.
2) At step s, set $\hat{W}_s = |\hat{\beta}|_{(m-s+1)}$ and obtain $\tilde{\boldsymbol{y}}_s = \tilde{\boldsymbol{y}}_{s-1} - \hat{\beta}_{m-s+2} \cdot \boldsymbol{x}_{m-s+2}$ with $\tilde{\boldsymbol{y}}_1 = \boldsymbol{y}$, $s = 1,2,\ldots,m$.
3) Select a large number, N (e.g., N = 5000), and repeat N times:

   (1) Obtain $\tilde{\boldsymbol{y}}_s^*$ through a random permutation of $\tilde{\boldsymbol{y}}_s$.

   (2) Compute $\tilde{\boldsymbol{\beta}}_s^*$ from $\tilde{\boldsymbol{y}}_s^*$, i.e.,

$$\begin{pmatrix} \tilde{\beta}_0^* \\ \tilde{\boldsymbol{\beta}}_s^* \end{pmatrix} = (X'X)^{-1} X' \tilde{\boldsymbol{y}}_s^*.$$

   (3) Obtain the largest absolute contrast $|\tilde{\beta}^*|_{(m)}$ from $\tilde{\boldsymbol{\beta}}_s^*$.

   (4) Compute

$$W_s^* = \left(\frac{m}{m+1-s}\right)^{\frac{1}{2}} |\tilde{\beta}^*|_{(m)}. \tag{2.1.31}$$

4) Compute the observed significance level (OSL) P_s for the test as

$$P_s = 1 - \left(\frac{\# W_s^* < \hat{W}_s}{N} \right)^{\frac{m+1-s}{m}} . \qquad (2.1.32)$$

5) Repeat steps 2) - 4) for as many contrasts as desired.

Let P_k^* be the smallest P_s. If $P_k^* < P_{\alpha,m}$, then the k contrasts with the highest magnitudes are declared active, regardless of their respective values of P_s. The critical values $P_{\alpha,m}$ are given in their Table 1.

Lawson, Grimshaw and Burt (1998)

Lawson, Grimshaw and Burt (1998) suggested a hybrid method based on the half-normal plot, which is a blend of Lenth's (1989) and Loh's (1992) methods. The method consists of fitting a simple least-squares line and prediction limits to the half-normal probability plot. Their procedure can be described as follows:

1) Fit to the model

$$|\hat{\beta}|_{(i)} = \hat{b}_1 \cdot |z|_i , \; i = 1, \ldots, m \qquad (2.1.33)$$

with all contrasts, where $|z|_i$ are the half-normal-order statistics approximated by

$$|z|_i = \Phi^{-1}\left(\frac{p_i + 1}{2} \right), \qquad (2.1.34)$$

in which $p_i = (i - 0.5)/m$ or $p_i = (i - 3/8)/(m + 5/8)$, $\Phi^{-1}(\cdot)$ is the inverse normal distribution and $\hat{b}_1$ is the estimated slope.

2) Fit to the model

$$|\hat{\beta}|_{(j)} = \hat{b}_2 \cdot |z|_j , \; j = 1, \ldots, m_{inactive} \qquad (2.1.35)$$

with the data $\left(|\hat{\beta}|_{(j)}, |z|_j \right)$, $j = 1, \ldots, m_{inactive}$, that consists of the remaining data after removing the datum pairs including $|\hat{\beta}_{i_j}|$ from the datum set $\left(|\hat{\beta}|_{(i)}, |z|_i \right)$, $i = 1, \ldots, m$, where $\hat{\beta}_{i_j}$ is declared active by $|\hat{\beta}_{i_j}| \geq 2.5 \cdot s_0$ (Lenth (1989)) and $\hat{b}_2$ is the estimated slope.

3) Calculate the statistic

$$R_{LGB98} = \frac{\hat{b}_1}{\hat{b}_2}. \qquad (2.1.36)$$

If $R_{LGB98} > R_{\alpha,m}$, it is assumed that there are active contrasts and then go to step 4); otherwise no active contrasts. Here, $R_{\alpha,m}$ is the critical value (see Lawson *et al.*'s (1998) Table 1).

4) Construct Scheffé prediction bands around the least-squares regression fit to the model (2.1.35) in step 2) to identify the active contrasts:

(1) Calculate Scheffé prediction bands of the least-squares line with slope $\hat{b}_2$ and non-intercept (see, e.g., Seber (1977), p.186).

$$B = \left\{(x,y):\ \left|y - \hat{b}_2 x\right| \le (m' \cdot F_{m',m_{inactive}-1;0.95})^{\frac{1}{2}} \cdot S_2 (1+w)^{\frac{1}{2}}\right\}, \qquad (2.1.37)$$

where $S_2^2 = \dfrac{1}{m_{inactive}-1}\sum_{j=1}^{m_{inactive}}\left(|\hat{\beta}|_{(j)} - \hat{b}_2 |z|_j\right)^2$ is the residual mean square, m' is the nearest integer to $m/4$ (Loh (1992)), $F_{m',m_{inactive}-1;0.95}$ is the 95% quantile of the F-distribution with m' degrees of freedom for numerator and $m_{inactive}$-1 degrees of freedom for denominator, and $w = x^2 \big/ \sum_{j=1}^{m_{inactive}} |z|_j^{\,2}$.

(2) Calculate the largest absolute contrast D that is within B.

(3) Calculate

$$C = (U\text{-}L)/2, \qquad (2.1.38)$$

where $U = Q_3 + 1.5 \cdot (Q_3 - Q_1)$, $L = Q_1 - 1.5 \cdot (Q_3 - Q_1)$, Q_1 and Q_3 are the first and third quartiles of the set of all contrasts (Loh (1992)). In fact, $C = 2 \cdot (Q_3 - Q_1)$.

(4) Calculate

$$C^* = \max(C, D). \qquad (2.1.39)$$

If $|\hat{\beta}_i| > C^*$, $\hat{\beta}_i$ is identified as active.

Al-Shiha and Yang (1999)

Al-Shiha and Yang (1999) suggested a multistage procedure based on the generalized likelihood ratio test statistic. For testing H_0: $\beta_1 = \beta_2 = \ldots = \beta_m = 0$ versus the alternative H_a*: exactly k of the* β_i*'s are not equal to zero*, the generalized likelihood ratio test statistic is of the form

$$R_{AY99} = \min_A (R_A), \qquad (2.1.40)$$

where $R_A = \left(1 + \dfrac{\sum_{\beta_i \in A} \hat{\beta}_i^2}{\sum_{\beta_i \notin A} \hat{\beta}_i^2}\right)^{-\frac{m}{2}}$ and A is a given subset of k distinct elements of $\beta_1, \beta_2, \ldots, \beta_m$.

Because of the fact that

$$R_{AY99} = \left(1 + \frac{k}{m-k} L_{m,k}\right)^{-\frac{m}{2}}, \qquad (2.1.41)$$

where

$$L_{m,k} = \frac{\sum_{i=m-k+1}^{m} |\hat{\beta}|_{(i)}^{2} / k}{\sum_{i=1}^{m-k} |\hat{\beta}|_{(i)}^{2} / (m-k)}, \tag{2.1.42}$$

the test based on R_{AY99} is equivalent to the one based on $L_{m,k}$. Al-Shiha and Yang (1999) recommended to reject H_0 and accept H_a when $L_{m,k}$ is very large. Some simulated critical values of $L_{m,k}$ are given in their Table 2.1 (Al-Shiha and Yang (1999)) and a more extensive table is available in Al-Shiha and Yang (2000). Their multistage procedure to identify active contrasts is as follows:

1) Select *k*, which is equal to or greater than the exact number of really active contrasts *r*.
2) Use $L_{m,k}$ to test H_0: *no active contrasts* vs. H_a :*exactly k active contrasts*.
3) If H_0 is rejected, then remove the largest absolute contrast. Reduce both *m* and *k* by 1 and repeat step 2) with the remaining contrasts.
4) If H_0 is not rejected then stop. The contrasts removed earlier cycles of step 3) are considered active.

CHAPTER 3

New Method

In this chapter, a new method for analyzing unreplicated fractional factorial designs will be suggested. The new method does not use an estimate of the error variance and has the potential to identify up to m – 1 active contrasts, where m is the number of contrasts in the study.

Among existing methods, many methods use the effect-sparsity assumption in order to have enough inactive contrasts to estimate the error variance σ^2, then using the estimate of the variance to test the significance of active contrasts. However, the effect-sparsity assumption is not always true. The proportion of active effects is sometimes greater than 20% and often as high as 40% or more (see, e.g., Hurley (1995), Haaland (Hamada and Balakrishnan (1998), comment)). When the number of active contrasts is too large, not all contrasts used to estimate the error variance are inert so that the estimate of the error variance is too large. In this case, these methods fail to test the significance of active contrasts.

Actually, we have two purposes in analyzing unreplicated factorial designs: one is to test the significance of active contrasts, and the other is to estimate the error variance σ^2. The directed methods attempt firstly to estimate σ (or τ), then to test the active contrasts using the estimate of σ (or τ). Most of such methods are unfeasible when there are too many active effects (more than 40%, say).

An opposing approach of the directed method is that we firstly test the active effects without using the estimate of σ (or τ), then estimate the error variance σ^2. The test proposed in section 3.1 is one of the methods that does not use the estimate of σ.

3.1 Test statistic $MaxU_r$

Assume that we have an experiment in n runs, and the observations $\boldsymbol{y} = (y_1, \ldots, y_n)'$ follow model (1.1) with conditions a) – d).

As a first step, assume we knew exactly which, if any, of $\hat{\beta}_1, \ldots, \hat{\beta}_m$ have nonzero expectation. Without loss of generality, assume $\beta_1 = \ldots = \beta_{m-k} = 0$ and $\beta_i \neq 0, m-k < i \leq m$. For testing the hypotheses

$$H_0 : \beta_1 = \ldots = \beta_m = 0 \text{ vs. } H_1 : \beta_1 = \ldots = \beta_{m-k} = 0, \beta_i \neq 0,\ m-k < i \leq m, \qquad (3.1.1)$$

we could use the F-statistic $\frac{1}{k}\sum_{i=m-k+1}^{m}(\hat{\beta}_i)^2 \Big/ \frac{1}{m-k}\sum_{j=1}^{m-k}(\hat{\beta}_j)^2$. We then would reject H_0 when the F-statistic is greater than the critical value of the F-distribution with k and $m-k$ degrees of freedom.

As a second step, we assume we know that exactly k, $1 \le k < m$, of the contrasts $\hat{\beta}_1,\ldots,\hat{\beta}_m$ are active, but we don't know which. In this case, we choose any k possibly active contrasts from $\hat{\beta}_1,\ldots,\hat{\beta}_m$. Let $\hat{\beta}_{i_1},\ldots,\hat{\beta}_{i_k}$ denote the chosen contrasts (the number of such choices is $\binom{m}{k}$). Then we calculate the statistics

$$v_k^{\{i_1,\ldots,i_k\}} = \frac{\frac{1}{k}\sum_{j=1}^{k}(\hat{\beta}_{i_j})^2}{\frac{1}{m-k}\sum_{j\notin\{i_1,\ldots,i_k\}}(\hat{\beta}_j)^2}, \tag{3.1.2}$$

where $\{i_1,\ldots,i_k\}$ denote a subset of the set $\{1,2,\ldots,m\}$. It is well known that under the global hypothesis H_0, all $v_k^{\{i_1,\ldots,i_k\}}$ have the same F-distribution with k and m-k degrees of freedom. In addition, if it is true that $\hat{\beta}_{j_1},\ldots,\hat{\beta}_{j_k}$ are active and the other contrasts are inactive, $v_k^{\{j_1,\ldots,j_k\}}$ should have the largest value among all of the $\binom{m}{k}$ possible statistics $v_k^{\{i_1,\ldots,i_k\}}$. That is, $v_k^{\{j_1,\ldots,j_k\}}$ $= \max_{\{i_1,\ldots,i_k\}\in P_k}\left(v_k^{\{i_1,\ldots,i_k\}}\right)$, where P_k is the set of all subsets of $\{1,2,\ldots,m\}$ that have exactly k elements. Hence, under the information in the second step, it is appropriate to use $\max_{\{i_1,\ldots,i_k\}\in P_k}\left(v_k^{\{i_1,\ldots,i_k\}}\right)$ to test H_0. We reject H_0 when the statistic $\max_{\{i_1,\ldots,i_k\}\in P_k}\left(v_k^{\{i_1,\ldots,i_k\}}\right)$ is greater than an appropriate critical value.

Finally, we get to the situation that we are really interested in and assume that we only know that there are at most r of the $\hat{\beta}_1,\ldots,\hat{\beta}_m$ being active.

Assume we can find transformation functions $g_k(\cdot)$, $1 \le k \le r$, such that all $g_k\left(v_k^{\{i_1,\ldots,i_k\}}\right)$, ($1 \le k \le r$, $\{i_1,\ldots,i_k\}\in P_k$) have an identical distribution under H_0 and that $g_{k^*}\left(v_{k^*}^{\{j_1,\ldots,j_{k^*}\}}\right)$ has the largest value when $\hat{\beta}_{j_1},\ldots,\hat{\beta}_{j_{k^*}}$ are active and the other contrasts are inactive. Then, similar to the last step, we could use $\max_{1\le k\le r}\left(\max_{\{i_1,\ldots,i_k\}\in P_k}\left(g_k\left(v_k^{\{i_1,\ldots,i_k\}}\right)\right)\right)$ to test H_0. We reject H_0 if $\max_{1\le k\le r}\left(\max_{\{i_1,\ldots,i_k\}\in P_k}\left(g_k\left(v_k^{\{i_1,\ldots,i_k\}}\right)\right)\right)$ is larger than an appropriate critical value.

One such transformation is the p-value. Let $g_k(\cdot) = F_{k,m-k}(\cdot)$, $1 \le k < m$, where $F_{k,m-k}(\cdot)$ is the distribution function of the F-distribution with k and m-k degrees of freedom. Then all statistics

$$u_k^{\{i_1,\ldots,i_k\}} = F_{k,m-k}(v_k^{\{i_1,\ldots,i_k\}}) \tag{3.1.3}$$

have the same uniform distribution $U[0,1]$ under the null hypothesis H_0. If, however, $\hat{\beta}_{j_1},\ldots,\hat{\beta}_{j_{k^*}}$ are active and the other contrasts are inactive, then the corresponding $u_{k^*}^{\{j_1,\ldots,j_{k^*}\}}$ should have the largest value among all $\sum_{k=1}^{r}\binom{m}{k}$ possible values $u_k^{\{i_1,\ldots,i_k\}}$.

According to the above idea, we therefore propose the following test statistic $MaxU_r$ for the analysis of unreplicated fractional factorial designs. The test statistic is defined by

$$MaxU_r = \max_{1\le k\le r}(MU_k)\,, \tag{3.1.4}$$

where

$$MU_k = \max_{\{i_1,\ldots,i_k\}\in P_k}(u_k^{\{i_1,\ldots,i_k\}})\,. \tag{3.1.5}$$

The procedure of using $MaxU_r$ to test the hypotheses (1.2) is proposed as follows:

Firstly, select an integer r which is not smaller than the number of truly active contrasts. When we do not have any information about the number of truly active contrasts, select

$$r = m - 1.$$

Then, calculate $MaxU_r$. If $MaxU_r > c_{\alpha;m,r}$, we reject H_0 and conclude that there are active contrasts. Here $c_{\alpha;m,r}$ is a critical value of $MaxU_r$, which satisfies

$$P\{MaxU_r > c_{\alpha;m,r}\} = \alpha \ \ (0<\alpha<1).$$

Some values $c_{\alpha;m,r}$ were derived by a Monte Carlo simulation and are given in appendix A3.

If H_0 is rejected, the value k^*, which satisfies the equation $MU_{k^*} = \max_{1\le k\le r}(MU_k)$, is the estimate of the number of active contrasts, and the corresponding contrasts $\hat{\beta}_{j_1},\ldots,\hat{\beta}_{j_{k^*}}$, which satisfy

$$MU_{k^*} = u_{k^*}^{\{j_1,\ldots,j_{k^*}\}} = F_{k^*,m-k^*}\left(\frac{\frac{1}{k^*}\sum_{i=1}^{k^*}(\hat{\beta}_{j_i})^2}{\frac{1}{m-k^*}\sum\limits_{i\ne j_1,\ldots,j_{k^*}}(\hat{\beta}_i)^2}\right),$$

are considered as being active.

Furthermore, it can be proven that

$$MaxU_r = \max_{1\le k\le r}\left(F_{k,m-k}\left(\frac{\frac{1}{k}\sum_{i=m-k+1}^{m}\left|\hat{\beta}\right|_{(i)}^2}{\frac{1}{m-k}\sum_{i=1}^{m-k}\left|\hat{\beta}\right|_{(i)}^2}\right)\right), \tag{3.1.6}$$

where $\left|\hat{\beta}\right|_{(1)},\ldots,\left|\hat{\beta}\right|_{(m)}$ are the order statistics of the absolute contrasts $\left|\hat{\beta}_1\right|,\ldots,\left|\hat{\beta}_m\right|$.

Theorem 3.1 *The statistic* $MaxU_r$ *defined by (3.1.2) – (3.1.5) is equivalent to the one defined by (3.1.6).*

Proof: First we prove

$$MU_k = F_{k,m-k}\left(L_{m,k}\right), \tag{3.1.7}$$

where

$$L_{m,k} := \frac{\frac{1}{k}\sum_{i=m-k+1}^{m}\left|\hat{\beta}\right|_{(i)}^{2}}{\frac{1}{m-k}\sum_{i=1}^{m-k}\left|\hat{\beta}\right|_{(i)}^{2}},\ k = 1,2,\ldots,r, \tag{3.1.8}$$

and $\left|\hat{\beta}\right|_{(1)},\ldots,\left|\hat{\beta}\right|_{(m)}$ are the order statistics of absolute contrasts $\left|\hat{\beta}_1\right|,\ldots,\left|\hat{\beta}_m\right|$.

For $1 \le k \le r$, using (3.1.2) and (3.1.3), we have

$$MU_k = \max_{\{i_1,\ldots,i_k\}\in P_k}\left(u_k^{\{i_1,\ldots,i_k\}}\right)$$

$$= \max_{\{i_1,\ldots,i_k\}\in P_k}\left(F_{k,m-k}\left(v_k^{\{i_1,\ldots,i_k\}}\right)\right).$$

Because of the increasing monotonicity of functions $F_{k,m-k}(\cdot),\ k = 1,2,\ldots,m-1$,

$$MU_k = F_{k,m-k}\left(\max_{\{i_1,\ldots,i_k\}\in P_k}\left(v_k^{\{i_1,\ldots,i_k\}}\right)\right).$$

Using the definition of $v_k^{\{i_1,\ldots,i_k\}}$, we have

$$MU_k = F_{k,m-k}\left(\max_{\{i_1,\ldots,i_k\}\in P_k}\left(\frac{\frac{1}{k}\sum_{j=1}^{k}(\hat{\beta}_{i_j})^2}{\frac{1}{m-k}\sum_{j\neq i_1,\ldots,i_k}(\hat{\beta}_j)^2}\right)\right)$$

$$= F_{k,m-k}\left(\max_{\{i_1,\ldots,i_k\}\in P_k}\left(\frac{\frac{1}{k}\sum_{j=1}^{k}(\hat{\beta}_{i_j})^2}{\frac{1}{m-k}\left(\sum_{j=1}^{m}(\hat{\beta}_j)^2 - \sum_{j=1}^{k}(\hat{\beta}_{i_j})^2\right)}\right)\right).$$

It is clear that

$$\max_{\{i_1,\ldots,i_k\}\in P_k}\left(\frac{\frac{1}{k}\sum_{j=1}^{k}(\hat{\beta}_{i_j})^2}{\frac{1}{m-k}\left(\sum_{j=1}^{m}(\hat{\beta}_j)^2 - \sum_{j=1}^{k}(\hat{\beta}_{i_j})^2\right)}\right) = \frac{\frac{1}{k}\max_{\{i_1,\ldots,i_k\}\in P_k}\left(\sum_{j=1}^{k}(\hat{\beta}_{i_j})^2\right)}{\frac{1}{m-k}\left(\sum_{j=1}^{m}(\hat{\beta}_j)^2 - \max_{\{i_1,\ldots,i_k\}\in P_k}\left(\sum_{j=1}^{k}(\hat{\beta}_{i_j})^2\right)\right)},$$

and

$$\max_{\{i_1,\ldots,i_k\}\in P_k}\left(\sum_{j=1}^{k}(\hat{\beta}_{i_j})^2\right) = \sum_{i=m-k+1}^{m}\left|\hat{\beta}\right|_{(i)}^{2},$$

where $\left|\hat{\beta}\right|_{(m-k+1)},\ldots,\left|\hat{\beta}\right|_{(m)}$ are the k largest order statistics of absolute contrasts $\left|\hat{\beta}_1\right|,\ldots,\left|\hat{\beta}_m\right|$.

Furthermore,

$$\sum_{j=1}^{m}(\hat{\beta}_j)^2 - \max_{\{i_1,\ldots,i_k\}\in P_k}\left(\sum_{j=1}^{k}(\hat{\beta}_{i_j})^2\right) = \sum_{i=1}^{m-k}\left|\hat{\beta}\right|_{(i)}^2,$$

where $\left|\hat{\beta}\right|_{(1)},\ldots,\left|\hat{\beta}\right|_{(m-k)}$ are the *m-k* smallest order statistics of $\left|\hat{\beta}_1\right|,\ldots,\left|\hat{\beta}_m\right|$.

Hence

$$\max_{\{i_1,\ldots,i_k\}\in P_k}\left(\frac{\frac{1}{k}\sum_{j=1}^{k}(\hat{\beta}_{i_j})^2}{\frac{1}{m-k}\left(\sum_{j=1}^{m}(\hat{\beta}_j)^2 - \sum_{j=1}^{k}(\hat{\beta}_{i_j})^2\right)}\right) = \frac{\frac{1}{k}\sum_{i=m-k+1}^{m}\left|\hat{\beta}\right|_{(i)}^2}{\frac{1}{m-k}\sum_{i=1}^{m-k}\left|\hat{\beta}\right|_{(i)}^2}. \qquad (3.1.9)$$

We define

$$L_{m,k} := \frac{\frac{1}{k}\sum_{i=m-k+1}^{m}\left|\hat{\beta}\right|_{(i)}^2}{\frac{1}{m-k}\sum_{i=1}^{m-k}\left|\hat{\beta}\right|_{(i)}^2},$$

then

$$MU_k = F_{k,m-k}\left(\frac{\frac{1}{k}\sum_{i=m-k+1}^{m}\left|\hat{\beta}\right|_{(i)}^2}{\frac{1}{m-k}\sum_{i=1}^{m-k}\left|\hat{\beta}\right|_{(i)}^2}\right)$$

$$= F_{k,m-k}\left(L_{m,k}\right).$$

The equation (3.1.7) is proved.

Finally using (3.1.4) and (3.1.7), we have

$$MaxU_r = \max_{1\le k\le r}\left(F_{k,m-k}\left(\frac{\frac{1}{k}\sum_{i=m-k+1}^{m}\left|\hat{\beta}\right|_{(i)}^2}{\frac{1}{m-k}\sum_{i=1}^{m-k}\left|\hat{\beta}\right|_{(i)}^2}\right)\right).$$

□

The statistic $L_{m,k}$ defined by (3.1.8) was also used by Al-Shiha and Yang (1999). It is derived from the generalized likelihood ratio test statistic under normality. However, our strategy to identify active contrasts is different from the multistage procedure proposed by Al-Shiha and Yang (1999). It will be shown in chapter 4 that our method is superior to Al-Shiha and Yang's (1999) in many respects.

Note that using (3.1.6) to calculate $MaxU_r$ needs much less computing time than using (3.1.4). For example, it needs $\sum_{k=1}^{r}\binom{m}{k}$ = 22818 times computations of the form

$F_{k,m-k}\left(\frac{\frac{1}{k}\sum_{j=1}^{k}(\hat{\beta}_{i_j})^2}{\frac{1}{m-k}\sum_{j\notin\{i_1,\ldots,i_k\}}(\hat{\beta}_j)^2}\right)$ for m =15 and r =8 using (3.1.4) to calculate $MaxU_r$; but it only needs 8 times computations of the form $F_{k,m-k}\left(\frac{\frac{1}{k}\sum_{i=m-k+1}^{m}\left|\hat{\beta}\right|_{(i)}^2}{\frac{1}{m-k}\sum_{i=1}^{m-k}\left|\hat{\beta}\right|_{(i)}^2}\right)$ and a computation of the order statistics $\left|\hat{\beta}\right|_{(1)},\ldots,\left|\hat{\beta}\right|_{(m)}$ using (3.1.6) to calculate $MaxU_r$. For larger *r*, (3.1.4) needs much more computing time. Hence, in practice we use (3.1.6) to calculate $MaxU_r$ instead of (3.1.4).

3.2 Theoretical derivation of distribution function etc.

In this section, we will show several theorems to get the distribution of $MaxU_r$ and the power function of the test $MaxU_r$ proposed in the last section under the condition that in model (1.1) errors are normally distributed.

In the following section, we are firstly going to develop the distribution of $MaxU_r$.

3.2.1 Distribution of $MaxU_r$

It is well known that a random variable X has a gamma distribution with parameter $\xi > 0$, written as $X \sim Ga(\xi)$, if X has the p.d.f

$$\left(\Gamma(\xi)\right)^{-1} x^{\xi-1} e^{-x}, \text{ for } x > 0. \qquad (3.2.1.1)$$

For $\xi = n/2$ in (3.2.1.1), $Y = 2X$ has a chi-square distribution with n degrees of freedom, i.e. $Y \sim \chi_n^2$.

Definition 3.1[1] *Let $X_1,\ldots X_{m-1}, X_m$ be independent random variables. If $X_i \sim Ga(\xi_i)$ (gamma distribution with parameter $\xi_i > 0$), $i = 1,\ldots,m-1,m$; set*

$$Y_j = X_j \Big/ \sum_{i=1}^{m} X_i, \; j = 1,\ldots,m; \qquad (3.2.1.2)$$

then the distribution of $(Y_1,\ldots,Y_{m-1})$ is called ***Dirichlet distribution*** *with parameters $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_{m-1},\xi_m)'$ and written as $(Y_1,\ldots,Y_{m-1}) \sim D_{m-1}(\boldsymbol{\xi}) = D_{m-1}(\xi_1,\ldots,\xi_{m-1},\xi_m)$ or $(Y_1,\ldots,Y_m) \sim D_m(\boldsymbol{\xi}) = D_m(\xi_1,\ldots,\xi_{m-1},\xi_m)$.*

[1] Fang *et al*. (1990), p17, Definition 1.4

Lemma 3.1 *The probability density function of* $(X_1,\ldots,X_{m-1})' \sim D_{m-1}(\xi_1,\ldots,\xi_{m-1},\xi_m)$ *is given by*

$$p(x_1,x_2,\ldots,x_{m-1}) = \frac{\Gamma(\sum_{i=1}^{m}\xi_i)}{\prod_{i=1}^{m}\Gamma(\xi_i)}\left(1-\sum_{i=1}^{m-1}x_i\right)^{\xi_m-1}\prod_{i=1}^{m-1}x_i^{\xi_i-1}, \qquad (3.2.1.3)$$

where $\sum_{i=1}^{m-1}x_i < 1$ *and* $x_i > 0$, $i = 1,\ldots,m-1$.

Proof: See Fang *et al*. (1990), Chapter 1, Theorem 1.2. □

Theorem 3.2 *Suppose there are no active contrasts, i.e.* $\beta_1,\ldots,\beta_m$ *in model (1.1) are zero and* $\boldsymbol{\varepsilon} \sim N(\mathbf{0},\sigma^2 I_{m+1})$. *Then the probability distribution function of statistic* $MaxU_r$, *which is defined by (3.1.4), is given by*

$$F_{MaxU_r}(x) = \frac{\Gamma(\frac{m}{2})}{\left[\Gamma(\frac{1}{2})\right]^m}\int\cdots\int_{\Omega}\left(1-\sum_{i=1}^{m-1}x_i\right)^{-\frac{1}{2}}\prod_{i=1}^{m-1}x_i^{-\frac{1}{2}}dx_i\,, \; for \; F_{1,m-1}(1) \le x < 1\,, \qquad (3.2.1.4)$$

$$F_{MaxU_r}(x) = 0, \; for \; x < F_{1,m-1}(1)\,,$$

and

$$F_{MaxU_r}(x) = 1, \; for \; x \ge 1\,;$$

where Ω

$$= \left\{(x_1,x_2,\ldots,x_{m-1}) : \textstyle\sum_{i=1}^{m}x_i = 1, x_i > 0,\; i = 1,\ldots,m; \sum_{j=1}^{k}x_{i_j} \le y_k, \{i_1,\ldots,i_k\} \in P_k, k = 1,\ldots,r < m\right\},$$

$$x_m = 1-\sum_{i=1}^{m-1}x_i\,,$$

$$y_k = \frac{k}{m-k}F_{k,m-k}^{-1}(x)\Big/\left(1+\frac{k}{m-k}F_{k,m-k}^{-1}(x)\right)$$

and $F_{k,m-k}^{-1}(\cdot)$ *is the inverse function of* $F_{k,m-k}(\cdot)$.

Proof: ***a)*** If $x \ge 1$, it is clear that the probability distribution function of $MaxU_r$

$$F_{MaxU_r}(x) = P\{MaxU_r \le x\} = 1$$

because the maximal value of $MaxU_r$ is equal to 1.

b) Next, we consider the case of $x < F_{1,m-1}(1)$.

According to the definitions of probability distribution function and the statistic $MaxU_r$, we have

$$F_{MaxU_r}(x) = P\{MaxU_r \le x\}$$

$$= P\left\{\max_{1\le k\le r}(MU_k) \le x\right\}$$

$$= P\{MU_1 \le x, MU_2 \le x, \ldots, MU_r \le x\}.$$

Using the definitions of MU_k and $u_k^{\{i_1,\ldots,i_k\}}, k = 1,\ldots,r$, we have

$$F_{MaxU_r}(x) = P\left\{\max_{\{i_1\}\in P_1}(u_1^{\{i_1\}}) \le x, \max_{\{i_1,i_2\}\in P_2}(u_2^{\{i_1,i_2\}}) \le x, \ldots, \max_{\{i_1,\ldots,i_r\}\in P_r}(u_r^{\{i_1,\ldots,i_r\}}) \le x\right\}$$

$$= P\left\{\max_{\{i_1\}\in P_1}\left(F_{1,m-1}(v_1^{\{i_1\}})\right) \le x, \max_{\{i_1,i_2\}\in P_2}\left(F_{2,m-2}(v_2^{\{i_1,i_2\}})\right) \le x, \ldots, \max_{\{i_1,\ldots,i_r\}\in P_r}\left(F_{r,m-r}(v_r^{\{i_1,\ldots,i_r\}})\right) \le x\right\}.$$

Because of the increasing monotonicity of functions $F_{k,m-k}^{-1}(x), k = 1,\ldots,r$,

$$F_{MaxU_r}(x) = P\left\{\max_{\{i_1\}\in P_1}\left(v_1^{\{i_1\}}\right) \le F_{1,m-1}^{-1}(x), \max_{\{i_1,i_2\}\in P_2}\left(v_2^{\{i_1,i_2\}}\right) \le F_{2,m-2}^{-1}(x), \ldots \right.$$

$$\left. \ldots, \max_{\{i_1,\ldots,i_r\}\in P_r}\left(v_r^{\{i_1,\ldots,i_r\}}\right) \le F_{r,m-r}^{-1}(x)\right\}.$$

Using the definitions of $v_k^{\{i_1,\ldots,i_k\}}, k = 1,\ldots,r$, we have

$$F_{MaxU_r}(x) = P\left\{\max_{\{i_1\}\in P_1}\left(\frac{\hat\beta_{i_1}^2}{\frac{1}{m-1}\sum_{j\ne i_1}\hat\beta_j^2}\right) \le F_{1,m-1}^{-1}(x), \max_{\{i_1,i_2\}\in P_2}\left(\frac{\frac{1}{2}\sum_{j=1}^{2}\hat\beta_{i_j}^2}{\frac{1}{m-2}\sum_{j\ne i_1,i_2}\hat\beta_j^2}\right) \le F_{2,m-2}^{-1}(x), \ldots\right.$$

$$\left. \ldots, \max_{\{i_1,\ldots,i_r\}\in P_r}\left(\frac{\frac{1}{r}\sum_{j=1}^{r}\hat\beta_{i_j}^2}{\frac{1}{m-r}\sum_{j\ne i_1,\ldots,i_r}\hat\beta_j^2}\right) \le F_{r,m-r}^{-1}(x)\right\}$$

$$= P\left\{\max_{\{i_1\}\in P_1}\left(\frac{\hat\beta_{i_1}^2}{\sum_{j\ne i_1}\hat\beta_j^2}\right) \le \frac{1}{m-1}F_{1,m-1}^{-1}(x), \max_{\{i_1,i_2\}\in P_2}\left(\frac{\sum_{j=1}^{2}\hat\beta_{i_j}^2}{\sum_{j\ne i_1,i_2}\hat\beta_j^2}\right) \le \frac{2}{m-2}F_{2,m-2}^{-1}(x), \ldots\right.$$

$$\left. \ldots, \max_{\{i_1,\ldots,i_r\}\in P_r}\left(\frac{\sum_{j=1}^{r}\hat\beta_{i_j}^2}{\sum_{j\ne i_1,\ldots,i_r}\hat\beta_j^2}\right) \le \frac{r}{m-r}F_{r,m-r}^{-1}(x)\right\}.$$

Let $x_k = \dfrac{k}{m-k}F_{k,m-k}^{-1}(x)$, $k = 1,2,\ldots,r$, then

$$F_{MaxU_r}(x) = P\left\{\max_{\{i_1\}\in P_1}\left(\frac{\hat\beta_{i_1}^2}{\sum_{j\ne i_1}\hat\beta_j^2}\right) \le x_1, \max_{\{i_1,i_2\}\in P_2}\left(\frac{\sum_{j=1}^{2}\hat\beta_{i_j}^2}{\sum_{j\ne i_1,i_2}\hat\beta_j^2}\right) \le x_2, \ldots\ldots, \max_{\{i_1,\ldots,i_r\}\in P_r}\left(\frac{\sum_{j=1}^{r}\hat\beta_{i_j}^2}{\sum_{j\ne i_1,\ldots,i_r}\hat\beta_j^2}\right) \le x_r\right\}.$$

Because of the increasing monotonicity of functions $f(x) = \dfrac{x}{1+x}$ and

$$f\left(\frac{\sum_{j=1}^{k}\hat{\beta}_{i_j}^2}{\sum_{j\neq i_1,\ldots,i_k}\hat{\beta}_j^2}\right) = \frac{\sum_{j=1}^{k}\hat{\beta}_{i_j}^2}{\sum_{j=1}^{m}\hat{\beta}_j^2},\ k = 1,\ldots,r,$$

$$F_{MaxU_r}(x) = P\left\{\max_{\{i_1\}\in P_1}\left(\frac{\hat{\beta}_{i_1}^2}{\sum_{j=1}^{m}\hat{\beta}_j^2}\right) \leq \frac{x_1}{1+x_1},\ \max_{\{i_1,i_2\}\in P_2}\left(\frac{\sum_{j=1}^{2}\hat{\beta}_{i_j}^2}{\sum_{j=1}^{m}\hat{\beta}_j^2}\right) \leq \frac{x_2}{1+x_2},\ldots\right.$$

$$\left.\ldots,\max_{\{i_1,\ldots,i_r\}\in P_r}\left(\frac{\sum_{j=1}^{r}\hat{\beta}_{i_j}^2}{\sum_{j=1}^{m}\hat{\beta}_j^2}\right) \leq \frac{x_r}{1+x_r}\right\}.$$

Set

$$X_i = \frac{\hat{\beta}_i^2}{\sum_{j=1}^{m}\hat{\beta}_j^2},\ i = 1,2,\ldots,m, \qquad (3.2.1.5)$$

and let $y_k = \dfrac{x_k}{1+x_k}$, $k = 1,2,\ldots,r$, thus

$$F_{MaxU_r}(x) = P\left\{\max_{\{i_1\}\in P_1}\left(X_{i_1}\right) \leq y_1,\ \max_{\{i_1,i_2\}\in P_2}\left(\sum_{j=1}^{2}X_{i_j}\right) \leq y_2,\ldots,\ \max_{\{i_1,\ldots,i_r\}\in P_r}\left(\sum_{j=1}^{r}X_{i_j}\right) \leq y_r\right\}. \qquad (3.2.1.6)$$

Since $y_k < \dfrac{k}{m}$ if $x < F_{k,m-k}(1)$, and

$$\min_{\sum_{i=1}^{m}X_i=1}\left(\max_{\{i_1,\ldots,i_k\}\in P_k}\left(\sum_{j=1}^{k}X_{i_j}\right)\right) = \frac{k}{m},$$

we have $F_{MaxU_r}(x)$ =0 when $x < F_{1,m-1}(1)$ because of the fact that

$$F_{1,m-1}(1) > F_{2,m-2}(1) > \ldots > F_{r,m-r}(1).$$

c) Let $F_{1,m-1}(1) \leq x < 1$.

Since there are no active contrasts, i.e. $\hat{\beta}_i \overset{i.i.d.}{\sim} N\left(0,\sigma_\beta^2\right)$, $i = 1,2,\ldots,m$, we have

$$\hat{\beta}_i^2 \Big/ (2\sigma_\beta^2) \overset{i.i.d.}{\sim} Ga(\tfrac{1}{2}).$$

By definition 3.1,

$$(X_1,\ldots,X_m)' \sim D_m(\tfrac{1}{2},\ldots,\tfrac{1}{2},\tfrac{1}{2})$$

because

$$X_i = \frac{\hat{\beta}_i^2}{\sum_{j=1}^{m}\hat{\beta}_j^2} = \frac{\hat{\beta}_i^2 \big/ 2\sigma_\beta^2}{\sum_{j=1}^{m}\hat{\beta}_j^2 \big/ 2\sigma_\beta^2}.$$

Finally, it follows from Lemma 3.1 and equation (3.2.1.6) that

$$F_{MaxU_r}(x) = P\{X_1 \le y_1, X_2 \le y_1, \ldots, X_m \le y_1; X_1 + X_2 \le y_2, X_1 + X_3 \le y_2, \ldots$$
$$\ldots, X_{m-1} + X_m \le y_2; \ldots\ldots; X_1 + \ldots + X_r \le y_r, \ldots, X_{m-r+1} + \ldots + X_m \le y_r\}$$

$$= \frac{\Gamma(\frac{m}{2})}{\left[\Gamma(\frac{1}{2})\right]^m} \int \cdots \int_{\Omega} \left(1 - \sum_{i=1}^{m-1} x_i\right)^{-\frac{1}{2}} \prod_{i=1}^{m-1} x_i^{-\frac{1}{2}} dx_i ,$$

where Ω

$$=\left\{(x_1, x_2, \ldots, x_{m-1}) : \sum_{i=1}^{m} x_i = 1, x_i > 0,\ i = 1, \ldots, m; \sum_{j=1}^{k} x_{i_j} \le y_k, \{i_1, \ldots, i_k\} \in P_k, k = 1, \ldots, r < m\right\},$$

$$x_m = 1 - \sum_{i=1}^{m-1} x_i ,$$

$$y_k = \frac{k}{m-k} F_{k,m-k}^{-1}(x) \Bigg/ \left(1 + \frac{k}{m-k} F_{k,m-k}^{-1}(x)\right)$$

and $F_{k,m-k}^{-1}(\cdot)$ is the inverse function of $F_{k,m-k}(\cdot)$. □

3.2.2 Power function of the test $MaxU_r$

Definition 3.2 *Let X be a random variable. If the probability density function of X is given by*

$$p(x; \xi, \lambda) = \sum_{i=0}^{\infty} P_\lambda(i) \cdot p(x; \xi + i), \qquad (3.2.2.1)$$

where $\xi > 0$, $\lambda \ge 0$, $P_\lambda(i) = e^{-\lambda} \lambda^i / i!$ and $p(x; \xi + i)$ is the p.d.f of $Ga(\xi + i)$, then the distribution of X is called ***non-central gamma distribution*** *with parameter $\xi > 0$ and non-centrality parameter $\lambda \ge 0$, and indicated by $X \sim Ga(\xi; \lambda)$.*

So, the non-central gamma distribution is a mixture of central gamma distribution $Ga(\xi + i)$ with Poisson weights $P_\lambda(i)$. When $\lambda = 0$, the non-central gamma distribution $Ga(\xi; 0)$ is the gamma distribution $Ga(\xi)$.

Lemma 3.2 *Set $X \sim Ga(\xi; \lambda)$. If $\xi = n/2$, where n is a positive integer, then $Y = 2X$ has a non-central chi-square distribution with n degrees of freedom and non-centrality parameter λ, i.e. $Y \sim \chi_n^2(\lambda)$.*

Proof: From (3.2.2.1), the p.d.f of Y is given by

$$p_Y(y;\frac{n}{2},\lambda) = \frac{1}{2}p_X(\frac{y}{2};\frac{n}{2},\lambda)$$

$$=\frac{1}{2}\sum_{i=0}^{\infty}P_\lambda(i)\cdot p(\frac{y}{2};\frac{n}{2}+i)$$

$$=\frac{1}{2}\sum_{i=0}^{\infty}P_\lambda(i)\cdot\frac{1}{\Gamma(\frac{n}{2}+i)}\left(\frac{y}{2}\right)^{\frac{n}{2}+i-1}e^{-\frac{1}{2}y}$$

$$=\sum_{i=0}^{\infty}P_\lambda(i)\cdot\frac{1}{2^{\frac{n}{2}+i}\Gamma(\frac{n}{2}+i)}y^{\frac{n}{2}+i-1}e^{-\frac{n}{2}y}\,. \qquad (3.2.2.2)$$

It is well known that this density is the p.d.f of the non-central chi-square distribution $\chi_n^2(\lambda)$.

□

Definition 3.3 *Let $X_1,\ldots X_{m-1},X_m$ be independent random variables. If $X_i \sim Ga(\xi_i;\lambda_i)$ with $\xi_i>0$ and $\lambda_i \ge 0$, $i=1,\ldots,m-1,m$, let $Y_1,\ldots,Y_m$ be defined by (3.2.1.2). Then the distribution of $(Y_1,\ldots,Y_{m-1})'$ is called* ***non-central Dirichlet distribution*** *with parameters $\boldsymbol{\xi}$ = $(\xi_1,\ldots,\xi_{m-1},\xi_m)'$ and non-centrality parameters $\boldsymbol{\lambda}$ = $(\lambda_1,\ldots,\lambda_{m-1},\lambda_m)'$, and written as $(Y_1,\ldots,Y_{m-1})' \sim D_{m-1}(\boldsymbol{\xi};\boldsymbol{\lambda})$ = $D_{m-1}(\xi_1,\ldots,\xi_{m-1},\xi_m;\lambda_1,\ldots,\lambda_{m-1},\lambda_m)$ or $(Y_1,\ldots,Y_m)' \sim D_m(\boldsymbol{\xi};\boldsymbol{\lambda})$ = $D_m(\xi_1,\ldots,\xi_{m-1},\xi_m;\lambda_1,\ldots,\lambda_{m-1},\lambda_m)$.*

Lemma 3.3 *The probability density function of $(Y_1,\ldots,Y_{m-1})' \sim D_{m-1}(\boldsymbol{\xi};\boldsymbol{\lambda})$ is given by*

$$p(\boldsymbol{y};\boldsymbol{\xi},\boldsymbol{\lambda}) = p(y_1,y_2,\ldots,y_{m-1};\xi_1,\ldots,\xi_{m-1},\xi_m;\lambda_1,\ldots,\lambda_{m-1},\lambda_m)$$

$$=\sum_{j_1=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\prod_{i=1}^{m}P_{\lambda_i}(j_i)\cdot p(\boldsymbol{y};\xi_1+j_1,\ldots,\xi_{m-1}+j_{m-1},\xi_m+j_m),$$

$$\text{for } \sum\nolimits_{i=1}^{m-1}y_i<1,\ y_i>0,\ i=1,\ldots,m-1, \qquad (3.2.2.3)$$

where $P_{\lambda_i}(j_i) = e^{-\lambda_i}\lambda_i^{j_i}/j_i!$ and $p(\boldsymbol{y};\xi_1+j_1,\ldots,\xi_{m-1}+j_{m-1},\xi_m+j_m)$ is the p.d.f of $D_{m-1}(\xi_1+j_1,\ldots,\xi_{m-1}+j_{m-1},\xi_m+j_m)$.

Proof: Assume that $X_1,\ldots X_{m-1},X_m$ are independent and $X_i \sim Ga(\xi_i;\lambda_i)$, $i=1,\ldots,m-1,m$. From (3.2.2.1), the joint density function of $X_1,\ldots X_{m-1},X_m$ is

$$\prod_{i=1}^{m}p(x_i;\xi_i,\lambda_i)=\prod_{i=1}^{m}\sum_{j_i=0}^{\infty}\left[P_{\lambda_i}(j_i)p(x_i;\xi_i+j_i)\right]=\sum_{j_1=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\prod_{i=1}^{m}\left[P_{\lambda_i}(j_i)\cdot p(x_i;\xi_i+j_i)\right],$$

where $P_{\lambda_i}(j_i) = e^{-\lambda_i}\lambda_i^{j_i}/j_i!$ and $p(x_i;\xi_i+j_i)$ is the p.d.f of $Ga(\xi_i+j_i)$.

Set

$$\begin{cases} Y_i = X_i \Big/ \sum_{j=1}^{m} X_j \,, & i = 1,\dots,m. \\ Z = \sum_{j=1}^{m} X_j \end{cases} \qquad (3.2.2.4)$$

Then the Jacobian of transformation (3.2.2.4) is Z^{m-1}. Therefore, the joint density function of $Y_1,\dots Y_{m-1}, Z$ is

$$z^{m-1}\sum_{j_1=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\prod_{i=1}^{m}\left[P_{\lambda_i}(j_i)\cdot p(z\cdot y_i;\xi_i+j_i)\right]$$

$$= z^{m-1}\sum_{j_1=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\prod_{i=1}^{m}\left[P_{\lambda_i}(j_i)\cdot\frac{1}{\Gamma(\xi_i+j_i)}\left(z\cdot y_i\right)^{\xi_i+j_i-1}e^{-z\cdot y_i}\right]$$

$$= z^{m-1}\sum_{j_1=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\prod_{i=1}^{m}\left[P_{\lambda_i}(j_i)\cdot\frac{y_i^{\xi_i+j_i-1}}{\Gamma(\xi_i+j_i)}\right]z^{\sum_{i=1}^{m}(\xi_i+j_i)-m}e^{-z\cdot\sum_{i=1}^{m}y_i}$$

$$= \sum_{j_1=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\prod_{i=1}^{m}\left[P_{\lambda_i}(j_i)\cdot\frac{y_i^{\xi_i+j_i-1}}{\Gamma(\xi_i+j_i)}\right]z^{\sum_{i=1}^{m}(\xi_i+j_i)-1}e^{-z}$$

$$= \sum_{j_1=0}^{\infty}\cdots\sum_{j_m=0}^{\infty}\prod_{i=1}^{m}P_{\lambda_i}(j_i)\left[\frac{\Gamma\left(\sum_{i=1}^{m}(\xi_i+j_i)\right)}{\prod_{i=1}^{m}\Gamma(\xi_i+j_i)}\prod_{i=1}^{m}y_i^{\xi_i+j_i-1}\right]\frac{z^{\sum_{i=1}^{m}(\xi_i+j_i)-1}e^{-z}}{\Gamma\left(\sum_{i=1}^{m}(\xi_i+j_i)\right)}.$$

Since $z^{\sum_{i=1}^{m}(\xi_i+j_i)-1}e^{-z}\Big/\Gamma\left(\sum_{i=1}^{m}(\xi_i+j_i)\right)$ is just the density function of $Ga\left(\sum_{i=1}^{m}(\xi_i+j_i)\right)$ and $\dfrac{\Gamma\left(\sum_{i=1}^{m}(\xi_i+j_i)\right)}{\prod_{i=1}^{m}\Gamma(\xi_i+j_i)}\prod_{i=1}^{m}y_i^{\xi_i+j_i-1}$ is the density function of $D_{m-1}(\xi_1+j_1,\dots,\xi_{m-1}+j_{m-1},\xi_m+j_m)$, the proof is completed by integrating out the density of z. □

Theorem 3.3 *For the test problem (1.2), the rejection region of the test (3.1.4) is* $\left\{MaxU_r > c_{m,r}\right\}$*, which is equivalent to*

$$\bigcup_{k=1}^{r}\ \bigcup_{\{i_1,\dots,i_k\}\in P_k} B^{\{i_1,\dots i_k\}}\,, \qquad (3.2.2.5)$$

where

$$B^{\{i_1,\dots i_k\}} = \left(\bigcap_{\substack{j=1\\ j\neq k}}^{r} A_j^{\{i_1,\dots,i_k\}}\right)\bigcap\left(\bigcap_{j=1}^{\min(k,m-k)} C_j^{\{i_1,\dots,i_k\}}\right)$$
$$\bigcap\left\{(x_1,\dots,x_m):\sum_{j=1}^{k}x_{i_j} > a_k;\sum_{i=1}^{m}x_i = 1; x_i > 0, 1\le i\le m\right\} \qquad (3.2.2.6)$$

is the just region of acceptation of

$$H_{i_1,\dots i_k}:\ \beta_{i_1}\neq 0,\dots,\beta_{i_k}\neq 0,\ \beta_j = 0, j\neq i_1,\dots,i_k\,, \qquad (3.2.2.7)$$

$$A_j^{\{i_1,\dots i_k\}} = \bigcap_{\substack{\{l_1,\dots,l_j\} \\ \subset\{1,\dots,m\}}} D_{\{l_1,\dots,l_j\}}^{\{i_1,\dots,i_k\}} , \tag{3.2.2.8}$$

$$C_j^{\{i_1,\dots i_k\}} = \bigcap_{\substack{\{l_1,\dots,l_j\}\subset \\ \{1,\dots,m\}-\{i_1,\dots,i_k\}}} E_{\{l_1,\dots,l_j\}}^{\{i_1,\dots,i_k\}} , \tag{3.2.2.9}$$

$$x_i = \hat{\beta}_i^2 \Big/ \sum\nolimits_{j=1}^{m} \hat{\beta}_j^2 ,\ i = 1,\dots,m , \tag{3.2.1.5}$$

$$a_k = \frac{\frac{k}{m-k} F_{k,m-k}^{-1}(c_{m,r})}{1 + \frac{k}{m-k} F_{k,m-k}^{-1}(c_{m,r})} , \tag{3.2.2.10}$$

$$D_{\{l_1,\dots,l_j\}}^{\{i_1,\dots i_k\}} = \left\{ (x_1,\dots,x_m):\ \sum\nolimits_{q=1}^{j} x_{l_q} < \frac{\frac{j}{m-j} F_{j,m-j}^{-1}\left(F_{k,m-k}\left(\frac{\frac{m-k}{k}\sum_{q=1}^{k} x_{i_q}}{1-\sum_{q=1}^{k} x_{i_q}} \right)\right)}{1 + \frac{j}{m-j} F_{j,m-j}^{-1}\left(F_{k,m-k}\left(\frac{\frac{m-k}{k}\sum_{q=1}^{k} x_{i_q}}{1-\sum_{q=1}^{k} x_{i_q}} \right)\right)} \right\}, \tag{3.2.2.11}$$

$$E_{\{l_1,\dots,l_j\}}^{\{i_1,\dots i_k\}} = \bigcap_{\substack{\{p_1,\dots,p_j\} \\ \subset\{i_1,\dots,i_k\}}} \left\{ (x_1,\dots,x_m):\ \sum\nolimits_{q=1}^{j} x_{l_q} < \sum\nolimits_{q=1}^{j} x_{p_q} \right\} . \tag{3.2.2.12}$$

If $j = k$, we have $E_{\{l_1,\dots,l_j\}}^{\{i_1,\dots i_k\}} = D_{\{l_1,\dots,l_j\}}^{\{i_1,\dots i_k\}}$.

Proof: It is clear that the event $\{\text{rejecting } H_0\} = \{MaxU_r > c_{m,r}\}$. Further, using the definitions of $MaxU_r$ and $MU_k, k = 1,\dots,r$, we have

$$\{MaxU_r > c_{m,r}\} = \left\{\max_{1\le k\le r}(MU_k) > c_{m,r}\right\} = \bigcup_{k=1}^{r}\left\{MU_k > c_{m,r} \text{ and } MU_j \le MU_k ,\ j = 1,\dots,r,\ j \ne k\right\},$$

and

$$\left\{MU_k > c_{m,r} \text{ and } MU_j \le MU_k ,\ j = 1,\dots,r,\ j \ne k\right\}$$

$$= \left\{ \max_{\{i_1,\dots,i_k\}\in P_k} (u_k^{\{i_1,\dots,i_k\}}) > c_{m,r} \text{ and } MU_j \le \max_{\{i_1,\dots,i_k\}\in P_k} (u_k^{\{i_1,\dots,i_k\}}), 1 \le j \ne k \le r \right\}$$

$$= \bigcup_{\{i_1,\dots,i\}\in P_k} \left\{ u_k^{\{i_1,\dots,i_k\}} = \max_{\{j_1,\dots,j_k\}\in P_k} (u_k^{\{j_1,\dots,j_k\}}),\ \max_{\{j_1,\dots,j_k\}\in P_k} (u_k^{\{j_1,\dots,j_k\}}) > c_{m,r} , \right.$$

$$\left. \text{and } MU_j \le \max_{\{j_1,\dots,j_k\}\in P_k} (u_k^{\{j_1,\dots,j_k\}}), 1 \le j \ne k \le r \right\}$$

$$= \bigcup_{\{i_1,\ldots,i_k\}\in P_k} \Big\{u_k^{\{j_1,\ldots,j_k\}} \le u_k^{\{i_1,\ldots,i_k\}}, \{j_1,\ldots,j_k\} \ne \{i_1,\ldots,i_k\}, u_k^{\{i_1,\ldots,i_k\}} > c_{m,r},$$
$$\text{and } MU_j \le u_k^{\{i_1,\ldots,i_k\}}, 1 \le j \ne k \le r\Big\},$$

where

$$\Big\{u_k^{\{i_1,\ldots,i_k\}} > c_{m,r}, u_k^{\{j_1,\ldots,j_k\}} \le u_k^{\{i_1,\ldots,i_k\}}, \{j_1,\ldots,j_k\} \ne \{i_1,\ldots,i_k\},\ MU_j \le u_k^{\{i_1,\ldots,i_k\}}, 1 \le j \ne k \le r\Big\}$$

is just the acceptation region of

$$H_{i_1,\ldots i_k}:\ \beta_{i_1} \ne 0,\ldots,\beta_{i_k} \ne 0,\ \beta_j = 0, j \ne i_1,\ldots,i_k,$$

denoted by $B^{\{i_1,\ldots i_k\}}$. Hence,

$$\{MaxU_r > c_{m,r}\} = \bigcup_{k=1}^{r} \bigcup_{\{i_1,\ldots,i_k\}\in P_k} B^{\{i_1,\ldots i_k\}}. \tag{3.2.2.13}$$

In addition, for fixed $\{i_1,\ldots,i_k\}$,

$$B^{\{i_1,\ldots i_k\}} = \Big\{MU_j \le u_k^{\{i_1,\ldots,i_k\}}, j = 1,\ldots,r, j \ne k\Big\}$$
$$\bigcap \Big\{u_k^{\{j_1,\ldots,j_k\}} \le u_k^{\{i_1,\ldots,i_k\}}, \{j_1,\ldots,j_k\} \ne \{i_1,\ldots,i_k\}\Big\} \bigcap \Big\{u_k^{\{i_1,\ldots,i_k\}} > c_{m,r}\Big\}. \tag{3.2.2.14}$$

First,

$$\Big\{MU_j \le u_k^{\{i_1,\ldots,i_k\}}, j = 1,\ldots,r, j \ne k\Big\} = \bigcap_{\substack{j=1\\ j\ne k}}^{r} \Big\{MU_j \le u_k^{\{i_1,\ldots,i_k\}}\Big\} = \bigcap_{\substack{j=1\\ j\ne k}}^{r} A_j^{\{i_1,\ldots i_k\}}, \tag{3.2.2.15}$$

where $A_j^{\{i_1,\ldots i_k\}} := \Big\{MU_j \le u_k^{\{i_1,\ldots,i_k\}}\Big\}$.

Using the definitions of MU_j and $u_j^{\{l_1,\ldots,l_j\}}$, we have

$$A_j^{\{i_1,\ldots i_k\}} = \left\{\max_{\{l_1,\ldots,l_j\}\in P_j} (u_j^{\{l_1,\ldots,l_j\}}) \le u_k^{\{i_1,\ldots,i_k\}}\right\}$$
$$= \Big\{u_j^{\{l_1,\ldots,l_j\}} \le u_k^{\{i_1,\ldots,i_k\}}, \{l_1,\ldots,l_j\} \in P_j\Big\}$$
$$= \bigcap_{\{l_1,\ldots,l_j\}\in P_j} \Big\{u_j^{\{l_1,\ldots,l_j\}} \le u_k^{\{i_1,\ldots,i_k\}}\Big\}$$
$$= \bigcap_{\{l_1,\ldots,l_j\}\in P_j} \Big\{F_{j,m-j}\Big(v_j^{\{l_1,\ldots,l_j\}}\Big) \le u_k^{\{i_1,\ldots,i_k\}}\Big\}.$$

Because of the increasing monotonicity of $F_{j,m-j}^{-1}(\cdot)$,

$$A_j^{\{i_1,\ldots i_k\}} = \bigcap_{\{l_1,\ldots,l_j\}\in P_j} \left\{ v_j^{\{l_1,\ldots,l_j\}} \le F_{j,m-j}^{-1}\left(u_k^{\{i_1,\ldots,i_k\}}\right)\right\}.$$

Using the definitions of $v_j^{\{l_1,\ldots,l_j\}}$, we have

$$A_j^{\{i_1,\ldots i_k\}} = \bigcap_{\{l_1,\ldots,l_j\}\in P_j} \left\{ \frac{\frac{1}{j}\sum_{q=1}^{j}\hat{\beta}_{l_q}^2}{\frac{1}{m-j}\sum\limits_{q\neq l_1,\ldots,l_j}\hat{\beta}_q^2} \le F_{j,m-j}^{-1}\left(u_k^{\{i_1,\ldots,i_k\}}\right)\right\}$$

$$= \bigcap_{\{l_1,\ldots,l_j\}\in P_j} \left\{ \frac{\sum_{q=1}^{j}\hat{\beta}_{l_q}^2}{\sum\limits_{q\neq l_1,\ldots,l_j}\hat{\beta}_q^2} \le \frac{j}{m-j} F_{j,m-j}^{-1}\left(u_k^{\{i_1,\ldots,i_k\}}\right)\right\}.$$

Because of the increasing monotonicity of function $f(x) = \dfrac{x}{1+x}$ and the fact

$$\sum_{q=1}^{m}\hat{\beta}_{j_q}^2 = \sum_{q=1}^{m}\hat{\beta}_q^2,$$

$$A_j^{\{i_1,\ldots i_k\}} = \bigcap_{\{l_1,\ldots,l_j\}\in P_j} \left\{ \frac{\sum_{q=1}^{j}\hat{\beta}_{l_q}^2}{\sum_{q=1}^{m}\hat{\beta}_q^2} \le \frac{\frac{j}{m-j} F_{j,m-j}^{-1}\left(u_k^{\{i_1,\ldots,i_k\}}\right)}{1+\frac{j}{m-j} F_{j,m-j}^{-1}\left(u_k^{\{i_1,\ldots,i_k\}}\right)}\right\}.$$

Under the transformations (3.2.1.5), i.e. $x_i = \hat{\beta}_i^2 \Big/ \sum_{j=1}^{m}\hat{\beta}_j^2$, $i = 1,\ldots,m$; we have

$$u_k^{\{i_1,\ldots i_k\}} = F_{k,m-k}\left(\frac{\frac{m-k}{k}\sum_{q=1}^{k}x_{i_q}}{1-\sum_{q=1}^{k}x_{i_q}}\right). \qquad (3.2.2.16)$$

This is because

$$u_k^{\{i_1,\ldots i_k\}} = F_{k,m-k}\left(v_k^{\{i_1,\ldots,i_k\}}\right)$$

$$= F_{k,m-k}\left(\frac{\frac{1}{k}\sum_{j=1}^{k}\hat{\beta}_{i_j}^2}{\frac{1}{m-k}\sum_{j=k+1}^{m}\hat{\beta}_{i_j}^2}\right)$$

$$= F_{k,m-k}\left(\frac{\frac{1}{k}\sum_{j=1}^{k}\hat{\beta}_{i_j}^2 \Big/ \sum_{i=1}^{m}\hat{\beta}_i^2}{\frac{1}{m-k}\sum_{j=k+1}^{m}\hat{\beta}_{i_j}^2 \Big/ \sum_{i=1}^{m}\hat{\beta}_i^2}\right)$$

$$= F_{k,m-k}\left(\frac{\frac{m-k}{k}\sum_{q=1}^{k}x_{i_q}}{\sum_{q=k+1}^{m}x_{i_q}}\right)$$

$$=F_{k,m-k}\left(\frac{\frac{m-k}{k}\sum_{q=1}^{k}x_{i_q}}{1-\sum_{q=1}^{k}x_{i_q}}\right),$$

where the last equation follows from $\sum_{q=1}^{m}x_{i_q}=\sum_{i=1}^{m}x_i=1$. Therefore,

$$A_j^{\{i_1,\dots i_k\}}=\bigcap_{\{l_1,\dots,l_j\}\in P_j}\left\{\sum_{q=1}^{j}x_{l_q}\le\frac{\frac{j}{m-j}F_{j,m-j}^{-1}\left(F_{k,m-k}\left(\frac{\frac{m-k}{k}\sum_{q=1}^{k}x_{i_q}}{1-\sum_{q=1}^{k}x_{i_q}}\right)\right)}{1+\frac{j}{m-j}F_{j,m-j}^{-1}\left(F_{k,m-k}\left(\frac{\frac{m-k}{k}\sum_{q=1}^{k}x_{i_q}}{1-\sum_{q=1}^{k}x_{i_q}}\right)\right)}\right\}.$$

Let

$$D_{\{l_1,\dots,l_j\}}^{\{i_1,\dots,i_k\}}=\left\{(x_1,\dots,x_m):\sum_{q=1}^{j}x_{l_q}\le\frac{\frac{j}{m-j}F_{j,m-j}^{-1}\left(F_{k,m-k}\left(\frac{\frac{m-k}{k}\sum_{q=1}^{k}x_{i_q}}{1-\sum_{q=1}^{k}x_{i_q}}\right)\right)}{1+\frac{j}{m-j}F_{j,m-j}^{-1}\left(F_{k,m-k}\left(\frac{\frac{m-k}{k}\sum_{q=1}^{k}x_{i_q}}{1-\sum_{q=1}^{k}x_{i_q}}\right)\right)}\right\},$$

then

$$A_j^{\{i_1,\dots i_k\}}=\bigcap_{\{l_1,\dots,l_j\}\in P_j}D_{\{l_1,\dots,l_j\}}^{\{i_1,\dots,i_k\}}. \qquad (3.2.2.17)$$

Second, using the definitions of $u_k^{\{i_1,\dots,i_k\}}$ and $v_k^{\{i_1,\dots,i_k\}}$, we have

$$\left\{u_k^{\{j_1,\dots,j_k\}}\le u_k^{\{i_1,\dots,i_k\}},\{j_1,\dots,j_k\}\ne\{i_1,\dots,i_k\}\right\}$$

$$=\left\{F_{k,m-k}\left(v_k^{\{j_1,\dots,j_k\}}\right)\le F_{k,m-k}\left(v_k^{\{i_1,\dots,i_k\}}\right),\{j_1,\dots,j_k\}\ne\{i_1,\dots,i_k\}\right\}$$

$$=\left\{v_k^{\{j_1,\dots,j_k\}}\le v_k^{\{i_1,\dots,i_k\}},\{j_1,\dots,j_k\}\ne\{i_1,\dots,i_k\}\right\}$$

$$=\left\{\frac{\frac{1}{k}\sum_{q=1}^{k}\hat\beta_{j_q}^2}{\frac{1}{m-k}\sum\limits_{j\ne j_1,\dots,j_k}\hat\beta_j^2}\le\frac{\frac{1}{k}\sum_{q=1}^{k}\hat\beta_{i_q}^2}{\frac{1}{m-k}\sum\limits_{j\ne i_1,\dots,i_k}\hat\beta_j^2},\{j_1,\dots,j_k\}\ne\{i_1,\dots,i_k\}\right\}$$

$$=\left\{\frac{\sum_{q=1}^{k}\hat\beta_{j_q}^2}{\sum\limits_{j\ne j_1,\dots,j_k}\hat\beta_j^2}\le\frac{\sum_{q=1}^{k}\hat\beta_{i_q}^2}{\sum\limits_{j\ne i_1,\dots,i_k}\hat\beta_j^2},\{j_1,\dots,j_k\}\ne\{i_1,\dots,i_k\}\right\}.$$

Because of the increasing monotonicity of function $f(x)=\dfrac{x}{1+x}$ and the fact

$$\sum_{q=1}^{m} \hat{\beta}_{j_q}^2 = \sum_{j=1}^{m} \hat{\beta}_j^2 ,$$

$$\left\{u_k^{\{j_1,\dots,j_k\}} \le u_k^{\{i_1,\dots,i_k\}}, \{j_1,\dots,j_k\} \ne \{i_1,\dots,i_k\}\right\}$$

$$= \left\{ \frac{\sum_{q=1}^{k} \hat{\beta}_{j_q}^2}{\sum_{j=1}^{m} \hat{\beta}_j^2} \le \frac{\sum_{q=1}^{k} \hat{\beta}_{i_q}^2}{\sum_{j=1}^{m} \hat{\beta}_j^2}, \{j_1,\dots,j_k\} \ne \{i_1,\dots,i_k\} \right\}.$$

It follows from (3.2.1.5) that

$$\left\{u_k^{\{j_1,\dots,j_k\}} \le u_k^{\{i_1,\dots,i_k\}}, \{j_1,\dots,j_k\} \ne \{i_1,\dots,i_k\}\right\}$$

$$= \left\{\sum_{q=1}^{k} x_{j_q} \le \sum_{q=1}^{k} x_{i_q}, \{j_1,\dots,j_k\} \ne \{i_1,\dots,i_k\}\right\}$$

$$= \bigcap_{j=1}^{\min(k,m-k)} \left\{\sum_{q=1}^{k} x_{j_q} \le \sum_{q=1}^{k} x_{i_q}, \ j \text{ and only } j \text{ elements of } \{j_1,\dots,j_k\} \text{ are not elements of } \{i_1,\dots,i_k\}\right\}$$

$$= \bigcap_{j=1}^{\min(k,m-k)} \left\{\sum_{q=1}^{j} x_{l_q} \le \sum_{q=1}^{j} x_{p_q}, \{l_1,\dots,l_j\} \subset \{1,\dots,m\} - \{i_1,\dots,i_k\}, \{p_1,\dots,p_j\} \subset \{i_1,\dots,i_k\}\right\}.$$

Let

$$C_j^{\{i_1,\dots i_k\}} = \left\{\sum_{q=1}^{j} x_{l_q} \le \sum_{q=1}^{j} x_{p_q}, \{l_1,\dots,l_j\} \subset \{1,\dots,m\} - \{i_1,\dots,i_k\}, \{p_1,\dots,p_j\} \subset \{i_1,\dots,i_k\}\right\},$$

then

$$\left\{u_k^{\{j_1,\dots,j_k\}} \le u_k^{\{i_1,\dots,i_k\}}, \{j_1,\dots,j_k\} \ne \{i_1,\dots,i_k\}\right\} = \bigcap_{j=1}^{\min(k,m-k)} C_j^{\{i_1,\dots i_k\}} . \qquad (3.2.2.18)$$

Further, for fixed j,

$$C_j^{\{i_1,\dots i_k\}} = \bigcap_{\substack{\{l_1,\dots,l_j\} \subset \\ \{1,\dots,m\} - \{i_1,\dots,i_k\}}} \left\{\sum_{q=1}^{j} x_{l_q} \le \sum_{q=1}^{j} x_{p_q}, \{p_1,\dots,p_j\} \subset \{i_1,\dots,i_k\}\right\}$$

$$= \bigcap_{\substack{\{l_1,\dots,l_j\} \subset \\ \{1,\dots,m\} - \{i_1,\dots,i_k\}}} E_{\{l_1,\dots,l_j\}}^{\{i_1,\dots,i_k\}}, \qquad (3.2.2.19)$$

where

$$E_{\{l_1,\dots,l_j\}}^{\{i_1,\dots,i_k\}} := \left\{\sum_{q=1}^{j} x_{l_q} \le \sum_{q=1}^{j} x_{p_q}, \{p_1,\dots,p_j\} \subset \{i_1,\dots,i_k\}\right\}$$

$$= \bigcap_{\substack{\{p_1,\ldots,p_j\} \\ \subset\{i_1,\ldots,i_k\}}} \left\{(x_1,\ldots,x_m): \sum_{q=1}^{j} x_{l_q} \le \sum_{q=1}^{j} x_{p_q}\right\}.$$

Finally, $\left\{u_k^{\{i_1,\ldots,i_k\}} > c_{m,r}\right\} = \left\{F_{k,m-k}\left(v_k^{\{i_1,\ldots,i_k\}}\right) > c_{m,r}\right\}$

$$= \left\{v_k^{\{i_1,\ldots,i_k\}} > F_{k,m-k}^{-1}(c_{m,r})\right\}$$

$$= \left\{\frac{\sum_{q=1}^{k}\hat{\beta}_{i_q}^2}{\sum_{j\neq i_1,\ldots,i_k}\hat{\beta}_j^2} > \tfrac{k}{m-k}F_{k,m-k}^{-1}(c_{m,r})\right\}$$

$$= \left\{\frac{\sum_{q=1}^{k}\hat{\beta}_{i_q}^2}{\sum_{j=1}^{m}\hat{\beta}_j^2} > \frac{\tfrac{k}{m-k}F_{k,m-k}^{-1}(c_{m,r})}{1+\tfrac{k}{m-k}F_{k,m-k}^{-1}(c_{m,r})}\right\},$$

and it follows from (3.2.1.5) that

$$\left\{u_k^{\{i_1,\ldots,i_k\}} > c_{m,r}\right\} = \left\{\sum_{j=1}^{k} x_{i_j} > \frac{\tfrac{k}{m-k}F_{k,m-k}^{-1}(c_{m,r})}{1+\tfrac{k}{m-k}F_{k,m-k}^{-1}(c_{m,r})}\right\}$$

$$= \left\{(x_1,\ldots,x_m): \sum_{j=1}^{k} x_{i_j} > a_k\right\},$$

where

$$a_k := \frac{\tfrac{k}{m-k}F_{k,m-k}^{-1}(c_{m,r})}{1+\tfrac{k}{m-k}F_{k,m-k}^{-1}(c_{m,r})}.$$

Note that $x_1,\ldots,x_m$ satisfy $x_i > 0$, $i = 1,\ldots,m$ and $\sum_{i=1}^{m} x_i = 1$. Hence,

$$\left\{u_k^{\{i_1,\ldots,i_k\}} > c_{m,r}\right\} = \left\{(x_1,\ldots,x_m): \sum_{j=1}^{k} x_{i_j} > a_k; \sum_{i=1}^{m} x_i = 1; x_i > 0, i = 1,\ldots,m\right\}. \quad (3.2.2.20)$$

Obviously, $E_{\{l_1,\ldots,l_j\}}^{\{i_1,\ldots,i_k\}} = D_{\{l_1,\ldots,l_j\}}^{\{i_1,\ldots,i_k\}}$ when $j = k$. The results of the theorem follow then from (3.2.2.13) – (3.2.2.15) and (3.2.2.17) – (3.2.2.20). □

Theorem 3.4 *When* $H_{j_1,\ldots,j_q}$: $\beta_{j_1} \neq 0,\ldots,\beta_{j_q} \neq 0, q < m; \beta_j = 0, j \neq j_1,\ldots,j_q$ *holds, if in model (1.1)* $\boldsymbol{\varepsilon} \sim N(\mathbf{0},\sigma^2 I_{m+1})$, *then the probability of accepting* $H_{i_1,\ldots,i_k}$ *is given by*

$$P\left\{\text{accepting } H_{i_1,\ldots,i_k} \middle| H_{j_1,\ldots,j_q} \text{ is true}\right\} = \int_{\boldsymbol{B}^{\{i_1,\ldots,i_k\}}} \ldots \int p_{H_{j_1,\ldots,j_q}}(\boldsymbol{x})d\boldsymbol{x}, \quad (3.2.2.21)$$

where $B^{\{i_1,\ldots,i_k\}}$ *is defined by (3.2.2.6) and* $p_{H_{j_1,\ldots,j_q}}(\boldsymbol{x})$ *is the density of* $D_m(\tfrac{1}{2},\ldots,\tfrac{1}{2};0,\ldots,0,\lambda_{j_1},0,\ldots,0,\lambda_{j_2},0,\ldots,0,\lambda_{j_q},0,\ldots,0)$, *in which* $\lambda_{j_i} = \beta_{j_i}^2 / (2\sigma_\beta^2)$, $i = 1,\ldots,q$.

Proof: Clearly, it follows from (3.2.2.6) of Theorem 3.3 that the probability of accepting $H_{i_1,\ldots i_k}$ is

$$P\left\{\text{accepting } H_{i_1,\ldots i_k}\right\} = P\left\{(X_1,\ldots,X_m) \in B^{\{i_1,\ldots,i_k\}}\right\}.$$

If

$$H_{j_1,\ldots j_q}:\ \beta_{j_1} \neq 0,\ldots,\beta_{j_q} \neq 0\,, q<m\,; \beta_j = 0, j \neq j_1,\ldots,j_q$$

holds and $\varepsilon \sim N(\mathbf{0},\sigma^2 I_{m+1})$, then

$$\hat{\beta}_{j_i}^2 \big/ \sigma_\beta^2 \sim \chi_1^2(\lambda_{j_i})\,,\ i = 1,\ldots,q, q<m$$

and

$$\hat{\beta}_j^2 \big/ \sigma_\beta^2 \sim \chi_1^2(0)\,,\ j \neq j_1,\ldots,j_q\,,$$

where $\lambda_{j_i} = \frac{1}{2}\beta_{j_i}^2 \big/ \sigma_\beta^2$.

From lemma 3.2, definition 3.3 and the fact that

$$X_i = \hat{\beta}_i^2 \Big/ \sum_{j=1}^m \hat{\beta}_j^2 = (\tfrac{1}{2}\hat{\beta}_i^2 \big/ \sigma_\beta^2) \Big/ \sum_{j=1}^m (\tfrac{1}{2}\hat{\beta}_j^2 \big/ \sigma_\beta^2)\,,$$

we have $\left(X_1,\ldots,X_m\right)' \sim D_m(\frac{1}{2},\ldots,\frac{1}{2};0,..,0,\lambda_{j_1},0,\ldots,0,\lambda_{j_2},0,\ldots,0,\lambda_{j_q},0,\ldots,0)$.

Hence,

$$P\left\{\text{accepting } H_{i_1,\ldots i_k} \middle| H_{j_1,\ldots j_q} \text{ is true}\right\} = P\left\{(X_1,\ldots,X_m) \in B^{\{i_1,\ldots,i_k\}} \middle| H_{j_1,\ldots j_q} \text{ is true}\right\}$$

$$= \int_{\boldsymbol{B}^{\{i_1,\ldots,i_k\}}} \cdots \int p_{H_{j_1,\ldots,j_q}}(\boldsymbol{x})d\boldsymbol{x}\,,$$

where $p_{H_{j_1,\ldots,j_q}}(\boldsymbol{x})$ is the density of $D_m(\frac{1}{2},\ldots,\frac{1}{2};0,..,0,\lambda_{j_1},0,\ldots,0,\lambda_{j_2},0,\ldots,0,\lambda_{j_q},0,\ldots,0)$. □

3.3 Derivation of critical values of $MaxU_r$

In this section, we will mainly introduce the procedures for estimating the critical values of the test statistic $MaxU_r$.

3.3.1 Simulation to derive critical values

In this subsection, we will use computer simulations to estimate the critical values of the $MaxU_r$-test for $m > 3$ and $r = 1, ..., m-1$ with the help of programs written in SAS/IML.

- *Approach of simulation to derive critical values*

The procedure of simulation to estimate the critical values of the $MaxU_r$-test is as follows:

1. Generate a random sample of size n from the standard normal distribution.
2. Compute the contrasts $\hat{\beta}_1,...,\hat{\beta}_m$ using (1.3) with the normal sample, where $m = n - 1$.
3. Compute the value of the statistic $MaxU_r$ using (3.1.6).
4. Save the value of statistic $MaxU_r$ computed in step 3 into an array variable.
5. Repeat step 1 – step 4 N times, then a sample of size N from the distribution of $MaxU_r$ is generated.
6. Use the $(1-\alpha)$-th quantiles of the above generated $MaxU_r$ sample as the estimates of the critical values of $MaxU_r$.

In fact, when the errors have identically independent standard normal distribution, instead of computing $\hat{\beta}_1,...,\hat{\beta}_m$ from a normal sample, one can generate the sample $\hat{\beta}_1,...,\hat{\beta}_m$ directly from the standard normal distribution in step 2, so that step 1 can be omitted.

- *Critical values for m = 7, 15 and 31*

Using the simulation procedure stated above, we got the critical values of $MaxU_r$ for α = 0.20, 0.10, 0.05, 0.025, 0.01 where N = 10 000. For m = 7, 15 and 31, the obtained critical values of $MaxU_r$ are shown in Table 3.3.1. A more extensive table is available in appendix A3.

Table 3.3.1: Critical values of $MaxU_r$ for m = 7, 15 and 31 through 10 000 simulations.

m	r	α				
		0.20	0.10	0.05	0.025	0.01
7	1	0.9709150	0.9847355	0.9926990	0.9968258	0.9987041
	2	0.9860312	0.9936809	0.9972449	0.9986215	0.9994449
	3	0.9917365	0.9965355	0.9984861	0.9992564	0.9997482
	4	0.9939139	0.9976750	0.9990135	0.9995253	0.9998187
	5	0.9951706	0.9981618	0.9991909	0.9996400	0.9998611
	6	0.9957065	0.9983210	0.9992564	0.9996765	0.9998700
15	1	0.9866033	0.9934845	0.9967511	0.9984956	0.9993982
	2	0.9955954	0.9980245	0.9991350	0.9995962	0.9998618
	3	0.9980299	0.9992156	0.9996795	0.9998601	0.9999538
	4	0.9989703	0.9996129	0.9998458	0.9999294	0.9999780
	5	0.9993896	0.9997834	0.9999110	0.9999641	0.9999880
	6	0.9995944	0.9998602	0.9999441	0.9999772	0.9999920
	7	0.9997044	0.9999015	0.9999630	0.9999853	0.9999948
	8	0.9997766	0.9999257	0.9999733	0.9999904	0.9999966
	9	0.9998207	0.9999406	0.9999789	0.9999922	0.9999973
	10	0.9998470	0.9999492	0.9999825	0.9999933	0.9999978
	11	0.9998614	0.9999545	0.9999847	0.9999941	0.9999982
	12	0.9998709	0.9999569	0.9999856	0.9999945	0.9999983
	13	0.9998761	0.9999579	0.9999860	0.9999946	0.9999983
	14	0.9998768	0.9999581	0.9999860	0.9999946	0.9999983
31	1	0.9933986	0.9968138	0.9985154	0.9992558	0.9997117
	2	0.9985660	0.9993731	0.9997371	0.9998828	0.9999570
	3	0.9995529	0.9998242	0.9999288	0.9999705	0.9999904
	4	0.9998348	0.9999399	0.9999768	0.9999917	0.9999972
	5	0.9999276	0.9999761	0.9999917	0.9999969	0.9999991
	6	0.9999649	0.9999896	0.9999965	0.9999987	0.9999997
	7	0.9999819935	0.9999950535	0.9999984094	0.9999994650	0.9999998510
	8	0.9999900783	0.9999973952	0.9999992076	0.9999997632	0.9999999273
	9	0.9999941849	0.9999985724	0.9999995750	0.9999998726	0.9999999616
	10	0.9999965063	0.9999991312	0.9999997565	0.9999999231	0.9999999808
	11	0.9999977940	0.9999994524	0.9999998594	0.9999999532	0.9999999873
	12	0.9999985216	0.9999996393	0.9999999082	0.9999999718	0.9999999923
	13	0.9999989685	0.9999997489	0.9999999366	0.9999999820	0.9999999950
	14	0.9999992364	0.9999998175	0.9999999528	0.9999999860	0.9999999967
	15	0.9999994141	0.9999998632	0.9999999647	0.9999999888	0.9999999979

3.3.2 Theoretical derivation of critical values

In this subsection, we will theoretically study the probability function and power function of $MaxU_r$ for *m* = 3. Then we compute the critical values and power using the software Mathematica.

• *The distribution function of* $MaxU_r$ *for m = 3*

Corollary 3.1 *Under the conditions of Theorem 3.2, the distribution function of* $MaxU_r$ *for m = 3 is given by*

$$F_{MaxU_r}(x) = \frac{1}{2\pi}\int\int_{\Omega} \left(1 - x_1 - x_2\right)^{-\frac{1}{2}} x_1^{-\frac{1}{2}} x_2^{-\frac{1}{2}} dx_1 dx_2 \text{, } for\ F_{1,2}(1) \le x < 1, \quad (3.3.2.1)$$

$$F_{MaxU_r}(x) = 0 \text{ for } x < F_{1,2}(1),$$

and

$$F_{MaxU_r}(x) = 1 \text{ for } x \ge 1,$$

where the constant $\pi \approx 3.14159$*;*

$\Omega = \left\{(x_1, x_2):\ 0 < x_1 \le y_1,\ 0 < x_2 \le y_1,\ 0 < 1 - x_1 - x_2 \le y_1\right\}$, *when r = 1;*

$\Omega = \left\{(x_1, x_2):\ 1 - y_2 < x_1 \le y_1,\ 1 - y_2 < x_2 \le y_1,\ 1 - y_1 < x_1 + x_2 \le y_2\right\}$, *when r =2;*

$$y_1 = \frac{1}{2}F_{1,2}^{-1}(x) \Big/ \left(1 + \frac{1}{2}F_{1,2}^{-1}(x)\right),$$

$$y_2 = 2F_{2,1}^{-1}(x) \Big/ \left(1 + 2F_{2,1}^{-1}(x)\right)$$

and $F_{k,3-k}^{-1}(\cdot)$, *k = 1,2 are the inverse functions of* $F_{k,3-k}(\cdot)$.

Proof: When *m* = 3, from Theorem 3.2 we know

$$F_{MaxU_r}(x) = 0 \text{ for } x < F_{1,2}(1),$$

$$F_{MaxU_r}(x) = 1 \text{ for } x \ge 1$$

and

$$F_{MaxU_r}(x) = \frac{\Gamma(\frac{3}{2})}{\left[\Gamma(\frac{1}{2})\right]^3}\int\cdots\int_{\Omega} \left(1 - \sum_{i=1}^{2} x_i\right)^{-\frac{1}{2}} \prod_{i=1}^{2} x_i^{-\frac{1}{2}} dx_i \text{, for } F_{1,2}(1) \le x < 1,$$

where $\Omega = \left\{(x_1, x_2): \sum_{i=1}^{3} x_i = 1, x_i > 0,\ i = 1,2,3; \sum_{j=1}^{k} x_{i_j} \le y_k, \{i_1, \ldots, i_k\} \in P_k, k = 1, \ldots, r < 3\right\}$,

$P_1 = \left\{\{1\}, \{2\}, \{3\}\right\}$ and $P_2 = \left\{\{1,2\}, \{2,3\}, \{1,3\}\right\}$.

Because $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2})$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$F_{MaxU_r}(x) = \frac{1}{2\pi}\int\int_{\Omega} \left(1 - x_1 - x_2\right)^{-\frac{1}{2}} x_1^{-\frac{1}{2}} x_2^{-\frac{1}{2}} dx_1 dx_2 \text{ , for } F_{1,2}(1) \le x < 1 .$$

For r =1,

$$\Omega = \left\{ (x_1, x_2) : \sum_{i=1}^{3} x_i = 1, x_i > 0,\ i = 1,2,3; x_{i_1} \le y_1, \{i_1\} \in P_1 \right\}$$

$$= \left\{ (x_1, x_2) :\ \sum_{i=1}^{3} x_i = 1,\ 0 < x_1 \le y_1,\ 0 < x_2 \le y_1,\ 0 < x_3 \le y_1 \right\}$$

$$= \left\{ (x_1, x_2) :\ 0 < x_1 \le y_1,\ 0 < x_2 \le y_1,\ 0 < 1 - x_1 - x_2 \le y_1 \right\}.$$

For r = 2,

$$\Omega = \left\{ (x_1, x_2) : \sum_{i=1}^{3} x_i = 1, x_i > 0,\ i = 1,2,3; \sum_{j=1}^{k} x_{i_j} \le y_k, \{i_1, \ldots, i_k\} \in P_k, k = 1,2 \right\}$$

$$= \left\{ (x_1, x_2) :\ \sum_{i=1}^{3} x_i = 1,\ 0 < x_{i_1} \le y_1,\ i_1 \in P_1;\ x_{i_1} + x_{i_2} \le y_2, \{i_1, i_2\} \in P_2 \right\}$$

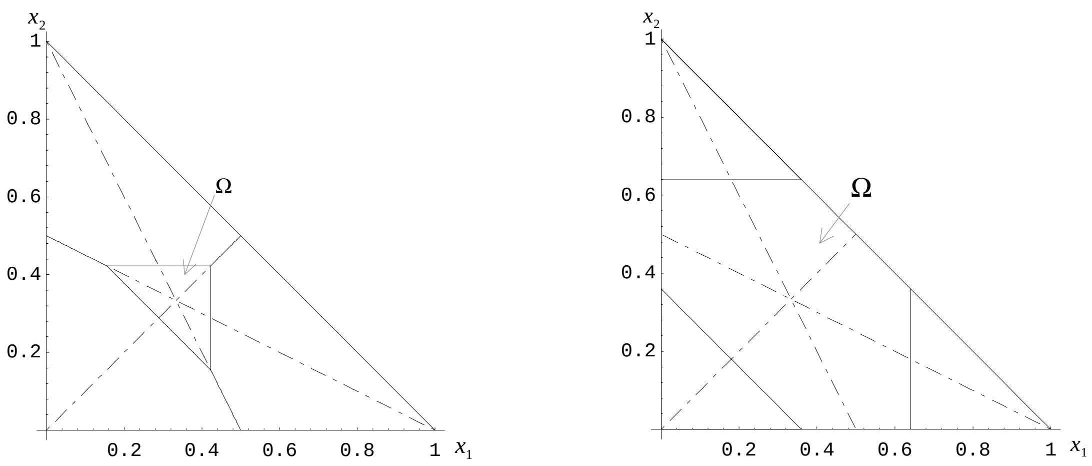
$$= \left\{ (x_1, x_2) :\ 0 < x_1 \le y_1, 0 < x_2 \le y_1, 0 < 1 - x_1 - x_2 \le y_1;\ x_1 + x_2 \le y_2, 1 - x_2 \le y_2, 1 - x_1 \le y_2 \right\}$$

$$= \left\{ (x_1, x_2) :\ 1 - y_2 < x_1 \le y_1,\ 1 - y_2 < x_2 \le y_1,\ 1 - y_1 < x_1 + x_2 \le y_2 \right\}.$$

□

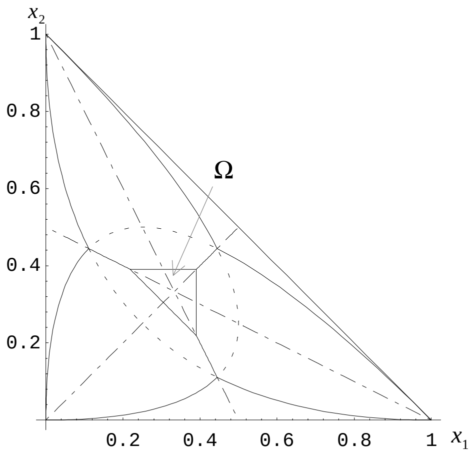
The integration region Ω in (3.3.2.1) for r = 1 is shown in Figure 3.3.1 (a) and (b). Figure 3.3.2 (a) and (b) show the integration region Ω for r = 2.

Figure 3.3.1: The integration region Ω for m = 3 and r = 1

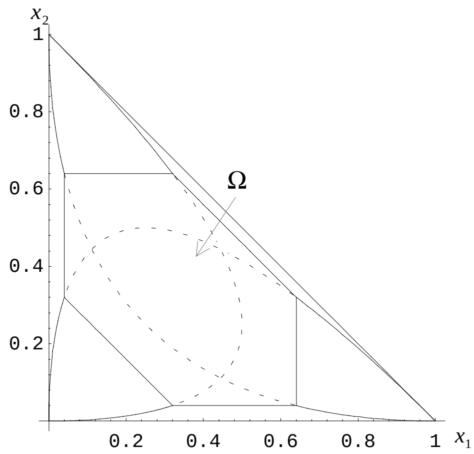


(a): $F_{1,2}(1) \le x < F_{1,2}(2)$ (b): $F_{1,2}(2) \le x < 1$

Figure 3.3.2: The integration region Ω for m = 3 and r = 2



(a): $F_{1,2}(1) \leq x < F_{1,2}(8/5)$



(b): $F_{1,2}(8/5) \leq x < 1$

- *Critical values for m = 3*

According to Corollary 3.1, we can calculate the critical values of $MaxU_r$ for the case m = 3 using the software Mathematica. The critical values for m = 3 calculated by Mathematica are shown in Table 3.3.2. The corresponding results from simulations are shown in Table 3.3.3.

Table 3.3.2: Critical values of $MaxU_r$ for m = 3 calculated with Mathematica.

m	r	α				
		0.20	0.10	0.05	0.025	0.01
3	1	0.9333333	0.9666668	0.9833333	0.9916667	0.9966667
	2	0.9592638	0.9809699	0.9908832	0.9955677	0.9982680

Table 3.3.3: Critical values of $MaxU_r$ for m = 3 through 10 000 simulations.

m	r	α				
		0.20	0.10	0.05	0.025	0.01
3	1	0.9336163	0.9671265	0.9836491	0.9915178	0.9964610
	2	0.9593938	0.9808001	0.9908787	0.9951778	0.9982718

The comparison of the values between Table 3.3.2 and Table 3.3.3 shows that the critical values are very close. This provides a partial check of the accuracy of our estimated critical values via Monte-Carlo method.

● *The power function of* $MaxU_r$ *for m = 3*

Corollary 3.2 *From Theorem 3.3, the regions of accepting* H_1 , H_2 *and* H_3 *for m = 3 and r = 1 are given by*

$$B^{\{1\}}=\{(x_1,x_2):x_1>a_1, 0<x_2<x_1, 1-x_2<2x_1, x_1+x_2<1\}, \tag{3.3.2.2}$$

$$B^{\{2\}}=\{(x_1,x_2):x_2>a_1, 0<x_1<x_2, 1-x_1<2x_2, x_1+x_2<1\}, \tag{3.3.2.3}$$

$$B^{\{3\}}=\{(x_1,x_2):x_1+x_2<1-a_1, 1-x_2>2x_1, 1-x_1>2x_2, x_1>0, x_2>0\}, \tag{3.3.2.4}$$

respectively, where $a_1=\dfrac{\frac{1}{2}F_{1,2}^{-1}(c_{3,1})}{1+\frac{1}{2}F_{1,2}^{-1}(c_{3,1})}$.

Proof: For m = 3 and r =1, it follows from Theorem 3.3 that the region of accepting H_{i_1}: $\beta_{i_1}\neq 0$, $\beta_j=0$, $1\le j\neq i_1\le m$ is

$$B^{\{i_1\}}=C_1^{\{i_1\}}\bigcap\{(x_1,x_2,x_3):x_{i_1}>a_1; x_1+x_2+x_3=1; x_i>0, i=1,2,3\}$$

$$=\bigcap_{\substack{\{l_1\}\subset\\ \{1,2,3\}-\{i_1\}}} E_{\{l_1\}}^{\{i_1\}}\bigcap\{(x_1,x_2,x_3):x_{i_1}>a_1; x_1+x_2+x_3=1; x_i>0, i=1,2,3\}$$

$$=\bigcap_{\substack{\{l_1\}\subset\\ \{1,2,3\}-\{i_1\}}}\{(x_1,x_2,x_3):\ x_{l_1}<x_{i_1}\}\bigcap\{(x_1,x_2,x_3):x_{i_1}>a_1; x_1+x_2+x_3=1; x_i>0, i=1,2,3\},$$

where $a_1=\dfrac{\frac{1}{2}F_{1,2}^{-1}(c_{3,1})}{1+\frac{1}{2}F_{1,2}^{-1}(c_{3,1})}$. Hence, for i_1 = 1, we have

$$B^{\{1\}}=\bigcap_{\substack{\{l_1\}\subset\\ \{2,3\}}}\{(x_1,x_2,x_3):\ x_{l_1}<x_1\}\bigcap\{(x_1,x_2,x_3):x_1>a_1; x_1+x_2+x_3=1; x_2>0, x_3>0\}$$

$$=\{(x_1,x_2,x_3):\ x_2<x_1, x_3<x_1\}\bigcap\{(x_1,x_2,x_3):x_1>a_1; x_1+x_2+x_3=1; x_2>0, x_3>0\}$$

$$=\{(x_1,x_2):x_1>a_1, 0<x_2<x_1, 1-x_2<2x_1, x_1+x_2<1\}.$$

Similarly, for i_1 = 2 and 3, we have

$$B^{\{2\}}=\{(x_1,x_2):x_2>a_1, 0<x_1<x_2, 1-x_1<2x_2, x_1+x_2<1\}$$

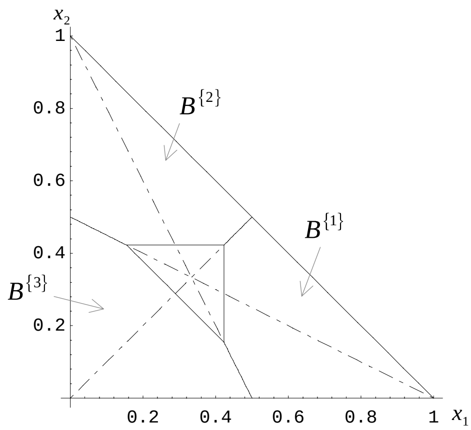
and

$$B^{\{3\}}=\{(x_1,x_2):x_1+x_2<1-a_1, 1-x_2>2x_1, 1-x_1>2x_2, x_1>0, x_2>0\},$$

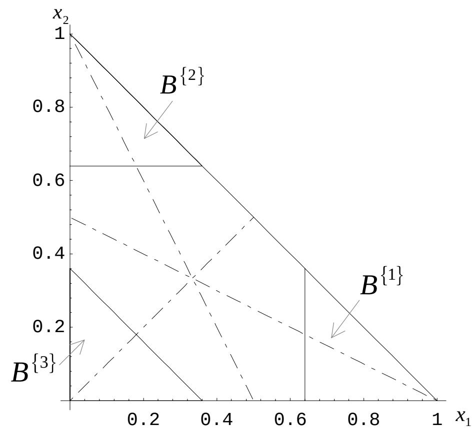
respectively. □

The graphical representation of the regions $B^{\{1\}}$, $B^{\{2\}}$ and $B^{\{3\}}$ for m = 3 and r = 1 is shown in Figure 3.3.3.

Figure 3.3.3: The regions $B^{\{1\}}$, $B^{\{2\}}$ and $B^{\{3\}}$ for m = 3 and r = 1.



(a): for $F_{1,2}(1) \le c_{3,1} < F_{1,2}(2)$.



(b): for $F_{1,2}(2) \le c_{3,1} < 1$.

Corollary 3.3 *From Theorem 3.3, the regions of accepting* H_1 , H_2 , H_3, H_{12}, H_{13} *and* H_{23} *for m = 3 and r = 2 are given by*

$$B^{\{1\}} = \left\{ (x_1, x_2) : \; x_1 > a_1, \; x_2 < x_1, \; 1 - x_2 < 2x_1, \right.$$

$$\left. x_2 < \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]}{1 + 2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]} - x_1, \; x_2 > 1 - \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]}{1 + 2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]} \right\}, \qquad (3.3.2.5)$$

$$B^{\{2\}} = \left\{ (x_1, x_2) : \; x_2 > a_1, \; x_1 < x_2, \; 1 - x_1 < 2x_2, \right.$$

$$\left. x_1 < \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_2}{1-x_2}\right)\right]}{1 + 2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_2}{1-x_2}\right)\right]} - x_2, \; x_2 > 1 - \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]}{1 + 2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]} \right\}, \qquad (3.3.2.6)$$

$$B^{\{3\}} = \left\{ (x_1,x_2):\ x_1 + x_2 < 1 - a_1,\ 1 - x_2 > 2x_1,\ 1 - x_1 > 2x_2, \right.$$

$$\left. x_1 < \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_2}{1-x_2}\right)\right]}{1+2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_2}{1-x_2}\right)\right]} - x_2,\ x_2 < \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]}{1+2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]} - x_1 \right\}, \quad (3.3.2.7)$$

$$B^{\{1,2\}} = \left\{ (x_1,x_2):\ \mathrm{a}_2 < x_1 + x_2 < 1, \right.$$

$$\left. x_1 > \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_2}{1-x_2}\right)\right]}{1+2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_2}{1-x_2}\right)\right]} - x_2,\ x_2 > \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]}{1+2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]} - x_1 \right\}, \quad (3.3.2.8)$$

$$B^{\{1,3\}} = \left\{ (x_1,x_2):\ 0 < x_2 < 1 - a_2, \right.$$

$$\left. x_1 > \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_2}{1-x_2}\right)\right]}{1+2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_2}{1-x_2}\right)\right]} - x_2,\ x_2 < 1 - \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]}{1+2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]} \right\}, \quad (3.3.2.9)$$

$$B^{\{2,3\}} = \left\{ (x_1,x_2):\ 0 < x_1 < 1 - a_2, \right.$$

$$\left. x_2 > \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]}{1+2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]} - x_1,\ x_2 < 1 - \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]}{1+2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]} \right\}, \quad (3.3.2.10)$$

respectively, where $a_1 = \dfrac{\frac{1}{2}F_{1,2}^{-1}(c_{3,2})}{1+\frac{1}{2}F_{1,2}^{-1}(c_{3,2})}$ *and* $a_2 = \dfrac{2F_{2,1}^{-1}(c_{3,2})}{1+2F_{2,1}^{-1}(c_{3,2})}$.

Proof: It follows from Theorem 3.3 that for m =3 and r = 2, if k =1,

$$B^{\{i_1\}} = A_2^{\{i_1\}} \bigcap C_1^{\{i_1\}} \bigcap \left\{ (x_1, x_2, x_3): x_{i_1} > a_1; \sum_{i=1}^{3} x_i = 1; x_i > 0, 1 \le i \le 3 \right\}$$

$$= \bigcap_{\substack{\{l_1, l_2\} \\ \subset \{1,2,3\}}} D_{\{l_1, l_2\}}^{\{i_1\}} \bigcap_{\substack{\{l_1\} \subset \\ \{1,2,3\} - \{i_1\}}} E_{\{l_1\}}^{\{i_1\}} \bigcap \left\{ (x_1, x_2, x_3): x_{i_1} > a_1; \sum_{i=1}^{3} x_i = 1; x_i > 0, 1 \le i \le 3 \right\},$$

where $a_1 = \dfrac{\frac{1}{2} F_{1,2}^{-1}(c_{3,2})}{1 + \frac{1}{2} F_{1,2}^{-1}(c_{3,2})}$, $E_{\{l_1\}}^{\{i_1\}} = \left\{ (x_1, x_2, x_3):\ x_{l_1} < x_{i_1} \right\}$ and

$$D_{\{l_1, l_2\}}^{\{i_1\}} = \left\{ (x_1, x_2, x_3):\ \sum\nolimits_{q=1}^{2} x_{l_q} < \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\dfrac{2x_{i_1}}{1 - x_{i_1}} \right) \right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\dfrac{2x_{i_1}}{1 - x_{i_1}} \right) \right)} \right\}.$$

If $k = 2$,

$$B^{\{i_1, i_2\}} = A_1^{\{i_1, i_2\}} \bigcap C_1^{\{i_1, i_2\}} \bigcap \left\{ (x_1, x_2, x_3): \sum_{j=1}^{2} x_{i_j} > a_2; \sum_{i=1}^{3} x_i = 1; x_i > 0, 1 \le i \le 3 \right\}$$

$$= \bigcap_{\substack{\{l_1\} \\ \subset \{1,2,3\}}} D_{\{l_1\}}^{\{i_1, i_2\}} \bigcap_{\substack{\{l_1\} \subset \\ \{1,2,3\} - \{i_1, i_2\}}} E_{\{l_1\}}^{\{i_1, i_2\}} \bigcap \left\{ (x_1, x_2, x_3): \sum_{j=1}^{2} x_{i_j} > a_2; \sum_{i=1}^{3} x_i = 1; x_i > 0, 1 \le i \le 3 \right\},$$

where $a_2 = \dfrac{2F_{2,1}^{-1}(c_{3,2})}{1 + 2F_{2,1}^{-1}(c_{3,2})}$, $E_{\{l_1\}}^{\{i_1, i_2\}} = \bigcap_{\substack{\{p_1\} \\ \subset \{i_1, i_2\}}} \left\{ (x_1, x_2, x_3):\ x_{l_1} < x_{p_1} \right\}$ and

$$D_{\{l_1\}}^{\{i_1, i_2\}} = \left\{ (x_1, x_2, x_3):\ x_{l_1} < \frac{2F_{1,2}^{-1}\left(F_{2,1}\left(\dfrac{\frac{1}{2}(x_{i_1} + x_{i_2})}{1 - (x_{i_1} + x_{i_2})} \right) \right)}{1 + 2F_{1,2}^{-1}\left(F_{2,1}\left(\dfrac{\frac{1}{2}(x_{i_1} + x_{i_2})}{1 - (x_{i_1} + x_{i_2})} \right) \right)} \right\}.$$

Therefore, when $i_1 = 1$, we have

$$B^{\{1\}} = \bigcap_{\substack{\{l_1, l_2\} \\ \subset \{1,2,3\}}} \left\{ (x_1, x_2, x_3):\ x_{l_1} + x_{l_2} < \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\dfrac{2x_1}{1 - x_1} \right) \right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\dfrac{2x_1}{1 - x_1} \right) \right)} \right\}$$

$$\bigcap \left\{ (x_1, x_2, x_3):\ x_2 < x_1, x_3 < x_1 \right\} \bigcap \left\{ (x_1, x_2, x_3): x_1 > a_1; \sum_{i=1}^{3} x_i = 1; x_i > 0, 1 \le i \le 3 \right\},$$

$$= \left\{ (x_1, x_2, x_3) \colon x_1 + x_2 < \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}, \right.$$

$$\left. x_1 + x_3 < \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}, x_2 + x_3 < \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)} \right\}$$

$$\bigcap \left\{ (x_1, x_2, x_3) \colon x_1 > a_1, 0 < x_2 < x_1, 0 < x_3 < x_1, \sum_{i=1}^{3} x_i = 1 \right\}$$

$$= \left\{ (x_1, x_2) \colon x_1 + x_2 < \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}, \right.$$

$$\left. 1 - x_2 < \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}, 1 - x_1 < \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)} \right\}$$

$$\bigcap \left\{ (x_1, x_2) \colon x_1 > a_1, 0 < x_2 < x_1, 0 < 1 - x_1 - x_2 < x_1 \right\}.$$

Since

$$1 - x_1 < \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)} \Leftrightarrow x_1 > \frac{\frac{1}{2}F_{1,2}^{-1}\left(\frac{1}{2}\right)}{1 + \frac{1}{2}F_{1,2}^{-1}\left(\frac{1}{2}\right)},$$

$$B^{\{1\}} = \left\{ (x_1, x_2) \colon x_2 < \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)} - x_1, x_2 > 1 - \frac{2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}{1 + 2F_{2,1}^{-1}\left(F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right)}, \right.$$

$$\left. x_1 > \frac{\frac{1}{2}F_{1,2}^{-1}\left(\frac{1}{2}\right)}{1 + \frac{1}{2}F_{1,2}^{-1}\left(\frac{1}{2}\right)}, x_1 > a_1, x_2 < x_1, 1 - x_2 < 2x_1 \right\}.$$

It follows from Corollary 3.1 that the critical value $c_{3,r}$ satisfy $F_{1,2}(1) \le c_{3,r} < 1$. In addition, $a_1 = \dfrac{\frac{1}{2}F_{1,2}^{-1}(c_{3,2})}{1+\frac{1}{2}F_{1,2}^{-1}(c_{3,2})}$ and $F_{1,2}(1) > \dfrac{1}{2}$. Hence, $a_1 > \dfrac{\frac{1}{2}F_{1,2}^{-1}(\frac{1}{2})}{1+\frac{1}{2}F_{1,2}^{-1}(\frac{1}{2})}$.

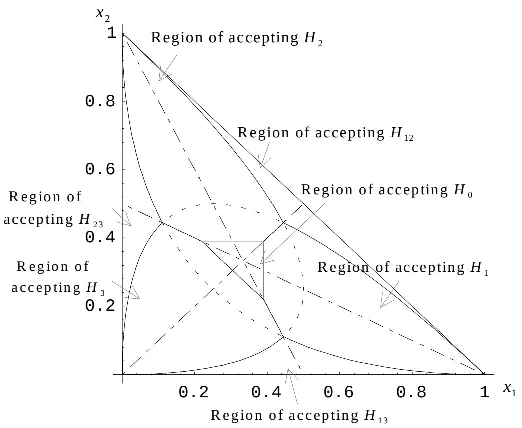
$$B^{\{1\}} = \left\{ (x_1,x_2):\ x_1 > a_1,\ x_2 < x_1,\ 1-x_2 < 2x_1,\right.$$

$$\left. x_2 < \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]}{1+2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]} - x_1,\ x_2 > 1 - \frac{2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]}{1+2F_{2,1}^{-1}\left[F_{1,2}\left(\frac{2x_1}{1-x_1}\right)\right]} \right\}.$$

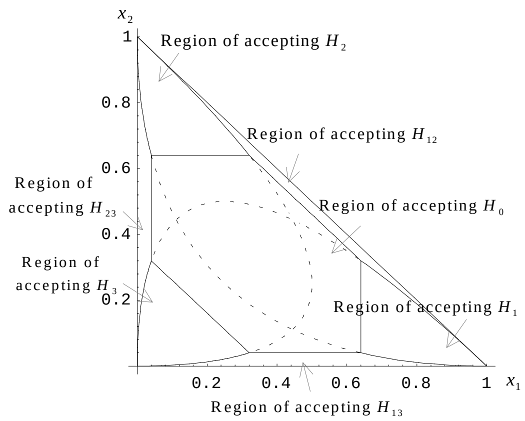
Equation (3.3.2.5) is proven. Similar to the proof of (3.3.2.5), equations (3.3.2.6) – (3.3.2.10) can also be proven. □

The graphical representation of the regions $B^{\{1\}}$, $B^{\{2\}}$, $B^{\{3\}}$, $B^{\{1,2\}}$, $B^{\{1,3\}}$ and $B^{\{2,3\}}$ for m = 3 and r = 2 is shown in Figure 3.3.4 (a) and (b).

Figure 3.3.4: Regions of accepting H_0, H_1, H_2, H_3, H_{12}, H_{13} and H_{23} for m = 3 and r = 2.



(a): for $F_{1,2}(1) \le c_{3,2} < F_{1,2}(8/5)$.



(b): for $F_{1,2}(8/5) \le c_{3,2} < 1$.

Corollary 3.4 *Under the conditions of Theorem 3.4, if* H_j: $\beta_j \ne 0, \beta_l = 0, 1 \le l \ne j \le 3$ *holds, the probabilities of accepting* H_i, *i=1,2,3 for m = 3 and r = 1 are given by*

$$P\left\{\text{accepting } H_i \middle| H_j \text{ is true}\right\} = \int\limits_{B^{\{i\}}}\int p_{H_j}(x_1,x_2)dx_1dx_2\,, \qquad (3.3.2.11)$$

where $B^{\{i\}}$, *i = 1,2,3 are defined by (3.3.2.2) – (3.3.2.4), respectively.*

For m = 3 and r = 2, the probabilities of accepting H_{i_1}, i_1 *=1,2,3 are given by*

$$P\left\{\text{accepting } H_{i_1} \middle| H_j \text{ is true}\right\} = \int\limits_{B^{\{i_1\}}}\int p_{H_j}(x_1, x_2)dx_1dx_2\,; \qquad (3.3.2.12)$$

and the probabilities of accepting H_{i_1,i_2}*,* $1 \le i_1 \ne i_2 \le 3$ *are given by*

$$P\left\{\text{accepting } H_{i_1,i_2} \middle| H_j \text{ is true}\right\} = \int\limits_{B^{\{i_1,i_2\}}}\int p_{H_j}(x_1, x_2)dx_1dx_2\,, \qquad (3.3.2.13)$$

where $B^{\{i_1\}}$*,* i_1 *= 1,2,3 are defined by (3.3.2.5) – (3.3.2.7), and* $B^{\{i_1,i_2\}}$*,* $1 \le i_1 \ne i_2 \le 3$ *are defined by (3.3.2.8) – (3.3.2.10), respectively. Here,* $p_{H_j}(x_1, x_2)$ *is the density of* $D_3(\frac{1}{2},\frac{1}{2},\frac{1}{2};0,\lambda_j,0)$*, where* $\lambda_j = \beta_j^2 \big/ (2\sigma_\beta^2)$*. Without loss of generality, let j =1. Then the density of* $D_3(\frac{1}{2},\frac{1}{2},\frac{1}{2};0,\lambda_j,0)$ *is given by*

$$p_{H_1}(x_1, x_2) = \frac{1}{2\pi} x_1^{-\frac{1}{2}}(2\lambda_1 x_1 + 1)e^{\lambda_1(x_1-1)} x_2^{-\frac{1}{2}}(1 - x_1 - x_2)^{-\frac{1}{2}}. \qquad (3.3.2.14)$$

Proof: Equations (3.3.2.11) – (3.3.2.13) directly follow from Theorem 3.4, Corollary 3.2 and Corollary 3.3. In what follows, we will focus our attention on the proof of equation (3.3.2.14). Firstly, it follows from Lemma 3.3 that the density of $D_3(\frac{1}{2},\frac{1}{2},\frac{1}{2};\lambda_1,0,0)$ is given by

$$p_{H_1}(x_1, x_2) = \sum_{j_1=0}^{\infty} P_{\lambda_1}(j_1)\cdot p(\boldsymbol{x};\tfrac{1}{2}+j_1,\tfrac{1}{2},\tfrac{1}{2}),$$

where $p(\boldsymbol{x};\ \frac{1}{2}+j_1,\frac{1}{2},\frac{1}{2}) = p(x_1, x_2;\ \frac{1}{2}+j_1,\frac{1}{2},\frac{1}{2})$ is the density of $D_2(\frac{1}{2}+j_1,\frac{1}{2},\frac{1}{2})$. Furthermore, it follows from Lemma 3.1 that

$$p(x_1, x_2;\ \tfrac{1}{2}+j_1,\tfrac{1}{2},\tfrac{1}{2}) = \frac{\Gamma(\frac{3}{2}+j_1)}{\Gamma(\frac{1}{2}+j_1)\left[\Gamma(\frac{1}{2})\right]^2} x_1^{j_1-\frac{1}{2}} x_2^{-\frac{1}{2}}(1 - x_1 - x_2)^{-\frac{1}{2}}.$$

Hence,

$$p_{H_1}(x_1, x_2) = \sum_{j_1=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^{j_1}}{j_1!}\cdot\frac{\Gamma(\frac{3}{2}+j_1)}{\Gamma(\frac{1}{2}+j_1)\left[\Gamma(\frac{1}{2})\right]^2} x_1^{j_1-\frac{1}{2}} x_2^{-\frac{1}{2}}(1 - x_1 - x_2)^{-\frac{1}{2}}.$$

Since $\Gamma(\frac{3}{2}+j_1) = (\frac{1}{2}+j_1)\Gamma(\frac{1}{2}+j_1)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we have

$$\begin{aligned} p_{H_1}(x_1, x_2) &= \sum_{j_1=0}^{\infty} \frac{e^{-\lambda_1}\lambda_1^{j_1}}{j_1!}\cdot\frac{\left(\frac{1}{2}+j_1\right)\Gamma\left(\frac{1}{2}+j_1\right)}{\Gamma\left(\frac{1}{2}+j_1\right)\left[\sqrt{\pi}\right]^2} x_1^{j_1-\frac{1}{2}} x_2^{-\frac{1}{2}}(1 - x_1 - x_2)^{-\frac{1}{2}} \\ &= \frac{1}{2\pi}e^{-\lambda_1}x_1^{-\frac{1}{2}}x_2^{-\frac{1}{2}}(1 - x_1 - x_2)^{-\frac{1}{2}}\sum_{j_1=0}^{\infty}\frac{\lambda_1^{j_1}}{j_1!}\cdot(1+2j_1)x_1^{j_1}. \end{aligned} \qquad (3.3.2.15)$$

Noting that

$$\sum_{j_1=0}^{\infty}\frac{\lambda_1^{j_1}}{j_1!}\cdot(1+2j_1)x_1^{j_1} = \sum_{j_1=0}^{\infty}\frac{\lambda_1^{j_1}}{j_1!}\cdot x_1^{j_1} + \sum_{j_1=0}^{\infty}\frac{\lambda_1^{j_1}}{j_1!}\cdot 2j_1x_1^{j_1}$$

and

$$\sum_{j_1=0}^{\infty}\frac{\lambda_1^{j_1}}{j_1!}\cdot 2j_1 x_1^{\ j_1} = 2\sum_{j_1=1}^{\infty}\frac{\lambda_1^{j_1}}{(j_1-1)!}\cdot x_1^{\ j_1}$$

$$= 2\lambda_1 x_1 \sum_{j_1=1}^{\infty}\frac{\lambda_1^{j_1-1}}{(j_1-1)!}\cdot x_1^{\ j_1-1}$$

$$= 2\lambda_1 x_1 \sum_{j_1=0}^{\infty}\frac{\lambda_1^{j_1}}{j_1!}\cdot x_1^{\ j_1},$$

equation (3.3.2.15) can be written as

$$p_{H_1}(x_1,x_2) = \frac{1}{2\pi}e^{-\lambda_1}x_1^{-\frac{1}{2}}x_2^{-\frac{1}{2}}\left(1-x_1-x_2\right)^{-\frac{1}{2}}\left(1+2\lambda_1 x_1\right)\sum_{j_1=0}^{\infty}\frac{\lambda_1^{j_1}}{j_1!}\cdot x_1^{\ j_1}. \quad (3.3.2.16)$$

Finally, from equation (3.3.2.16) and $\sum_{j_1=0}^{\infty}\frac{\lambda_1^{j_1}}{j_1!}\cdot x_1^{\ j_1} = \sum_{j_1=0}^{\infty}\frac{1}{j_1!}\cdot(\lambda_1 x_1)^{j_1} = e^{\lambda_1 x_1}$, we have

$$p_{H_1}(x_1,x_2) = \frac{1}{2\pi}e^{-\lambda_1}x_1^{-\frac{1}{2}}x_2^{-\frac{1}{2}}\left(1-x_1-x_2\right)^{-\frac{1}{2}}\left(1+2\lambda_1 x_1\right)e^{\lambda_1 x_1}$$

$$=\frac{1}{2\pi}x_1^{-\frac{1}{2}}(2\lambda_1 x_1+1)e^{\lambda_1(x_1-1)}x_2^{-\frac{1}{2}}(1-x_1-x_2)^{-\frac{1}{2}}.$$

□

From Corollary 3.4, we can calculate the power of $MaxU_r$ for m = 3 with Mathematica for the case of only one active contrast. The powers for $\beta_1 \neq 0$, $\beta_2 = \beta_3 = 0$, m = 3 and r = 2 calculated by Mathematica and determined in simulations are shown in Table 3.3.4 and Table 3.3.5, respectively.

Table 3.3.4: Probabilities of accepting H_0, H_1 and H_{12} for *m* = 3, *r* = 2 obtained with Mathematica and through 10 000 simulations, given one active contrast at position 1.

Size of the active contrast	Probability of accepting					
	H_0		H_1		H_{12}	
	Mathematica	Simulation	Mathematica	Simulation	Mathematica	Simulation
0	0.9500	0.9499	0.0086	0.0087	0.0081	0.0078
0.5	0.9477	0.9486	0.0170	0.0179	0.0100	0.0097
1.0	0.9268	0.9317	0.0419	0.0419	0.0139	0.0122
2.0	0.8221	0.8292	0.1350	0.1324	0.0215	0.0203
3.0	0.6781	0.6875	0.2696	0.2644	0.0262	0.0252
4.0	0.5216	0.5243	0.4218	0.4232	0.0283	0.0271
5.0	0.3740	0.3754	0.5691	0.5705	0.0285	0.0278
10	0.0242	0.0237	0.9364	0.9402	0.0197	0.0194
20	10^{-6}	0	0.9800	0.9821	0.0100	0.0101
50	10^{-39}	0	0.9920	0.9925	0.0040	0.0045

Note: the probability of accepting H_{13}, which is not listed here, should be the same as the probability of accepting H_{12}.

Table 3.3.5: Probabilities of accepting H_2, H_3 and H_{23} for *m* = 3, *r* = 2 obtained with Mathematica and through 10 000 simulations, given one active contrast at position 1.

Size of the active contrast	Probability of accepting					
	H_2		H_3		H_{23}	
	Mathematica	Simulation	Mathematica	Simulation	Mathematica	Simulation
0	0.0086	0.0086	0.0086	0.0086	0.0081	0.0096
0.5	0.0052	0.0056	0.0052	0.0046	0.0049	0.0051
1.0	0.0012	0.0009	0.0012	0.0009	0.0011	0.0013
2.0	10^{-5}	0	10^{-5}	0	10^{-5}	0
3.0	10^{-9}	0	10^{-9}	0	10^{-9}	0
4.0	10^{-15}	0	10^{-15}	0	10^{-15}	0
5.0	10^{-23}	0	10^{-23}	0	10^{-23}	0
10	10^{-87}	0	10^{-87}	0	10^{-88}	0
20	0	0	0	0	0	0
50	0	0	0	0	0	0

Note: the probability of accepting H_3 should be the same as the probability of accepting H_2.

3.4 Examples

To illustrate the test procedure of $MaxU_r$, we present several examples and solve them using the new method proposed in section 3.1.

3.4.1 Example 1

We use Example II of Box and Meyer (1986). The data are from Taguchi and Wu (1980) and listed in Table 3.4.1. This example contains 15 contrasts. Our analysis procedure using the new method $MaxU_r$ is also shown in Table 3.4.1. We assume that we have not any information about how many active contrasts there are and use the version r = 14 of $MaxU_r$ to test if some of the contrasts are active. Two contrasts were declared as active by Box and Meyer (1986) using the assumption of effect-sparsity with EER being about 0.25, while the same result is obtained by $MaxU_r$ with EER = 0.01 only. In fact, the same result can also be obtained when the value of r is set to any integer from 2 to 14. In addition, the largest absolute contrast can be identified as active by $MaxU_r$ even if the version r = 1 is used.

Table 3.4.1: Data and analysis of example 1

k	$L_{15,k}$	MU_k	r	$MaxU_r$	$c_{0.01;15,r}$	Significant	Contrast	Column
1	24.103336	0.9997698	1	0.9997698	0.9993982	Yes	3.10	15
2	96.436203	1.0000000*	2	1.0000000	0.9998618	Yes	2.15	14
3	73.618087	0.9999999	3	1.0000000	0.9999538	No	0.42	10
4	64.319273	0.9999999	4	1.0000000	0.9999780	No	0.40	8
5	63.637503	0.9999997	5	1.0000000	0.9999880	No	0.40	5
6	68.400920	0.9999995	6	1.0000000	0.9999920	No	-0.37	13
7	90.671300	0.9999995	7	1.0000000	0.9999948	No	0.37	7
8	133.26047	0.9999994	8	1.0000000	0.9999966	No	0.30	3
9	131.55091	0.9999966	9	1.0000000	0.9999973	No	0.15	4
10	139.90481	0.9999824	10	1.0000000	0.9999978	No	-0.15	2
11	148.13881	0.9998936	11	1.0000000	0.9999982	No	0.13	12
12	186.84113	0.9994293	12	1.0000000	0.9999983	No	0.13	11
13	687.25792	0.9985462	13	1.0000000	0.9999983	No	0.13	1
14	1205.6270	0.9774306	14	1.0000000	0.9999983	No	-0.05	9
						No	-0.03	6

3.4.2 Example 2

In the second example, we use Example IV of Box and Meyer (1986). This example also contains 15 effects. The data are from Davies (1954) and presented in Table 3.4.2. This table also contains our results of analysis of the experiment using $MaxU_r$.

For this example, the analysis of Box and Meyer (1986) with α_{active} = 0.20 and k =10 suggested no active contrasts at all while their analysis with α_{active} = 0.30 and k =10 suggested contrasts 1, 7, 8, 9, 10, 11 and 14 being active. Our analysis using $MaxU_r$ with r = 14 and EER = 0.10 shows that there are 9 active contrasts, i.e. contrasts 1, 4, 6, 7, 8, 9, 10, 11 and 14.

We feel that the assumption of effect-sparsity may be incorrect for the experiment. Then we used α_{active} = 0.40 and k = 10 to recalculate the posterior probabilities using Box and Meyer's (1986) Bayesian method. The result shows that the column contrasts 1, 4, 6, 7, 8, 9, 10, 11 and 14 all have posterior probabilities greater than 0.80, while the posterior probabilities of the remaining contrasts are smaller than 0.27. This leads to the same conclusions as those obtained by our new method.

Table 3.4.2: Data and analysis of example 2

k	$L_{15,k}$	MU_k	r	$MaxU_r$	$c_{0.10;15,r}$	Significant	Contrast	Column
1	5.6513679	0.9677585	1	0.9677585	0.9934845	Yes	0.274	8
2	7.2977607	0.9924979	2	0.9924979	0.9980245	Yes	-0.251	10
3	8.0718596	0.9967172	3	0.9967172	0.9992156	Yes	-0.191	1
4	9.1004977	0.9983066	4	0.9983066	0.9996129	Yes	-0.161	9
5	11.60396	0.9993376	5	0.9993376	0.9997834	Yes	0.149	7
6	15.522692	0.9997245	6	0.9997245	0.9998602	Yes	0.124	14
7	22.16061	0.9998772	7	0.9998772	0.9999015	Yes	-0.101	11
8	31.632115	0.9999195	8	0.9999195	0.9999257	Yes	-0.076	4
9	64.49245	0.9999720	9	0.9999720	0.9999406	Yes	-0.066	6
10	85.658416	0.9999408	10	0.9999720	0.9999492	No	0.034	5
11	112.78405	0.9998171	11	0.9999720	0.9999545	No	-0.026	12
12	163.73241	0.9993049	12	0.9999720	0.9999569	No	-0.021	2
13	1085.3347	0.9990791	13	0.9999720	0.9999579	No	0.019	15
14	18647.071	0.9942604	14	0.9999720	0.9999581	No	-0.006	13
						No	-0.001	3

3.4.3 Example 3

In this example, the data are from the artificial experiment of Ye *et al*. (2001). There are 15 effects in the experiment. The really active effects are β_9, β_{10}, β_{11}, β_{12}, β_{13}, β_{14} and β_{15} (column 9 – 15). The data and our results of analysis of the experiment using $MaxU_r$ are shown in Table 3.4.3.

For this example, one out of seven really active contrasts was declared as active by the original Lenth method; and five out of seven really active contrasts were identified by the step-down Lenth method proposed by Ye *et al.*(2001). The test $MaxU_r$ with r = 8 reports that there are 8 active contrasts, in which 6 of them are really active and 2 of them are really inactive. The version r = 14 of $MaxU_r$ reports the same result. The new method $MaxU_r$

identifies one really active contrast more than the step-down Lenth method, but it misidentifies 2 inactive contrasts as active. The active effect β_{11} can not be declared as active because the contrast $\hat{\beta}_{11}$ = 0.28 is too small to be identified by any of the three methods.

Table 3.4.3: Data and analysis of example 3

k	$L_{15,k}$	MU_k	r	$MaxU_r$	$c_{0.05;15,r}$	Significant	Contrast	Column
1	8.7595658	0.9896538	1	0.9896538	0.9967511	Yes	5.52	15
2	9.0483958	0.9965487	2	0.9965487	0.9991350	Yes	3.95	14
3	9.4126427	0.9982205	3	0.9982205	0.9996795	Yes	3.08	12
4	10.679287	0.9991244	4	0.9991244	0.9998458	Yes	2.72	13
5	14.751791	0.9997568	5	0.9997568	0.9999110	Yes	2.60	10
6	21.858517	0.9999308	6	0.9999308	0.9999441	Yes	-2.09	5
7	29.810505	0.9999595	7	0.9999595	0.9999630	Yes	-1.47	8
8	63.655578	0.9999925	8	0.9999925	0.9999733	Yes	1.36	9
9	83.835201	*0.9999871*	*9*	*0.9999925*	*0.9999789*	No	0.67	7
10	105.17245	*0.9999644*	*10*	*0.9999925*	*0.9999825*	No	0.50	4
11	151.39713	*0.9998981*	*11*	*0.9999925*	*0.9999847*	No	0.43	2
12	177.57974	*0.9993843*	*12*	*0.9999925*	*0.9999856*	No	0.28	11
13	249.43569	*0.9960002*	*13*	*0.9999925*	*0.9999860*	No	-0.25	6
14	14137.393	*0.9934082*	*14*	*0.9999925*	*0.9999860*	No	0.22	1
						No	-0.02	3

Furthermore, it is worth noting that our results are obtained at the level 0.05 (EER) while the results of Ye *et al.* (2001) were obtained at the level 0.10 (EER).

3.4.4 Example 4

The data are from Quinlan's (1985) work, in which he used a saturated 16-run experiment to study the effects of 15 variables on the amount of post extrusion shrinkage (y) of speedometer cables, assigning factors *A*-*O* in order to the 15 columns of an L_{16}orthogonal array. For each run, shrinkage measurements were made at four locations on a single 3000-ft length of cable. His analysis showed that there were eight active effects (*E*, *G*, *A*, *C*, *D*, *F*, *H*, and *K*).

The data were reanalyzed by Box (1988). His analysis was based on the average of ln(y), which he showed to be an appropriate transformation. He concluded that there were only two active effects by using the method of Box and Meyer (1986) and a subjective analysis of a normal probability plot. Schneider *et al.* (1993) also concluded the same results as those by Box (1988) after analyzing Quinlan's (1985) data with four different analysing methods (Box and Meyer (1986), Lenth (1989), Benski (1989), and Schneider *et al.* (1993)).

As done by Box (1988), our analysis using $MaxU_r$ with r = 14 is also based on the average of ln(y), namely, -0.73, -0.55, -2.45, -1.75, -1.66, -1.94, -1.50, -1.76, -2.13, -2.31, -0.79, -0.63, -1.79, -1.29, -1.08, and -0.54 for each of the 16 runs, respectively. The calculated contrasts and analysis results are listed in Table 3.4.4. Our results support the Quinlan's (1985) conclusion which was supported by the confirmation experiments of Quinlan (1985). Note that our results are obtained at EER = 0.10 (IER = 0.055) while Quinlan's (1995) results were obtained at IER = 0.363 when all contrasts are inactive (cf. Hurley (1995) for details).

Table 3.4.4: Data and analysis of example 4

k	$L_{15,k}$	MU_k	r	$MaxU_r$	$c_{0.10;15,r}$	Significant	Contrast	Column
1	14.084433	0.9978590	1	0.9978590	0.9934845	Yes	0.44125	5
2	18.052097	0.9998228	2	0.9998228	0.9980245	Yes	0.30125	7
3	16.016998	0.9998300	3	0.9998300	0.9992156	Yes	0.15875	3
4	17.185093	0.9998935	4	0.9998935	0.9996129	Yes	-0.15500	11
5	17.680092	0.9998886	5	0.9998935	0.9997834	Yes	-0.11875	4
6	20.006434	0.9999007	6	0.9999007	0.9998602	Yes	-0.11125	1
7	26.958762	0.9999409	7	0.9999409	0.9999015	Yes	0.10625	6
8	38.791507	0.9999594	8	0.9999594	0.9999257	Yes	-0.08500	8
9	48.559881	0.9999357	9	0.9999594	0.9999406	No	0.05750	12
10	69.888385	0.9999022	10	0.9999594	0.9999492	No	0.05000	14
11	147.99940	0.9998934	11	0.9999594	0.9999545	No	0.04250	9
12	217.70702	0.9995459	12	0.9999594	0.9999569	No	-0.02250	10
13	232.93340	0.9957175	13	0.9999594	0.9999579	No	0.01375	2
14	277.23996	0.9529582	14	0.9999594	0.9999581	No	0.01250	15
						No	0.01000	13

CHAPTER 4

Simulation Study to Compare the Methods

In this chapter, we will compare 13 quantitative methods with the help of computer simulations. In our simulations we use slightly modified versions of the methods in order to put them all on the same global level and thus allow comparisons among them. We therefore begin the report on the study by describing the versions that we have used in some detail. Next, details of the computer simulation procedure are given, and the criteria are discussed that we use for the comparison of the methods. Finally, the results are presented with the help of some tables and graphics. In section 4.1, we study only the orthogonal case.

4.1 Orthogonal case

In order to examine the performance of the new method proposed in chapter 3, we used an extensive simulation study based on 10000 samples for each of the parameter combinations considered. All were done under the condition of normal distribution. We compared the new method with 12 existing methods.

Several other simulation studies comparing some of the available methods for the orthogonal case have been published by, e.g., Zahn (1975b), Voss (1988), Berk and Picard (1991), Loh (1992), Dong (1993), Benski and Caban (1995), Haaland and O'Connell (1995), Loughin and Noble (1997), Hamada and Balakrishnan (1998), Lawson *et al*. (1998), Al-Shiha and Yang (1999), Aboukalam and Al-Shiha (2001). The study by Hamada and Balakrishnan (1998) is one of the most detailed ones. They reviewed many existing methods for objectively analyzing unreplicated fractional factorial designs and compared them via an extensive simulation study.

In contrast to the simulation study of Hamada and Balakrishnan (1998), we control the global error rate of falsely rejecting the global hypothesis at 0.05, instead of the individual error rate (IER). Additionally, we do not restrict attention to the cases where all active contrasts have the same magnitudes. It is much more realistic to assume that the magnitudes of the active contrasts are different. As evaluation standard we use the power, as well as the loss of decision (explained below).

The results show that the compared methods perform differently in different cases. Furthermore, the outcomes depend on the criteria used.

4.1.1 Compared methods

The compared methods are denoted as follows:

Al-SY99	Al-Shiha and Yang (1999),
Ben89	Benski (1989),
Bi89	Bissell (1989),
BM86	Box and Meyer (1986),
Dong93	Dong (1993),
JP92	Juan and Pena (1992),
JTuk87	Johnson and Tukey (1987),
LGB98	Lawson, Grimshaw and Burt (1998),
Len89	Lenth (1989),
LN97	Loughin and Nobel (1997),
MaxUr	The new method,
MLZ92	Modified Le and Zamar (1992),
STuk82	Seheult and Tukey (1982).

Each of the methods has an upper limit on the number of factors that it can declare active. With a 16 run design this upper limit is at least 6, except for **LGB98**, for which it is only 4. The methods **AL_SY99**, **LGB98**, **LN97** and **MaxUr** have not been studied in the paper by Hamada and Balakrishnan (1998), while the other methods were considered in that paper.

In order to fairly compare the methods we have to standardize them somehow. We have chosen to control the probability of falsely rejecting the null-hypothesis of no active contrasts at all. Table 4.1 gives the off-the-shelf performance of these standardized versions, when all contrasts are inactive. The table is based on 10000 simulations of a 16 run experiment and normal errors. The row with 0 contrasts should contain 95% of the experiments for each method. Table 4.1 shows that this standardization seems to have worked out.

Table 4.1: Off-the-shelf performance of existing methods. p_i = observed proportion of simulations detecting i contrasts under all inactive contrasts for a 16 run experiment

Number of contrasts declared active	Methods											
	Al-SY99	Ben89	Bi89	BM86	Dong93	JP92	JTuk87	Lenth89	LGB98	LN97	MaxUr	MLZ92
0	0.9495	0.9481	0.9485	0.9491	0.9488	0.9500	0.9518	0.9499	0.9567	0.9493	0.9500	0.9500
1	0.0260	0.0342	0.0194	0.0397	0.0394	0.0252	0.0344	0.0308	0.0249	0.0403	0.0000	0.0228
2	0.0091	0.0085	0.0065	0.0064	0.0067	0.0086	0.0075	0.0094	0.0141	0.0063	0.0007	0.0114
3	0.0045	0.0038	0.0049	0.0024	0.0033	0.0061	0.0027	0.0058	0.0032	0.0029	0.0007	0.0068
4	0.0032	0.0019	0.0031	0.0012	0.0011	0.0028	0.0019	0.0021	0.0011	0.0004	0.0031	0.0031
5	0.0031	0.0009	0.0018	0.0004	0.0004	0.0023	0.0008	0.0009	0	0.0005	0.0058	0.0019
6	0.0019	0.0004	0.0015	0.0003	0.0002	0.0020	0.0003	0.0007	0	0.0000	0.0080	0.0017
7	0.0027	0.0008	0.0026	0.0005	0.0001	0.0016	0.0006	0.0004	0	0.0003	0.0107	0.0016
8	0	0.0001	0.0020	0	0	0.0005	0	0	0	0	0.0210	0.0004
9	0	0.0007	0.0010	0	0	0.0004	0	0	0	0	0	0.0001
10	0	0.0006	0.0015	0	0	0.0003	0	0	0	0	0	0.0001
11	0	0	0.0019	0	0	0.0001	0	0	0	0	0	0.0001
12	0	0	0.0011	0	0	0.0001	0	0	0	0	0	0
13	0	0	0.0014	0	0	0	0	0	0	0	0	0
14	0	0	0.0028	0	0	0	0	0	0	0	0	0
EER	0.0505	0.0519	0.0515	0.0509	0.0512	0.0500	0.0482	0.0501	0.0433	0.0507	0.0500	0.0500
IER	0.00775	0.00639	0.01515	0.00479	0.00473	0.00797	0.00501	0.00579	0.00447	0.00452	0.02239	0.00770

To control the global error rate (EER) for the case of no active contrasts at 0.05, some of the methods had to be modified rather than using the original versions. For **MaxUr**, the version with $r = 8$ is used. The critical values of **MaxUr** are given in appendix A3.

For **Al-SY99**, we use the version with $k = 7$ and the 95% quantiles given in Al-Shiha and Yang's (2000) Table 2.2 as critical values in order to control EER at 0.05.

For **Ben89**, we use 0.05 as the significance level of the test W'. That is, if $P_1 < 0.05$, then reject the null hypothesis. For the combined test, we reject the null hypothesis if $P_c <$ 0.05, otherwise we stop the procedure. In fact, P_c is always smaller than 0.05 when $P_1 < 0.05$ and $m \geq 3$.

For **Bi89**, in the original paper, the critical values were given based on an approximation using an *F*-distribution. The critical values of **Bi89** that we used were obtained through 10000 simulations and are given in Table 4.2.

Table 4.2: Critical values $C_{\alpha,k}$ for **Bi89**

Sample sizes k	Level α 0.10	0.05	0.01
2	1.8413986	2.5256635	4.1879632
3	2.0178575	2.6008280	3.8407102
4	2.0287096	2.5086685	3.6319083
5	2.0047258	2.4386931	3.2654993
6	1.9844773	2.3221077	3.1430306
7	1.9363890	2.3109622	3.0162070
8	1.9409944	2.2400427	2.8835145
9	1.8979222	2.1898496	2.7443001
10	1.8668126	2.1375358	2.6983155
11	1.8561824	2.1050430	2.6137084
12	1.8325930	2.0768316	2.5585057
13	1.8253131	2.0341312	2.4888220
14	1.8074164	2.0222958	2.4924073
15	1.7896837	2.0015712	2.4350308

In the original version of **BM86**, it was recommended that effect i is considered active if the calculated marginal posterior probability $p_i > p_c = 0.5$. In this case, however, EER is about 0.25 rather than 0.05. To control EER at 0.05, we increase the value of p_c from 0.5 to 0.8872372, which is obtained with $\alpha = 0.2$ and $k = 10$ by 10000 simulations for a 16 runs factorial experiment with all contrasts being inactive. In addition, we computed the marginal posterior probability p_i by summing over 2^m combinations instead of calculating the integral (2.1.4) through numerical integration which is difficult with SAS.

For **Dong93**, we use the same version of ASE as Kunert (1997) and the statistic

$$\frac{|\hat{\beta}_i|}{\text{ASE}} > c \qquad (4.1.1)$$

to test whether a contrast $\hat{\beta}_i$ is active, where *c* is a critical value for the distribution of $\max_i\left(\left|\hat{\beta}_i\right|/\text{ASE}\right)$ under H_0 that was derived via Monte-Carlo simulations. For our study with 15 contrasts and a level 0.05 we used 3.794. This is slightly larger than 3.68 that Kunert (1994) proposed on the basis of his approximations.

For **JP92**, the values of w_c used here are shown in Table 4.3. They are obtained with *w* = 3.5 through 10000 simulations.

Table 4.3: the values of w_c for **JP92** when *w* =3.5

level	Runs of design		
α	8	16	32
0.10	5.9619572	5.7103877	5.4165376
0.05	8.7767630	7.2300000	6.3488788
0.01	21.732811	13.101929	9.2673505

For **JTuk87**, we use the 95% quantile given in Johnson and Tukey's (1987, pp. 201) Table 12 to control EER at 0.05. The approximate formulas in their Table 12 are used as well.

For **Len89**, we use SME to determine the active contrasts, i.e., if $|\hat{\beta}_i|$ > SME = $t_{\gamma;d}\times$PSE, the contrast $\hat{\beta}_i$ is considered as active, where $d = m/3$ and $\gamma = \frac{1}{2}\left(1+(0.883)^{\frac{1}{m}}\right)$ rather than the original $\gamma = \frac{1}{2}\left(1+(0.95)^{\frac{1}{m}}\right)$. For n = 16, the $t_{\gamma;d}$ used is equal to 4.23. Note that this is considerably larger than the corresponding value for **Dong93**.

The description of **LGB98** by Lawson, et al. (1998) is not very precise. They refer to Loh (1992) for details. We tried to follow the description in four steps that they gave in section 3.1 and section 4 of their paper. To calculate *C*, we use the smallest integers that are not smaller than $\frac{1}{4}m$ and $\frac{3}{4}m$ to derive the first and third quartiles Q_1 and Q_3, respectively. This is the same approach as was used in Lawson, et al.'s (1998) example. It should be noted that the 95% Scheffé prediction band that we computed according to their description in section 3.1 is different from the band given in their example (see Table 4.4). Their results may have been calculated with the SAS procedure REG with options MODEL y = x / NOINT CLI, which does not produce Scheffé prediction limits (see, e.g., Seber, 1977, p. 186). To keep the original performances of **LGB98**, we used the 75.8% quantile $F_{m',m_{inactive}-1;0.758}$ (see Loh (1992) for the notation) instead of the proposed 95% quantile to compute the Scheffé prediction bands. The upper Scheffé prediction limits calculated in such a way approximate the limits given in their example. In addition, we found that after the null hypothesis has been rejected by the test R_n, sometimes no contrasts could be declared as active by using $|\hat{\beta}_i| > C^*$ in the last step of **LGB98**'s procedure. A study of 10000 simulations using the critical value 1.201 showed that the null hypothesis was rejected 517 times by the test R_n. Among these there are 84 cases where no contrasts fulfilled $|\hat{\beta}_i| > C^*$, however. This is the reason why the EER of **LGB98** in Table 4.1 is only 0.0433.

Table 4.4: Upper prediction limits of Scheffé prediction bands calculated with the data in Lawson, Grimshaw and Burt's (1998) example using various approaches.

Label	Contrast	Ordered absolute contrast	z-score $\Phi^{-1}((p+1)/2)$	Lawson *et al.*'s 95% pred. limit	Pred. limit using quant. $F_{m',m_{inactive}-1;0.758}$	Pred. limit using quant. $F_{m',m_{inactive}-1;0.95}$
AC	0.00	0.00	0.0417893	0.0152	0.0152244	0.0219055
ABC	0.00	0.00	0.1256613	0.0152*	0.0184345	0.0251246
AB	-0.01	0.01	0.2104284	0.0216	0.0216975	0.0284057
BD	-0.01	0.01	0.2967378	0.0250	0.0250388	0.0317748
BCD	0.01	0.01	0.3853205	0.0284	0.0284878	0.0352619
BC	-0.02	0.02	0.4770404	0.0321	0.0320795	0.0389033
ABD	0.02	0.02	0.5729675	0.0358	0.0358580	0.0427444
ACD	0.02	0.02	0.6744898	0.0399	0.0398807	0.0468451
ABCD	0.02	0.02	0.7835004	0.0442	0.0442265	0.0512875
AD	0.03	0.03	0.9027348	0.0490	0.0490098	0.0561912
CD	0.04	0.04	1.0364334	0.0544	0.0544088	0.0617425
A	0.06	0.06	1.1918162	0.0607	0.0607274	0.0682597
D	0.14	0.14	1.3829941	0.0686	0.0685615	0.0763675
B	0.25	0.25	1.6448536	0.0794	0.0793875	0.0876152
C	0.50	0.50	2.1280452	0.0996	0.0996045	0.1087282

* the prediction limit 0.0152 given in the original table 3 equals to the one of the smallest absolute contrast. This can be assumed to be due to mistyping.

For **LN97**, as in Loughin and Nobel's (1997) study, we select N = 1000 for the permutation test in the step 3) of the testing procedure and use the function UNIFORM( ) of SAS/IML to generate the random permutations. Additionally, we used the 95% quantile (0.042) to control the EER at 0.05 rather than 0.135 to control the EER at 0.20.

For **MLZ92**, we use the version that was modified by Hamada and Balakrishnan (1998), i.e. the ρ-function $\rho_2(x)=x^2$ rather than the original $\rho_2(x)=\rho_1+b\cdot(x^4-6x^2)$ proposed by Le and Zamar because of the practical problem that it has two roots. Besides, we used T = 0 for the situation when the location parameter is known. Under the above conditions, we obtain

$$S_1=\sqrt{\frac{1}{a\cdot m}\sum_{|\hat{\beta}_i|<cS_1}\hat{\beta}_i^2} \tag{4.1.2}$$

and

$$S_2=\sqrt{\frac{1}{m}\sum_{i=1}^{m}\hat{\beta}_i^2}\,, \tag{4.1.3}$$

where $a=\Gamma\left(\frac{c^2}{2},\frac{3}{2}\right)+2c^2\left[1-\Phi(c)\right]-\frac{N_{S_1}}{m}c^2$, c = 0.9, $\Gamma\left(x,\frac{3}{2}\right)$ is the Gamma-distribution function with parameter 1.5, i.e. $\Gamma\left(\frac{c^2}{2},\frac{3}{2}\right)=\frac{1}{\Gamma(\frac{3}{2})}\int_0^{\frac{c^2}{2}}x^{\frac{1}{2}}e^{-x}dx$, $\Phi(\cdot)$ is the standard normal distribution and N_{S_1} is the number of $\hat{\beta}_i$ that satisfy $|\hat{\beta}_i/c|\geq|S_1|$. The critical values for **MLZ92** were obtained through 10000 simulations and are given in Table 4.5.

Table 4.5: Critical values $R_{\alpha,k}$ for **MLZ92**

Sample sizes k	Level α 0.10	0.05	0.01
2	2.6991601	5.3632900	28.294023
3	1.6279101	2.3360387	5.1284433
4	1.7518075	2.4191278	5.3970638
5	1.5575284	1.9883191	3.5217565
6	1.5394742	1.9424356	3.1579346
7	1.4884376	1.8111432	2.9259179
8	1.4570464	1.7620456	2.6587059
9	1.4223239	1.6682851	2.3625043
10	1.3919250	1.6276035	2.2397940
11	1.3735141	1.5942978	2.1514033
12	1.3645874	1.5674985	2.0678256
13	1.3527291	1.5258623	1.9779860
14	1.3373584	1.5023086	1.9541169
15	1.3192506	1.4799562	1.8902463

For **STuk82**, the EER of the original version is about 0.27 for 16-runs designs. To control the EER at 0.05, instead of the original critical value 4, we use the critical values C_m which were obtained through 10000 simulations and are listed in Table 4.6.

Table 4.6: Critical values C_m for **STuk82**

Sample sizes k	Level α 0.10	0.05	0.01
2	12.530899	24.958304	131.76932
3	5.8932323	8.6034664	19.413810
4	7.3298856	10.552486	24.620596
5	5.7276718	7.4537168	13.831914
6	6.1918139	8.0524351	14.379224
7	5.5673294	6.9006177	11.325000
8	5.7546699	7.1922633	11.729675
9	5.3525109	6.5083849	9.9647511
10	5.4849873	6.6532008	9.9044538
11	5.2834265	6.2594431	8.9685121
12	5.3964665	6.4368943	8.9622199
13	5.1766706	6.0975776	8.2513403
14	5.2812877	6.1557258	8.4509843
15	5.1267314	5.9890203	7.9098891

4.1.2 Description of the simulation study

The simulation study was based on 10000 samples and normal errors using the function RANNOR from SAS/IML. For n = 16 runs, one, two, four, six and eight active contrasts all having the same magnitude varying between 0.5σ and 4.0σ (0.5σ, σ, 1.5σ, 2σ, 2.5σ, 3σ, 4σ) were studied. In addition, two, four, six and eight active contrasts with different magnitudes, in which the magnitude of the smallest active contrasts equals a fourth of the magnitude of the largest, were studied. More precisely, we used β_1= 0.2*Size*, β_2 = 0.8*Size* for the case of two active contrasts, β_1=β_2=0.2*Size*, β_3=β_4=0.8*Size* for four active contrasts, β_1=β_2=0.2*Size*, β_3=β_4=0.5*Size*, β_5=β_6=0.8*Size* for six active contrasts, and β_1=β_2=0.2*Size*, β_3=β_4= β_5=β_6=0.5*Size*, β_7=β_8= 0.8*Size* for eight active contrasts. Here *Size* = 2.0σ, 4.0σ, 6.0σ, 8.0σ, 10σ, 20σ. The magnitudes of the contrasts that are not listed above was equal to zero.

We first generated a standard normal sample $\varepsilon_1,\ldots,\varepsilon_n$ of size n = 16. Then, we obtained the observation vector $\boldsymbol{y}$ by $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{\varepsilon}$ and the contrasts $\hat{\beta}_1,\ldots,\hat{\beta}_m$ with $\hat{\boldsymbol{\beta}}$ = $\left(\hat{\beta}_0,\hat{\beta}_1,\ldots,\hat{\beta}_m\right)'$ = $(\boldsymbol{X'X})^{-1}\boldsymbol{X'y}$, where $\boldsymbol{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_n)'$ and $m = n - 1 = 15$. On these data, we applied the respective methods to identify active factors. The above procedure was repeated 10000 times for each of thirteen compared methods, and the contrast combinations which were declared active were recorded. From this some results are listed below.

4.1.3 Evaluation standards

It is clear that the evaluation standards will largely influence the results. Therefore, we had to carefully select an appropriate standard. Some evaluation standards have been proposed in the literature. For example, the experimentwise error rate (EER) and the average proportion of inactive effects declared active (IER) have been used (Loughin & Noble (1997), Hamada & Balakrishnan (1998), etc.). Benski and Cabau (1995) proposed a Figure of *Merit Q*. In the comment on the paper by Hamada & Balakrishnan (1998), Lenth suggested the ”decent-chance detection capability” (DCDC), which is the effect size that can be detected with 50% power when the size of the test is α. Aboukalam and Al-Shiha (2001) suggested using the probability that an effect is considered inactive given that it is really an inactive effect (pow_{II}), which is in fact nothing than the compliment (1 – IER) that was used by Hamada and Balakrishnan (1998).

We use empirical power and the loss of decision (LD) to evaluate these methods. The power was studied in five forms – **power, power I**, **power II**, **power III** and **power IV**. Three versions of the loss of decision **LD2L**, **LD1L0** and **LD1L9** were also studied.

Power is defined by the expected fraction of active contrasts that are declared being active, which was used by many authors (e.g., Hamada and Balakrishnan (1998), Ye *et al.* (2001).). **Power I** is defined by the probability of rejecting H_0. **Power II** is defined by the probability that the contrasts declared being active include all really active contrasts. **Power III** is defined by the probability that the contrasts declared being active are exactly the really active contrasts. **Power IV** is defined by the probability that the contrasts declared being active include all of the really active contrasts except for the smallest really active contrasts, i.e. the probability that all really active contrasts with the magnitude 0.5*Size* and 0.8*Size* are identified.

We use **power** and **power I** as evaluation standards because they are two of the most often used standards to evaluate a test.

Unreplicated fractional experiments are frequently used in the screening stage of industrial experimentation. The main objective of a screening experiment is to identify among many factors a few significant factors for further studies. As said earlier, misjudging inactive effects as being active may not be a serious mistake in this case because we have the opportunity of making a confirmation experiment to check our predictions. Hence, **power II** seems to be a sensible evaluation standard.

Clearly, the most ideal method is that would exactly choose the truly active contrasts with probability 1. **Power III** measures the ability of exactly identifying the really active contrasts. **Power IV** measures the ability of identifying the relatively large active contrasts from all active contrasts. Therefore, we use it as an evaluation standard for the case of active contrasts with different magnitudes.

One of the disadvantages of the criteria **power**, **power I ~ IV** is that they only consider the ability of a method to detect the active contrasts, but do not measure the error rate of a method to misjudge inactive contrasts as active. The *Merit Q* proposed by Benski and Cabau (1995) seems to be the only known criterion which considers both measurements described above at the same time. The *Merit Q* is defined by

$$Q = \frac{n^+}{N^+} \cdot \left(1 - \frac{n^-}{N^-}\right), \tag{4.1.4}$$

where N^+ is the total number of really active effects that could potentially be detected, n^+ is the number of real effects actually detected, N^- is the total number of inactive effects present and n^- is the actual number of inactive effects wrongly identified as active.

However, *Merit Q* gives the same measurement when an active effect is declared as inactive as when an inactive effect is declared as active. That might be not suitable in the screening stage. The main objective of a screening experiment is to identify a few significant factors for further studies. Misidentification of inactive effects as being active may not be a serious mistake, as long as all important factors have been identified. But failure to find an important factor might be fatal for our aim to optimize the process (see also the comment by Haaland in Hamada & Balakrishnan (1998)). Hence, the above two cases should be measured differently.

One of the possible approaches that satisfy the above conditions is to use the loss of decision (LD).

Obviously, we do not just have a simple hypothesis problem — only to reject the null hypothesis (no active contrasts exist). We want to judge how many active contrasts there are and which contrasts are active. In all, the inference problem of identification of active contrasts in unreplicated designs is not a simple hypothesis test, but a multi-decision problem (Birnbaum (1961), Benski (Hamada & Balakrishnan (1998), Comment)). Formally, the inference problem can be described as a choice among the following set of hypotheses:

H_0: β_i=0; for $i = 1,\ldots,m$ (the global null-hypothesis of no active effects)

For $1 \le i_1 \le m$: H_{i_1} : $\beta_{i_1} \ne 0, \beta_i$=0, for $i \ne i_1$ and $i = 1,\ldots,m$;

For $1 \le i_1 < i_2 \le m$: $H_{i_1 i_2}$: $\beta_{i_1} \ne 0$, $\beta_{i_2} \ne 0, \beta_i$=0, for $i \ne i_1, i_2$, and $i = 1,\ldots,m$; (4.1.5)

⋮

For $1 \le i_1 < i_2 < \ldots < i_r \le m$: $H_{i_1 i_2 \ldots i_r}$: β_{i_1} β_{i_2} ... $\beta_{i_r} \ne 0$, β_i=0, for $i \ne i_1,\ldots,i_r$, and $i = 1,\ldots,m$;

where $r\,(\le m)$ is the largest number of effects which are possibly active. Clearly, there are $\sum_{i=1}^{r}\binom{m}{i}$ alternative hypotheses vs. the global null hypothesis. Our objective is to decide, on the basis of one observation vector $(\hat\beta_1,\ldots,\hat\beta_m)'$, which of the above $\sum_{i=0}^{r}\binom{m}{i}$ hypotheses is true. Note that there is some resemblance to multiple testing procedures (see e.g. Sonnemann (1982)). However, as said before, a test which keeps the multiple level α does not seem appropriate here. This “multiple level α” is the same as the EER.

The loss of decision is defined as the expectation of a loss function. It is given by

$$\mathbf{LD} = E\left[L(d_T, H_{j_1\cdots j_q})\right]. \qquad (4.1.6)$$

When $H_{j_1\cdots j_q}$ is true and the decision d_T is given, the loss function is defined by

$$L(d_T, H_{j_1\cdots j_q}) = a \times \left\|\Theta_{j_1\cdots j_q} - d_T\right\| + b \times \left\|d_T - \Theta_{j_1\cdots j_q}\right\|, \qquad (4.1.7)$$

where $\|S\|$ denotes the number of elements in set S, $\Theta_{j_1\cdots j_q}$ denotes the set $\{j_1, j_2,\ldots, j_q\!: \; 1 \le j_1 < j_2 < \ldots < j_q \le r\}$, a is the coefficient of loss when missing an important effect and b is the coefficient of loss when misjudging an inactive effect as active. The decision function is given by

$$d_T = \begin{cases} d_0 = \Theta_0 = \varnothing & \text{when no effect is identified as active by } T \\ d_{i_1\ldots i_k} = \Theta_{i_1\ldots i_k} & \text{when } \beta_{i_1}\ldots\beta_{i_k} \text{ are identified as active by } T \end{cases}, \qquad (4.1.8)$$

which depends on the test statistic T.

In our simulation study, we used the loss functions

$$L_{1L0}(d_{i_1\ldots i_k}, H_{j_1\cdots j_q}) = \left\|\Theta_{j_1\cdots j_q} - \Theta_{i_1\ldots i_k}\right\| \qquad (4.1.9)$$

and

$$L_{1L9}(d_{i_1\ldots i_k}, H_{j_1\cdots j_q}) = 0.9 \times \left\|\Theta_{j_1\cdots j_q} - \Theta_{i_1\ldots i_k}\right\| + 0.1 \times \left\|\Theta_{i_1\ldots i_k} - \Theta_{j_1\cdots j_q}\right\|, \qquad (4.1.10)$$

where $L_{1L0}(d_{i_1 \dots i_k}, H_{j_1 \dots j_q})$ gives a loss only when important factors are missed. $L_{1L9}(d_{i_1 \dots i_k}, H_{j_1 \dots j_q})$ gives loss 0.9 for each important factor that is missed, and gives loss 0.1 for each inactive contrast misidentified as being active.

In our simulation study, we also studied the following squared loss function

$$L_{2L}(d_{i_1 \dots i_k}, H_{j_1 \dots j_q}) = \left\| \Theta_{i_1 \dots i_k} \, \Delta \, \Theta_{j_1 \dots j_q} \right\|^2, \qquad (4.1.11)$$

where Δ is the symmetric difference operator. $L_{2L}(d_{i_1 \dots i_k}, H_{j_1 \dots j_q})$ gives the same loss for misidentifying an inactive contrast as being active as for an important factor being missed. The LDs used by us are then given by

$$\textbf{LD1L0} = E\left[L_{1L0}(d_T, H_{j_1 \dots j_q})\right], \qquad (4.1.12)$$

$$\textbf{LD1L9} = E\left[L_{1L9}(d_T, H_{j_1 \dots j_q})\right]. \qquad (4.1.13)$$

$$\textbf{LD2L} = E\left[L_{2L}(d_T, H_{j_1 \dots j_q})\right], \qquad (4.1.14)$$

It is worth noting that in principle criterion **LD1L0** is equivalent to **power** and **LD2L** is nearly equivalent to *Merit Q*.

4.1.4 Conclusions from the simulation study

We now discuss some results from our simulation study. We obtain the following results:

- Only **Ben89**, **Bi89**, **BM86**, **MaxUr** and **MLZ92** are working properly for the case where there are eight active contrasts with the same magnitude, while the others fail to reject the null hypothesis in this case.

- According to **power**, **MaxUr** is absolutely more powerful than **Al-SY99** in all studied cases. For the case of active contrasts with different magnitudes, **MaxUr** is the most powerful among the compared methods over all studied cases. For the cases of all active contrasts having the same magnitude, **MaxUr** performs best if the number of active contrasts is large (i.e. more than 2).

- According to **power I**, **MaxUr** is more powerful than **Al-SY99** except that **Al-SY99** performs a little better than **MaxUr** in the case of six active contrasts with the same magnitude. However, **Al-SY99** is one of the worst three methods when the number of active contrasts is not larger than 2.

- It is clear that there are mainly two groups among these methods. The first group is made up of **Al-SY99**, **JP92**, **MaxUr** and **MLZ92**. The second group consists of the rest of the methods except for **Bi89**, which is a special case and does not perform well in the range from one to six active contrasts. Overall, the first group performs better than the second if the number of active contrasts is large (larger than two for the same magnitude, larger than four for different magnitudes, say); however, the second does better than the first if there is a small number of active contrasts. In the first group, although none has an absolute superiority over the others and there are not much differences among them in the range up to six active contrasts, the method **MaxUr** always stands at least the second best position.

● According to **power II**, **MaxUr** is also absolutely more powerful than **Al-SY99** in all of the studied cases. **Al-SY99** is one of the worst three methods in the cases where the number of active contrasts is not larger than 2.

• **MaxUr** is the most powerful among the compared methods except for two special cases. The first case is that there is only one active contrast. The second is that there are two active contrasts with the same magnitude 1.5σ.

● According to **power III**, in the cases where active contrasts have different magnitudes, **MaxUr** is more powerful than **Al-SY99** except for the cases of two active contrasts with *Size* 10σ, four active contrasts with *Size* 10σ and 20σ, and six active contrasts with *Size* not smaller than 8.0σ.

• For the cases where active contrasts all have the same magnitude, **Al-SY99** is more powerful than **MaxUr** except for the case where there are eight active contrasts, in which **MaxUr** is the most powerful among the compared methods.

● According to **power IV**, **MaxUr** is more powerful than **Al-SY99**. **MaxUr** also performs best in the cases where there are six or eight active contrasts. For four active contrasts, **MaxUr** is one of the best three methods while **Al-SY99** is one of the worst three methods when the number of active contrasts is 2 or 4.

● According to **LD1L0**, **MaxUr** gives absolute lower loss than **Al-SY99** in all of the studied cases. **Al-SY99** is one of the worst three methods when the number of active contrasts is not larger than 2, but **MaxUr** is not. Besides, **MaxUr** is almost the best one except for the cases where the number of active contrasts is not larger than 2 and all active contrasts have the same magnitude.

● According to **LD1L9**, in the cases where active contrasts have different magnitudes, **MaxUr** has lower loss than **Al-SY99** except for the cases of six active contrasts with *Size* equal to 10σ and 20σ, in which case **MaxUr** gives a slightly higher loss than **Al-SY99**. In the cases where active contrasts all have the same magnitude, **MaxUr** has lower loss than **Al-SY99** except for the cases of one active contrast with magnitude $0.5\sigma \sim 4.0\sigma$, two and four active contrasts with magnitude $2.5\sigma \sim 4.0\sigma$ and six active contrasts with magnitude $2.0\sigma \sim 4.0\sigma$.

● According to **LD2L**, when the number of active contrasts is not larger than 4 and all active contrasts have the same magnitude, **Al-SY99** gives absolute lower loss than **MaxUr**. In the cases of 2 active contrasts with different magnitude, **MaxUr** has lower loss than **Al-SY99** except when the *Size* is not larger than 6.0σ. In the cases of 4 active contrasts with different magnitude, **MaxUr** is better than **Al-SY99** except when the *Size* equals 2.0σ or 10σ. For 6 active contrasts with the same magnitude, **MaxUr** is better than **Al-SY99** when the magnitude is 0.5σ or 1.0σ. For 6 active contrasts with different magnitude, **MaxUr** performs better than **Al-SY99** when the *Size* is not larger than 6.0σ. For 8 active contrasts, **MaxUr** is absolutely better than **Al-SY99**.

For a more detailed impression, see the figures and tables in appendix A1 and A2.

Appendix A1.1 shows the diagrams of the simulated powers for the active contrasts with the same magnitude, in which Figures A1.1.1 – A1.1.5 display **power,** Figures A1.1.6 – A1.1.10 display **power I**, Figures A1.1.11 – A1.1.15 display **power II**, and Figures A1.1.16 – A1.1.20 display **power III**. Appendix A1.2 shows the diagrams of the simulated powers for the active contrasts with different magnitudes, in which Figures A1.2.1 – A1.2.4 display **power,** Figures A1.2.5 – A1.2.8 display **power I**, Figures A1.2.9 – A1.2.12 display **power II**, Figures A1.2.13 – A1.1.16 display **power III**, and Figures A1.2.17 – A1.2.20 display **power IV**. Appendix A1.3 shows the tables of the simulated powers for active contrasts with the same magnitude. Appendix A1.4 shows the tables of the simulated powers for active contrasts with different magnitudes.

Appendix A2.1 shows the diagrams of the simulated LDs for the active contrasts with the same magnitude, in which Figures A2.1.1 – A2.1.5 display **LD1L0**, Figures A2.1.6 – A2.1.10 display **LD1L9**, and Figures A2.1.11 – A2.1.15 display **LD2L**. Appendix A2.2 shows the diagrams of the simulated LDs for the active contrasts with different magnitudes, in which Figures A2.2.1 – A2.2.4 display **LD1L0**, Figures A2.2.5 – A2.2.8 display **LD1L9**, and Figures A2.2.9 – A2.2.12 display **LD2L**. Note that these figures practically show the values of the function *C* – LD instead of the LD so that they can be shown more clearly, where *C* is a constant.

To sum up, **MaxUr** performs better than **Al-SY99** in most of the studied cases. **MaxUr** also performs very well over a wide range of numbers of active contrasts according to most of the used criteria: from 2 to 8 active contrasts by **power**, from 4 to 8 active contrasts by **power I**, from 2 to 8 active contrasts by **power II**, from 4 to 8 active contrasts by **power IV**, from 2 to 8 active contrasts by **LD1L0**, and from 4 to 8 active contrasts by **LD1L9**.

CHAPTER 5

Conclusion and Discussion

➢ Obviously, there is not one of the compared methods that performs very well over all of the studied cases. Some methods are good for small numbers of active effects (smaller than 4 out of 15, say) but very bad if the number of active effects gets large, e.g., **BM86**, **Dong93** and **LN97**. **MaxUr** has the advantage that it performs reasonably well for all numbers of active effects up to 8.

➢ There is little difference between the methods for small size contrasts, say 0.5σ, which exhibit little power. Also, there is not much of a difference for large size contrasts, say 3.0 and 4.0σ. There are marked differences between the methods for intermediate size contrasts (1.0σ – 2.5σ) which become more pronounced when the number of active contrasts increases.

➢ For most of the studied cases, **MaxUr** is more powerful than **Al-SY99** which is based on the same statistics as **MaxUr**. Al-Shiha and Yang (1999) use a number k, which is the largest number of active contrasts that can be identified by **Al-SY99**. From Al-Shiha and Yang's (1999) and our simulation studies, we see that **Al-SY99** has the largest power when k equals the true number of active effects and its power decreases if k becomes larger than the true number of active effects. Similarly, **MaxUr** has a number r, which is the largest number of active contrasts that can be identified by it. Again, **MaxUr** has the largest power when r equals the true number of active effects. Since we allowed for up to 8 active effects in our simulation study, $k = 7$ and $r =8$ were used. (We used $k = 7$ because we did not have critical values for $k = 8$.)

If **Al-SY99** with $k = 8$ was more powerful than **MaxUr**, then **Al-SY99** with $k = 7$ should have been more powerful than **MaxUr** whenever the true number of active effects was not larger than 7. Hence, we can conclude that the superiority of **MaxUr** is not due to the choice of $k = 7$.

➢ **MaxUr** with $r = 8$ is almost the best one among the compared methods by the criteria **power**, **power II** and **LD1L0**. Compared to other methods, **power II** of **MaxUr** is relatively higher than its **power I** and **III**. Note the following inequalities

$$\textbf{power I} \geq \textbf{power IV} \geq \textbf{power II} \geq \textbf{power III.} \qquad (5.1)$$

Hence, **MaxUr** can more efficiently identify all really active contrasts than the others, but it often declares more active contrasts than given. Since in the screening stage, as said before, it is more important to find active contrasts than to avoid misidentifying inactive contrasts as active. Therefore, **MaxUr** is still very valuable in practice.

➢ There was a large difference between the performance of **LGB98** reported by Lawson, et al. (1998) and what we observed in our study. This can be explained by the fact that maybe in their simulation study, Lawson, et al. (1998) used a different version of their method, than what they described in their section 3.1 and section 4. It seems that Lawson, et al. (1998) in step 4 did not use the bound *C** that they gave in their paper, but the statistic *D* instead, which is a much less conservative method.

➢ There are cases, when the **power** did not decrease as the number of active effects increases (see Figures A1.1.21 – A1.1.27). This result is different from that reported by Hamada and Balakrishnan (1998). The reason may be that we controlled EER under H_0 at 0.05 while they controlled IER at 0.044 and the methods **AL_SY99**, **LGB98**, **LN97** and **MaxUr** were not studied by them.

➢ The performance of some methods are dependent on how the levels of EER or IER are controlled. For example, although **LN97** can identify up to 12 out of 15 active contrasts at the level 0.20 (EER), it hardly ever rejects the null hypothesis at the level 0.05 when the number of active contrasts is greater than 7. **Ben89** performs well for large number of active contrasts at the level 0.05 (EER) while it works poorly at the level 0.002 (IER) (see Hamada and Balakrishnan (1998)).

➢ The performance of some methods also depends on whether all active contrasts are of the same magnitude or not. For example, **LN97** is one of the best three methods by criterion **power III** for 1~6 active contrasts in the former case, but it is one of the worst three in the latter case. **JTuk87** performs good in the latter case but badly in the former. **BM86** performs better in the former case than in the latter by criterion **LD1L0**. **MaxUr** in general performs worse in the former case than in the latter. In practice, the case of the active contrasts having different magnitudes is much more likely. Hence, differently sized active contrasts should be considered in such a simulation study. But how to reasonably choose the mix of active contrasts of different magnitudes is still an open problem. Maybe the sizes of active contrasts of different magnitudes should be selected as a random sample of normally distributed variables (see, e.g., Benski and Cabau (1995), Haaland and O'Connell (1995)).

➢ It is also clear that the criteria play an important role in evaluating the methods. The methods perform differently by different criteria. Table 5.1 and 5.2 show the best three and worst three methods with respect to different criteria from the simulation study, respectively. It must be pointed out that the sequences listed in Table 5.1 and Table 5.2 are sometimes not absolute. That means the method listed at the first position may in some cases perform worse (Table 5.1) / better (Table 5.2) than that listed at the second position, but in most cases the first is better (Table 5.1) / worse (Table5.2) than the second. For example, Table 5.1 shows that **LN97** is the best one, **BM86** is the second best and **Dong93** the third for one active contrast by the criterion **power**; the **power** of **LN97** and **BM86** is smaller than **Dong93**'s when the size of the active contrast is 2.0σ, however. For more details, see Table A1.3.1 – Table A1.3.20, Table A1.4.1 – Table A1.4.20, Table A2.3.1 – Table A2.3.15 and Table A2.4.1 – Table A2.4.12 in the appendices.

➢ According to **power**, **MaxUr** is the most powerful among the compared methods in all of the studied cases except for the cases that all active contrasts have the same magnitude and the number of active contrasts is not larger than 2. When the number of active contrasts is one, **LN97** is the best one; **BM86** and **Dong93** perform also very well; in this case the worst one is **Bi89**, the second worst is **Al-SY99**. In the case of two active contrasts with the same magnitude, **LGB98** is the most powerful; **BM86** and **Dong93** perform also very well; the worst and the second worst are still **Bi89** and **Al-SY99**, respectively. Except for **LGB98**, the **power** increases as the size of active contrasts increases provided the number of active contrasts is not larger than six.

➢ According to **power I**, only in the case of six active contrasts with the same magnitude, **Al-SY99** performs a little better than **MaxUr**. In the cases where the number of active contrasts is not larger than 2 and there are 4 active contrasts with different magnitudes, **BM86** and **Dong93** mostly stand at the first two positions; while **Bi89** and **Al-SY99** mostly stand at the last. In the remaining cases, besides **MaxUr**, **Al-SY99** and **MLZ92** perform also very well, while **LGB98** and **JTuk87** perform very badly. **Ben89** is the best one for 8 active contrasts with the same magnitude.

➢ According to **power II**, in the case where there is only one active contrast, **LN97** is the best one, **BM86** and **Dong93** are also quite competitive. **MaxUr** is the most powerful in the remaining cases. Besides, **Al-SY99** also works very well for 6 active contrasts, **Ben89** also very well for large number of active contrasts (greater than 4, say), **JTuk87** performs also well for between 2 and 6 active contrasts with different magnitudes. **Bi89**, **Al-SY99** and **MLZ92** show the lowest power when the number of active contrasts is not larger than 2. If the number of active contrasts is between 4 and 6, **LGB98**, **Bi89** and **LN97** are the worst. For 8 active contrasts, **Al-SY99**, **Dong93** and **LGB98** perform worst. Except for **Al-SY99**, **LGB98** and **MaxUr**, **power II** of most of these methods is a monotone decreasing function of the number of active contrasts (see Figures A1.1.35 – A1.1.41 and Tables A1.3.35 – A1.3.41 in appendix A1.1 and A1.3, respectively).

➢ According to **power III**, **Dong93** and **BM86** perform very well for up to four active contrasts while **Bi89** does very bad in these cases. **LN97** almost performs best for up to 6 active contrasts with the same magnitude, but it is almost the worst in the cases where the active contrasts do not have the same magnitude. **MaxUr** is the most powerful in the case of 6 active contrasts with different magnitudes and the cases of 8 active contrasts, it is almost the lest powerful when all active contrasts have the same magnitude and the number of active contrasts is not larger than 6, however. **Al-SY99** performs very well for 6 active contrasts, but does very bad when the number of active contrasts is not larger than 2. In addition, for 8 active contrasts, **Ben89** is the second best, while **Al-SY99**, **Dong93** and **LGB98** are the worst three methods.

Power III of **Al-SY99** and **LGB98** is a non-monotone function of the number of active contrasts (see Figures A1.1.42 – A1.1.48 and Tables A1.3.42 – A1.3.48). **LGB98** reaches its highest power when the number of active contrasts is two and the size is greater than 1.0σ. **Al-SY99** arrives at its highest power when the number of active contrasts is six. It is interesting that **power III** of **MaxUr** is a monotonically increasing function of the number of active contrasts and the others are monotonically decreasing functions.

Table 5.1: The best three methods for different cases according to variant evaluation standards from the simulation study.

Evaluation Standard	The Number of Active Contrasts								
	1	2		4		6		8	
		Magnitude		Magnitude		Magnitude		Magnitude	
		Same	Diffe.	Same	Diffe.	Same	Diffe.	Same	Diffe.
Power	**LN97 BM86 Dong93**	**LGB98 BM86 Dong93**	**MaxUr LGB98 JTuk87**	**MaxUr BM86 Dong93**	**MaxUr JTuk87 Ben89**	**MaxUr Al-SY99 MLZ92**	**MaxUr Al-SY99 JTuk87**	**MaxUr Ben89 Bi89**	**MaxUr MLZ92 JP92**
Power I	**LN97 BM86 Dong93**	**BM86 Dong93 LGB98**	**BM86 LN97 Dong93**	**MLZ92 MaxUr Al-SY99**	**BM86 Dong93 LGB98**	**MLZ92 Al-SY99 MaxUr**	**MLZ92 MaxUr Al-SY99**	**Ben89 MaxUr Bi89**	**MaxUr Al-SY99 MLZ92**
Power II	**LN97 BM86 Dong93**	**MaxUr LGB98 BM86**	**MaxUr LGB98 JTuk87**	**MaxUr BM86 LN97**	**MaxUr JTuk87 Dong93**	**MaxUr Al-SY99 Ben89**	**MaxUr Al-SY99 JTuk87**	**MaxUr Ben89 Bi89**	**MaxUr Ben89 JP92**
Power III	**LN97 BM86 Dong93**	**BM86 Dong93 LN97**	**LGB98 Dong93 JTuk87**	**LN97 BM86 Dong93**	**Dong93 JTuk87 Ben89**	**Al-SY99 LN97 Ben89**	**MaxUr Al-SY99 JTuk87**	**MaxUr Ben89 Bi89**	**MaxUr Ben89 JP92**
Power IV	**--**	**--**	**BM86 LN97 Dong93**	**--**	**BM86 Dong93 MaxUr**	**--**	**MaxUr BM86 JTuk87**	**--**	**MaxUr MLZ92 Al-SY99**
LD1L0	**LN97 BM86 Dong93**	**BM86 Dong93 LGB98**	**MaxUr LGB98 JTuk87**	**MaxUr BM86 Dong93**	**MaxUr JTuk87 Ben89**	**MaxUr Al-SY99 MLZ92**	**MaxUr Al-SY99 JTuk87**	**MaxUr Ben89 Bi89**	**MaxUr MLZ92 JP92**
LD1L9	**LN97 BM86 Dong93**	**BM86 Dong93 LGB98**	**MaxUr LGB98 JTuk87**	**BM86 LN97 Dong93**	**MaxUr JTuk87 Ben89**	**Al-SY99 MaxUr MLZ92**	**MaxUr Al-SY99 JTuk87**	**MaxUr Ben89 Bi89**	**MaxUr MLZ92 JP92**
LD2L	**LN97 BM86 Dong93**	**BM86 LN97 LGB98**	**BM86 LGB98 Dong93**	**BM86 LN97 Dong93**	**Dong93 JTuk87 MaxUr**	**Al-SY99 MLZ92 MaxUr**	**MaxUr Al-SY99 JTuk87**	**MaxUr Ben89 Bi89**	**MaxUr MLZ92 JP92**

- According to **power IV**, in the cases where the number of active contrasts is not larger than 4, **BM86** and **Dong93** perform best while **Bi89**, **Al-SY99** and **MLZ92** do worst. Besides, **LN97** also performs very well for 2 active contrasts, and **MaxUr** also does very well for 4 active contrasts. **MaxUr** is the most powerful for 6 and 8 active contrasts while **LGB98** is the worst one in these cases. In addition, for 6 active contrasts, **BM86** and **JTuk87** also work very powerfully while **LN97** and **Bi89** do very poorly. For 8 active contrasts, **MLZ92** and **Al-SY99** are also very powerful while **Dong93** and **LN97** are very bad.

Table 5.2: The worst three methods for different cases according to variant evaluation standards from the simulation study.

Evaluation Standard	The Number of Active Contrasts								
	1	2		4		6		8	
		Magnitude		Magnitude		Magnitude		Magnitude	
		Same	Diffe.	Same	Diffe.	Same	Diffe.	Same	Diffe.
Power	**Bi89 Al-SY99 MLZ92**	**Bi89 Al-SY99 JP92**	**LN97 Bi89 Al-SY99**	**LGB98 Bi89 JP92**	**LGB98 LN97 Bi89**	**LGB98 JTuk87 Bi89**	**LGB98 LN97 Bi89**	**LGB98 JTuk87 STuk82**	**LGB98 Dong93 JTuk87**
Power I	**Bi89 Al-SY99 MLZ92**	**Bi89 JP92 Al-SY99**	**Bi89 Al-SY99 MLZ92**	**LGB98 Bi89 JTuk87**	**Bi89 JP92 Al-SY99**	**LGB98 JTuk87 Bi89**	**LGB98 LN97 Ben89**	**LGB98 JTuk87 Dong93**	**LGB98 Dong93 JTuk87**
Power II	**Bi89 Al-SY99 MLZ92**	**Bi89 Al-SY99 MLZ92**	**LN97 Bi89 Al-SY99**	**LGB98 Bi89 MLZ92**	**LGB98 LN97 Bi89**	**LGB98 Bi89 JTuk87**	**LGB98 LN97 BM86**	**Al-SY99 Dong93 JTuk87 Lenth89 LGB98**	**Al-SY99 Dong93 LGB98**
Power III	**MaxUr Bi89 Al-SY99**	**MaxUr Bi89 Al-SY99**	**LN97 Bi89 Al-SY99**	**LGB98 MaxUr Bi89**	**LGB98 LN97 Bi89**	**LGB98 MaxUr Bi89**	**LGB98 LN97 BM86**	**Al-SY99 Dong93 JTuk87 Lenth89 LGB98**	**Al-SY99 Dong93 LGB98**
Power IV	**--**	**--**	**Bi89 Al-SY99 MLZ92**	**--**	**Bi89 Al-SY99 MLZ92**	**--**	**LGB98 LN97 Bi89**	**--**	**LGB98 Dong93 LN97**
LD1L0	**Bi89 Al-SY99 MLZ92**	**Bi89 Al-SY99 MLZ92**	**LN97 Bi89 Al-SY99**	**LGB98 Bi89 JP92**	**LGB98 LN97 Bi89**	**LGB98 JTuk87 Bi89**	**LGB98 LN97 Bi89**	**LGB98 JTuk87 STuk82**	**LGB98 Dong93 JTuk87**
LD1L9	**MaxUr Bi89 Al-SY99**	**Bi89 Al-SY99 MaxUr**	**LN97 Bi89 Al-SY99**	**LGB98 Bi89 JP92**	**LGB98 LN97 Bi89**	**LGB98 JTuk87 Bi89**	**LGB98 LN97 Bi89**	**LGB98 JTuk87 STuk82**	**LGB98 Dong98 JTuk87**
LD2L	**MaxUr Bi89 JP92**	**MaxUr Bi89 JP92**	**Bi89 MaxUr JP92**	**LGB98 Bi89 MaxUr**	**LGB98 LN97 Bi89**	**LGB98 JTuk87 Bi89**	**LGB98 LN97 Bi89**	**LGB98 JTuk87 STuk82**	**LGB98 Dong93 JTuk87**

- According to **LD1L0**, the performances of the compared methods are almost the same as those according to **power**. When the number of active contrasts with the same magnitude is not greater than six, except for **LGB98**, **LD1L0** of these methods are monotone decreasing functions of the size of the active contrast. **LD1L0** of **LGB98** is a monotonically increasing function of the active contrast magnitude for four and six active contrasts. When there are eight active contrasts, **LD1L0** of **Ben89**, **Bi89**, **BM86**, **MaxUr** and **MLZ92** is a monotone decreasing function of the active contrast magnitude; **LD1L0** of **LN97** is a non-monotone function of the active contrast magnitude; and **LD1L0** of the others are monotone increasing functions. The criterion **LD1L0** is in fact equivalent to the criterion **power**.

➢ According to **LD1L9**, in general, **BM86** and **Dong93** perform very well for small number of active contrasts (not greater than 4, say), while **MaxUr** does very good for large number of active contrasts (not smaller than 4, say). In the cases where the number of active contrasts is not larger than 2 and all active contrasts have the same magnitude, **Al-SY99**, **Bi89** and **MaxUr** perform very badly. **Al-SY99** and **Bi89** do also very badly for 2 active contrasts with different magnitude. For 4 and 6 active contrasts, **LGB98** gives the highest loss and **Bi89** also gives very high loss. Besides, **LN97** works very poorly for 2 up to 6 active contrasts with different magnitude. For eight active contrasts, the worst two methods are **LGB98** and **JTuk87**.

➢ According to **LD2L**, in general, **Dong93**, **BM86** and **LN97** give very low loss when the number of active contrasts is not greater than 4, while **MaxUr** gives very low loss when the number of active contrasts is between 4 and 8. **MaxUr**, **Bi89** and **JP92** are three of the methods giving the highest loss if the number of active contrasts is not larger than 2. **LGB98** is the worst one when there are more than 3 active contrasts.

➢ To summarize, **LN97** performs best when there is only one active contrast. **BM86** and **Dong93** work very good when the number of active contrasts is not large (not larger than 4, say). **MaxUr** does very well when there are many active contrasts (not less than 4, say).

➢ Note that α_{active} for **BM86** was set at 0.20, which was designed for 3 active contrasts. If α_{active} had been set at 0.4, its performance for large numbers of active contrasts must have been better. For the same reason, **MaxUr** must have performed better for small numbers of active contrasts if the version of r = 4 instead of r = 8 had been used. Of course, if this had been done, the performances of **BM86** for small numbers of active contrasts and that of **MaxUr** for large numbers of active contrasts would have become worse.

➢ As said by Haaland (Hamada and Balakrishnan (1998), Comment), the optimal method depends on the true number of active contrasts that is unknown. It would be more meaningful in practice to find the test methods that provide good performance over a range of numbers of active contrasts than to find the best method only for a fixed number of active contrasts. Consequently, we would recommend **Dong93** for small number of active contrasts because of its simple computation and **MaxUr** with r = 8 for large number of active contrasts.

➢ Since the criteria largely influence the results, we have to carefully select an appropriate criterion. Which criterion is more suitable? It depends on the practical circumstances. If we only attend to finding the active contrasts, then **power II** may be suitable. If we want not only to detect the active contrasts but also to avoid misidentifying inactive contrasts as active,. the LD may be more suitable because it is flexible and has the potential of measuring the whole performance of a method. Then, how should we choose the loss function? This needs to be studied further.

Appendix

A1 Simulated Power, Power I~IV

A1.1 Figures of power, power I, II and III for a 16-run experiment and 1~8 active contrasts with the same magnitude

Figure A1.1.1: power for n = 16 and one active contrast.

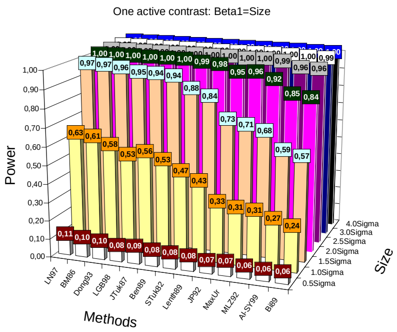


Figure A1.1.4: power for n = 16 and six active contrasts with the same magnitude.

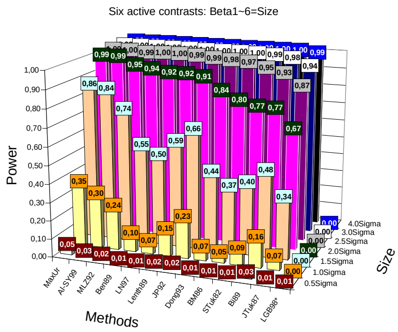


Figure A1.1.2: power for n = 16 and two active contrasts with the same magnitude.

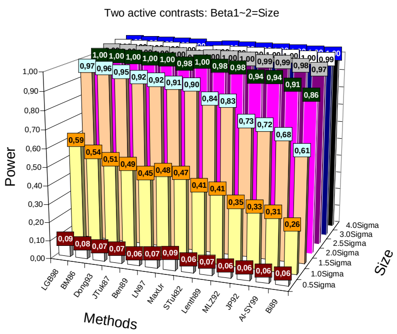


Figure A1.1.5: power for n = 16 and eight active contrasts with the same magnitude.

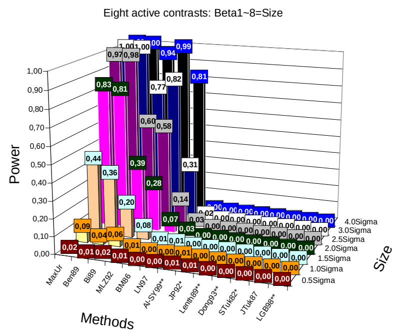


Figure A1.1.3: power for n = 16 and four active contrasts with the same magnitude.

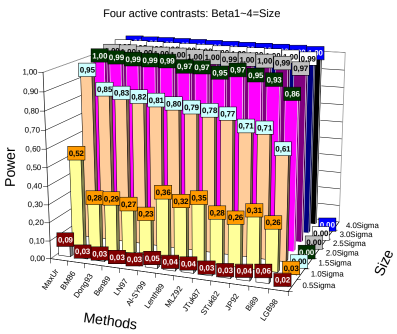


Figure A1.1.6: power I for n = 16 and one active contrast.

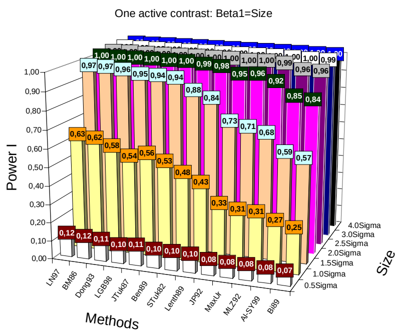


Figure A1.1.7: power I for *n* = 16 and two active contrasts with the same magnitude.

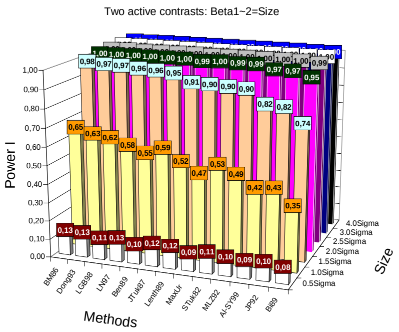


Figure A1.1.8: power I for *n* = 16 and four active contrasts with the same magnitude.

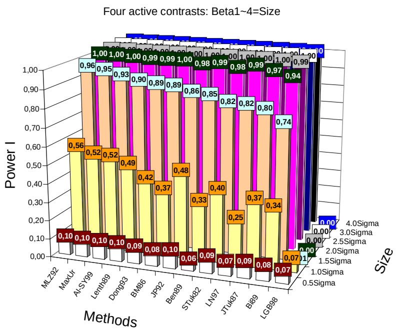


Figure A1.1.9: power I for *n* = 16 and six active contrasts with the same magnitude.

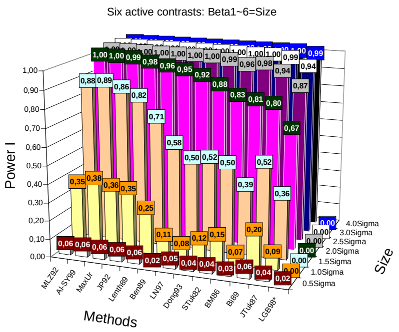


Figure A1.1.10: power I for *n* = 16 and eight active contrasts with the same magnitude.

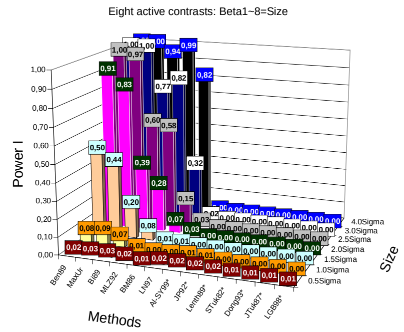


Figure A1.1.11: power II for *n* = 16 and one active contrast.

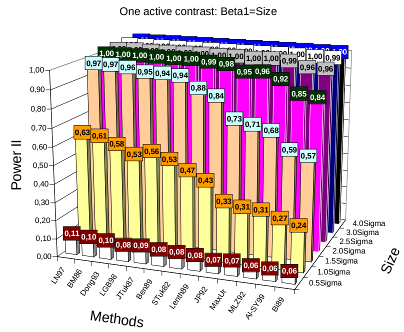


Figure A1.1.12: power II for *n* = 16 and two active contrasts with the same magnitude.

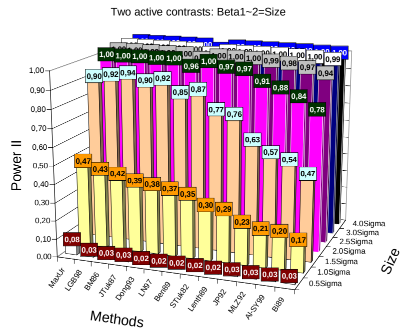


Figure A1.1.13: power II for n = 16 and four active contrasts with the same magnitude.

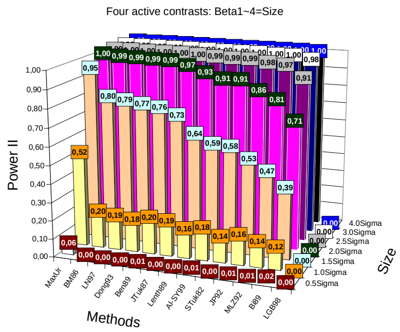


Figure A1.1.16: power III for n = 16 and one active contrast.

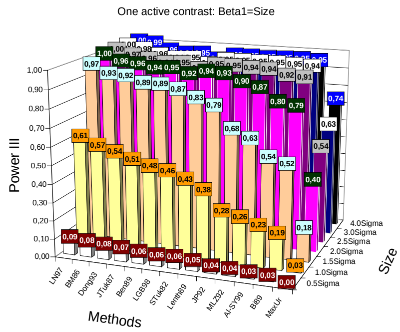


Figure A1.1.14: power II for n = 16 and six active contrasts with the same magnitude.

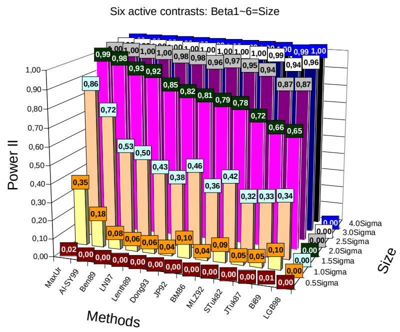


Figure A1.1.17: power III for n = 16 and two active contrasts with the same magnitude.

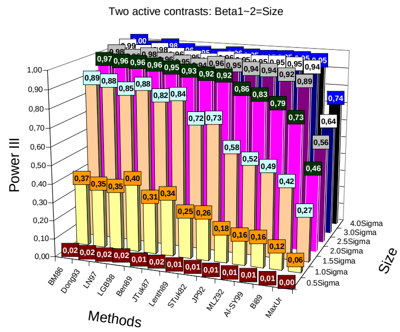


Figure A1.1.15: power II for n = 16 and eight active contrasts with the same magnitude.

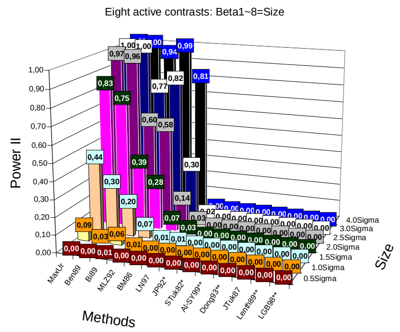


Figure A1.1.18: power III for n = 16 and four active contrasts with the same magnitude.

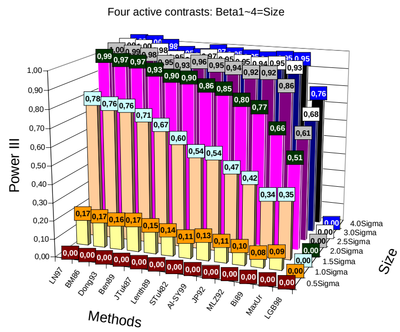


Figure A1.1.19: power III for n = 16 and six active contrasts with the same magnitude.

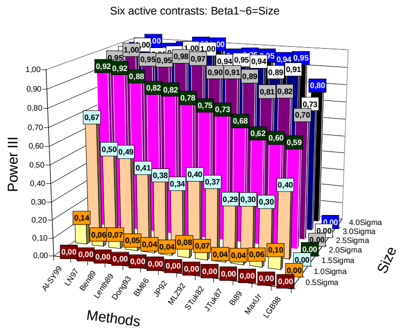


Figure A1.1.20: power III for n = 16 and eight active contrasts with the same magnitude.

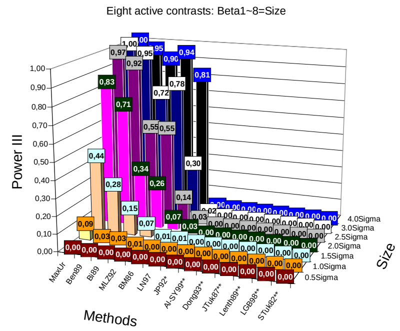


Figure A1.1.21: power for n = 16 and active contrasts with the same size: 0.5 sigma

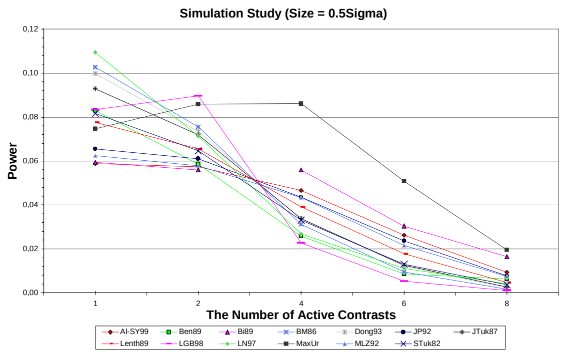


Figure A1.1.23: power for n = 16 and active contrasts with the same size: 1.5 sigma

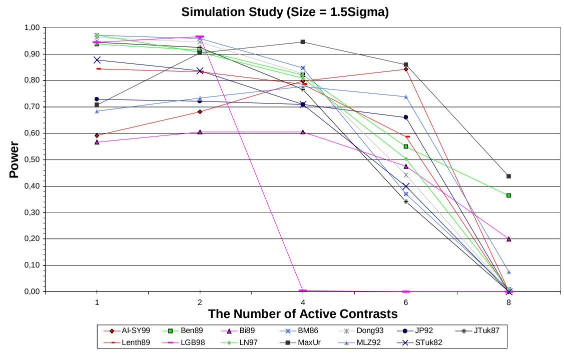


Figure A1.1.22: power for n = 16 and active contrasts with the same size: 1.0 sigma

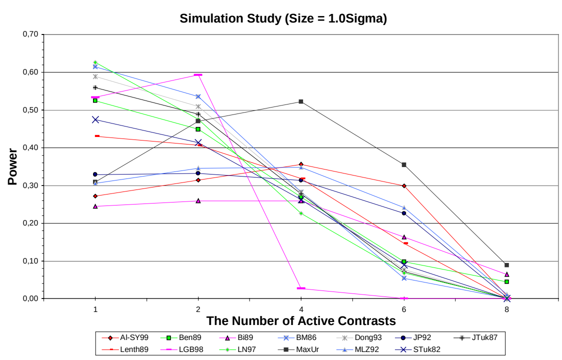


Figure A1.1.24: power for n = 16 and active contrasts with the same size: 2.0 sigma

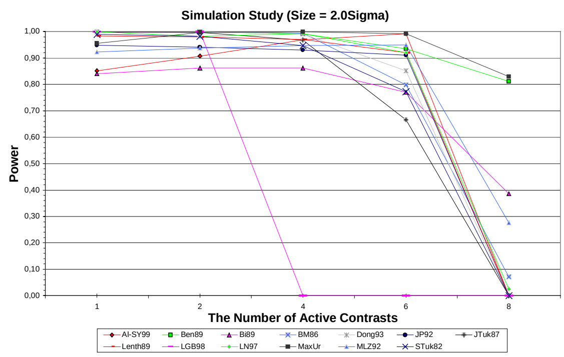


Figure A1.1.25: power for n = 16 and active contrasts with the same size: 2.5 sigma

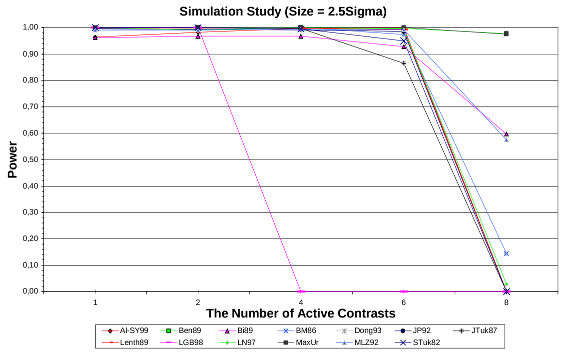


Figure A1.1.26: power for n = 16 and active contrasts with the same size: 3.0 sigma

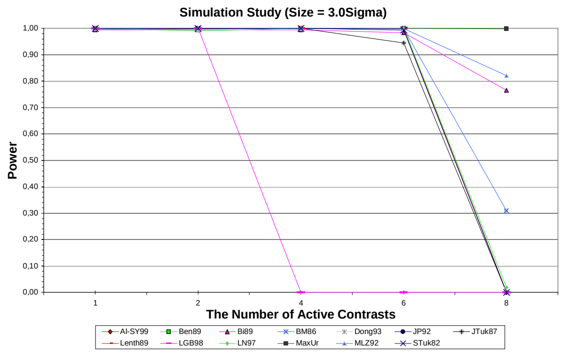


Figure A1.1.27: power for n = 16 and active contrasts with the same size: 4.0 sigma

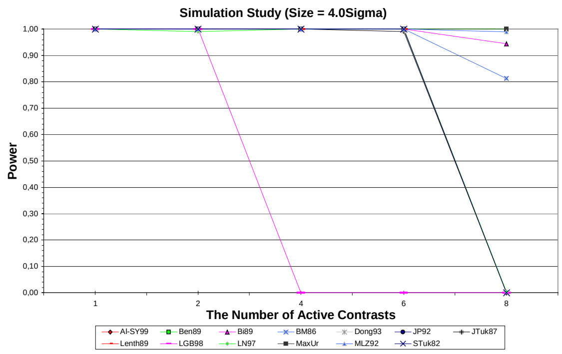


Figure A1.1.28: power I for n = 16 and active contrasts with the same size: 0.5 sigma

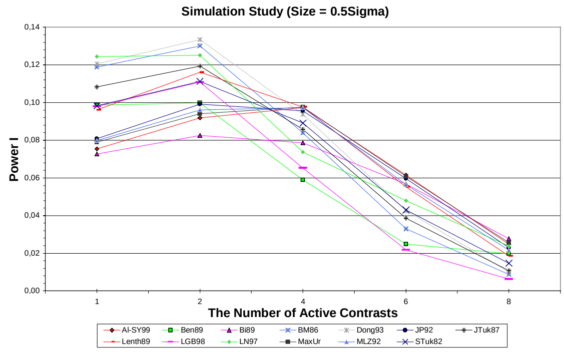


Figure A1.1.29: power I for n = 16 and active contrasts with the same size: 1.0 sigma

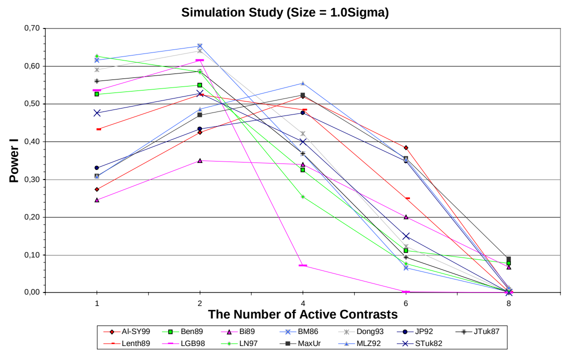


Figure A1.1.30: power I for n = 16 and active contrasts with the same size: 1.5 sigma

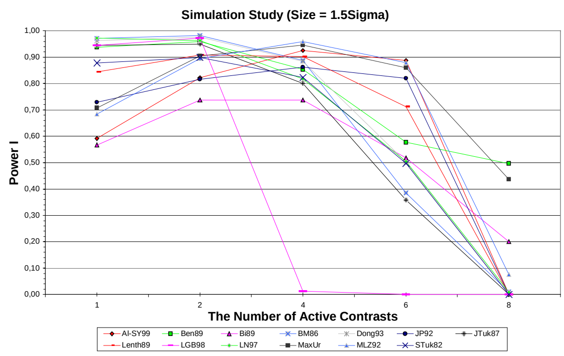


Figure A1.1.31: power I for *n* = 16 and active contrasts with the same size: 2.0 sigma

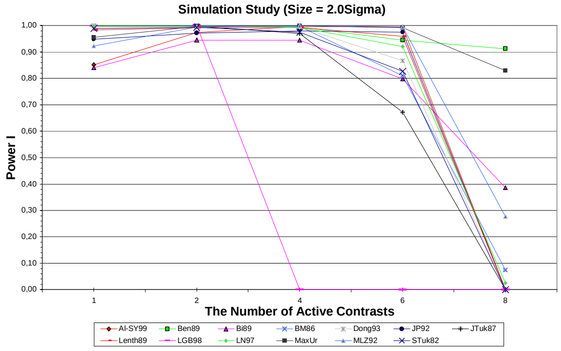


Figure A1.1.32: power I for *n* = 16 and active contrasts with the same size: 2.5 sigma

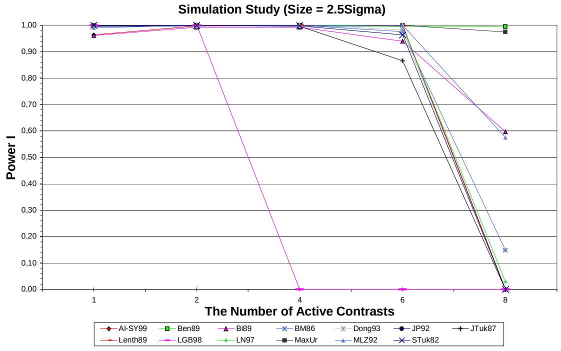


Figure A1.1.33: power I for *n* = 16 and active contrasts with the same size: 3.0 sigma

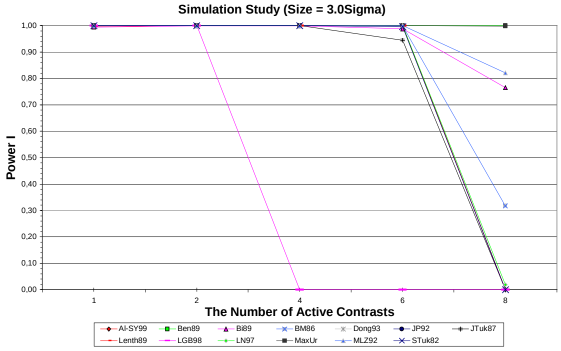


Figure A1.1.34: power I for *n* = 16 and active contrasts with the same size: 4.0 sigma

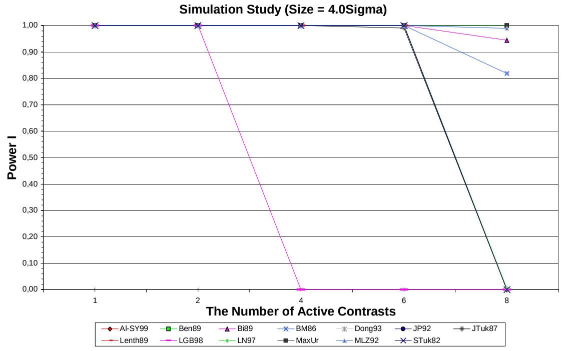


Figure A1.1.35: power II for *n* = 16 and active contrasts with the same size: 0.5 sigma

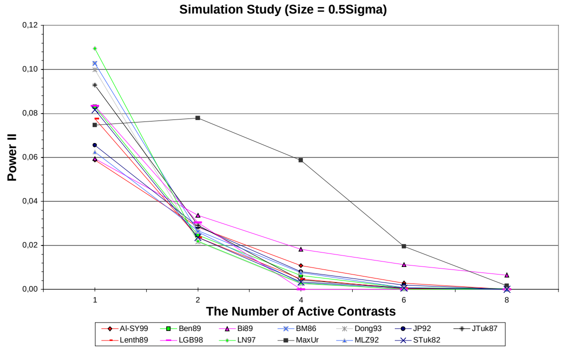


Figure A1.1.36: power II for *n* = 16 and active contrasts with the same size: 1.0 sigma

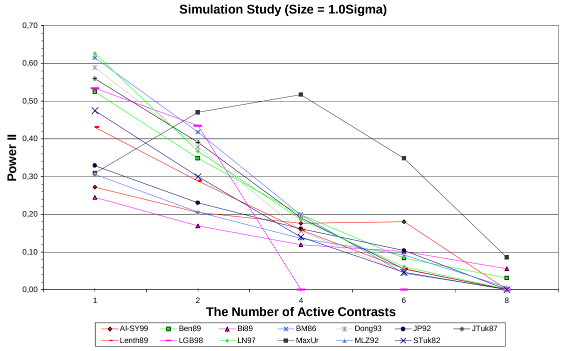


Figure A1.1.37: power II for *n* = 16 and active contrasts with the same size: 1.5 sigma

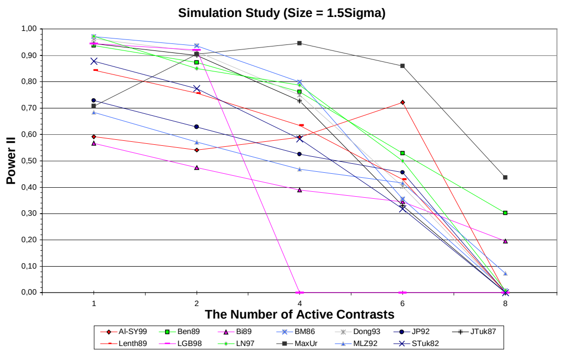


Figure A1.1.38: power II for *n* = 16 and active contrasts with the same size: 2.0 sigma

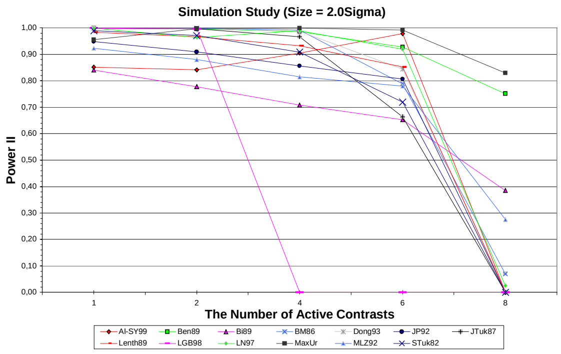


Figure A1.1.39: power II for *n* = 16 and active contrasts with the same size: 2.5 sigma

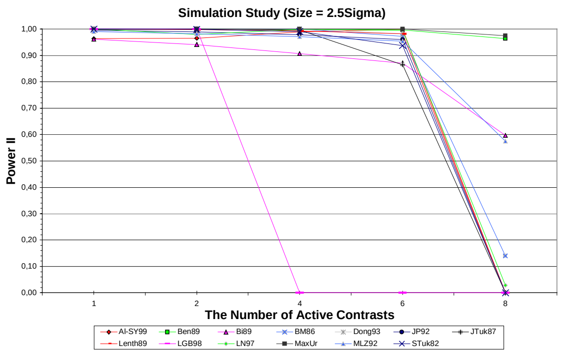


Figure A1.1.40: power II for *n* = 16 and active contrasts with the same size: 3.0 sigma

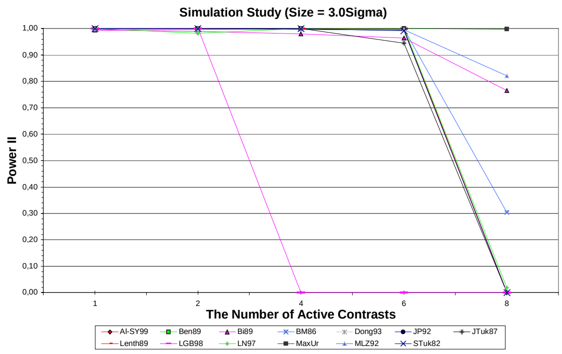


Figure A1.1.41: power II for *n* = 16 and active contrasts with the same size: 4.0 sigma

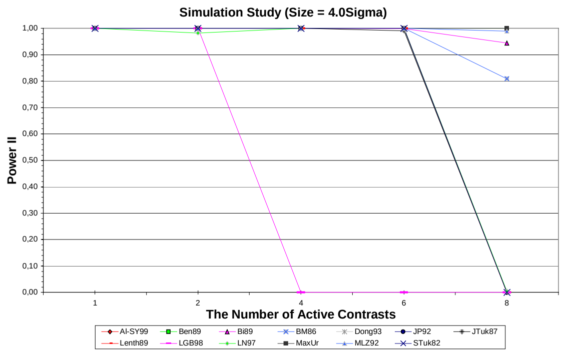


Figure A1.1.42: power III for *n* = 16 and active contrasts with the same size: 0.5 sigma

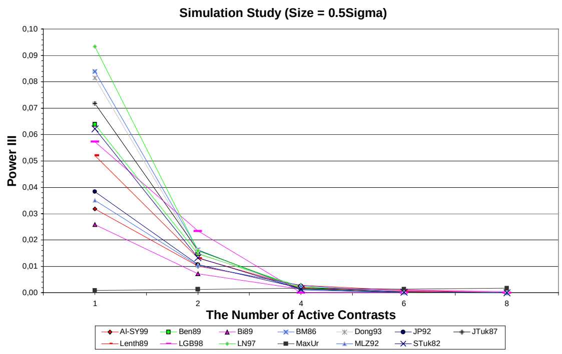


Figure A1.1.43: **power III** for *n* = 16 and active contrasts with the same size: 1.0 sigma

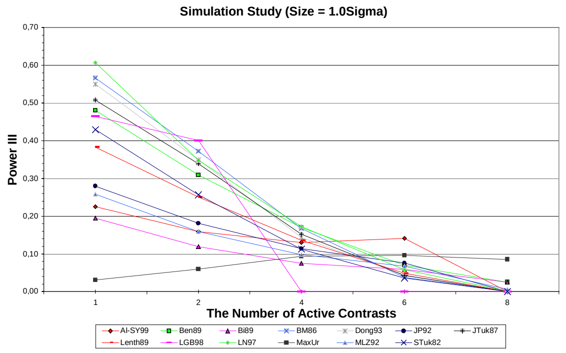


Figure A1.1.44: **power III** for *n* = 16 and active contrasts with the same size: 1.5 sigma

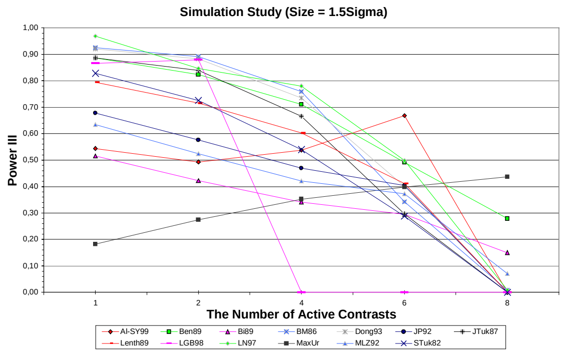


Figure A1.1.45: **power III** for *n* = 16 and active contrasts with the same size: 2.0 sigma

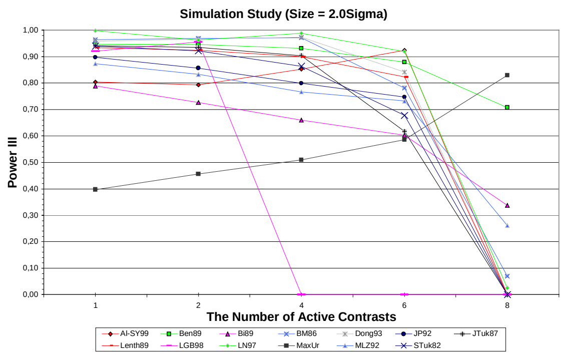


Figure A1.1.46: **power III** for *n* = 16 and active contrasts with the same size: 2.5 sigma

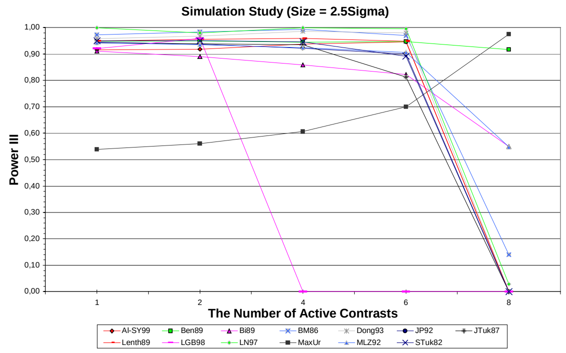


Figure A1.1.47: **power III** for *n* = 16 and active contrasts with the same size: 3.0 sigma

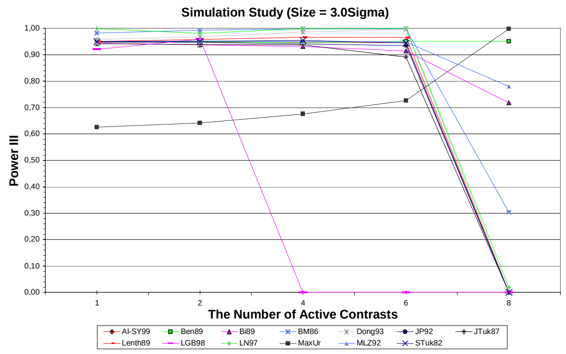
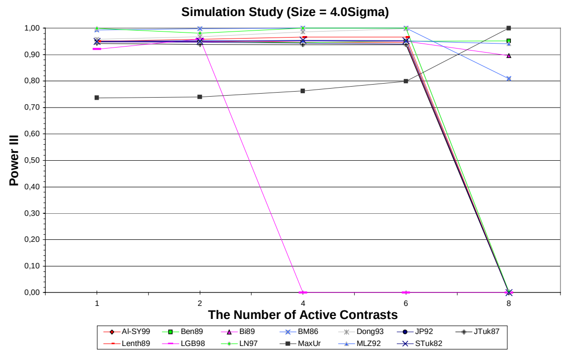


Figure A1.1.48: **power III** for *n* = 16 and active contrasts with the same size: 4.0 sigma



A1.2 Figures of power, power I, II, III and IV for a 16-run experiment and 2~8 active contrasts with different magnitudes

Figure A1.2.1: power for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

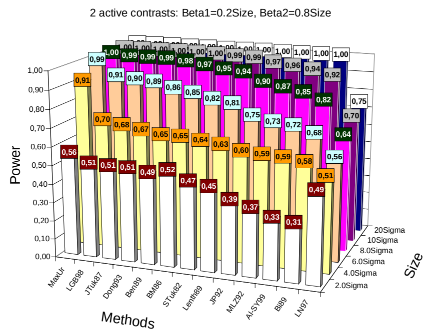


Figure A1.2.2: power for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

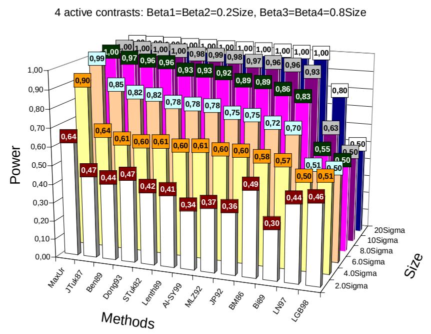


Figure A1.2.3: power for *n* = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

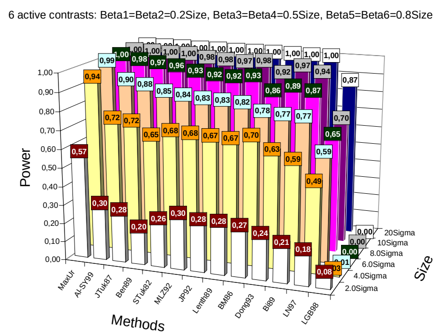


Figure A1.2.4: power for *n* = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

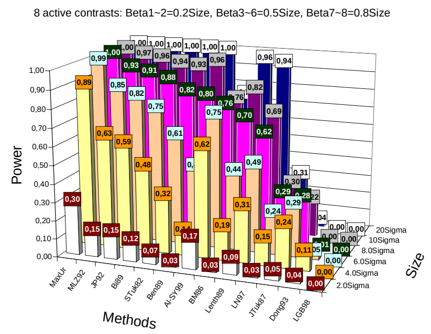


Figure A1.2.5: power I for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

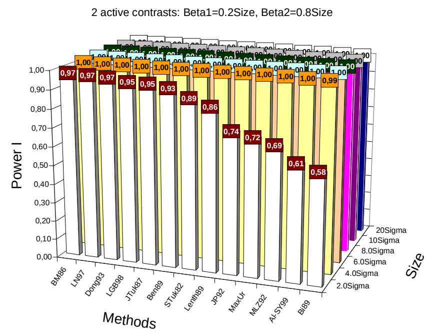


Figure A1.2.6: power I for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

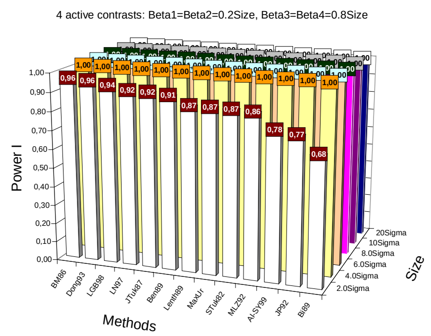


Figure A1.2.7: power I for *n* = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

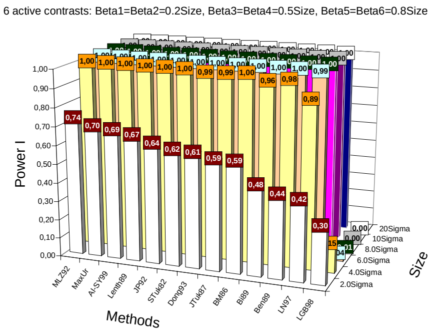


Figure A1.2.10: power II for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

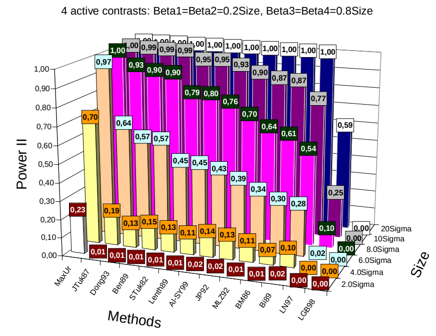


Figure A1.2.8: power I for *n* = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

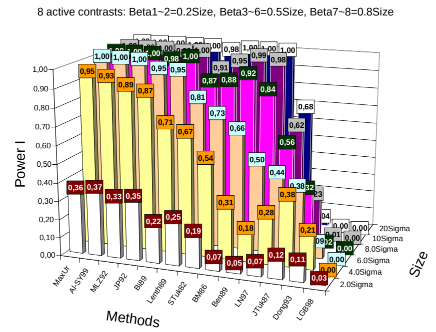


Figure A1.2.11: power II for *n* = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

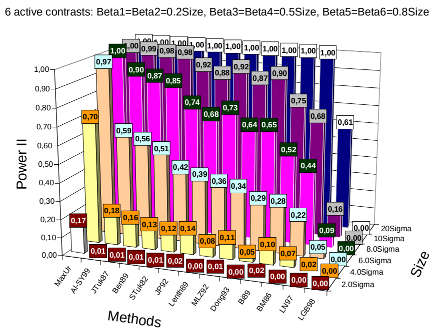


Figure A1.2.9: power II for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

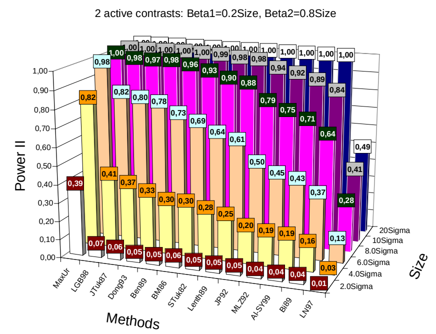


Figure A1.2.12: power II for *n* = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

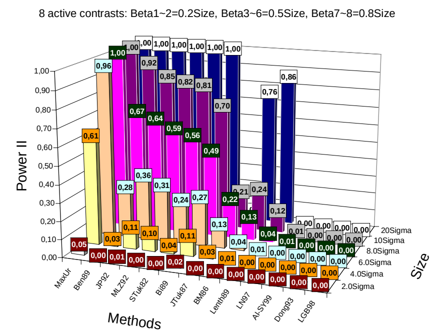


Figure A1.2.13: power III for n = 16 and two active contrast with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

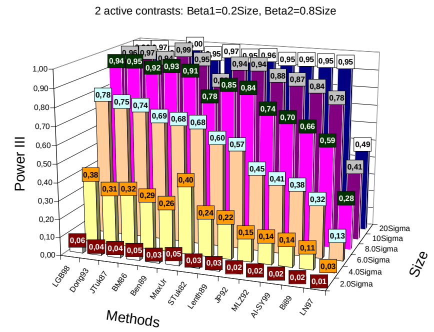


Figure A1.2.14: power III for n = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

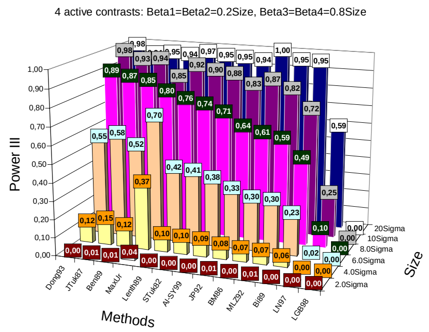


Figure A1.2.15: power III for n = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

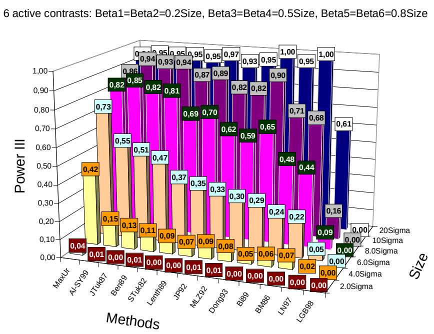


Figure A1.2.16: power III for n = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

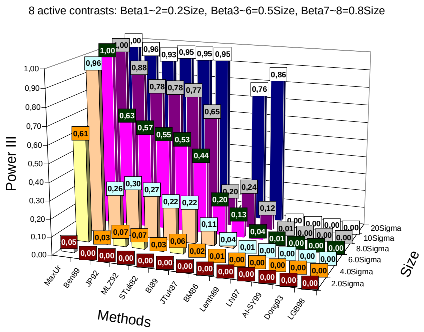


Figure A1.2.17: power IV for n = 16 and two active contrast with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

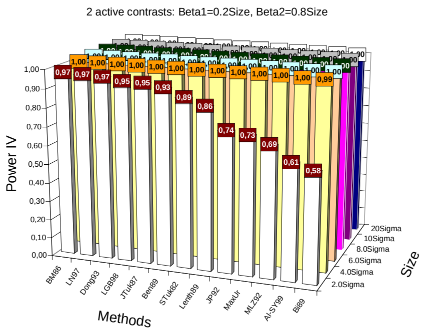


Figure A1.2.18: power IV for n = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

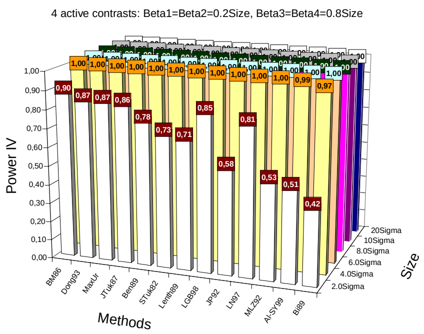


Figure A1.2.19: power IV for n = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

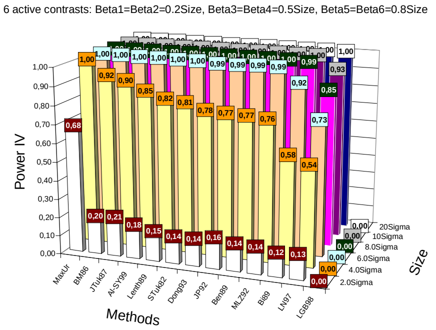
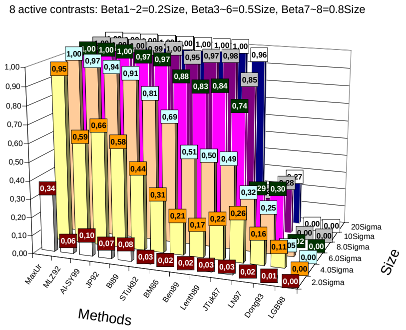


Figure A1.2.20: power IV for n = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.



A1.3 Tables of power, power I, II and III for a 16-run experiment and 1~8 active contrasts with the same magnitude

Table A1.3.1: power for n = 16 and one active contrast.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,05890	0,27170	0,59170	0,85200	0,96380	0,99550	1,00000
Ben89	0,08300	0,52510	0,93790	0,99900	1,00000	1,00000	1,00000
Bi89	0,05940	0,24450	0,56720	0,84040	0,96210	0,99430	1,00000
BM86	0,10270	0,61490	0,97120	0,99960	1,00000	1,00000	1,00000
Dong93	0,09990	0,58900	0,96320	0,99980	1,00000	1,00000	1,00000
JP92	0,06550	0,32910	0,72910	0,94880	0,99540	1,00000	1,00000
JTuk87	0,09290	0,55980	0,94440	0,99800	1,00000	1,00000	1,00000
Lenth89	0,07760	0,43070	0,84420	0,98380	0,99930	1,00000	1,00000
LGB98	0,08340	0,53400	0,94500	0,99920	1,00000	1,00000	1,00000
LN97	0,10950	0,62610	0,97300	0,99960	1,00000	1,00000	1,00000
MaxUr	0,07470	0,30960	0,70790	0,95600	0,99850	1,00000	1,00000
MLZ92	0,06250	0,30590	0,68390	0,92280	0,99140	0,99950	1,00000
STuk82	0,08160	0,47470	0,87820	0,98980	0,99990	1,00000	1,00000

Table A1.3.2: power for n = 16 and two active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,05735	0,31455	0,6815	0,90755	0,9823	0,99825	1
Ben89	0,0586	0,44905	0,91615	0,99795	1	1	1
Bi89	0,05595	0,25955	0,60595	0,86175	0,9675	0,99465	1
BM86	0,07555	0,536	0,95945	0,9995	1	1	1
Dong93	0,07345	0,50965	0,94565	0,9991	1	1	1
JP92	0,06105	0,33185	0,7224	0,94055	0,99365	0,9998	1
JTuk87	0,0719	0,48875	0,92495	0,99765	1	1	1
Lenth89	0,0655	0,40635	0,8327	0,97975	0,9987	1	1
LGB98	0,08965	0,59305	0,9664	0,9996	1	1	1
LN97	0,07135	0,4756	0,9069	0,98205	0,9903	0,991	0,99105
MaxUr	0,0859	0,47035	0,9047	0,9976	1	1	1
MLZ92	0,0578	0,346	0,7332	0,9374	0,9915	0,99985	1
STuk82	0,06465	0,41385	0,8369	0,98225	0,99955	1	1

Table A1.3.3: power for n = 16 and four active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0466	0,356125	0,79865	0,96885	0,997425	0,999925	1
Ben89	0,02585	0,26905	0,82125	0,9918	0,99995	1	1
Bi89	0,05595	0,25955	0,60595	0,86175	0,9675	0,99465	1
BM86	0,031175	0,2828	0,847275	0,994425	1	1	1
Dong93	0,03355	0,2823	0,825725	0,99155	0,999925	1	1
JP92	0,04355	0,3132	0,709175	0,93035	0,991075	0,999125	1
JTuk87	0,033875	0,27785	0,766525	0,96825	0,99745	1	1
Lenth89	0,039175	0,318675	0,786225	0,97025	0,997325	0,9999	1
LGB98	0,02275	0,026725	0,004125	0,00015	0	0	0
LN97	0,0268	0,226175	0,80925	0,992025	0,9999	1	1
MaxUr	0,086125	0,52195	0,9463	0,9993	1	1	1
MLZ92	0,043325	0,348425	0,77735	0,94775	0,992875	0,999425	1
STuk82	0,03305	0,26245	0,709525	0,947475	0,99555	0,999875	1

Table A1.3.4: power for n = 16 and six active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,02625	0,2993333	0,8435667	0,9917667	0,99995	1	1
Ben89	0,0086833	0,09795	0,5497667	0,9353	0,99715	1	1
Bi89	0,0303833	0,1632667	0,47515	0,7697667	0,9281667	0,9839833	0,9999333
BM86	0,00955	0,0541167	0,3699167	0,7983333	0,9743833	0,9987167	1
Dong93	0,0122167	0,07665	0,4423	0,85195	0,9822667	0,9993	1
JP92	0,0236833	0,2259667	0,6606333	0,9111667	0,9855667	0,9985167	1
JTuk87	0,01255	0,0716167	0,34075	0,66625	0,8655333	0,9447167	0,9912
Lenth89	0,0177667	0,1467167	0,5872	0,9202667	0,9930667	0,9996667	1
LGB98*	0,0052667	0,0003333	0	0	0	0	0
LN97	0,0111667	0,0678833	0,5037333	0,9207333	0,9961	0,9999	1
MaxUr	0,05085	0,35475	0,8600667	0,9923	1	1	1
MLZ92	0,0216333	0,2414833	0,7387167	0,9500833	0,9921667	0,9993667	1
STuk82	0,0130333	0,08945	0,3994333	0,7729	0,9495	0,9935833	1

Table A1.3.5: power for n = 16 and eight active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99**	0,0094	0,0080125	0,0014125	0	0	0	0
Ben89	0,0063625	0,044475	0,3644875	0,8112625	0,977625	0,9990125	1
Bi89	0,0165375	0,0645	0,1998625	0,3863625	0,5968875	0,7657	0,9441
BM86	0,0017	0,0014125	0,00865	0,07145	0,144025	0,309675	0,81335
Dong93**	0,0021625	0	0	0	0	0	0
JP92*	0,0077875	0,0039625	0,0000625	0	0	0	0
JTuk87	0,0026875	0,0000125	0	0	0	0	0
Lenth89**	0,0046375	0,000625	0,00005	0	0	0	0
LGB98**	0,0011125	0	0	0	0	0	0
LN97	0,00415	0,001675	0,009025	0,025125	0,0287	0,0183	0,0045
MaxUr	0,019575	0,0885125	0,4369875	0,8297	0,9747	0,9985	1
MLZ92	0,0075	0,0116375	0,075575	0,2764125	0,575575	0,8211	0,9892
STuk82*	0,00365	0,0001875	0	0	0	0	0

Table A1.3.6: power I for n = 16 and one active contrast.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0754	0,2734	0,5917	0,852	0,9638	0,9955	1
Ben89	0,0987	0,526	0,938	0,999	1	1	1
Bi89	0,0727	0,2459	0,5672	0,8404	0,9621	0,9943	1
BM86	0,1188	0,6159	0,9712	0,9996	1	1	1
Dong93	0,1206	0,5911	0,9632	0,9998	1	1	1
JP92	0,0809	0,3305	0,7291	0,9488	0,9954	1	1
JTuk87	0,1083	0,5608	0,9445	0,998	1	1	1
Lenth89	0,0961	0,4326	0,8442	0,9838	0,9993	1	1
LGB98	0,098	0,5359	0,945	0,9992	1	1	1
LN97	0,1245	0,6268	0,973	0,9996	1	1	1
MaxUr	0,079	0,3096	0,7079	0,956	0,9985	1	1
MLZ92	0,0798	0,3078	0,6839	0,9228	0,9914	0,9995	1
STuk82	0,0983	0,4763	0,8782	0,9898	0,9999	1	1

Table A1.3.7: power I for n = 16 and two active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0919	0,4246	0,8221	0,9738	0,9987	1	1
Ben89	0,0998	0,5499	0,9588	0,9996	1	1	1
Bi89	0,0826	0,3498	0,7378	0,9454	0,9932	0,9998	1
BM86	0,1301	0,654	0,9824	0,9999	1	1	1
Dong93	0,1335	0,6412	0,9764	0,9998	1	1	1
JP92	0,0992	0,4338	0,816	0,972	0,9978	1	1
JTuk87	0,1194	0,5864	0,9497	0,9987	1	1	1
Lenth89	0,1161	0,5244	0,9084	0,9927	0,9996	1	1
LGB98	0,1109	0,6158	0,9722	0,9998	1	1	1
LN97	0,1252	0,5846	0,9631	0,9998	1	1	1
MaxUr	0,0941	0,4707	0,9047	0,9976	1	1	1
MLZ92	0,0962	0,4862	0,8953	0,9949	1	1	1
STuk82	0,1113	0,5282	0,8993	0,9934	1	1	1

Table A1.3.8: power I for n = 16 and four active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0973	0,5196	0,9257	0,9969	1	1	1
Ben89	0,059	0,325	0,8534	0,994	1	1	1
Bi89	0,0787	0,3398	0,7379	0,9448	0,9931	0,9998	1
BM86	0,0839	0,3681	0,8874	0,9963	1	1	1
Dong93	0,0937	0,4215	0,8846	0,9945	1	1	1
JP92	0,0956	0,4762	0,8625	0,9805	0,9983	0,9999	1
JTuk87	0,0858	0,3688	0,8013	0,9702	0,9975	1	1
Lenth89	0,0977	0,485	0,902	0,9932	0,9995	1	1
LGB98	0,0653	0,0717	0,0124	0,0004	0	0	0
LN97	0,0737	0,2543	0,8178	0,9928	1	1	1
MaxUr	0,0977	0,5236	0,9463	0,9993	1	1	1
MLZ92	0,0974	0,5556	0,9594	0,9999	1	1	1
STuk82	0,0891	0,3996	0,8231	0,9751	0,9985	1	1

Table A1.3.9: power I for *n* = 16 and six active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0615	0,3844	0,8876	0,995	1	1	1
Ben89	0,0248	0,1111	0,577	0,9453	0,9976	1	1
Bi89	0,0568	0,2007	0,518	0,798	0,9403	0,988	1
BM86	0,0329	0,0655	0,3853	0,8101	0,9761	0,9988	1
Dong93	0,0426	0,1227	0,5019	0,8676	0,9835	0,9993	1
JP92	0,0596	0,3483	0,8199	0,9753	0,9976	0,9998	1
JTuk87	0,0387	0,0938	0,3579	0,6715	0,8662	0,9448	0,9912
Lenth89	0,0553	0,2497	0,7124	0,9596	0,9983	1	1
LGB98*	0,0218	0,0017	0	0	0	0	0
LN97	0,0479	0,0762	0,5047	0,9209	0,9961	0,9999	1
MaxUr	0,0608	0,356	0,8601	0,9923	1	1	1
MLZ92	0,0565	0,3536	0,8792	0,9954	1	1	1
STuk82	0,043	0,1497	0,497	0,8267	0,9637	0,996	1

Table A1.3.10: power I for *n* = 16 and eight active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99*	0,0248	0,0118	0,0017	0	0	0	0
Ben89	0,0198	0,0773	0,4968	0,9125	0,9951	0,9999	1
Bi89	0,0278	0,0677	0,2006	0,3865	0,5969	0,7657	0,9441
BM86	0,0088	0,0018	0,0097	0,0741	0,1489	0,3173	0,8185
Dong93*	0,0101	0	0	0	0	0	0
JP92*	0,0229	0,0079	0,0001	0	0	0	0
JTuk87*	0,0107	0,0001	0	0	0	0	0
Lenth89*	0,0186	0,0017	0,0001	0	0	0	0
LGB98*	0,0063	0	0	0	0	0	0
LN97	0,0238	0,0038	0,0091	0,0252	0,0287	0,0183	0,0045
MaxUr	0,0257	0,089	0,437	0,8297	0,9747	0,9985	1
MLZ92	0,0216	0,0139	0,0759	0,2766	0,5756	0,8211	0,9892
STuk82*	0,0147	0,0005	0	0	0	0	0

Table A1.3.11: power II for *n* = 16 and one active contrast.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0589	0,2717	0,5917	0,852	0,9638	0,9955	1
Ben89	0,083	0,5251	0,9379	0,999	1	1	1
Bi89	0,0594	0,2445	0,5672	0,8404	0,9621	0,9943	1
BM86	0,1027	0,6149	0,9712	0,9996	1	1	1
Dong93	0,0999	0,589	0,9632	0,9998	1	1	1
JP92	0,0655	0,3291	0,7291	0,9488	0,9954	1	1
JTuk87	0,0929	0,5598	0,9444	0,998	1	1	1
Lenth89	0,0776	0,4307	0,8442	0,9838	0,9993	1	1
LGB98	0,0834	0,534	0,945	0,9992	1	1	1
LN97	0,1095	0,6261	0,973	0,9996	1	1	1
MaxUr	0,0747	0,3096	0,7079	0,956	0,9985	1	1
MLZ92	0,0625	0,3059	0,6839	0,9228	0,9914	0,9995	1
STuk82	0,0816	0,4747	0,8782	0,9898	0,9999	1	1

Table A1.3.12: power II for *n* = 16 and two active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0282	0,2046	0,5409	0,8413	0,9659	0,9965	1
Ben89	0,0247	0,3482	0,8735	0,9963	1	1	1
Bi89	0,0337	0,1693	0,4741	0,7781	0,9418	0,9895	1
BM86	0,0261	0,4182	0,9365	0,9991	1	1	1
Dong93	0,0216	0,3782	0,9149	0,9984	1	1	1
JP92	0,0285	0,2302	0,6288	0,9091	0,9895	0,9996	1
JTuk87	0,0291	0,3911	0,9002	0,9966	1	1	1
Lenth89	0,0236	0,2884	0,757	0,9668	0,9978	1	1
LGB98*	0,0304	0,4344	0,9204	0,9965	1	1	1
LN97	0,0219	0,3667	0,8507	0,9643	0,9806	0,982	0,9821
MaxUr	0,0779	0,47	0,9047	0,9976	1	1	1
MLZ92	0,0267	0,2061	0,5711	0,8799	0,983	0,9997	1
STuk82	0,0236	0,2996	0,7745	0,9711	0,9991	1	1

Table A1.3.13: power II for *n* = 16 and four active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0108	0,176	0,5899	0,9053	0,9899	0,9997	1
Ben89	0,0063	0,1992	0,7619	0,9863	0,9998	1	1
Bi89	0,0182	0,1192	0,3895	0,7081	0,907	0,9797	0,9999
BM86	0,003	0,1983	0,7989	0,9919	1	1	1
Dong93	0,0026	0,1553	0,7493	0,9876	0,9999	1	1
JP92	0,0081	0,1617	0,5257	0,8566	0,9787	0,9975	1
JTuk87	0,0042	0,1905	0,7272	0,9663	0,9973	1	1
Lenth89	0,0047	0,1584	0,635	0,933	0,9929	0,9997	1
LGB98	0	0	0	0	0	0	0
LN97	0,0029	0,186	0,7876	0,9897	0,9996	1	1
MaxUr	0,0587	0,5171	0,9463	0,9993	1	1	1
MLZ92	0,0076	0,1361	0,468	0,8147	0,9717	0,9977	1
STuk82	0,0034	0,1397	0,583	0,9088	0,9913	0,9995	1

Table A1.3.14: power II for *n* = 16 and six active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0029	0,1799	0,7224	0,9782	0,9997	1	1
Ben89	0,0008	0,0833	0,5291	0,9283	0,9969	1	1
Bi89	0,0113	0,1017	0,3443	0,6523	0,8716	0,9641	0,9996
BM86	0,0002	0,0432	0,3554	0,7887	0,9725	0,9987	1
Dong93	0,0003	0,0454	0,4031	0,844	0,9819	0,9993	1
JP92	0,0019	0,1042	0,4565	0,8069	0,9602	0,9952	1
JTuk87	0,0007	0,0545	0,3299	0,6642	0,8654	0,9447	0,9912
Lenth89	0,0004	0,0554	0,4302	0,8524	0,982	0,9988	1
LGB98	0	0	0	0	0	0	0
LN97	0,0001	0,0609	0,5005	0,9199	0,9961	0,9999	1
MaxUr	0,0196	0,3486	0,8599	0,9923	1	1	1
MLZ92	0,0008	0,092	0,4167	0,7798	0,9543	0,9962	1
STuk82	0,0006	0,0459	0,3183	0,7196	0,938	0,9918	1

Table A1.3.15: power II for *n* = 16 and eight active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99**	0	0	0	0	0	0	0
Ben89	0,0004	0,0308	0,3026	0,7514	0,9643	0,998	1
Bi89	0,0065	0,0558	0,1957	0,3854	0,5968	0,7657	0,9441
BM86	0	0,0011	0,0081	0,0698	0,1399	0,3038	0,8097
Dong93**	0	0	0	0	0	0	0
JP92*	0	0,0004	0	0	0	0	0
JTuk87	0	0	0	0	0	0	0
Lenth89**	0	0	0	0	0	0	0
LGB98**	0	0	0	0	0	0	0
LN97	0	0,0012	0,009	0,0251	0,0287	0,0183	0,0045
MaxUr	0,0017	0,0854	0,4369	0,8297	0,9747	0,9985	1
MLZ92	0	0,0066	0,0733	0,2751	0,5754	0,8211	0,9892
STuk82*	0,0001	0	0	0	0	0	0

Table A1.3.16: power III for *n* = 16 and one active contrast.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0318	0,2255	0,5441	0,8044	0,9162	0,9479	0,9524
Ben89	0,0639	0,4807	0,8865	0,9475	0,9485	0,9485	0,9485
Bi89	0,0259	0,1946	0,5160	0,7892	0,9109	0,9431	0,9488
BM86	0,0839	0,5666	0,9250	0,9637	0,9730	0,9819	0,9929
Dong93	0,0815	0,5498	0,9212	0,9578	0,9580	0,9580	0,9580
JP92	0,0383	0,2798	0,6780	0,8977	0,9443	0,9489	0,9489
JTuk87	0,0718	0,5080	0,8872	0,9407	0,9427	0,9427	0,9427
Lenth89	0,0521	0,3834	0,7945	0,9341	0,9496	0,9503	0,9503
LGB98	0,0573	0,4649	0,8665	0,9204	0,9212	0,9212	0,9212
LN97	0,0934	0,6072	0,9693	0,9987	0,9995	0,9995	0,9995
MaxUr	0,0009	0,0309	0,1825	0,3975	0,5387	0,6256	0,7367
MLZ92	0,0351	0,2582	0,6347	0,8736	0,9422	0,9503	0,9508
STuk82	0,0622	0,4295	0,8284	0,9399	0,9500	0,9501	0,9501

Table A1.3.17: power III for n = 16 and two active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0101	0,1592	0,4933	0,7937	0,9183	0,9489	0,9524
Ben89	0,0147	0,3090	0,8238	0,9460	0,9497	0,9497	0,9497
Bi89	0,0072	0,1193	0,4221	0,7261	0,8898	0,9375	0,9480
BM86	0,0163	0,3724	0,8918	0,9694	0,9840	0,9928	0,9981
Dong93	0,0151	0,3496	0,8830	0,9665	0,9681	0,9681	0,9681
JP92	0,0107	0,1812	0,5767	0,8569	0,9373	0,9474	0,9478
JTuk87	0,0161	0,3387	0,8387	0,9348	0,9382	0,9382	0,9382
Lenth89	0,0131	0,2506	0,7151	0,9248	0,9558	0,9580	0,9580
LGB98	0,0234	0,4006	0,8797	0,9557	0,9592	0,9592	0,9592
LN97	0,0159	0,3483	0,8470	0,9634	0,9801	0,9815	0,9816
MaxUr	0,0013	0,0596	0,2740	0,4566	0,5606	0,6416	0,7400
MLZ92	0,0104	0,1588	0,5239	0,8327	0,9358	0,9525	0,9528
STuk82	0,0133	0,2566	0,7259	0,9222	0,9502	0,9511	0,9511

Table A1.3.21: power for n = 16 and active contrasts with the same magnitude: 0.5 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,05890	0,05735	0,0466	0,02625	0,0094
Ben89	0,08300	0,0586	0,02585	0,0086833	0,0063625
Bi89	0,05940	0,05595	0,05595	0,0303833	0,0165375
BM86	0,10270	0,07555	0,031175	0,00955	0,0017
Dong93	0,09990	0,07345	0,03355	0,0122167	0,0021625
JP92	0,06550	0,06105	0,04355	0,0236833	0,0077875
JTuk87	0,09290	0,0719	0,033875	0,01255	0,0026875
Lenth89	0,07760	0,0655	0,039175	0,0177667	0,0046375
LGB98	0,08340	0,08965	0,02275	0,0052667	0,0011125
LN97	0,10950	0,07135	0,0268	0,0111667	0,00415
MaxUr	0,07470	0,0859	0,086125	0,05085	0,019575
MLZ92	0,06250	0,0578	0,043325	0,0216333	0,0075
STuk82	0,08160	0,06465	0,03305	0,0130333	0,00365

Table A1.3.18: power III for n = 16 and four active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0028	0,1304	0,5375	0,8529	0,9375	0,9473	0,9476
Ben89	0,0023	0,1703	0,7109	0,9319	0,9454	0,9456	0,9456
Bi89	0,0014	0,0753	0,3416	0,6602	0,8591	0,9318	0,952
BM86	0,001	0,1675	0,7592	0,9718	0,9931	0,998	1
Dong93	0,0013	0,1441	0,7354	0,9737	0,986	0,9861	0,9861
JP92	0,0017	0,1141	0,4693	0,7997	0,9218	0,9406	0,9431
JTuk87	0,0015	0,1519	0,6662	0,904	0,9349	0,9376	0,9376
Lenth89	0,002	0,1358	0,6026	0,8995	0,9593	0,9661	0,9664
LGB98	0	0	0	0	0	0	0
LN97	0,0016	0,1725	0,7804	0,9888	0,9993	0,9998	0,9999
MaxUr	0,0017	0,0936	0,3525	0,5091	0,6061	0,6763	0,7623
MLZ92	0,0027	0,098	0,421	0,7672	0,9242	0,9502	0,9525
STuk82	0,0014	0,1125	0,5398	0,8637	0,946	0,9541	0,9546

Table A1.3.22: power for n = 16 and active contrasts with the same magnitude: 1.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,27170	0,31455	0,356125	0,2993333	0,0080125
Ben89	0,52510	0,44905	0,26905	0,09795	0,044475
Bi89	0,24450	0,25955	0,25955	0,1632667	0,0645
BM86	0,61490	0,536	0,2828	0,0541167	0,0014125
Dong93	0,58900	0,50965	0,2823	0,07665	0
JP92	0,32910	0,33185	0,3132	0,2259667	0,0039625
JTuk87	0,55980	0,48875	0,27785	0,0716167	0,0000125
Lenth89	0,43070	0,40635	0,318675	0,1467167	0,000625
LGB98	0,53400	0,59305	0,026725	0,0003333	0
LN97	0,62610	0,4756	0,226175	0,0678833	0,001675
MaxUr	0,30960	0,47035	0,52195	0,35475	0,0885125
MLZ92	0,30590	0,346	0,348425	0,2414833	0,0116375
STuk82	0,47470	0,41385	0,26245	0,08945	0,0001875

Table A1.3.19: power III for n = 16 and six active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,0009	0,1414	0,6684	0,9241	0,9456	0,9459	0,9459
Ben89	0,0001	0,0698	0,4919	0,8795	0,9474	0,9504	0,9504
Bi89	0,0011	0,0588	0,295	0,6029	0,8222	0,9147	0,9502
BM86	0,0001	0,0379	0,3425	0,7811	0,9697	0,9981	1
Dong93	0,0003	0,0438	0,401	0,8419	0,9798	0,9972	0,9979
JP92	0,0005	0,0756	0,4041	0,7472	0,9004	0,9354	0,9402
JTuk87	0,0001	0,0424	0,2969	0,6174	0,8133	0,8915	0,9376
Lenth89	0,0004	0,0484	0,4114	0,8231	0,9489	0,965	0,9662
LGB98	0	0	0	0	0	0	0
LN97	0,0001	0,0582	0,4982	0,9194	0,9961	0,9999	1
MaxUr	0,0014	0,0959	0,3986	0,5864	0,6995	0,7267	0,7998
MLZ92	0,0002	0,0675	0,3728	0,7323	0,9067	0,9486	0,9524
STuk82	0,0002	0,0357	0,2884	0,6775	0,8918	0,9446	0,9527

Table A1.3.23: power for n = 16 and active contrasts with the same magnitude: 1.5 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,59170	0,6815	0,79865	0,8435667	0,0014125
Ben89	0,93790	0,91615	0,82125	0,5497667	0,3644875
Bi89	0,56720	0,60595	0,60595	0,47515	0,1998625
BM86	0,97120	0,95945	0,847275	0,3699167	0,00865
Dong93	0,96320	0,94565	0,825725	0,4423	0
JP92	0,72910	0,7224	0,709175	0,6606333	0,0000625
JTuk87	0,94440	0,92495	0,766525	0,34075	0
Lenth89	0,84420	0,8327	0,786225	0,5872	0,00005
LGB98	0,94500	0,9664	0,004125	0	0
LN97	0,97300	0,9069	0,80925	0,5037333	0,009025
MaxUr	0,70790	0,9047	0,9463	0,8600667	0,4369875
MLZ92	0,68390	0,7332	0,77735	0,7387167	0,075575
STuk82	0,87820	0,8369	0,709525	0,3994333	0

Table A1.3.20: power III for n = 16 and eight active contrasts with the same magnitude.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99**	0	0	0	0	0	0	0
Ben89	0	0,0254	0,2787	0,7076	0,9173	0,9506	0,9525
Bi89	0,0004	0,0252	0,1502	0,3378	0,5492	0,7181	0,8965
BM86	0	0,0011	0,0081	0,0698	0,1399	0,3038	0,8097
Dong93**	0	0	0	0	0	0	0
JP92*	0	0,0002	0	0	0	0	0
JTuk87**	0	0	0	0	0	0	0
Lenth89**	0	0	0	0	0	0	0
LGB98**	0	0	0	0	0	0	0
LN97	0	0,0012	0,009	0,0251	0,0287	0,0183	0,0045
MaxUr	0,0017	0,0854	0,4369	0,8297	0,9747	0,9985	1
MLZ92	0	0,0061	0,0707	0,2611	0,5478	0,7805	0,9416
STuk82**	0	0	0	0	0	0	0

Table A1.3.24: power for n = 16 and active contrasts with the same magnitude: 2.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,85200	0,90755	0,96885	0,9917667	0
Ben89	0,99900	0,99795	0,9918	0,9353	0,8112625
Bi89	0,84040	0,86175	0,86175	0,7697667	0,3863625
BM86	0,99960	0,9995	0,994425	0,7983333	0,07145
Dong93	0,99980	0,9991	0,99155	0,85195	0
JP92	0,94880	0,94055	0,93035	0,9111667	0
JTuk87	0,99800	0,99765	0,96825	0,66625	0
Lenth89	0,98380	0,97975	0,97025	0,9202667	0
LGB98	0,99920	0,9996	0,00015	0	0
LN97	0,99960	0,98205	0,992025	0,9207333	0,025125
MaxUr	0,95600	0,9976	0,9993	0,9923	0,8297
MLZ92	0,92280	0,9374	0,94775	0,9500833	0,2764125
STuk82	0,98980	0,98225	0,947475	0,7729	0

Table A1.3.25: power for *n* = 16 and active contrasts with the same magnitude: 2.5 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,96380	0,9823	0,997425	0,99995	0
Ben89	1,00000	1	0,99995	0,99715	0,977625
Bi89	0,96210	0,9675	0,9675	0,9281667	0,5968875
BM86	1,00000	1	1	0,9743833	0,144025
Dong93	1,00000	1	0,999925	0,9822667	0
JP92	0,99540	0,99365	0,991075	0,9855667	0
JTuk87	1,00000	1	0,99745	0,8655333	0
Lenth89	0,99930	0,9987	0,997325	0,9930667	0
LGB98	1,00000	1	0	0	0
LN97	1,00000	0,9903	0,9999	0,9961	0,0287
MaxUr	0,99850	1	1	1	0,9747
MLZ92	0,99140	0,9915	0,992875	0,9921667	0,575575
STuk82	0,99990	0,99955	0,99555	0,9495	0

Table A1.3.26: power for *n* = 16 and active contrasts with the same magnitude: 3.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,99550	0,99825	0,999925	1	0
Ben89	1,00000	1	1	1	0,9990125
Bi89	0,99430	0,99465	0,99465	0,9839833	0,7657
BM86	1,00000	1	1	0,9987167	0,309675
Dong93	1,00000	1	1	0,9993	0
JP92	1,00000	0,9998	0,999125	0,9985167	0
JTuk87	1,00000	1	1	0,9447167	0
Lenth89	1,00000	1	0,9999	0,9996667	0
LGB98	1,00000	1	0	0	0
LN97	1,00000	0,991	1	0,9999	0,0183
MaxUr	1,00000	1	1	1	0,9985
MLZ92	0,99950	0,99985	0,999425	0,9993667	0,8211
STuk82	1,00000	1	0,999875	0,9935833	0

Table A1.3.27: power for *n* = 16 and active contrasts with the same magnitude: 4.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	1,00000	1	1	1	0
Ben89	1,00000	1	1	1	1
Bi89	1,00000	1	1	0,9999333	0,9441
BM86	1,00000	1	1	1	0,81335
Dong93	1,00000	1	1	1	0
JP92	1,00000	1	1	1	0
JTuk87	1,00000	1	1	0,9912	0
Lenth89	1,00000	1	1	1	0
LGB98	1,00000	1	0	0	0
LN97	1,00000	0,99105	1	1	0,0045
MaxUr	1,00000	1	1	1	1
MLZ92	1,00000	1	1	1	0,9892
STuk82	1,00000	1	1	1	0

Table A1.3.28: power I for *n* = 16 and active contrasts with the same magnitude: 0.5 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,0754	0,0919	0,0973	0,0615	0,0248
Ben89	0,0987	0,0998	0,059	0,0248	0,0198
Bi89	0,0727	0,0826	0,0787	0,0568	0,0278
BM86	0,1188	0,1301	0,0839	0,0329	0,0088
Dong93	0,1206	0,1335	0,0937	0,0426	0,0101
JP92	0,0809	0,0992	0,0956	0,0596	0,0229
JTuk87	0,1083	0,1194	0,0858	0,0387	0,0107
Lenth89	0,0961	0,1161	0,0977	0,0553	0,0186
LGB98	0,098	0,1109	0,0653	0,0218	0,0063
LN97	0,1245	0,1252	0,0737	0,0479	0,0238
MaxUr	0,079	0,0941	0,0977	0,0608	0,0257
MLZ92	0,0798	0,0962	0,0974	0,0565	0,0216
STuk82	0,0983	0,1113	0,0891	0,043	0,0147

Table A1.3.29: power I for *n* = 16 and active contrasts with the same magnitude: 1.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,0754	0,0919	0,0973	0,0615	0,0248
Ben89	0,0987	0,0998	0,059	0,0248	0,0198
Bi89	0,0727	0,0826	0,0787	0,0568	0,0278
BM86	0,1188	0,1301	0,0839	0,0329	0,0088
Dong93	0,1206	0,1335	0,0937	0,0426	0,0101
JP92	0,0809	0,0992	0,0956	0,0596	0,0229
JTuk87	0,1083	0,1194	0,0858	0,0387	0,0107
Lenth89	0,0961	0,1161	0,0977	0,0553	0,0186
LGB98	0,098	0,1109	0,0653	0,0218	0,0063
LN97	0,1245	0,1252	0,0737	0,0479	0,0238
MaxUr	0,079	0,0941	0,0977	0,0608	0,0257
MLZ92	0,0798	0,0962	0,0974	0,0565	0,0216
STuk82	0,0983	0,1113	0,0891	0,043	0,0147

Table A1.3.30: power I for *n* = 16 and active contrasts with the same magnitude: 1.5 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,2734	0,4246	0,5196	0,3844	0,0118
Ben89	0,526	0,5499	0,325	0,1111	0,0773
Bi89	0,2459	0,3498	0,3398	0,2007	0,0677
BM86	0,6159	0,654	0,3681	0,0655	0,0018
Dong93	0,5911	0,6412	0,4215	0,1227	0
JP92	0,3305	0,4338	0,4762	0,3483	0,0079
JTuk87	0,5608	0,5864	0,3688	0,0938	0,0001
Lenth89	0,4326	0,5244	0,485	0,2497	0,0017
LGB98	0,5359	0,6158	0,0717	0,0017	0
LN97	0,6268	0,5846	0,2543	0,0762	0,0038
MaxUr	0,3096	0,4707	0,5236	0,356	0,089
MLZ92	0,3078	0,4862	0,5556	0,3536	0,0139
STuk82	0,4763	0,5282	0,3996	0,1497	0,0005

Table A1.3.31: power I for *n* = 16 and active contrasts with the same magnitude: 2.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,852	0,9738	0,9969	0,995	0
Ben89	0,999	0,9996	0,994	0,9453	0,9125
Bi89	0,8404	0,9454	0,9448	0,798	0,3865
BM86	0,9996	0,9999	0,9963	0,8101	0,0741
Dong93	0,9998	0,9998	0,9945	0,8676	0
JP92	0,9488	0,972	0,9805	0,9753	0
JTuk87	0,998	0,9987	0,9702	0,6715	0
Lenth89	0,9838	0,9927	0,9932	0,9596	0
LGB98	0,9992	0,9998	0,0004	0	0
LN97	0,9996	0,9998	0,9928	0,9209	0,0252
MaxUr	0,956	0,9976	0,9993	0,9923	0,8297
MLZ92	0,9228	0,9949	0,9999	0,9954	0,2766
STuk82	0,9898	0,9934	0,9751	0,8267	0

Table A1.3.32: power I for *n* = 16 and active contrasts with the same magnitude: 2.5 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,9638	0,9987	1	1	0
Ben89	1	1	1	0,9976	0,9951
Bi89	0,9621	0,9932	0,9931	0,9403	0,5969
BM86	1	1	1	0,9761	0,1489
Dong93	1	1	1	0,9835	0
JP92	0,9954	0,9978	0,9983	0,9976	0
JTuk87	1	1	0,9975	0,8662	0
Lenth89	0,9993	0,9996	0,9995	0,9983	0
LGB98	1	1	0	0	0
LN97	1	1	1	0,9961	0,0287
MaxUr	0,9985	1	1	1	0,9747
MLZ92	0,9914	1	1	1	0,5756
STuk82	0,9999	1	0,9985	0,9637	0

Table A1.3.33: power I for *n* = 16 and active contrasts with the same magnitude: 3.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,9955	1	1	1	0
Ben89	1	1	1	1	0,9999
Bi89	0,9943	0,9998	0,9998	0,988	0,7657
BM86	1	1	1	0,9988	0,3173
Dong93	1	1	1	0,9993	0
JP92	1	1	0,9999	0,9998	0
JTuk87	1	1	1	0,9448	0
Lenth89	1	1	1	1	0
LGB98	1	1	0	0	0
LN97	1	1	1	0,9999	0,0183
MaxUr	1	1	1	1	0,9985
MLZ92	0,9995	1	1	1	0,8211
STuk82	1	1	1	0,996	0

Table A1.3.34: power I for *n* = 16 and active contrasts with the same magnitude: 4.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	1	1	1	1	0
Ben89	1	1	1	1	1
Bi89	1	1	1	1	0,9441
BM86	1	1	1	1	0,8185
Dong93	1	1	1	1	0
JP92	1	1	1	1	0
JTuk87	1	1	1	0,9912	0
Lenth89	1	1	1	1	0
LGB98	1	1	0	0	0
LN97	1	1	1	1	0,0045
MaxUr	1	1	1	1	1
MLZ92	1	1	1	1	0,9892
STuk82	1	1	1	1	0

Table A1.3.35: power II for *n* = 16 and active contrasts with the same magnitude: 0.5 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,0589	0,0282	0,0108	0,0029	0
Ben89	0,083	0,0247	0,0063	0,0008	0,0004
Bi89	0,0594	0,0337	0,0182	0,0113	0,0065
BM86	0,1027	0,0261	0,003	0,0002	0
Dong93	0,0999	0,0216	0,0026	0,0003	0
JP92	0,0655	0,0285	0,0081	0,0019	0
JTuk87	0,0929	0,0291	0,0042	0,0007	0
Lenth89	0,0776	0,0236	0,0047	0,0004	0
LGB98	0,0834	0,0304	0	0	0
LN97	0,1095	0,0219	0,0029	0,0001	0
MaxUr	0,0747	0,0779	0,0587	0,0196	0,0017
MLZ92	0,0625	0,0267	0,0076	0,0008	0
STuk82	0,0816	0,0236	0,0034	0,0006	0,0001

Table A1.3.36: power II for *n* = 16 and active contrasts with the same magnitude: 1.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,2717	0,2046	0,176	0,1799	0
Ben89	0,5251	0,3482	0,1992	0,0833	0,0308
Bi89	0,2445	0,1693	0,1192	0,1017	0,0558
BM86	0,6149	0,4182	0,1983	0,0432	0,0011
Dong93	0,589	0,3782	0,1553	0,0454	0
JP92	0,3291	0,2302	0,1617	0,1042	0,0004
JTuk87	0,5598	0,3911	0,1905	0,0545	0
Lenth89	0,4307	0,2884	0,1584	0,0554	0
LGB98	0,534	0,4344	0	0	0
LN97	0,6261	0,3667	0,186	0,0609	0,0012
MaxUr	0,3096	0,47	0,5171	0,3486	0,0854
MLZ92	0,3059	0,2061	0,1361	0,092	0,0066
STuk82	0,4747	0,2996	0,1397	0,0459	0

Table A1.3.37: power II for *n* = 16 and active contrasts with the same magnitude: 1.5 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,852	0,8413	0,9053	0,9782	0
Ben89	0,999	0,9963	0,9863	0,9283	0,7514
Bi89	0,8404	0,7781	0,7081	0,6523	0,3854
BM86	0,9996	0,9991	0,9919	0,7887	0,0698
Dong93	0,9998	0,9984	0,9876	0,844	0
JP92	0,9488	0,9091	0,8566	0,8069	0
JTuk87	0,998	0,9966	0,9663	0,6642	0
Lenth89	0,9838	0,9668	0,933	0,8524	0
LGB98	0,9992	0,9965	0	0	0
LN97	0,9996	0,9643	0,9897	0,9199	0,0251
MaxUr	0,956	0,9976	0,9993	0,9923	0,8297
MLZ92	0,9228	0,8799	0,8147	0,7798	0,2751
STuk82	0,9898	0,9711	0,9088	0,7196	0

Table A1.3.38: power II for *n* = 16 and active contrasts with the same magnitude: 2.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,852	0,8413	0,9053	0,9782	0
Ben89	0,999	0,9963	0,9863	0,9283	0,7514
Bi89	0,8404	0,7781	0,7081	0,6523	0,3854
BM86	0,9996	0,9991	0,9919	0,7887	0,0698
Dong93	0,9998	0,9984	0,9876	0,844	0
JP92	0,9488	0,9091	0,8566	0,8069	0
JTuk87	0,998	0,9966	0,9663	0,6642	0
Lenth89	0,9838	0,9668	0,933	0,8524	0
LGB98	0,9992	0,9965	0	0	0
LN97	0,9996	0,9643	0,9897	0,9199	0,0251
MaxUr	0,956	0,9976	0,9993	0,9923	0,8297
MLZ92	0,9228	0,8799	0,8147	0,7798	0,2751
STuk82	0,9898	0,9711	0,9088	0,7196	0

Table A1.3.39: power II for *n* = 16 and active contrasts with the same magnitude: 2.5 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,9638	0,9659	0,9899	0,9997	0
Ben89	1	1	0,9998	0,9969	0,9643
Bi89	0,9621	0,9418	0,907	0,8716	0,5968
BM86	1	1	1	0,9725	0,1399
Dong93	1	1	0,9999	0,9819	0
JP92	0,9954	0,9895	0,9787	0,9602	0
JTuk87	1	1	0,9973	0,8654	0
Lenth89	0,9993	0,9978	0,9929	0,982	0
LGB98	1	1	0	0	0
LN97	1	0,9806	0,9996	0,9961	0,0287
MaxUr	0,9985	1	1	1	0,9747
MLZ92	0,9914	0,983	0,9717	0,9543	0,5754
STuk82	0,9999	0,9991	0,9913	0,938	0

Table A1.3.40: power II for *n* = 16 and active contrasts with the same magnitude: 3.0 sigma.

#AC / Methods	1	2	4	6	8
Al-SY99	0,9955	0,9965	0,9997	1	0
Ben89	1	1	1	1	0,998
Bi89	0,9943	0,9895	0,9797	0,9641	0,7657
BM86	1	1	1	0,9987	0,3038
Dong93	1	1	1	0,9993	0
JP92	1	0,9996	0,9975	0,9952	0
JTuk87	1	1	1	0,9447	0
Lenth89	1	1	0,9997	0,9988	0
LGB98	1	1	0	0	0
LN97	1	0,982	1	0,9999	0,0183
MaxUr	1	1	1	1	0,9985
MLZ92	0,9995	0,9997	0,9977	0,9962	0,8211
STuk82	1	1	0,9995	0,9918	0

Table A1.3.41: power II for *n* = 16 and active contrasts with the same magnitude: 4.0 sigma.

Methods \ #AC	1	2	4	6	8
Al-SY99	1	1	1	1	0
Ben89	1	1	1	1	1
Bi89	1	1	0,9999	0,9996	0,9441
BM86	1	1	1	1	0,8097
Dong93	1	1	1	1	0
JP92	1	1	1	1	0
JTuk87	1	1	1	0,9912	0
Lenth89	1	1	1	1	0
LGB98	1	1	0	0	0
LN97	1	0,9821	1	1	0,0045
MaxUr	1	1	1	1	1
MLZ92	1	1	1	1	0,9892
STuk82	1	1	1	1	0

Table A1.3.42: power III for *n* = 16 and active contrasts with the same magnitude: 0.5 sigma.

Methods \ #AC	1	2	4	6	8
Al-SY99	0,0318	0,0101	0,0028	0,0009	0
Ben89	0,0639	0,0147	0,0023	0,0001	0
Bi89	0,0259	0,0072	0,0014	0,0011	0,0004
BM86	0,0839	0,0163	0,001	0,0001	0
Dong93	0,0815	0,0151	0,0013	0,0003	0
JP92	0,0383	0,0107	0,0017	0,0005	0
JTuk87	0,0718	0,0161	0,0015	0,0001	0
Lenth89	0,0521	0,0131	0,002	0,0004	0
LGB98	0,0573	0,0234	0	0	0
LN97	0,0934	0,0159	0,0016	0,0001	0
MaxUr	0,0009	0,0013	0,0017	0,0014	0,0017
MLZ92	0,0351	0,0104	0,0027	0,0002	0
STuk82	0,0622	0,0133	0,0014	0,0002	0

Table A1.3.43: power III for *n* = 16 and active contrasts with the same magnitude: 1.0 sigma.

Methods \ #AC	1	2	4	6	8
Al-SY99	0,2255	0,1592	0,1304	0,1414	0
Ben89	0,4807	0,309	0,1703	0,0698	0,0254
Bi89	0,1946	0,1193	0,0753	0,0588	0,0252
BM86	0,5666	0,3724	0,1675	0,0379	0,0011
Dong93	0,5498	0,3496	0,1441	0,0438	0
JP92	0,2798	0,1812	0,1141	0,0756	0,0002
JTuk87	0,508	0,3387	0,1519	0,0424	0
Lenth89	0,3834	0,2506	0,1358	0,0484	0
LGB98	0,4649	0,4006	0	0	0
LN97	0,6072	0,3483	0,1725	0,0582	0,0012
MaxUr	0,0309	0,0596	0,0936	0,0959	0,0854
MLZ92	0,2582	0,1588	0,098	0,0675	0,0061
STuk82	0,4295	0,2566	0,1125	0,0357	0

Table A1.3.44: power III for *n* = 16 and active contrasts with the same magnitude: 1.5 sigma.

Methods \ #AC	1	2	4	6	8
Al-SY99	0,5441	0,4933	0,5375	0,6684	0
Ben89	0,8865	0,8238	0,7109	0,4919	0,2787
Bi89	0,516	0,4221	0,3416	0,295	0,1502
BM86	0,925	0,8918	0,7592	0,3425	0,0081
Dong93	0,9212	0,883	0,7354	0,401	0
JP92	0,678	0,5767	0,4693	0,4041	0
JTuk87	0,8872	0,8387	0,6662	0,2969	0
Lenth89	0,7945	0,7151	0,6026	0,4114	0
LGB98	0,8665	0,8797	0	0	0
LN97	0,9693	0,847	0,7804	0,4982	0,009
MaxUr	0,1825	0,274	0,3525	0,3986	0,4369
MLZ92	0,6347	0,5239	0,421	0,3728	0,0707
STuk82	0,8284	0,7259	0,5398	0,2884	0

Table A1.3.45: power III for *n* = 16 and active contrasts with the same magnitude: 2.0 sigma.

Methods \ #AC	1	2	4	6	8
Al-SY99	0,8044	0,7937	0,8529	0,9241	0
Ben89	0,9475	0,946	0,9319	0,8795	0,7076
Bi89	0,7892	0,7261	0,6602	0,6029	0,3378
BM86	0,9637	0,9694	0,9718	0,7811	0,0698
Dong93	0,9578	0,9665	0,9737	0,8419	0
JP92	0,8977	0,8569	0,7997	0,7472	0
JTuk87	0,9407	0,9348	0,904	0,6174	0
Lenth89	0,9341	0,9248	0,8995	0,8231	0
LGB98	0,9204	0,9557	0	0	0
LN97	0,9987	0,9634	0,9888	0,9194	0,0251
MaxUr	0,3975	0,4566	0,5091	0,5864	0,8297
MLZ92	0,8736	0,8327	0,7672	0,7323	0,2611
STuk82	0,9399	0,9222	0,8637	0,6775	0

Table A1.3.46: power III for *n* = 16 and active contrasts with the same magnitude: 2.5 sigma.

Methods \ #AC	1	2	4	6	8
Al-SY99	0,9162	0,9183	0,9375	0,9456	0
Ben89	0,9485	0,9497	0,9454	0,9474	0,9173
Bi89	0,9109	0,8898	0,8591	0,8222	0,5492
BM86	0,973	0,984	0,9931	0,9697	0,1399
Dong93	0,958	0,9681	0,986	0,9798	0
JP92	0,9443	0,9373	0,9218	0,9004	0
JTuk87	0,9427	0,9382	0,9349	0,8133	0
Lenth89	0,9496	0,9558	0,9593	0,9489	0
LGB98	0,9212	0,9592	0	0	0
LN97	0,9995	0,9801	0,9993	0,9961	0,0287
MaxUr	0,5387	0,5606	0,6061	0,6995	0,9747
MLZ92	0,9422	0,9358	0,9242	0,9067	0,5478
STuk82	0,95	0,9502	0,946	0,8918	0

Table A1.3.47: power III for *n* = 16 and active contrasts with the same magnitude: 3.0 sigma.

Methods \ #AC	1	2	4	6	8
Al-SY99	0,9479	0,9489	0,9473	0,9459	0
Ben89	0,9485	0,9497	0,9456	0,9504	0,9506
Bi89	0,9431	0,9375	0,9318	0,9147	0,7181
BM86	0,9819	0,9928	0,998	0,9981	0,3038
Dong93	0,958	0,9681	0,9861	0,9972	0
JP92	0,9489	0,9474	0,9406	0,9354	0
JTuk87	0,9427	0,9382	0,9376	0,8915	0
Lenth89	0,9503	0,958	0,9661	0,965	0
LGB98	0,9212	0,9592	0	0	0
LN97	0,9995	0,9815	0,9998	0,9999	0,0183
MaxUr	0,6256	0,6416	0,6763	0,7267	0,9985
MLZ92	0,9503	0,9525	0,9502	0,9486	0,7805
STuk82	0,9501	0,9511	0,9541	0,9446	0

Table A1.3.48: power III for *n* = 16 and active contrasts with the same magnitude: 4.0 sigma.

Methods \ #AC	1	2	4	6	8
Al-SY99	0,9524	0,9524	0,9476	0,9459	0
Ben89	0,9485	0,9497	0,9456	0,9504	0,9525
Bi89	0,9488	0,948	0,952	0,9502	0,8965
BM86	0,9929	0,9981	1	1	0,8097
Dong93	0,958	0,9681	0,9861	0,9979	0
JP92	0,9489	0,9478	0,9431	0,9402	0
JTuk87	0,9427	0,9382	0,9376	0,9376	0
Lenth89	0,9503	0,958	0,9664	0,9662	0
LGB98	0,9212	0,9592	0	0	0
LN97	0,9995	0,9816	0,9999	1	0,0045
MaxUr	0,7367	0,74	0,7623	0,7998	1
MLZ92	0,9508	0,9528	0,9525	0,9524	0,9416
STuk82	0,9501	0,9511	0,9546	0,9527	0

A1.4 Tables of power, power I, II , III and IV for a 16-run experiment and 2~8 active contrasts with different magnitudes

Table A1.4.1: power for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,32645	0,59025	0,71505	0,85470	0,94495	1,00000
Ben89	0,49235	0,64935	0,86455	0,98015	0,99895	1,00000
Bi89	0,31190	0,57590	0,68330	0,81865	0,91910	1,00000
BM86	0,51570	0,65080	0,84665	0,96666	0,99700	1,00000
Dong93	0,50585	0,65775	0,88555	0,98685	0,99935	1,00000
JP92	0,39475	0,60245	0,75000	0,89715	0,97010	1,00000
JTuk87	0,50645	0,68450	0,89905	0,98690	0,99955	1,00000
Lenth89	0,45190	0,62670	0,80565	0,94110	0,99000	1,00000
LGB98	0,51015	0,70265	0,90980	0,98835	0,99920	1,00000
LN97	0,49080	0,51280	0,56435	0,64070	0,70365	0,74530
MaxUr	0,55850	0,90810	0,98945	0,99970	1,00000	1,00000
MLZ92	0,36680	0,59460	0,72575	0,87455	0,95965	1,00000
STuk82	0,47005	0,63815	0,82140	0,95090	0,99210	1,00000

Table A1.4.4: power for *n* = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,17370	0,61625	0,74575	0,75951	0,75640	0,75010
Ben89	0,02584	0,14174	0,44865	0,80284	0,96156	1,00000
Bi89	0,11934	0,47531	0,75369	0,88009	0,93956	0,99964
BM86	0,03135	0,18800	0,44406	0,70039	0,82120	0,96079
Dong93	0,03576	0,09915	0,05173	0,01573	0,00319	0,00000
JP92	0,15116	0,59054	0,81558	0,91028	0,96439	0,99988
JTuk87	0,05119	0,24423	0,29474	0,28251	0,22166	0,04483
Lenth89	0,08793	0,31434	0,49409	0,61875	0,69495	0,94466
LGB98	0,00476	0,00051	0,00000	0,00000	0,00000	0,00000
LN97	0,03088	0,15174	0,23623	0,29346	0,29868	0,30553
MaxUr	0,29940	0,89478	0,99346	0,99975	1,00000	1,00000
MLZ92	0,14570	0,62553	0,85461	0,93125	0,97471	1,00000
STuk82	0,06699	0,32468	0,61290	0,81894	0,92881	0,99999

Table A1.4.2: power for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,34295	0,60670	0,78055	0,92283	0,98050	1,00000
Ben89	0,44123	0,60620	0,82068	0,96313	0,99743	1,00000
Bi89	0,29538	0,57060	0,69670	0,82710	0,92618	0,99998
BM86	0,48595	0,58133	0,71530	0,85893	0,95738	1,00000
Dong93	0,46913	0,59045	0,80035	0,96050	0,99755	1,00000
JP92	0,35815	0,60035	0,74745	0,88890	0,96450	1,00000
JTuk87	0,46933	0,63985	0,85493	0,97250	0,99793	1,00000
Lenth89	0,41138	0,59743	0,78115	0,92898	0,98505	1,00000
LGB98	0,45695	0,51193	0,50350	0,50045	0,50005	0,50000
LN97	0,43875	0,50123	0,51105	0,55120	0,62590	0,79630
MaxUr	0,64135	0,90380	0,99033	0,99978	1,00000	1,00000
MLZ92	0,36810	0,59695	0,75088	0,89078	0,96590	1,00000
STuk82	0,41728	0,60958	0,78358	0,92738	0,98363	1,00000

Table A1.4.5: power I for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,61170	0,99540	1,00000	1,00000	1,00000	1,00000
Ben89	0,93430	1,00000	1,00000	1,00000	1,00000	1,00000
Bi89	0,57950	0,99340	1,00000	1,00000	1,00000	1,00000
BM86	0,97470	1,00000	1,00000	1,00000	1,00000	1,00000
Dong93	0,96350	1,00000	1,00000	1,00000	1,00000	1,00000
JP92	0,74340	1,00000	1,00000	1,00000	1,00000	1,00000
JTuk87	0,94960	1,00000	1,00000	1,00000	1,00000	1,00000
Lenth89	0,85670	1,00000	1,00000	1,00000	1,00000	1,00000
LGB98	0,95090	1,00000	1,00000	1,00000	1,00000	1,00000
LN97	0,97160	1,00000	1,00000	1,00000	1,00000	1,00000
MaxUr	0,72180	1,00000	1,00000	1,00000	1,00000	1,00000
MLZ92	0,69180	0,99980	1,00000	1,00000	1,00000	1,00000
STuk82	0,89240	1,00000	1,00000	1,00000	1,00000	1,00000

Table A1.4.3: power for *n* = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,30357	0,72418	0,89822	0,97917	0,99828	1,00000
Ben89	0,20015	0,65137	0,85083	0,95738	0,99518	1,00000
Bi89	0,20925	0,59010	0,77182	0,87222	0,94103	0,99993
BM86	0,27188	0,69525	0,78295	0,85772	0,92492	0,99973
Dong93	0,23620	0,61102	0,76605	0,89342	0,96902	1,00000
JP92	0,27758	0,67425	0,82595	0,91918	0,97168	1,00000
JTuk87	0,28048	0,71752	0,87823	0,96802	0,99580	1,00000
Lenth89	0,28112	0,66982	0,82075	0,93083	0,98208	1,00000
LGB98	0,08175	0,03480	0,00772	0,00128	0,00005	0,00000
LN97	0,18473	0,48685	0,59443	0,64702	0,69567	0,87000
MaxUr	0,57370	0,93768	0,99393	0,99987	1,00000	1,00000
MLZ92	0,29825	0,67887	0,82793	0,92255	0,97512	1,00000
STuk82	0,25718	0,68350	0,83777	0,93467	0,98153	1,00000

Table A1.4.6: power I for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,78240	0,99980	1,00000	1,00000	1,00000	1,00000
Ben89	0,91220	1,00000	1,00000	1,00000	1,00000	1,00000
Bi89	0,68090	0,99680	1,00000	1,00000	1,00000	1,00000
BM86	0,96300	1,00000	1,00000	1,00000	1,00000	1,00000
Dong93	0,95930	1,00000	1,00000	1,00000	1,00000	1,00000
JP92	0,76850	0,99960	1,00000	1,00000	1,00000	1,00000
JTuk87	0,92140	1,00000	1,00000	1,00000	1,00000	1,00000
Lenth89	0,86990	1,00000	1,00000	1,00000	1,00000	1,00000
LGB98	0,94050	0,99980	1,00000	1,00000	1,00000	1,00000
LN97	0,92290	1,00000	1,00000	1,00000	1,00000	1,00000
MaxUr	0,86910	1,00000	1,00000	1,00000	1,00000	1,00000
MLZ92	0,86260	1,00000	1,00000	1,00000	1,00000	1,00000
STuk82	0,86800	1,00000	1,00000	1,00000	1,00000	1,00000

Table A1.4.7: power I for *n* = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,69100	0,99990	1,00000	1,00000	1,00000	1,00000
Ben89	0,44010	0,97990	1,00000	1,00000	1,00000	1,00000
Bi89	0,47600	0,96430	0,99990	1,00000	1,00000	1,00000
BM86	0,58750	0,99650	1,00000	1,00000	1,00000	1,00000
Dong93	0,60330	0,98710	1,00000	1,00000	1,00000	1,00000
JP92	0,63890	0,99790	1,00000	1,00000	1,00000	1,00000
JTuk87	0,58840	0,98910	0,99930	1,00000	1,00000	1,00000
Lenth89	0,67260	0,99790	1,00000	1,00000	1,00000	1,00000
LGB98	0,30190	0,15140	0,03630	0,00670	0,00030	0,00000
LN97	0,42390	0,88780	0,99430	1,00000	1,00000	1,00000
MaxUr	0,70120	0,99970	1,00000	1,00000	1,00000	1,00000
MLZ92	0,73520	1,00000	1,00000	1,00000	1,00000	1,00000
STuk82	0,61640	0,99660	1,00000	1,00000	1,00000	1,00000

Table A1.4.8: power I for *n* = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,37110	0,92970	0,99850	1,00000	1,00000	1,00000
Ben89	0,04760	0,17810	0,50170	0,84370	0,97570	1,00000
Bi89	0,22080	0,70700	0,95420	0,99700	0,99980	1,00000
BM86	0,06570	0,30530	0,66100	0,92460	0,99210	1,00000
Dong93	0,10550	0,19620	0,09170	0,02840	0,00560	0,00000
JP92	0,34560	0,86550	0,95450	0,97620	0,99040	0,99990
JTuk87	0,12180	0,38450	0,38030	0,31800	0,23280	0,04490
Lenth89	0,25490	0,66700	0,81260	0,86930	0,90830	0,98350
LGB98	0,02680	0,00400	0,00000	0,00000	0,00000	0,00000
LN97	0,07220	0,27660	0,44250	0,56180	0,61680	0,68120
MaxUr	0,35580	0,94880	1,00000	1,00000	1,00000	1,00000
MLZ92	0,33100	0,89150	0,99590	1,00000	1,00000	1,00000
STuk82	0,19420	0,53550	0,73440	0,88120	0,95400	1,00000

Table A1.4.9: power II for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,04120	0,18510	0,43010	0,70940	0,88990	1,00000
Ben89	0,05040	0,29870	0,72910	0,96030	0,99790	1,00000
Bi89	0,04430	0,15840	0,36660	0,63730	0,83820	1,00000
BM86	0,05670	0,30160	0,69330	0,93310	0,99400	1,00000
Dong93	0,04820	0,31550	0,77110	0,97370	0,99870	1,00000
JP92	0,04610	0,20490	0,50000	0,79430	0,94020	1,00000
JTuk87	0,06330	0,36900	0,79810	0,97380	0,99910	1,00000
Lenth89	0,04710	0,25340	0,61130	0,88220	0,98000	1,00000
LGB98	0,06940	0,40530	0,81960	0,97670	0,99840	1,00000
LN97	0,01000	0,02560	0,12870	0,28140	0,40730	0,49060
MaxUr	0,38890	0,81620	0,97890	0,99940	1,00000	1,00000
MLZ92	0,04180	0,18940	0,45150	0,74910	0,91930	1,00000
STuk82	0,04770	0,27630	0,64280	0,90180	0,98420	1,00000

Table A1.4.10: power II for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,01510	0,13540	0,43070	0,75870	0,93040	1,00000
Ben89	0,01110	0,14820	0,56670	0,89860	0,99170	1,00000
Bi89	0,02290	0,09820	0,28200	0,53580	0,77200	0,99990
BM86	0,00640	0,07180	0,29700	0,60870	0,86660	1,00000
Dong93	0,00460	0,11130	0,52370	0,89860	0,99310	1,00000
JP92	0,01750	0,12880	0,38670	0,69930	0,89550	1,00000
JTuk87	0,01250	0,18600	0,63510	0,92520	0,99450	1,00000
Lenth89	0,00740	0,11410	0,45120	0,79670	0,95260	1,00000
LGB98	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
LN97	0,00140	0,00230	0,02130	0,10190	0,25170	0,59260
MaxUr	0,23440	0,69830	0,96630	0,99910	1,00000	1,00000
MLZ92	0,01400	0,10960	0,33740	0,63930	0,87220	1,00000
STuk82	0,00980	0,13080	0,45340	0,79350	0,94780	1,00000

Table A1.4.11: power II for *n* = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,01430	0,18490	0,59350	0,90150	0,99050	1,00000
Ben89	0,00930	0,13330	0,51240	0,85290	0,98260	1,00000
Bi89	0,02190	0,10300	0,28480	0,52230	0,75410	0,99960
BM86	0,00390	0,06650	0,22070	0,44170	0,67620	0,99850
Dong93	0,00160	0,04680	0,28630	0,66710	0,90480	1,00000
JP92	0,01720	0,13790	0,38680	0,68100	0,87990	1,00000
JTuk87	0,00830	0,15870	0,55530	0,87370	0,98300	1,00000
Lenth89	0,00350	0,07840	0,36150	0,72610	0,92210	1,00000
LGB98	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
LN97	0,00140	0,02080	0,05370	0,08850	0,15570	0,61080
MaxUr	0,17250	0,70040	0,96810	0,99920	1,00000	1,00000
MLZ92	0,01310	0,11240	0,34350	0,64010	0,86520	1,00000
STuk82	0,00770	0,12350	0,41580	0,73860	0,91880	1,00000

Table A1.4.12: power II for *n* = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
Ben89	0,00010	0,03450	0,27910	0,67390	0,91750	1,00000
Bi89	0,02290	0,10590	0,26850	0,48520	0,70240	0,99730
BM86	0,00000	0,00670	0,04020	0,12620	0,23570	0,76290
Dong93	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
JP92	0,00680	0,11360	0,35910	0,63970	0,84760	0,99980
JTuk87	0,00050	0,03020	0,12570	0,21720	0,20820	0,04480
Lenth89	0,00010	0,00050	0,01020	0,04030	0,12050	0,85620
LGB98	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
LN97	0,00000	0,00080	0,00460	0,01150	0,01170	0,00130
MaxUr	0,04900	0,60510	0,95500	0,99810	1,00000	1,00000
MLZ92	0,00400	0,09520	0,31390	0,59290	0,82450	1,00000
STuk82	0,00090	0,03930	0,24050	0,56430	0,81220	0,99990

Table A1.4.13: power III for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,01950	0,13860	0,38380	0,65660	0,83960	0,94980
Ben89	0,03470	0,26440	0,68260	0,91270	0,94720	0,94970
Bi89	0,01530	0,11150	0,31690	0,58930	0,78400	0,94980
BM86	0,04730	0,29410	0,69040	0,93300	0,99390	1,00000
Dong93	0,03940	0,29440	0,74030	0,94590	0,97050	0,97290
JP92	0,02380	0,15290	0,44910	0,73570	0,88340	0,94540
JTuk87	0,04370	0,32000	0,74140	0,91610	0,93950	0,94120
Lenth89	0,03260	0,22150	0,57240	0,84160	0,93950	0,96070
LGB98	0,05850	0,37690	0,78340	0,93900	0,95710	0,96160
LN97	0,00870	0,02540	0,12860	0,28130	0,40720	0,49020
MaxUr	0,05270	0,40210	0,67510	0,77840	0,83400	0,97030
MLZ92	0,02140	0,14380	0,40560	0,69830	0,86910	0,95020
STuk82	0,03230	0,23560	0,59730	0,85010	0,93510	0,94990

Table A1.4.14: power III for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,00430	0,09270	0,38140	0,70630	0,88400	0,94760
Ben89	0,00550	0,12100	0,52470	0,84970	0,94230	0,94970
Bi89	0,00300	0,05500	0,23180	0,48860	0,71890	0,95200
BM86	0,00480	0,07120	0,29690	0,60870	0,86660	1,00000
Dong93	0,00290	0,10270	0,50880	0,88470	0,98080	0,98910
JP92	0,00520	0,07910	0,33320	0,63610	0,83350	0,93820
JTuk87	0,00550	0,14540	0,58210	0,86590	0,93310	0,93960
Lenth89	0,00410	0,09620	0,41880	0,76310	0,91850	0,96660
LGB98	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
LN97	0,00080	0,00230	0,02130	0,10190	0,25170	0,59260
MaxUr	0,03790	0,37430	0,69690	0,79860	0,84860	0,94440
MLZ92	0,00550	0,07130	0,29530	0,58730	0,82380	0,95230
STuk82	0,00470	0,09510	0,41170	0,74120	0,89750	0,95180

Table A1.4.15: power III for n = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,00760	0,15050	0,54620	0,84890	0,94030	0,94880
Ben89	0,00500	0,11050	0,47110	0,80660	0,93560	0,95360
Bi89	0,00440	0,06100	0,23590	0,47870	0,70640	0,95280
BM86	0,00370	0,06620	0,22070	0,44170	0,67620	0,99850
Dong93	0,00120	0,04490	0,28480	0,66540	0,90270	0,99820
JP92	0,00690	0,08920	0,32510	0,61660	0,81550	0,93260
JTuk87	0,00470	0,12690	0,50590	0,81910	0,93020	0,94810
Lenth89	0,00250	0,07070	0,34620	0,70310	0,88690	0,96640
LGB98	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
LN97	0,00090	0,02030	0,05370	0,08840	0,15570	0,61080
MaxUr	0,04290	0,41940	0,72900	0,81710	0,86290	0,93530
MLZ92	0,00550	0,07520	0,29770	0,58920	0,81510	0,94820
STuk82	0,00380	0,09160	0,37060	0,68560	0,87040	0,94870

Table A1.4.16: power III for n = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
Ben89	0,00070	0,02920	0,25590	0,63490	0,87580	0,95580
Bi89	0,00360	0,06370	0,22390	0,44220	0,65370	0,95270
BM86	0,00000	0,00670	0,04020	0,12620	0,23570	0,76290
Dong93	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
JP92	0,00270	0,07420	0,29980	0,57400	0,77530	0,93050
JTuk87	0,00000	0,02220	0,11260	0,20060	0,19600	0,04160
Lenth89	0,00000	0,00020	0,00980	0,04030	0,12040	0,85620
LGB98	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
LN97	0,00000	0,00080	0,00460	0,01150	0,01170	0,00130
MaxUr	0,04900	0,60510	0,95500	0,99810	1,00000	1,00000
MLZ92	0,00220	0,06820	0,27420	0,54560	0,77590	0,95120
STuk82	0,00050	0,03090	0,21820	0,52510	0,76840	0,94820

Table A1.4.17: power IV for n = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,61160	0,99540	1,00000	1,00000	1,00000	1,00000
Ben89	0,93430	1,00000	1,00000	1,00000	1,00000	1,00000
Bi89	0,57950	0,99340	1,00000	1,00000	1,00000	1,00000
BM86	0,97470	1,00000	1,00000	1,00000	1,00000	1,00000
Dong93	0,96340	1,00000	1,00000	1,00000	1,00000	1,00000
JP92	0,74330	1,00000	1,00000	1,00000	1,00000	1,00000
JTuk87	0,94960	1,00000	1,00000	1,00000	1,00000	1,00000
Lenth89	0,85670	1,00000	1,00000	1,00000	1,00000	1,00000
LGB98	0,95090	1,00000	1,00000	1,00000	1,00000	1,00000
LN97	0,97160	1,00000	1,00000	1,00000	1,00000	1,00000
MaxUr	0,72810	1,00000	1,00000	1,00000	1,00000	1,00000
MLZ92	0,69170	0,99980	1,00000	1,00000	1,00000	1,00000
STuk82	0,89240	1,00000	1,00000	1,00000	1,00000	1,00000

Table A1.4.18: power IV for n = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,51100	0,99260	1,00000	1,00000	1,00000	1,00000
Ben89	0,78410	1,00000	1,00000	1,00000	1,00000	1,00000
Bi89	0,42190	0,96980	1,00000	1,00000	1,00000	1,00000
BM86	0,90200	1,00000	1,00000	1,00000	1,00000	1,00000
Dong93	0,86420	1,00000	1,00000	1,00000	1,00000	1,00000
JP92	0,57990	0,99790	1,00000	1,00000	1,00000	1,00000
JTuk87	0,86240	1,00000	1,00000	1,00000	1,00000	1,00000
Lenth89	0,71080	0,99970	1,00000	1,00000	1,00000	1,00000
LGB98	0,85440	0,99920	1,00000	1,00000	1,00000	1,00000
LN97	0,81410	0,99710	1,00000	1,00000	1,00000	1,00000
MaxUr	0,86910	1,00000	1,00000	1,00000	1,00000	1,00000
MLZ92	0,53470	0,99710	1,00000	1,00000	1,00000	1,00000
STuk82	0,72630	0,99990	1,00000	1,00000	1,00000	1,00000

Table A1.4.19: power IV for n = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,17750	0,85410	0,99840	1,00000	1,00000	1,00000
Ben89	0,14310	0,76670	0,99300	1,00000	1,00000	1,00000
Bi89	0,12230	0,58140	0,91650	0,99390	0,99970	1,00000
BM86	0,19940	0,92200	0,99960	1,00000	1,00000	1,00000
Dong93	0,12670	0,70490	0,96870	0,99910	1,00000	1,00000
JP92	0,16230	0,77140	0,98860	0,99990	1,00000	1,00000
JTuk87	0,20660	0,90190	0,99740	1,00000	1,00000	1,00000
Lenth89	0,15480	0,81990	0,99610	1,00000	1,00000	1,00000
LGB98	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
LN97	0,12880	0,54100	0,73250	0,85230	0,93130	0,99920
MaxUr	0,68250	0,99970	1,00000	1,00000	1,00000	1,00000
MLZ92	0,14010	0,75950	0,99160	1,00000	1,00000	1,00000
STuk82	0,14040	0,81020	0,99530	1,00000	1,00000	1,00000

Table A1.4.20: power IV for n = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	0,10400	0,65890	0,94150	0,99630	0,99970	1,00000
Ben89	0,02180	0,17130	0,50030	0,84370	0,97570	1,00000
Bi89	0,08230	0,44490	0,81240	0,96680	0,99690	1,00000
BM86	0,02420	0,20710	0,51030	0,83120	0,96980	1,00000
Dong93	0,01670	0,09220	0,05610	0,01740	0,00360	0,00000
JP92	0,07240	0,58030	0,91420	0,97490	0,99040	0,99990
JTuk87	0,03280	0,26160	0,32050	0,29130	0,22190	0,04480
Lenth89	0,02800	0,21690	0,49420	0,74020	0,85040	0,95980
LGB98	0,00000	0,00000	0,00000	0,00000	0,00000	0,00000
LN97	0,02150	0,16260	0,24840	0,30020	0,28310	0,26980
MaxUr	0,34000	0,94880	1,00000	1,00000	1,00000	1,00000
MLZ92	0,05940	0,58740	0,96740	0,99960	1,00000	1,00000
STuk82	0,02690	0,31370	0,69460	0,87940	0,95400	1,00000

A2 Simulated LD1L0, LD1L9 and LD2L

A2.1. LD1L0, LD1L9 and LD2L for a 16-run experiment and 1~8 active contrasts with the same magnitude

Figure A2.1.1: LD1L0 for n = 16 and one active contrast.

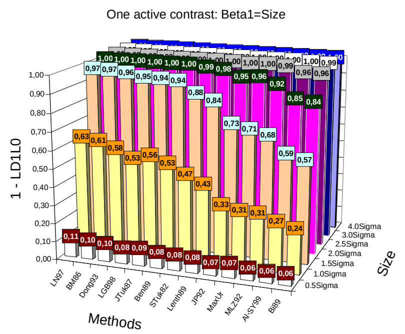


Figure A2.1.4: LD1L0 for n = 16 and six active contrasts with the same magnitudes.

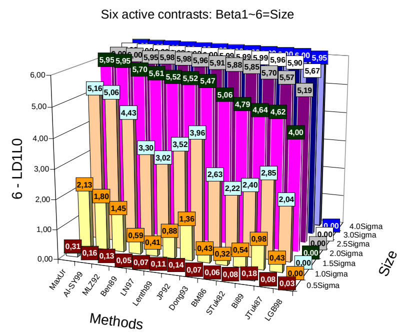


Figure A2.1.2: LD1L0 for n = 16 and two active contrasts with the same magnitudes.

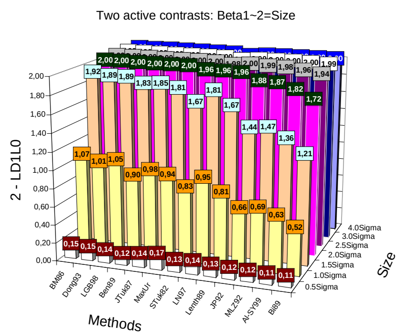


Figure A2.1.5: LD1L0 for n = 16 and eight active contrasts with the same magnitudes.

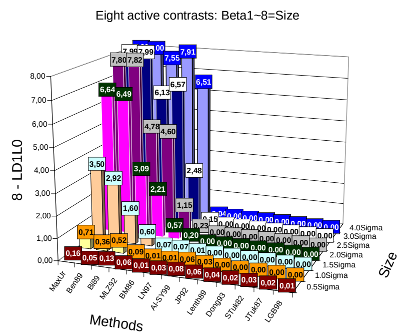


Figure A2.1.3: LD1L0 for n = 16 and four active contrasts with the same magnitudes.

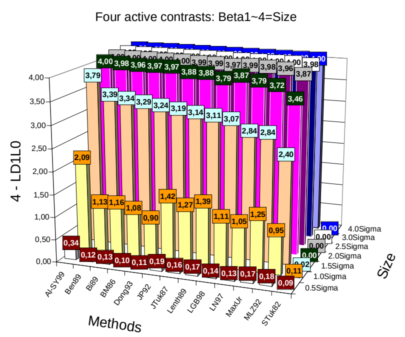


Figure A2.1.6: LD1L9 for n = 16 and one active contrast.

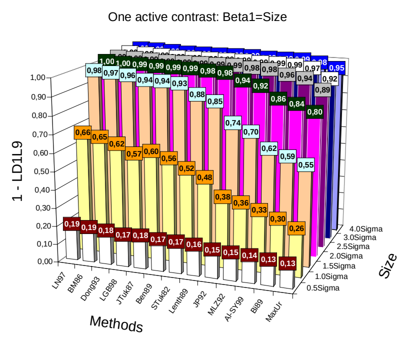


Figure A2.1.7: LD1L9 for *n* = 16 and two active contrasts with the same magnitudes.

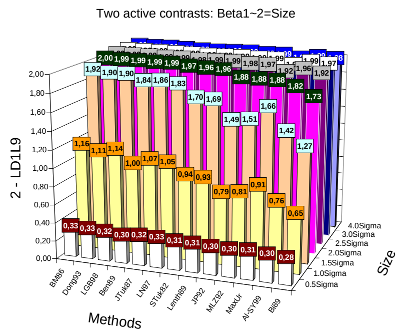


Figure A2.1.10: LD1L9 for *n* = 16 and eight active contrasts with the same magnitudes.

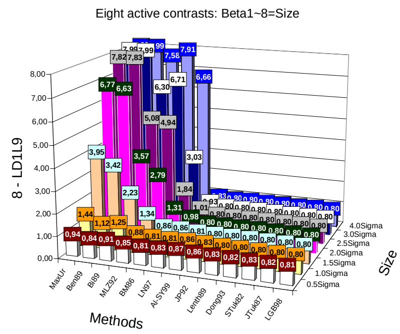


Figure A2.1.8: LD1L9 for *n* = 16 and four active contrasts with the same magnitudes.

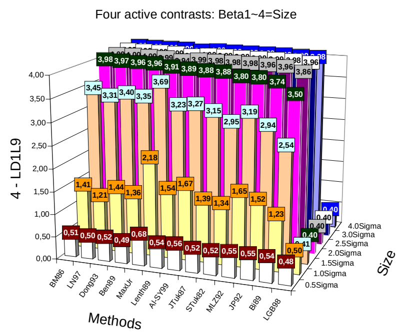


Figure A2.1.11: LD2L for *n* = 16 and one active contrast.

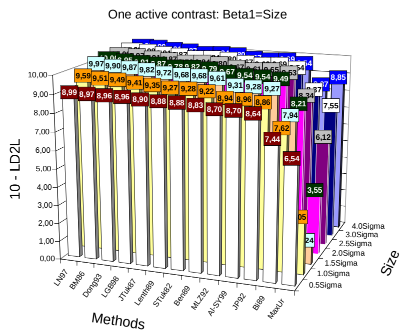


Figure A2.1.9: LD1L9 for *n* = 16 and six active contrasts with the same magnitudes.

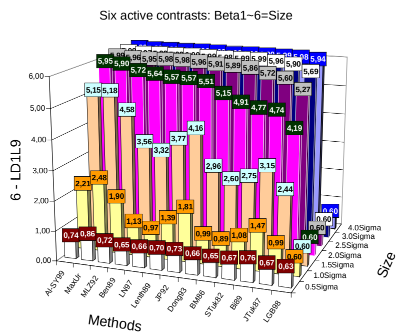


Figure A2.1.12: LD2L for *n* = 16 and two active contrasts with the same magnitudes.

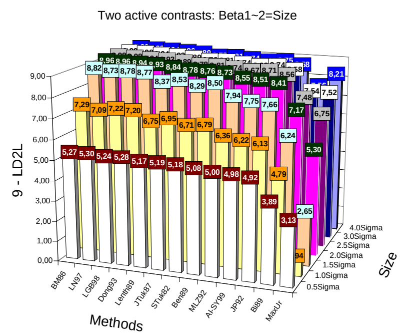


Figure A2.1.13: LD2L for *n* = 16 and four active contrasts with the same magnitudes.

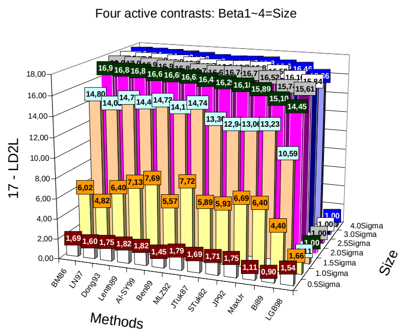


Figure A2.1.14: LD2L for *n* = 16 and six active contrasts with the same magnitudes.

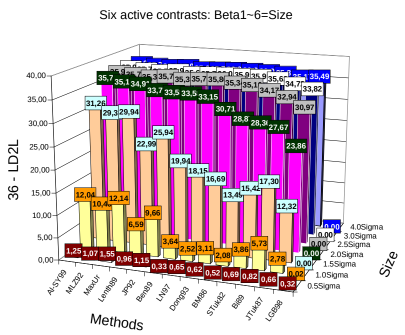
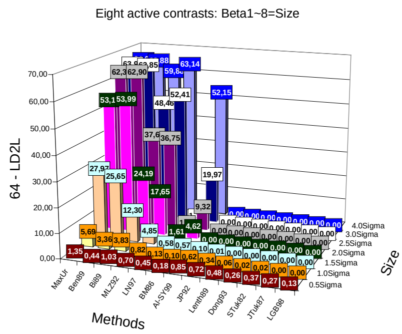


Figure A2.1.15: LD2L for *n* = 16 and eight active contrasts with the same magnitudes.



A2.2 LD1L0, LD1L9 and LD2L for a 16-run experiment and 2~8 active contrasts with different magnitudes

Figure A2.2.1: LD1L0 for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

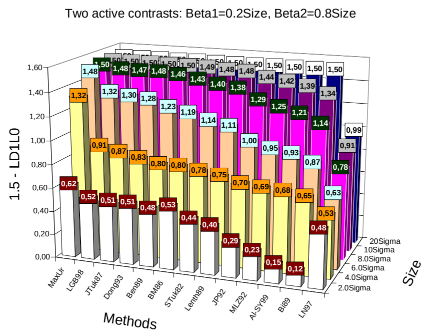


Figure A2.2.4: LD1L0 for *n* = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

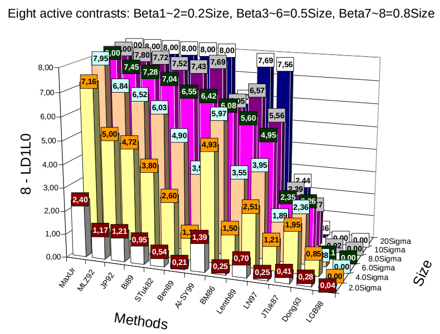


Figure A2.2.2: LD1L0 for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

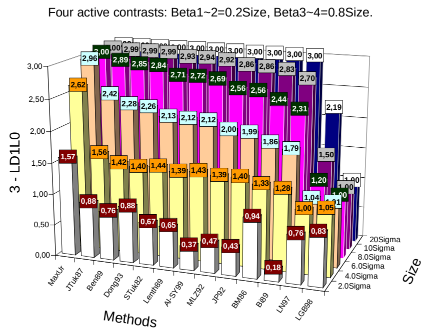


Figure A2.2.5: LD1L9 for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

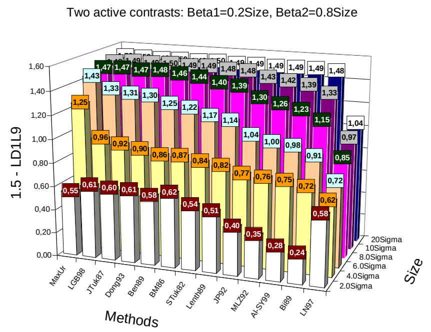


Figure A2.2.3: LD1L0 for *n* = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

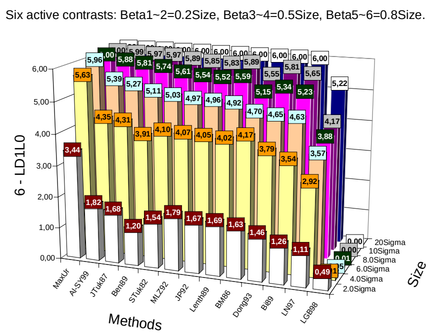


Figure A2.2.6: LD1L9 for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

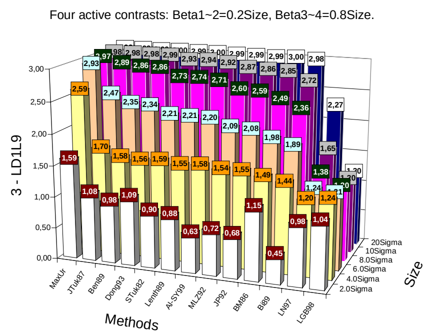


Figure A2.2.7: LD1L9 for n = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

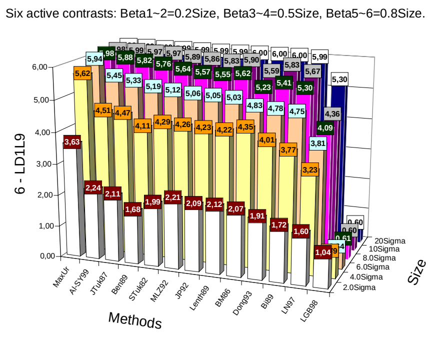


Figure A2.2.10: LD2L for n = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

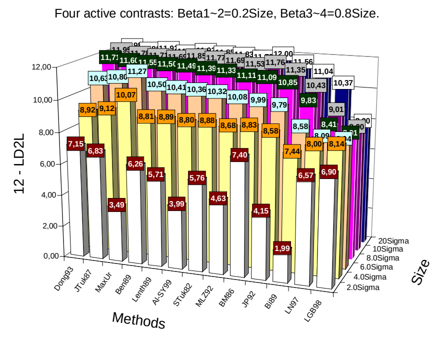


Figure A2.2.8: LD1L9 for n = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

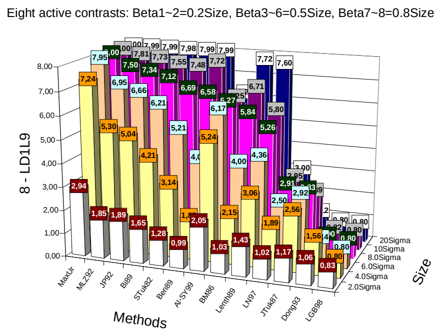


Figure A2.2.11: LD2L for n = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

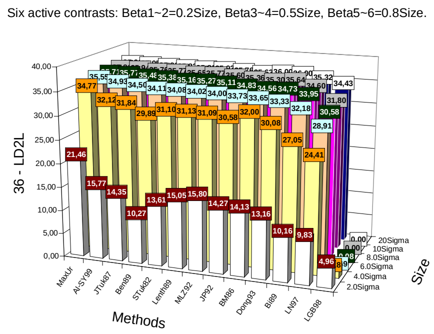


Figure A2.2.9: LD2L for n = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

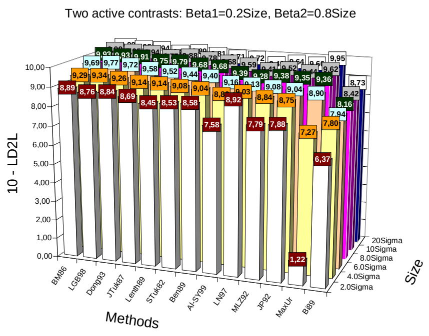
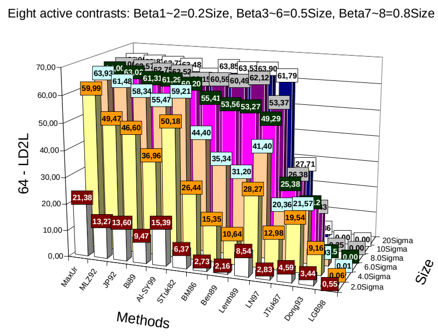


Figure A2.2.12: LD2L for n = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.



A2.3 Tables of LD1L0, LD1L9 and LD2L for a 16-run experiment and 1~8 active contrasts with the same magnitude

Table A2.3.1: LD1L0 for n = 16 and one active contrast.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,9411	0,7283	0,4083	0,148	0,0362	0,0045	0
Ben89	0,917	0,4749	0,0621	0,001	0	0	0
Bi89	0,9406	0,7555	0,4328	0,1596	0,0379	0,0057	0
BM86	0,8973	0,3851	0,0288	0,0004	0	0	0
Dong93	0,9001	0,411	0,0368	0,0002	0	0	0
JP92	0,9345	0,6709	0,2709	0,0512	0,0046	0	0
JTuk87	0,9071	0,4402	0,0556	0,002	0	0	0
Lenth89	0,9224	0,5693	0,1558	0,0162	0,0007	0	0
LGB98	0,9166	0,466	0,055	0,0008	0	0	0
LN97	0,8905	0,3739	0,027	0,0004	0	0	0
MaxUr	0,9253	0,6904	0,2921	0,044	0,0015	0	0
MLZ92	0,9375	0,6941	0,3161	0,0772	0,0086	0,0005	0
STuk82	0,9184	0,5253	0,1218	0,0102	0,0001	0	0

Table A2.3.2: LD1L0 for n = 16 and two active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	1,8853	1,3709	0,637	0,1849	0,0354	0,0035	0
Ben89	1,8828	1,1019	0,1677	0,0041	0	0	0
Bi89	1,8881	1,4809	0,7881	0,2765	0,065	0,0107	0
BM86	1,8489	0,928	0,0811	0,001	0	0	0
Dong93	1,8531	0,9807	0,1087	0,0018	0	0	0
JP92	1,8779	1,3363	0,5552	0,1189	0,0127	0,0004	0
JTuk87	1,8562	1,0225	0,1501	0,0047	0	0	0
Lenth89	1,869	1,1873	0,3346	0,0405	0,0026	0	0
LGB98	1,8632	0,9498	0,1074	0,0037	0	0	0
LN97	1,8573	1,0488	0,1862	0,0359	0,0194	0,018	0,0179
MaxUr	1,8282	1,0593	0,1906	0,0048	0	0	0
MLZ92	1,8844	1,308	0,5336	0,1252	0,017	0,0003	0
STuk82	1,8707	1,1723	0,3262	0,0355	0,0009	0	0

Table A2.3.3: LD1L0 for n = 16 and four active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	3,8136	2,5755	0,8054	0,1246	0,0103	0,0003	0
Ben89	3,8966	2,9238	0,715	0,0328	0,0002	0	0
Bi89	3,825	3,0542	1,6018	0,5407	0,1304	0,0224	0,0001
BM86	3,8753	2,8688	0,6109	0,0223	0	0	0
Dong93	3,8658	2,8708	0,6971	0,0338	0,0003	0	0
JP92	3,8258	2,7472	1,1633	0,2786	0,0357	0,0035	0
JTuk87	3,8645	2,8886	0,9339	0,127	0,0102	0	0
Lenth89	3,8433	2,7253	0,8551	0,119	0,0107	0,0004	0
LGB98	3,909	3,8931	3,9835	3,9994	4	4	4
LN97	3,8928	3,0953	0,763	0,0319	0,0004	0	0
MaxUr	3,6555	1,9122	0,2148	0,0028	0	0	0
MLZ92	3,8267	2,6063	0,8906	0,209	0,0285	0,0023	0
STuk82	3,8678	2,9502	1,1619	0,2101	0,0178	0,0005	0

Table A2.3.4: LD1L0 for n = 16 and six active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	5,8425	4,204	0,9386	0,0494	0,0003	0	0
Ben89	5,9479	5,4123	2,7014	0,3882	0,0171	0	0
Bi89	5,8177	5,0204	3,1491	1,3814	0,431	0,0961	0,0004
BM86	5,9427	5,6753	3,7805	1,21	0,1537	0,0077	0
Dong93	5,9267	5,5401	3,3462	0,8883	0,1064	0,0042	0
JP92	5,8579	4,6442	2,0362	0,533	0,0866	0,0089	0
JTuk87	5,9247	5,5703	3,9555	2,0025	0,8068	0,3317	0,0528
Lenth89	5,8934	5,1197	2,4768	0,4784	0,0416	0,002	0
LGB98	5,9684	5,998	6	6	6	6	6
LN97	5,933	5,5927	2,9776	0,4756	0,0234	0,0006	0
MaxUr	5,6949	3,8715	0,8396	0,0462	0	0	0
MLZ92	5,8702	4,5511	1,5677	0,2995	0,047	0,0038	0
STuk82	5,9218	5,4633	3,6034	1,3626	0,303	0,0385	0

Table A2.3.5: LD1L0 for n = 16 and eight active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	7,9248	7,9359	7,9887	8	8	8	8
Ben89	7,9491	7,6442	5,0841	1,5099	0,179	0,0079	0
Bi89	7,8677	7,484	6,4011	4,9091	3,2249	1,8744	0,4472
BM86	7,9864	7,9887	7,9308	7,4284	6,8478	5,5226	1,4932
Dong93	7,9827	8	8	8	8	8	8
JP92	7,9377	7,9683	7,9995	8	8	8	8
JTuk87	7,9785	7,9999	8	8	8	8	8
Lenth89	7,9629	7,995	7,9996	8	8	8	8
LGB98	7,9911	8	8	8	8	8	8
LN97	7,9668	7,9866	7,9278	7,799	7,7704	7,8536	7,964
MaxUr	7,8434	7,2919	4,5041	1,3624	0,2024	0,012	0
MLZ92	7,94	7,9069	7,3954	5,7887	3,3954	1,4312	0,0864
STuk82	7,9708	7,9985	8	8	8	8	8

Table A2.3.6: LD1L9 for n = 16 and one active contrast.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	0,85652	0,66552	0,37751	0,14324	0,04262	0,01409	0,01004
Ben89	0,83164	0,43591	0,06539	0,0104	0,0095	0,0095	0,0095
Bi89	0,86651	0,70103	0,41063	0,16475	0,05522	0,02624	0,02111
BM86	0,81249	0,35293	0,03133	0,00433	0,00289	0,00189	0,00074
Dong93	0,81531	0,37539	0,0387	0,00576	0,00558	0,00558	0,00558
JP92	0,85098	0,61576	0,25581	0,05808	0,01614	0,012	0,012
JTuk87	0,82249	0,40469	0,05915	0,01091	0,00911	0,00911	0,00911
Lenth89	0,83705	0,51986	0,14779	0,02215	0,0082	0,00757	0,00757
LGB98	0,83026	0,428	0,05888	0,01013	0,00941	0,00941	0,00941
LN97	0,80572	0,33895	0,02476	0,00045	0,00005	0,00005	0,00005
MaxUr	0,8749	0,7418	0,44672	0,19937	0,11233	0,07935	0,04622
MLZ92	0,85313	0,63545	0,29544	0,08043	0,01869	0,0114	0,01095
STuk82	0,83268	0,48041	0,1177	0,01727	0,00818	0,00809	0,00809

Table A2.3.7: LD1L9 for n = 16 and two active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	1,70463	1,24308	0,58259	0,17569	0,04114	0,01243	0,00928
Ben89	1,69922	0,99863	0,15958	0,01247	0,00878	0,00878	0,00878
Bi89	1,71691	1,35268	0,72937	0,26893	0,07858	0,02971	0,02008
BM86	1,66774	0,84108	0,078	0,00403	0,00166	0,00073	0,00019
Dong93	1,67106	0,88668	0,102	0,00579	0,00417	0,00417	0,00417
JP92	1,69881	1,21402	0,51164	0,11897	0,02339	0,01232	0,01196
JTuk87	1,67557	0,92864	0,1446	0,01376	0,00953	0,00953	0,00953
Lenth89	1,68699	1,07433	0,30711	0,04244	0,00833	0,00599	0,00599
LGB98	1,68023	0,85893	0,10117	0,00785	0,00452	0,00452	0,00452
LN97	1,67406	0,94622	0,16801	0,0324	0,01751	0,01625	0,01616
MaxUr	1,68667	1,09319	0,33813	0,1237	0,084	0,06184	0,03919
MLZ92	1,70321	1,18721	0,49	0,12243	0,02505	0,01002	0,00975
STuk82	1,6879	1,06222	0,30144	0,03985	0,00871	0,0079	0,0079

Table A2.3.8: LD1L9 for n = 16 and four active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	3,43677	2,32538	0,7326	0,11988	0,01701	0,00801	0,00774
Ben89	3,50935	2,63648	0,6525	0,03898	0,00964	0,00946	0,00946
Bi89	3,45593	2,76626	1,45954	0,50455	0,13528	0,03808	0,01801
BM86	3,48921	2,58578	0,55407	0,02214	0,00069	0,0002	0
Dong93	3,48039	2,58511	0,62899	0,03202	0,00187	0,0016	0,0016
JP92	3,4477	2,48346	1,05936	0,26322	0,04461	0,01563	0,01248
JTuk87	3,48034	2,60586	0,84968	0,12374	0,01863	0,00945	0,00945
Lenth89	3,4611	2,45575	0,77359	0,11124	0,01379	0,00452	0,00416
LGB98	3,51869	3,5038	3,58515	3,59946	3,6	3,6	3,6
LN97	3,50459	2,7874	0,68752	0,0288	0,00039	0,00002	0,00001
MaxUr	3,31987	1,82272	0,30934	0,08709	0,06223	0,04783	0,03211
MLZ92	3,44794	2,35388	0,81129	0,19798	0,03553	0,01195	0,00988
STuk82	3,48274	2,65977	1,05286	0,19663	0,02358	0,00802	0,00757

Table A2.3.9: LD1L9 for *n* = 16 and six active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	5,25995	3,78764	0,85015	0,04987	0,00568	0,00541	0,00541
Ben89	5,35449	4,87336	2,43725	0,35746	0,02362	0,00824	0,00824
Bi89	5,24444	4,53337	2,85027	1,25935	0,40399	0,10258	0,01645
BM86	5,34873	5,10833	3,40376	1,08976	0,13861	0,00699	0
Dong93	5,3343	4,98626	3,01179	0,79968	0,09597	0,00399	0,00021
JP92	5,27367	4,18508	1,84278	0,49158	0,08986	0,01933	0,01192
JTuk87	5,33279	5,015	3,56489	1,80958	0,73453	0,30713	0,05617
Lenth89	5,30461	4,60854	2,23134	0,43426	0,04175	0,0062	0,0044
LGB98	5,37163	5,3982	5,4	5,4	5,4	5,4	5,4
LN97	5,33996	5,0337	2,68007	0,42809	0,02106	0,00054	0
MaxUr	5,13686	3,52338	0,82059	0,09575	0,04222	0,03395	0,02401
MLZ92	5,28451	4,10041	1,41871	0,27805	0,05084	0,01196	0,00854
STuk82	5,33024	4,91834	3,24736	1,23282	0,28002	0,04219	0,00758

Table A2.3.10: LD1L9 for *n* = 16 and eight active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	7,13273	7,14231	7,18983	7,2	7,2	7,2	7,2
Ben89	7,15566	6,88063	4,57908	1,36505	0,16765	0,01372	0,00664
Bi89	7,08531	6,74514	5,77399	4,43174	2,91596	1,70051	0,41603
BM86	7,18779	7,18983	7,13772	6,68556	6,16302	4,97034	1,34388
Dong93	7,18445	7,2	7,2	7,2	7,2	7,2	7,2
JP92	7,14437	7,1715	7,19955	7,2	7,2	7,2	7,2
JTuk87	7,18072	7,19991	7,2	7,2	7,2	7,2	7,2
Lenth89	7,16665	7,1955	7,19964	7,2	7,2	7,2	7,2
LGB98	7,192	7,2	7,2	7,2	7,2	7,2	7,2
LN97	7,17017	7,18794	7,13502	7,0191	6,99336	7,06824	7,1676
MaxUr	7,06178	6,56286	4,0537	1,22616	0,18216	0,0108	0
MLZ92	7,14641	7,11634	6,6563	5,21216	3,0603	1,2947	0,08556
STuk82	7,17386	7,19865	7,2	7,2	7,2	7,2	7,2

Table A2.3.11: LD2L for *n* = 16 and one active contrast.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	1,3012	1,0416	0,7177	0,4574	0,3456	0,3139	0,3094
Ben89	1,1746	0,7803	0,3935	0,3322	0,3312	0,3312	0,3312
Bi89	2,5573	2,3823	2,0587	1,7855	1,6638	1,6316	1,6259
BM86	1,0285	0,4915	0,1037	0,0485	0,0327	0,0205	0,008
Dong93	1,0389	0,5089	0,1318	0,0952	0,095	0,095	0,095
JP92	1,3616	1,1356	0,7325	0,5128	0,4662	0,4616	0,4616
JTuk87	1,1037	0,6523	0,2773	0,2235	0,2215	0,2215	0,2215
Lenth89	1,1245	0,7324	0,3163	0,1767	0,1612	0,1605	0,1605
LGB98	1,039	0,592	0,185	0,1311	0,1303	0,1303	0,1303
LN97	1,0052	0,4133	0,0334	0,0013	0,0005	0,0005	0,0005
MaxUr	3,456	6,9532	8,7584	6,4529	3,8785	2,4499	1,1544
MLZ92	1,2965	1,0601	0,694	0,4551	0,3865	0,3784	0,3779
STuk82	1,1206	0,7189	0,3206	0,2091	0,199	0,1989	0,1989

Table A2.3.12: LD2L for *n* = 16 and two active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	4,0241	2,782	1,2479	0,4921	0,2928	0,2583	0,2548
Ben89	3,9168	2,2065	0,5046	0,2663	0,2614	0,2614	0,2614
Bi89	5,1071	4,2112	2,7567	1,8299	1,5228	1,4553	1,4442
BM86	3,727	1,7118	0,1796	0,0357	0,0178	0,0075	0,0019
Dong93	3,7228	1,7702	0,2238	0,0701	0,0679	0,0679	0,0679
JP92	4,0841	2,867	1,3434	0,5949	0,4371	0,4204	0,42
JTuk87	3,8105	2,0454	0,4666	0,2232	0,2159	0,2159	0,2159
Lenth89	3,8313	2,2497	0,6267	0,1644	0,1127	0,1093	0,1093
LGB98	3,7609	1,7753	0,2181	0,0593	0,0552	0,0552	0,0552
LN97	3,7034	1,9148	0,2655	0,0372	0,0199	0,0185	0,0184
MaxUr	5,8673	8,0575	6,3529	3,699	2,2524	1,4778	0,7879
MLZ92	3,9969	2,6433	1,061	0,4531	0,3347	0,318	0,3177
STuk82	3,8196	2,2908	0,7146	0,2363	0,1885	0,1876	0,1876

Table A2.3.13: LD2L for *n* = 16 and four active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	15,1757	9,31	2,2742	0,351	0,1523	0,1419	0,1416
Ben89	15,5527	11,4322	2,8318	0,3782	0,2708	0,2706	0,2706
Bi89	16,1029	12,6042	6,415	2,5493	1,39	1,1648	1,1367
BM86	15,3127	10,977	2,2033	0,0964	0,0069	0,002	0
Dong93	15,2343	10,6289	2,3737	0,1346	0,0215	0,0206	0,0206
JP92	15,2468	10,3062	3,9408	1,1112	0,4801	0,4065	0,4002
JTuk87	15,312	11,111	3,7048	0,7114	0,2561	0,2159	0,2159
Lenth89	15,1794	9,8665	2,6031	0,3404	0,0829	0,0618	0,0612
LGB98	15,4591	15,3376	15,8947	15,9962	16	16	16
LN97	15,3985	12,1844	2,9696	0,1192	0,0007	0,0002	0,0001
MaxUr	15,8873	10,604	3,7726	1,9047	1,2649	0,8979	0,5379
MLZ92	15,2112	9,2758	2,2585	0,566	0,3317	0,3051	0,3028
STuk82	15,2876	11,0683	4,0574	0,8175	0,233	0,1852	0,1847

Table A2.3.14: LD2L for *n* = 16 and six active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	34,7501	23,9566	4,7377	0,2631	0,0544	0,0541	0,0541
Ben89	35,6687	32,3628	16,0561	2,4246	0,3022	0,2052	0,2052
Bi89	35,1838	30,2655	18,7023	8,3323	3,0583	1,284	0,8279
BM86	35,4775	33,9167	22,5052	7,1272	0,9031	0,0463	0
Dong93	35,3236	32,7342	19,4427	5,1806	0,6313	0,0273	0,0021
JP92	34,8453	26,3366	10,0616	2,4158	0,618	0,3595	0,3334
JTuk87	35,3405	33,2226	23,6807	12,1382	5,0279	2,1847	0,5125
Lenth89	35,0415	29,4132	13,0052	2,2316	0,2165	0,0736	0,0696
LGB98	35,6811	35,9788	36	36	36	36	36
LN97	35,3492	33,4798	17,8469	2,8491	0,1404	0,0036	0
MaxUr	34,4512	23,863	6,0625	1,0905	0,6056	0,4719	0,3199
MLZ92	34,9277	25,6047	6,6093	0,8055	0,2686	0,2226	0,2188
STuk82	35,3052	32,141	20,5806	7,6398	1,8264	0,3787	0,1754

Table A2.3.15: LD2L for *n* = 16 and eight active contrasts with the same magnitudes.

Size / Methods	0.5Sigma	1.0Sigma	1.5Sigma	2.0Sigma	2.5Sigma	3.0Sigma	4.0Sigma
Al-SY99	63,1509	63,3783	63,8977	64	64	64	64
Ben89	63,5618	60,6419	38,3472	10,0117	1,0991	0,1496	0,1164
Bi89	62,9691	60,1694	51,6983	39,8116	26,345	15,5417	4,1241
BM86	63,8165	63,9023	63,4284	59,3816	54,6836	44,0328	11,8478
Dong93	63,7719	64	64	64	64	64	64
JP92	63,2771	63,6622	63,9945	64	64	64	64
JTuk87	63,731	63,9985	64	64	64	64	64
Lenth89	63,5165	63,9402	63,9952	64	64	64	64
LGB98	63,8738	64	64	64	64	64	64
LN97	63,5471	63,8716	63,4212	62,3908	62,1632	62,8288	63,712
MaxUr	62,6478	58,313	36,0324	10,8992	1,6192	0,096	0
MLZ92	63,3001	63,178	59,1546	46,35	27,2548	11,5896	0,8572
STuk82	63,63	63,9837	64	64	64	64	64

A2.4 Tables of LD1L0, LD1L9 and LD2L for a 16-run experiment and 2~8 active contrasts with different magnitudes

Table A2.4.1: LD1L0 for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	1,3471	0,8195	0,5699	0,2906	0,1101	0
Ben89	1,0153	0,7013	0,2709	0,0397	0,0021	0
Bi89	1,3762	0,8482	0,6334	0,3627	0,1618	0
BM86	0,9686	0,6984	0,3067	0,0669	0,006	0
Dong93	0,9883	0,6845	0,2289	0,0263	0,0013	0
JP92	1,2105	0,7951	0,5	0,2057	0,0598	0
JTuk87	0,9871	0,631	0,2019	0,0262	0,0009	0
Lenth89	1,0962	0,7466	0,3887	0,1178	0,02	0
LGB98	0,9797	0,5947	0,1804	0,0233	0,0016	0
LN97	1,0184	0,9744	0,8713	0,7186	0,5927	0,5094
MaxUr	0,883	0,1838	0,0211	0,0006	0	0
MLZ92	1,2664	0,8108	0,5485	0,2509	0,0807	0
STuk82	1,0599	0,7237	0,3572	0,0982	0,0158	0

Table A2.4.2: LD1L0 for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	2,6282	1,5732	0,8778	0,3087	0,078	0
Ben89	2,2351	1,5752	0,7173	0,1475	0,0103	0
Bi89	2,8185	1,7176	1,2132	0,6916	0,2953	0,0001
BM86	2,0562	1,6747	1,1388	0,5643	0,1705	0
Dong93	2,1235	1,6382	0,7986	0,158	0,0098	0
JP92	2,5674	1,5986	1,0102	0,4444	0,142	0
JTuk87	2,1227	1,4406	0,5803	0,11	0,0083	0
Lenth89	2,3545	1,6103	0,8754	0,2841	0,0598	0
LGB98	2,1722	1,9523	1,986	1,9982	1,9998	2
LN97	2,245	1,9951	1,9558	1,7952	1,4964	0,8148
MaxUr	1,4346	0,3848	0,0387	0,0009	0	0
MLZ92	2,5276	1,6122	0,9965	0,4369	0,1364	0
STuk82	2,3309	1,5617	0,8657	0,2905	0,0655	0

Table A2.4.3: LD1L0 for *n* = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	4,1786	1,6549	0,6107	0,125	0,0103	0
Ben89	4,7991	2,0918	0,895	0,2557	0,0289	0
Bi89	4,7445	2,4594	1,3691	0,7667	0,3538	0,0004
BM86	4,3687	1,8285	1,3023	0,8537	0,4505	0,0016
Dong93	4,5828	2,3339	1,4037	0,6395	0,1859	0
JP92	4,3345	1,9545	1,0443	0,4849	0,1699	0
JTuk87	4,3171	1,6949	0,7306	0,1919	0,0252	0
Lenth89	4,3133	1,9811	1,0755	0,415	0,1075	0
LGB98	5,5095	5,7912	5,9537	5,9923	5,9997	6
LN97	4,8916	3,0789	2,4334	2,1179	1,826	0,78
MaxUr	2,5578	0,3739	0,0364	0,0008	0	0
MLZ92	4,2105	1,9268	1,0324	0,4647	0,1493	0
STuk82	4,4569	1,899	0,9734	0,392	0,1108	0

Table A2.4.4: LD1L0 for *n* = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	6,6104	3,07	2,034	1,9239	1,9488	1,9992
Ben89	7,7933	6,8661	4,4108	1,5773	0,3075	0
Bi89	7,0453	4,1975	1,9705	0,9593	0,4835	0,0029
BM86	7,7492	6,496	4,4475	2,3969	1,4304	0,3137
Dong93	7,7139	7,2068	7,5862	7,8742	7,9745	8
JP92	6,7907	3,2757	1,4754	0,7178	0,2849	0,001
JTuk87	7,5905	6,0462	5,6421	5,7399	6,2267	7,6414
Lenth89	7,2966	5,4853	4,0473	3,05	2,4404	0,4427
LGB98	7,9619	7,9959	7,9996	8	8	8
LN97	7,753	6,7861	6,1102	5,6523	5,6106	5,5558
MaxUr	5,6048	0,8418	0,0523	0,002	0	0
MLZ92	6,8344	2,9958	1,1631	0,55	0,2023	0
STuk82	7,4641	5,4026	3,0968	1,4485	0,5695	0,0001

Table A2.4.5: LD1L9 for *n* = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	1,22069	0,74869	0,52232	0,2725	0,1095	0,01001
Ben89	0,9205	0,63841	0,2528	0,04453	0,01136	0,00906
Bi89	1,257	0,78212	0,58955	0,34636	0,1662	0,0185
BM86	0,87526	0,62954	0,27633	0,06022	0,00541	0
Dong93	0,89317	0,61927	0,2097	0,02725	0,00491	0,00339
JP92	1,09894	0,72847	0,46241	0,19816	0,06757	0,01193
JTuk87	0,89535	0,57677	0,19057	0,03267	0,01027	0,009
Lenth89	0,99211	0,67731	0,35515	0,11146	0,02368	0,00566
LGB98	0,88752	0,5393	0,16626	0,02504	0,00592	0,00415
LN97	0,91687	0,87698	0,78418	0,64675	0,53344	0,4585
MaxUr	0,95395	0,25313	0,06803	0,03315	0,02218	0,00359
MLZ92	1,14811	0,74048	0,50425	0,23684	0,08411	0,01084
STuk82	0,95992	0,65876	0,32908	0,09701	0,02227	0,00799

Table A2.4.6: LD1L9 for *n* = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	2,37154	1,42439	0,79773	0,28613	0,07731	0,00807
Ben89	2,01632	1,42326	0,65309	0,14165	0,01859	0,00867
Bi89	2,55169	1,56371	1,11101	0,63956	0,28486	0,0167
BM86	1,85242	1,50739	1,02493	0,50787	0,15345	0
Dong93	1,91271	1,47554	0,7204	0,14373	0,01024	0,00116
JP92	2,31855	1,45041	0,9213	0,41347	0,14148	0,0135
JTuk87	1,91504	1,3039	0,53042	0,10831	0,01679	0,00913
Lenth89	2,12189	1,45199	0,79188	0,25993	0,0582	0,00425
LGB98	1,95604	1,75712	1,7874	1,79838	1,79982	1,8
LN97	2,02086	1,79559	1,76022	1,61568	1,34676	0,73332
MaxUr	1,40658	0,40617	0,07312	0,02774	0,01912	0,00643
MLZ92	2,28117	1,45981	0,90561	0,40365	0,13327	0,00969
STuk82	2,10159	1,41222	0,78627	0,27023	0,06756	0,00766

Table A2.4.7: LD1L9 for n = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	3,76408	1,49333	0,55437	0,11776	0,01429	0,00512
Ben89	4,3217	1,88668	0,81234	0,23711	0,03371	0,00736
Bi89	4,28124	2,22939	1,24874	0,70387	0,33448	0,01463
BM86	3,93277	1,64571	1,17207	0,76833	0,40545	0,00144
Dong93	4,12477	2,10071	1,26348	0,57572	0,16752	0,00018
JP92	3,90731	1,76903	0,95181	0,44911	0,16609	0,01345
JTuk87	3,88833	1,53112	0,66541	0,18154	0,03133	0,00871
Lenth89	3,88301	1,78406	0,96961	0,37614	0,10107	0,00425
LGB98	4,95855	5,21208	5,35833	5,39307	5,39973	5,4
LN97	4,40277	2,77106	2,19006	1,90612	1,6434	0,702
MaxUr	2,36894	0,3779	0,06204	0,02255	0,01563	0,00716
MLZ92	3,7939	1,74142	0,93729	0,42702	0,14348	0,00951
STuk82	4,0136	1,71456	0,8832	0,36116	0,10772	0,0084

Table A2.4.8: LD1L9 for n = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	5,94986	2,763	1,8306	1,73151	1,75392	1,79928
Ben89	7,01425	6,18035	3,9729	1,4249	0,28249	0,00598
Bi89	6,35044	3,79099	1,78694	0,87575	0,44968	0,01563
BM86	6,97437	5,8464	4,00275	2,15721	1,28736	0,28233
Dong93	6,94251	6,48612	6,82758	7,08678	7,17705	7,2
JP92	6,11422	2,95555	1,33831	0,65728	0,26942	0,01306
JTuk87	6,83184	5,44299	5,07979	5,16842	5,60589	6,87767
Lenth89	6,56702	4,93683	3,64261	2,745	2,19637	0,39843
LGB98	7,16571	7,19631	7,19964	7,2	7,2	7,2
LN97	6,97771	6,10749	5,49918	5,08707	5,04954	5,00022
MaxUr	5,05983	0,76164	0,04717	0,0018	0	0
MLZ92	6,15253	2,70066	1,053	0,50221	0,19011	0,00801
STuk82	6,71812	4,86379	2,79047	1,30931	0,5194	0,0079

Table A2.4.9: LD2L for n = 16 and two active contrasts with different magnitudes, the magnitude of smallest contrast equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	2,4181	1,1761	0,8376	0,608	0,4142	0,2787
Ben89	1,4152	0,9623	0,5976	0,3241	0,3164	0,2856
Bi89	3,63	2,1972	2,0597	1,8412	1,5758	1,2692
BM86	1,1132	0,7128	0,3099	0,067	0,0061	0
Dong93	1,1701	0,7445	0,2814	0,0841	0,0635	0,0535
JP92	2,1154	1,2465	0,9643	0,6494	0,5641	0,3985
JTuk87	1,3095	0,8601	0,4155	0,2483	0,2333	0,2048
Lenth89	1,5521	0,8629	0,4819	0,2118	0,1226	0,1058
LGB98	1,2416	0,6624	0,2254	0,0706	0,055	0,0481
LN97	1,0825	0,9746	0,8714	0,7187	0,5928	0,5098
MaxUr	8,7845	2,7347	1,0967	0,6371	0,3808	0,0521
MLZ92	2,2085	1,1598	0,9239	0,6204	0,4803	0,3578
STuk82	1,4688	0,9168	0,5554	0,3159	0,2155	0,1913

Table A2.4.10: LD2L for n = 16 and four active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	8,0112	3,2031	1,6431	0,6055	0,2303	0,1515
Ben89	5,7412	3,1914	1,5007	0,4967	0,3051	0,2323
Bi89	10,0149	4,5615	3,4203	2,1744	1,5744	0,9556
BM86	4,6008	3,1717	2,0105	0,9103	0,2447	0
Dong93	4,8805	3,1558	1,4638	0,2893	0,0344	0,013
JP92	7,8495	3,4165	2,2078	1,1517	0,652	0,4362
JTuk87	5,1732	2,8774	1,1962	0,4001	0,2181	0,1951
Lenth89	6,2939	3,1085	1,5872	0,5105	0,1524	0,0675
LGB98	5,0968	3,8614	3,958	3,9946	3,9994	4
LN97	5,4308	3,9957	3,91	3,5894	2,9926	1,6296
MaxUr	8,511	1,9327	0,7318	0,4472	0,2928	0,0835
MLZ92	7,3699	3,3213	1,9245	0,8943	0,4671	0,2779
STuk82	6,2449	3,1172	1,6821	0,6745	0,3064	0,1742

Table A2.4.11: LD2L for n = 16 and six active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	20,226	3,8773	1,0699	0,2306	0,0621	0,0512
Ben89	25,7256	6,1108	1,8948	0,6217	0,2349	0,1676
Bi89	25,8392	8,9527	3,8194	2,0457	1,4028	0,6779
BM86	21,8693	3,9995	2,3499	1,4445	0,7039	0,0018
Dong93	23,1437	6,5215	2,8736	1,257	0,3694	0,0018
JP92	21,7341	5,4229	2,2705	1,1739	0,6377	0,3703
JTuk87	21,6477	4,1586	1,5001	0,5224	0,2367	0,2019
Lenth89	20,9515	4,869	1,9809	0,7324	0,2291	0,0653
LGB98	31,0371	33,8184	35,5107	35,9173	35,9967	36
LN97	26,1677	11,5946	7,0904	5,417	4,2016	1,5664
MaxUr	14,5362	1,2282	0,4456	0,2915	0,1947	0,0854
MLZ92	20,2	4,9138	2,0021	0,8884	0,4028	0,2397
STuk82	22,3906	4,901	1,923	0,8396	0,3512	0,1902

Table A2.4.12: LD2L for n = 16 and eight active contrasts with different magnitudes, the magnitude of smallest contrasts equals 1/4 the magnitude of the largest.

Size / Methods	2.0Sigma	4.0Sigma	6.0Sigma	8.0Sigma	10Sigma	20Sigma
Al-SY99	48,606	13,8192	4,7854	3,8047	3,8506	3,9976
Ben89	61,8407	53,3629	32,804	10,735	1,8777	0,1002
Bi89	54,528	27,0389	8,5332	2,7055	1,4782	0,5201
BM86	61,2693	48,6452	28,6559	10,4397	3,5104	0,4669
Dong93	60,5709	55,3504	59,6008	62,6582	63,7293	64
JP92	50,3994	17,4011	5,6559	2,6912	1,2538	0,281
JTuk87	59,4074	44,4627	42,4287	44,8784	49,5737	61,1363
Lenth89	55,4636	35,7343	22,5991	14,7132	10,6301	2,2105
LGB98	63,4525	63,9387	63,9942	64	64	64
LN97	61,1707	51,0209	43,6368	38,6163	37,6236	36,2876
MaxUr	42,6173	4,011	0,0699	0,0022	0	0
MLZ92	50,7337	14,5322	2,5202	0,9777	0,4255	0,1705
STuk82	57,6252	37,5559	19,6023	8,5945	3,4076	0,155

A3 Tables of Critical Values for $MaxU_r$

Table A3.1.1: Critical values of $MaxU_r$ for $2 \le m \le 9$ through 10 000 simulations.

m	r	α				
		0.20	0.10	0.05	0.025	0.01
2	1	0.8999835	0.9493034	0.9745063	0.9877042	0.9951688
3	1	0.9313855	0.9652261	0.9832665	0.9913436	0.9965511
	2	0.9577377	0.9808187	0.9908710	0.9955862	0.9981815
4	1	0.9504775	0.9748824	0.9878541	0.9934903	0.9971137
	2	0.9708046	0.9875912	0.9938884	0.9969313	0.9987545
	3	0.9792087	0.9908944	0.9956872	0.9981283	0.9991959
5	1	0.9608247	0.9803927	0.9905788	0.9953829	0.9982574
	2	0.9782969	0.9903121	0.9955114	0.9980193	0.9992433
	3	0.9854659	0.9937394	0.9972438	0.9987114	0.9995125
	4	0.9881919	0.9949333	0.9978835	0.9989767	0.9995942
6	1	0.9675291	0.9841156	0.9920399	0.9959464	0.9984267
	2	0.9834245	0.9924763	0.9963203	0.9984004	0.9993395
	3	0.9889392	0.9953467	0.9979555	0.9990657	0.9996053
	4	0.9919333	0.9965658	0.9985064	0.9993287	0.9997309
	5	0.9929774	0.9971057	0.9987390	0.9994179	0.9997886
7	1	0.9709150	0.9847355	0.9926990	0.9968258	0.9987041
	2	0.9860312	0.9936809	0.9972449	0.9986215	0.9994449
	3	0.9917365	0.9965355	0.9984861	0.9992564	0.9997482
	4	0.9939139	0.9976750	0.9990135	0.9995253	0.9998187
	5	0.9951706	0.9981618	0.9991909	0.9996400	0.9998611
	6	0.9957065	0.9983210	0.9992564	0.9996765	0.9998700
8	1	0.9747526	0.9873443	0.9938646	0.9971847	0.9989308
	2	0.9884873	0.9949420	0.9976842	0.9989530	0.9996541
	3	0.9932500	0.9973399	0.9988399	0.9995019	0.9998224
	4	0.9954130	0.9981834	0.9992804	0.9996844	0.9998855
	5	0.9965362	0.9987034	0.9994942	0.9997811	0.9999117
	6	0.9970633	0.9989287	0.9995690	0.9998197	0.9999324
	7	0.9973640	0.9990373	0.9995896	0.9998295	0.9999347
9	1	0.9776760	0.9892877	0.9944806	0.9974404	0.9989139
	2	0.9905945	0.9958680	0.9981448	0.9991068	0.9996011
	3	0.9946660	0.9978722	0.9990371	0.9995289	0.9998520
	4	0.9964096	0.9986392	0.9994197	0.9997511	0.9999103
	5	0.9973669	0.9990412	0.9995881	0.9998330	0.9999421
	6	0.9979090	0.9992216	0.9996874	0.9998655	0.9999556
	7	0.9981688	0.9993265	0.9997439	0.9998840	0.9999584
	8	0.9983116	0.9993635	0.9997541	0.9998897	0.9999612

Table A3.1.2: Critical values of $MaxU_r$ for $10 \le m \le 12$ through 10 000 simulations..

m	r	α 0.20	0.10	0.05	0.025	0.01
10	1	0.9800326	0.9902236	0.9948804	0.9974786	0.9989945
	2	0.9917575	0.9962114	0.9983427	0.9992603	0.9997071
	3	0.9956123	0.9981376	0.9991869	0.9996435	0.9998730
	4	0.9971658	0.9988908	0.9995573	0.9998119	0.9999362
	5	0.9979861	0.9992512	0.9997103	0.9998688	0.9999531
	6	0.9984374	0.9994327	0.9997817	0.9998982	0.9999645
	7	0.9987257	0.9995415	0.9998158	0.9999197	0.9999714
	8	0.9988621	0.9995939	0.9998354	0.9999268	0.9999727
	9	0.9989233	0.9996073	0.9998437	0.9999299	0.9999744
11	1	0.9824784	0.9912833	0.9954692	0.9978646	0.9991495
	2	0.9930438	0.9968914	0.9985847	0.9993530	0.9997763
	3	0.9963898	0.9985047	0.9993265	0.9997111	0.9998951
	4	0.9978440	0.9991176	0.9996630	0.9998441	0.9999489
	5	0.9985244	0.9994382	0.9997826	0.9999060	0.9999691
	6	0.9988618	0.9996020	0.9998474	0.9999373	0.9999779
	7	0.9990805	0.9996872	0.9998765	0.9999509	0.9999824
	8	0.9992187	0.9997346	0.9998926	0.9999585	0.9999847
	9	0.9992999	0.9997601	0.9999026	0.9999608	0.9999850
	10	0.9993227	0.9997673	0.9999050	0.9999625	0.9999851
12	1	0.9838530	0.9920499	0.9961314	0.9980414	0.9992604
	2	0.9939386	0.9972112	0.9986877	0.9994279	0.9998028
	3	0.9969852	0.9987353	0.9994631	0.9997565	0.9999146
	4	0.9982481	0.9993155	0.9997346	0.9998788	0.9999569
	5	0.9988035	0.9995859	0.9998412	0.9999312	0.9999756
	6	0.9991474	0.9997112	0.9998903	0.9999531	0.9999854
	7	0.9993223	0.9997756	0.9999199	0.9999629	0.9999877
	8	0.9994376	0.9998220	0.9999328	0.9999713	0.9999900
	9	0.9995160	0.9998444	0.9999388	0.9999750	0.9999910
	10	0.9995539675	0.9998533718	0.9999412501	0.9999765483	0.9999912549
	11	0.9995668334	0.9998560451	0.9999431974	0.9999768925	0.9999913488

Table A3.1.3: Critical values of $MaxU_r$ for $13 \le m \le 15$ through 10 000 simulations.

m	*r*	α				
		0.20	0.10	0.05	0.025	0.01
13	1	0.9847782	0.9924934	0.9964059	0.9982283	0.9992700
	2	0.9944997	0.9976504	0.9989028	0.9995264	0.9998339
	3	0.9974662	0.9989663	0.9995455	0.9998047	0.9999402
	4	0.9985735	0.9994533	0.9997709	0.9999071	0.9999691
	5	0.9990959	0.9996700	0.9998720	0.9999450	0.9999810
	6	0.9993669	0.9997791	0.9999128	0.9999649	0.9999884
	7	0.9995218	0.9998382	0.9999386	0.9999727	0.9999913
	8	0.9996019	0.9998726	0.9999519	0.9999796	0.9999934
	9	0.9996543	0.9998893	0.9999584	0.9999829	0.9999945
	10	0.9996835	0.9999005	0.9999621	0.9999846	0.9999951
	11	0.9997038069	0.9999030532	0.9999633823	0.9999853200	0.9999953911
	12	0.9997106322	0.9999060001	0.9999641717	0.9999860214	0.9999954728
14	1	0.9861304	0.9929962	0.9964026	0.9982485	0.9992832
	2	0.9951047	0.9979179	0.9990505	0.9995685	0.9998417
	3	0.9977539	0.9991215	0.9996336	0.9998425	0.9999472
	4	0.9988098	0.9995567	0.9998271	0.9999275	0.9999755
	5	0.9992645	0.9997355	0.9999029	0.9999606	0.9999850
	6	0.9994998	0.9998297	0.9999367	0.9999746	0.9999917
	7	0.9996349	0.9998764	0.9999547	0.9999822	0.9999944
	8	0.9997094	0.9999059	0.9999635	0.9999859	0.9999960
	9	0.9997577	0.9999230	0.9999718	0.9999890	0.9999968
	10	0.9997862	0.9999323	0.9999751	0.9999905	0.9999972
	11	0.9998034781	0.9999373514	0.9999765041	0.9999909287	0.9999974227
	12	0.9998098202	0.9999385257	0.9999771756	0.9999912147	0.9999974305
	13	0.9998131445	0.9999392114	0.9999773393	0.9999912509	0.9999974395
15	1	0.9866033	0.9934845	0.9967511	0.9984956	0.9993982
	2	0.9955954	0.9980245	0.9991350	0.9995962	0.9998618
	3	0.9980299	0.9992156	0.9996795	0.9998601	0.9999538
	4	0.9989703	0.9996129	0.9998458	0.9999294	0.9999780
	5	0.9993896	0.9997834	0.9999110	0.9999641	0.9999880
	6	0.9995944	0.9998602	0.9999441	0.9999772	0.9999920
	7	0.9997044	0.9999015	0.9999630	0.9999853	0.9999948
	8	0.9997766	0.9999257	0.9999733	0.9999904	0.9999966
	9	0.9998207	0.9999406	0.9999789	0.9999922	0.9999973
	10	0.9998470	0.9999492	0.9999825	0.9999933	0.9999978
	11	0.9998614	0.9999545	0.9999847	0.9999941	0.9999982
	12	0.9998708815	0.9999568667	0.9999856314	0.9999945245	0.9999983141
	13	0.9998761293	0.9999579010	0.9999859613	0.9999945695	0.9999983228
	14	0.9998767742	0.9999581489	0.9999859613	0.9999945695	0.9999983228

Table A3.1.4: Critical values of $MaxU_r$ for m = 16 and 17 through 10 000 simulations.

m	r	α				
		0.20	0.10	0.05	0.025	0.01
16	1	0.9874910	0.9939795	0.9970191	0.9985773	0.9995023
	2	0.9958562	0.9982808	0.9992585	0.9996554	0.9998638
	3	0.9982559	0.9993132	0.9997122	0.9998810	0.9999597
	4	0.9991123	0.9996712	0.9998747	0.9999437	0.9999814
	5	0.9994841	0.9998254	0.9999362	0.9999710	0.9999899
	6	0.9996730	0.9998927	0.9999609	0.9999826	0.9999940
	7	0.9997756	0.9999285	0.9999737	0.9999893	0.9999967
	8	0.9998376	0.9999473	0.9999813	0.9999924	0.9999978
	9	0.9998688	0.9999609	0.9999852	0.9999945	0.9999984
	10	0.9998917	0.9999669	0.9999881	0.9999955	0.9999986
	11	0.9999065804	0.9999720647	0.9999897974	0.9999959592	0.9999987077
	12	0.9999151011	0.9999742800	0.9999904225	0.9999963479	0.9999988325
	13	0.9999197355	0.9999756809	0.9999909221	0.9999965562	0.9999989030
	14	0.9999217417	0.9999762403	0.9999910660	0.9999966020	0.9999989454
	15	0.9999226086	0.9999765667	0.9999911141	0.9999966105	0.9999989646
17	1	0.9880125	0.9943096	0.9971242	0.9985531	0.9994553
	2	0.9962699	0.9983953	0.9992816	0.9996722	0.9998764
	3	0.9984665	0.9993880	0.9997403	0.9998832	0.9999650
	4	0.9992567	0.9997222	0.9998790	0.9999579	0.9999841
	5	0.9995720	0.9998471	0.9999423	0.9999767	0.9999925
	6	0.9997323	0.9999104	0.9999666	0.9999870	0.9999963
	7	0.9998231	0.9999416	0.9999794	0.9999918	0.9999978
	8	0.9998710	0.9999604	0.9999855	0.9999944	0.9999985
	9	0.9998997	0.9999704	0.9999894	0.9999960	0.9999988
	10	0.9999199	0.9999765	0.9999918	0.9999970	0.9999991
	11	0.9999308545	0.9999795527	0.9999932407	0.9999974301	0.9999992925
	12	0.9999377225	0.9999821366	0.9999939947	0.9999978607	0.9999993654
	13	0.9999426901	0.9999833070	0.9999941870	0.9999979589	0.9999993831
	14	0.9999457855	0.9999840673	0.9999944625	0.9999980402	0.9999994394
	15	0.9999472106	0.9999842659	0.9999944778	0.9999980626	0.9999994394
	16	0.9999477098	0.9999843532	0.9999945015	0.9999980681	0.9999994394

Table A3.1.5: Critical values of $MaxU_r$ for m = 18 and 19 through 10 000 simulations.

m	r	α				
		0.20	0.10	0.05	0.025	0.01
18	1	0.9889960	0.9945512	0.9973700	0.9986641	0.9993930
	2	0.9965228	0.9985250	0.9993623	0.9997010	0.9998914
	3	0.9986172	0.9994525	0.9997676	0.9999085	0.9999723
	4	0.9993546	0.9997566	0.9999037	0.9999599	0.9999868
	5	0.9996545	0.9998749	0.9999521	0.9999819	0.9999940
	6	0.9997846	0.9999298	0.9999742	0.9999895	0.9999969
	7	0.9998574	0.9999544	0.9999834	0.9999937	0.9999982
	8	0.9999001	0.9999680	0.9999889	0.9999958	0.9999988
	9	0.9999238	0.9999766	0.9999919	0.9999971	0.9999992
	10	0.9999387	0.9999816	0.9999936	0.9999978	0.9999993
	11	0.9999491283	0.9999851657	0.9999950744	0.9999982482	0.9999994121
	12	0.9999551289	0.9999868829	0.9999956276	0.9999984375	0.9999994404
	13	0.9999601640	0.9999884394	0.9999961840	0.9999985364	0.9999994847
	14	0.9999625786	0.9999891383	0.9999964043	0.9999985946	0.9999995264
	15	0.9999643493	0.9999895969	0.9999965587	0.9999986353	0.9999995355
	16	0.9999648934	0.9999897304	0.9999966021	0.9999986357	0.9999995355
	17	0.9999651654	0.9999897884	0.9999966054	0.9999986449	0.9999995355
19	1	0.9894540	0.9948108	0.9974648	0.9987307	0.9993798
	2	0.9968378	0.9986490	0.9993916	0.9997044	0.9999022
	3	0.9987432	0.9995122	0.9997860	0.9999138	0.9999743
	4	0.9994295	0.9997772	0.9999168	0.9999652	0.9999903
	5	0.9996883	0.9998926	0.9999600	0.9999831	0.9999946
	6	0.9998188	0.9999394	0.9999789	0.9999913	0.9999973
	7	0.9998856	0.9999639	0.9999874	0.9999947	0.9999985
	8	0.9999204	0.9999757	0.9999912	0.9999963	0.9999991
	9	0.9999407	0.9999826	0.9999937	0.9999977	0.9999994
	10	0.9999556	0.9999870	0.9999955	0.9999983	0.9999995
	11	0.9999627820	0.9999895917	0.9999963708	0.9999986647	0.9999996812
	12	0.9999686466	0.9999909025	0.9999969060	0.9999988557	0.9999997036
	13	0.9999726779	0.9999919454	0.9999972950	0.9999989976	0.9999997212
	14	0.9999752269	0.9999928530	0.9999975215	0.9999990776	0.9999997579
	15	0.9999763694	0.9999932040	0.9999976565	0.9999991015	0.9999997638
	16	0.9999768479	0.9999932733	0.9999976957	0.9999991244	0.9999997638
	17	0.9999773576	0.9999933062	0.9999977071	0.9999991245	0.9999997752
	18	0.9999775492	0.9999933218	0.9999977243	0.9999991258	0.9999997752

Table A3.1.6: Critical values of $MaxU_r$ for m = 20 and 21 through 10 000 simulations.

m	r	α				
		0.20	0.10	0.05	0.025	0.01
20	1	0.9897086	0.9949093	0.9975153	0.9987603	0.9995363
	2	0.9970863	0.9987466	0.9994447	0.9997336	0.9998936
	3	0.9989022	0.9995466	0.9998136	0.9999214	0.9999755
	4	0.9994886	0.9998013	0.9999241	0.9999696	0.9999906
	5	0.9997316	0.9999074	0.9999648	0.9999855	0.9999956
	6	0.9998483	0.9999478	0.9999809	0.9999922	0.9999979
	7	0.9999066	0.9999693	0.9999885	0.9999958	0.9999989
	8	0.9999372	0.9999801	0.9999927	0.9999972	0.9999992
	9	0.9999555	0.9999863	0.9999954	0.9999983	0.9999996
	10	0.9999660	0.9999897	0.9999966	0.9999987	0.9999997
	11	0.9999721831	0.9999917052	0.9999972968	0.9999989827	0.9999997541
	12	0.9999768085	0.9999932094	0.9999977135	0.9999992564	0.9999997956
	13	0.9999801283	0.9999943260	0.9999981488	0.9999993624	0.9999998140
	14	0.9999821203	0.9999948475	0.9999983431	0.9999994450	0.9999998450
	15	0.9999833733	0.9999952462	0.9999984460	0.9999994829	0.9999998508
	16	0.9999844014	0.9999954216	0.9999985181	0.9999994965	0.9999998558
	17	0.9999848522	0.9999955727	0.9999985201	0.9999994984	0.9999998558
	18	0.9999849906	0.9999955946	0.9999985208	0.9999994987	0.9999998558
	19	0.9999850265	0.9999955966	0.9999985209	0.9999994987	0.9999998558
21	1	0.9903029	0.9951063	0.9976164	0.9988141	0.9995264
	2	0.9972642	0.9988356	0.9994678	0.9997668	0.9999072
	3	0.9990117	0.9996070	0.9998314	0.9999306	0.9999758
	4	0.9995741	0.9998404	0.9999400	0.9999736	0.9999920
	5	0.9997835	0.9999253	0.9999712	0.9999894	0.9999966
	6	0.9998800	0.9999599	0.9999857	0.9999950	0.9999986
	7	0.9999260	0.9999759	0.9999917	0.9999972	0.9999992
	8	0.9999510	0.9999848	0.9999951	0.9999984	0.9999995
	9	0.9999663	0.9999900	0.9999968	0.9999989	0.9999997
	10	0.9999748106	0.9999928661	0.9999977347	0.9999991859	0.9999998124
	11	0.9999799856	0.9999943730	0.9999982540	0.9999993778	0.9999998565
	12	0.9999833487	0.9999954222	0.9999985799	0.9999995297	0.9999998756
	13	0.9999856986	0.9999961168	0.9999987308	0.9999996199	0.9999998974
	14	0.9999877424	0.9999966410	0.9999988792	0.9999996434	0.9999999075
	15	0.9999887375	0.9999969646	0.9999990003	0.9999996826	0.9999999192
	16	0.9999894256	0.9999971426	0.9999990410	0.9999997169	0.9999999254
	17	0.9999898948	0.9999972084	0.9999990508	0.9999997181	0.9999999254
	18	0.9999900653	0.9999972605	0.9999990688	0.9999997202	0.9999999254
	19	0.9999900824	0.9999972843	0.9999990719	0.9999997219	0.9999999254
	20	0.9999901028	0.9999972957	0.9999990831	0.9999997219	0.9999999254

Table A3.1.7: Critical values of $MaxU_r$ for $22 \leq m \leq= 26$ through 10 000 simulations.

m	r	α				
		0.20	0.10	0.05	0.025	0.01
22	1	0.9908004	0.9954815	0.9977861	0.9988730	0.9995783
	2	0.9975081	0.9989166	0.9995064	0.9997723	0.9999226
	3	0.9991101	0.9996467	0.9998382	0.9999345	0.9999790
	4	0.9996163	0.9998513	0.9999406	0.9999750	0.9999919
	5	0.9998102	0.9999310	0.9999742	0.9999893	0.9999963
	6	0.9998969	0.9999649	0.9999871	0.9999948	0.9999984
	7	0.9999381	0.9999799	0.9999930	0.9999971	0.9999992
	8	0.9999606	0.9999881	0.9999958	0.9999983	0.9999995
	9	0.9999730458	0.9999921176	0.9999971761	0.9999990147	0.9999997008
	10	0.9999804064	0.9999944662	0.9999980317	0.9999993052	0.9999998175
	11	0.9999847729	0.9999955900	0.9999985200	0.9999995249	0.9999998662
	12	0.9999879690	0.9999965572	0.9999988716	0.9999996219	0.9999999087
	13	0.9999899015	0.9999971768	0.9999990965	0.9999996837	0.9999999258
	14	0.9999913264	0.9999974668	0.9999992188	0.9999997189	0.9999999325
	15	0.9999924895	0.9999977212	0.9999993127	0.9999997494	0.9999999399
	16	0.9999932025	0.9999979101	0.9999993526	0.9999997663	0.9999999432
	17	0.9999934953	0.9999980240	0.9999993741	0.9999997788	0.9999999451
	18	0.9999936979	0.9999980640	0.9999993837	0.9999997835	0.9999999466
	19	0.9999937981	0.9999981305	0.9999994078	0.9999997842	0.9999999466
	20	0.9999938127	0.9999981330	0.9999994101	0.9999997842	0.9999999466
	21	0.9999938134	0.9999981331	0.9999994101	0.9999997842	0.9999999466
23	1	0.9913348	0.9957906	0.9979061	0.9989452	0.9995964
	2	0.9977182	0.9990108	0.9995574	0.9997981	0.9999173
	3	0.9992029	0.9996736	0.9998660	0.9999420	0.9999796
	4	0.9996581	0.9998754	0.9999477	0.9999808	0.9999946
	5	0.9998326	0.9999411	0.9999776	0.9999921	0.9999978
	6	0.9999089	0.9999702	0.9999900	0.9999963	0.9999990
	7	0.9999469	0.9999835	0.9999944	0.9999978	0.9999995
	8	0.9999671	0.9999900	0.9999967	0.9999988	0.9999997
	9	0.9999775944	0.9999936316	0.9999978451	0.9999992721	0.9999998032
	10	0.9999840523	0.9999956842	0.9999986031	0.9999995193	0.9999998652
	11	0.9999882394	0.9999968940	0.9999989916	0.9999997067	0.9999999169
	12	0.9999908719	0.9999974997	0.9999992552	0.9999997745	0.9999999362
	13	0.9999924647	0.9999979987	0.9999993984	0.9999998094	0.9999999536
	14	0.9999935360	0.9999982821	0.9999994894	0.9999998419	0.9999999625
	15	0.9999943135	0.9999984635	0.9999995745	0.9999998622	0.9999999682
	16	0.9999948894	0.9999985988	0.9999996141	0.9999998728	0.9999999711
	17	0.9999952939	0.9999986889	0.9999996516	0.9999998782	0.9999999730
	18	0.9999954857	0.9999987620	0.9999996670	0.9999998830	0.9999999749
	19	0.9999955736	0.9999987769	0.9999996723	0.9999998832	0.9999999749

Table A3.1.7: Critical values of $MaxU_r$ for $22 \leq m \leq= 26$ through 10 000 simulations. (*continued*).

m	*r*	*α*				
		0.20	0.10	0.05	0.025	0.01
23	20	0.9999956554	0.9999987867	0.9999996859	0.9999998843	0.9999999763
	21	0.9999956842	0.9999987877	0.9999996863	0.9999998843	0.9999999763
	22	0.9999956869	0.9999987878	0.9999996863	0.9999998843	0.9999999763
24	1	0.9915439	0.9958165	0.9979014	0.9988866	0.9995633
	2	0.9978473	0.9990510	0.9995659	0.9998011	0.9999301
	3	0.9992465	0.9997014	0.9998759	0.9999485	0.9999812
	4	0.9996796	0.9998867	0.9999554	0.9999817	0.9999937
	5	0.9998468	0.9999491	0.9999812	0.9999923	0.9999976
	6	0.9999185	0.9999745	0.9999909	0.9999962	0.9999989
	7	0.9999538	0.9999863	0.9999951	0.9999981	0.9999995
	8	0.9999726	0.9999917	0.9999972	0.9999989	0.9999997
	9	0.9999817	0.9999948	0.9999982	0.9999993	0.9999998
	10	0.9999873339	0.9999964386	0.9999988139	0.9999995924	0.9999999125
	11	0.9999907062	0.9999974432	0.9999991782	0.9999997174	0.9999999414
	12	0.9999930581	0.9999980727	0.9999993834	0.9999998117	0.9999999543
	13	0.9999943662	0.9999984412	0.9999995474	0.9999998502	0.9999999662
	14	0.9999951747	0.9999987520	0.9999996387	0.9999998782	0.9999999729
	15	0.9999958762	0.9999989109	0.9999996853	0.9999999042	0.9999999754
	16	0.9999963371	0.9999990117	0.9999997245	0.9999999127	0.9999999786
	17	0.9999966496	0.9999991200	0.9999997436	0.9999999188	0.9999999803
	18	0.9999968175	0.9999991577	0.9999997599	0.9999999229	0.9999999811
	19	0.9999969599	0.9999991765	0.9999997618	0.9999999259	0.9999999820
	20	0.9999970112	0.9999991943	0.9999997664	0.9999999259	0.9999999820
	21	0.9999970489	0.9999992110	0.9999997698	0.9999999275	0.9999999820
	22	0.9999970516	0.9999992112	0.9999997698	0.9999999275	0.9999999820
	23	0.9999970542	0.9999992112	0.9999997698	0.9999999275	0.9999999820
25	1	0.9920032	0.9960399	0.9980782	0.9990218	0.9995339
	2	0.9980155	0.9991658	0.9996193	0.9998102	0.9999333
	3	0.9993186	0.9997426	0.9998851	0.9999511	0.9999817
	4	0.9997188	0.9998981	0.9999566	0.9999833	0.9999941
	5	0.9998676	0.9999541	0.9999825	0.9999928	0.9999978
	6	0.9999306	0.9999774	0.9999920	0.9999968	0.9999990
	7	0.9999607	0.9999883	0.9999959	0.9999984	0.9999995
	8	0.9999761	0.9999930	0.9999977	0.9999991	0.9999997
	9	0.9999851838	0.9999956091	0.9999985937	0.9999995011	0.9999998393
	10	0.9999899416	0.9999972561	0.9999990884	0.9999996830	0.9999999116
	11	0.9999928842	0.9999980507	0.9999993661	0.9999997628	0.9999999490
	12	0.9999947413	0.9999985718	0.9999995372	0.9999998393	0.9999999636

Table A3.1.7: Critical values of $MaxU_r$ for $22 \leq m \leq= 26$ through 10 000 simulations. (*continued*).

m	r	α				
		0.20	0.10	0.05	0.025	0.01
25	13	0.9999958087	0.9999988610	0.9999996459	0.9999998881	0.9999999716
	14	0.9999965349	0.9999990759	0.9999997077	0.9999999079	0.9999999761
	15	0.9999970461	0.9999992210	0.9999997551	0.9999999232	0.9999999800
	16	0.9999973771	0.9999993128	0.9999997896	0.9999999398	0.9999999837
	17	0.9999975828	0.9999993897	0.9999998054	0.9999999432	0.9999999852
	18	0.9999977554	0.9999994376	0.9999998222	0.9999999441	0.999999986282
	19	0.9999978919	0.9999994656	0.9999998360	0.9999999457	0.999999986289
	20	0.9999979592	0.9999994785	0.9999998393	0.9999999501	0.999999986523
	21	0.9999979881	0.9999994826	0.9999998424	0.9999999501	0.999999986523
	22	0.9999980138	0.9999994871	0.9999998429	0.9999999501	0.999999986523
	23	0.9999980167	0.9999994871	0.9999998429	0.9999999501	0.999999986523
	24	0.9999980174	0.9999994871	0.9999998429	0.9999999501	0.999999986523
26	1	0.9922294	0.9962090	0.9980953	0.9990005	0.9996064
	2	0.9980449	0.9991506	0.9996219	0.9998298	0.9999391
	3	0.9993433	0.9997486	0.9998938	0.9999518	0.9999835
	4	0.9997348	0.9999087	0.9999611	0.9999831	0.9999955
	5	0.9998802	0.9999602	0.9999850	0.9999936	0.9999981
	6	0.9999377	0.9999800	0.9999926	0.9999973	0.9999991
	7	0.9999661	0.9999893	0.9999962	0.9999987	0.9999996
	8	0.9999801	0.9999941	0.9999979	0.9999992	0.9999998
	9	0.9999872589	0.9999964000	0.9999988240	0.9999995523	0.9999998667
	10	0.9999916312	0.9999976535	0.9999992784	0.9999997609	0.9999999242
	11	0.9999941688	0.9999983638	0.9999995055	0.9999998317	0.9999999542
	12	0.9999957243	0.9999988936	0.9999996501	0.9999998852	0.9999999696
	13	0.9999967220	0.9999991745	0.9999997482	0.9999999219	0.9999999790
	14	0.9999973128	0.9999993378	0.9999998017	0.9999999398	0.9999999831
	15	0.9999976946	0.9999994462	0.9999998332	0.9999999487	0.9999999870
	16	0.9999979832	0.9999995304	0.9999998528	0.9999999560	0.9999999898
	17	0.9999982439	0.9999995701	0.9999998714	0.9999999613	0.9999999901
	18	0.9999983594	0.9999996095	0.9999998851	0.9999999641	0.9999999908
	19	0.9999984717	0.9999996331	0.9999998903	0.9999999671	0.9999999915
	20	0.9999985409	0.9999996425	0.9999998962	0.9999999676	0.9999999920
	21	0.9999985914	0.9999996528	0.9999998983	0.9999999677	0.9999999925
	22	0.9999986032	0.9999996584	0.9999998985	0.9999999677	0.9999999925
	23	0.9999986179	0.9999996618	0.9999998989	0.9999999679	0.9999999925
	24	0.9999986224	0.9999996627	0.9999998989	0.9999999679	0.9999999925
	25	0.9999986224	0.9999996627	0.9999998989	0.9999999679	0.9999999925

Table A3.1.8: Critical values of $MaxU_r$ for $27 \leq m \leq 30$ through 10 000 simulations..

m	r	α				
		0.20	0.10	0.05	0.025	0.01
27	1	0.9926093	0.9962660	0.9980795	0.9990652	0.9995836
	2	0.9981605	0.9992205	0.9996421	0.9998424	0.9999408
	3	0.9994050	0.9997626	0.9999008	0.9999581	0.9999852
	4	0.9997651	0.9999162	0.9999663	0.9999858	0.9999959
	5	0.9998929	0.9999641	0.9999870	0.9999951	0.9999985
	6	0.9999467	0.9999829	0.9999941	0.9999980	0.9999993
	7	0.9999710	0.9999910	0.9999970	0.9999990	0.9999997
	8	0.9999833735	0.9999950579	0.9999984321	0.9999993808	0.9999998191
	9	0.9999896196	0.9999970687	0.9999991075	0.9999996802	0.9999999061
	10	0.9999932994	0.9999981862	0.9999994366	0.9999998039	0.9999999424
	11	0.9999953862	0.9999987736	0.9999996263	0.9999998761	0.9999999652
	12	0.9999966480	0.9999991419	0.9999997356	0.9999999162	0.9999999762
	13	0.9999974536	0.9999993620	0.9999998078	0.9999999394	0.9999999838
	14	0.9999980109	0.9999994920	0.9999998489	0.9999999529	0.9999999879
	15	0.9999983639	0.9999995801	0.9999998817	0.9999999614	0.9999999908
	16	0.9999986026	0.9999996348	0.9999999024	0.9999999679	0.9999999923
	17	0.9999987555	0.9999996767	0.9999999155	0.9999999720	0.9999999938
	18	0.9999989116	0.9999997186	0.9999999251	0.9999999754	0.9999999943
	19	0.9999990196	0.9999997380	0.9999999304	0.9999999769	0.9999999945
	20	0.9999990756	0.9999997561	0.9999999355	0.9999999788	0.9999999946
	21	0.9999991044	0.9999997627	0.9999999377	0.9999999793	0.9999999949
	22	0.9999991186	0.9999997696	0.9999999387	0.9999999795	0.9999999949
	23	0.9999991419	0.9999997744	0.9999999395	0.9999999796	0.9999999949
	24	0.9999991495	0.9999997764	0.9999999400	0.9999999797	0.9999999949
	25	0.9999991495	0.9999997764	0.9999999400	0.9999999797	0.9999999949
	26	0.9999991496	0.9999997764	0.9999999400	0.9999999797	0.9999999949
28	1	0.9926704	0.9963928	0.9981168	0.9990383	0.9996021
	2	0.9982933	0.9992661	0.9996737	0.9998490	0.9999400
	3	0.9994507	0.9997883	0.9999109	0.9999619	0.9999863
	4	0.9997875	0.9999238	0.9999711	0.9999875	0.9999959
	5	0.9999031	0.9999693	0.9999881	0.9999951	0.9999986
	6	0.9999530	0.9999855	0.9999947	0.9999980	0.9999994
	7	0.9999755	0.9999923	0.9999974	0.9999990	0.9999997
	8	0.9999859951	0.9999959025	0.9999986332	0.999999516027	0.999999850465
	9	0.9999914518	0.9999976099	0.9999992468	0.999999734767	0.999999917387
	10	0.9999944034	0.9999985122	0.9999995388	0.999999843821	0.999999954482
	11	0.9999961346	0.9999990160	0.9999997044	0.999999901737	0.999999970378
	12	0.9999972892	0.9999993105	0.9999997945	0.999999931720	0.999999979514

Table A3.1.8: Critical values of $MaxU_r$ for $27 \le m \le 30$ through 10 000 simulations. (*continued*).

m	r	α				
		0.20	0.10	0.05	0.025	0.01
28	13	0.9999980030	0.9999994990	0.9999998515	0.999999953705	0.999999986199
	14	0.9999984547	0.9999996154	0.9999998897	0.999999964442	0.999999990111
	15	0.9999987551	0.9999996861	0.9999999152	0.999999970384	0.999999993153
	16	0.9999989355	0.9999997503	0.9999999290	0.999999975468	0.999999994247
	17	0.9999990750	0.9999997801	0.9999999412	0.999999979301	0.999999995164
	18	0.9999992004	0.9999998011	0.9999999470	0.999999982069	0.999999995938
	19	0.9999992808	0.9999998158	0.9999999523	0.999999984081	0.999999996358
	20	0.9999993229	0.9999998302	0.9999999556	0.999999985105	0.999999996710
	21	0.9999993625	0.9999998388	0.9999999573	0.999999985859	0.999999996987
	22	0.9999993835	0.9999998449	0.9999999587	0.999999985913	0.999999997023
	23	0.9999993943	0.9999998480	0.9999999592	0.999999986526	0.999999997023
	24	0.9999993992	0.9999998490	0.9999999592	0.999999986615	0.999999997023
	25	0.9999994053	0.9999998494	0.9999999596	0.999999986615	0.999999997023
	26	0.9999994062	0.9999998494	0.9999999596	0.999999986615	0.999999997023
	27	0.9999994062	0.9999998494	0.9999999596	0.999999986615	0.999999997023
29	1	0.9929163	0.9964780	0.9982793	0.9991388	0.9996801
	2	0.9984089	0.9993087	0.9996891	0.9998632	0.9999532
	3	0.9994889	0.9998059	0.9999156	0.9999674	0.9999886
	4	0.9998051	0.9999313	0.9999734	0.9999891	0.9999964
	5	0.9999157	0.9999720	0.9999896	0.9999960	0.9999986
	6	0.9999582	0.9999868	0.9999953	0.9999982	0.9999995
	7	0.9999784	0.9999932	0.9999977	0.9999992	0.9999998
	8	0.9999879429	0.9999962936	0.9999987466	0.9999995585	0.9999998784
	9	0.9999928007	0.9999979022	0.9999993067	0.9999997556	0.9999999377
	10	0.9999952886	0.9999987396	0.9999995804	0.9999998529	0.9999999564
	11	0.9999968983	0.9999991985	0.9999997408	0.9999999107	0.9999999724
	12	0.9999978909	0.9999994549	0.9999998379	0.9999999384	0.9999999834
	13	0.9999984637	0.9999996139	0.9999998840	0.9999999597	0.9999999884
	14	0.9999988358	0.9999997130	0.9999999145	0.9999999695	0.9999999919
	15	0.9999990695	0.9999997816	0.9999999352	0.9999999781	0.9999999940
	16	0.9999992371	0.9999998250	0.9999999469	0.9999999834	0.9999999951
	17	0.9999993408	0.9999998545	0.9999999552	0.9999999858	0.9999999961
	18	0.9999994302	0.9999998699	0.9999999603	0.9999999881	0.9999999966
	19	0.9999994823	0.9999998817	0.9999999636	0.9999999894	0.9999999970
	20	0.9999995195	0.9999998933	0.9999999677	0.9999999901	0.9999999974
	21	0.9999995517	0.9999998994	0.9999999699	0.9999999907	0.999999997734
	22	0.9999995688	0.9999999040	0.9999999712	0.9999999910	0.999999997782

Table A3.1.8: Critical values of $MaxU_r$ for $27 \le m \le 30$ through 10 000 simulations. (*continued*).

m	r	α 0.20	0.10	0.05	0.025	0.01
29	23	0.9999995882	0.9999999053	0.9999999723	0.9999999914	0.999999997824
	24	0.9999995967	0.9999999069	0.999999973119	0.999999991589	0.999999997824
	25	0.9999996003	0.9999999091	0.999999973192	0.999999991589	0.999999997824
	26	0.9999996022	0.9999999094	0.999999973306	0.999999991589	0.999999997824
	27	0.9999996022	0.9999999094	0.999999973306	0.999999991589	0.999999997824
	28	0.9999996022	0.9999999094	0.999999973306	0.999999991589	0.999999997824
30	1	0.9932405	0.9966856	0.9983631	0.9991240	0.9996588
	2	0.9984625	0.9993408	0.9996998	0.9998826	0.9999538
	3	0.9995158	0.9998134	0.9999254	0.9999712	0.9999906
	4	0.9998176	0.9999355	0.9999763	0.9999916	0.9999975
	5	0.9999199	0.9999745	0.9999908	0.9999969	0.9999991
	6	0.9999620	0.9999884	0.9999960	0.9999987	0.9999997
	7	0.9999805	0.9999942	0.9999981	0.9999994	0.9999998
	8	0.9999893016	0.9999969991	0.9999990570	0.9999996861	0.9999999187
	9	0.9999935921	0.9999982956	0.9999994896	0.9999998347	0.9999999507
	10	0.9999961145	0.9999990119	0.9999997048	0.9999999084	0.9999999720
	11	0.9999975150	0.9999993907	0.9999998193	0.9999999411	0.9999999825
	12	0.9999982628	0.9999995872	0.9999998861	0.9999999622	0.9999999890
	13	0.9999987684	0.9999997073	0.9999999234	0.9999999745	0.9999999921
	14	0.9999990644	0.9999997874	0.9999999418	0.9999999810	0.9999999949
	15	0.9999992752	0.9999998274	0.9999999546	0.9999999851	0.9999999966
	16	0.9999994159	0.9999998637	0.9999999653	0.9999999886	0.9999999970
	17	0.9999994990	0.9999998852	0.9999999692	0.9999999901	0.9999999978
	18	0.9999995719	0.9999999030	0.9999999729	0.9999999912	0.9999999982
	19	0.9999996198	0.9999999123	0.9999999765	0.9999999922	0.999999998501
	20	0.9999996479	0.9999999216	0.9999999786	0.9999999932	0.999999998576
	21	0.9999996777	0.9999999286	0.9999999800	0.999999993858	0.999999998764
	22	0.9999996958	0.9999999324	0.9999999810	0.999999994057	0.999999998787
	23	0.9999997076	0.9999999348	0.9999999819	0.999999994210	0.999999998816
	24	0.9999997162	0.9999999364	0.9999999825	0.999999994292	0.999999998816
	25	0.9999997197	0.9999999367	0.9999999827	0.999999994312	0.999999998816
	26	0.9999997228	0.9999999370	0.9999999827	0.999999994312	0.999999998816
	27	0.9999997237	0.9999999370	0.9999999827	0.999999994312	0.999999998816
	28	0.9999997242	0.9999999370	0.9999999827	0.999999994312	0.999999998816
	29	0.9999997242	0.9999999370	0.9999999827	0.999999994312	0.999999998816

Table A3.1.9: Critical values of $MaxU_r$ for m = 31 through 10 000 simulations.

m	r	α				
		0.20	0.10	0.05	0.025	0.01
31	1	0.9933986	0.9968138	0.9985154	0.9992558	0.9997117
	2	0.9985660	0.9993731	0.9997371	0.9998828	0.9999570
	3	0.9995529	0.9998242	0.9999288	0.9999705	0.9999904
	4	0.9998348	0.9999399	0.9999768	0.9999917	0.9999972
	5	0.9999276	0.9999761	0.9999917	0.9999969	0.9999991
	6	0.9999649	0.9999896	0.9999965	0.9999987	0.9999997
	7	0.9999819935	0.9999950535	0.9999984094	0.9999994650	0.9999998510
	8	0.9999900783	0.9999973952	0.9999992076	0.9999997632	0.9999999273
	9	0.9999941849	0.9999985724	0.9999995750	0.9999998726	0.9999999616
	10	0.9999965063	0.9999991312	0.9999997565	0.9999999231	0.9999999808
	11	0.9999977940	0.9999994524	0.9999998594	0.9999999532	0.9999999873
	12	0.9999985216	0.9999996393	0.9999999082	0.9999999718	0.9999999923
	13	0.9999989685	0.9999997489	0.9999999366	0.9999999820	0.9999999950
	14	0.9999992364	0.9999998175	0.9999999528	0.9999999860	0.9999999967
	15	0.9999994141	0.9999998632	0.9999999647	0.9999999888	0.9999999979
	16	0.9999995288	0.9999998921	0.9999999708	0.9999999912	0.9999999983
	17	0.9999996001	0.9999999123	0.9999999757	0.9999999931	0.9999999988
	18	0.9999996600	0.9999999256	0.9999999804	0.9999999943	0.999999998994
	19	0.9999997079	0.9999999364	0.9999999834	0.9999999951	0.999999999093
	20	0.9999997436	0.9999999436	0.9999999852	0.9999999957	0.999999999192
	21	0.9999997705	0.9999999485	0.9999999860	0.999999996186	0.999999999263
	22	0.9999997873	0.9999999512	0.9999999868	0.999999996378	0.999999999341
	23	0.9999997962	0.9999999549	0.9999999877	0.999999996586	0.999999999353
	24	0.9999998061	0.9999999567	0.9999999880	0.999999996736	0.999999999376
	25	0.9999998113	0.9999999578	0.999999988419	0.999999996854	0.999999999376
	26	0.9999998123	0.9999999581	0.999999988431	0.999999996857	0.999999999376
	27	0.9999998135	0.9999999587	0.999999988560	0.999999996857	0.999999999376
	28	0.999999813978	0.999999958847	0.999999988560	0.999999996857	0.999999999376
	29	0.999999813978	0.999999958847	0.999999988560	0.999999996857	0.999999999376
	30	0.999999813978	0.999999958847	0.999999988560	0.999999996857	0.999999999376

A4 Notations

In this part of the appendix, the symbols and notations which are repeatedly used in the previous chapters are summarized and briefly explained.

Vectors and matrices are printed in bold face type. The prime indicates transposition of a vector or matrix, e.g. $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)'$. Further, we employ the following symbols:

$Cov(\cdot)$	covariance
$E(\cdot)$	expectation
$E(\cdot\|\cdot)$	conditional expectation
$F(\cdot)$	cumulative distribution function
$F^{-1}(\cdot)$	inverse function
$\boldsymbol{I}$	identity matrix
$N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$
$P(\cdot)$	probability function
$P(\cdot\|\cdot)$	conditional probability
$p(\cdot)$	probability density function
$Var(\cdot)$	variance
$\mathbf{1}_n$	the n dimensional vector of ones
$(\ )^{-1}$	inverse matrix
$\|\ \|$	absolute value
$\Gamma(\xi)$	gamma function, i.e. $\int_0^\infty x^{\xi-1} e^{-x} dx$
$\hat{\beta}$	estimate of β
$\varepsilon_{(1)}, \varepsilon_{(2)}, \ldots, \varepsilon_{(n)}$	order statistics of $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$
$\lVert S \rVert$	the number of elements in set S
Δ	the symmetric difference operator

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Curriculum vitae

Ying Chen was born in 1966 in Zhejiang, China. He graduated from East China Normal University (ECNU) in Shanghai with a Master of Science in Statistics in 1994. For the following four years he worked for the Department of Statistics at the ECNU as a lecturer. With a scholarship from the “Friedrich-Ebert-Stiftung”, he went to Germany and started his advanced studies in 1998. Afterwards he was granted with a scholarship of the “Deutsche Forschungsgemeinschaft” at the Graduate College of Applied Statistics at the University of Dortmund as a Ph.D. student.

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