Ensembles, altered dispersion relations and CPT violation in neutrino oscillations and charged lepton decays

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Sebastian Hollenberg
aus Wilhelmshaven

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- Hollenberg S., Micu O. and Pal P. *Lepton-number violating effects in neutrino oscillations* Phys.Rev. **D85**, (2012), 053004. arXiv:1112.1523.

Vorausgesetzt ist aber ferner: daß das, was bei wissenschaftlicher Arbeit herauskommt, wichtig im Sinn von 'wissenswert' sei. Und da stecken nun offenbar alle unsere Probleme darin. Denn diese Voraussetzung ist nicht wieder ihrerseits mit den Mitteln der Wissenschaft beweisbar. Sie läßt sich nur auf ihren letzten Sinn deuten, den man dann ablehnen oder annehmen muss, je nach der eigenen letzten Stellungnahme zum Leben.

Max Weber [1]

To
HEINZ–GERD
who is no more

Abstract

This thesis introduces a description of altered dispersion relations and CPT-violating effects in neutrino oscillations and charged lepton decays. Such phenomena can arise, e.g., in the early Universe and models with extra spatial dimensions.
We develop a perturbation theory for the coherence vector description of neutrino oscillations and propose a unified approach to adiabatic and nonadiabatic two-flavor oscillations in neutrino ensembles with finite temperature and generic potential terms. We neglect ensemble decoherence and solve the associated quantum kinetic equations. Eventually, we apply the methodology of said perturbation theory to neutrino ensembles in the presence of decohering interactions.
Moreover, we develop a nonadiabatic perturbation theory for oscillations involving an arbitrary number of neutrinos and antineutrinos. We include lepton-number violation in the approach and treat it as a small perturbation parameter. We find that small lepton-number violation in vacuo can be enhanced in CP-odd matter and apply the formalism to the two-generation case.
Finally, we investigate low-energy CPT-violating modifications in charged current weak interactions. We analyze muon and antimuon decays and, using the difference in their lifetimes, put bounds on the CPT-violating parameters. We also elaborate on muon and antimuon differential decay rates.

Zusammenfassung

Die vorliegende Dissertation behandelt eine Beschreibung veränderter Dispersionsrelationen und CPT-verletzender Effekte in Neutrinooszillationen und dem Zerfall geladener Leptonen. Solche Phänomene können unter anderem im frühen Universum und in Modellen mit zusätzlichen Raumzeitdimensionen auftreten.
Wir entwickeln eine Störungstheorie für den Kohärenzvektorformalismus von Neutrinooszillationen und einen vereinheitlichten Zugang für adiabatische und nichtadiabatische Neutrinooszillationen zwischen zwei Generationen in Neutrinoensembles mit endlicher Temperatur und generischen Potenzialen. Dabei vernachlässigen wir Dekohärenz des Ensembles und lösen die quantenkinetischen Gleichungen. Schließlich wenden wir die Methodik der Störungstheorie auf Neutrinoensembles an, in denen Dekohärenz auftritt.
Darüberhinaus entwickeln wir eine nichtadiabatische Störungstheorie für Oszillationen zwischen einer beliebigen Anzahl von Neutrinos und Antineutrinos. Wir beziehen dabei Verletzung der Leptonzahl mit ein und behandeln diese als kleinen Störungsparameter. Es zeigt sich, dass eine kleine Leptonzahlverletzung im Vakuum verstärkt werden kann in Materie, welche die CP-Symmetrie bricht. Wir untersuchen weiterhin den Fall von Oszillationen zwischen zwei Neutrinogenerationen.
Schließlich betrachten wir Modifikationen von geladenen schwachen Strömen durch Verletzung der CPT-Symmetrie bei niedrigen Energien. Wir untersuchen Zerfälle von Myonen und Antimyonen und verwenden die Vorhersage für deren unterschiedliche Lebensdauern, um die Größenordnung der CPT-verletzenden Parameter einzuschränken. Weiterhin untersuchen wir die differentiellen Zerfallsraten von Myonen und Antimyonen.

Contents

1 Introduction

The physics of neutrino oscillations must *per se* be considered physics beyond the standard model of particle physics, since the latter does not allow for neutrino masses. Neutrino masses, however, are essential for neutrino-flavor oscillations. This is one of the shortcomings of the standard model of particle physics, which leads us to explore novel pathways beyond the established picture.

One promising environment to study the ramifications of nonvanishing neutrino masses is the early Universe: elementary particles form a hot, interacting plasma, in which frequent scattering among its constituents is taking place. In such an environment neutrinos are considered as a quantum ensemble with finite temperature [2]. The frequent scattering of neutrinos off particles from the background plasma results in a small mean free path of neutrinos and introduces decoherence into the ensemble. The time evolution of such neutrino ensembles is encoded in their coherence vector [3].

The possibility of CPT violation can be thought of as another such endeavor to understand physics beyond the standard model and it comes in close conjunction with the endeavor of understanding neutrino oscillation anomalies, which we shall discuss shortly. Much theoretical prejudice goes in the favor of CPT invariance. This may be attributed to the fact that all Poincaré-invariant field theories are found to be CPT-invariant [4]. Other consequences of CPT invariance, such as the equality of masses and lifetimes for particles and antiparticles, have been scrutinized experimentally to a fairly good accuracy [5]. However, the question of whether CPT violation manifests in observables is again ultimately an experimental one and it is conceivable that effects of CPT violation are disguised by the current experimental threshold of accuracy. Also there are attempts to explain neutrino oscillation anomalies by means of CPT-violating neutrino oscillations. It is therefore an interesting exercise to consider effective quantum field theories which are not invariant under CPT as well as ramifications of such theories in neutrino oscillation phenomena [6].

Altered dispersion relations are a feature which both neutrino ensembles in the early Universe as well as CPT-violating effects in neutrino oscillations have in common. Dispersion relations describe the connection between the energy E, momentum $\vec{p}$ and mass m of a particle. In their standard form they are just given by the relativistic expression $E^2 = \vec{p}^2 + m^2$. This simple expression is modified in the early Universe and in models with CPT violation. In the former case such altered dispersion relations arise, e.g., because of effective potentials neutrinos are subject to due to their scattering off background particles [7]; in the latter case violation of CPT introduces nonstandard energy dependences into the oscillation Hamiltonian and thereby gives rise to altered dispersion relations [6].

We shall elaborate further on the questions of neutrino ensembles in the early Universe, violation of the CPT symmetry and consequences of altered dispersion relations in the course of this thesis. Before doing so, however, let us recapitulate the history of the neutrino as it dates as far back as the year 1930 despite it being such an elusive particle.

In the year 1930 Pauli postulated the existence of a neutral particle in order to reconcile the findings by Chadwick that electrons emitted in radioactive β decay reveal a continuous energy spectrum with the principle of energy conservation. It took, however, another four years before Fermi developed a theory of β decays [8] which introduced the *electroweak* scale and overcame the misconception that the neutrino should be a particle bound in atomic nuclei rather than being created in a decay process. After more than twenty years later the first direct detection of neutrinos was performed [9] and the observed neutrinos were identified to be left-handed particles [10] in 1958. In the 1960s Pontecorvo suggested the idea of neutrino oscillations [11] and as of the 1970s the Homestake experiment began to measure the solar neutrino flux [12]. Eventually, by the end of the 1980s first evidence for atmospheric neutrino oscillations began to show in the data of the Kamiokande experiment [13].

In fact atmospheric [14, 15] and solar [16, 17] neutrino oscillation data are, besides reactor experiments [18, 19, 20, 21], still the main source for the extraction of neutrino oscillation parameters. The *atmospheric* (Δm^2_{23}) and *solar mass splitting* (Δm^2_{12}) as well as the *solar* (θ_{12}), *atmospheric* (θ_{23}) and *reactor* (θ_{13}) *mixing angles* are being measured with an ever increasing precision [22]. Recent measurements of the Daya Bay reactor experiment [21] even suggest the first conclusive evidence for a nonvanishing mixing angle θ_{13}, and therefore pave the way for exploring the possibility of CP violation in the lepton sector. The CP-violating Dirac phase δ cannot currently be accessed by existing experiments.

The picture of neutrino oscillation phenomena is, however, far from being complete or consistent. It is, for instance, yet unknown whether neutrinos obey a *normal* or *inverted hierarchy*, i.e., whether the first or third mass eigenstate is the lightest. There is currently no information about possible CP violation in the lepton sector, i.e., the value of the CP-violating phase in neutrino oscillations is completely unknown. Moreover, there are long-standing *neutrino oscillation anomalies*. The latter notion refers to neutrino oscillation experiments which cannot be reconciled with the standard description of neutrino oscillation phenomena. Among those exceptional experiments are LSND [23] and MiniBooNE [24, 25]. The LSND collaboration finds a third mass splitting for neutrinos, whereas the MiniBooNE collaboration observes as yet unexplained excesses in both the neutrino and antineutrino oscillation data. The latter data are also consistent with the LSND findings but not with expectations in the standard neutrino oscillation picture. As deviations from an expected outcome are always interesting since they may point towards new physics, there has been a plethora of theoretical speculations as to the nature of such neutrino oscillation anomalies. These speculations range from additional light sterile neutrino species, nonstandard neutrino interactions [26], and extra spatial dimensions [27, 28] to CPT violation in the lepton sector [6, 29]. It is possible that the reported neutrino oscillation anomalies might not last and may eventually be refuted by future oscillation experiments. The question, however, whether physics beyond the established picture of flavor oscillations in neutrinos with similar signatures exists, is ultimately an experimental one. It is therefore a sensible exercise to further our understanding of neutrino oscillations and possible explanations of *anomalies* in this sector on the theoretical frontier.

In this thesis we pursue different strategies. We provide and develop novel techniques to treat neutrino oscillations by means of a perturbation theory in a suitably small quantity, which we specify in each application correspondingly. We then utilize these novel techniques to study a framework for neutrino oscillations in which lepton-number is not conserved and also CPT violation is possible. The possibility of CPT violation in neutrinos is then extended to charged current weak interactions and we study the impact of CPT nonconservation on muon and antimuon decays.

The structure of this thesis hence assumes the following form. In chapter 2 we gather the necessary preliminaries for the subsequent studies. We focus on the introduction of the standard picture of neutrino oscillations, mathematical means such as the Magnus expansion and the coherence vector as a convenient formalism to describe neutrino oscillations. We conclude this chapter with an account on CPT violation in both the neutrino sector and charged current weak interactions. In chapter 3 we develop an adiabatic and nonadiabatic perturbation theory for the coherence vector description of neutrino oscillations. We focus on a two-flavor system and pay special attention to the methodology of this novel technique. We also briefly present an application of the perturbation theory methods in an early Universe framework. In chapter 4 we turn our attention to lepton-number violation in neutrino oscillations and develop a novel perturbation theory, which treats lepton-number violation as a small effect relative to the standard description of neutrino oscillations. We derive neutrino oscillation probabilities for a system of an arbitrary number of neutrino flavors and exemplify how lepton-number violation affects one- and two-generation oscillations. In order to furnish the to some extent abstract considerations of developing perturbation theories with some phenomenological thoughts, we briefly present two different models for neutrino oscillations with altered dispersion relations in section 4.4. Eventually, in chapter 5 we study CPT-violating modifications in charged current weak interactions and apply them to muon and antimuon decays. We derive expressions for the difference in lifetimes and the differential decay rates. We conclude our analysis in chapter 6.

2 Physical and mathematical preliminaries

In the following sections we collect both mathematical and physical preliminaries which are essential for an understanding of the work presented in this thesis. In order to keep the notational burden to a minimum and facilitate readability, we shall mostly entertain the idea of a descriptive introduction to the topic rather than dwelling upon details of derivations and mathematical proofs. We give references where the latter can be found and restrict our attention to conveying the basic concepts.

2.1 Neutrino oscillations and the standard model

Let us begin our introductory remarks on physical and mathematical preliminaries with a concise review of the aspects of the standard model which are essential for our understanding of neutrino oscillation phenomena. While we shall briefly discuss the issues of neutrino masses in the standard model and the associated difference between mass and weak eigenbases, we shall not give a detailed description of neutrino mass models and their possible origins in physics beyond the standard model.

The standard model of particle physics is based on the concept of gauge invariance. The three relevant types of interactions – electromagnetic, weak and strong – are associated with three different gauge groups. The strong interaction features a $SU(3)_{\mathrm{C}}$ color symmetry, the weak interaction comes with a left-chiral $SU(2)_{\mathrm{L}}$ and the electromagnetic interaction is described by a $U(1)_{\mathrm{Y}}$ with hypercharge Y as the associated quantum number. Such is the situation before spontaneous symmetry breaking via the Higgs mechanism [30, 31, 32, 33]. The question of whether fermions are affected by any of the three fundamental interactions then is one of their representation under the aforementioned groups. Neutrinos are neutral fermions which neither participate in the electromagnetic nor the strong interactions; they are singlets under the associated gauge groups of the standard model. There are three active neutrinos in the standard model, which are left-handed and are part of the lepton doublet in the chiral representation of $SU(2)_{\mathrm{L}}$. We denote the doublets of neutrino ν_i and electrically charged lepton l_i^- Dirac spinors as

$$L_i = \begin{pmatrix} \nu_i \\ l_i^- \end{pmatrix}_{\mathrm{L}} , \tag{2.1}$$

where $i = 1, 2, 3$ is a flavor index labeling the different lepton flavors e, μ, τ and the subscript L denotes a projection onto left-chiral states via the projection operator $P_{\mathrm{L}} = \frac{1}{2}(1 - \gamma_5)$. Since the electrically charged fermions are part of the chiral representation of the gauge group, their mass terms are generated via Yukawa interactions. We write the Lagrangian density for the leptonic Yukawa couplings as

$$\mathcal{L}_{\rm Yukawa} = -Y^l_{ij}\,\overline{L_i}\phi R_j + {\rm h.c.}\,, \tag{2.2}$$

where Y^l_{ij} is the leptonic Yukawa coupling, ϕ is the standard model Higgs doublet and R_j are the three right-handed charged fermion singlets under the chiral gauge group $SU(2)_{\rm L}$. The Higgs field assumes a vacuum expectation value $\langle\phi\rangle = (0, v/\sqrt{2})^\top$ and thereby breaks the $SU(3)_{\rm C} \times SU(2)_{\rm L} \times U(1)_{\rm Y}$ symmetry down to $SU(3)_{\rm C} \times U(1)_{\rm em}$. In the process of spontaneous symmetry breaking, the Yukawa interactions give rise to charged fermion masses; neutrinos, however, remain massless [34].

The standard model also has several shortcomings such as the fine-tuning problem of the Higgs mass and unification of gauge couplings to mention only a few [35]. Therefore, it is sensible to think of the standard model of particle physics as an effective low-energy theory, which is valid up to some scale Λ which characterizes the onset of physics beyond the standard model. If we take this viewpoint [36, 37, 38], we have to write down also nonrenormalizable terms in the Lagrangian. There is a dimension-five operator, which is consistent with the gauge symmetry of the standard model,

$$\frac{Z^\nu_{ij}}{\Lambda}\phi\phi L_i L_j, \tag{2.3}$$

and which leads to neutrino masses given by

$$M^{\rm M}_{ij} = \frac{Z^\nu_{ij}}{2}\frac{v^2}{\Lambda} \tag{2.4}$$

upon symmetry breaking. One might expect similar terms induced by loop corrections in a purely standard model framework. Such terms, however, do not arise since lepton-number conservation is an accidental symmetry of the standard model. From the theoretical point of view such neutrino mass terms are very attractive. Since the dimension-five operator is a generic one, we conclude that neutrino masses are likely to appear in extensions of the standard model and they will be suppressed by v/Λ, which may serve as an explanation of the smallness of neutrino masses. Eventually, such a term also breaks total lepton-number as well as lepton-flavor symmetry, if Z^ν_{ij} has nondiagonal entries. Violation of CP and lepton mixing are therefore likely to occur in the lepton sector. The most prominent example, that gives rise to the kind of dimension-five operator discussed here, are seesaw models [39, 40].

Such is the situation if we consider the standard model as a low-energy effective theory in which only left-handed neutrino states (and their associated antiparticles) are present. Those neutrinos can be given a mass via the nonrenormalizable operator discussed above. The picture changes if the existence of new fields beyond the standard model particle content is assumed. In the most natural extension, the right-handed counterpart $\nu_{{\rm R}i}$ for each left-handed neutrino $\nu_{{\rm L}i}$ of flavor i is added. Two different types of mass terms arise

$$\mathcal{L}_{\text{mass}} = -\overline{\nu_{\text{L}i}}\, M^{\text{D}}_{ij}\, \nu_{\text{R}j} - \overline{\nu^{\text{c}}_{\text{L}i}}\, M^{\text{D}}_{ij}\, \nu^{\text{c}}_{\text{R}j} - \frac{1}{2}\overline{\nu^{\text{c}}_{\text{L}i}} M^{\text{M,R}}_{ij} \nu_{\text{R}j} - \frac{1}{2}\overline{\nu_{\text{L}i}} M^{\text{M,L}}_{ij} \nu^{\text{c}}_{\text{R}j} + \text{h.c.} \tag{2.5}$$

They are labeled Dirac (M^{D}_{ij}) and Majorana ($M^{\text{M,L}}_{ij}$), ($M^{\text{M,R}}_{ij}$) mass terms, respectively, and ν^{c} denotes the charge-conjugated field $\nu^{\text{c}} = C\bar{\nu}^{\top}$, where C is the charge conjugation operator. The mass terms in Eq. (2.5) represent the most general situation in which both left- and right-handed neutrinos are present and in which neutrinos have both a Dirac and a Majorana mass. The Dirac mass term can be generated from Yukawa couplings connecting the left-handed lepton doublets, the charge-conjugated Higgs field, as well as right-handed neutrinos after spontaneous symmetry breaking in analogy to charged fermion masses. Furthermore, it conserves total lepton-number. The Majorana mass term can be generated from nonrenormalizable Yukawa interactions as indicated above. On the other hand, such a mass term can be generated effectively by integrating out heavy right-handed ν_{R}, giving rise to various versions of the seesaw mechanism. Such models have a number of attractive features, which we shall not discuss here as they do not add any insight where lepton mixing is concerned. A Majorana mass term violates lepton-number by two units and hence such terms are only allowed if neutrinos do not carry any conserved (additive) charge.

Let us focus on the Dirac mass term for the time being. Suppose the charged lepton mass term has been diagonalized. We then need only consider diagonalization of the Dirac neutrino mass term by means of a rotation in flavor space. This rotation is different for left- and right-handed neutrinos

$$\nu_{\text{L}i} = U_{ij}\, \nu'_{\text{L}j}, \tag{2.6}$$

$$\nu_{\text{R}i} = V_{ij}\, \nu'_{\text{R}j} \tag{2.7}$$

implying that the diagonal mass term $M^{\text{D}}_{\text{diag}}$ is given as

$$M^{\text{D}}_{\text{diag}} = V^{\dagger} M^{\text{D}} U. \tag{2.8}$$

The diagonalization of the Majorana mass term can be done along similar lines with the exception that only one diagonalization matrix is needed [34].

The fact that weak interaction (or flavor) and mass eigenstates are two distinct concepts for neutrinos has consequences both for charged current weak interactions as well as neutrino propagation in vacuo.
Using the mass eigenstates ν', the leptonic charged current weak interaction Lagrangian [41, 42] reads

$$\mathcal{L}_{\text{cc}} = -\frac{g}{\sqrt{2}}\overline{\nu'_{\text{L}i}}(U^{\dagger})_{ij}\gamma^{\mu} l^{-}_{\text{L}j} W^{+}_{\mu} + \text{h.c.}\,, \tag{2.9}$$

where $l^{-}_{\text{L}j}$ are the charged left-handed leptons. Now, $U^{\dagger}$ is the unitary lepton mixing matrix, which is named *PMNS* matrix after *Pontecorvo, Maki, Nakagawa, Sakata* [43]. The mixing can be parametrized using three independent rotations in flavor space each through an Euler angle θ_{jk} and one CP-violating Dirac phase δ. Choosing the abbreviatory notation $c_{jk} \equiv \cos\theta_{jk}$ and $s_{jk} \equiv \sin\theta_{jk}$ one possible parametrization for U is found to be

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}s_{13} \end{pmatrix}, \tag{2.10}$$

where we have omitted two Majorana phases since those do not affect neutrino oscillation probabilities [44, 45].

The different mixing angles have different *common names* for historical reasons. The angle θ_{12} is termed solar mixing angle since this angle has been measured in solar neutrino oscillation experiments for the first time. The energy source of the sun is nuclear fusion. In the sun's core neutrinos are produced, e.g., via the ^{4}He reaction: $4p + 2e^- \to {}^4\text{He} + 2\nu_e$. The total neutrino spectrum emitted by the sun is complex due to the fact that different reaction cycles exist which produce neutrinos [38].

The angle θ_{23} is also commonly known as the atmospheric angle. Again this is because it was measured for the first time in an experiment designed to study atmospheric neutrinos. Atmospheric neutrinos are produced in collisions of cosmic rays with the nuclei of air in the Earth's atmosphere [38]. These collisions produce pions, which decay into muons and neutrinos (e.g., via the reaction $\pi^+ \to \mu^+\nu_\mu$). Due to the relativistic dilatation factor the muons travel some distance before they decay into electrons and neutrinos (e.g., via the reaction $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$).

Eventually, θ_{13} is the most elusive of the three neutrino mixing angles and has only recently been determined to be nonvanishing [21]. Reactor mixing angle is another name for θ_{13} since reactor experiments turn out to be most suitable to measure it. The idea behind reactor experiments is that nuclear fissions in the reactor's core produce low-energy $\bar{\nu}_e$ abundantly, which then can be measured by a detector near the nuclear power plant.

In experiments involving neutrinos the flavor eigenstates are most relevant. A neutrino produced by electroweak interactions is not a mass but a weak interaction eigenstate. Let us therefore consider the propagation of a neutrino flavor eigenstate in vacuo [11, 34, 46]. In the case of off-diagonal entries in the neutrino mass matrix, a flavor eigenstate ν_α in vacuo is a linear combination of mass eigenstates ν_i', which evolve in time. Therefore, we can write the flavor eigenstate ν_α at time t according to

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i}\, e^{-iE_i t}\, |\nu_i'\rangle \,. \tag{2.11}$$

We can then construct the transition probability [46] for a flavor ν_α converting into a flavor ν_β via

$$P(\nu_\alpha \to \nu_\beta) = |\,\langle \nu_\beta | \nu_\alpha(t)\rangle\,|^2 = \sum_{ij} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} e^{-i(E_i - E_j)t}, \tag{2.12}$$

where we use the usual approximation of the neutrino wave functions as plane waves. There are some subtleties concerning the formulation of quantum mechanics for neutrino oscillations, which we do not discuss here. The reason for our omission of such a discussion is that the expressions for the neutrino oscillation probabilities obtained from a plane wave ansatz are reproduced in a more careful treatment using, e.g., wave packets [47, 48]. The energies E_i of the neutrino states can be taken to be ultra-relativistic in all relevant cases

$$E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2p_i}. \tag{2.13}$$

Furthermore, it is justified to assume $p_i \simeq p_j \equiv p$ for the neutrino momenta such that the parameters determining neutrino oscillations are the ingredients of the lepton mixing matrix (three mixing angles and one CP-violating phase) and two mass-squared differences. It is important to understand that neutrino oscillation phenomena do not discriminate between Dirac and Majorana neutrinos. For both cases the formalism describing neutrino oscillations is identical [44, 45]. We can further rewrite [49] the vacuum neutrino oscillation probability $P(\nu_\alpha \to \nu_\beta)$ making use of assumption (2.13). We have

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - 4\sum_{i>j} \mathrm{Re} J_{ij}^{\alpha\beta} \sin^2\left(\frac{\Delta m_{ij}^2 L}{4E}\right) + 2\sum_{i>j} \mathrm{Im} J_{ij}^{\alpha\beta} \sin\left(\frac{\Delta m_{ij}^2 L}{2E}\right). \tag{2.14}$$

We have introduced the expression $J_{ij}^{\alpha\beta} = U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^*$, which has been termed *quartic rephasing invariant* [50]. Furthermore, we have written the mass-squared differences between the neutrino mass eigenstates as $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ and replaced the travel time t of the neutrinos by the distance L between the source and the detector, which is justified for ultra-relativistic particles. Note, eventually, that the term in the oscillation probability proportional to the real part of the rephasing invariant is CP-conserving, whereas the one proportional to its imaginary part violates CP. Neutrino flavor oscillations are hence sensitive to neutrino mass-squared differences, mixing angles and leptonic CP violation. The different mixing angles have already been commented on. Let us add that there are two linearly independent mass-squared differences, namely the solar mass splitting Δm_{12}^2 and the atmospheric mass splitting Δm_{23}^2. The sign of the mass splittings determines the hierarchy of the neutrino mass eigenstates. It is known from solar neutrino experiments that the mass splitting Δm_{12}^2 has positive sign and that hence the first mass eigenstate is lighter than the second one. The sign for the atmospheric splitting, however, still eludes us. It is therefore as yet unknown whether neutrinos obey a normal (m_1 being the lightest mass eigenstate) or inverted (m_3 being the lightest mass eigenstate) hierarchy.

Moreover, the standard picture of neutrino vacuum oscillations, as described here, predicts a certain *spectral dependence* of the oscillation probability on L and E, which is a direct consequence of the neutrino *dispersion relation*. The latter gives the connection between neutrino momentum, energy and mass and for vacuum oscillations leads to a spectral dependence, which is just L/E. This simple relation need not hold for, e.g., CPT-violating extensions of the standard picture of neutrino oscillations as we shall see in due course.

We shall not delve into further details of neutrino oscillations in vacuo or matter since the upcoming considerations entail sufficient information and formal developments in this direction.

2.2 The Magnus expansion

The bulk of the work presented in this thesis heavily relies upon a certain mathematical technique which is termed *Magnus expansion* and which we shall use to provide novel calculational tools for different areas in the physics of neutrino oscillations. In this section we shall briefly review the main ideas of this perturbative expansion and provide the necessary means for its application to the physics scenarios studied in the course of the upcoming sections. The Magnus expansion is a mathematical tool used for obtaining perturbative solutions to nonautonomous linear differential equations for linear operators. One example for this type of equations is the Schrödinger equation for neutrino oscillations. According to Magnus' theorem the solution to such an equation can be written as a matrix exponential for the Magnus operator. The latter is calculated from the evolution matrix of the system by means of a series expansion as shall be seen shortly. We shall mention some reasons why the Magnus expansion is a convenient way of treating neutrino oscillations in this section; among other things it allows to treat systems with an arbitrary number of neutrino generations as we shall show in chapter 4.

In the following consideration we shall abstain from mathematical rigor when it comes to proving the results which we present. The proofs of the theorems to be stated can be found in the literature [51] and do not add any additional insight as far as applications of and the assignment of physical meaning to the theorems are concerned.

In order to achieve compact appearance and readability of the upcoming expressions, we introduce some convenient notations first. For our purposes it shall suffice to consider a finite n-dimensional Lie algebra $\mathfrak{g}$ of matrices. In this algebra we define the associated Lie bracket as the usual matrix commutator $[A, B] = AB - BA$, where $A, B \in \mathfrak{g}$. We can then associate with any element $A \in \mathfrak{g}$ a linear operator $\mathrm{ad}_A : \mathfrak{g} \to \mathfrak{g}$ which acts upon any element B of the Lie algebra as

$$\mathrm{ad}_A B = [A, B]\,, \quad \mathrm{ad}_A^i B = \left[A, \mathrm{ad}_A^{i-1} B\right]\,, \quad \mathrm{ad}_A^0 B = B, \qquad i \in \mathbb{N},\ A, B \in \mathfrak{g}. \tag{2.15}$$

Moreover, it is important to have the notion of some type of matrix norm at one's disposal. A matrix norm is a real, non-negative number $||A||$ ascribed to any matrix $A \in \mathbb{C}^{n\times n}$, which satisfies the following properties

(i) $||A|| > 0 \quad \forall A \in \mathfrak{g}$ as well as $||A|| = 0$ iff $A = 0$.

(ii) $||\alpha A|| = |\alpha|\,||A|| \quad \forall \alpha \in \mathbb{C}$.

(iii) $||A + B|| \leq ||A|| + ||B||$.

Different types of matrix norms exist. However, in a Lie algebra of finite-dimensional matrices, all such norms are found to be equivalent and related by certain inequalities. We shall hence focus on one particularly useful matrix norm, the Frobenius norm

$$||A||_{\mathrm{F}}^2 = \mathrm{tr}\left(A^\dagger A\right) \tag{2.16}$$

for any $A \in \mathfrak{g}$ and where $\mathrm{tr}A$ is the trace of the matrix A. An interesting feature of the Frobenius norm is that it is unitarily invariant. By this we mean $||UAV|| = ||A||$ for unitary matrices U and V as can be inferred from the norm's definition.

After this preliminary introduction of terminology and notation let us start describing the nature of the Magnus expansion and its applications. The Magnus expansion is a tool for perturbatively solving nonautonomous linear differential equations for linear operators. To be more precise, consider the linear evolution equation

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = A(t)Y(t) \tag{2.17}$$

for the operator $Y(t)$ supplied with the initial condition $Y(t = t_0) = 1$ and in which $A(t)$, henceforth termed the *evolution matrix*, is a known function of t. The Magnus theorem [52] then states that $Y(t)$ can be written as a true exponential of the Magnus operator $\Omega(t)$ according to

$$Y(t) = \exp \Omega(t) \tag{2.18}$$

and that under this assumption $\Omega(t)$ satisfies the following differential equation

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \sum_{i=0}^{\infty} \frac{1}{i!}\, B_i\, \mathrm{ad}_\Omega^i A, \tag{2.19}$$

where B_i are the Bernoulli numbers. The first few nonvanishing Bernoulli numbers are $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$. It is understood that Magnus' proposal immediately poses the question of existence of a true exponential solution for certain values of t and certain operators A. Existence of a true exponential solution can be ensured under fairly general assumptions, which we shall state in due course. Let us for now elaborate on how to solve the subsidiary evolution equation for the Magnus operator by means of a perturbative expansion. To this end, suppose the operator A contains a *smallness parameter* ε, which can be factored out in such a way that substituting A by εA is possible. In this case suppose furthermore the Magnus operator can be written as a series expansion according to

$$\Omega(t) = \sum_{i=1}^{\infty} \varepsilon^i \Omega_i(t), \tag{2.20}$$

such that each *Magnus approximant* $\Omega_i(t)$ is of order ε^i. This series expansion for the Magnus operator can now be substituted into the underlying subsidiary differential equation for $\Omega(t)$ along with the replacement of A by εA. Doing so and equating corresponding powers in ε on the right hand side and left hand side of the resulting equation reveals explicit expressions for the different approximants. The original Magnus expansion is then obviously recovered by setting $\varepsilon = 1$. The first two approximants are then found to obey

$$\Omega_1(t) = \int_{t_0}^{t} \mathrm{d}\tau \, A(\tau), \tag{2.21}$$

$$\Omega_2(t) = \frac{1}{2} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \, \left[A(t_1), A(t_2)\right]. \tag{2.22}$$

Higher-order terms can be obtained and have been worked out in the literature [51, 53]. We leave the question untouched how to efficiently construct higher-order terms and whether a closed-form expression for $\Omega_i(t)$ exists. We shall also not state higher-order terms here since the upcoming considerations only involve the Magnus expansion up to second order in the approximants. We shall, however, remark that all such terms contain nested commutators of the evolution matrix evaluated at different times. Hence, the Magnus expansion gives the exact solution to the differential equation under consideration already in first order if the evolution matrix is self-commuting at different times (e.g., if the time dependence can be factored out of the matrix). This is, for instance, the case for a time-independent evolution matrix. The latter fact motivates a manner of speaking in which the Magnus expansion for nonautonomous linear differential equations can be regarded as a (time-dependent) perturbation expansion around the autonomous (static) solution (the case in which the evolution matrix is time-independent and hence the differential equation easily solved by mere exponentiation).

The fact that we write the Magnus operator as a series immediately leads to another issue, which has to be addressed in a rigorous treatment, namely for which values of t and for which operators A does the series actually converge. We do not go into detail when it comes to addressing the questions of existence and convergence of the Magnus expansion but simply state the answer to these issues for the case, which is of interest for the purposes which concern us here [51]. Let us therefore consider the differential equation (2.17) on a finite-dimensional Hilbert space with the initial condition $Y(t = t_0) = 1$, and suppose $A(t)$ is a bounded operator on this space, i.e., its matrix norm is finite. In this case the Magnus series $\Omega(t) = \sum_{i=1}^{\infty} \Omega_i(t)$ with the approximants $\Omega_i(t)$ (as given above for the first two orders), converges in the interval $t \in [t_0, t_\mathrm{c}[$ such that

$$\int_{t_0}^{t_\mathrm{c}} \mathrm{d}\tau \, ||A(\tau)|| < \pi \tag{2.23}$$

and the sum of the approximants, i.e., the Magnus operator satisfies $Y(t) = \exp \Omega(t)$. This integral expression involving the *convergence bound* t_c is particularly helpful if one aims at discriminating between two solutions for reasons of their convergence properties. Note, also, at this stage of our analysis that using the Frobenius norm for the matrix norm implies that unitary transformations of $A(t)$, e.g., a change of basis in the differential equation determining $Y(t)$, does not alter the convergence properties. The statement about convergence and existence of the Magnus expansion on finite-dimensional Hilbert spaces is, of course, easily applied to the space of finite-dimensional matrices we are concerned with.

As a matter of fact, there is some virtue in a change of basis prior to attempting to solve the actual differential equation [51]. Suppose the matrix $A(t)$ can be decomposed into an exactly solvable part $A_0(t)$ and a perturbation $\varepsilon A_1(t)$ according to $A(t) = A_0(t) + \varepsilon A_1(t)$ with a small perturbation parameter ε. In such a situation it is possible to *integrate out* the exactly solvable part. In general, one may decompose the matrix $A(t)$ by means of a linear transformation $F(t)$ according to

$$A(t) = F(t)A_{\mathrm{F}}(t)F^{\dagger}(t_0). \tag{2.24}$$

In the new basis, which we shall refer to as the *F-picture* henceforth, the evolution equation for $Y_{\mathrm{F}}(t)$ reads

$$\frac{\mathrm{d}Y_{\mathrm{F}}(t)}{\mathrm{d}t} = A_{\mathrm{F}}(t)Y_{\mathrm{F}}(t) \qquad \text{with} \qquad A_{\mathrm{F}}(t) = F^{\dagger}(t)A(t)F(t) - F^{\dagger}(t)\frac{\mathrm{d}F(t)}{\mathrm{d}t}. \tag{2.25}$$

Suppose for now $F(t)$ is a unitary transformation. In that case the matrix norm of the first term in the expression for $A_{\mathrm{F}}(t)$ is clearly identical to the matrix norm for $A(t)$. It is only by virtue of the time dependence of the linear transformation $F(t)$ that the matrix norm of $A_{\mathrm{F}}(t)$ is altered with respect to the matrix norm of the evolution matrix in the original basis – as can be seen from the second term in the expression for $A_{\mathrm{F}}(t)$. Hence, it is seen that a linear transformation prior to solving the differential equation can in fact improve the domain of convergence as well as the accuracy of the perturbation expansion. We shall also see in due course that such changes of basis can add to understanding physical properties of the system it is applied to and reveal quantities suitable to characterize the physics of the system under consideration. It shall then also become clear that a succession of different linear transformations prior to solving the differential equation has its merits, too.

There is another attractive feature about the Magnus expansion, which shall not go unmentioned here. Suppose we were to apply it to a Schrödinger-type evolution equation, as we shall do extensively in the upcoming analysis, of the form

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = -iH(t)U(t), \tag{2.26}$$

where $U(t)$ is some time evolution operator for the system and $H(t)$ its Hamiltonian. The Magnus approximants for this system are easily read off from Eqs. (2.21 – 2.22). It is also obvious from these equations that each Magnus approximant is anti-Hermitian, $\Omega_i^{\dagger}(t) = -\Omega_i(t)$. Therefore, the solution for the time evolution operator $U(t)$ of the system is *unitary to each order* of the perturbation expansion. Put another way, using the

Magnus expansion for Schrödinger-type evolution equations guarantees conservation of probability to each order of the perturbation series regardless of the order one chooses to truncate it at. We shall explicitly show in one of the upcoming analyses that this is indeed the case.

Let us finally observe that the Magnus expansion presents an interesting alternative to the customary diagonalization procedure in standard Schrödinger quantum mechanics. In the area of neutrino physics, the latter typically seeks a solution to the Schrödinger equation by diagonalizing the Hamiltonian by means of an effective mixing matrix from which oscillation probabilities can be constructed using eigenvalues and eigenvectors of the Hamiltonian. The former, however, does not rely on such notions as it transmutes this issue of diagonalization into integrating the Hamiltonian with respect to time and the calculation of a matrix exponential of an operator while retaining the essential features of the customary solution. We shall further elaborate on this later on.

We have mentioned many an application and advantage of the Magnus expansion when it comes to analyzing neutrino oscillations in different contexts and we shall exploit these features and furnish them with a physical interpretation.

2.3 The coherence vector

Apart from the Magnus expansion a good deal of the following analysis relies on the notion of a *coherence vector* and that of a *density matrix* in neutrino oscillations. We shall therefore briefly introduce said notions in this section. At the same time those means typically find application in, e.g., the early Universe. We shall use the early Universe framework as motivation for the novel formal calculational tools provided hereafter. We shall see how to apply parts of them in such a physics context. The coherence vector is a quantity that comes into play when the density matrix of a quantum mechanical system is rewritten in a certain way. The dynamics of quantum ensembles are governed by a von Neumann equation for the density matrix of the system. This equation can be given a geometrical interpretation by rewriting it with the help of the coherence vector which, instead of a von Neumann equation, obeys a gyroscope-type equation. The latter is well-known from classical mechanics and can be studied in close analogy.

A convenient and established way to deal with neutrino flavor oscillations is to encode this effect in a Hamiltonian formulation in which the oscillatory behavior is captured in a Schrödinger-type equation for a wave function in neutrino flavor space. This formalism, in principle, applies to an arbitrary number of neutrino generations and is also capable of incorporating medium effects on neutrino propagation such as coherent elastic forward scattering in, e.g., stellar matter [54, 55, 56] encoding it in an effective potential in the Schrödinger equation. It was soon realized that the Hamiltonian formalism for neutrino oscillations can be given a geometrical interpretation in a N^2-1-dimensional space for

N neutrino flavors [57]. This approach to neutrino oscillations sees equations of motion for a coherence vector[1] in that the Schrödinger-type equation of motion can be rephrased as a gyroscope equation, i.e., a formal equivalent to, e.g., the precession of a magnetic moment in an external magnetic field. Besides its apparent usefulness when it comes to picture neutrino oscillations there is also a purely formal merit to the gyroscope-type equations in that they are introduced by means of decomposing the Hamiltonian in terms of the generators of the associated $SU(N)$, e.g., the Pauli matrices for a two-flavor system. We shall see to the derivation of this gyroscope-type equation for the coherence vector from the density matrix in a moment.

The decomposition procedure is also most convenient when the notion of a wave function is not suitable any more to describe the physics of neutrino oscillations. A typical situation in which the breakdown of the wave function formalism is expected are quantum ensembles with a finite temperature or neutrino ensembles with a finite temperature and an interacting background plasma [58, 59]. The latter situation is encountered, e.g., in the early Universe prior to big bang nucleosynthesis [60, 61, 62]. Let us dwell upon this subject further and qualitatively sketch the main features of neutrino interactions in the presence of a background plasma. First of all we understand that neutrinos only very weakly interact with matter. However, it is those interactions, which give rise to interesting physics. It is hence most convenient to treat them by means of a perturbative expansion, which assumes that to first order the neutrino and background fields are not correlated at all and hence the evolution factorizes into a neutrino part and a background part. This part of the perturbation gives the *forward scattering* or *refractive* part of the interaction. To second order in this perturbation theory neutrinos do interact with the background particles and are produced and absorbed by the medium as well as scattered off its ingredients. This part of the perturbation is termed *non-forward scattering* or *collision* part of the interaction. In order to consistently describe a neutrino ensemble, e.g., in the early Universe in full generality one must account for both oscillation and collision effects at the same time. Interaction terms of neutrinos with the background plasma must include scattering $\nu_p X \to \nu_{p'} X'$, production (mediated via charged currents) $X \to X' \nu_p$, absorption (mediated via charged currents) $\nu_p X \to X'$, pair production (mediated via neutral currents) $X \to X' \nu_p \bar{\nu}_{p'}$ and pair absorption (mediated via neutral currents) $\nu_p \nu_{p'} X \to X'$ as well as self-interactions (mediated via neutral currents) $\nu_p \nu_{p'} \to \nu_q \nu_{q'}$, where X and X' each represent a number of particles from the background before and after the interactions, respectively; a subscript p, p', q, q' denotes the neutrino four-momentum. It is understood that each of these processes has an antineutrino equivalent and that interactions obtained from *crossing* are also to be taken into account (such as pair creation $\nu_{p'} \bar{\nu}_{q'} \to \nu_p \bar{\nu}_q$). The frequent interactions of neutrinos in the presence of a background plasma give rise to a small mean free path of the neutrinos and hence

[1] Different names exist for the coherence vector depending on the area of physics it is used in. Bloch vector is most common in solid state physics, whereas polarization or coherence vector is more common in particle physics applications.

the breaking of coherence of the ensemble. It is essentially for this reason that the wave function formalism must fail in describing neutrino oscillations and interactions now.

The appropriate description is then given by the density matrix formalism. The density matrix $\rho(p,t)$ of the neutrino ensemble obeys a von Neumann equation and the different contributions to the effective Hamiltonian are given by collisions (non-forward scattering), with particles from the background medium, which introduce decoherence, the vacuum oscillatory part and eventually the refractive part. Avoiding any unnecessary notational burden, we shall briefly sketch the composition of the equations of motion in a pictorial manner

$$\begin{aligned} \dot{\rho}(p,t) &= \text{standard oscillatory (vacuum) terms} \\ &+ \text{refractive terms} \\ &+ \text{forward and non} - \text{forward scattering terms}, \end{aligned} \tag{2.27}$$

where the dot represents a derivative with respect to time. There are three distinct cases for neutrino interactions in a plasma apart from the oscillatory and the refractive part. Firstly, charged current interactions exchange lepton-number between the background plasma and the neutrino ensemble; secondly, neutral current interactions conserve lepton-number for the ensemble and the background separately; and thirdly, neutrino self-interactions mediated by neutral currents can occur even without the presence of a background of interacting particles.

There is another technical subtlety when it comes to the equations of motion for a neutrino ensemble in the presence of a background plasma – the distinction between *quantum kinetic equations* and *quantum rate equations*. We shall, however, not reiterate the details of the derivation of the former, since they do not contribute much to the understanding of the upcoming analysis [59].

The dynamics of the neutrino ensemble are determined by the quantum kinetic equations, which present a generalization of the Pauli-Boltzmann equations. The former evolve quantum amplitudes as is indispensable if a consistent description of particle oscillation phenomena, which are inherently nonclassical, is sought. The latter evolve probabilities rather than amplitudes. This procedure is essentially classical since quantum mechanics only enters the problem when it comes to calculating cross sections for the various possible reaction channels. The resultant quantum rate equations are inappropriate when neutrino oscillations occur. Thus, in order to obtain the quantum kinetic equations the full density matrix for all particles in the plasma is evolved forward in time by means of the S-matrix and tracing over all degrees of freedom other than the neutrinos under consideration yields the equation of motion for the system's density matrix, the quantum kinetic equations, which do reduce to quantum rate equations in the appropriate limit [59]. The variable of interest in the quantum kinetic equations is, as described above, the one-body reduced momentum-dependent density operator, which is conveniently decomposed in terms of the generators of the associated $SU(N)$.

The purpose of the analysis to come is to describe the evolution of a two-flavor neutrino ensemble with generic potentials at a finite temperature. We account for a brief motivation on how to obtain the quantum kinetic equations from the density matrix formalism in the case of coherent forward scattering (which also dominates the bulk of the studies to follow on this subject) and how to relate the solutions of the quantum kinetic equations, i.e., the coherence vector, to physical observables of interest [3].

Let us denote the density matrix by $\rho(p,t)$. It obeys a von Neumann equation

$$\dot{\rho}(p,t) = -i\left[H(p,t), \rho(p,t)\right], \tag{2.28}$$

where p is the neutrino four-momentum, H a generic Hamiltonian for the system. We can now decompose both the density matrix and the Hamiltonian making use of the $SU(2)$ generators, which form a basis for two-dimensional matrices. Denoting the Pauli matrices by σ_i, this yields

$$\rho = \frac{1}{2}\,\mathrm{tr}\,\rho\left[1 + \mathrm{tr}\left(\rho\sigma_i\right)\sigma_i\right] \equiv \frac{1}{2}P_0\left[1 + \vec{P}\vec{\sigma}\right], \tag{2.29}$$

$$H = \frac{1}{2}\,\mathrm{tr}\,H\left[1 + \mathrm{tr}\left(H\sigma_i\right)\sigma_i\right] \equiv \frac{1}{2}V_0\left[1 + \vec{V}\vec{\sigma}\right], \tag{2.30}$$

where repeated indices are to be summed over and we have introduced the following shorthand identifications

$$P_0 = \mathrm{tr}\,\rho, \qquad \vec{P} = \mathrm{tr}\left(\rho\vec{\sigma}\right), \tag{2.31}$$

$$V_0 = \mathrm{tr}\,H, \qquad \vec{V} = \mathrm{tr}\left(H\vec{\sigma}\right), \tag{2.32}$$

and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. Here $\vec{P}$ is the coherence vector [63, 64]. Using this notation, it is straightforward to recast the von Neumann equation as a differential equation for the coherence vector

$$\dot{P}_i = \left[-V_0\varepsilon_{ilk}V^k\right]P^l. \tag{2.33}$$

In this expression we identify the evolution matrix of the neutrino ensemble

$$S_{il} \equiv -V_0\varepsilon_{ilk}V^k \qquad \text{or} \qquad S = V_0\begin{pmatrix} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{pmatrix}. \tag{2.34}$$

Note, that the matrix S as defined here does not coincide with the S-matrix of the plasma mentioned before. Since we will not refer to the latter explicitly any more, no confusion should arise from this notation. The quantum kinetic equations can now be written in matrix notation as

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{P}(t) = S(t)\vec{P}(t). \tag{2.35}$$

Note, that this equation can also be given as $\frac{\mathrm{d}}{\mathrm{d}t}\vec{P} = V_0\,\vec{V} \times \vec{P}$ using the cross product in three-dimensional Euclidean space. We remark that this aforementioned gyroscope-type equation for neutrino oscillations presents a convenient way to interpret neutrino oscillations geometrically. However, when it comes to solving the quantum kinetic equations it is appropriate to treat them as a nonautonomous system of coupled differential equations as shall be seen in due course and has been alluded to in the discussion of the Magnus expansion.

The entries of the *effective potential vector* $\vec{V} = (V_x, V_y, V_z)$ are functions of the elements of the effective Hamiltonian of the system. They can be obtained straightforwardly by making use of its defining equation as well as the two-dimensional neutrino oscillation Hamiltonian. We find

$$V_0 V_x \equiv \beta = 2\mathrm{Re}H_{12} = \frac{\Delta m^2}{2p}\sin 2\theta_0, \tag{2.36}$$

$$V_0 V_y = 2\mathrm{Im}H_{12} = 0, \tag{2.37}$$

$$V_0 V_z \equiv \lambda = (H_{11} - H_{22}) = -\frac{\Delta m^2}{2p}\cos 2\theta_0 + V_\alpha, \tag{2.38}$$

where H_{ij} are the elements of the Hamiltonian H, Δm^2 is the mass-squared difference of the two neutrino states. The vacuum mixing angle between the two flavors, which we shall denote as ν_a and ν_b for definiteness, is written as θ_0. The difference of potential terms affecting ν_a and ν_b, respectively, is V_α. Using the diagonal part of the 2×2 neutrino oscillation Hamiltonian in flavor space,

$$\frac{\Delta m^2}{4p}\begin{pmatrix} -\cos 2\theta_0 + V_\alpha & \sin 2\theta_0 \\ \sin 2\theta_0 & \cos 2\theta_0 \end{pmatrix}, \tag{2.39}$$

for H we have obtained the last equality. The component V_y is set to zero since it is proportional to the imaginary part of the off-diagonal entry of the Hamiltonian. There are no physical phases in a two-flavor system. Note, also, that all possible time dependences for the effective potential vector have been suppressed for reasons of notational convenience. It is, however, understood that all components of $\vec{V}$ depend on time in general and we shall in fact use $V_x = V_x(t)$ and $V_z = V_z(t)$ for the upcoming analysis.

In the coherence vector description the expectation values of the generators of the associated $SU(2)$ are promoted to observables of interest. All information about the system can thus, in principle, be extracted from a solution to Eq. (2.35) for $\vec{P}(t)$. Some comments related to this issue are in order. The diagonal entries of the density matrix simply give the probability to find the system in one or the other state, i.e.,

$$\mathrm{prob}\,(\nu_a \to \nu_a) = \frac{1}{2}P_0\,[1 + P_z]\,, \tag{2.40}$$

$$\mathrm{prob}\,(\nu_a \to \nu_b) = \frac{1}{2}P_0\,[1 - P_z]\,, \tag{2.41}$$

if the density matrix is considered in its one-particle interpretation. It can be easily seen how these relations emerge. Suppose we decompose the density matrix $\rho(p,t)$ in terms of the neutrino wave function $|\psi(p,t)\rangle$ (in the one-particle interpretation the notion of a wave function still is a sensible one) according to

$$\rho(p,t) = \frac{1}{N^{\mathrm{EQ}}(p,0)} \, |\psi(p,t)\rangle \, \langle\psi(p,t)| \, , \tag{2.42}$$

where $N^{\mathrm{EQ}}(p,0)$ is some time-independent normalization function which we shall interprete shortly. Furthermore, for a two-state system we can additionally decompose the wave function itself as $|\psi(p,t)\rangle = a(p,t)\,|\nu_a\rangle + b(p,t)\,|\nu_b\rangle$. With this definition the diagonal entries of the density matrix are given by $|a(p,t)|^2$ as well as $|b(p,t)|^2$, which is but the probability to find the system in one state or the other. Using the decomposition of the density matrix in terms of the coherence vector then results in the expressions for the oscillation probabilities as given above.

In the ensemble interpretation of the density matrix the diagonal entries give the relative number densities $N_a(p)$ and $N_b(p)$ for the different neutrino flavors. These number densities are normalized to the equilibrium Fermi-Dirac number distribution at zero chemical potential μ according to

$$N_a(p) = \frac{1}{2} P_0 \left[1 + P_z\right] N^{\mathrm{EQ}}(p,0), \tag{2.43}$$

$$N_b(p) = \frac{1}{2} P_0 \left[1 - P_z\right] N^{\mathrm{EQ}}(p,0), \tag{2.44}$$

$$N^{\mathrm{EQ}}(p,\mu) = \frac{1}{2\pi^2} \frac{p^2}{1 + e^{\frac{p-\mu}{T}}}, \tag{2.45}$$

where T is the temperature of the ensemble [65]. Inverting these relations, it is possible to ascribe a physical meaning to both P_0 and P_z. These quantities are found to read

$$P_0 = \frac{N_a + N_b}{N^{\mathrm{EQ}}}, \tag{2.46}$$

$$P_z = \frac{N_a - N_b}{N_a + N_b}, \tag{2.47}$$

and hence P_0 is connected to conservation of probability and, in a broader context, also lepton-number [66]. Note, also, that $P_0 = 1$ corresponds to a closed thermodynamic system, whereas $P_0 \neq 1$ describes an open thermodynamic system such as for a neutrino ensemble in the presence of a background plasma. It is important to note that oscillations merely swap neutrinos from one flavor to another such that P_0 does not evolve in time, unless repopulation effects from some background plasma have to be taken into account as is the case, e.g., in the early Universe. On the other hand P_z parametrizes the asymmetry of the system. It gives the excess of ν_a over ν_b. The latter fact also motivates the way of speaking in which P_x, P_y are called *coherences*, which encode the amount of decoherence in the system. Therefore, the time evolution of P_z is of special interest in most applications.

2.4 CPT violation

Besides the seemingly abstract and formal considerations about the Magnus expansion and the coherence vector, we now delve into a more phenomenologically relevant part. The discussion shall focus on the topic of Lorentz and CPT violation in neutrinos. We shall also briefly comment on how possible CPT violation manifests in charged current weak interactions.

Needless to say that CPT invariance is one of the cornerstones of any relativistic field theory. It can be shown under fairly general assumptions that all Poincaré-invariant field theories are also CPT-invariant [4]. As for the opposite case, theories in which CPT invariance is violated, also violate Poincaré invariance [67]. There are certain consequences of CPT invariance, such as the equality of masses and lifetimes for particles and antiparticles, which have been tested experimentally to a fairly good accuracy [5].

And yet, regardless of how much prejudice goes in the favor of CPT conservation, the question of whether CPT is violated in nature should ultimately be an experimental one. It is conceivable that the ramifications of CPT violation are detectable in observables, which have been measured already, below the current experimental threshold of accuracy. On the other hand, also as yet unseen channels might exist, in which CPT violation could be present at an observable level. Even if one considers such possibilities as remote, there still is a great deal to be learned from pushing the boundaries of our theoretical understanding of physics and its fashions in model building beyond the beaten tracks.

Let us therefore plunge into discussing how violation of Lorentz and CPT invariance affects neutrino oscillation phenomena [6]. In order to account for Lorentz- and CPT-violating interactions, the standard model Lagrangian is augmented with all conceivable terms, which can be constructed with standard model fields and which introduce breaking of the Lorentz and CPT symmetries. Such additional terms in the form of Lorentz- and CPT-violating operators with Lorentz indices are coupled to newly introduced coefficients with Lorentz indices. These coefficients (or rather linear combinations thereof) are observables in a standard model extension with Lorentz and CPT violation. However, the standard model is a rather successful description of most of particle physics at scales well below the Planck scale. Therefore, any signals of Lorentz and CPT violation appearing at low energies must come from an effective quantum field theory, which contains the standard model. Different mechanisms can be held responsible for the generation of low-energy Lorentz- and CPT-violating operators. Here, we only mention a few, since the provenience of such terms is of no further relevance when it comes to their phenomenological implications. In string theory the spontaneous violation of Lorentz or even CPT symmetry presents a generic mechanism [68, 69]; noncommutative field theory [70, 71, 72] and non-string approaches to quantum gravity [73, 74, 75] also feature the breakdown of Lorentz and CPT symmetry.

Focussing on renormalizable Lorentz- and CPT-violating operators and studying their impact on neutrino physics, we can write [6] all possible Dirac and Majorana couplings of freely propagating left- and right-handed neutrinos in a Dirac equation of the form

$$\left(i\Gamma^{\mu}_{ab}\partial_{\mu} - M_{ab}\right)\boldsymbol{\nu}_b = 0. \tag{2.48}$$

In this notation we assemble both the neutrino fields and their CPT conjugates in the symbol $\boldsymbol{\nu} = (\nu_e, \nu_\mu, \nu_\tau, \dots, \bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau, \dots)$ and the subscripts a, b hence label both the flavor of neutrinos as well as antineutrinos. This equation of motion is a generalization of the usual Dirac equation for freely propagating neutrino and antineutrino states. The matrices Γ^{μ}_{ab} and M_{ab} can be expanded in the basis of Dirac matrices as

$$\Gamma^{\mu}_{ab} = \gamma^{\mu}\delta_{ab} + c^{\nu\mu}_{ab}\gamma_{\nu} + d^{\nu\mu}_{ab}\gamma_5\gamma_{\nu} + e^{\mu}_{ab} + if^{\mu}_{ab}\gamma_5 + \frac{1}{2}g^{\mu\alpha\beta}_{ab}\sigma_{\alpha\beta}, \tag{2.49}$$

$$M_{ab} = m_{ab} + im_{5ab}\gamma_5 + a^{\nu}_{ab}\gamma_{\nu} + b^{\nu}_{ab}\gamma_5\gamma_{\nu} + \frac{1}{2}H^{\alpha\beta}_{ab}\sigma_{\alpha\beta}. \tag{2.50}$$

In these expressions all tensors of even rank (m, m_5, c, d, H) are CPT-conserving, whereas all remaining tensors of odd rank are CPT-violating. However, the coefficients c, d, H are Lorentz-violating. Additionally, we impose Hermiticity of the equations (2.49) and (2.50), which translates to the fact that all coefficients are Hermitian in flavor space. Hermiticity is assumed in order to be able to construct meaningful observables from the Hamiltonian. We note that assembling neutrino fields and their CPT conjugates in one symbol generates interdependencies between the Lorentz- and CPT-violating coefficients under charge conjugation. Using the enhanced Dirac equation, we arrive at the following Hamiltonian

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \delta\mathcal{H} \\ &= -\gamma^0\left(i\gamma^k\partial_k - M_0\right) - \frac{1}{2}\left(\gamma^0\delta\Gamma^0\mathcal{H}_0 + \mathcal{H}_0\gamma^0\delta\Gamma^0\right) - \gamma^0\left(i\delta\Gamma^k\partial_k - \delta M\right). \end{aligned} \tag{2.51}$$

In this expression only leading order terms $\delta\mathcal{H}$ in the Lorentz- and CPT-violating coefficients have been kept. This expression therefore constitutes the basis for studying leading-order effects of Lorentz and CPT violation in neutrinos. It can be shown that the mass terms m and m_5 in the Lorentz- and CPT-conserving Hamiltonian $\mathcal{H}_0$ do indeed reproduce the usual Dirac and Majorana masses [6].

With the Hamiltonian $\mathcal{H}$ at our disposal, we can eventually begin deriving the neutrino (antineutrino) oscillation Hamiltonian h_{eff}. To this end, a few more assumptions are convenient in order to simplify the as yet cumbrous expressions. We assume that there are only three propagating neutrino (antineutrino) flavors and that neutrino (antineutrino) masses are generated via a standard seesaw mechanism. The latter assumption hampers the propagation of heavy, sterile neutrino (antineutrino) states. Restricting attention to the light, active (left-handed) states, we can decompose the neutrino (antineutrino) fields into their Fourier modes and use the Hamiltonian $\mathcal{H}$ to obtain the oscillation Hamiltonian h_{eff} in its mass-diagonal basis. The calculations are cumbersome [6] and we shall not reproduce them here. Instead we give the neutrino (antineutrino) oscillation Hamiltonian to first order in the Lorentz- and CPT-violating coefficients.

The neutrino (antineutrino) oscillation Hamiltonian for freely propagating states of three flavors in the presence of Lorentz- and CPT-violating operators of renormalizable dimension is given by

$$\begin{aligned}(h_{\text{eff}})_{ab} &= p\,\delta_{ab}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{2p}\begin{pmatrix}(\tilde{m}^2)_{ab} & 0\\ 0 & (\tilde{m}^2)^*_{ab}\end{pmatrix} \\ &+ \frac{1}{p}\begin{pmatrix}[(a_L)^\mu p_\mu - (c_L)^{\mu\nu}p_\mu p_\nu]_{ab} & -i\sqrt{2}p_\mu(\varepsilon_+)_\nu\,[(g^{\mu\nu\alpha}p_\alpha - H^{\mu\nu})\mathcal{C}]_{ab}\\ +i\sqrt{2}p_\mu(\varepsilon_+)^*_\nu\,[(g^{\mu\nu\alpha}p_\alpha + H^{\mu\nu})\mathcal{C}]^*_{ab} & [-(a_L)^\mu p_\mu - (c_L)^{\mu\nu}p_\mu p_\nu]^*_{ab}\end{pmatrix}.\end{aligned} \tag{2.52}$$

Obviously, a few comments concerning notation are in order. The matrix structure in this equation resembles the neutrino-antineutrino basis, which we adopted in writing down the equations of motion. As a substructure in these matrices, the Lorentz- and CPT-violating coefficients also bear flavor indices a, b. The neutrino three-momentum is denoted by $p = |\vec{p}|$ and shorthand notations for $(c_L)^{\mu\nu}_{ab} = (c+d)^{\mu\nu}_{ab}$ and $(a_L)^{\mu}_{ab} = (a+b)^{\mu}_{ab}$ have been introduced. The neutrino four-momentum may be written as $p_\mu = (p, \vec{p})$ to first order and the Lorentz-conserving mass term stems from the usual seesaw mechanism such that $\tilde{m}^2 = m_l m_l^\dagger$, where m_l is the light-mass matrix. The vectors $(\varepsilon_+)^\mu$ and $(\varepsilon_-)^\mu = (\varepsilon_+)^{\mu *}$ emerge in analogy to the common photon helicity basis. This reflects that the active neutrinos (antineutrinos), being almost massless, have a near-definite helicity. Eventually, $\mathcal{C}$ is the charge conjugation in flavor space. The coefficients a_L and c_L are the leading-order Lorentz- and CPT-violating contributions to neutrino (antineutrino) mixing. These coefficients respect the standard model gauge invariance. The coefficients $g\mathcal{C}$ and $H\mathcal{C}$ violate the gauge invariance of the standard model and induce lepton-number violating neutrino-antineutrino mixing.

A few more comments as to the nature of the oscillation Hamiltonian are in order. It reveals a nonstandard energy dependence. Except for the correction of $1/p$ which multiplies both the mass term as well as the Lorentz- and CPT-violating part, the latter scales as p and p^2; the conventional oscillation term originating from the mass difference in neutrinos (antineutrinos), however, scales as p^0. This altered energy dependence can give rise to resonances in neutrino (antineutrino) vacuum oscillations [27, 29, 76], but it also covers the well-known *Mikheev-Smirnov-Wolfenstein* resonance in matter. Besides the nonstandard neutrino (antineutrino) dispersion relations, the oscillation Hamiltonian also features a direction dependence via the polarization vector $(\varepsilon_+)^\mu$. The appearance of such a term gives rise to direction-dependent oscillation phenomena as well as, e.g., siderial variations in neutrino (antineutrino) oscillation experiments [6, 77, 78]. Moreover, in CPT-violating extensions of the standard model, conversions between active neutrinos and active antineutrinos become possible. The latter violate lepton-number. We remark that even for CPT-conserving oscillations there can be terms in the oscillation Hamiltonian, which violate lepton-number. Moreover, the conservation of CPT results in certain interdependencies between the oscillation probabilities for neutrinos and antineutrinos.

We shall derive said properties in the course of our analysis.

We shall study the ramifications of a Lorentz- and CPT-violating neutrino oscillation Hamiltonian and aspects concerning its solution in greater detail in the upcoming sections. Let us now turn our attention to CPT violation in charged current weak interation processes. To this end, we commence by giving a brief sketch of the CPT theorem [4]. Let us preface these considerations with the observation that violation of the CPT symmetry (such as dropping the assumption of fields governed by a local, Lorentz-invariant Lagrangian or the fact that fields with half-integer spin obey Fermi-Dirac statistics [79]) would lead to a complete reformulation of quantum field theory [80, 81, 82]. It would also include the emergence of novel fields and associated particles.

The upcoming analysis of CPT-violating charged current weak interactions in the standard model takes a different viewpoint. A vector field, such as the photon, is odd under CPT. Fermion fields in the Lagrangian must appear in bilinear combinations of the spinor fields. Studying the CPT properties [83] of such bilinear combinations reveals that those with an even number of Dirac matrices are even under CPT; those with an odd number of Dirac matrices are odd under CPT. Put another way, field operators with an even (odd) number of Lorentz indices are even (odd) under CPT. In order to obtain a Lorentz scalar for the Lagrangian density such combinations of operators have to be contracted with some tensors inherent of spacetime. In the common Minkowski spacetime the only tensors at hand are the metric tensor $\eta_{\mu\nu}$ and the totally antisymmetric tensor $\varepsilon_{\mu\nu\alpha\beta}$. Therefore, the number of indices carried by fields and bilinears must be even. This, in turn, implies that also the Lagrangian must be even under CPT. The situation changes, however, if the assumption is put forward that spacetime is additionally endowed with characteristic tensors of odd rank. If such tensors do exist, they can appear in the Dirac structure of various currents in the standard model, such as the charged current weak interactions, which we shall study. There are two possible approaches to such novel terms. On the one hand, one might assume such terms to be present in the Lagrangian density of the standard model and derive the consequences of such an assumption as has been sketched for the neutrino sector above. On the other hand, one can assume that the free Dirac equation is not altered in the presence of CPT violation, but only some interactions are. In our case, these *preferred* interactions are the charged current weak interactions. This assumption introduces additional terms in the leptonic current of the charged current interactions and is hence responsible for different lifetimes and decay rates for particles and antiparticles as shall be seen in our analysis.

3 Adiabatic and nonadiabatic perturbation theory for coherence vectors

The discussion of the Magnus expansion and the coherence vector in the previous paragraphs now puts us into a position to develop on these grounds a novel adiabatic and nonadiabatic perturbation theory for the coherence vector description of neutrino oscillations [84]. Both the adiabatic and the nonadiabatic regimes can be treated on the same grounds; there is no need for different perturbation theories for these distinct physical situations. As we have discussed in conjunction with the Magnus expansion, it is sensible to change the quantum mechanical picture of the evolution equation before solving them explicitly. We shall use this fact in order to find a suitable expansion parameter for the system under consideration here; we also understand that different changes of the quantum mechanical picture reveal certain quantities (such as an adiabaticity parameter) which are useful to characterize the physics of the system. We shall solve the quantum kinetic equations explicitly for the coherence vector for a collision-free neutrino ensemble with finite temperature and also extend our formalism to early Universe applications.

3.1 Adiabatic perturbation theory

The quantum kinetic equations for a two-flavor ensemble of neutrinos shall be written as

$$\frac{\partial}{\partial t}\vec{P}(t) = S(t)\vec{P}(t), \tag{3.1}$$

$$S(t) = \begin{pmatrix} 0 & -\lambda(t) & 0 \\ \lambda(t) & 0 & -\beta(t) \\ 0 & \beta(t) & 0 \end{pmatrix}, \tag{3.2}$$

for the upcoming analysis. Having mentioned that the matrix norm of the evolution matrix of a system can add to its physical understanding, we calculate the Frobenius norm to be

$$||S||_{\mathrm{F}}^2 = \mathrm{tr}\left(S^\dagger S\right) = 2\omega_{\text{eff}}^2. \tag{3.3}$$

It is straightforward to show that the effective oscillation length of the system is indeed given by

$$\frac{2\pi}{l_{\text{osc}}^{\text{eff}}} \equiv \omega_{\text{eff}} = \sqrt{\lambda^2 + \beta^2} \tag{3.4}$$

and that obviously the norm and the effective oscillation frequency are related or, as a matter of fact, proportional to one another. If it was only for this statement, it might seem academic to calculate the matrix norm of the evolution equation under consideration. However, a comparison between matrix norms in different quantum mechanical pictures can provide even further physical insight into the nature of the neutrino ensemble. We shall see this as we proceed. Note, moreover, that both β and λ are taken to be time-dependent quantities. Two different time dependences for β and λ can be identified for the purpose of our analysis. On the one hand, both quantities scale as p^{-1} in momentum and in early Universe applications the neutrino momenta redshift as the Universe expands and hence introduce a time dependence. On the other hand, λ can also depend on time via the potential term V_α. In an early Universe environment, a time dependence can be converted into a temperature T dependence using a standard paradigm from cosmology, which derives a time-temperature relation using conservation of comoving entropy and the Friedmann equations to obtain

$$\mathrm{d}t(T) = -2\bar{m}_{\mathrm{Pl}}\frac{\mathrm{d}T}{T^3} \tag{3.5}$$

with $\bar{m}_{\mathrm{Pl}} = \sqrt{\frac{90}{32\pi^3 g_*}}m_{\mathrm{Pl}}$, where m_{Pl} is the Planck mass and g_* gives the effective degrees of freedom at the epoch under consideration. Keeping this in mind all time dependences occuring throughout our analysis can easily be converted into temperature dependences, which might be more convenient in, for instance, early Universe applications. We shall sketch how to treat such situations later on.

Furthermore, in order to get a grasp on how the oscillation length can be understood physically, we transform the quantum kinetic equations to a basis, which resembles the commonly encountered mass eigenbasis in the Mikheev-Smirnov-Wolfenstein framework. To this end, it is only sensible to consider a generic time-dependent rotation in the xz-plane by an angle $\Theta(t)$ as the only nonvanishing entries of the effective potential vector are V_x and V_z. We parametrize the time-dependent rotation via

$$\vec{P}(t) = R(t)\vec{Q}(t) \qquad \text{with} \qquad R[\Theta(t)] = \begin{pmatrix} \cos\Theta(t) & 0 & \sin\Theta(t) \\ 0 & 1 & 0 \\ -\sin\Theta(t) & 0 & \cos\Theta(t) \end{pmatrix}, \tag{3.6}$$

where $\vec{Q}$ is the coherence vector in the new *corotating frame* and $R(t)$ is the time-dependent rotation matrix. The quantum kinetic equations in the new basis appear as

$$\begin{aligned} \frac{\partial}{\partial t}\vec{Q}(t) &= S_Q(t)\vec{Q}(t), \\ S_Q(t) &= \begin{pmatrix} 0 & -\lambda\cos\Theta - \beta\sin\Theta & -\frac{\mathrm{d}\Theta}{\mathrm{d}t} \\ \lambda\cos\Theta + \beta\sin\Theta & 0 & -\beta\cos\Theta + \lambda\sin\Theta \\ \frac{\mathrm{d}\Theta}{\mathrm{d}t} & \beta\cos\Theta - \lambda\sin\Theta & 0 \end{pmatrix}. \end{aligned} \tag{3.7}$$

Since we introduced $\Theta(t)$ as a generic time-dependent mixing angle, we may define it according to our needs. It is seen that the $(S_Q)_{23}$ and $(S_Q)_{32}$ elements of the evolution matrix can be eliminated by an appropriate choice of the mixing angle $\Theta(t)$. The advantage of this choice is the geometrical interpretation. In the Q-picture the motion of the coherence vector is confined to the xy-plane, if there was not the additional perturbation by the time derivative of the effective angle, which introduces a nonzero z-component to the problem and forces the motion to exit the xy-plane as the ensemble evolves. The smaller the change of the effective mixing with time, the smaller the urge of the coherence vector to exit the xy-plane. Now, in order to eliminate said entries of the evolution matrix, we fix the effective mixing angle by implicitly defining it via

$$\cos\Theta(t) = \frac{\lambda(t)}{\sqrt{\lambda^2(t)+\beta^2(t)}}, \qquad \sin\Theta(t) = \frac{\beta(t)}{\sqrt{\lambda^2(t)+\beta^2(t)}}. \tag{3.8}$$

Although the transformation to the Q-picture is done on the level of a coherence vector equation rather than a Hamiltonian, the effective mixing angle reveals that mixing becomes maximal ($\Theta = \pi/2$) if the condition

$$\lambda(t_{\text{res}}) = 0 \tag{3.9}$$

is satisfied for the *resonant time* t_{res}. A vanishing $\lambda(t)$, i.e., maximal effective mixing, hence coincides with the existence of a resonance in neutrino conversions, which can also be equivalently rephrased for a resonant temperature T_{res}, depending on the application one has in mind. This behavior closely resembles what is expected for the Hamiltonian treatment of neutrino oscillations. It is, nonetheless, a completely different framework and thus notions such as that for resonant mixing have to be established anew.

We now recast the evolution matrix in the Q-picture as

$$S_Q = \begin{pmatrix} 0 & -\omega_{\text{eff}} & -\frac{\mathrm{d}\Theta}{\mathrm{d}t} \\ \omega_{\text{eff}} & 0 & 0 \\ \frac{\mathrm{d}\Theta}{\mathrm{d}t} & 0 & 0 \end{pmatrix}. \tag{3.10}$$

Consider the matrix norm of this evolution matrix in the new quantum mechanical picture

$$||S_Q||_{\text{F}}^2 = 2\omega_{\text{eff}}^2 \left[1 + \left(\frac{1}{\omega_{\text{eff}}}\frac{\mathrm{d}\Theta}{\mathrm{d}t}\right)^2\right]. \tag{3.11}$$

At first glance the above transformation seems to worsen the convergence properties due to the appearance of the additional

$$\gamma \equiv \frac{1}{\omega_{\text{eff}}}\frac{\mathrm{d}\Theta}{\mathrm{d}t} \tag{3.12}$$

term. However, if this term is sufficiently small, $\gamma \ll 1$, the convergence will only be marginally altered. Moreover, the smallness condition can be understood physically as well. The characteristic time scale of the system under study is $\tau_{\text{sys}} = 1/\omega_{\text{eff}}$, whereas the characteristic time scale of the interaction can be identified as $\tau_{\text{int}} = (\mathrm{d}\Theta/\mathrm{d}t)^{-1}$. Hence, the parameter γ simply compares the characteristic time scale of the system to the characteristic time scale of the interaction. A small γ can be paraphrased as the system's time scale being much smaller than the interaction's time scale. Put another way, the interaction is adiabatic. The parameter γ is thus interpreted and henceforth referred to as the adiabaticity parameter for the system. We adopt here the notion of adiabaticity versus nonadiabaticity of Refs. [85, 86, 87, 88]. The emergence of an adiabaticity parameter from the quantum kinetic equations and by means of defining an effective mixing angle through a change of the quantum mechanical picture, is a novel feature of our analysis and shall come in handy once collision-affected systems are studied later on.

Introducing the adiabaticity parameter into the evolution matrix of the system leaves us with

$$S_Q = \begin{pmatrix} 0 & -\omega_{\text{eff}} & -\gamma\omega_{\text{eff}} \\ \omega_{\text{eff}} & 0 & 0 \\ \gamma\omega_{\text{eff}} & 0 & 0 \end{pmatrix}. \tag{3.13}$$

There is one more minor caveat concerning the adiabaticity parameter, which we have to comment on briefly before moving on with our analysis. It has been noticed that the effective mixing angle as defined above essentially resembles the one known from a Mikheev-Smirnov-Wolfenstein framework for matter-affected neutrino oscillations. Note, however, that the effective mixing angle has a completely different interpretation in the coherence vector description. It is for this reason that the mixing angle as defined here does not feature the common factor of two encountered in the Hamiltonian formulation. Moreover, we understand that defining $1/\gamma$ as the adiabaticity parameter is also quite common in the literature. The physics, however, is not altered by this convention. Also, when comparing our analysis to other work it is important to notice that the adiabaticity parameter lacks a factor of two as well due to our definition of the effective mixing angle. We shall analyze the adiabaticity parameter further in section 3.2.

Note, that the concept of a Hamiltonian, in general, ceases to exist when quantum kinetic equations of thermal neutrino ensembles are considered. It is important to keep in mind this point in order to fully appreciate our paradigm. In the adiabatic limit of taking $\gamma \to 0$ the time-dependent rotation of the coherence vector by an angle $\Theta(t)$ establishes a basis in which the motion of the coherence vector is confined to the xy-plane. Apart from its geometric interpretation, defining a mixing angle and an adiabaticity parameter for the ensemble bears some similarity to the diagonalization of the underlying Hamiltonian for single neutrino states. The benefit of considering the equations of motion in a coherence vector framework and determining effective mixing and hence adiabaticity, however, is that it does not explicitly rely on the form of the time evolution matrix. The adiabaticity parameter for the system under consideration can be found by employing a suitable change of basis for the quantum kinetic equations without making any reference to an underlying Hamiltonian of the system. This point can prove especially useful when collision-affected neutrino conversions are considered; a system in which the Hamiltonian formulation ceases to be applicable as mentioned above.

Furnished with both its geometric interpretation as well as its appearance in the matrix norm of the evolution equation in the Q-picture, γ can serve as a small perturbation parameter in the adiabatic regime of neutrino conversions. It thus feels harmonious to struggle through just another transformation before solving the quantum kinetic equations. The additional change of basis is introduced to isolate the perturbation parameter γ in a convenient way and such that fast convergence of the expansion to come is assured. We work with the following transformation

$$\vec{Q}(t) = U(t)\vec{X}(t), \qquad \text{where} \qquad \frac{\partial}{\partial t}U(t) = S_Q^\omega(t)U(t), \qquad U(t_0) = \mathbf{1}, \tag{3.14}$$

in which we have decomposed the evolution matrix via

$$S_Q = S_Q^\omega + S_Q^\gamma = \begin{pmatrix} 0 & -\omega_{\text{eff}} & 0 \\ \omega_{\text{eff}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\gamma\omega_{\text{eff}} \\ 0 & 0 & 0 \\ \gamma\omega_{\text{eff}} & 0 & 0 \end{pmatrix} \tag{3.15}$$

in self-obvious notation. The subsidiary evolution equation for $U(t)$ is solved and exhibits an oscillatory behavior. We have

$$U(t) = \begin{pmatrix} \cos\tilde{\omega}_{\text{eff}} & -\sin\tilde{\omega}_{\text{eff}} & 0 \\ \sin\tilde{\omega}_{\text{eff}} & \cos\tilde{\omega}_{\text{eff}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \tilde{\omega}_{\text{eff}}(t) = \int_{t_0}^{t} \mathrm{d}\tau\, \omega_{\text{eff}}(\tau). \tag{3.16}$$

The quantum kinetic equations take a new form, which is identified as

$$\begin{aligned} \frac{\partial}{\partial t}\vec{X}(t) &= S_X(t)\vec{X}(t), \\ S_X(t) &= \begin{pmatrix} 0 & 0 & -\gamma\omega_{\text{eff}}\cos\tilde{\omega}_{\text{eff}} \\ 0 & 0 & \gamma\omega_{\text{eff}}\sin\tilde{\omega}_{\text{eff}} \\ \gamma\omega_{\text{eff}}\cos\tilde{\omega}_{\text{eff}} & -\gamma\omega_{\text{eff}}\sin\tilde{\omega}_{\text{eff}} & 0 \end{pmatrix} \end{aligned} \tag{3.17}$$

and calculating the matrix norm yields

$$||S_X||_{\text{F}}^2 = 2\gamma^2\omega_{\text{eff}}^2. \tag{3.18}$$

It is evident now that the small parameter in the adiabatic regime, namely γ, has been isolated and hence good convergence of the sought-after perturbation theory can be expected. This motivates a perturbative expansion in this basis (which is but an interaction picture for the Q-basis).

The considerations in this section deal with two linear transformations $R(t)$, $U(t)$ from the original P-basis to the Q- and X-basis. The reason for those transformations is twofold. On the one hand changing the basis for the quantum kinetic equations discloses the physics of the system we are dealing with and on the other hand it seems advisable to find a basis for the quantum kinetic equations in which an approximate solution gives accurate results. For convenience, we shall now recapitulate the meaning of the transformations introduced so far.

The first transformation ($\vec{P} \stackrel{R}{\to} \vec{Q}$) is inherently physical. It gives a recipe how to establish the concept of a *mass eigenbasis* in the coherence vector description of neutrino oscillations. The effective mixing angle defined in this way differs from the effective mixing angle encountered in the common Mikheev-Smirnov-Wolfenstein formalism by a conventional factor of two. In this quantum mechanical picture a clear path of approaching the resonance in neutrino oscillations is found. A resonant conversion of neutrino flavors is encountered for $\lambda(t) = 0$. Moreover, the transformation to the matter eigenbasis sees the introduction of an effective mixing angle, which is useful for defining the adiabaticity parameter γ subsequently layed open. Adiabatic neutrino conversion occurs for $\gamma \ll 1$, when the time scale of the system is much smaller than the time scale of the interaction. The mathematical benefit of this transformation is that we get a grasp on the convergence properties of the approximation we want to employ and we can give it physical meaning. The convergence properties of the expansion get worse as the amount of adiabaticity violation increases. A fact which later will be useful to construct a nonadiabatic perturbation theory. Also the change to the matter eigenbasis suggests that the adiabaticity parameter γ should be the appropriate small quantity to expand in.

The second transformation ($\vec{Q} \stackrel{U}{\to} \vec{X}$) is convenient from a mathematical point of view. It removes an exactly integrable part of the evolution matrix and thus the matrix norm for the latter is directly proportional to the small expansion parameter γ. This truly renders γ into the sought-after perturbation parameter and the envisaged expansion converges fast. Put another way, already the first approximant should provide a good approximation for the exact solution.

However, one final comment regarding the terminology is in order. The terms *quantum mechanical picture* or rather *change of the quantum mechanical picture* are used to denote a distinct basis for the time evolution matrix of the quantum kinetic equations or a basis change from one basis of the quantum kinetic equations to another, respectively. This is to be understood as a manner of speaking motivated by standard Schrödinger quantum mechanics.

Note, eventually, that no attempt for solving the quantum kinetic equations has been made so far. We have merely changed the quantum mechanical pictures to unfold the underlying physics. The paradigm of our analysis is that a careful treatment of the quantum kinetic equations, i.e., a succession of different changes of the quantum mechanical pictures already allows to extract important information about the system under consideration without explicitly solving the quantum kinetic equations.

In the newly established X-basis the quantum kinetic equations are finally solved to first order in the Magnus expansion by

$$\vec{X}^{(1)}(t) = \exp\left[\int_{t_0}^{t} \mathrm{d}\tau\; S_X(\tau)\right] \vec{X}(t_0). \tag{3.19}$$

The formal solution for the coherence vector $\vec{P}(t)$ to first order in the Magnus expansion is thus obtained as

$$\vec{P}^{(1)}(t) = R(t)U(t)\exp\left[\int_{t_0}^{t} \mathrm{d}\tau\; S_X(\tau)\right]\; R^{-1}(t_0)\vec{P}(t_0). \tag{3.20}$$

In order to streamline notation, we write the terms contained in the matrix exponential as

$$J_s(t) \;=\; \int_{t_0}^{t} \mathrm{d}\tau\; \gamma\omega_{\text{eff}} \sin\tilde{\omega}_{\text{eff}}, \tag{3.21}$$

$$J_c(t) \;=\; \int_{t_0}^{t} \mathrm{d}\tau\; \gamma\omega_{\text{eff}} \cos\tilde{\omega}_{\text{eff}} \tag{3.22}$$

as well as

$$|J| \equiv \sqrt{J_c^2 + J_s^2}. \tag{3.23}$$

The resultant expression for the coherence vector assumes a form

$$\begin{aligned}
\vec{P}^{(1)}(t) = \;& \begin{pmatrix} \cos\Theta(t) & 0 & \sin\Theta(t) \\ 0 & 1 & 0 \\ -\sin\Theta(t) & 0 & \cos\Theta \end{pmatrix} \begin{pmatrix} \cos\tilde{\omega}_{\text{eff}} & -\sin\tilde{\omega}_{\text{eff}} & 0 \\ \sin\tilde{\omega}_{\text{eff}} & \cos\tilde{\omega}_{\text{eff}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \\
& \times \frac{1}{|J|^2} \begin{pmatrix} J_s^2 + J_c^2\cos|J| & -J_cJ_s\left(-1+\cos|J|\right) & -J_c\,|J|^2\,\mathrm{sinc}\,|J| \\ -J_cJ_s\left(-1+\cos|J|\right) & J_c^2 + J_s^2\cos|J| & J_s\,|J|^2\,\mathrm{sinc}\,|J| \\ J_c\,|J|^2\,\mathrm{sinc}\,|J| & -J_s\,|J|^2\,\mathrm{sinc}\,|J| & |J|^2\cos|J| \end{pmatrix} \times \\
& \times \begin{pmatrix} \cos\Theta_0 & 0 & -\sin\Theta_0 \\ 0 & 1 & 0 \\ \sin\Theta_0 & 0 & \cos\Theta_0 \end{pmatrix} \vec{P}(t_0).
\end{aligned} \tag{3.24}$$

Additionally $\Theta(t_0) \equiv \Theta_0$ and sinc$x \equiv \frac{\sin x}{x}$ have been defined. This expression looks cumbersome at a first glance. It does, however, present an analytic, yet perturbative, solution to the quantum kinetic equations as given in Eqs. (3.1 – 3.2) for a generic potential, i.e., for a generic time (or equivalently temperature in early Universe applications) dependence of both β and λ, as long as the transition can be considered adiabatic ($\gamma \ll 1$). In the one-particle interpretation oscillation probabilities can be extracted from this formal solution and oscillating contributions to this very probability can be studied since there is no inherent averaging over rapidly oscillating contributions as is usually considered in the derivation of the oscillation probability in the Mikheev-Smirnov-Wolfenstein framework. Still, to fully appreciate this result a thorough discussion of various limiting cases, such as the adiabatic limit, is called for [85]. We postpone this endeavor until section 3.3.

3.2 Nonadiabatic perturbation theory

We proceed by developing a nonadiabatic perturbation theory. It shall be seen that such a perturbation theory for the nonadiabatic regime of neutrino conversions can be formulated on the same grounds as the foregoing adiabatic formalism. To this end, it is instructive to further elaborate on the adiabaticity parameter as defined in Eq. (3.12) and write it explicitly as a function of the parameters β and λ. Such rephrasing of the notion of adiabaticity allows for a better understanding of the underlying physics. It is straightforward to express γ in terms of β and λ. We find

$$\gamma(t) = \frac{\beta\lambda}{\omega_{\text{eff}}^3} \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\beta}{\lambda}. \tag{3.25}$$

From this relation also the adiabaticity parameter at the neutrino conversion resonance ($\lambda = 0$) can be obtained. It is this quantity, which is of foremost interest in physical applications and it is calculated to be

$$\gamma_{\text{res}} \equiv \gamma(t_{\text{res}}) = -\frac{1}{\beta^2(t_{\text{res}})} \left.\frac{\mathrm{d}\lambda}{\mathrm{d}t}\right|_{t=t_{\text{res}}}. \tag{3.26}$$

Two important pieces of information can be gathered. The adiabaticity of the system is determined by the time variation of the matter profile $\mathrm{d}\lambda/\mathrm{d}t$ (or the *shape* of the matter potential, if it is interpreted as a function of the experimental baseline). This is expected from the Mikheev-Smirnov-Wolfenstein framework and recovered in our treatment for neutrino ensembles. Large variations of the matter profile with time clearly drive the system towards nonadiabaticity. Besides this contribution, also the term $1/\beta^2$ is familiar. It states that a small β at the neutrino conversion resonance is incompatible with an adiabatic perturbation expansion to some extent. This observation, however, does not hold true when repopulation of the neutrino ensemble from a background plasma is considered as we shall see shortly. In this case, obviously, the application of adiabatic and nonadiabatic perturbation theory as developed here runs into difficulties and a thorough treatment of such cases is called for. However, for the purposes of this section, a small β indicates nonadiabatic transitions and we shall also refer to this regime as the *sudden regime* henceforth.

It is then obvious that β itself can be adopted as a small perturbation parameter in a nonadiabatic perturbation expansion. Recalling the quantum kinetic equations according to Eqs. (3.1 – 3.2), we split the evolution matrix S into two different submatrices thereby isolating the parameter in which we seek a perturbative solution to the quantum kinetic equations. The evolution matrix reads

$$S(t) = S_\lambda(t) + S_\beta(t) = \begin{pmatrix} 0 & -\lambda(t) & 0 \\ \lambda(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\beta(t) \\ 0 & \beta(t) & 0 \end{pmatrix} \tag{3.27}$$

in obvious notation. The S_λ subsystem of this evolution equation can be integrated exactly, which again gives rise to an oscillatory behavior – this time in the time-integrated coefficient λ. Integrating out the exactly solvable part can be achieved by changing the quantum mechanical picture for the evolution equation by means of the following transformation

$$\vec{P}(t) = V(t)\vec{Y}(t) \qquad \text{with} \qquad \frac{\partial}{\partial t}V(t) = S_\lambda(t)V(t). \tag{3.28}$$

Solving the subsidiary evolution equation gives

$$V(t) = \begin{pmatrix} \cos\tilde{\lambda} & -\sin\tilde{\lambda} & 0 \\ \sin\tilde{\lambda} & \cos\tilde{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \tilde{\lambda}(t) = \int_{t_0}^{t} \mathrm{d}\tau\, \lambda(\tau), \tag{3.29}$$

which rephrases the quantum kinetic equations as

$$\begin{aligned} \frac{\partial}{\partial t}\vec{Y}(t) &= S_Y(t)\vec{Y}(t), \qquad (3.30) \\ S_Y(t) &= \begin{pmatrix} 0 & 0 & -\beta\sin\tilde{\lambda} \\ 0 & 0 & -\beta\cos\tilde{\lambda} \\ \beta\sin\tilde{\lambda} & \beta\cos\tilde{\lambda} & 0 \end{pmatrix}. \end{aligned}$$

The new quantum mechanical picture is just the interaction picture and calculation of the matrix norm reveals isolation of the small perturbation parameter

$$||S_Y||_{\mathrm{F}}^2 = 2\beta^2. \tag{3.31}$$

Since the smallness parameter has been isolated by one change of basis only, we can start solving the quantum kinetic equations for $\vec{Y}(t)$. To first order, the Magnus expansion gives the solution for the coherence vector as

$$\vec{P}^{(1)}(t) = V(t)\,\exp\left[\int_{t_0}^{t} \mathrm{d}\tau\; S_Y(\tau)\right]\,\vec{P}(t_0) \tag{3.32}$$

and it is thus sensible to define time-integrated quantities in analogy to the adiabatic case and in order to streamline notation

$$K_s(t) = \int_{t_0}^{t} \mathrm{d}\tau\; \beta\sin\tilde{\lambda}, \tag{3.33}$$

$$K_c(t) = \int_{t_0}^{t} \mathrm{d}\tau\; \beta\cos\tilde{\lambda}. \tag{3.34}$$

Also the notation

$$|K| \equiv \sqrt{K_c^2 + K_s^2} \tag{3.35}$$

comes in handy as it mimics the notation introduced above for the adiabatic perturbation theory. Making use of said notations, the coherence vector to first order in the Magnus expansion is calculated to be

$$\vec{P}^{(1)}(t) = \begin{pmatrix} \cos\tilde{\lambda} & -\sin\tilde{\lambda} & 0 \\ \sin\tilde{\lambda} & \cos\tilde{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \times \frac{1}{|K|^2} \begin{pmatrix} K_c^2 + K_s^2 \cos|K| & K_c K_s \left(-1+\cos|K|\right) & -K_s |K|^2 \operatorname{sinc}|K| \\ K_c K_s \left(-1+\cos|K|\right) & K_s^2 + K_c^2 \cos|K| & -K_c |K|^2 \operatorname{sinc}|K| \\ K_s |K|^2 \operatorname{sinc}|K| & K_c |K|^2 \operatorname{sinc}|K| & |K|^2 \cos|K| \end{pmatrix} \vec{P}(t_0). \tag{3.36}$$

This is the analytic perturbative solution to the quantum kinetic equations in Eqs. (3.1 – 3.2) with generic potential and time dependence for β and λ as long as the evolution can be considered nonadiabatic. This is equivalent to saying that β is a small quantity to expand in. Again, this result can only be fully appreciated once the associated limiting cases are recovered. We shall see to this in the next section [86, 87, 88].

3.3 Perturbation theory ingredients and limiting cases

We understand that our method for obtaining adiabatic and nonadiabatic solutions to the quantum kinetic equations in Eqs. (3.1 – 3.2) for a two-flavor neutrino ensemble is a generic one: it is model-independent in so far as it does not make any explicit reference to the physics scenarios it can be applied to. For this reason, it is also apparent that the main ingredients of the perturbation theory, namely the integrals $J_{c/s}(t)$, $K_{c/s}(t)$, have to be evaluated in each application separately. Both integrals do depend on time via the time dependence of $\lambda(t)$ and $\beta(t)$. General statements about these integrals are obscured by the model-independent formulation of the perturbation theory, but not entirely impossible. This is due to the fact that neutrino conversions reveal a resonance at $\lambda = 0$.

In any case, however, it is still necessary to demonstrate that the Magnus expansion does give exact results in the various physical limits. We shall do so by evaluating the different ingredients of the perturbation theory for certain physics scenarios.

3.3.1 The integrals $J_{c/s}(t)$ and $K_{c/s}(t)$

The integrals $J_{c/s}(t)$ and $K_{c/s}(t)$, for the adiabatic and nonadiabatic case, respectively, have in common certain characteristic features. For the adiabatic case the integrand in $J_{c/s}(t)$ is proportional to the expansion parameter $\gamma\omega_{\text{eff}}$ safe for an oscillatory function. This also holds true for the nonadiabatic case in which $K_{c/s}(t)$ is proportional to β. So at a first glance both integrals seem to be similar. On second thought, however, there is a crucial difference encoded in the oscillatory term in $K_{c/s}$. Because of the neutrino conversion resonance, it has a stationary phase ($\mathrm{d}\tilde{\lambda}/\mathrm{d}t = 0$ at resonance), whereas $J_{c/s}$ does not.

Let us therefore evaluate $K_{c/s}$ by means of the stationary phase method. Suppose the two main assumptions needed to apply the stationary phase method are realized in the physics scenario under consideration. This assumption does, of course, rely on the nature of the ensemble under investigation and hence assumes an explicit underlying model. However, we shall assume that the oscillatory behavior of the integrand is rapid enough to suppress all large contributions to the integral, which might come from $\beta(t)$. If this is so, the latter can simply be evaluated at resonance. There is yet another requirement, which has to be met so that the stationary phase method can be applied. The resonance has to happen in a *small* region around t_{res}. Put another way, the smallness of the aforementioned region is determined by whether the substitution $t_0 \to -\infty$ and $t \to \infty$ is justified in this region or not. For the sake of discussion, we shall suppose that these two requirements are met. We can then evaluate the integrals in question to find

$$|K_s| = 0, \tag{3.37}$$

$$|K_c| \simeq \sqrt{\frac{2\pi}{\gamma_{\text{res}}}}. \tag{3.38}$$

Using these results also allows determining their linear combination termed $|K|$. It obviously obeys

$$|K| \simeq \sqrt{\frac{2\pi}{\gamma_{\text{res}}}}. \tag{3.39}$$

This result is reassuring, since it gives the main ingredient of the nonadiabatic perturbation theory as the reciprocal value of the adiabaticity parameter at resonance. The latter expression is a small quantity in the sudden regime.

Let us turn to the integrals $J_{c/s}$ now. The first step that comes to mind here is integration by parts. We get

$$J_s(t) = -\gamma \cos\tilde{\omega}_{\text{eff}}\Big|_{t_0}^{t} + \int_{t_0}^{t} \mathrm{d}\tau\, \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} \cos\tilde{\omega}_{\text{eff}}, \tag{3.40}$$

$$J_c(t) = \gamma \sin\tilde{\omega}_{\text{eff}}\Big|_{t_0}^{t} - \int_{t_0}^{t} \mathrm{d}\tau\, \frac{\mathrm{d}\gamma}{\mathrm{d}\tau} \sin\tilde{\omega}_{\text{eff}}. \tag{3.41}$$

If the variation of γ in the interval $[t_0, t]$ is sufficiently mild, the main contribution to the integrals is expected to come from the first term on the right hand side. This signifies that also the integrals $J_{c/s}$ are small in an appropriate sense, since in an adiabatic regime γ is a small quantity (and the additional oscillatory part is bounded by one).

Given these arguments the integrals $K_{c/s}$, $J_{c/s}$ reveal a common trademark. Both integrals turn out to be small. In the nonadiabatic case $K_{c/s}$ is proportional to the inverse of γ_{res}, which, in a nonadiabatic perturbation theory is a large quantity. Likewise, in the adiabatic perturbation theory γ is the small quantity to expand in and again the integrals $J_{c/s}$ turn out to be proportional to γ. This first check for consistency obviously works out fine.

3.3.2 Limiting cases

Yet another check for consistency of the perturbation theory is whether the appropriate limits are recovered. Let us see to studying this now and begin with the adiabatic case.

1. *Adiabatic perturbation theory: The vacuum limit*
   This limit to Eq. (3.24) is probably the most intuitive one. We confine ourselves to the one-particle interpretation and understand that for an exactly solvable system the first-order Magnus term should already give the exact result. This implies $\vec{P}^{(1)}(t) \equiv \vec{P}(t)$. Let us examine how this works out here. Firstly, we discard the potential term V_α. This does away with the potential term in λ and leaves us at $\lambda \to -\frac{\Delta m^2}{2p}\cos 2\theta_0$. The quantity β is left unchanged by this modification. It still reads $\beta = \frac{\Delta m^2}{2p}\sin 2\theta_0$. Hence, for consistency, we must take the adiabatic limit of $\gamma \to 0$. It immediately implies $J_{c/s} \to 0$. Moreover, we can define the common oscillation frequency $\omega = \frac{\Delta m^2}{2p}$ and set $t_0 = 0$ (as no resonance time exists, the choice of t_0 is arbitrary). This means $\tilde{\omega}_{\text{eff}} \to \omega t$. Finally, it follows directly from the definition of the effective mixing angles that $\Theta(t) \to 2\theta_0$ for all times. All this reduces the coherence vector to

$$\vec{P}^{(1)}(t) = \begin{pmatrix} s^2 + c^2\cos\omega t & c\sin\omega t & 2sc\sin^2\frac{\omega t}{2} \\ c\sin\omega t & \cos\omega t & -s\sin\omega t \\ 2sc\sin^2\frac{\omega t}{2} & -s\sin\omega t & c^2 + s^2\cos\omega t \end{pmatrix} \vec{P}(t_0), \tag{3.42}$$

   where we have used $c \equiv \cos 2\theta_0$ and $s \equiv \sin 2\theta_0$. Suppose we start with a ν_a flavor such that $\vec{P}(t_0) = (0,0,1)$. The probability to find the neutrino in the same/the other state after time t is then using Eqs. (2.40 – 2.41) as well as Eq. (3.42) given by

$$\text{prob}(\nu_a \to \nu_a) = 1 - \sin^2 2\theta_0 \sin^2\frac{\omega t}{2}, \tag{3.43}$$

$$\text{prob}(\nu_a \to \nu_b) = \sin^2 2\theta_0 \sin^2\frac{\omega t}{2}. \tag{3.44}$$

   This is just the common probability for neutrino oscillations in vacuo. Note, moreover, that this result was obtained solely by using the truncated Magnus expansion as given above and that it accounts for probability conservation. Put another way, unitarity is guaranteed by means of the expansion itself and does not have to be imposed by hand.

We next turn our attention to the nonadiabatic case, which also has some interesting limiting cases. Two interesting limits can be identified.

1. *Nonadiabatic perturbation theory: The sudden limit*
   The sudden limit of taking $\beta \to 0$ renders the quantum kinetic equations (3.1 – 3.2) into formally exactly solvable differential equations such that the Magnus expansion should give an exact result. Let us see how this works out here. The limit $\beta \to 0$ enforces $\omega_{\text{eff}} = |\lambda|$. Therefore, we must have $\cos\Theta = 1$, $\sin\Theta = 0$, which in

turn implies $\gamma \to 0$. The coherence vector assumes a form

$$\vec{P}(t) = \begin{pmatrix} \cos\tilde{\lambda} & -\sin\tilde{\lambda} & 0 \\ \sin\tilde{\lambda} & \cos\tilde{\lambda} & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{P}(t_0). \tag{3.45}$$

The coherences of the ensemble are oscillating as a function of time (the ensemble is incoherent) and the flavor is frozen to its initial value. Put another way, in physical situations in which the evolution of the ensemble happens in a way that with increasing time also β increases, the unfreezing of the ensemble can be studied using nonadiabatic perturbation theory since it treats β as a small perturbation. We shall point out in section 3.3.3 that this scenario of unfreezing is typically of interest in early Universe applications. There is, however, a twist when it comes to early Universe applications. In such systems collisions become a dominant source of decoherence at high temperatures and thus the notion of adiabaticity is expected to get modified. Put another way, a small β in early Universe environments augmented by the presence of decohering collisions might as well allow for an adiabatic perturbation theory (see section 3.3.3 for some more details).

2. *Nonadiabatic perturbation theory: The slab model limit*
Let us consider a one-particle application also for nonadiabatic situations. To this end, suppose that we start the evolution of the neutrino ensemble from a purely ν_a state. This means $\vec{P}(t_0) = (0, 0, 1)$. We then obtain for the coherence vector

$$\vec{P}^{(1)}(t) = \begin{pmatrix} (-K_s \cos\tilde{\lambda} + K_c \sin\tilde{\lambda}) \text{ sinc } |K| \\ -(K_c \cos\tilde{\lambda} + K_s \sin\tilde{\lambda}) \text{ sinc } |K| \\ \cos|K| \end{pmatrix}, \tag{3.46}$$

and the flavor oscillation probability is written as

$$\text{prob}(\nu_a \to \nu_a) = \cos^2 \frac{1}{2} |K| \,. \tag{3.47}$$

Let us reshape this result further in order to render it into a well-known result in neutrino oscillations in the nonadiabatic regime. To do so, suppose the situation as described above to estimate the K-type integrals holds, i.e., the resonance in neutrino conversions happens in a narrow time interval centered around t_{res}. Applying the stationary phase approximation, we then obtain

$$\text{prob}(\nu_a \to \nu_a) = \cos^2 \sqrt{\frac{\pi}{2} \frac{1}{\gamma_{\text{res}}}}. \tag{3.48}$$

This result is the oscillation probability for the slab model as outlined in Ref. [89]. The slab model was originally conceived in order to account for neutrino flavor conversions in the sun via the Mikheev-Smirnov-Wolfenstein effect. It assumes a small mixing angle in vacuo and that the neutrino conversion occurs within a thin region (the *slab*) around the resonant matter density region. It is found that this model describes the solar neutrino data reasonably well within the region of its validity which is just given by the nonadiabatic regime. So the relevant limit is respected in this situation.

3.3.3 Applications and higher order corrections

It is an easy task now to extend the Magnus expansion to higher orders by means of summing the associated approximants according to $\Omega(t) = \Omega_1(t) + \Omega_2(t) + \ldots$. In order to a grasp on how this prescription unfolds, we take a look at the second-order Magnus approximant and briefly comment on its shape. We find

$$\Omega_2(t) = \begin{pmatrix} 0 & \mathcal{J}(t) & 0 \\ -\mathcal{J}(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.49}$$

where $\mathcal{J}(t)$ is given by

$$\mathcal{J}_{\text{ad}}(t) = \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2\ \gamma(t_1)\gamma(t_2)\omega_{\text{eff}}(t_1)\omega_{\text{eff}}(t_2)\sin\left[\tilde{\omega}_{\text{eff}}(t_2) - \tilde{\omega}_{\text{eff}}(t_1)\right] \tag{3.50}$$

for the adiabatic case and

$$\mathcal{J}_{\text{nad}}(t) = \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2\ \beta(t_1)\beta(t_2)\sin\left[\tilde{\lambda}(t_2) - \tilde{\lambda}(t_1)\right] \tag{3.51}$$

for the nonadiabatic case, respectively. The calculations are performed in the X-picture for adiabatic transitions and in the Y-picture for nonadiabatic transitions. Two things are easily inferred. The second-order approximant is indeed $\mathcal{O}(\gamma^2)$ and $\mathcal{O}(\beta^2)$ for adiabatic and nonadiabatic corrections, respectively, as is expected. Moreover, it is seen that the second order populates the (23) and (32) entries of the Magnus operator Ω.

We shall now briefly elaborate on the complications which arise when ensemble decoherence is to be taken into account. This typically happens in early Universe applications in which the time evolution of neutrinos is governed by three distinct physical processes: firstly, the expansion of the Universe; secondly, coherent oscillations governed by a matter-dependent effective Hamiltonian, which results from coherent forward scattering processes of neutrinos off the background particles; thirdly, scattering processes with the background plasma of elementary particles. These collisions, or non-forward scattering processes, with particles from the background medium typically introduce decoherence effects into the neutrino ensemble. In our analysis we have neglected the ensemble decoherence due to non-forward scattering.

The epoch of foremost interest in studying neutrino oscillations in the early Universe is the one between muon decoupling at $T \sim m_\mu \sim 100$ MeV and neutrino decoupling, i.e., prior to big bang nucleosynthesis, at about $T \sim 1$ MeV, since during this time the initial conditions for nucleosynthesis, the electron neutrino abundance, are set, which then directly influence the neutron-to-proton ratio at the onset of big bang nucleosynthesis via β processes $p + e^- \rightleftharpoons n + \nu_e$. The primordial plasma during this epoch thus consists of electrons, positrons, neutrinos and antineutrinos [60, 61, 62]. Note, however, that an additional restriction in the derivation of the quantum kinetic equations emerges. If a treatment using the S-matrix is employed, it is understood that the initial and final scattering states evolve as *free states*. This is only the case if the plasma is sufficiently dilute and the quanta do not spend most of their time interacting. For the case of a weakly interacting gas of relativistic particles a rough estimate yields that the approach using the S-matrix should be valid for temperatures $T \ll 100$ GeV [59]. The temperature regime under consideration here has $T < 100$ MeV, so this requirement is fulfilled and the formalism we are dealing with is applicable.

The density matrix ρ for the system of interacting and oscillating neutrinos encodes ratios of number density distributions and hence the expansion of the Universe does not directly contribute to the time evolution of the density matrix. However, the momenta of the particles are redshifted and the equilibrium number distributions $N^{\mathrm{EQ}}(p, 0)$ thus depend on time through this redshifting.

An interesting application of the approach developed here exists in scenarios as discussed in, e.g., Refs. [65, 90, 91, 92], namely active-sterile flavor oscillations. The latter are interesting since active-sterile oscillations would populate the additional sterile species and thus contribute significant additional energy density, which in turn would trigger an accelerated expansion of the Universe and hence lead to a higher weak freeze-out temperature [93, 2, 94]. This again would result in a higher neutron abundance and therefore a higher abundance of $^4\mathrm{He}$. However, the correct prediction of the primordial Helium abundance is one of the cornerstones of big bang nucleosynthesis and therefore can constrain such models as discussed here.

The coherent part of matter-affected active-sterile oscillations splits into two contributions. One is just the leading order density-dependent contribution (the Mikheev-Smirnov-Wolfenstein [54, 55, 56] part). This part is only temperature-dependent indirectly via the cosmological redshifting of fermion number density. The second contribution comes from leading order finite temperature gauge boson effects, which cannot be neglected at the temperatures considered here.

The loss of coherence is due to neutrino collisions with the background medium. The decoherence (or damping) function for this process in thermal equilibrium turns out to be proportional to the total collision rate for the neutrino with momentum p under consideration.

The epoch of interest can now be decomposed into three distinct domains. At high temperatures finite temperature gauge boson effects [7] dominate. Repopulation effects from the background plasma can be neglected since at high temperatures thermal equilibrium for all relevant species is rapidly established. At intermediate temperatures lepton-number production starts and the forward scattering contribution comes into play as a small perturbation. Finally, prior to the onset of big bang nucleosynthesis, at low temperatures, collisional effects and finite temperature gauge boson contributions cease to be important and coherent neutrino oscillations are the dominant process.

In each of the aforementioned temperature domains the quantum kinetic equations should be solved to determine the evolution of the neutrino ensemble. It is obvious that decoherence and repopulation effects modify the underlying quantum kinetic equations and complicate their analysis by introducing new physical scales into the system.

While the early Universe framework in principle deserves a more careful treatment, we do, however, understand that our paradigms for obtaining the oscillation frequency, the effective mixing as well as the adiabaticity parameter of the system do not necessitate a full solution for the coherence vector. Hence, we take a first glance at such collision-affected neutrino conversions using our formalism.

The time evolution matrix for a collision-affected two-flavor active-sterile neutrino ensemble in the early Universe is given by

$$S(t) = \begin{pmatrix} -D(t) & -\lambda(t) & 0 \\ \lambda(t) & -D(t) & -\beta(t) \\ 0 & \beta(t) & 0 \end{pmatrix} \tag{3.52}$$

in the framework described above. The potential term $\lambda(t)$ now contains the Mikheev-Smirnov-Wolfenstein potential from coherent elastic forward scattering of neutrinos on the background plasma as well as a finite temperature W-boson contribution. Moreover, $D(t)$ gives the decoherence (damping) function, which is proportional to the total collision rate for the active neutrino flavor. Applying our paradigm to this system, we find for the effective oscillation frequency

$$\omega_{\text{eff}} \equiv \sqrt{\lambda^2 + \beta^2 + D^2}, \tag{3.53}$$

which reduces to the well-known expression in the collision-less limit of taking $D \to 0$. Moreover, we demand that effective mixing still have the property of being maximal at the resonance ($\lambda(t_{\text{res}}) = 0$) and give way to the standard coherent oscillation results as discussed in our analysis above. We thus find

$$\cos\Theta(t) = \frac{\lambda(t)}{\sqrt{\lambda^2(t) + \beta^2(t) + D^2(t)}}, \tag{3.54}$$

$$\sin\Theta(t) = \frac{\sqrt{\beta^2(t) + D^2(t)}}{\sqrt{\lambda^2(t) + \beta^2(t) + D^2(t)}}. \tag{3.55}$$

Since now we have an effective mixing angle at our disposal, we can identify an adiabaticity parameter via

$$\gamma \equiv \frac{1}{\omega_{\text{eff}}} \frac{\mathrm{d}\Theta}{\mathrm{d}t}, \tag{3.56}$$

which can be calculated to yield

$$\gamma = \frac{1}{\omega_{\text{eff}}^3} \left[\frac{\beta\lambda}{\sqrt{\beta^2 + D^2}} \frac{\mathrm{d}\beta}{\mathrm{d}t} + \frac{D\lambda}{\sqrt{\beta^2 + D^2}} \frac{\mathrm{d}D}{\mathrm{d}t} - \sqrt{\beta^2 + D^2} \frac{\mathrm{d}\lambda}{\mathrm{d}t} \right] \tag{3.57}$$

or rather at the resonance

$$\gamma_{\text{res}} = -\frac{1}{\beta^2(t_{\text{res}}) + D^2(t_{\text{res}})} \left. \frac{\mathrm{d}\lambda}{\mathrm{d}t} \right|_{t=t_{\text{res}}}. \tag{3.58}$$

Two things are observed. Firstly, this result reduces to the standard paradigm of an adiabaticity parameter in the collision-unaffected regime ($D \to 0$) and secondly, it entails an intriguing modification of adiabaticity in the presence of collisions. The latter means that a small β does not necessarily coincide with nonadiabatic neutrino conversions any more.

As has been mentioned before, an explicit solution for the coherence vector in a collision-affected regime demands good care and is beyond the scope of this work.

4 Lepton-number violating effects in neutrino oscillations

In the following considerations we study a system of an arbitrary number of neutrino flavors and introduce lepton-number violating terms into the oscillation Hamiltonian. Note, in this context, that there is a crucial difference between lepton-number violation in the Lagrangian of a model (e.g., in Majorana mass terms) and lepton-number violation in the oscillation Hamiltonian for neutrinos. Lepton-number violation in either of them does not necessarily result in lepton-number violation in the other. We also comment on the CPT properties of the underlying Hamiltonian and derive explicit expressions for the various oscillation probabilities. Thereafter, we exemplify how lepton-number violation manifests in one- and two-flavor oscillations and add further insight as to the nature of the developed perturbation theory [95].

4.1 Perturbation theory

Inspired by the structure of the Lorentz- and CPT-violating neutrino oscillation Hamiltonian in section 2.4, we write the most general Schrödinger equation for oscillations involving both active neutrino and active antineutrino states in flavor space. The left-chiral neutrino fields are collectively denoted by $\boldsymbol{\nu} = (\nu_e, \nu_\mu, \nu_\tau, \dots)$; the abbreviatory notation for right-chiral antineutrino fields is $\bar{\boldsymbol{\nu}} = (\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau, \dots)$. With this notation, we have for the Schrödinger equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{\nu}(t)\\ \bar{\boldsymbol{\nu}}(t)\end{pmatrix} = -i\mathbb{H}(t)\begin{pmatrix}\boldsymbol{\nu}(t)\\ \bar{\boldsymbol{\nu}}(t)\end{pmatrix}, \tag{4.1}$$

where $\mathbb{H}$ is the Hamiltonian. Suppose a seesaw mechanism is held responsible for neutrino mass generation. In this case the energy of the neutrino beams as encountered in, e.g., Earth-bound oscillation experiments is too small to excite the heavy sterile neutrino states and thus those states dynamically decouple and do not participate in neutrino flavor oscillations. Given the form of the equations of motion in flavor space, it is convenient to split the Hamiltonian into block-diagonal and block off-diagonal parts. We hence write the Hamiltonian as

$$\begin{aligned}\mathbb{H}(t) &= \begin{pmatrix}H(t) & 0\\ 0 & \hat{H}(t)\end{pmatrix} + \begin{pmatrix}0 & B(t)\\ B^\dagger & 0\end{pmatrix}\\ &= \mathbb{H}_0(t) + \delta\mathbb{H}(t).\end{aligned} \tag{4.2}$$

The states in the vector $\boldsymbol{\nu}$ are assigned lepton-number equal to $+1$, whereas the associated antineutrino fields $\bar{\boldsymbol{\nu}}$ have opposite lepton-number -1. Therefore, our notation separates the lepton-number conserving part $\mathbb{H}_0$ of the Hamiltonian from the lepton-number violating part $\delta\mathbb{H}$. Moreover, we assume that all neutrino states are stable and do not decay. This implies that the Hamiltonian $\mathbb{H}$ is Hermitian. Hermiticity of the Hamiltonian $\mathbb{H}$ is then also handed down to its block-diagonal entries H and $\hat{H}$, which are also both Hermitian. No further information on B can be gained from Hermiticity, though. Moreover, H and $\hat{H}$ are generically different since neutrino and antineutrino states can experience, e.g., CP-nonconserving interactions as is the case, for instance, in elastic forward scattering of neutrinos off background particles [54].

Further constraints on the elements of the Hamiltonian $\mathbb{H}$ can be obtained by supposing that CPT be an exact symmetry of the system. Keeping in mind that the active neutrino states are but the CPT transforms of the active antineutrino states and denoting the CPT operator as Θ, we can study matrix elements of a given operator $\mathcal{O}(t)$. Let $|a\rangle$ and $|b\rangle$ be arbitrary state vectors, then

$$\langle\Theta a|\,\mathcal{O}(t)\,|\Theta b\rangle = \langle b|\,\mathcal{O}^\dagger(t)\,|a\rangle = \langle a|\,\mathcal{O}(t)\,|b\rangle^* \tag{4.3}$$

under the action of CPT and writing $\Theta\,|b\rangle$ for $|\Theta b\rangle$. We are concerned with active neutral fermions in our studies and hence we can choose the phases as well as the ordering of the states such that

$$\Theta\,|\nu_a\rangle = |\bar{\nu}_a\rangle\,. \tag{4.4}$$

Let us now apply these relations to the Hamiltonian $\mathbb{H}$. Making use of Hermiticity we find

$$\langle\nu_a|\,\mathbb{H}(t)\,|\nu_b\rangle = \langle\Theta\nu_b|\,\mathbb{H}(t)\,|\Theta\nu_a\rangle = \langle\bar{\nu}_b|\,\mathbb{H}(t)\,|\bar{\nu}_a\rangle\,. \tag{4.5}$$

Making use of the block-diagonal structure of the Hamiltonian introduced in Eq. (4.2), we understand that CPT imposes additional restrictions on the block entries according to

$$\hat{H}(t) = H^\top(t), \tag{4.6}$$

thence relating neutrinos and antineutrinos in the lepton-number conserving part of the Hamiltonian. Similar steps yield a similar restriction for the block off-diagonal lepton-number violating terms

$$B(t) = B^\top(t), \tag{4.7}$$

i.e., the block $B(t)$ is symmetric. It is interesting to note that conservation of CPT does not imply that there cannot be lepton-number violating terms in the oscillation Hamiltonian. Even if CPT is an exact symmetry of the Hamiltonian, lepton-number violating neutrino oscillations are still possible [96, 97, 98], but obviously have to have a different origin than CPT violation. In the one-generation case, we shall see that the mere existence of lepton-number violation is not enough for neutrino-antineutrino oscillations to develop; one additionally needs CP or even CPT violation in neutrinos and antineutrinos. We shall discuss this issue in due course. Of course lepton-number violating neutrino-antineutrino oscillations can occur in scenarios with CPT violation, in which case the identities above need not hold.

Different origins for genuine lepton-number violation are conceivable. In the upcoming analysis we predominantly have in mind Lorentz- and CPT-violating extensions of the standard model as the underlying physics as described above and in, e.g., Ref. [6]. We shall, however, stick to a model-independent parametrization of lepton-number violating effects and analyze different genuine ramifications thereof, rather than delving into specific physics models.

Instead of dealing with the Schrödinger equation for state vectors, we find it to be more convenient to introduce the time evolution operator $\mathbb{U}(t, t_0)$ for the system according to

$$\begin{pmatrix} \boldsymbol{\nu}(t) \\ \bar{\boldsymbol{\nu}}(t) \end{pmatrix} = \mathbb{U}(t, t_0) \begin{pmatrix} \boldsymbol{\nu}(t_0) \\ \bar{\boldsymbol{\nu}}(t_0) \end{pmatrix}. \tag{4.8}$$

The equations of motion in flavor space take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{U}(t, t_0) = -i\mathbb{H}(t)\mathbb{U}(t, t_0). \tag{4.9}$$

Henceforth, we shall often take $t_0 = 0$ and omit it in the notation. The CPT properties of the time evolution operator are worth looking at since they permit further insight into the CPT properties of neutrino oscillation probabilities. The CPT symmetry for the time evolution operator implies the following condition

$$\Theta\mathbb{U}(t) = \mathbb{U}(-t)\Theta \tag{4.10}$$

such that we find

$$\langle \Theta a | \, \mathbb{U}(t) \, | \Theta b \rangle = \langle a | \, \mathbb{U}(-t) \, | b \rangle^* \tag{4.11}$$

for expectation values. Applied to neutrino states this relation translates to

$$\langle \bar{\nu}_b | \bar{\nu}_a(t) \rangle = \langle \nu_b | \nu_a(-t) \rangle^* = \langle \nu_a | \nu_b(t) \rangle \,. \tag{4.12}$$

Put another way, the oscillation probability of an $\bar{\nu}_a$ going to $\bar{\nu}_b$ is the same as that of a ν_b going to ν_a such that

$$P(\bar{\nu}_a \to \bar{\nu}_b; t) = P(\nu_b \to \nu_a; t). \tag{4.13}$$

Additionally, we find a similar expression for lepton-number violating oscillations, which reads

$$P(\nu_a \to \bar{\nu}_b; t) = P(\nu_b \to \bar{\nu}_a; t). \tag{4.14}$$

We adopt the paradigm that lepton-number violating neutrino-antineutrino couplings B are small compared to the lepton-number conserving terms H and $\hat{H}$ and we shall therefore treat B as a small quantity for a perturbative solution of the equations of motion [77]. To this end, we decouple the evolution of neutrinos and antineutrinos from the evolution of the neutrino-antineutrino system by introducing a (subsidiary) time evolution operator $\mathbb{G}(t)$ for the neutrino and antineutrino systems. The latter is then used to transform the time evolution equation for $\mathbb{U}(t, t_0)$ to what we shall, henceforth, refer to as the interaction picture. Establishing this line of action, we write

$$\mathbb{U}(t, t_0) = \mathbb{G}(t, t_0)\mathbb{U}_{\mathrm{I}}(t, t_0) \tag{4.15}$$

with the underlying equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{G}(t, t_0) = -i\mathbb{H}_0(t, t_0)\mathbb{G}(t, t_0), \tag{4.16}$$

where the subscript I indicates the associated quantity in the interaction picture. It is seen that the time evolution equation in the interaction picture can be recast as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{U}_{\mathrm{I}}(t, t_0) = -i\mathbb{H}_{\mathrm{I}}(t)\mathbb{U}_{\mathrm{I}}(t, t_0) \tag{4.17}$$

with the Hamiltonian in the interaction picture given by

$$\mathbb{H}_{\mathrm{I}}(t) = \mathbb{G}^{-1}(t, t_0)\delta\mathbb{H}(t)\mathbb{G}(t, t_0). \tag{4.18}$$

The Hamiltonian $\mathbb{H}_0$ is lepton-number conserving, and therefore the solution to Eq. (4.16) obviously takes the form

$$\mathbb{G}(t, t_0) = \begin{pmatrix} G(t, t_0) & 0 \\ 0 & \hat{G}(t, t_0) \end{pmatrix}. \tag{4.19}$$

Making use of this expression, we can write the interaction Hamiltonian according to

$$\mathbb{H}_{\mathrm{I}}(t, t_0) = \begin{pmatrix} 0 & G^{-1}(t, t_0)B(t)\hat{G}(t, t_0) \\ \hat{G}^{-1}(t, t_0)B^{\dagger}(t)G(t, t_0) & 0 \end{pmatrix}. \tag{4.20}$$

Perturbatively solving the time evolution equation in the interaction picture can now be easily achieved by employing the Magnus expansion. We use a matrix exponential for its solution and write

$$\mathbb{U}_{\mathrm{I}}(t, t_0) = e^{\Omega(t, t_0)}, \tag{4.21}$$

where the Magnus operator is the sum of the Magnus approximants, as discussed at length in chapter 2. We shall see in our analysis that neutrino-antineutrino oscillation phenomena only enter the oscillation probabilities in second-order perturbation theory. Suppose that the time evolution for the lepton-number conserving oscillations, which have been separated into $\mathbb{G}$, can be obtained by solving its evolution equation, such that we have it at our disposal for further calculations. Our objective here is to comment on how lepton-number violating effects manifest in neutrino (antineutrino) oscillations. Reversing the transformation to the interaction picture for the time evolution operator $\mathbb{U}(t, t_0)$ now yields

$$\mathbb{U}(t, t_0) = \mathbb{G}(t, t_0) e^{\Omega(t,t_0)}. \tag{4.22}$$

We can use the block structure of the interaction Hamiltonian to calculate the Magnus operator up to second-order perturbation theory. We find

$$\Omega^{(2)}(t, t_0) = \begin{pmatrix} \tilde{C}(t, t_0) & -i\tilde{B}(t, t_0) \\ -i\tilde{B}^\dagger(t, t_0) & \tilde{D}(t, t_0) \end{pmatrix}, \tag{4.23}$$

where we have made use of the shorthand notations

$$\tilde{B}(t, t_0) = \int_{t_0}^{t} \mathrm{d}\tau \, G^{-1}(\tau, \tau_0) B(\tau) \hat{G}(\tau, \tau_0), \tag{4.24}$$

$$\tilde{C}(t, t_0) = -\frac{1}{2} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \left\{ \dot{\tilde{B}}(t_1) \dot{\tilde{B}}^\dagger(t_2) - \mathrm{h.c.} \right\}, \tag{4.25}$$

$$\tilde{D}(t, t_0) = -\frac{1}{2} \int_{t_0}^{t} \mathrm{d}t_1 \int_{t_0}^{t_1} \mathrm{d}t_2 \left\{ \dot{\tilde{B}}^\dagger(t_1) \dot{\tilde{B}}(t_2) - \mathrm{h.c.} \right\}. \tag{4.26}$$

A dot indicates a derivative with respect to time. Note, that the quantities $\tilde{C}(t, t_0)$ as well as $\tilde{D}(t, t_0)$, which appear on the diagonal of the Magnus operator, are quadratic in $\tilde{B}(t, t_0)$ and therefore in the lepton-number violating parameter B in the Hamiltonian. Then, up to second-order expressions in B, we can obtain the matrix exponential of the Magnus operator by truncating its power series. We obtain

$$\begin{aligned} e^{\Omega(t,t_0)} &\simeq 1 + \Omega(t, t_0) + \frac{1}{2}\Omega^2(t, t_0) + \mathcal{O}(\Omega^3) \\ &= \begin{pmatrix} 1 + \mathcal{O}^{(2)}(t, t_0) & -i\tilde{B}(t, t_0) \\ -i\tilde{B}^\dagger(t, t_0) & 1 + \hat{\mathcal{O}}^{(2)}(t, t_0) \end{pmatrix} + \ldots . \end{aligned} \tag{4.27}$$

In this expression we have introduced second-order operators for neutrinos $\mathcal{O}^{(2)}(t, t_0)$ and antineutrinos $\hat{\mathcal{O}}^{(2)}(t, t_0)$, which decode the deviation from unity (lepton-number conserving oscillations) in the diagonal entries of the exponential of the Magnus operator. Those read

$$\mathcal{O}^{(2)}(t, t_0) = \tilde{C}(t, t_0) - \frac{1}{2}\tilde{B}(t, t_0)\tilde{B}^\dagger(t, t_0), \tag{4.28}$$

$$\hat{\mathcal{O}}^{(2)}(t, t_0) = \tilde{D}(t, t_0) - \frac{1}{2}\tilde{B}^\dagger(t, t_0)\tilde{B}(t, t_0). \tag{4.29}$$

These operators reveal how lepton-number violating neutrino-antineutrino mixing as introduced by $B(t)$ affects the neutrino and antineutrino sectors, respectively. We have now assembled all ingredients which are necessary to give the various oscillation probabilities. The oscillation probabilities for common lepton-number conserving oscillations are denoted by

$$P_0(\nu_a \to \nu_b; t) = |G_{ba}(t)|^2, \tag{4.30}$$

$$P_0(\bar{\nu}_a \to \bar{\nu}_b; t) = |\hat{G}_{ba}(t)|^2. \tag{4.31}$$

Up to second order in lepton-number violating terms, the lepton-number preserving probabilities get modified according to

$$P(\nu_a \to \nu_b; t) \;=\; P_0(\nu_a \to \nu_b; t) + 2\mathrm{Re}\Big\{G^*_{ba}(t) \times (G(t)\mathcal{O}^{(2)}(t))_{ba}\Big\}, \tag{4.32}$$

$$P(\bar{\nu}_a \to \bar{\nu}_b; t) \;=\; P_0(\bar{\nu}_a \to \bar{\nu}_b; t) + 2\mathrm{Re}\Big\{\hat{G}^*_{ba}(t) \times (\hat{G}(t)\hat{\mathcal{O}}^{(2)}(t))_{ba}\Big\}. \tag{4.33}$$

There is no summation of recurring indices implied here or elsewhere in this chapter. Lepton-number violation in the Hamiltonian also gives rise to lepton-number violating oscillations for which the probabilities now read

$$P(\nu_a \to \bar{\nu}_b; t) \;=\; |(\hat{G}(t)\tilde{B}^\dagger(t))_{ba}|^2, \tag{4.34}$$

$$P(\bar{\nu}_a \to \nu_b; t) \;=\; |(G(t)\tilde{B}(t))_{ba}|^2. \tag{4.35}$$

A few comments concerning the nature of these expressions for the oscillation probabilities are in order.

As has already been pointed out in the discussion of the virtues of the Magnus expansion, the oscillation probabilities can be obtained without resorting to the effective mixing matrix at any stage of the calculations. This can be considered a convenient feature, especially for systems of three or more flavors. It is, at least in principle, also possible to derive oscillation probabilities by diagonalizing the entire lepton-number violating Hamiltonian and to extract oscillation probabilities from the eigenvalues and eigenstates. The essential features of the solution, however, are retained in our approach. Another notable characteristic is found in the fact that diagonalization of a $2N \times 2N$ system (N denoting the number of neutrino flavors) is reduced to a matrix multiplication for $N \times N$ matrices, since the perturbation expansion only involves block entries from the original Hamiltonian $\mathbb{H}(t)$.

One would expect that truncation of the exponential series of the Magnus operator results in loss of unitarity. However, truncating the series in a way that only keeps second-order terms in the small perturbation $B(t)$, does in fact imply that also the time evolution operator is unitary up to second order in this quantity. It is straightforward to verify this by invoking the unitarity condition on $\mathbb{U}(t, t_0)$. Yet another way to explicitly test for unitarity is to study the oscillation probabilities given above. Let us begin by taking the sum on all final flavor states in the oscillation probability $P(\nu_a \to \nu_b; t)$. We find

$$\begin{aligned}\sum_b P(\nu_a \to \nu_b; t) \;&=\; 1 + 2\sum_b \mathrm{Re}\Big\{G^*_{ba}(t) \times (G(t)\mathcal{O}^{(2)}(t))_{ba}\Big\} \\ &=\; 1 + 2\mathrm{Re}\Big\{G^\dagger(t)G(t)\mathcal{O}^{(2)}(t)\Big\}_{aa} \\ &=\; 1 + 2\mathrm{Re}\Big\{\mathcal{O}^{(2)}(t)\Big\}_{aa},\end{aligned} \tag{4.36}$$

where we have used that $G(t)$ is a time evolution operator and hence unitary. Further simplifications can be made. The expression for $\mathcal{O}^{(2)}(t)$ reveals that the diagonal entries of this matrix must have a negative real part. This is due to the fact that the diagonal elements of $\tilde{C}$ are purely imaginary and those of $\tilde{B}\tilde{B}^\dagger$ must be real and positive. We can hence write the sum on all final state flavors for lepton-number conserving oscillations according to

$$\sum_b P(\nu_a \to \nu_b; t) = 1 - \left\{\tilde{B}(t)\tilde{B}^\dagger(t)\right\}_{aa} . \tag{4.37}$$

This sum is less than unity as one would expect since also lepton-number violating oscillations are possible in our scenario. Let us therefore look at the lepton-number violating oscillations from a flavor ν_a to all possible antineutrino flavors. It is found to be

$$\begin{aligned}\sum_b P(\nu_a \to \bar{\nu}_b; t) &= \sum_b \left(\hat{G}(t)\tilde{B}^\dagger(t)\right)^*_{ba} \left(\hat{G}(t)\tilde{B}^\dagger(t)\right)_{ba} \\ &= \left(\tilde{B}(t)\hat{G}^\dagger(t)\hat{G}(t)\tilde{B}^\dagger(t)\right)_{aa} \\ &= \left(\tilde{B}(t)\tilde{B}^\dagger(t)\right)_{aa} ,\end{aligned} \tag{4.38}$$

again using the unitarity of $\hat{G}$. Assembling the lepton-number violating and lepton-number conserving part, we obtain

$$\sum_b P(\nu_a \to \nu_b; t) + \sum_b P(\nu_a \to \bar{\nu}_b; t) = 1. \tag{4.39}$$

In the presence of lepton-number violation, the lepton-number conserving oscillation probabilities decrease, making room for lepton-number violating oscillations.

The Magnus expansion allows us to give an approximate solution to the time evolution operator, which implies that the structure of the oscillation probabilities is also of a perturbative manner. It is therefore a viable question for which baselines (or times t) the approximation can actually be considered a good one. We find that this is the case if the relation $|B|t \ll 1$ is satisfied. Here $|B|$ indicates the magnitude of any nonzero eigenvalue of B. We will see later how this condition is realized, when we study a simple two-flavor model, which can be solved exactly and by means of the perturbation theory introduced here.

Let us, however, put aside such technicalities and begin a description of the physics of the oscillation probabilities for now. The oscillation probabilities are illustrative in a way that they are reducible to the standard lepton-number conserving neutrino oscillation results in the limiting case of vanishing neutrino-antineutrino coupling, i.e., $B(t) \to 0$. The appearance of the oscillation probabilities for $\nu_a \to \bar{\nu}_b$ and $\bar{\nu}_a \to \nu_b$ can also be interpreted in an intuitive way. Reading Eqs. (4.34) and (4.35) from right to left they state that $\tilde{B}^\dagger(t)$ $[\tilde{B}(t)]$ switches the initial neutrino (antineutrino) state to the associated antiparticle and $\hat{G}(t)$ $[G(t)]$ then evolves the antiparticle (particle) state until its detection.

So, neutrino-antineutrino oscillations are clearly a signal for lepton-number violation in the neutrino sector. Note, that this statement is a general one. All oscillation probabilities come with $\tilde{B}(t)$ and derivatives thereof. Put another way, in an approach in which neutrino-antineutrino couplings are treated as a small perturbation to all other potential enhancements in the neutrino and antineutrino sector respectively, one cannot have modifications in the neutrino-neutrino and antineutrino-antineutrino probabilities without introducing neutrino-antineutrino conversions at the same time. The oscillation probabilities for $\nu_a \to \bar{\nu}_b$ and $\bar{\nu}_a \to \nu_b$ are generically different by virtue of the generic difference between $\hat{G}(t)$ and $G(t)$. Even if the respective Hamiltonians H and $\hat{H}$ do not discriminate between particles and antiparticles [e.g., $H(t) = \hat{H}(t)$] there still is a difference due to the fact that neutrino-antineutrino coupling can be complex and not self-adjoint, i.e., $\tilde{B}(t) \neq \tilde{B}^{\dagger}(t)$ in general.

A similar situation applies for $\nu_a \to \nu_b$ and $\bar{\nu}_a \to \bar{\nu}_b$ oscillations. They differ in the time evolution operators for neutrinos and antineutrinos but also in the second-order operators $\mathcal{O}^{(2)}(t)$ and $\hat{\mathcal{O}}^{(2)}(t)$. Those operators are, in general, not identical, which is again due to the fact that $\tilde{B}(t)$ does not have to be self-adjoint. This statement translates to the fact that lepton-number violating neutrino oscillations discriminate between particles and antiparticles even if there is no difference in the respective Hamiltonians H and $\hat{H}$ for neutrinos and antineutrinos.

4.2 The one-generation case

Having established an approach to neutrino oscillations in which lepton-number violation is considered a *small* effect, let us now elaborate on the physical notion underlying this framework. For such purposes, we begin by studying the one-generation case of neutrino oscillations, since already in this case lepton-number violation allows neutrino-antineutrino oscillations to develop. Making use of the formalism, we obtain

$$\tilde{B}(t, t_0) = \int_{t_0}^{t} \mathrm{d}\tau \; B(\tau)\, e^{i\Delta\tilde{H}(\tau)}, \tag{4.40}$$

where we have defined

$$\Delta\tilde{H}(t) \equiv \int_{t_0}^{t} \mathrm{d}\tau \; \Delta H(\tau) = \int_{t_0}^{t} \mathrm{d}\tau \left[H(\tau) - \hat{H}(\tau)\right] \tag{4.41}$$

as the difference between (CP-nonconserving) *potential* terms in the neutrino and antineutrino sectors. Clearly, if the difference in such potential terms vanishes for some time $t = t_{\mathrm{res}}$, the integral (4.40) has a stationary phase and we can evaluate it by means of the saddle point approximation. This yields

$$\tilde{B} \simeq e^{i\Delta\tilde{H}(t_{\mathrm{res}})} \sqrt{\frac{\pi}{2}\frac{1}{\gamma_{\mathrm{res}}}}. \tag{4.42}$$

Here we have introduced the *adiabaticity parameter at resonance* γ_{res}, which we shall properly define and explain shortly. To this end, let us start from the lepton-number violating Hamiltonian and notice that for a simple two-dimensional case, we can always easily find a unitary transformation (for a two-dimensional problem, phases are irrelevant and the unitary transformation amounts to a time-dependent rotation in flavor space), which entails an *effective mixing angle* $\Theta(t)$ fixed via

$$\cos\Theta(t) = \frac{\Delta H(t)}{\omega_{\text{eff}}(t)}, \qquad \sin\Theta(t) = \frac{2|B(t)|}{\omega_{\text{eff}}(t)}. \tag{4.43}$$

The *effective oscillation frequency* $\omega_{\text{eff}}(t)$ of the system is here given by

$$\omega_{\text{eff}}(t) = \sqrt{4|B(t)|^2 + \Delta H^2(t)}. \tag{4.44}$$

It is interesting to note that the effective mixing becomes maximal and neutrino conversions undergo a resonance, in the case of vanishing $\Delta H(t)$, i.e., $\Delta H(t_{\text{res}}) = 0$ at some resonance time t_{res}. Put another way, mixing between neutrinos and antineutrinos becomes maximal when the difference between potential terms for the two species is minimal (or in fact vanishing at the resonance time). This is also corroborated by the fact that for identical neutrino and antineutrino potentials mixing is always maximal. Obviously, for the case of CP-conserving neutrino and antineutrino potentials the difference between those vanishes at all times since they are identical to begin with. However, in the case of CP-nonconserving matter potentials, the difference is generally nonzero and can be varying with time – the latter effect is crucial for some physics implications of the resonance structure. It is also conceivable to allow for CPT-violating potential terms, which then generate a nonvanishing difference $\Delta H(t)$. Note, eventually, that both a lepton-number violating coupling $B(t)$ as well as a CP-violating difference in neutrinos and antineutrinos has to be assumed in order for neutrino-antineutrino oscillations to develop. This can be understood by comparing the one-generation case of neutrino-antineutrino oscillations with standard flavor oscillations between two neutrinos in vacuo. In the latter case oscillations can only develop if there is a difference in neutrino masses for the two states; in the former situation there also has to be some difference between neutrinos and antineutrinos (which obviously cannot be the mass) which mimicks the effect of neutrino masses in vacuum flavor oscillations. CP or CPT violation takes this role here and along with lepton-number violation neutrino-antineutrino oscillations can develop.

With the effective mixing at our disposal, we write the adiabaticity parameter at resonance as

$$\gamma(t_{\text{res}}) \equiv \gamma_{\text{res}} = \frac{1}{\omega_{\text{eff}}} \left.\frac{\mathrm{d}\Theta(t)}{\mathrm{d}t}\right|_{t=t_{\text{res}}} = -\frac{1}{4|B(t)|^2} \left.\frac{\mathrm{d}\Delta H(t)}{\mathrm{d}t}\right|_{t=t_{\text{res}}}. \tag{4.45}$$

We have already discussed in chapter 3 how this expression for the adiabaticity parameter can be understood physically by comparing the system's time scale to the interaction's time scale.

For $\gamma_{\rm res} \gg 1$ we encounter nonadiabatic neutrino-antineutrino conversions; $\gamma_{\rm res} \ll 1$ gives the adiabatic case. So roughly speaking (neglecting the derivative of the potential term for the time being) the smaller $B(t)$, the larger the nonadiabaticity of the neutrino-antineutrino system and hence if we start the evolution with only ν_a ($\bar{\nu}_a$) states present the transition $\nu_a \to \nu_a$ ($\bar{\nu}_a \to \bar{\nu}_a$) prevails. The lepton-number violating oscillation channel $\nu_a \to \bar{\nu}_a$ ($\bar{\nu}_a \to \nu_a$) gets more and more suppressed as the nonadiabaticity increases. If, however, the change of the difference in matter potentials is small with time, this effect can partially compensate a small neutrino-antineutrino coupling, driving the evolution of the system towards adiabatic transitions opening the lepton-number violating oscillation channel $\nu_a \to \bar{\nu}_a$ ($\bar{\nu}_a \to \nu_a$) again. Note, also, that the suppression of the lepton-number violating oscillation channel depends on the time dependence of the potential terms in the Hamiltonian.

Substituting the result for the adiabaticity parameter $\gamma_{\rm res}$ in the expression for $\tilde{B}$ and at the same time keeping in mind that the oscillation probability $P(\nu_a \to \bar{\nu}_a; t)$ is directly proportional to $\tilde{B}^\dagger$, it is seen that interpreting lepton-number violating (neutrino-antineutrino) couplings as a small perturbation to lepton-number conserving neutrino oscillations in the one-generation case is equivalent to assuming that neutrino-antineutrino oscillations occur nonadiabatically. Nonadiabaticity hence *closes* the lepton-number violating oscillation channel $\nu_a \to \bar{\nu}_a$ but at the same time improves the perturbative expansion as outlined above.

These results for the one-generation framework make the case for referring to the perturbation theory as a *nonadiabatic perturbation expansion*.

From the definition of the adiabaticity parameter $\gamma_{\rm res}$ another interesting feature emerges. Suppose we assume *small* lepton-number violating couplings $B(t)$ between neutrinos and antineutrinos in the oscillation Hamiltonian. This means that oscillations between particles and antiparticles are suppressed by the large nonadiabaticity of the transitions giving rise mostly to the lepton-number conserving oscillation channel $\nu_a \to \nu_a$. If, however, the time variation of the difference in matter potentials at the resonance is sufficiently mild, it is seen that the presence of such matter along the neutrino (antineutrino) propagation path can drive the system towards adiabaticity thus opening the oscillation channel $\nu_a \to \bar{\nu}_a$. Put another way, the presence of CP- (or even CPT-) nonconserving matter of a varying density (easily reinterpreted as a time dependence of the potential terms) can enhance lepton-number violating neutrino oscillations as compared to the case in vacuo. Note that this statement holds true regardless of the nature of the perturbation theory since it is merely a result obtained from the adiabaticity parameter $\gamma_{\rm res}$ and the latter does only need the Hamiltonian of the system as a prerequisite.

We have not explicitly written down the steps leading to a solution for the Schrödinger equation by means of the standard procedure of diagonalizing the effective Hamiltonian since this does not add to the understanding of either the perturbation theory developed here or the physics of the neutrino-antineutrino system under consideration.

We shall, however, give a simple geometrical interpretation of neutrino-antineutrino oscillations in the one-generation case using the methods developed for treating neutrino oscillations in the coherence vector framework. In this approach, as presented in chapter 3, we were dealing with a density matrix in flavor space for neutrinos since the system under consideration reveals flavor oscillations. However, if we are interested in neutrino-antineutrino oscillations, we write the Hamiltonian and thus also the density matrix in a *state space* in which the distinct states stand for particles or antiparticles of a specific flavor. This entails a reinterpretation of the *density matrix* in this context. In the one-particle interpretation of the density matrix its diagonal elements now give the probability to find the system in the particle or antiparticle state (rather than in one flavor or the other) and thus give the occupation numbers for particles and antiparticles (at a given momentum), respectively. This amounts to saying that the density matrix (up to a time-independent normalization) can be decomposed as $\rho(p,t) \sim |\psi(p,t)\rangle \langle\psi(p,t)|$ using the states $|\psi(p,t)\rangle = a(p,t)\,|\nu\rangle + b(p,t)\,|\bar{\nu}\rangle$ consisting of particles and antiparticles. We omit the flavor index since we are dealing with a one-flavor system after all.

The time evolution equation for the coherence vector $\vec{P}(t)$ of the system can then be obtained in analogy to the analysis in section 2.3. We find for the equations of motions

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\vec{P}(t) &= S(t)\vec{P}(t), \\ S(t) &= S_{\Delta H}(t) + S_{\nu\bar{\nu}}(t) = \begin{pmatrix} 0 & \Delta H(t) & 2\mathrm{Im}B(t) \\ \Delta H(t) & 0 & -2\mathrm{Re}B(t) \\ -2\mathrm{Im}B(t) & 2\mathrm{Re}B(t) & 0 \end{pmatrix}, \end{aligned} \tag{4.46}$$

where we have decomposed $S(t)$ into its matter-affected, lepton-number conserving part $S_{\Delta H}(t)$ and its lepton-number violating neutrino-antineutrino part $S_{\nu\bar{\nu}}(t)$. Note, that this equation is written in a basis of particle and antiparticle states rather than flavor states.

Let us further analyze these equations of motion. Since the lepton-number conserving term $S_{\Delta H}$ is self-commuting at different times, it can be integrated exactly, hence providing a solution for the time evolution of a system of matter-affected neutrinos and antineutrinos in the absence of lepton-number violation. We write

$$\frac{\partial}{\partial t}T(t) = S_{\Delta H}(t)T(t) \tag{4.47}$$

and have for its solution

$$T(t) = \begin{pmatrix} \cos\Delta\tilde{H}(t) & -\sin\Delta\tilde{H}(t) & 0 \\ \sin\Delta\tilde{H}(t) & \cos\Delta\tilde{H}(t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.48}$$

Such a shape for the solution in $T(t)$ is easily interpreted, keeping in mind that we are operating in a state space of particles and antiparticles rather than flavor space. The oscillatory part in the upper left 2×2 matrix states that the system is oscillatory in P_x and P_y and hence completely incoherent. The one in the lower right entry of the matrix tells us that the system is also *frozen* to its initial $\nu - \bar{\nu}$ distribution. No transitions between neutrinos and antineutrinos can occur as is, of course, expected if lepton-number violating terms are inexistent. Put another way, a neutrino (antineutrino) ensemble with vanishing (or rather negligible) neutrino-antineutrino couplings is frozen to this configuration and completely incoherent.

We can use $T(t)$ to introduce a new coherence vector $\vec{Q}(t)$ via $\vec{P}(t) = T(t)\vec{Q}(t)$, which yields equations of motion

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\vec{Q}(t) &= S_Q(t)\vec{Q}(t), \\ S_Q(t) &= \begin{pmatrix} 0 & 0 & 2\mathrm{Im}\dot{\tilde{B}}(t) \\ 0 & 0 & -2\mathrm{Re}\dot{\tilde{B}}(t) \\ -2\mathrm{Im}\dot{\tilde{B}}(t) & 2\mathrm{Re}\dot{\tilde{B}}(t) & 0 \end{pmatrix}, \end{aligned} \tag{4.49}$$

where we recover the integral quantity $\dot{\tilde{B}}(t, t_0)$ as given above for the one-generation case.

The solution for $\vec{P}(t)$ to first order in a perturbation theory in $B(t)$ and around the stationary solution for the equations of motion is given by

$$\vec{P}^{(1)}(t) = T(t) \exp\left[\int_{t_0}^{t} \mathrm{d}\tau\, S_Q(\tau)\right] \vec{P}(t_0), \tag{4.50}$$

wherein $\vec{P}(t_0) = (0, 0, \pm 1)$ is the initial neutrino-antineutrino distribution in which the plus sign describes an ensemble of pure neutrino states and the minus sign one with only antineutrino states present. The observable of interest thus is $P_z(t)$ as it gives the time evolution of the neutrino-antineutrino ensemble or rather the excess of particle over antiparticle states. We find

$$P_z^{(1)}(t) = \mathrm{sgn}\left[P_z(t_0)\right] \left(1 - 2\sin^2|\tilde{B}(t)|\right). \tag{4.51}$$

This expression can be interpreted geometrically. The z-component of the coherence vector is fixed to its initial value except for a correction that goes with the absolute value of the integral quantity $\tilde{B}(t)$. We have seen in the foregoing considerations that $\tilde{B}$ can be related to the adiabaticity parameter of the system at resonance. Substituting this result in the expression for the time evolution of $P_z^{(1)}(t)$, it is seen that interpreting neutrino-antineutrino coupling as a small perturbation to lepton-number conserving neutrino oscillations is equivalent to assuming that neutrino-antineutrino oscillations occur nonadiabatically.

The oscillation probabilities are now easily obtained. We have

$$\begin{aligned} \text{prob}(\nu \to \nu; t) &= \frac{1}{2}\left[1 + P_z(t)\right] = \cos^2 \sqrt{\frac{\pi}{2}\frac{1}{\gamma_{\text{res}}}}, \\ \text{prob}(\nu \to \bar{\nu}; t) &= \frac{1}{2}\left[1 - P_z(t)\right] = 1 - \cos^2 \sqrt{\frac{\pi}{2}\frac{1}{\gamma_{\text{res}}}}, \end{aligned} \tag{4.52}$$

where obviously we have set $P_0 = 1$ since we are dealing with a closed system. We see that unitarity is also preserved in this solution and in accordance with the discussion above, neutrino-antineutrino conversions cease to exist as the system gets more and more nonadiabatic.

4.3 The two-generation case

After having analyzed the one-generation case for neutrino-antineutrino oscillations, let us turn our attention to the two-generation case in vacuo. Neglecting terms proportional to the unit matrix, the Hamiltonian for flavor oscillations in neutrinos as well as antineutrinos can be decomposed in terms of the Pauli matrices σ_i $(i = 1, 2, 3)$ and the mixing angle θ_0 in the absence of lepton-number violating terms

$$\begin{aligned} H = \hat{H} &= \frac{\Delta m^2}{4p}\left(\sigma_x \sin 2\theta_0 - \sigma_z \cos 2\theta_0\right) \\ &\equiv \omega\left(\sigma_x \sin 2\theta_0 - \sigma_z \cos 2\theta_0\right). \end{aligned} \tag{4.53}$$

It directly follows that also the time-evolution operators for neutrino and antineutrino states are identical. They are found to obey

$$G(t, t_0 = 0) = \hat{G}(t, t_0 = 0) = \cos \omega t - i\left(\sigma_x \sin 2\theta_0 - \sigma_z \cos 2\theta_0\right) \sin \omega t. \tag{4.54}$$

In order to encorporate effects of lepton-number violation, let us parametrize the matrix $B(t)$ using the Pauli matrices via

$$B(t) = b_0(t) + b_i(t)\sigma_i, \tag{4.55}$$

where

$$b_0(t) = \frac{1}{2}\text{tr } B(t), \qquad b_i(t) = \frac{1}{2}\text{tr } \left(\sigma_i B(t)\right). \tag{4.56}$$

We have $b_y(t) = 0$ for the case of CPT-conserving oscillations. It is now straightforward to see

$$\dot{\hat{B}}(t) = \beta_0(t) + \beta_i(t)\sigma_i, \tag{4.57}$$

where

$$
\begin{aligned}
\beta_0(t) &= b_0(t)\,, \\
\beta_x(t) &= \sin 2\theta_0\Big[b_x(t)\sin 2\theta_0 - b_z(t)\cos 2\theta_0\Big] + \cos 2\omega t\cos 2\theta_0\Big[b_x(t)\cos 2\theta_0 + b_z(t)\sin 2\theta_0\Big] \\
&\quad -b_y(t)\sin 2\omega t\cos 2\theta_0\,, \\
\beta_y(t) &= b_y(t)\cos 2\omega t + \sin 2\omega t\Big[b_x(t)\cos 2\theta_0 + b_z(t)\sin 2\theta_0\Big]\,, \\
\beta_z(t) &= -\cos 2\theta_0\Big[b_x(t)\sin 2\theta_0 - b_z(t)\cos 2\theta_0\Big] + \cos 2\omega t\sin 2\theta_0\Big[b_x(t)\cos 2\theta_0 + \\
&\quad +b_z(t)\sin 2\theta_0\Big] - b_y(t)\sin 2\omega t\sin 2\theta_0\,.
\end{aligned}
\tag{4.58}
$$

Let us assume that the lepton-number violating part $B(t)$ of the Hamiltonian is time-independent and hence also independent of baseline. The above expression can then be easily integrated to give $\tilde{B}(t)$ and the resulting expressions can be used to calculate various oscillation probabilities.

For the most general B, however, the expressions are cumbrous. So rather than working through the full expressions, we focus on a simple example, which shall further clarify the nature of the perturbation theory at hand since it can be tackled both analytically, which gives an exact solution, and perturbatively. Suppose B is proportional to the unit matrix, which means that lepton-number violation is flavor blind. This implies that all $b_i = 0$ such that we are left with a nonzero contribution from only b_0. We hence find that $\tilde{B}$ is simply given by $b_0 t$ times the unit matrix. Using the formulae for the various oscillation probabilities of the neutrino-antineutrino conversions, we find

$$P(\nu_a \to \bar{\nu}_b; t) = |b_0|^2 t^2 \times P_0(\nu_a \to \nu_b; t), \tag{4.59}$$

whereas for the lepton-number conserving oscillations we have

$$P(\nu_a \to \nu_b; t) = \big(1 - |b_0|^2 t^2\big) \times P_0(\nu_a \to \nu_b; t). \tag{4.60}$$

It is only just to comment on the nature of these oscillation probabilities. In the absence of lepton-number violating contributions to the Hamiltonian, the oscillation probabilities exhibit an oscillatory nature with respect to time (or baseline for that matter). The periodic maxima and minima of this oscillation all come with the same amplitude. If, however, lepton-number violating effects come into play, this periodicity is modulated by a time-dependent function $\big(1 - |b_0|^2 t^2\big)$ and as the propagation distance of the neutrino on its path from the source to the detector increases, the successive maxima and minima of the oscillation get more and more suppressed. Keeping in mind, though, that the perturbation expansion is only valid in a regime where $|b_0 t| \ll 1$, the oscillation probability is not liable to ebb away completely.

As a matter of fact, for the two-state system under consideration here the oscillation probabilities can be obtained by exactly solving the underlying Schrödinger equation. For such purposes we hearken back to the fact that $G = \hat{G}$ for the Hamiltonian $\mathbb{H}_0$ and hence the 2×2 block structure of the Hamiltonian $\mathbb{H}_\mathrm{I}$ in the interaction picture can be written as

$$\mathbb{H}_\mathrm{I} = \begin{pmatrix} 0 & b_0 \\ b_0^* & 0 \end{pmatrix} = |b_0| \begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}, \tag{4.61}$$

where $b_0 = |b_0| e^{i\alpha}$. Moreover, out of all Magnus approximants only the first term Ω_1 is actually nonzero. We find that it is given by

$$\Omega(t) = \Omega_1(t) = -i\mathbb{H}_\mathrm{I} t. \tag{4.62}$$

The matrix exponential for this expression can be done exactly. We have

$$e^{\Omega(t)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos|b_0|t - i \begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix} \sin|b_0|t. \tag{4.63}$$

With the exponential of the Magnus operator at our disposal, the evolution operator of the system is easily obtained and we find for the oscillation probabilities

$$\begin{aligned} P(\nu_a \to \bar{\nu}_b; t) &= \sin^2|b_0|t \times P_0(\nu_a \to \nu_b; t), \\ P(\nu_a \to \nu_b; t) &= \cos^2|b_0|t \times P_0(\nu_a \to \nu_b; t). \end{aligned} \tag{4.64}$$

The expressions given in Eqs. (4.59) and (4.60) are nothing but the first order approximations of the exact oscillation probabilities for $|b_0|t \ll 1$. The expressions for the oscillation probabilities given in Eq. (4.64) are valid for all t. We finally remark that the general formulae for the oscillation probabilities as given in Eqs. (4.32 - 4.35) hold for time-dependent systems as well, e.g., for matter-induced oscillations involving neutrinos and antineutrinos.

4.4 Some phenomenology of models with altered dispersion relations

Before we further our analysis by means of beginning a discussion of CPT violation in decays of charged fermions, we pause with the intention of furnishing the seemingly abstract considerations concerning the development of perturbation theories for neutrino oscillations in models with altered dispersion relations with some thoughts on the phenomenology of said class of models.

To this end, we shall focus the discussion on the work previously published in Refs. [28, 29] by the author of this thesis and used for his Diploma thesis. Let it be decisively clear that the results previously published in the articles [28, 29] are not a part of the results of this Ph.D. thesis. They are, however, closely related and may thus serve as an illustrative example for possible phenomenologies of the wide range of models to which the perturbation theories developed here apply.

In those works, we are dealing with two very different kinds of models for neutrino oscillations with altered dispersion relations. On the one hand, we shall discuss the impact of an asymmetrically-warped spacetime geometry on active-sterile neutrino oscillations and the resultant phenomenological implications for neutrino oscillation experiments [28]. This model serves as an example of Lorentz-violating, but CPT-conserving scenarios. On the other hand, we shall present a model with generic violation of CPT which gives rise to neutrino-antineutrino oscillations between electron and muon neutrino flavors [29].

Let us begin our endeavor with the discussion of baseline-dependent neutrino oscillations with extra-dimensional shortcuts. In models with large extra spatial dimensions spacetime is typically composed of a 3+1-dimensional Minkowskian brane, which is embedded in a bulk formed by the additional dimensions. In such models particles which are singlets under the gauge group of the standard model are assumed to be allowed to travel freely within the brane as well as the bulk. The remaining standard model particles are then confined to the Minkowskian brane. Suppose now we introduce an asymmetrically-warped metric

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - e^{-2k|u|}\mathrm{d}x_i\mathrm{d}x^i - \mathrm{d}u^2 \tag{4.65}$$

for the brane-bulk system, such that $e^{-2k|u|}$ is the warp factor with k some presently unknown constant of nature and u denotes the coordinate of the extra spatial dimension. It is easily understood that active and sterile neutrinos behave differently in such an asymmetrically-warped spacetime. While the propagation of the active neutrino is confined to the brane, the sterile state can have off-brane trajectories. Since the extra dimension is warped (the spatial coordinates of the brane are *shrunken* relative to the time and bulk coordinates t and u), longer travel times (or baselines for that matter) for a neutrino on the brane leave room for the geodesic of the sterile state to penetrate deeper into the bulk. The deeper the penetration of the bulk, the greater the warp experienced by the sterile state. For a brane-bound observer it will appear as if the sterile neutrino is taking a temporial (i.e., superluminal) *shortcut* through the bulk. As the travel times for active and sterile states are different in an asymmetrically-warped spacetime, a new phase difference in neutrino propagation emerges. On the one hand, there is the common phase difference of $t\delta H = L\Delta m^2/2E$, where H is the Hamiltonian of the system, L the baseline, E the neutrino energy and Δm^2 the mass-squared difference. On the other hand, there now is a second phase difference $H\delta t = Ht(\delta t/t)$ due to the difference in travel times of the active and sterile states. It is convenient to capture the relative difference in travel times in the *shortcut parameter* $\varepsilon = \delta t/t = (t^{\mathrm{brane}} - t^{\mathrm{bulk}})/t^{\mathrm{brane}}$. The two phase differences may conspire in a way as to produce resonant oscillation phenomena. As the shortcut parameter depends on the neutrino baseline, so does the resonance behavior in neutrino oscillations.

The derivation of the probability of oscillation between active and sterile states is a subtle affair and we shall sketch its main ingredients here without going into too much detail. A detailed derivation can be found in Ref. [28]. An analysis of the geodesic equations for the spacetime under consideration reveals that there exists a countably infinite number of geodesics for the sterile neutrino connecting two points on the brane. In a semi-classical approach to path-integral quantum mechanics a path-integral weighted sum over all possible classical trajectories must be performed. Multiple geodesics contribute and therefore also multiple resonances are expected in the oscillation probability. Whether these additional resonances contribute significantly depends on the initial distribution of sterile neutrino velocities transverse to the brane. The fact that there can be initial momentum components of the neutrino transverse to the brane is a consequence of the uncertainty principle. We assume that the distribution of such sterile neutrino momenta along the extra dimension obeys a Gaußian shape. The width of this Gaußian distribution is then related to the brane thickness.

It turns out that for resonant oscillations to develop, the experimental baselines have to be short on the scale of the warp factor k^{-1}. In this case, it is seen that the resonance condition is that the product of baseline and neutrino energy, LE, be an integer multiple of the *resonant product* $(LE)_{\text{res}} \propto k^{-1}\sqrt{\Delta m^2}$. In this way, brane-bulk resonances are determined by a product of baseline and energy rather than energy alone. The phenomenology of such novel LE-resonances can be understood as follows. Above the resonance active and sterile states decouple; at the resonance effective mixing becomes maximal and below the resonance vacuum oscillations are recovered. As higher-resonances are suppressed, also active-sterile mixing is suppressed for values of LE above its resonant value. In long-baseline experiments sterile states taking shortcuts through the extra-dimensional bulk decouple from active states on the brane. Consequently, there is no signal of active-sterile neutrino mixing expected in atmospheric data, for instance.

The model for active-sterile neutrino oscillations in asymmetrically-warped spacetimes predicts a modified spectral dependence of the oscillation probability. Instead of a L/E-dependence we find a LE-dependence. This fact can be traced back to the observation that we are dealing here with a model with altered dispersion relations. The effective Hamiltonian of the neutrino system contains an additional effective potential term which scales as εE in energy. The apparently superluminal behavior of sterile neutrinos also indicates that the model constructed here obviously violates Lorentz invariance. However, the model is CP-symmetric as the origin of the altered dispersion relations is a purely gravitational one and gravity does not discriminate between particles and antiparticles.

Let us in addition also explore a model with altered dispersion relations based on the CPT-violating extension presented in section 2.4. We begin by considering the effective Hamiltonian of Eq. (2.52) for the first two neutrino generations, electron and muon neutrinos. We allow for Lorentz- and CPT-violating interactions which modify both the neutrino and antineutrino sectors and additionally introduce neutrino-antineutrino mixing. The effective potential term $V(E)$ in the Hamiltonian assumes a form

$$V(E) = \frac{E}{2} \begin{pmatrix} -c_{ee} & 0 & b_e & 0 \\ 0 & -c_{\mu\mu} & 0 & b_\mu \\ b_e & 0 & -c_{ee} & 0 \\ 0 & b_\mu & 0 & -c_{\mu\mu} \end{pmatrix} . \tag{4.66}$$

This potential is written in a flavor basis $(\nu_e, \nu_\mu, \bar{\nu}_e, \bar{\nu}_\mu)$. The c_{ee}, $c_{\mu\mu}$ coefficients are Lorentz-violating parameters, whereas b_e, b_μ are both Lorentz- and CPT-violating. The b-type coefficients introduce neutrino-antineutrino mixing; the c-type coefficients are responsible for altered dispersion relations in the neutrino and antineutrino sectors. Using this potential term and the vacuum oscillation Hamiltonian for neutrinos and antineutrinos, it is seen that switching from the flavor basis to a basis of charge-conjugation eigenstates block-diagonalizes the four-dimensional effective Hamiltonian. Instead of flavor eigenstates ν_i and $\bar{\nu}_i$, the charge-conjugation states $\nu_i^- = \frac{1}{\sqrt{2}}(\nu_i - \bar{\nu}_i)$ and $\nu_i^+ = \frac{1}{\sqrt{2}}(\nu_i + \bar{\nu}_i)$ are used. Here ν_i^- is odd under charge conjugation, whereas ν_i^+ is even. Each sector of charge-conjugation eigenstates can then be diagonalized separately. This introduces an effective mixing angle for each sector and depending on the values of the Lorentz- and CPT-violating parameters, resonances do occur in one or both sectors. The resulting resonance energies in the charge-conjugation odd and even sectors are different in general. Keeping in mind that CPT violation discriminates between particle and antiparticle states, this is easily understood. Another interesting observation is that introducing matter effects into the system under consideration invalidates the approach via charge-conjugation eigenstates. Matter effects explicitly break the charge-conjugation symmetry between neutrinos and antineutrinos.

This simple CPT-violating framework for neutrino-antineutrino mixing allows for new resonances in neutrino (antineutrino) oscillations in vacuo, which are suitably analyzed in terms of charge-conjugation even and odd states. The linear dependence of the effective potential on energy generates altered dispersion relations. The model also predicts daily and seasonal variations of the neutrino (antineutrino) oscillation probabilities via the coefficients b and c. In Lorentz-violating models the Sun-centered celestial equatorial frame (in which the Earth's rotational axis is lying along the z-direction) is introduced as a standard frame of reference in which neutrino oscillation experiments can compare their findings. The coefficients of the model presented here only depend on the celestial colatitude Θ such that $c_{ab} \propto 1 + \cos^2\Theta$ and $b_a \propto \sin\Theta$. Therefore CPT-violating models do not only modify the dispersion relations of neutrinos (antineutrinos), but typically they also introduce a directional dependence. It is obvious that the model introduced here is one example for lepton-number violating neutrino oscillations as discussed in this chapter.

5 CPT-violating effects in muon decay

Up until now we have been occupied with studying effects of lepton-number and also CPT violation in neutral fermions. However, standard model interactions involving charged fermions might also be subject to CPT violation. In the following, we shall consider CPT-violating effects in low-energy muon and antimuon decays. To this end, we augment the standard model charged current interactions by CPT-violating vertex operators. This enhancement results in different lifetimes for muons and antimuons. Making use of the experimental bounds on those lifetimes, we constrain the CPT-violating couplings introduced in our approach. In addition to this, we study decay rates differential in electron energy and and spatial angles. Those quantities provide suitable new observables, which have the potential to further constrain effects of CPT violation in charged lepton decays [99].

5.1 Lifetimes

Let us begin our considerations by writing the charged current part of the standard model Lagrangian

$$\mathcal{L}_{\mathrm{cc}} = -\frac{g}{\sqrt{2}} J^{\lambda} W_{\lambda} + \mathrm{h.c.}\,, \tag{5.1}$$

where the leptonic current is given by

$$J^{\lambda} = \sum_{l=e,\mu,\tau} \bar{l}\,\gamma^{\lambda} P_{\mathrm{L}}\,\nu_l. \tag{5.2}$$

The chirality projection operator is defined as usual via

$$P_{\mathrm{L}} = \frac{1}{2}\left(1-\gamma_5\right). \tag{5.3}$$

We entertain the idea that the expression for the leptonic current is enhanced by additional tensors of odd rank in order to establish CPT violation in the charged current interactions. Hence, we substitute

$$J^{\lambda} \to J^{\lambda} \equiv \sum_{l=e,\mu,\tau} \bar{l}\left(\gamma^{\lambda} P_{\mathrm{L}} + \delta\Gamma^{\lambda}\right)\nu_l \tag{5.4}$$

and suppose that some of these extra terms $\delta\Gamma^{\lambda}$ are CPT-violating. The effects of such additional terms on, e.g., the lifetime of muons and antimuons can be considered to be very small due to the existing experimental constraints [5]. This fact motivates that only first order contributions of $\delta\Gamma^\lambda$ can be considered relevant and thus we neglect all terms $\mathcal{O}\left(\delta\Gamma^2\right)$. We shall see how to explicitly parametrize the additional couplings $\delta\Gamma^\lambda$ and actually quantify the smallness of the odd rank tensors entailed therein. Having laid down our paradigm for CPT violation in leptonic currents, we can now delve into applications and explore its consequences on observables such as muon and antimuon lifetimes. We begin by assigning momenta for the particles involved in muon decays as follows

$$\mu^-(p) \to e^-(p') + \nu_\mu(k) + \bar{\nu}_e(k'). \tag{5.5}$$

For the corresponding particles in the antimuon decay the momenta are assumed to be the same. We can write the matrix element of the muon decay process in a four Fermi ansatz since all masses involved in the process are much smaller than the W-boson mass. We have

$$\mathcal{M}(\mu^- \to e^-\nu_\mu\bar{\nu}_e) = 2\sqrt{2}G_\mathrm{F}\Big[\bar{u}(p')\Gamma^\lambda v(k')\Big]\Big[\bar{u}(k)\Gamma_\lambda u(p)\Big], \tag{5.6}$$

using a shorthand notation $\Gamma^\lambda \equiv \gamma^\lambda P_\mathrm{L} + \delta\Gamma^\lambda$ in order to streamline notation. The spin-averaged squared matrix element for the muon decay can then be written as

$$\begin{aligned}\left\langle|\mathcal{M}(\mu^- \to e^-\bar{\nu}_e\nu_\mu)|^2\right\rangle &\equiv \frac{1}{2}\sum_{\text{spins}}|\mathcal{M}(\mu^- \to e^-\bar{\nu}_e\nu_\mu)|^2 \\ &= 4G_\mathrm{F}^2\,\mathrm{tr}\left\{\Gamma^\lambda(\not{p} + m_\mu)\overline{\Gamma}^\rho\not{k}\right\}\mathrm{tr}\left\{\Gamma_\lambda\not{k}'\overline{\Gamma}_\rho\not{p}'\right\}\,, \end{aligned}\tag{5.7}$$

where $\overline{\Gamma}^\lambda = \gamma_0{\Gamma^\lambda}^\dagger\gamma_0$ is the Dirac adjoint, m_μ is the muon mass. We neglect the masses of all final state decay particles in the following considerations. The corresponding matrix element for the antimuon decay can be obtained by swapping any u-spinor with the associated v-spinor and vice versa. The only difference for the spin-averaged squared matrix element for the antimuon is a switch in the sign of the mass term. This observation suggests that standard model interactions can be *subtracted* by taking the difference in decay rates for muon and antimuon. This procedure isolates the CPT-violating effects. Put another way, CPT violation manifests in the different lifetimes for the muon and antimuon. Following this train of thought, we introduce the difference in spin-averaged matrix elements squared $\delta\mathcal{M}^2$ for muon and antimuon decays. We obtain

$$\begin{aligned}\delta\mathcal{M}^2 &\equiv \left\langle|\mathcal{M}(\mu^- \to e^-\bar{\nu}_e\nu_\mu)|^2\right\rangle - \left\langle|\mathcal{M}(\mu^+ \to e^+\nu_e\bar{\nu}_\mu)|^2\right\rangle \\ &= 8G_\mathrm{F}^2 m_\mu\,\mathrm{tr}\left\{\Gamma^\lambda\overline{\Gamma}^\rho\not{k}\right\}\mathrm{tr}\left\{\Gamma_\lambda\not{k}'\overline{\Gamma}_\rho\not{p}'\right\}\,. \end{aligned}\tag{5.8}$$

This expression clearly vanishes for standard model interactions in which $\delta\Gamma^\lambda = 0$, since the first trace in this case only contains an odd number of Dirac matrices.

However, interference between terms with an even number of Dirac matrices and the usual (V$-$A) structure of the current can give rise to a nonvanishing trace in the expression above. Hence, we parametrize the CPT-violating part of the current according to

$$\delta\Gamma^\lambda = A^\lambda + {B^\lambda}_{\alpha\beta}\sigma^{\alpha\beta}, \tag{5.9}$$

where A^λ and $B^\lambda{}_{\alpha\beta}$ are a set of real constants. Apparently, by definition, $B^\lambda{}_{\alpha\beta} = -B^\lambda{}_{\beta\alpha}$. As for nomenclature, we shall refer to A^λ as the *vector part* and to $B^\lambda{}_{\alpha\beta}$ as the *dipole part* of the CPT-violating terms. Given this parametrization for CPT violation, we can write the difference in spin-averaged squared matrix elements to first order in $\delta\Gamma^\lambda$ as

$$\begin{aligned}\delta\mathcal{M}^2 &= 16G_\mathrm{F}^2 m_\mu \left(k'_\lambda p'_\rho + k'_\rho p'_\lambda - k'\cdot p'\, g_{\lambda\rho} - i\varepsilon_{\lambda\alpha\rho\beta}k'^\alpha p'^\beta\right) \\ &\quad \times \mathrm{tr}\left\{\delta\Gamma^\rho\gamma^\lambda P_\mathrm{L}\not{k} + \gamma^\rho P_\mathrm{L}\delta\Gamma^\lambda\not{k}\right\}\,, \end{aligned} \tag{5.10}$$

wherein the Levi-Civita tensor is used with the convention $\varepsilon_{0123} = +1$. Calculating the remaining trace does not pose too big a problem; it contains but a single power of the neutrino four-momentum k. The difference in decay rates $\Delta\Gamma$ can now be easily obtained by phase space integration. The latter factors into different tensorial contributions and we write it as

$$\Delta\Gamma = \frac{G_\mathrm{F}^2}{2\pi^5}\int\frac{\mathrm{d}^3p'}{2E'} I_{\alpha\beta}(q)\left[T_A^{\alpha\beta}(p') + T_B^{\alpha\beta}(p')\right]. \tag{5.11}$$

There is one tensorial expression for each of the CPT-violating terms given as

$$T_A^{\alpha\beta}(p') = p'\cdot A\, g^{\alpha\beta}\,, \tag{5.12}$$

$$T_B^{\alpha\beta}(p') = \epsilon_{\lambda\rho\mu\nu}\left(2B^{\lambda\alpha\rho}g^{\beta\mu}p'^\nu + (B^{\beta\mu\nu}p'^\rho - B^{\rho\mu\nu}p'^\beta)g^{\lambda\alpha}\right)\,, \tag{5.13}$$

and the integration over neutrino and antineutrino momenta is contained in

$$I_{\alpha\beta}(q) = \int\frac{\mathrm{d}^3k}{2k_0}\int\frac{\mathrm{d}^3k'}{2k'_0}\,\delta^4(q-k-k')k_\alpha k'_\beta\,, \tag{5.14}$$

in which $q = p - p'$. This neutrino phase space integral is well-known from the calculation of muon decay rates in the standard model [4]. Taking into account that the expression for the difference in spin-averaged matrix elements is already linear in the CPT-violating terms, we can use the usual expression [4] for the neutrino phase space

$$I_{\alpha\beta}(q) = \frac{\pi}{24}(q^2 g_{\alpha\beta} + 2q_\alpha q_\beta)\,. \tag{5.15}$$

This tensor is symmetric in its indices and we have already made use of this property to get rid of the antisymmetric parts in the tensorial expressions $T_A^{\alpha\beta}(p')$ and $T_B^{\alpha\beta}(p')$.

Given these prerequisites the rest of the calculations is straightforward. We arrive at the following form for the difference in muon and antimuon decay rates in their rest frame

$$\Delta\Gamma = \frac{G_\mathrm{F}^2 m_\mu^5}{192\pi^3}\left(A_0 - \varepsilon_{0ijk}B^{ijk}\right)\,. \tag{5.16}$$

Clearly, both the vector and the dipole part violate CPT invariance and thus trigger the difference in muon and antimuon lifetimes. Although the tensor $B^\lambda{}_{\alpha\beta}$ is antisymmetric in its last two indices only, its completely antisymmetric part contributes to the decay rate.

It is seen that for the case in which spacetime is endowed with characteristic odd-rank tensors, there is CPT violation. We can restrict the magnitude of the CPT-violating couplings, which we introduced, by making use of the known bounds on lifetime differences of the muon and antimuon. To the 1σ level [5] we have

$$\frac{\tau(\mu^+)}{\tau(\mu^-)} = 1.00002 \pm 0.00008, \tag{5.17}$$

which implies

$$A_0 < 10^{-4}\,, \qquad \varepsilon_{0ijk} B^{ijk} < 10^{-4}\,. \tag{5.18}$$

The experimental data for the lifetime of the tau also provide similar constraints. However, those constraints are less restrictive.

5.2 Differential decay rates

We can gain more insight when it comes to the CPT-violating parameters if we study the differential decay rate with respect to the energy of the charged particle in the final state of the process. For such purposes, we integrate out the angular variables in the difference in decay rates. We first consider the vector part. It is given by

$$\frac{\mathrm{d}\Delta\Gamma_\mathrm{A}}{\mathrm{d}x} = \frac{G_\mathrm{F}^2 m_\mu^5}{16\pi^3} x^2 \left(1-x\right) A_0, \tag{5.19}$$

with x a dimensionless energy variable defined via

$$x = \frac{2E'}{m_\mu}\,. \tag{5.20}$$

The distribution vanishes at the kinematic boundaries of $x = 0$ and $x = 1$. It attains a maximal value at $x_\mathrm{peak} = \frac{2}{3}$. Both these properties are independent of the explicit CPT-violating parameter A_0 and yet for $A_0 = 0$, i.e., in the absence of CPT violation, the energy dependence of the difference in muon and antimuon decay rates does not exist. Put another way, CPT-violating effects (here: a preferred direction) also shift the energy spectra of electrons and positrons emergent from muon and antimuon decays relative to one another. This difference is proportional to the time component of the preferred four-vector of spacetime. Irrespective of the value of A_0, the difference in spectra peaks at $x_\mathrm{peak} = \frac{2}{3}$ or equivalently $E'_\mathrm{peak} = \frac{m_\mu}{3}$ provided the only CPT-violating effects are coming from A^λ.

The differential decay rate for the dipole part can be obtained in a similar way. We find

$$\frac{\mathrm{d}\Delta\Gamma_\mathrm{B}}{\mathrm{d}x} = -\frac{G_\mathrm{F}^2 m_\mu^5}{48\pi^3} x^2 \left(1 - \frac{1}{3}x\right) \varepsilon_{0ijk} B^{ijk}. \tag{5.21}$$

The difference in energy distributions stemming from the dipole part does vanish at $x = 0$, but within the kinematic region it does neither vanish nor peak anywhere else.

The sum of both the vector part and the dipole part contributions leaves us with another observation. The difference in electron and positron energy spectra from muon and antimuon decay definitely vanishes at $x = 0$. It may also vanish at $x = \frac{9A_0 - 3\eta}{9A_0 - \eta}$ where $\eta = \varepsilon_{0ijk}B^{ijk}$ provided this value of x is within the kinematic region $0 < x < 1$. The difference is largest at $x_{\text{peak}} = \frac{6A_0 - \eta}{9A_0 - \eta}$, if this is within the kinematic region. Otherwise, it will be largest for $x = 1$. We eventually notice that the total decay rate cannot restrict in any way the spatial components of A^λ and the components of $B^{\lambda\alpha\beta}$ with any of the indices equal to the time component. We have, however, not yet fully exploited all information, which can be gained from the decay rates. It is also possible to integrate out the momentum variable p' and arrive at decay rates differential in the spatial angle $\mathrm{d}\Omega$.

Let us now state the result for the decay rate differential in the spatial angle first for the vector part. We find

$$\frac{\mathrm{d}\Delta\Gamma_{\mathrm{A}}}{\mathrm{d}\Omega} = \frac{G_{\mathrm{F}}^2 m_\mu^5}{768\pi^4}\left(A_0 - |\vec{A}|\cos\vartheta\right), \tag{5.22}$$

where ϑ is the angle between the electrons (positrons) emergent from the muon (antimuon) decays and the preferred direction $\vec{A}$. Put another way, not only does CPT violation enforce a slight difference in energy spectra for electrons and positrons, but it also alters their angular distributions with respect to one another. The angular dependence is proportional to the spatial components of A^λ. The direction and magnitude of $\vec{A}$ can then in principle be determined from the angular dependence. The angular dependence for the dipole part is found to intricately depend on both the azimuth as well as the zenith angle

$$\begin{aligned}\frac{\mathrm{d}\Delta\Gamma_{\mathrm{B}}}{\mathrm{d}\Omega} = & -\frac{G_{\mathrm{F}}^2 m_\mu^5}{192\pi^4}\left[\frac{5}{24}\varepsilon_{0ijk}B^{ijk} + \frac{5}{24}\varepsilon_{0ijk}B^{0jk}\hat{p}'^i\right.\\ & \left. - \frac{1}{8}\varepsilon_{i\kappa\lambda\rho}B^{\kappa\lambda\rho}\hat{p}'^i - \frac{1}{8}\varepsilon_{0ijk}B^{ljk}\hat{p}'_l\hat{p}'^i\right],\end{aligned} \tag{5.23}$$

where $\hat{p}'^i$ is a unit vector which can be written in spherical coordinates according to $\hat{p}'^i = (\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta)$. Two observations are made. The dipole part shows a rich angular dependence. Statements about the time components of $B^\lambda{}_{\alpha\beta}$ now become possible by analyzing the decay rate differential in the spatial angles.

Both the vector and the dipole part reveal interesting phenomenological consequences on their own. If both effects are to be considered simultaneously, one again simply adds the respective contributions.

6 Conclusion

Let us summarize the results of this thesis.

The first part of the thesis is presented in chapter 3. Our analysis contained in this chapter treats flavor oscillations in neutrino ensembles with a finite temperature by means of the coherence vector formalism. In situations with quantum ensembles with a finite temperature or systems in which the constituents of the ensemble are subject to scattering processes with a background plasma, the usual approach to flavor oscillations using particle wave functions fails. It is replaced by the coherence vector description. We began describing such systems by studying a two-flavor neutrino ensemble with generic matter potentials. We solved the underlying quantum kinetic equations perturbatively for adiabatic and nonadiabatic neutrino conversions leaving aside effects of ensemble decoherence for the time being.

Our approach for solving the quantum kinetic equations is model-independent and allows defining effective mixing angles and an adiabaticity parameter without explicitly solving for the coherence vector. The fact that our analysis is model-independent also entails as a consequence that the adiabatic and nonadiabatic perturbation theory are treated on the same ground; the two distinct physical situations do not have to be studied with two distinct perturbation theories.

Eventually, we illustrate how the ideas developed for a system unaffected by collisions can easily be generalized for systems of collision-affected neutrino conversions in the early Universe.

In the second part of the thesis in chapter 4, we developed a novel way of dealing with lepton-number violation in neutrino and antineutrino flavor oscillations.

We treat lepton-number violation as a small effect as compared to common lepton-number conserving flavor oscillations and develop a practical and efficient nonadiabatic perturbation theory, which gives the various oscillation probabilities up to second order in the perturbation. Assuming lepton-number violation in neutrino oscillations to be a small effect is justified by and compatible with experimental findings. We therefore give illustrative examples of how lepton-number violation affects neutrino oscillations for one and two neutrino generations.

We found that the one-generation case already allows for neutrino-antineutrino oscillations to develop. These oscillations can be resonant in a CP-nonconserving matter environment and therefore a potentially small lepton-number violation in vacuo can be enhanced.

For the exactly solvable two-generation case of time-independent and flavor-blind lepton-number violation in vacuo, we understand that in the presence of lepton-number nonconservation the common flavor oscillation probabilities are periodically modulated with a frequency given by the lepton-number violating coefficient.

Moreover, we have introduced a coherence vector description for the one-generation case of neutrino-antineutrino oscillations and perturbatively solved this system by means of the Magnus expansion. For such purposes, we have used the perturbation theory developed in chapter 3.

In the last part of this thesis, we were concerned with CPT violation in charged current weak interactions. CPT violation in muon and antimuon decays can manifest in many different ways. For instance the masses of muon and antimuon could be different, which would result in different decay rates.

The paradigm of the considerations in chapter 5, however, has been a different one. We assume that CPT violation occurs only through interactions and that the free part of the Lagrangian is left untouched by CPT-violating modifications. To this end, we enhance the standard model charged current weak interactions by additional nonstandard couplings. As spacetime is now endowed with additional characteristic odd-rank tensors, namely a vector and a dipole contribution, the decay rates of charged fermions are altered with respect to the standard model results. We calculated the decay rates to first order in these novel CPT-violating parameters and find that comparing decay rates for particles and antiparticles isolates effects of CPT violation. The difference in decay rates and the difference in decay rates differential in the energy of the final state charged fermion can constrain the time component of the vector part as well as the totally antisymmetric part of the dipole part. In order to obtain further information on the remaining components of the characteristic tensors, the angular-dependent differential decay rate can be considered.

Our approach is an effective one, though, and the newly introduced currents do not respect the gauge invariance of the standard model. We infer from our analysis that in principle the CPT-violating couplings can be determined by measuring the total as well as differential rates of the decay of both muon and antimuon.

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It may strike the reader as a curious thing or inappropriate even to acknowledge contributions to this thesis which neither stem from a collaboration with someone nor from an indirect influence on the author when it comes to his outlook on physics. It is, however, the author's wish to emphasize the seemingly naïve statement that it is precisely in those activities and social relationships, which are not connected to one's own work, that we find and regain the strength and stamina to venture on undertaking our journeys into the abstract relations governing inanimate nature.

Friends like these, huh, Gary? That's right, Dude [100]. It is true that you will seldom find a human being which you allow yourself to call an eminent friend. It takes a match in character, interests, outlook on life and maybe even similar experiences in the past to build a deep, personally enriching and meaningful friendship on. It is also a very pleasing and reassuring experience to find such mutual bonds unwavering in the face of necessary, but occasionally radical, changes in your friend's and your own character as well as outlook on life. At this point it is too ambitious an endeavor to attempt a description adequate to and worthy of my friendship with Jan Schröder. Suffice it to say that I love this man for all he is for me and that I am deeply grateful for his love and support throughout the course of our friendship.

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