Symbolic dynamics and scattering theory for localized magnetic fields

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Frank Schulz

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Contents

Introduction

Overview

The motion of a particle in a magnetic field has been object of research in many areas of mathematics, which include dynamical systems, mathematical physics, symplectic topology, differential geometry and mathematical billiards. Some examples of the discussed topics are the interpretation of the magnetic flow as the geodesic flow of a non-reversible Finsler metric [8, 46], the comparison of magnetic and Riemannian geodesic flows [39] and, most prominently, the question of the existence of closed orbits [9, 16, 30].

In this work, we shall focus on magnetic fields that vanish at infinity, which we summarize by the term “localized”. For the motion in these magnetic fields two types of orbits can occur: bounded and unbounded ones. The study of the unbounded orbits is the topic of scattering theory, while the bounded ones shall be examined by techniques of symbolic dynamics. In the following, we will give an introduction to these subjects and provide an outline of this work’s results.

The main goal in scattering theory is to study the asymptotic behaviour of the motion. This theory has its origin in physics, where Rutherford’s scattering experiments with a gold foil gave rise to a new atomic model. The solid mathematical foundations were laid from the 1950s onwards, e.g. in [10], and much effort was devoted to scattering by a potential in the classical as well as in the quantum mechanical setting [21, 24, 44]. Scattering of a single classical particle in a non-constant magnetic field has, as far as we know, only been treated by M. Loss and B. Thaller [33], who focused on the quantum mechanical case of scattering in $\mathbb{R}^d$ and considered the classical particle only for the special case $d = 3$. Consequently, further studies have only been conducted for quantum particles, see e.g. [34, 48]. The information about the asymptotic behaviour is contained in so-called “wave transformations”. This allows the study of the inverse problem, i.e. the reconstruction of the magnetic field from the asymptotic data, which is often called inverse scattering and has been treated by A. Jollivet for magnetic fields, e.g. in [26]. We will consider the motion of a classical particle and generalize the results obtained by M. Loss and B. Thaller to arbitrary dimensions $d \geq 2$. Furthermore, we will extend them in the sense that we obtain stronger results for the regularity of the wave transformations, even for the case $d = 3$. In addition, we consider scattering in a simplified time-dependent magnetic field, which, to our knowledge, has not been studied before.

Symbolic dynamics is used to analyze dynamical systems by discretizing the underlying space. The key idea is to assign symbols to certain subsets of the space and label all trajectories by the successive visits of these sets. Then, the state of the dynamical system is described by an infinite sequence of symbols and the evolution is given by a shift map. These techniques have first been used by J. Hadamard in 1898 for the analysis of geodesic flows on surfaces of negative curvature [20]. Symbolic dynamics received its first formal treatment as well as its name in 1938 by G. Hedlund and M. Morse [38]. Their study of abstract symbolic systems was not only motivated by pure mathematical interest in these systems, it was also necessary to be able to use symbolic techniques for the study of continuous systems. However, the formal notion of shift spaces was first introduced in the 1960s by S. Smale, who contributed the most notable advancements to this theory [45]. One of the most important results involving the use of symbolic dynamics is Sharkovsky's theorem about periodic points of continuous self-maps of an interval [42], but note that these techniques were not only used for the analysis of dynamical systems: The most prominent example is C. Shannon's use of methods from symbolic dynamics for the mathematical foundation of communication theory, which has provided the basis for information theory [41]. For a more detailed presentation of the history of symbolic dynamics we refer to S. Williams' article [47] as well as the book by D. Lind and B. Marcus [31].

Outline of the thesis

This work consists of three main sections.

First, in Chapter 1, we present preliminary results and gather necessary basics to perform the analysis. We start by reviewing the foundations of classical mechanics, which are Lagrange's and Hamilton's principles of motion and their connection by the Legendre transformation. Based on this, we introduce the notion of a magnetic field and, in particular, the notation we will use throughout this work. After this, we present a deep result from symplectic topology, the symplectic rigidity. Although we will need it only once, due to its importance and complexity we outline the proof instead of simply quoting the theorem. Finally, we introduce the concept of topological entropy as a way to measure the complexity of a dynamical system.

In Chapter 2 we examine the unbounded orbits by using ideas of scattering theory, where it is the aim to compare the magnetic flow to the free flow. We start by considering a time-dependent magnetic field whose strength decays in time, which turns out to be easier to handle than spatial decay and will provide a useful tool for the calculations in the time-independent case. We consider the asymptotic velocity and the asymptotic position of the motion separately and then combine them to define the wave transformations, whose regularity depends on the rate of decay of the magnetic field. After that, we turn to time-independent magnetic fields which decay at infinity. Our main tool for their analysis will be the construction of a time-dependent magnetic field whose decay in time corresponds to the spatial decay of the original magnetic field. Using this and following the same plan as before, we obtain similar results for the regularity of the wave transformations. Finally, we consider the magnetic motion on the cotangent bundle where we also construct wave transformations. They turn out to be symplectic under certain assumptions, which is useful for fixed point problems like the question if there is an initial value with the same asymptotic velocity as the initial velocity.

Chapter 3 is devoted to the study of bounded orbits and focuses on symbolic dynamics. In a magnetic field that consists of rotationally symmetric components, the question is if one can prescribe the itinerary of a trajectory, i.e. the order in which the components' supports are visited. To answer this, we start by examining the motion in a single rotationally symmetric magnetic field. For sufficiently low energies, it admits circular orbits for which we analyze whether they are hyperbolic or elliptic. Furthermore, we find an integral of motion and use it to study the motion of trajectories that stay outside the largest circular orbit. For a magnetic field consisting of several rotationally symmetric components we choose a Poincaré section in each support and consider the corresponding Poincaré (first return) map. Using the results obtained for single components we show that the Poincaré map is semi-conjugated to the shift map. In particular, it has positive topological entropy and is chaotic. Finally, we show that the integral of motion is not necessary: The same result holds if we drop the assumption of rotational symmetry of the magnetic field.

Note that in Chapter 1 we present known results and concepts, while the ones we shall obtain in Chapter 2 and Chapter 3 are new unless explicitly marked otherwise.

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CHAPTER 1

Preliminaries

In this chapter, we are going to present basic results that are needed for the following considerations. In Section 1.1 we will give a short overview over the principles of classical mechanics. This is the basis for Section 1.2, the main part of this chapter, where we shall give the definition of a magnetic field and the magnetic flow. Furthermore, we will derive essential properties of the magnetic flow and introduce the corresponding notation we will constantly use. Afterwards, in Section 1.3 and Section 1.4, we are going to present two concepts, symplectic rigidity and topological entropy, that are required at certain points. However, they are rarely needed, while the notion of a magnetic field is essential for this work.

1.1 Principles of classical mechanics

We want to describe the motion of a particle moving on a d-dimensional (smooth) Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$. This can be modelled on the tangent bundle TM, where $(q, v) \in TM$ stands for the particle being at position $q \in M$ and moving with velocity $v \in T_qM$. The manifold M is called the *configuration space* and the tangent bundle TM the *(velocity) phase space.*

One way of describing the motion is the Lagrangian formulation, which we will sketch in the following. A more detailed treatment can be found in [7, 15]. Motivated by physical observations, the Lagrangian formulation is based on the calculus of variations, and the whole information about the dynamics is encoded in a single function, the Lagrangian.

Definition 1.1.1 A *Lagrangian* $L\colon TM \to \mathbb{R}$ is a C^3-function such that the following two conditions hold:

(*i*) The Hessian

$$\frac{\partial^2 L}{\partial v^2} := \left(\frac{\partial^2 L}{\partial v_i \partial v_j} \right)_{i,j=1,\dots,d},$$

calculated in linear coordinates on each fibre T_qM, is positive definite for all points $(q,v) \in TM$, i.e. L is fibrewise strictly convex.

(*ii*) L has superlinear growth, i.e.

$$\lim_{|v|\to\infty} \frac{L(q,v)}{|v|} \to \infty$$

for every $q \in M$. □

Note that, since later in this work we shall only consider the case of $\mathbb{R}^d$, we will use local coordinates wherever possible, so one can think of $\langle \cdot, \cdot \rangle$ as the Euclidean inner product. In particular, we use the vector notation

$$\frac{\partial L}{\partial v} := \left(\frac{\partial L}{\partial v_1}, \dots, \frac{\partial L}{\partial v_d} \right),$$

and similarly for $\frac{\partial L}{\partial q}$.

Given a Lagrangian L, for a C^1-curve $\gamma\colon [t_1, t_2] \to M$ its *action functional* is defined as

$$\mathcal{A}_L(\gamma) := \int_{t_1}^{t_2} L(\gamma(s), \dot\gamma(s))\, ds\,.$$

Lagrange's principle states that the motion of a particle between two points $q_1, q_2 \in M$ is given by the curve $\gamma : [t_1, t_2] \to M$ that satisfies $\gamma(t_1) = q_1$, $\gamma(t_2) = q_2$ and minimizes the value of $\mathcal{A}_L$, which is why Lagrange's principle is also called the principle of least action. A curve γ is a minimum of $\mathcal{A}_L$ if and only if γ satisfies the so called *Euler-Lagrange equation*

$$\frac{\partial L}{\partial q}\Big(\gamma(t), \dot\gamma(t)\Big) = \frac{d}{dt}\frac{\partial L}{\partial v}\Big(\gamma(t), \dot\gamma(t)\Big),$$

which is exactly the case if

$$\frac{\partial L}{\partial q}\Big(\gamma(t), \dot\gamma(t)\Big) = \frac{\partial^2 L}{\partial q \partial v}\Big(\gamma(t), \dot\gamma(t)\Big)\dot\gamma(t) + \frac{\partial^2 L}{\partial v^2}\Big(\gamma(t), \dot\gamma(t)\Big)\ddot\gamma(t)$$

holds. Due to the non-degeneracy of $\frac{\partial^2 L}{\partial v^2}$, this equation can be solved for $\ddot\gamma$ and yields a vector field X_L on TM.

Definition 1.1.2 The vector field X_L defines a flow φ^t on the tangent bundle TM, which we call the *Euler-Lagrange flow* of L. □

Note that if two Lagrangians L_1 and L_2 satisfy $L_1(q,v) - L_2(q,v) = \alpha_q(v)$ for a closed 1-form α, then their action functionals differ only by an additive constant and therefore their minimizing curves coincide. Hence, the Euler-Lagrange flow of a Lagrangian L remains unchanged if we add a closed 1-form to L.

A different way to describe the motion of a particle is by using Hamilton's principle, which is linked to the Lagrangian formulation by the Legendre transformation. Again, we will give a brief introduction to these concepts and refer to [1] for a detailed treatment. The Hamiltonian formulation describes the motion on the cotangent bundle or *momentum phase space* T^*M. Together with the standard symplectic form $\omega_0 := d\lambda \in \Omega^2(M)$, where $\lambda \in \Omega^1(M)$ denotes the Liouville form, T^*M is a symplectic manifold (i.e. a manifold together with a closed, non-degenerate 2-form which is called symplectic form). Note that by $\Omega^k(M)$ we describe the set of k-forms on M. In local coordinates q, p for T^*M, the Liouville form λ satisfies

$$\lambda = \sum_{i=1}^{d} p_i dq_i$$

and the standard symplectic form equals

$$\omega_0 = \sum_{i=1}^{d} dp_i \wedge dq_i \,.$$

Definition 1.1.3 A function $H \in C^2(T^*M, \mathbb{R})$ is called a *Hamiltonian* and the corresponding vector field X_H given by

$$\omega_0(X_H, \cdot) = -dH$$

is said to be the *Hamiltonian vector field* of H. This vector field X_H induces the *Hamiltonian flow* φ_*^t of H on T^*M. □

The connection between the Lagrangian and the Hamiltonian formulation is specified by the Legendre transformation and the fibre derivative. The *Legendre transformation* of some Lagrangian $L\colon TM \to \mathbb{R}$ is defined as the Hamiltonian $H\colon T^*M \to \mathbb{R}$ determined in coordinates by

$$H(q,p) := \langle p, v\rangle - L(q,v)$$

with the *momentum* $p := \frac{\partial L}{\partial v}(q,v)$. Let us point out that this implicit definition is possible due to the convexity assumption on the Lagrangian L. The fibre derivative

$\Psi\colon TM \to T^*M$ of L, which is given by

$$\Psi(q,v) := \left(q, \frac{\partial L}{\partial v}(q,v)\right) ,$$

conjugates the flows φ^t and φ^t_* corresponding to L and H, respectively, i.e. the diagram

$$\begin{tikzcd} T^*M \arrow[r, "\varphi^t_*"] & T^*M \\ TM \arrow[u, "\Psi"] \arrow[r, "\varphi^t"'] & TM \arrow[u, "\Psi"'] \end{tikzcd}$$

commutes. Furthermore, the projections of the flows φ^t and φ^t_* to the configuration space M coincide, i.e. they describe the same motion in the configuration space.

1.2 Magnetic fields and the magnetic flow

We start by introducing the notion of a magnetic field and its magnetic flow. Although later we will only consider magnetic fields on $\mathbb{R}^d$, we give the definition in the general case of some manifold M. The defined flow models the motion of a charged particle on M under the influence of a magnetic field.

Definition 1.2.1 A *magnetic field* on a d-dimensional Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is given by an exact 2-form $\beta \in \Omega^2(M)$ with C^2-coefficients. Given some *magnetic potential* $\alpha \in \Omega^1(M)$ such that $d\alpha = \beta$, the *magnetic flow* of β on the tangent bundle TM is defined as the Euler-Lagrange flow with respect to the Lagrangian

$$L(q,v) = \frac{1}{2}|v|^2 + \alpha_q(v)\,.$$

□

As we have seen in the previous section, the flow is independent of the choice of α. In coordinates we associate a vector field $A := (A_1, \dots, A_d)$ to α by $\alpha_q = \sum_{i=1}^d A_i(q)dq_i$. Using the Legendre transformation, we get

$$p = \frac{\partial L}{\partial v}(q,v) = v + A(q)$$

and obtain the Hamiltonian

$$H(q,p) = \langle v + A(q), v\rangle - L(q,v) = \frac{1}{2}|v|^2 = \frac{1}{2}|p - A(q)|^2$$

on T^*M. Moreover, the magnetic Euler-Lagrange flow is conjugated to the Hamiltonian flow of H with respect to the standard symplectic form $\omega_0 = d\lambda$ on T^*M. This flow is equivalent to the Hamiltonian flow generated by

$$H(q,p) = \frac{1}{2}|p|^2$$

on T^*M with respect to the twisted symplectic form

$$\omega = \omega_0 + \pi^*\beta\,,$$

where $\pi\colon T^*M \to M$ denotes the canonical projection. In particular, this also shows that the flow is independent of the choice of α. Using this last formulation, the magnetic flow can be generalized to closed forms $\beta \in \Omega^2(M)$ which are not exact.

Since from now on we will consider magnetic fields on $\mathbb{R}^d$ and every closed 2-form on $\mathbb{R}^d$ is exact, we do not need the generalized definition but work with the Lagrangian formulation instead. Choosing $\alpha \in \Omega^1(\mathbb{R}^d)$ such that $d\alpha = \beta$ and using linear, global coordinates $q_1, \ldots, q_n$ for $\mathbb{R}^d$, we have the global representations

$$\beta = \sum_{\substack{i,j=1\\ i<j}}^{d} B_{ij}(q)dq_i \wedge dq_j$$

and

$$\alpha = \sum_{i=1}^{d} A_i(q)dq_i$$

with $B_{ij} \in C^2(\mathbb{R}^d, \mathbb{R})$ for $i < j \in \{1, \ldots, d\}$ as well as $A_i \in C^3(\mathbb{R}^d, \mathbb{R})$ for $i \in \{1, \ldots, d\}$. Hence, we obtain the relation

$$\begin{aligned} d\alpha &= \sum_{i=1}^{d} d(A_i dq_i) \\ &= \sum_{i=1}^{d} \left(\sum_{j=1}^{d} \frac{\partial A_i}{\partial q_j} dq_j \right) \wedge dq_i \\ &= \sum_{i,j=1}^{d} \frac{\partial A_i}{\partial q_j} dq_j \wedge dq_i \\ &= \sum_{\substack{i,j=1\\ i<j}}^{d} \left(\frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j} \right) dq_i \wedge dq_j\,, \end{aligned} \tag{1.1}$$

i.e.

$$B_{ij} = \frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j} \quad (i < j \in \{1, \ldots, d\})\,.$$

In canonical coordinates q, v for $T\mathbb{R}^d$ and with the vector field $A\colon \mathbb{R}^d \to T\mathbb{R}^d$ corresponding to α, the magnetic Lagrangian equals

$$L(q,v) = \frac{1}{2}|v|^2 + \langle A(q), v\rangle = \frac{1}{2}|v|^2 + \sum_{j=1}^{d} A_j(q)v_j\,,$$

where from now on $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product and $|\cdot|$ the Euclidean norm on $\mathbb{R}^d$ as well as the absolute value in $\mathbb{R}$. Then, a curve $q\colon (a,b) \to \mathbb{R}^d$ solves the Euler-Lagrange equation if and only if for all $i \in \{1,\ldots,d\}$ we have

$$\begin{aligned} 0 &= \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial v_i}\right)(q,\dot q) \\ &= \sum_{j=1}^{d} \frac{\partial A_j}{\partial q_i}\dot q_j - \frac{d}{dt}(\dot q_i + A_i(q)) \\ &= \sum_{j=1}^{d} \frac{\partial A_j}{\partial q_i}\dot q_j - \left(\ddot q_i + \sum_{j=1}^{d}\frac{\partial A_i}{\partial q_j}\dot q_j\right) \\ &= \left(\sum_{j=1}^{d} \frac{\partial A_j}{\partial q_i} - \sum_{j=1}^{d}\frac{\partial A_i}{\partial q_j}\right)\dot q_j - \ddot q_i \\ &= \sum_{j=1}^{d} B_{ij}(q)\dot q_j - \ddot q_i \end{aligned} \tag{1.2}$$

with $B_{ji} \coloneqq -B_{ij}$ for $j > i$ and $B_{ii} \coloneqq 0$. With the skew-symmetric matrix $B \coloneqq (B_{ij})_{i,j}$ this yields the differential equation

$$\ddot q = B(q)\dot q \tag{1.3}$$

or equivalently

$$\begin{cases} \dot q = v\,, \\ \dot v = B(q)v\,. \end{cases} \tag{1.4}$$

Therefore, on $\mathbb{R}^d$, we can generalize the definition of a magnetic field.

Definition 1.2.2 A *magnetic field* on $\mathbb{R}^d$ is a locally Lipschitz continuous map

$$B = (B_{ij})_{i,j=1,\ldots,d}\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$$

such that $B(q)$ is skew-symmetric for all $q \in \mathbb{R}^d$. Equation (1.4) defines the corresponding *magnetic flow*

$$\varphi^t = (q^t, v^t)\colon \mathbb{P} \to \mathbb{P}$$

on the phase space

$$\mathbb{P} \coloneqq T\mathbb{R}^d \cong \mathbb{R}^d_q \times \mathbb{R}^d_v\,.$$

□

Note that we do not assume the 1-form given by B to be closed anymore. However, since the skew-symmetry of the magnetic field is still required, we have $\langle v, B(q)v \rangle = 0$ for all $(q, v) \in \mathbb{P}$. This yields

$$\frac{d}{dt}\frac{1}{2}|v^t(x)|^2 = \langle v^t(x), B(q^t(x))v^t(x) \rangle \equiv 0$$

for all $x \in \mathbb{P}$ and therefore the *kinetic energy*

$$\begin{aligned} \mathcal{E} \colon \mathbb{P} &\to \mathbb{R}\,, \\ (q, v) &\mapsto \frac{1}{2}|v|^2 \end{aligned}$$

is constant along trajectories, i.e. $\mathcal{E}$ is an integral of motion. In particular, we have the estimate

$$|q^t(x)| \leq |q^0(x)| + \int_0^{|t|} |v^s(x)|\, ds \leq |q^0(x)| + \sqrt{2\mathcal{E}(x)}|t| \quad (x \in \mathbb{P},\ t \in \mathbb{R}) \tag{1.5}$$

and therefore the magnetic flow is complete, i.e. $\varphi \colon \mathbb{R} \times \mathbb{P} \to \mathbb{P}$. Furthermore, we can consider the energy surfaces

$$\mathbb{P}_E \coloneqq \mathcal{E}^{-1}(E)\,,$$

which are diffeomorphic to $\mathbb{R}^d \times S^{d-1}$ for $E > 0$. In analogy, for $I \subseteq [0, \infty)$ we define $\mathbb{P}_I \coloneqq \mathcal{E}^{-1}(I)$ as the set of points $x \in \mathbb{P}$ with energy $\mathcal{E}(x) \in I$. Finally, note that the Euclidean norm on $\mathbb{R}^d$ induces the canonical operator norm $\|\cdot\|$ on the space $\mathbb{R}^{d\times d}$ of matrices, which we shall use to measure magnetic fields.

Let us conclude the introduction of magnetic fields with an explanation of the definition:

Remark 1.2.3 In the context of physics one often defines a magnetic field on $\mathbb{R}^3$ as a (Lipschitz continuous) vector field

$$\vec{B} = (b_1, b_2, b_3) \colon \mathbb{R}^3 \to \mathbb{R}^3\,.$$

The motion of a particle with unit charge and unit mass is modelled by the differential equation

$$\ddot{q} = \dot{q} \times \vec{B}(q)\,,$$

where $\dot{q} \times \vec{B}(q)$ describes the Lorentz force influencing the particle, and $\times$ denotes the vector or cross product. There is a one-to-one correspondence between this setting and our definition: A straightforward computation shows that the flow given by the vector field (b_1, b_2, b_3) coincides with the flow given by the 3×3-matrix (B_{ij}) with the identification $b_1 = B_{23} = -B_{32}$, $b_2 = -B_{13} = B_{31}$ and $b_3 = B_{12} = -B_{21}$. □

1.3 Symplectic rigidity

After having introduced the basic notion of a magnetic field, we will proceed with the presentation of a central result from symplectic topology. For basic concepts and results of symplectic geometry and topology we refer to [36]. In Section 2.4 we shall consider a sequence of symplectic maps, and the natural question will be whether the limit is also symplectic. Since the condition of symplecticity involves the first derivative, the C^1-limit of symplectic maps is also symplectic, but surprisingly, the property to be symplectic also remains intact under limits in the C^0-topology. This result, which is often called *symplectic rigidity*, is due to Y. Eliashberg [13, 14] and M. Gromov [19].

Definition 1.3.1 For manifolds M_1, M_2 let $\mathrm{Diff}(M_1, M_2)$ denote the set of C^1-diffeomorphisms $f\colon M_1 \to M_2$. If (M_1, ω_1) and (M_2, ω_2) are symplectic manifolds, we call a diffeomorphism $f \in \mathrm{Diff}(M_1, M_2)$ with $f^*\omega_2 = \omega_1$ a *symplectic diffeomorphism* or *symplectomorphism.* The set of all symplectomorphisms from (M_1, ω_1) to (M_2, ω_2) is denoted by $\mathrm{Symp}(M_1, M_2; \omega_1, \omega_2)$. □

Theorem 1.3.2 (Y. Eliashberg, M. Gromov) *Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds. Furthermore, let*

$$f_n \in \mathrm{Symp}(M_1, M_2; \omega_1, \omega_2) \quad (n \in \mathbb{N})$$

be a sequence of symplectic C^1-diffeomorphisms and

$$f \in \mathrm{Diff}(M_1, M_2)$$

a C^1-diffeomorphism, such that f_n converges to f uniformly on compact subsets of M_1. Then f is symplectic, i.e. $f \in Symp(M_1, M_2; \omega_1, \omega_2)$.

Let us point out that we do not require the manifolds to be compact or the mappings to have compact support.
Note that the literature provides detailed proofs of the Eliashberg-Gromov theorem for the special case $(\mathbb{R}^{2d}, \widehat{\omega}_0)$ with the standard symplectic form

$$\widehat{\omega}_0 := \sum_{i=1}^{d} dx_i \wedge dy_i$$

on $\mathbb{R}^{2d}$, while for the general setting it is only mentioned that one obtains the result by using local coordinates (e.g. in [23] and [36]). We will conduct this argument later, but before that, we state the result for the special case $(\mathbb{R}^{2d}, \widehat{\omega}_0)$ (see Proposition 1.3.6) and sketch the corresponding proof as given in [23]. This particular proof relies on the existence of symplectic capacities of $\mathbb{R}^{2d}$, which are symplectic invariants other than the volume. Although there is a more general definition of symplectic capacities for any symplectic manifold, the following one will be sufficient for our purposes.

Definition 1.3.3 A *symplectic capacity* c of $\mathbb{R}^{2d}$ is a map $c\colon \{A \mid A \subseteq \mathbb{R}^{2d}\} \to [0,\infty]$ satisfying the following properties:

(*i*) Monotonicity: $c(A) \leq c(B)$ holds for any subsets $A, B \subseteq \mathbb{R}^{2d}$ such that there is a symplectic embedding $\Phi\colon A \to \mathbb{R}^{2d}$ with $\Phi(A) \subseteq B$. By a symplectic embedding of some arbitrary subset $A \subseteq \mathbb{R}^{2d}$ we mean that Φ can be extended to a symplectic embedding defined on some open set containing A.

(*ii*) Conformality: $c(\mu A) = \mu^2 c(A)$ holds for all $A \subseteq \mathbb{R}^{2d}$ and every $\mu \in \mathbb{R}$.

(*iii*) Non-triviality: $c(B(1)) = c(Z(1)) = \pi$, where

$$B(r) := \left\{(x,y) \in \mathbb{R}^{2d} \mid |x|^2 + |y|^2 < r^2\right\}$$

is the open ball of radius $r > 0$ and

$$Z(r) := \left\{(x,y) \in \mathbb{R}^{2d} \mid x_1^2 + y_1^2 < r^2\right\}$$

denotes the standard symplectic cylinder of radius $r > 0$ in $\mathbb{R}^{2d}$. □

Note that the non-triviality condition excludes the trivial choices where c is the symplectic volume or $c \equiv 0$. Let us assume the existence of a capacity. Then, as an immediate consequence of the axioms we obtain Gromov's non-squeezing theorem, which originally had motivated the concept of symplectic capacities.

Theorem 1.3.4 (Gromov's non-squeezing theorem) *There exists a symplectic embedding $B(r) \to Z(R)$ from the ball of radius $r > 0$ into the cylinder of radius $R > 0$ if and only if $r \leq R$.*

In fact, the existence of a symplectic capacity is a highly non-trivial result which we will not investigate further, but refer to [22]. A detailed study of symplectic capacities, including applications in the context of dynamical systems, can be found in [49].

We will sketch the proof of the Eliashberg-Gromov theorem for the special case of the standard symplectic space $(\mathbb{R}^{2d}, \widehat{\omega}_0)$ (Proposition 1.3.6) according to [23]. For this, we need the following technical lemma, whose proof can be found in [23] as well.

Lemma 1.3.5 *Let $A\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ be an isomorphism such that $A^*\widehat{\omega}_0 \neq \mu\widehat{\omega}_0$ applies for all $\mu \in \mathbb{R}$. Then, for any $a > 0$ there are symplectic matrices S and T such that*

$$S^{-1}AT = \begin{pmatrix} \begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix} & 0 \\ * & * \end{pmatrix}$$

holds with respect to the splitting $\mathbb{R}^{2d} = \mathbb{R}^2 \oplus \mathbb{R}^{2d-2}$ into symplectic subspaces.

This means that $S^{-1}AT$ maps the unit ball $B(1)$ into the cylinder $Z(a)$ of radius a, which now allows us to sketch the proof for symplectic rigidity in $\mathbb{R}^{2d}$ as given in [23].

Proposition 1.3.6 *Let*

$$\Phi_n\colon (B(r), \widehat{\omega}_0) \to (\mathbb{R}^{2d}, \widehat{\omega}_0) \quad (n \in \mathbb{N})$$

be a sequence of symplectic embeddings converging locally uniformly to a map

$$\Phi\colon B(r) \to \mathbb{R}^{2d}\,.$$

If Φ is differentiable at $x = 0$, then

$$A := D\Phi(0)\colon \mathbb{R}^{2d} \to \mathbb{R}^{2d}$$

is symplectic with respect to $\widehat{\omega}_0$.

Proof (sketch) The proof breaks down to the following three claims:

Claim A A is an isomorphism.

Claim B $A^*\omega_0 = \mu\omega_0$ for some $\mu \neq 0$.

Claim C $A^*\omega_0 = \mu\omega_0$ for $\mu \neq 0 \Rightarrow \mu = 1$.

The main difficulties are hidden in Claim B whose proof is based on the existence of a capacity, in particular on its consequence, the non-squeezing theorem. For the proof of this claim we will argue by contradiction and make use of Lemma 1.3.5 to show that we can map some ball into a smaller cylinder, which violates the non-squeezing theorem.

Proof of Claim A: We assume $\Phi(0) = 0$. First, note that the Lebesgue measure $\lambda = \lambda^{2d}$ on $\mathbb{R}^{2d}$ coincides with the symplectic measure given by $\frac{1}{d!}(\widehat{\omega}_0)^d$, i.e.

$$\lambda(A) = \frac{1}{d!}\int\limits_A (\widehat{\omega}_0)^d \quad (A \subseteq \mathbb{R}^{2d} \text{ open})\,.$$

The maps Φ_n are measure preserving and, since they converge locally uniformly to Φ, we obtain

$$\lambda(\Phi(B(\varepsilon))) = \lambda(B(\varepsilon))$$

for every $\varepsilon > 0$. On the other hand, we have

$$\frac{\lambda(\Phi(B(\varepsilon)))}{\lambda(B(\varepsilon))} \to |\det A| \quad (\varepsilon \to 0)\,,$$

which yields $|\det A| = 1$ and implies that A is an isomorphism.

Proof of Claim B: We now show that $A^*\omega_0 = \mu\omega_0$ for some $\mu \neq 0$ by using Lemma 1.3.5. If we assume that $A^*\omega_0 \neq \mu\omega_0$ holds for all $\mu \neq 0$, then for the constant $a = \frac{1}{8}$ we find symplectic matrices S, T such that

$$S^{-1}AT(B(r)) \subseteq Z\left(\tfrac{r}{8}\right)\,. \tag{1.6}$$

Defining $\psi_n \coloneqq S^{-1}\Phi_n T$, we obtain that $\psi_n \to \psi \coloneqq S^{-1}\Phi T$ converges locally uniformly and the derivative of the limit satisfies $D\psi(0) = S^{-1}AT$. Because of (1.6) and since $D\psi(0)$ approximates ψ around the origin, we have $\psi(B(\varepsilon)) \subseteq Z(\frac{\varepsilon}{4})$ for $\varepsilon > 0$ small enough. Consequently, the locally uniform convergence $\psi_n \to \psi$ implies that the relation

$$\psi_n(B(\varepsilon)) \subseteq Z\left(\tfrac{\varepsilon}{2}\right)$$

holds for sufficiently large values of $n \in \mathbb{N}$. Since the maps $\psi_n = S^{-1}\Phi_n T$ are symplectic, this contradicts Gromov's non-squeezing theorem and hence $A^*\omega_0 = \mu\omega_0$ holds for some $\mu \neq 0$.

Proof of Claim C: For $n \in \mathbb{N}$ we consider the symplectic embeddings

$$(\Phi_n, \mathrm{id})\colon (B(r) \times \mathbb{R}^{2d}, \widehat{\omega}_0 \oplus \widehat{\omega}_0) \to (\mathbb{R}^{2d} \times \mathbb{R}^{2d}, \widehat{\omega}_0 \oplus \widehat{\omega}_0)\,.$$

By applying the same arguments as above to this sequence, we obtain that the derivative $\bar{A} \coloneqq D(\Phi, \mathrm{id})(0,0)$ of the limit (Φ, id) at $(0,0)$ satisfies

$$\bar{A}^*(\widehat{\omega}_0 \oplus \widehat{\omega}_0) = \nu(\widehat{\omega}_0 \oplus \widehat{\omega}_0)$$

for some $\nu \neq 0$. On the other hand, since $\bar{A} = (A, \mathbb{1})$, we have

$$\bar{A}^*(\widehat{\omega}_0 \oplus \widehat{\omega}_0) = (\mu\widehat{\omega}_0) \oplus \widehat{\omega}_0$$

and therefore $\mu = 1$. ∎

By introducing local coordinates on the manifolds M_1 and M_2 one can transfer this result to arbitrary manifolds, as stated in Theorem 1.3.2.

Proof (of Theorem 1.3.2) For an arbitrary point $x \in M_1$ we have to show that the derivative $Df(x)\colon T_xM_1 \to T_{f(x)}M_2$ is symplectic. By Darboux's theorem, each symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2d}, \widehat{\omega}_0)$. Thus, around x and $f(x)$ there are Darboux charts, i.e. there are open sets U_1, U_2 around x and $f(x)$, respectively, and coordinates $\psi_i\colon U_i \to \mathbb{R}^{2d}$ such that $\psi_i^*\widehat{\omega}_0 = \omega_i$ on U_i for $i = 1, 2$. Without loss of generality we can assume that $\psi_1(x) = 0$. Furthermore, we can choose $r > 0$ such that U_2 is an open environment of

$$\overline{f(\psi_1^{-1}(B(r)))}\,.$$

Since $\overline{\psi_1^{-1}(B(r))}$ is compact and $f_n \to f$ converges uniformly on compact sets, there is $N \in \mathbb{N}$ such that

$$f_n(\psi_1^{-1}(B(r))) \subseteq U_2 \quad (n \geq N)\,.$$

Hence, for $n \geq N$ the maps $\Phi_n\colon B(r) \to \mathbb{R}^{2d}$ given by

$$\Phi_n \coloneqq \psi_2 \circ f_n \circ \psi_1^{-1}$$

are well defined, symplectic with respect to $\widehat{\omega}_0$ and converge locally uniformly to

$$\Phi \coloneqq \psi_2 \circ f \circ \psi_1^{-1}\,.$$

By Proposition 1.3.6 we obtain that $D\Phi(0)$ is symplectic, and therefore

$$Df(x)\colon T_xM_1 \to T_{f(x)}M_2$$

is symplectic. This holds for any $x \in M_1$ and therefore f is symplectic. ∎

1.4 Topological entropy

In this section, we will give a brief introduction to the concept of topological entropy, which can be thought of as a tool to measure how sensitive the motion depends on changes of the initial value. The notion of topological entropy was introduced by R. Adler, A. Konheim and M. McAndrew in 1965 for topological spaces [2], while in 1971 R. Bowen gave a different definition for metric spaces [4], which he proved to be equivalent to the previous one [5]. We shall use Bowen's definition, which is introduced in the following. For a thorough discussion we refer to [27] and [40].

Definition 1.4.1 Let $f\colon X \to X$ be a continuous map on a compact metric space (X, d). For $n \in \mathbb{N}$ we introduce a new metric d_f^n on X by

$$d_f^n(x,y) := \max\left\{d(f^i(x), f^i(y)) \mid 0 \le i < n\right\} \quad (x, y \in X)\,,$$

which measures the distance between the two orbit segments $\{x, \dots, f^{n-1}(x)\}$ and $\{y, \dots, f^{n-1}(y)\}$. Given $n \in \mathbb{N}$ and $\varepsilon > 0$, a subset $A \subseteq X$ is called (n, ε)-*separated for* f if $d_f^n(x,y) > \varepsilon$ holds for any $x, y \in A, x \neq y$. By

$$r(n, \varepsilon, f) := \max\left\{|A| \mid A \subseteq X \text{ is an } (n,\varepsilon)\text{-separated set for } f\right\}$$

we denote the largest cardinality of an (n, ε)-separated set for f. We consider the growth rate of r as n increases and define

$$h(\varepsilon, f) := \limsup_{n\to\infty} \frac{\log r(n, \varepsilon, f)}{n}\,.$$

Finally, the *topological entropy* $h_{\text{top}}(f)$ of f is defined by

$$h_{\text{top}}(f) := \lim_{\varepsilon \searrow 0} h(\varepsilon, f)\,.$$

□

Note that although the construction depends on the specific metric, the value of the topological entropy does not. Two metrics defining the same topology yield the same value of $h_{\text{top}}(f)$, which justifies the name "topological" entropy. According to [40], one can use the topological entropy to define chaos.

Definition 1.4.2 Let (f, X, d) be a *(discrete) dynamical system*, meaning that the map $f\colon X \to X$ is a homeomorphism on the metric space (X, d). If the topological entropy $h_{\text{top}}(f) > 0$ is strictly positive, the system (f, X, d) is said to be *chaotic*. □

As an important example of a chaotic dynamical system, we consider the *full shift on* N *symbols*, or simply the *shift*:
For $N \in \mathbb{N}$ let

$$\Sigma_N := \{1, \dots, N\}^{\mathbb{Z}}$$

denote the space of bi-infinite sequences in N symbols. On this space we define a metric $d\colon \Sigma_N \times \Sigma_N \to [0, \infty)$ by

$$d(x,y) := \max\left\{\frac{1}{2^{|j|}} \mid j \in \mathbb{Z}\colon x_j \neq y_j\right\}\,,$$

which describes the first index where $x, y \in \Sigma_N$ differ. In this metric, Σ_N is compact and the (left) shift map $\sigma\colon \Sigma_N \to \Sigma_N$ given by

$$\sigma((s_i)_{i\in\mathbb{Z}}) \coloneqq (s_{i+1})_{i\in\mathbb{Z}}\,,$$

shifting a sequence one position to the left, is a homeomorphism. For the dynamical system (σ, Σ_N, d) we will now compute the topological entropy. For this, we fix $\varepsilon > 0$ and let $k \in \mathbb{N}_0$ such that $2^{-k-1} \leq \varepsilon < 2^{-k}$. For $n \in \mathbb{N}$ we have $d_\sigma^n(x, y) > \varepsilon$ if and only if there exists an index $j \in \{-k, \ldots, k + n - 1\}$ such that $x_j \neq y_j$, which means that x and y have to differ at least at one of these $2k + n$ indices. Hence, we have

$$r(n, \varepsilon, \sigma) = N^{2k+n}$$

and obtain

$$h(\varepsilon, \sigma) = \limsup_{n\to\infty} \frac{(2k + n)\log N}{n} = \log N\,,$$

which implies

$$h_{\mathrm{top}}(\sigma) = \log N\,.$$

In many other examples, including the ones studied in this work, it is complicated to compute the topological entropy. Therefore, we need estimates for $h_{\mathrm{top}}(f)$ and $h_{\mathrm{top}}(g)$ depending on how the homeomorphisms f and g are related.

Definition 1.4.3 Let (f, X, d_X) and (g, Y, d_Y) be dynamical systems. Then f and g are *conjugated* if there is a homeomorphism $h\colon X \to Y$ such that $h \circ f = g \circ h$. If the map h is only continuous and surjective, then f is said to be *semi-conjugated* to g. □

Now we can formulate the following result, whose proof can be found in [27] and [40].

Proposition 1.4.4 *Let $f\colon X \to X$ and $g\colon Y \to Y$ be homeomorphisms of the compact metric spaces (X, d_X) and (Y, d_Y), respectively. If f and g are conjugated, then their topological entropies satisfy $h_{\mathrm{top}}(f) = h_{\mathrm{top}}(g)$. If f is only semi-conjugated to g, we still have the estimate $h_{\mathrm{top}}(f) \geq h_{\mathrm{top}}(g)$.*

In order to show that a dynamical system is chaotic, it is therefore sufficient to establish a semi-conjugacy to the shift.

CHAPTER 2

Scattering theory

In this chapter, we consider a magnetic field B on $\mathbb{R}^d$ that vanishes at infinity and study its influence on the motion of a charged particle, where we focus on the analysis of the asymptotic behaviour for the so-called scattering states, i.e. the unbounded trajectories. In particular, we compare the magnetic flow to the free flow $\varphi_0^t \colon \mathbb{P} \to \mathbb{P}$ given by

$$\varphi_0^t(q, v) \coloneqq (q + tv, v)$$

and derive conditions under which the limit

$$\lim_{t \to \infty} \varphi_0^{-t} \circ \varphi^t \tag{2.1}$$

exists. Under certain assumptions about the decay of the magnetic field the existence of this limit is equivalent to the fact that for any given x there is some y such that the magnetic trajectory of x and the free trajectory of y are asymptotic to each other for $t \to \infty$. Moreover, we will figure out that the limit (2.1) conjugates the free flow φ_0^t and the magnetic flow φ^t (restricted to the scattering states), and we shall study its regularity depending on how fast the magnetic field decays for $|q| \to \infty$. In particular, we do not consider a fixed rate of decay in this work, which is the reason for not giving a precise definition of the term "localized" magnetic field.

A useful tool for these examinations for a time-independent magnetic field vanishing at infinity will be the consideration of (a simplified model for) a time-dependent magnetic field, whose strength decays uniformly in time. Here, due to the time dependence, we cannot ask the question of conjugacy, but we can still check if the limit

$$\lim_{t\to\infty} \varphi_0^{-t} \circ \varphi^{t,0} \tag{2.2}$$

exists and study its properties. At first sight it might seem odd to consider the time-dependent case first, but the time decay is easier to handle than the spatial decay, and we will trace back the latter case to the first one. The reason behind this is that the decay of the magnetic field occurs in the same variable as the one the motion evolves in, and therefore we directly know estimates on the strength of the magnetic field along the trajectory, independent of the position. In contrast, if there is spatial decay, we first have to translate the time evolution of the trajectory into estimates on the position to be able to make use of the decay of the magnetic field. This will be done by constructing a time-dependent magnetic field whose decay in time corresponds to the spatial decay of the original magnetic field.

Most of the arguments in this section are based on B. Simon's approach for the examination of scattering in potentials [44], while the method of discussing time decay as a model for spatial decay was introduced and used by G. Graf [17] and J. Dereziński [11]. A comprehensive summary of the results and techniques for the case of potential scattering is given by J. Dereziński and C. Gérard [12], which inspired the consideration of scattering by magnetic fields in this chapter.

2.1 The virial radius

As mentioned in the introduction, we will examine bounded and unbounded orbits. Before we can start with the study of the asymptotic behaviour, we need to derive conditions when orbits are unbounded and we have to obtain estimates on how fast they escape to infinity. The key value for this is the virial radius which we will introduce later in this section. Before that, we shall start with the following definition of bounded and scattering subsets of the phase space which goes back to W. Hunziker [25]. At this point, recall that for a magnetic field $B\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ the magnetic flow

$$\varphi^t = (q^t, v^t)\colon \mathbb{P} \to \mathbb{P}$$

was given by the magnetic differential equation (1.3) on the phase space $\mathbb{P} = T\mathbb{R}^d$. The kinetic energy $\mathcal{E}(q,v) = \frac{1}{2}|v|^2$ is constant along the trajectories and, in particular, the energy surfaces $\mathbb{P}_E = \mathcal{E}^{-1}(E)$ are invariant with respect to φ^t.

Definition 2.1.1 By
$$b^\pm := \Big\{x \in \mathbb{P} \mid \sup_{\pm t \geq 0} |q^t(x)| < \infty\Big\}$$
we denote the set of points whose trajectories are bounded for $t \to \pm\infty$, and by
$$b := b^+ \cap b^-$$
we describe the set of *bounded states.* The set
$$s := s^+ \cap s^-$$
with
$$s^\pm := \mathbb{P} \setminus b^\pm = \Big\{x \in \mathbb{P} \mid \sup_{\pm t \geq 0} |q^t(x)| = \infty\Big\}$$
will be called the set of *scattering states.* For an energy $E \geq 0$ we denote the sets of bounded and scattering states with energy E by
$$b^\pm_E := b^\pm \cap \mathbb{P}_E$$
and
$$s^\pm_E := s^\pm \cap \mathbb{P}_E\,.$$
□

Note that we will often describe points in $b^\pm$ and $s^\pm$ as bounded and scattering states, if there are no ambiguities about the case $t \to \pm\infty$. We will see shortly that for those magnetic fields considered in this work the term scattering states is justified, i.e. that points $x \in s^\pm$ satisfy
$$\lim_{t\to\pm\infty} |q^t(x)| = \infty$$
(see Proposition 2.1.5). Before we turn to the result, we remark that is is sufficient to study the case $t \to +\infty$:

Remark 2.1.2 Note that the flow is not reversible, which means that in general the curve $t \mapsto \varphi^{-t}(x)$ is not a trajectory. This suggests that the case $t \to -\infty$ has to be treated separately. However, a straightforward computation shows that for any initial value $x = (q, v) \in \mathbb{P}$ the backward trajectory $t \mapsto q^{-t}(x)$ coincides with the forward trajectory of the flow induced by the magnetic field $-B(q)$ with respect to the initial value $(q, -v)$. Thus, it is sufficient to consider the limit $t \to +\infty$ and we shall give all proofs only for this case. □

We will study magnetic fields that satisfy at least the following condition (2.3). This means they decay faster than $\frac{1}{|q|}$ for $|q| \to \infty$.

Lemma 2.1.3 *Assume the magnetic field $B\colon \mathbb{R}^d \to \mathbb{R}^{d\times d}$ satisfies*

$$\int_0^\infty \sup_{|q|\geq r} \|B(q)\|\, dr < \infty\,. \tag{2.3}$$

Then we have the convergence

$$|q|\cdot\|B(q)\| \to 0 \quad (|q|\to\infty)\,.$$

Proof Suppose the assertion does not hold. Then there exist $\varepsilon > 0$ and a sequence $(q_n)_{n\in\mathbb{N}}$ with $|q_n|\to\infty$ such that $|q_n|\cdot\|B(q_n)\|\geq\varepsilon$. We can assume $|q_{n+1}|\geq 2|q_n|$ and together with the inequality

$$\sup_{|q|\geq|q_n|}\|B(q)\| \geq \frac{\varepsilon}{|q_n|}$$

we obtain

$$\begin{aligned}
\int_0^\infty \sup_{|q|\geq r}\|B(q)\|\,dr &\geq \sum_{n=1}^\infty \int_{|q_n|}^{|q_{n+1}|} \sup_{|q|\geq|q_{n+1}|}\|B(q)\|\,dr \\
&\geq \varepsilon\sum_{n=1}^\infty \frac{|q_{n+1}|-|q_n|}{|q_{n+1}|} \\
&\geq \varepsilon\sum_{n=1}^\infty \frac{1}{2}\,.
\end{aligned}$$

This contradicts the premise (2.3). ■

For the analysis of scattering states the following quantity is crucial.

Definition 2.1.4 For a magnetic field B satisfying condition (2.3), the value

$$R_{\mathrm{vir}}(E) := \max\left\{r\geq 0 \mid |q|\cdot\|B(q)\|\geq\sqrt{2E} \text{ for } |q| = r\right\}$$

is said to be the *virial radius* of B with respect to the energy E. If the set on the right hand side is empty for an energy $E>0$, we set $R_{\mathrm{vir}}(E) := 0$. □

The virial radius plays an important role for the dynamics: Outside the ball of radius $R_{\mathrm{vir}}(E)$ the magnetic field is too weak to capture orbits and prevent them from escaping to infinity. This is expressed by the following result which, in particular, justifies the term scattering states. A visualization of the assumptions is given in Figure 2.1.

Proposition 2.1.5 *Assume the magnetic field B satisfies condition* (2.3)*. Let $E > 0$ and $x_0 = (q_0, v_0) \in \mathbb{P}_E$ with $|q_0| > R_{\mathrm{vir}}(E)$ and $\langle q_0, v_0\rangle \geq 0$. Then there exists $\delta > 0$ such that*

$$|q^t(x_0)|^2 \geq |q_0|^2 + \delta t^2$$

holds for all $t \geq 0$. In particular, the set s^+ of scattering states is open and satisfies

$$s^+ = \left\{ x \in \mathbb{P} \mid \lim_{t\to\infty} |q^t(x)| = \infty \right\}.$$

For s^- we obtain the analogous result if we assume $\langle q_0, v_0\rangle \leq 0$ instead of $\langle q_0, v_0\rangle \geq 0$.

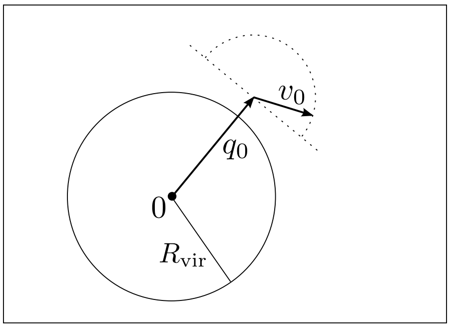


Figure 2.1: Visualization of the scattering condition

Proof For $|q| \geq |q_0| > R_{\mathrm{vir}}(E)$ the inequality $|q| \cdot \|B(q)\| < \sqrt{2E}$ holds, so we can define

$$\delta := E - \frac{\sqrt{E}}{\sqrt{2}} \max_{|q|\geq|q_0|} \Big(|q| \cdot \|B(q)\| \Big) > 0\,,$$

where the maximum exists due to the convergence $|q| \cdot \|B(q)\| \to 0$ for $|q| \to \infty$. In particular, we have the estimate

$$|q| \cdot \|B(q)\| \sqrt{2E} \leq 2E - 2\delta \quad (|q| \geq |q_0|)\,.$$

We set $(q(t), v(t)) := \varphi^t(x_0)$ and consider the function $f(t) := \frac{1}{2}|q(t)|^2$ with derivatives

$$f'(t) = \langle q(t), v(t)\rangle$$

and

$$f''(t) = 2E + \langle q(t), B(q(t))v(t)\rangle \geq 2E - |q(t)| \cdot \|B(q(t))\| \sqrt{2E}.$$

As long as $|q(t)| \geq |q_0|$, the second derivative satisfies the inequality

$$f''(t) \geq 2E - (2E - 2\delta) = 2\delta. \tag{2.4}$$

Since $f'(0) \geq 0$, this holds for any $t \geq 0$ and hence the claim follows. ∎

Remark 2.1.6 The radius R_{vir} is the best possible value for this result in the sense that the statement does not hold if one replaces the condition $|q_0| > R_{\mathrm{vir}}(E)$ by $|q_0| \geq R_{\mathrm{vir}}(E)$. For the case $d = 2$ we will see in Chapter 3 that there is a circular orbit of radius $r > 0$ around the origin if $|q| \cdot \|B(q)\| = \sqrt{2E}$ holds for all $|q| = r$. Hence, if for an energy $E > 0$ we have $|q| \cdot \|B(q)\| = \sqrt{2E}$ for all $|q| = R_{\mathrm{vir}}(E)$, then an initial value $x_0 = (q_0, v_0) \in \mathbb{P}_E$ with $|q_0| = R_{\mathrm{vir}}(E)$ and $\langle q_0, v_0 \rangle = 0$ (where v_0 points into the right direction) yields a circular orbit of radius $R_{\mathrm{vir}}(E)$. In particular, $|q^t(x_0)| = R_{\mathrm{vir}}(E)$ holds for all $t \in \mathbb{R}$ and therefore, the estimate on the escape rate does not apply. Moreover, x_0 is not even a scattering state. $\square$

From Proposition 2.1.5 we immediately get the following corollary which says that for sufficiently large energies there are only scattering orbits.

Corollary 2.1.7 *For any energy $E > \widetilde{E^\circ}$ with*

$$\widetilde{E^\circ} := \max_{q \in \mathbb{R}^d} \frac{(|q| \cdot \|B(q)\|)^2}{2}$$

we have $R_{\mathrm{vir}}(E) = 0$ and, in particular, the energy surface $\mathbb{P}_E$ consists only of scattering states, i.e. $\mathbb{P}_E = s_E$. Furthermore, as a function of the energy E, R_{vir} is strictly decreasing for energies $E \leq \widetilde{E^\circ}$.

Since $|q^t(x_0)| \leq |q_0| + |t||v_0|$ as in (1.5), the solution curve can escape to infinity at most at linear speed and Proposition 2.1.5 states that the rate is not less than linear. In fact, for compact sets of scattering states there is a uniform lower bound on the escape speed:

Lemma 2.1.8 *Let $K \subseteq s^+$ be compact. Then there exist constants $C, T > 0$ such that $|q^t(x)| \geq Ct$ holds for all $x \in K$ and $t \geq T$. The analogous result applies for s^-.*

Proof Let

$$E_{\min} := \min_{x \in K} \mathcal{E}(x)$$

denote the minimal energy of initial values in K. There is a time $T > 0$ and a radius $R > 0$ such that

$$|q^t(x)| \geq R > R_{\mathrm{vir}}(E_{\min}) \geq R_{\mathrm{vir}}(\mathcal{E}(x)) \quad (t \geq T)$$

and

$$\langle q^T(x), v^T(x) \rangle \geq 0$$

hold for all $x \in K$. From Proposition 2.1.5 we obtain the estimate

$$|q^t(x)|^2 \geq |q^T(x)|^2 + \delta(\mathcal{E}(x))(t - T)^2 \,,$$

where we can choose

$$\delta(E) = E - \frac{\sqrt{E}}{\sqrt{2}} \max_{|q|\geq R} \Big(|q| \cdot \|B(q)\| \Big) \, ,$$

as the proof showed. Due to

$$\delta'(E) = 1 - \frac{1}{2\sqrt{2E}} \max_{|q|\geq R} \Big(|q| \cdot \|B(q)\| \Big) > 1 - \frac{1}{2\sqrt{2E}} \sqrt{2E} = \frac{1}{2} \quad (E \geq E_{\min})$$

we have $\delta(E) \geq \delta(E_{\min})$ for $E \geq E_{\min}$ and with $C \coloneqq \sqrt{\delta(E_{\min})}$ we obtain

$$|q^t(x)|^2 \geq |q^T(x)|^2 + C^2(t-T)^2 \geq C^2(t-T)^2$$

for all $x \in K$ and $t \geq T$. Hence, the inequality

$$|q^t(x)| \geq \frac{C}{2} t$$

holds for all $t \geq 2T$. ■

In a later section we need to divide by the time t. In order to do this, we have to work around the singularity at $t = 0$ and therefore introduce the function $\langle t \rangle \coloneqq \sqrt{1+t^2}$ for $t \in \mathbb{R}$, which can be seen as a smooth modification of the absolute value.

Lemma 2.1.9 *The function $\langle t \rangle \coloneqq \sqrt{1+t^2}$ for $t \in \mathbb{R}$ satisfies the following properties:*

(i) $\displaystyle\lim_{t\to\infty} \frac{t}{\langle t \rangle} = 1$.

(ii) Given $T > 0$, there exists a constant $C > 0$ such that $t \geq C\langle t \rangle$ holds for $t \geq T$.

Proof The first assertion is obvious and the second one holds since $\frac{t}{\langle t \rangle}$ is bounded away from 0 for $t \geq T > 0$. ■

Using this function, we immediately obtain an analogue for the escape rate in Lemma 2.1.8:

Lemma 2.1.10 *Let $K \subseteq s^+$ be compact. Then there exist constants $C, T > 0$ such that $|q^t(x)| \geq C\langle t \rangle$ holds for all $x \in K$ and $t \geq T$. Again, the same applies for s^-.*

2.2 Time decay in a simplified time-dependent magnetic field

Now we turn to the analysis of the asymptotic behaviour of the motion, where we are interested in the limits (2.1) and (2.2), respectively. Before we analyze a time-independent magnetic field in Section 2.3, we start with a simplified version of the motion in a time-dependent magnetic field. In particular, we consider the equation

$$\ddot{q} = B(t,q)\dot{q} \tag{2.5}$$

or equivalently

$$\begin{cases} \dot{q} = v\,, \\ \dot{v} = B(t,q)v\,, \end{cases} \tag{2.6}$$

with a skew-symmetric matrix $B(t,q)$.

Definition 2.2.1 A *(time-dependent) magnetic field* is a continuous map

$$\begin{aligned} B\colon \mathbb{R}\times\mathbb{R}^d &\to \mathbb{R}^{d\times d} \\ (t,q) &\mapsto B(t,q) \end{aligned}$$

which is locally Lipschitz continuous with respect to q and satisfies $B(t,q)^T = -B(t,q)$ for each $(t,q)\in\mathbb{R}\times\mathbb{R}^d$. The *(time-dependent) magnetic flow* is induced by the time-dependent magnetic equation (2.6) and is denoted by

$$\varphi^{t,t_0}(x) = (q^{t,t_0}(x), v^{t,t_0}(x))\,,$$

which means that $\varphi^{t,t_0}(x_0)$ solves the differential equation (2.6) with initial time t_0 and initial value $\varphi^{t_0,t_0}(x_0) = x_0$. □

Note that this is a slightly different definition of the motion in a time-dependent magnetic field than the one we would obtain by the Lagrangian formulation. On the one hand it is simplified in the way described in the following remark, on the other hand it is more general since it does not require the magnetic field to define an exact 1-form.

Remark 2.2.2 For a given time-dependent magnetic field

$$\beta_t = \sum_{\substack{i,j=1 \\ i<j}}^{d} B_{ij}(t,q)dq_i \wedge dq_j \in \Omega^2(\mathbb{R}^d)$$

and a corresponding magnetic potential

$$\alpha_t = \sum_{i=1}^{d} A_i(t,q)dq_i \in \Omega^1(\mathbb{R}^d)\,,$$

the time-dependent magnetic Lagrangian $L\colon \mathbb{R}\times T\mathbb{R}^d \to \mathbb{R}$ is given by

$$L(t,q,v) = \frac{1}{2}|v|^2 + \langle A(t,q), v\rangle - \Phi(t,q)\,,$$

with a suitable function $\Phi\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$, as described below. The flow induced by this Lagrangian models the motion of a charged particle in a magnetic field when an effect called electromagnetic induction is taken into account, which is described by the Maxwell-Faraday equations. The function $\Phi\colon \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ satisfies

$$\frac{\partial \Phi}{\partial q_i} = E_i + \frac{\partial A_i}{\partial t} \quad (i = 1, \dots, d)$$

with the induced electric field

$$E(t,q)dq \coloneqq \sum_{i=1}^{d} E_i(t,q)dq_i$$

given by

$$-d(Edq) = \frac{\partial \beta_t}{\partial t}\,.$$

In fact, E is not uniquely determined by this condition. However, this is achieved by additional assumptions in the model, which we shall not investigate here. Uniqueness does not hold for Φ, though, but the corresponding motion will not depend on the choice. Note that this construction does not contradict the one in the time-independent case, since for a time-independent magnetic field β we can choose $\Phi \equiv 0$. Similar to (1.2), for the motion we obtain

$$\begin{aligned}
0 &= \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial v_i}\right)(t,q,\dot q) \\
&= \sum_{j=1}^{d} \frac{\partial A_j}{\partial q_i}\dot q_j - \frac{\partial \Phi}{\partial q_i} - \left(\ddot q_i + \frac{\partial A_i}{\partial t} + \sum_{j=1}^{d}\frac{\partial A_i}{\partial q_j}\dot q_j\right) \\
&= \left(\sum_{j=1}^{d} \frac{\partial A_j}{\partial q_i} - \sum_{j=1}^{d}\frac{\partial A_i}{\partial q_j}\right)\dot q_j - \frac{\partial \Phi}{\partial q_i} - \frac{\partial A_i}{\partial t} - \ddot q_i \\
&= \sum_{j=1}^{d} B_{ij}(t,q)\dot q_j - E_i(t,q) - \ddot q_i
\end{aligned}$$

and hence the differential equation

$$\ddot q = B(t,q)\dot q - E(t,q)\,. \tag{2.7}$$

□

For our purposes it is more suitable to consider the simplified time-dependent magnetic flow instead of the one described in the previous remark. Since the magnetic field is skew-symmetric, we still have $\langle v, B(t,q)v\rangle = 0$ for all $t \in \mathbb{R}$ and $(q,v) \in \mathbb{P}$. This yields

$$\frac{d}{dt}\frac{1}{2}|v^{t,t_0}(x)|^2 = \langle v^{t,t_0}(x), \frac{d}{dt}v^{t,t_0}(x)\rangle = \langle v^{t,t_0}(x), B(t,q^{t,t_0}(x))v^{t,t_0}(x)\rangle = 0$$

and hence the kinetic energy

$$\mathcal{E}\colon \mathbb{P} \to \mathbb{R},\ (q,v) \mapsto \frac{1}{2}|v|^2$$

is constant along trajectories. Note that this would not be the case for trajectories of the flow given by (2.7). We observe that the divergence of the vector field associated to the magnetic equation (2.6) vanishes and therefore, by Liouville's theorem, the generated flow preserves the volume. Finally, we claim that although the flow is also not reversible, it suffices to consider the asymptotic behaviour of the motion and the limit of $\varphi_0^{-t} \circ \varphi^{t,0}$ for $t \to +\infty$, similarly to the time-independent case described in Remark 2.1.2:

Remark 2.2.3 A straightforward computation shows that for any given initial value $x = (q,v) \in \mathbb{P}$, the backward trajectory $t \mapsto q^{-t,0}(x)$ coincides with the forward trajectory of the flow induced by the magnetic field $-B(-t,q)$ with respect to the initial value $(q,-v)$. Thus, it is sufficient to study the limit $t \to +\infty$ and we shall give all proofs only for this case. In particular, for magnetic fields decaying for $t \to -\infty$, we can consider the limit

$$\lim_{t\to-\infty} \varphi_0^{-t} \circ \varphi^{t,0}$$

and obtain analogous results as for $t \to +\infty$. □

2.2.1 Asymptotic velocity and asymptotic position

To describe the asymptotic behaviour of the motion we start by considering the velocity. We shall examine time-dependent magnetic fields that decay uniformly in time, faster than $\frac{1}{t}$ for $t \to \infty$.

Proposition 2.2.4 *Let the magnetic field satisfy*

$$\int_0^\infty \|B(t,\cdot)\|_\infty \, dt < \infty\,. \tag{2.8}$$

Then the following statements hold:

(i) For any $x \in \mathbb{P}$ the limit

$$v^+(x) := \lim_{t\to\infty} v^{t,0}(x)$$

exists and is called the asymptotic velocity.

(ii) The asymptotic velocity satisfies

$$v^+(x) = \lim_{t\to\infty} \frac{q^{t,0}(x)}{t}$$

for all $x \in \mathbb{P}$.

(iii) The mapping $v^+ \colon \mathbb{P} \to \mathbb{R}^d$ is continuous. Moreover, the limits in (i) and (ii) are uniform on the sets $\mathbb{P}_{[0,E]} = \mathcal{E}^{-1}([0,E])$ for any energy $E > 0$.

The analogous results hold for $v^- : \mathbb{P} \to \mathbb{R}^d$ given by

$$v^-(x) \coloneqq \lim_{t \to -\infty} v^{t,0}(x) \,.$$

Proof For fixed $x_0 = (q_0, v_0) \in \mathbb{P}$ we have

$$v^{t,0}(x_0) = v_0 + \int_0^t \frac{d}{ds} v^{s,0}(x_0)\, ds = v_0 + \int_0^t B(s, q^{s,0}(x_0)) v^{s,0}(x_0)\, ds \,.$$

With the kinetic energy $\mathcal{E}(x_0)$ of x_0 the inequality

$$\int_0^t |B(s, q^{s,0}(x_0)) v^{s,0}(x_0)|\, ds \leq \sqrt{2\mathcal{E}(x_0)} \int_0^t \|B(s,\cdot)\|_\infty\, ds$$

holds for all $t \in \mathbb{R}$, and because of (2.8) the limit on the left hand side exists for $t \to \infty$. Hence,

$$v^+(x_0) \coloneqq \lim_{t \to \infty} v^{t,0}(x_0) = v_0 + \int_0^\infty B(s, q^{s,0}(x_0)) v^{s,0}(x_0)\, ds$$

exists. Since for any given $E > 0$ and all $x \in \mathbb{P}_{[0,E]}$ we have the convergence

$$|v^+(x) - v^{t,0}(x)| = |\int_t^\infty B(s, q^{s,0}(x)) v^{s,0}(x)\, ds| \leq \sqrt{2E} \int_t^\infty \|B(s,\cdot)\|_\infty\, ds \to 0 \tag{2.9}$$

for $t \to \infty$, v^+ is the uniform limit of continuous functions on $\mathbb{P}_{[0,E]}$ and hence it is continuous on the interior $\mathbb{P}_{[0,E)}$. This implies that v^+ is continuous on $\mathbb{P} = \cup_{E>0} \mathbb{P}_{[0,E)}$.

To show assertion *(ii)* we also fix $x_0 = (q_0, v_0) \in \mathbb{P}$ and set $(q(t), v(t)) \coloneqq \varphi^{t,0}(x_0)$. We have

$$\frac{d}{ds}\Big(q(s) - s v^+(x_0)\Big) = v(s) - v^+(x_0) = -\int_s^\infty B(u, q(u)) v(u)\, du \tag{2.10}$$

and, using

$$\frac{d}{ds} \int_s^\infty B(u, q(u)) v(u)\, du = -B(s, q(s)) v(s) \,,$$

we can compute the integral $\int_0^t ds$ of (2.10) through integration by parts. This yields

$$\begin{aligned} q(t) - tv^+(x_0) &= q_0 - \int_0^t 1 \cdot \left(\int_s^\infty B(u, q(u))v(u)\, du \right) ds \\ &= q_0 - \left(s \int_s^\infty B(u, q(u))v(u)\, du \Big|_0^t - \int_0^t s(-B(s, q(s))v(s))\, ds \right) \qquad (2.11) \\ &= q_0 - t \int_t^\infty B(u, q(u))v(u)\, du - \int_0^t uB(u, q(u))v(u)\, du \end{aligned}$$

and we obtain the equation

$$\frac{q(t)}{t} - v^+(x_0) = \frac{q_0}{t} - \int_t^\infty B(u, q(u))v(u)\, du - \int_0^t \frac{u}{t} B(u, q(u))v(u)\, du\,.$$

We now show that this expression converges to zero uniformly on $\mathbb{P}_{[0,E]}$. With $\mathbb{1}_{[0,t]}$ denoting the characteristic function of the interval $[0, t]$ we have the pointwise convergence

$$\frac{u}{t} B(u, q(u))v(u)\mathbb{1}_{[0,t]}(u) \to 0 \quad (t \to \infty)$$

and due to

$$|\frac{u}{t} B(u, q(u))v(u)\mathbb{1}_{[0,t]}(u)| \leq \sqrt{2\mathcal{E}(x_0)} \|B(u, \cdot)\|_\infty \quad (u \geq 0)\,,$$

there is an integrable majorizing function. Thus, Lebesgue's dominated convergence theorem yields

$$\int_0^t \frac{u}{t} B(u, q(u))v(u)\, du \to 0 \quad (t \to \infty)\,.$$

Furthermore, we have

$$|\int_t^\infty B(u, q(u))v(u)\, du| \leq \sqrt{2\mathcal{E}(x_0)} \int_t^\infty \|B(u, \cdot)\|_\infty\, du \to 0 \quad (t \to \infty)\,,$$

which finally implies that

$$\frac{q^t(x)}{t} - v^+(x) \to 0 \quad (t \to \infty)$$

converges uniformly on $\mathbb{P}_{[0,E]}$ and hence completes the proof. ■

After the velocity we now study the position component $q^{t,0}(x) - tv^{t,0}(x)$ of the term $\varphi_0^{-t} \circ \varphi^{t,0}$ in (2.2). For this we need a faster decay than for the asymptotic velocity.

Proposition 2.2.5 *Let the magnetic field satisfy*

$$\int_0^\infty t\|B(t,\cdot)\|_\infty \, dt < \infty \, . \tag{2.12}$$

Then the following statements hold:

(i) For any $x \in \mathbb{P}$ the limit

$$q^+(x) \coloneqq \lim_{t\to\infty} \Big(q^{t,0}(x) - tv^+(x) \Big)$$

exists and is called the asymptotic position*.*

(ii) The asymptotic position satisfies the equation

$$q^+(x) = \lim_{t\to\infty} \Big(q^{t,0}(x) - tv^{t,0}(x) \Big) \tag{2.13}$$

for all $x \in \mathbb{P}$.

(iii) The mapping $q^+ : \mathbb{P} \to \mathbb{R}^d$ is continuous. Moreover, both limits are uniform on the subsets $\mathbb{P}_{[0,E]} \subseteq \mathbb{P}$.

The analogous results hold for $q^- : \mathbb{P} \to \mathbb{R}^d$ given by

$$q^-(x) \coloneqq \lim_{t\to-\infty} \Big(q^{t,0}(x) - tv^-(x) \Big) \, .$$

Proof We fix $x_0 = (q_0, v_0) \in \mathbb{P}$ and make use of the previously obtained identity (2.11) for $q^{t,0}(x_0) - tv^+(x_0)$, namely

$$q^{t,0}(x_0) - tv^+(x_0) = q_0 - t \int_t^\infty B(s, q^{s,0}(x_0))v^{s,0}(x_0)\, ds - \int_0^t sB(s, q^{s,0}(x_0))v^{s,0}(x_0)\, ds \, .$$

The integral $\int_0^t sB(s, q^{s,0}(x_0))\, ds$ converges for $t \to \infty$ by Lebesgue's dominated convergence theorem. Since furthermore the estimate

$$|t \int_t^\infty B(s, q^{s,0}(x_0))v^{s,0}(x_0)\, ds| \le \sqrt{2\mathcal{E}(x_0)} \int_t^\infty s\|B(s,\cdot)\|_\infty \, ds \to 0 \quad (t \to \infty)$$

holds, we obtain the existence of the asymptotic position $q^+(x_0)$. The limit is uniform in x_0 on all subsets of $\mathbb{P}$ with bounded energy and hence, q^+ is continuous on $\mathbb{P}$. Similarly to (2.9) we have $t(v^+(x_0) - v^{t,0}(x_0)) \to 0$ for $t \to \infty$ and hence

$$q^{t,0}(x_0) - tv^{t,0}(x_0) = q^{t,0}(x_0) - tv^+(x_0) + t(v^+(x_0) - v^{t,0}(x_0)) \to q^+(x_0) \quad (t \to \infty)$$

converges uniformly on $\mathbb{P}_{[0,E]}$. ∎

2.2.2 Wave transformations

Asymptotic velocity and asymptotic position together allow the following definition.

Definition 2.2.6 For a magnetic field satisfying condition (2.12), namely

$$\int_0^\infty t\|B(t,\cdot)\|_\infty \, dt < \infty \,,$$

the mappings

$$\begin{aligned} \Omega^\pm \colon \mathbb{P} &\to \mathbb{P} \\ x &\mapsto (q^\pm(x), v^\pm(x)) \end{aligned}$$

are well defined. These are called the *(velocity) wave transformations* and, assuming (2.12), they coincide with the limit (2.2), i.e.

$$\Omega^\pm = \lim_{t\to\pm\infty} \varphi_0^{-t} \circ \varphi^{t,0} \,.$$

□

The name wave transformation (sometimes also called wave operator or Møller operator) shows that scattering theory has its roots in quantum mechanics, where the state of a system is described by a so-called wave function. It has been adopted to classical mechanics, for example in [12, 44], and we shall follow this terminology.

If for once we drop any assumptions on the magnetic field, then we can still define the wave transformations $\Omega^\pm := (q^\pm, v^\pm)$ as formal limits. The following proposition shows that the existence of the limit $\Omega^+(x) = y$ is equivalent to the fact that there is a point $y \in \mathbb{P}$ such that the free trajectory of y is asymptotic to the magnetic trajectory of $x \in \mathbb{P}$. Note that points $x \in \mathbb{P}_0$ with zero energy are fixed points and $\Omega^\pm|_{\mathbb{P}_0} = \mathrm{id}$ holds, but for those points $x \in \mathbb{P}_{(0,\infty)}$ with positive energy such that the limits $\Omega^\pm(x)$ exist, this implies that the magnetic trajectories are asymptotically straight lines.

Proposition 2.2.7 *Let B be any time-dependent magnetic field. Then, for $x \in \mathbb{P}$ the following statements are equivalent:*

(i) $\Omega^+(x) = (q^+(x), v^+(x)) = y \in \mathbb{P}$ *(i.e. the limit exists and equals y).*

(ii) $\varphi^{t,0}(x) - \varphi_0^t(y) \to 0$ *for $t \to \infty$.*

The analogous result holds for Ω^-.

Proof Let $x \in \mathbb{P}$. Then for a point $y = (q_\infty, v_\infty)$ we have

$$
\begin{aligned}
\Omega^+(x) = y &\iff \lim_{t\to\infty} v^{t,0}(x) = v_\infty \quad \text{and} \quad \lim_{t\to\infty} \left(q^{t,0}(x) - tv_\infty\right) = q_\infty \\
&\iff \lim_{t\to\infty} \left(q^{t,0}(x) - tv_\infty - q_\infty, v^{t,0}(x) - v_\infty\right) = 0 \\
&\iff \lim_{t\to\infty} \left(q^{t,0}(x), v^{t,0}(x)\right) - \left(q_\infty + tv_\infty, v_\infty\right) = 0 \\
&\iff \lim_{t\to\infty} \varphi^{t,0}(x) - \varphi_0^t(y) = 0\,. \qquad \blacksquare
\end{aligned}
$$

Remark 2.2.8 We can relate the incoming asymptotic straight line to the outgoing one: The wave transformations give rise to the *scattering transformation*

$$\mathcal{S} := \Omega^+ \circ (\Omega^-)^{-1} \colon \mathbb{P} \to \mathbb{P}\,,$$

which is relevant when considering inverse scattering. In the following, we will focus on the analysis of the wave transformations $\Omega^\pm$, whose properties transfer to $\mathcal{S}$. $\square$

From now on we shall assume that the magnetic field satisfies at least condition (2.12) on the decay. Then the previous results on asymptotic velocity and asymptotic position imply that the wave transformations $\Omega^\pm$ exist and are continuous mappings. The remainder of this section is devoted to the study of their regularity. Naturally, we begin with the examination if they are bijections and, moreover, homeomorphisms. For this, we start with a technical lemma which relates the asymptotic values to an integral equation for the trajectory.

Lemma 2.2.9 *Let $(q_\infty, v_\infty) \in \mathbb{P}$ and let $\gamma \in C^1([T,\infty), \mathbb{R}^d)$ for some $T > 0$. Furthermore, let the magnetic field satisfy*

$$\int_0^\infty t\|B(t,\cdot)\|_\infty \, dt < \infty\,.$$

Then the following statements are equivalent:

(i) γ is a solution of the magnetic equation (2.5) for $t \geq T$ with

$$\Omega^+(\varphi^{0,T}(\gamma(T), \dot\gamma(T))) = (q_\infty, v_\infty)\,.$$

(ii) γ satisfies the integral equation

$$\gamma(t) = q_\infty + tv_\infty + \int_t^\infty (s-t) B(s, \gamma(s)) \dot\gamma(s)\, ds \tag{2.14}$$

for $t \geq T$ and $\dot\gamma$ is bounded.

The analogous result holds for Ω^-.

Proof "*(i) ⇒ (ii)*": For $t, u \geq T$ we have

$$\begin{aligned}\gamma(t) &= \gamma(u) - \int_t^u \dot\gamma(s)\,ds \\ &= \gamma(u) - \int_t^u \left(\dot\gamma(u) - \int_s^u \ddot\gamma(\tau)\,d\tau\right)\,ds \\ &= \gamma(u) - (u-t)\dot\gamma(u) + \int_t^u \int_s^u B(\tau,\gamma(\tau))\dot\gamma(\tau)\,d\tau\,ds\,.\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}&\int_t^u 1\cdot\left(\int_s^u B(\tau,\gamma(\tau))\dot\gamma(\tau)\,d\tau\right)\,ds \\ &\quad= s\int_s^u B(\tau,\gamma(\tau))\dot\gamma(\tau)\,d\tau\bigg|_t^u - \int_t^u s(-B(s,\gamma(s))\dot\gamma(s))\,ds \\ &\quad= -t\int_t^u B(\tau,\gamma(\tau))\dot\gamma(\tau)\,d\tau + \int_t^u sB(s,\gamma(s))\dot\gamma(s)\,ds \\ &\quad= \int_t^u (s-t)B(s,\gamma(s))\dot\gamma(s)\,ds\end{aligned}$$

and hence we obtain

$$\gamma(t) = \gamma(u) - u\dot\gamma(u) + t\dot\gamma(u) + \int_t^u (s-t)B(s,\gamma(s))\dot\gamma(s)\,ds\,.$$

Furthermore, since γ is a solution of the magnetic equation (2.5), we have

$$\gamma(t) = q^{t,T}(\gamma(T),\dot\gamma(T)) = q^{t,0}(\varphi^{0,T}(\gamma(T),\dot\gamma(T)))$$

and similarly

$$\dot\gamma(t) = v^{t,0}(\varphi^{0,T}(\gamma(T),\dot\gamma(T)))$$

for $t \geq T$. Together with the assumption on the decay of the magnetic field, this implies

$$\dot\gamma(u) \to v_\infty \quad (u\to\infty)$$

and, using property (2.13) of the asymptotic position, also the convergence

$$\gamma(u) - u\dot\gamma(u) \to q_\infty \quad (u\to\infty)\,.$$

Therefore we obtain

$$\gamma(t) = q_\infty + tv_\infty + \int_t^\infty (s-t)B(s,\gamma(s))\dot\gamma(s)\,ds\,,$$

i.e. γ satisfies the integral equation. Since $(\gamma,\dot\gamma)$ solves the magnetic equation (2.5), $|\dot\gamma(t)|$ is constant and, in particular, $\dot\gamma$ is bounded.

"*(ii)* $\Rightarrow$ *(i)*": Let $\gamma\colon [T,\infty) \to \mathbb{R}^d$ satisfy the integral equation. For the derivatives of the integral with respect to t we have

$$\begin{aligned}
&\frac{d}{dt}\int_t^\infty (s-t)B(s,\gamma(s))\dot\gamma(s)\,ds \\
&\qquad = -tB(t,\gamma(t))\dot\gamma(t) - \left(\int_t^\infty B(s,\gamma(s))\dot\gamma(s)\,ds - tB(t,\gamma(t))\dot\gamma(t)\right) \qquad (2.15)\\
&\qquad = -\int_t^\infty B(s,\gamma(s))\dot\gamma(s)\,ds
\end{aligned}$$

and

$$\frac{d^2}{dt^2}\int_t^\infty (s-t)B(s,\gamma(s))\dot\gamma(s)\,ds = B(t,\gamma(t))\dot\gamma(t)\,.$$

Therefore, γ is a solution of the magnetic equation (2.5) and satisfies

$$\dot\gamma(t) = v_\infty - \int_t^\infty B(s,\gamma(s))\dot\gamma(s)\,ds \to v_\infty \quad (t\to\infty)$$

as well as

$$\gamma(t) - tv_\infty = q_\infty + \int_t^\infty (s-t)B(s,\gamma(s))\dot\gamma(s)\,ds \to q_\infty \quad (t\to\infty)\,.$$

Hence, for

$$x := \varphi^{0,T}(\gamma(T),\dot\gamma(T))$$

we have $\varphi^{t,0}(x) = (\gamma(t),\dot\gamma(t))$ for $t \ge T$ and therefore

$$\Omega^+(x) = (q_\infty, v_\infty)\,. \qquad \blacksquare$$

In order to show that Ω^+ and Ω^- are bijections, we will use this lemma to construct an operator with a unique fixed point. To prove the continuity of $(\Omega^\pm)^{-1}$, we need the following lemma that states when solutions of a fixed point equation depend continuously on an additional parameter. A version of this result with similar assumptions is given in Theorem 4.9.2 in [32].

Lemma 2.2.10 *Let (X, d) be a complete metric space and Y a topological space. Furthermore, let $I \colon Y \times X \to X$ satisfy the following two properties:*

(i) There is some $\lambda < 1$ such that the inequality $d(I(y, x_1), I(y, x_2)) \leq \lambda d(x_1, x_2)$ holds for all $y \in Y$ and all $x_1, x_2 \in X$.

(ii) For any $x \in X$ the map $Y \to X,\ y \mapsto I(y, x)$ is continuous.

For any $y \in Y$ let $x_\infty(y) \in X$ denote the unique solution of $x = I(y, x)$ as given by the Banach fixed point theorem. Then the mapping $Y \to X,\ y \mapsto x_\infty(y)$ is continuous.

Proof For $y_0 \in Y$ we have

$$\begin{aligned} d(x_\infty(y), x_\infty(y_0)) &= d(I(y, x_\infty(y)), I(y_0, x_\infty(y_0))) \\ &\leq d(I(y, x_\infty(y)), I(y, x_\infty(y_0))) + d(I(y, x_\infty(y_0)), I(y_0, x_\infty(y_0))) \\ &\leq \lambda d(x_\infty(y), x_\infty(y_0)) + d(I(y, x_\infty(y_0)), I(y_0, x_\infty(y_0)))\,. \end{aligned}$$

This implies

$$d(x_\infty(y), x_\infty(y_0)) \leq \frac{1}{1-\lambda} d(I(y, x_\infty(y_0)), I(y_0, x_\infty(y_0)))$$

and therefore the continuity of the mapping $y \mapsto I(y, x_\infty(y_0))$ yields the continuity of $y \mapsto x_\infty(y)$ in y_0. ∎

Finally, this allows us to show when Ω^+ and Ω^- are homeomorphisms.

Theorem 2.2.11 *Let the magnetic field satisfy*

$$\int_0^\infty t \|B(t, \cdot)\|_\infty \, dt < \infty$$

and assume there is a continuous function $\ell \colon [0, \infty) \to [0, \infty)$ with

$$\|B(t, q_1) - B(t, q_2)\| \leq \ell(t) |q_1 - q_2| \quad (q_1, q_2 \in \mathbb{R}^d,\ t \geq 0)\,.$$

(i) If ℓ satisfies the condition

$$\int_0^\infty t\ell(t)\,dt < \infty\,,$$

then Ω^+ is a bijection.

(ii) If ℓ decays faster, such that

$$\int_0^\infty t^2\ell(t)\,dt < \infty$$

holds, then Ω^+ is a homeomorphism.

The analogous results hold for Ω^-.

Proof *(i)* We show that for any fixed energy $E > 0$ the wave transformation Ω^+ maps the set $\mathbb{P}_{[0,E]} = \{(q,v) \in \mathbb{P} \mid |v| \leq \sqrt{2E}\}$ bijectively to itself. Since $|v^+(x_0)| = |v_0|$, we already know the relation $\Omega^+(\mathbb{P}_{[0,E]}) \subseteq \mathbb{P}_{[0,E]}$ and it remains to show that $\Omega^+|_{\mathbb{P}_{[0,E]}}$ attains every value $(q_\infty, v_\infty) \in \mathbb{P}_{[0,E]}$. This is achieved by using Lemma 2.2.9 and a contracting operator that depends on the parameters q_∞, v_∞ and whose unique fixed point yields the desired solution curve asymptotic to $q_\infty + tv_\infty$. For this, we note that γ satisfies

$$\gamma(t) = \int_t^\infty (s-t)B(s, \gamma(s) + q_\infty + sv_\infty)(\dot\gamma(s) + v_\infty)\,ds \tag{2.16}$$

if and only if the mapping $t \mapsto \gamma(t) + q_\infty + tv_\infty$ satisfies the integral equation (2.14). We will show that the operator given by the right hand side of (2.16) is a contraction on some complete metric space. For any $E > 0$ this will be a function space of curves γ, with the only difference being the domain of definition.

We fix an energy $E > 0$ and some $\lambda < \frac{1}{2}$. Furthermore, let $T = T(E) > 0$ satisfy

$$\int_T^\infty t\|B(t,\cdot)\|_\infty\,dt \leq \frac{\lambda}{1+\sqrt{2E}} \tag{2.17}$$

as well as

$$\int_T^\infty t\ell(t)\,dt \leq \frac{\lambda}{1+\sqrt{2E}}\,. \tag{2.18}$$

We define

$$X = X(E) \coloneqq \left\{\gamma \in C^1([T(E),\infty), \mathbb{R}^d) \mid |\gamma(t)| + |\dot\gamma(t)| \to 0 \text{ for } t \to \infty\right\}$$

and

$$X_1 \coloneqq \Big\{\gamma \in X \mid \|\dot\gamma\|_\infty \leq 1\Big\}.$$

Then X, together with the C^1-norm $\|\gamma\|_{C^1} \coloneqq \|\gamma\|_\infty + \|\dot\gamma\|_\infty$, is a closed subspace of the Banach space $(C_b^1([T,\infty),\mathbb{R}^d), \|\cdot\|_{C^1})$ of bounded C^1-functions with bounded derivative, and hence itself a Banach space. As a closed subset, $X_1 \subseteq X$ is a complete metric space. For the desired asymptotic values $(q_\infty, v_\infty) \in \mathbb{P}_{[0,E]}$ and for $\gamma \in X$ we define $\mathcal{I}(\gamma) = \mathcal{I}_E(q_\infty, v_\infty, \gamma)\colon [T,\infty) \to \mathbb{R}^d$ by

$$\mathcal{I}(\gamma)(t) \coloneqq \int_t^\infty (s-t) B(s, \gamma(s) + q_\infty + s v_\infty)(\dot\gamma(s) + v_\infty)\, ds\,. \tag{2.19}$$

We claim that $\mathcal{I}$ maps X_1 into itself and is a contraction. We first show $\mathcal{I}(X_1) \subseteq X_1$ and for this, let $\gamma \in X_1$. Similar to (2.15) we obtain

$$\frac{d}{dt}\mathcal{I}(\gamma)(t) = -\int_t^\infty B(s, \gamma(s) + q_\infty + s v_\infty)(\dot\gamma(s) + v_\infty)\, ds \quad (t \geq T) \tag{2.20}$$

and due to $\|\dot\gamma\|_\infty \leq 1$ we have

$$\begin{aligned}|\mathcal{I}(\gamma)(t)| + |\frac{d}{dt}\mathcal{I}(\gamma)(t)| &\leq (1+\sqrt{2E})\int_t^\infty s\|B(s,\cdot)\|_\infty\, ds + (1+\sqrt{2E})\int_t^\infty \|B(s,\cdot)\|_\infty\, ds \\ &\to 0 \quad (t \to \infty)\,.\end{aligned}$$

Furthermore, the inequality

$$\|\frac{d}{dt}\mathcal{I}(\gamma)\|_\infty \leq (1+\sqrt{2E})\int_T^\infty \|B(s,\cdot)\|_\infty\, ds \leq \lambda < 1\,, \tag{2.21}$$

applies, which yields, together with the previous computation, that $\mathcal{I}(\gamma) \in X_1$ holds.

It remains to show the contracting property of $\mathcal{I}$ on X_1. For this, let $\gamma_1, \gamma_2 \in X_1$. Then, for $s \in [T, \infty)$ we have

$$\begin{aligned}
&|B(s, \gamma_1(s) + q_\infty + sv_\infty)(\dot\gamma_1(s) + v_\infty) - B(s, \gamma_2(s) + q_\infty + sv_\infty)(\dot\gamma_2(s) + v_\infty)| \\
&\quad \le |B(s, \gamma_1(s) + q_\infty + sv_\infty) - B(s, \gamma_2(s) + q_\infty + sv_\infty)| \cdot |v_\infty| \\
&\qquad + |B(s, \gamma_1(s) + q_\infty + sv_\infty)\dot\gamma_1(s) - B(s, \gamma_2(s) + q_\infty + sv_\infty)\dot\gamma_2(s)| \\
&\quad \le \ell(s)|\gamma_1(s) - \gamma_2(s)| \cdot |v_\infty| \\
&\qquad + |B(s, \gamma_1(s) + q_\infty + sv_\infty)\dot\gamma_1(s) - B(s, \gamma_2(s) + q_\infty + sv_\infty)\dot\gamma_1(s)| \\
&\qquad + |B(s, \gamma_2(s) + q_\infty + sv_\infty)\dot\gamma_1(s) - B(s, \gamma_2(s) + q_\infty + sv_\infty)\dot\gamma_2(s)| \\
&\quad \le \ell(s)|\gamma_1(s) - \gamma_2(s)|\sqrt{2E} \\
&\qquad + |B(s, \gamma_1(s) + q_\infty + sv_\infty) - B(s, \gamma_2(s) + q_\infty + sv_\infty)| \\
&\qquad + |B(s, \gamma_2(s) + q_\infty + sv_\infty)| \cdot |\dot\gamma_1(s) - \dot\gamma_2(s)| \\
&\quad \le (1 + \sqrt{2E})\ell(s)\|\gamma_1 - \gamma_2\|_\infty + \|B(s, \cdot)\|_\infty \|\dot\gamma_1 - \dot\gamma_2\|_\infty .
\end{aligned} \tag{2.22}$$

Together with the inequalities (2.17) and (2.18) this implies

$$\begin{aligned}
|\Big(\mathcal{I}(\gamma_1) - \mathcal{I}(\gamma_2)\Big)(t)| &\le \int_t^\infty (s - t)|B(s, \gamma_1(s) + q_\infty + sv_\infty)(\dot\gamma_1(s) + v_\infty) \\
&\qquad - B(s, \gamma_2(s) + q_\infty + sv_\infty)(\dot\gamma_2(s) + v_\infty)|\, ds \\
&\le (1 + \sqrt{2E}) \int_T^\infty s\ell(s)\, ds\, \|\gamma_1 - \gamma_2\|_\infty \\
&\qquad + (1 + \sqrt{2E}) \int_T^\infty s\|B(s, \cdot)\|_\infty\, ds\, \|\dot\gamma_1 - \dot\gamma_2\|_\infty \\
&\le \lambda \|\gamma_1 - \gamma_2\|_{C^1} .
\end{aligned}$$

Similarly, by using (2.20) and (2.22) again, we have

$$\begin{aligned}
|\frac{d}{dt}\Big(\mathcal{I}(\gamma_1) - \mathcal{I}(\gamma_2)\Big)(t)| &\le \int_t^\infty |B(s, \gamma_1(s) + q_\infty + sv_\infty)(\dot\gamma_1(s) + v_\infty) \\
&\qquad - B(s, \gamma_2(s) + q_\infty + sv_\infty)(\dot\gamma_2(s) + v_\infty)|\, ds \\
&\le (1 + \sqrt{2E}) \int_T^\infty \ell(s)\, ds\, \|\gamma_1 - \gamma_2\|_\infty \\
&\qquad + (1 + \sqrt{2E}) \int_T^\infty \|B(s, \cdot)\|_\infty\, ds\, \|\dot\gamma_1 - \dot\gamma_2\|_\infty \\
&\le \lambda \|\gamma_1 - \gamma_2\|_{C^1} .
\end{aligned}$$

Therefore, the inequality

$$\|\mathcal{I}(\gamma_1) - \mathcal{I}(\gamma_2)\|_{C^1} \leq 2\lambda \|\gamma_1 - \gamma_2\|_{C^1} \tag{2.23}$$

holds with $2\lambda < 1$, independent of $\gamma_1, \gamma_2 \in X_1$. Hence, restricting $\mathcal{I}$ to X_1 we obtain that $\mathcal{I}\colon X_1 \to X_1$ is a contraction. Therefore, there exists exactly one curve

$$\gamma_\infty = \gamma_\infty(q_\infty, v_\infty) \in X_1 \subseteq C_b^1([T,\infty), \mathbb{R}^d)$$

with $\mathcal{I}(q_\infty, v_\infty, \gamma_\infty) = \gamma_\infty$, and Lemma 2.2.9 yields that

$$q(t) := \gamma_\infty(t) + q_\infty + t v_\infty \quad (t \geq T)$$

solves the magnetic differential equation (2.5) on $[T, \infty)$. This curve can be extended to a solution $q\colon \mathbb{R} \to \mathbb{R}^d$ which satisfies $\Omega^+(q(0), \dot{q}(0)) = (q_\infty, v_\infty)$. Since q as well as the extension are uniquely determined by q_∞ and v_∞, the wave transformation Ω^+ maps $\mathbb{P}_{[0,E]}$ bijectively to itself for any $E > 0$. Thus, the inverse $(\Omega^+)^{-1}$ exists.

(ii) To show the continuity of $(\Omega^+)^{-1}$ we need a more formal description of the previously mentioned extension process. With the evaluation functional

$$\delta_T\colon C_b^1([T,\infty), \mathbb{R}^d) \to \mathbb{P}, \;\; \delta_T(\gamma) = (\gamma(T), \dot{\gamma}(T))$$

the result of Lemma 2.2.9 yields

$$(q_\infty, v_\infty) = \Omega^+(\varphi^{0,T}(\delta_T(\gamma_\infty(q_\infty, v_\infty)) + (q_\infty + T v_\infty, v_\infty)))\,.$$

Hence, on $\mathbb{P}_{[0,E]}$ the map $(\Omega^+)^{-1}$ is given by

$$(\Omega^+)^{-1}(q_\infty, v_\infty) = \varphi^{0,T}(\delta_T(\gamma_\infty(q_\infty, v_\infty)) + (q_\infty + T v_\infty, v_\infty))\,. \tag{2.24}$$

The evaluation functional is continuous and therefore it remains to show that the unique fixed point $\gamma_\infty(q_\infty, v_\infty)$ of $\mathcal{I}(q_\infty, v_\infty, \cdot)$ depends continuously on the parameters q_∞ and v_∞. To obtain this we will apply Lemma 2.2.10. Since the inequality (2.23) holds for all points $(q_\infty, v_\infty) \in \mathbb{P}_{[0,E]}$, it suffices to verify that the mapping $(q_\infty, v_\infty) \mapsto \mathcal{I}(q_\infty, v_\infty, \gamma)$ is continuous for fixed $\gamma \in X_1$, which is achieved by using a similar estimate as (2.22) for the contraction property. For any points (q_1, v_1), $(q_2, v_2) \in \mathbb{P}_{[0,E]}$ and all $s \in [T, \infty)$

we have

$$
\begin{aligned}
&|B(s,\gamma(s)+q_1+sv_1)(\dot\gamma(s)+v_1)-B(s,\gamma(s)+q_2+sv_2)(\dot\gamma(s)+v_2)| \\
&\qquad \le |B(s,\gamma(s)+q_1+sv_1)-B(s,\gamma(s)+q_2+sv_2)|\cdot|\dot\gamma(s)| \\
&\qquad\qquad + |B(s,\gamma(s)+q_1+sv_1)v_1-B(s,\gamma(s)+q_2+sv_2)v_2| \\
&\qquad \le |B(s,\gamma(s)+q_1+sv_1)-B(s,\gamma(s)+q_2+sv_2)| \\
&\qquad\qquad + |B(s,\gamma(s)+q_1+sv_1)v_1-B(s,\gamma(s)+q_2+sv_2)v_1| \\
&\qquad\qquad + |B(s,\gamma(s)+q_2+sv_2)v_1-B(s,\gamma(s)+q_2+sv_2)v_2| \\
&\qquad \le \ell(s)|q_1+sv_1-q_2-sv_2| \\
&\qquad\qquad + |B(s,\gamma(s)+q_1+sv_1)-B(s,\gamma(s)+q_2+sv_2)|\cdot|v_1| \\
&\qquad\qquad + |B(s,\gamma(s)+q_2+sv_2)|\cdot|v_1-v_2| \\
&\qquad \le \ell(s)(1+|v_1|)|q_1+sv_1-q_2-sv_2| + \|B(s,\cdot)\|_\infty|v_1-v_2| \\
&\qquad \le \ell(s)(1+\sqrt{2E})|q_1-q_2| + s\ell(s)(1+\sqrt{2E})|v_1-v_2| + \|B(s,\cdot)\|_\infty|v_1-v_2|\,.
\end{aligned}
$$

Hence, since the integrals

$$\int_0^\infty s^2\ell(s)\,ds<\infty$$

and

$$\int_0^\infty s\|B(s,\cdot)\|_\infty\,ds<\infty$$

exist by assumption, both the inequalities

$$
\begin{aligned}
&|\Big(\mathcal{I}(q_1,v_1,\gamma)-\mathcal{I}(q_2,v_2,\gamma)\Big)(t)| \\
&\qquad \le \int_t^\infty (s-t)|B(s,\gamma(s)+q_1+sv_1)(\dot\gamma(s)+v_1) \\
&\qquad\qquad - B(s,\gamma(s)+q_2+sv_2)(\dot\gamma(s)+v_2)|\,ds \\
&\qquad \le \int_T^\infty s\ell(s)\,ds\,(1+\sqrt{2E})|q_1-q_2| \\
&\qquad\qquad + \int_T^\infty s^2\ell(s)\,ds\,(1+\sqrt{2E})|v_1-v_2| \\
&\qquad\qquad + \int_T^\infty s\|B(s,\cdot)\|_\infty\,ds\,|v_1-v_2| \\
&\qquad \le \mathrm{const}(|q_1-q_2|+|v_1-v_2|)
\end{aligned}
$$

and

$$
\begin{aligned}
&|\frac{d}{dt}\Big(\mathcal{I}(q_1,v_1,\gamma)-\mathcal{I}(q_2,v_2,\gamma)\Big)(t)| \\
&\quad\leq \int\limits_T^\infty \ell(s)\,ds\,(1+\sqrt{2E})|q_1-q_2| \\
&\qquad + \int\limits_T^\infty s\ell(s)\,ds\,(1+\sqrt{2E})|v_1-v_2| \\
&\qquad + \int\limits_T^\infty \|B(s,\cdot)\|_\infty\,ds\,|v_1-v_2| \\
&\quad\leq \mathrm{const}(|q_1-q_2|+|v_1-v_2|)
\end{aligned}
$$

hold for all $t \geq T$. Consequently, this yields

$$
\|\mathcal{I}(q_1,v_1,\gamma)-\mathcal{I}(q_2,v_2,\gamma)\|_{C^1} \leq \mathrm{const}(|q_1-q_2|+|v_1-v_2|)\,,
$$

which implies the continuity of $\mathcal{I}$ with respect to q_∞ and v_∞. By using Lemma 2.2.10 we obtain the continuity of the mapping $(q_\infty, v_\infty) \mapsto \gamma_\infty(q_\infty, v_\infty)$, which yields the continuity of $(\Omega^+)^{-1}$ on $\mathbb{P}_{[0,E)}$. Since this applies for all $E > 0$, the inverse $(\Omega^+)^{-1}$ is continuous and, thus, Ω^+ is a homeomorphism. ■

As Ω^+ and Ω^- are the locally uniform limits of the volume preserving maps $\varphi_0^{-t} \circ \varphi^{t,0}$, we expect that this property also holds for Ω^+ and Ω^-. Before we are able to show that this is indeed the case, we need the following condition for a homeomorphism to be volume preserving, which is a consequence of the transformation formula for integrals. In fact, this is not a new result – for example B. Simon also used it to show the volume preserving property when studying potential scattering [44]. Due to the lack of a reference to a precise statement, though, we give the proof ourselves.

Lemma 2.2.12 *For any homeomorphism $\Phi\colon \mathbb{R}^n \to \mathbb{R}^n$ the following statements are equivalent:*

(i) *Φ is volume preserving, i.e. $\Phi(\lambda) = \lambda$, with the Lebesgue measure $\lambda = \lambda^n$ and the pushforward measure $\Phi(\lambda)(A) := \lambda(\Phi^{-1}(A))$.*

(ii) *The equality*

$$
\int\limits_{\mathbb{R}^n} f \circ \Phi \, d\lambda = \int\limits_{\mathbb{R}^n} f \, d\lambda
$$

holds for all smooth and compactly supported functions $f \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$.

Proof "*(i) ⇒ (ii)*": By the transformation formula for integrals, the equality

$$\int_{\mathbb{R}^n} f \circ \Phi \, d\lambda = \int_{\mathbb{R}^n} f \, d\Phi(\lambda) = \int_{\mathbb{R}^n} f \, d\lambda$$

holds for any measurable function $f \colon \mathbb{R}^n \to \mathbb{R}^n$.

"*(ii) ⇒ (i)*": We show this statement by contradiction and assume $\Phi(\lambda) \neq \lambda$. Then there exists a cuboid $A \subseteq \mathbb{R}^n$ such that

$$\Phi(\lambda)(A) \neq \lambda(A) \, .$$

For A, there is an increasing sequence of non-negative, smooth functions $f_k \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$ such that $f_k \to \mathbb{1}_A$ converges pointwise to the characteristic function $\mathbb{1}_A$ of A. By assumption, we have

$$\int_{\mathbb{R}^n} f_k \circ \Phi \, d\lambda = \int_{\mathbb{R}^n} f_k \, d\lambda$$

for all $k \in \mathbb{N}$, and the monotone convergence theorem yields

$$\int_{\mathbb{R}^n} f_k \, d\lambda \to \int_{\mathbb{R}^n} \mathbb{1}_A \, d\lambda = \lambda(A)$$

as well as

$$\int_{\mathbb{R}^n} f_k \circ \Phi \, d\lambda \to \int_{\mathbb{R}^n} \mathbb{1}_A \circ \Phi \, d\lambda = \int_{\mathbb{R}^n} \mathbb{1}_{\Phi^{-1}(A)} \, d\lambda = \lambda(\Phi^{-1}(A)) = \Phi(\lambda)(A) \, .$$

Since the limit is unique, this contradicts the assumption. ∎

Finally, this allows us to show that the wave transformations $\Omega^\pm$ are volume preserving.

Proposition 2.2.13 *Let the magnetic field satisfy the assumptions of Theorem 2.2.11, such that the wave transformations $\Omega^\pm \colon \mathbb{P} \to \mathbb{P}$ are homeomorphisms. Then the maps $\Omega^\pm$ are volume preserving.*

Proof Let $f \in C_c^\infty(\mathbb{R}^{2d}, \mathbb{R})$. We claim that

$$f \circ \varphi_0^{-t} \circ \varphi^{t,0} \to f \circ \Omega^+ \tag{2.25}$$

converges uniformly on $\mathbb{P} \cong \mathbb{R}^{2d}$. Due to the compact support of f there is some $E_0 > 0$ such that $\operatorname{supp} f \subseteq \mathbb{P}_{[0,E_0]}$. Since f is uniformly continuous and $\varphi_0^{-t} \circ \varphi^{t,0} \to \Omega^+$ converges uniformly on $\mathbb{P}_{[0,E_0]}$, the convergence in (2.25) is uniform on $\mathbb{P}_{[0,E_0]}$. The energy surfaces

$\mathbb{P}_E$ are invariant under the flow and therefore we have $f \circ \varphi_0^{-t} \circ \varphi^{t,0}(x) = f \circ \Omega^+(x) = 0$ for any $x \notin \mathbb{P}_{[0,E_0]}$. Thus, the convergence in (2.25) is uniform on $\mathbb{P}$ and we obtain

$$\int_{\mathbb{P}} f \circ \Omega^+ \, d\lambda = \lim_{t\to\infty} \int_{\mathbb{P}} f \circ \varphi_0^{-t} \circ \varphi^{t,0} \, d\lambda \,. \tag{2.26}$$

Now we can make use of Lemma 2.2.12: The maps $\varphi_0^{-t} \circ \varphi^{t,0}$ are volume preserving and therefore

$$\int_{\mathbb{P}} f \circ \varphi_0^{-t} \circ \varphi^{t,0} \, d\lambda = \int_{\mathbb{P}} f \, d\lambda$$

holds for all $t \in \mathbb{R}$. According to (2.26) this implies

$$\int_{\mathbb{P}} f \circ \Omega^+ \, d\lambda = \int_{\mathbb{P}} f \, d\lambda$$

and, by applying Lemma 2.2.12 again, we obtain that Ω^+ is volume preserving. ∎

This shows that we do not require any additional assumptions on the rate of decay for the homeomorphisms $\Omega^\pm$ to be volume preserving. The following considerations will be devoted to the question whether a higher rate of decay yields differentiability of the wave transformations $\Omega^\pm$. One way of approaching this problem would be to try to show convergence in the C^1-topology. Instead, we proceed similarly to the proof of the continuity of $(\Omega^\pm)^{-1}$ and start with an analogous result to Lemma 2.2.10, which states when a fixed point of a parametrized equation depends differentiably on the additional parameters. For a similar case, this result can be found in Theorem 4.9.4 in [32].

Lemma 2.2.14 *Let V, W be Banach spaces and $A \subseteq V$, $B \subseteq W$ be open. Furthermore, let $I \colon A \times B \to W$ be a C^1-map and $\lambda < 1$ such that*

$$\|I(v, w_1) - I(v, w_2)\| \leq \lambda \|w_1 - w_2\| \quad (v \in V, \;\; w_1, w_2 \in W) \,.$$

Furthermore, assume that for each $v \in A$ the contraction $w \mapsto I(v, w)$ has a fixed point in B. Then the map $v \mapsto w_\infty(v)$ denoting this (unique) fixed point of $I(v, \cdot)$ is a C^1-map.

Proof We apply the implicit function theorem. Note that $w = w_\infty(v)$ holds if and only if $\widehat{I}(v, w) := I(v, w) - w = 0$. The map $\widehat{I}$ is continuously differentiable and its derivative $D_w\widehat{I}$ with respect to w equals $D_w\widehat{I} = D_w I - \mathrm{id}$. Due to $\|D_w I\| \leq \lambda < 1$ the map $D_w\widehat{I}$ is invertible, and hence the statement follows from the implicit function theorem. ∎

Having obtained this lemma, we can proceed to show the differentiability of the wave transformations. For this result the magnetic field has to be differentiable and we need to make assumptions on the decay of the derivative of B, which we will describe in terms of its components B_{ij}.

Theorem 2.2.15 *Let the components B_{ij} of the magnetic field $B \in C^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$ satisfy*

$$\int_0^\infty t \|B_{ij}(t,\cdot)\|_\infty \, dt < \infty$$

and

$$\int_0^\infty t^2 \|\nabla_q B_{ij}(t,\cdot)\|_\infty \, dt < \infty$$

for all $i,j \in \{1,\dots,d\}$. Furthermore, assume that for every pair $i,j \in \{1,\dots,d\}$ there exists a continuous function $\ell\colon [0,\infty) \to [0,\infty)$ with

$$|\nabla_q B_{ij}(t,q_1) - \nabla_q B_{ij}(t,q_2)| \le \ell(t)|q_1 - q_2| \quad (q_1,q_2 \in \mathbb{R}^d,\ t \ge 0)$$

and

$$\int_0^\infty t^3 \ell(t)\, dt < \infty\,.$$

Then the wave transformation Ω^+ is a C^1-diffeomorphism. The analogue holds for Ω^-.

Proof We fix an energy $E > 0$ and show that $(\Omega^+)^{-1}$ is a C^1-map on each set $\mathbb{P}_{[0,E)}$, from which we then deduce that this also holds for Ω^+. The proof works similarly to the one that Ω^+ is a homeomorphism in Theorem 2.2.11. Again, we use the map

$$\mathcal{I} = \mathcal{I}(E)\colon \mathbb{P} \times X \to X = X(E)$$

which is given by (2.19), namely

$$\mathcal{I}(\gamma)(t) = \int_t^\infty (s-t) B(s, \gamma(s) + q_\infty + s v_\infty)(\dot\gamma(s) + v_\infty)\, ds\,, \tag{2.27}$$

on the space $\mathbb{P} \times X \cong \mathbb{R}^d \times \mathbb{R}^d \times X$ with

$$X = X(E) = \Big\{ \gamma \in C_b^1([T,\infty), \mathbb{R}^d) \mid |\gamma(t)| + |\dot\gamma(t)| \to 0 \text{ for } t \to \infty \Big\}\,.$$

The proof of Theorem 2.2.11 yields that $\mathcal{I}$ is a contraction on $\mathbb{P}_{[0,E)} \times \mathring{X}_1$ with

$$\mathring{X}_1 \coloneqq \Big\{ \gamma \in X \mid \|\dot\gamma\|_\infty < 1 \Big\}\,,$$

and due to inequality (2.21) the unique fixed point $\gamma_\infty = \gamma_\infty(q_\infty, v_\infty)$ of $\mathcal{I}(q_\infty, v_\infty, \cdot)$ satisfies $\|\dot\gamma_\infty\|_\infty < 1$, i.e. $\gamma_\infty \in \mathring{X}_1$. Moreover, for $(q_\infty, v_\infty) \in \mathbb{P}_{[0,E)}$ we obtain the equation (2.24) for $(\Omega^+)^{-1}$, namely

$$(\Omega^+)^{-1}(q_\infty, v_\infty) = \varphi^{0,T}(\delta_T(\gamma_\infty(q_\infty, v_\infty)) + (q_\infty + Tv_\infty, v_\infty))\,.$$

Thus, it suffices to show that the unique fixed point $\gamma_\infty(q_\infty, v_\infty)$ of $\mathcal{I}(q_\infty, v_\infty, \cdot)$ depends in a continuously differentiable way on the parameters. We obtain this fact by Lemma 2.2.14, and to apply its result it remains to verify that $\mathcal{I}$ is continuously differentiable on $\mathbb{P}_{[0,E)} \times \mathring{X}_1$ with respect to the norms

$$\|(q, v, \gamma)\|_{\mathbb{P}\times C^1} \coloneqq |q| + |v| + \|\gamma\|_{C^1} \quad ((q, v, \gamma) \in \mathbb{P} \times X)$$

on $\mathbb{P} \times X$ and $\|\cdot\|_{C^1}$ on X. We postpone this verification to Lemma 2.2.16, where we shall show that $\mathcal{I}$ is a C^1-map, even on the whole space $\mathbb{P} \times X$. Already assuming that this holds, we obtain that the unique fixed point $\gamma_\infty(q_\infty, v_\infty)$ depends in a continuously differentiable way on the parameters q_∞, v_∞ and the equality above shows that $(\Omega^+)^{-1}$ is a C^1-map. By Proposition 2.2.13, the wave transformation Ω^+ is volume preserving, which of course also holds for its inverse $(\Omega^+)^{-1}$. Therefore $(\Omega^+)^{-1}$ satisfies $|\det D(\Omega^+)^{-1}| \equiv 1$ and the inverse mapping theorem yields that Ω^+ is a C^1-map as well. ■

It remains to show that the contraction $\mathcal{I}$ given by (2.27) is a C^1-map. Unfortunately, it is not easily argued that $\mathcal{I}$ is a composition of differentiable functions since the map $f \mapsto (t \mapsto \int_t^\infty (s-t) f(s)\, ds)$ is linear but unbounded and hence not differentiable. To obtain the result nonetheless, we shall therefore use the definition explicitly.

Lemma 2.2.16 *If the assumptions of Theorem 2.2.15 hold, then for any $E > 0$ the map $\mathcal{I} = \mathcal{I}(E)\colon \mathbb{P} \times X \to X = X(E)$ given by (2.27) is a C^1-map.*

Proof In the following we will denote the i-th component of $w \in \mathbb{R}^d$ by $[w]_i$. Because of

$$[\mathcal{I}(q, v, \gamma)(t)]_i = \sum_{j=1}^d \int_t^\infty (s-t) B_{ij}(s, \gamma(s) + q + sv)[\dot\gamma(s) + v]_j\, ds$$

for $(q, v, \gamma) \in \mathbb{P} \times X$ and $t \geq T = T(E)$, it is sufficient to show the assertions for the maps $\mathcal{I}_{ij}\colon \mathbb{P} \times X \to C_b^1([T, \infty), \mathbb{R})$ given by

$$\mathcal{I}_{ij}(q, v, \gamma)(t) \coloneqq \int_t^\infty (s-t) B_{ij}(s, \gamma(s) + q + sv)[\dot\gamma(s) + v]_j\, ds\,.$$

For fixed $i, j \in \{1, \ldots, d\}$ we will show that $\mathcal{I}_{ij}$ is differentiable and the derivative

$$D\mathcal{I}_{ij}(q, v, \gamma) \colon \mathbb{P} \times X \cong \mathbb{R}^d \times \mathbb{R}^d \times X \to C^1_b([T, \infty), \mathbb{R})$$

at $(q, v, \gamma) \in \mathbb{P} \times X$ is given by

$$\begin{aligned} &D\mathcal{I}_{ij}(q, v, \gamma)(\rho, \nu, \eta)(t) \\ &\quad = \int_t^\infty (s - t)\Big(\langle \nabla_q B_{ij}(s, \gamma(s) + q + sv), \eta(s) + \rho + s\nu\rangle [\dot\gamma(s) + v]_j \\ &\qquad\quad + B_{ij}(s, \gamma(s) + q + sv)[\dot\eta(s) + \nu]_j\Big)\, ds \end{aligned} \tag{2.28}$$

for $(\rho, \nu, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \times X$ and $t \geq T$. We will explain later how this formula is obtained (see Remark 2.2.17). The proof of this statement consists of two parts.

Claim A For any fixed $(q, v, \gamma) \in \mathbb{P} \times X$, the linear map

$$\Phi \colon \mathbb{R}^d \times \mathbb{R}^d \times X \to C^1_b([T, \infty), \mathbb{R})$$

given by the right hand side of (2.28) satisfies

$$\|\mathcal{I}_{ij}(q + \rho, v + \nu, \gamma + \eta) - \mathcal{I}_{ij}(q, v, \gamma) - \Phi(\rho, \nu, \eta)\|_{C^1} = o(\|(\rho, \nu, \eta)\|) \tag{2.29}$$

for $(\rho, \nu, \eta) \to 0$ with

$$\|(\rho, \nu, \eta)\| := |\rho| + |\nu| + \|\eta\|_{C^1} \quad ((\rho, \nu, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \times X).$$

Claim B Φ is bounded.

We start with the proof of Claim A. We set $C := |v| + \|\dot\gamma\|_\infty$ and let $(\rho, \nu, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \times X$. To estimate the C^1-norm in (2.29) we need a preliminary calculation for the integrand. For all $s \in [T, \infty)$ we have

$$\begin{aligned} &B_{ij}(s, \gamma(s) + \eta(s) + q + \rho + sv + s\nu)[\dot\gamma(s) + \dot\eta(s) + v + \nu]_j \\ &\qquad - B_{ij}(s, \gamma(s) + q + sv)[\dot\gamma(s) + v]_j \\ &\qquad - \langle \nabla_q B_{ij}(s, \gamma(s) + q + sv), \eta(s) + \rho + s\nu\rangle [\dot\gamma(s) + v]_j \\ &\qquad - B_{ij}(s, \gamma(s) + q + sv)[\dot\eta(s) + \nu]_j \\ &\quad = \Big(B_{ij}(s, \gamma(s) + \eta(s) + q + \rho + sv + s\nu) - B_{ij}(s, \gamma(s) + q + sv) \\ &\qquad - \langle \nabla_q B_{ij}(s, \gamma(s) + q + sv), \eta(s) + \rho + s\nu\rangle\Big)[\dot\gamma(s) + v]_j \\ &\qquad + \Big(B_{ij}(s, \gamma(s) + \eta(s) + q + \rho + sv + s\nu) - B_{ij}(s, \gamma(s) + q + sv)\Big)[\dot\eta(s) + \nu]_j \\ &\quad = \langle \nabla_q B_{ij}(s, \xi_s) - \nabla_q B_{ij}(s, \gamma(s) + q + sv), \eta(s) + \rho + s\nu\rangle [\dot\gamma(s) + v]_j \\ &\qquad + \langle \nabla_q B_{ij}(s, \xi_s), \eta(s) + \rho + s\nu\rangle [\dot\eta(s) + \nu]_j \end{aligned}$$

for some point $\xi_s \in \mathbb{R}^d$ between $\gamma(s) + q + sv$ and $\gamma(s) + \eta(s) + q + \rho + sv + s\nu$. Since

$$|\xi_s - (\gamma(s) + q + sv)| \leq |\eta(s) + \rho + s\nu|$$

holds, we can estimate the absolute value of the previous equation and obtain the inequality

$$\begin{aligned}
&|B_{ij}(s, \gamma(s) + \eta(s) + q + \rho + sv + s\nu)[\dot\gamma(s) + \dot\eta(s) + v + \nu]_j \\
&\qquad - B_{ij}(s, \gamma(s) + q + sv)[\dot\gamma(s) + v]_j \\
&\qquad - \langle \nabla_q B_{ij}(s, \gamma(s) + q + sv), \eta(s) + \rho + s\nu\rangle [\dot\gamma(s) + v]_j \\
&\qquad - B_{ij}(s, \gamma(s) + q + sv)[\dot\eta(s) + \nu]_j| \\
&\quad \leq \ell(s)|\eta(s) + \rho + s\nu|^2 |\dot\gamma(s) + v| \\
&\qquad + \|\nabla_q B_{ij}(s, \cdot)\|_\infty |\eta(s) + \rho + s\nu| \cdot |\dot\eta(s) + \nu| \\
&\quad \leq C\ell(s)(\|\eta\|_\infty + |\rho| + s|\nu|)^2 \\
&\qquad + \|\nabla_q B_{ij}(s, \cdot)\|_\infty (\|\eta\|_\infty + |\rho| + s|\nu|)(\|\dot\eta\|_\infty + |\nu|) \\
&\quad \leq \Big(Cs^2\ell(s) + s\|\nabla_q B_{ij}(s, \cdot)\|_\infty\Big)(\|\eta\|_\infty + \|\dot\eta\|_\infty + |\rho| + |\nu|)^2 \\
&\quad = \Big(Cs^2\ell(s) + s\|\nabla_q B_{ij}(s, \cdot)\|_\infty\Big)\|(\rho, \nu, \eta)\|^2 .
\end{aligned} \tag{2.30}$$

Finally, we can consider the C^1-norm in (2.29). Inequality (2.30) yields

$$\begin{aligned}
&|\Big(\mathcal{I}_{ij}(q + \rho, v + \nu, \gamma + \eta) - \mathcal{I}_{ij}(q, v, \gamma) - \Phi(\rho, \nu, \eta)\Big)(t)| \\
&\quad = |\int_t^\infty (s - t)\Big(B_{ij}(s, \gamma(s) + \eta(s) + q + \rho + sv + s\nu)[\dot\gamma(s) + \dot\eta(s) + v + \nu]_j \\
&\qquad - B_{ij}(s, \gamma(s) + q + sv)[\dot\gamma(s) + v]_j \\
&\qquad - \langle \nabla_q B_{ij}(s, \gamma(s) + q + sv), \eta(s) + \rho + s\nu\rangle [\dot\gamma(s) + v]_j \\
&\qquad - B_{ij}(s, \gamma(s) + q + sv)[\dot\eta(s) + \nu]_j\Big)\, ds| \\
&\quad \leq \int_T^\infty Cs^3\ell(s) + s^2\|\nabla_q B_{ij}(s, \cdot)\|_\infty \, ds \, \|(\rho, \nu, \eta)\|^2 .
\end{aligned} \tag{2.31}$$

Furthermore, by using (2.30) again, we obtain

$$
\begin{aligned}
&|\frac{d}{dt}\Big(\mathcal{I}_{ij}(q+\rho, v+\nu, \gamma+\eta) - \mathcal{I}_{ij}(q, v, \gamma) - \Phi(\rho, \nu, \eta)\Big)(t)| \\
&\quad = | - \int_t^\infty B_{ij}(s, \gamma(s) + \eta(s) + q + \rho + sv + s\nu)[\dot\gamma(s) + \dot\eta(s) + v + \nu]_j \\
&\qquad - B_{ij}(s, \gamma(s) + q + sv)[\dot\gamma(s) + v]_j \\
&\qquad - \langle \nabla_q B_{ij}(s, \gamma(s) + q + sv), \eta(s) + \rho + s\nu\rangle [\dot\gamma(s) + v]_j \\
&\qquad - B_{ij}(s, \gamma(s) + q + sv)[\dot\eta(s) + \nu]_j \, ds| \\
&\quad \leq \int_T^\infty Cs^2 \ell(s) + s\|\nabla_q B_{ij}(s, \cdot)\|_\infty \, ds \, \|(\rho, \nu, \eta)\|^2 .
\end{aligned} \tag{2.32}
$$

The inequalities (2.31) and (2.32) imply that

$$\|\mathcal{I}_{ij}(q+\rho, v+\nu, \gamma+\eta) - \mathcal{I}_{ij}(q, v, \gamma) - \Phi(\rho, \nu, \eta)\|_{C^1} = \mathrm{const}\|(\rho, \nu, \eta)\|^2$$

holds, which shows statement (2.29) and hence proves Claim A.

We now turn to the proof of Claim B. For $(\rho, \nu, \eta) \in \mathbb{R}^d \times \mathbb{R}^d \times X$ and $t \in [T, \infty)$ we have

$$
\begin{aligned}
|\Phi(\rho, \nu, \eta)(t)| &\leq \int_T^\infty s\Big(C\|\nabla_q B_{ij}(s, \cdot)\|_\infty |\eta(s) + \rho + s\nu| + \|B_{ij}(s, \cdot)\|_\infty |\dot\eta(s) + \nu|\Big) \, ds \\
&\leq \int_T^\infty Cs^2 \|\nabla_q B_{ij}(s, \cdot)\|_\infty + s\|B_{ij}(s, \cdot)\|_\infty \, ds \, \Big(\|\eta\|_\infty + \|\dot\eta\|_\infty + |\rho| + |\nu|\Big)
\end{aligned}
$$

and

$$|\frac{d}{dt}\Phi(\rho, \nu, \eta)(t)| \leq \int_T^\infty Cs\|\nabla_q B_{ij}(s, \cdot)\|_\infty + \|B_{ij}(s, \cdot)\|_\infty \, ds \, \Big(\|\eta\|_\infty + \|\dot\eta\|_\infty + |\rho| + |\nu|\Big) .$$

Together, these inequalities yield

$$\|\Phi(\rho, \nu, \eta)\|_{C^1} \leq \mathrm{const}(\|\eta\|_\infty + \|\dot\eta\|_\infty + |\rho| + |\nu|) = \mathrm{const}\|(\rho, \nu, \eta)\| ,$$

which implies that Φ is bounded, i.e. proves Claim B. Therefore, $\mathcal{I}_{ij}$ is differentiable with $D\mathcal{I}_{ij}(q, v, \gamma) = \Phi$.

It remains to show the continuity of $D\mathcal{I}_{ij}$. Moreover, we will show that $D\mathcal{I}_{ij}$ is even Lipschitz continuous on those subsets of $\mathbb{P} \times X$ where $|v|$ and $\|\dot\gamma\|_\infty$ are bounded. For this, we have to check if

$$\|D\mathcal{I}_{ij}(q_1, v_1, \gamma_1) - D\mathcal{I}_{ij}(q_2, v_2, \gamma_2)\|_{Op} \leq \mathrm{const}\|(q_1, v_1, \gamma_1) - (q_2, v_2, \gamma_2)\|$$

holds for $(q_1,v_1,\gamma_1),(q_2,v_2,\gamma_2)\in\mathbb{P}\times X$ with $|v_i|+\|\dot\gamma_i\|_\infty\le C$ $(i=1,2)$, where $\|\cdot\|_{Op}$ denotes the canonical operator norm on the space

$$\mathcal{L}(\mathbb{R}^d\times\mathbb{R}^d\times X, C_b^1([T,\infty),\mathbb{R}^d))$$

of linear bounded maps from $\mathbb{R}^d\times\mathbb{R}^d\times X$ to $C_b^1([T,\infty),\mathbb{R}^d)$. We start with a preliminary estimation. For $(\rho,\nu,\eta)\in\mathbb{R}^d\times\mathbb{R}^d\times X$ and $s\ge T$ we have

$$\begin{aligned}
&|\langle\nabla_q B_{ij}(s,\gamma_1(s)+q_1+sv_1),\eta(s)+\rho+s\nu\rangle[\dot\gamma_1(s)+v_1]_j\\
&\qquad+B_{ij}(s,\gamma_1(s)+q_1+sv_1)[\dot\eta(s)+\nu]_j\\
&\qquad-\langle\nabla_q B_{ij}(s,\gamma_2(s)+q_2+sv_2),\eta(s)+\rho+s\nu\rangle[\dot\gamma_2(s)+v_2]_j\\
&\qquad-B_{ij}(s,\gamma_2(s)+q_2+sv_2)[\dot\eta(s)+\nu]_j|\\
&\quad=|\langle\nabla_q B_{ij}(s,\gamma_1(s)+q_1+sv_1)-\nabla_q B_{ij}(s,\gamma_2(s)+q_2+sv_2),\eta(s)+\rho+s\nu\rangle\\
&\qquad\cdot[\dot\gamma_2(s)+v_2]_j\\
&\qquad+\langle\nabla_q B_{ij}(s,\gamma_1(s)+q_1+sv_1),\eta(s)+\rho+s\nu\rangle[\dot\gamma_1(s)-\dot\gamma_2(s)+v_1-v_2]_j\\
&\qquad+\Big(B_{ij}(s,\gamma_1(s)+q_1+sv_1)-B_{ij}(s,\gamma_2(s)+q_2+sv_2)\Big)[\dot\eta(s)+\nu]_j|\\
&\quad\le\ell(s)|\gamma_1(s)-\gamma_2(s)+q_1-q_2+sv_1-sv_2|\cdot|\eta(s)+\rho+s\nu|\cdot|\dot\gamma_2(s)+v_2|\\
&\qquad+\|\nabla_q B_{ij}(s,\cdot)\|_\infty|\eta(s)+\rho+s\nu|\cdot|\dot\gamma_1(s)-\dot\gamma_2(s)+v_1-v_2|\\
&\qquad+\|\nabla_q B_{ij}(s,\cdot)\|_\infty|\gamma_1(s)-\gamma_2(s)+q_1-q_2+sv_1-sv_2|\cdot|\dot\eta(s)+\nu|\\
&\quad\le\Big(Cs^2\ell(s)+2s\|\nabla_q B_{ij}(s,\cdot)\|_\infty\Big)\|(q_1,v_1,\gamma_1)-(q_2,v_2,\gamma_2)\|\|(\rho,\nu,\eta)\|
\end{aligned}\tag{2.33}$$

and therefore

$$\begin{aligned}
&|\Big(D\mathcal{I}_{ij}(q_1,v_1,\gamma_1)-D\mathcal{I}_{ij}(q_2,v_2,\gamma_2)\Big)(\rho,\nu,\eta)(t)|\\
&\quad=|\int_t^\infty(s-t)\Big(\langle\nabla_q B_{ij}(s,\gamma_1(s)+q_1+sv_1),\eta(s)+\rho+s\nu\rangle[\dot\gamma_1(s)+v_1]_j\\
&\qquad+B_{ij}(s,\gamma_1(s)+q_1+sv_1)[\dot\eta(s)+\nu]_j\\
&\qquad-\langle\nabla_q B_{ij}(s,\gamma_2(s)+q_2+sv_2),\eta(s)+\rho+s\nu\rangle[\dot\gamma_2(s)+v_2]_j\\
&\qquad-B_{ij}(s,\gamma_2(s)+q_2+sv_2)[\dot\eta(s)+\nu]_j\Big)\,ds|\\
&\quad\le\int_T^\infty Cs^3\ell(s)+2s^2\|\nabla_q B_{ij}(s,\cdot)\|_\infty\,ds\,\|(q_1,v_1,\gamma_1)-(q_2,v_2,\gamma_2)\|\|(\rho,\nu,\eta)\|\,.
\end{aligned}$$

Using estimate (2.33) again, we obtain

$$
\begin{aligned}
&|\frac{d}{dt}\Big(D\mathcal{I}_{ij}(q_1,v_1,\gamma_1)-D\mathcal{I}_{ij}(q_2,v_2,\gamma_2)\Big)(\rho,\nu,\eta)(t)| \\
&\qquad \le \int_T^\infty Cs^2\ell(s)+2s\|\nabla_q B_{ij}(s,\cdot)\|_\infty \,ds\,\|(q_1,v_1,\gamma_1)-(q_2,v_2,\gamma_2)\|\|(\rho,\nu,\eta)\|\,.
\end{aligned}
$$

The last two inequalities show that

$$
\|\Big(D\mathcal{I}_{ij}(q_1,v_1,\gamma_1)-D\mathcal{I}_{ij}(q_2,v_2,\gamma_2)\Big)(\rho,\nu,\eta)\|_{C^1}\le \text{const}\|(q_1,v_1,\gamma_1)-(q_2,v_2,\gamma_2)\|\|(\rho,\nu,\eta)\|
$$

and therefore the estimate

$$
\|D\mathcal{I}_{ij}(q_1,v_1,\gamma_1)-D\mathcal{I}_{ij}(q_2,v_2,\gamma_2)\|_{Op}\le \text{const}\|(q_1,v_1,\gamma_1)-(q_2,v_2,\gamma_2)\|
$$

holds, as claimed. ■

Remark 2.2.17 Instead of the direct computation in the previous lemma we could have made use of a modified chain rule, but this would have required even more calculations. However, this approach provides a better insight of how the formula (2.28) for the derivative of $\mathcal{I}_{ij}$ is obtained, so we will outline the arguments in the following.

Modified chain rule *Let X,Y,Z be Banach spaces. Let $H\colon \operatorname{dom} H\subseteq Y\to Z$ be a linear, but not necessarily bounded map, where $\operatorname{dom} H$ denotes the domain of definition of H. Furthermore, let $G\colon X\to Y$ be differentiable at $x\in X$ and let $r\colon X\to Y$ be the corresponding remainder given by*

$$
r(\xi)\coloneqq G(x+\xi)-G(x)-DG(x)\xi=o(\|\xi\|)
$$

for $\xi\to 0$. Assume furthermore that the following conditions hold:

(i) The images of G, $DG(x)$ and r are contained in $\operatorname{dom} H$.

(ii) $H\circ DG(x)\colon X\to Z$ is bounded.

(iii) $H\circ r\colon X\to Z$ satisfies $H(r(\xi))=o(\|\xi\|)$ for $\xi\to 0$.

Then

$$
(H\circ G)(x+\xi)-(H\circ G)(x)-(H\circ DG(x))\xi=(H\circ r)(\xi)=o(\|\xi\|)
$$

holds for $\xi\to 0$ and, thus, $H\circ G$ is differentiable at x with

$$
D(H\circ G)(x)=H\circ DG(x)\,.
$$

In the proof of the previous lemma we have considered the map $\mathcal{I}_{ij} = H \circ G$ which is composed of

$$\begin{aligned} H\colon \operatorname{dom} H \subseteq C_b^0([T,\infty),\mathbb{R}) &\to C_b^1([T,\infty),\mathbb{R})\,, \\ f &\mapsto \left(t \mapsto \int_t^\infty (s-t) f(s)\, ds \right) \end{aligned}$$

and

$$\begin{aligned} G\colon \mathbb{P} \times X &\to C_b^0([T,\infty),\mathbb{R})\,, \\ (q,v,\gamma) &\mapsto \Big(s \mapsto B_{ij}(s, \gamma(s) + q + sv)[\dot\gamma(s) + v]_j \Big)\,. \end{aligned}$$

Let us assume that the derivative $DG(q,v,\gamma)\colon \mathbb{P} \times X \to C_b^0([T,\infty),\mathbb{R})$ of G at a point $(q,v,\gamma) \in \mathbb{P} \times X$ is already known and given by

$$\begin{aligned} DG(q,v,\gamma)(\rho,\nu,\eta)(s) = \langle \nabla_q B_{ij}(s, \gamma(s) + q + sv), \eta(s) + \rho + s\nu \rangle [\dot\gamma(s) + v]_j \\ + B_{ij}(s, \gamma(s) + q + sv)[\dot\eta(s) + \nu]_j\,. \end{aligned} \tag{2.34}$$

Then we can apply the modified chain rule: Claim A assures that condition *(iii)* holds, Claim B yields condition *(ii)*, and the assumptions on the decay of the magnetic field imply that the integrals exist, i.e. condition *(i)* holds. Thus, we obtain the formula (2.28) for the derivative of $\mathcal{I}_{ij} = H \circ G$ at the point $(q,v,\gamma) \in \mathbb{P} \times X$.

However, we have not shown the formula for the derivative of G yet. This can be done by using the product rule for the Banach algebra $C_b^0([T,\infty),\mathbb{R})$: The map $G = G_1 \cdot G_2$ is the product of the maps $G_1, G_2\colon \mathbb{P} \times X \to C_b^0([T,\infty),\mathbb{R})$ given by

$$G_1(q,v,\gamma)(s) = B_{ij}(s, \gamma(s) + q + sv)$$

and

$$G_2(q,v,\gamma)(s) = [\dot\gamma(s) + v]_j\,.$$

A computation shows that the derivatives $DG_1(q,v,\gamma), DG_2(q,v,\gamma)\colon \mathbb{P}\times X \to C_b^0([T,\infty)$ at a point $(q,v,\gamma) \in \mathbb{P} \times X$ are given by

$$DG_1(q,v,\gamma)(\rho,\nu,\eta)(s) = \langle \nabla_q B_{ij}(s, \gamma(s) + q + sv), \eta(s) + \rho + s\nu \rangle$$

and, since G_2 is linear,

$$DG_2(q,v,\gamma)(\rho,\nu,\eta)(s) = [\dot\eta(s) + \nu]_j\,.$$

Thus, by using the product rule in Banach algebras we obtain formula (2.34) for the derivative of G, and consequently formula (2.28) for the derivative of $\mathcal{I}_{ij}$. □

With this observation we conclude the analysis of the asymptotic behaviour of a charged particle in a time-dependent magnetic field. Let us point out that every trajectory (apart from the fixed points) is asymptotically straight, provided that the magnetic field decays sufficiently fast as the time increases. This is a main difference to the motion in a time-independent magnetic field, which we will examine in the following.

2.3 Spatial decay in a time-independent magnetic field

In this section we can finally consider scattering in magnetic fields that vanish at infinity, i.e. study the existence and the properties of the limit (2.1). To do so, we take the same approach as in the time-dependent case and consider the asymptotic velocity and the asymptotic position. Fortunately, we do not have to start from scratch to show their existence and continuity, but can build upon the results for time-decaying magnetic fields and combine them with the results obtained for the escape rate in Section 2.1. The main tool we shall use is the introduction of a time-dependent magnetic field $B^\chi(t,q)$ related to the time-independent one $B(q)$, where the decay of B^χ in time corresponds to the decay of B in space. We will now describe this process in detail, so in the following proofs we simply refer to this remark:

Remark 2.3.1 We shall start with a compact set $K \subseteq s^+$. Assume that there are constants $C > 0$ and $T \in \mathbb{R}$ such that

$$|q^t(x)| \geq C\langle t\rangle \quad (x \in K,\ t \geq T)\,, \tag{2.35}$$

where $\langle t\rangle = \sqrt{1+t^2}$ denotes the smooth modification of the absolute value. Furthermore, let $\chi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ satisfy the following three properties:

(i) $\chi(q) = 0$ for $|q| \leq \varepsilon$ for some $\varepsilon > 0$.

(ii) $\chi(q) = 1$ for $|q| \geq C$.

(iii) $0 \leq \chi(q) \leq 1$ for all $q \in \mathbb{R}^d$.

We call χ a *cut-off function with respect to C* and use it to define the time-dependent magnetic field

$$B^\chi(t,q) \coloneqq \chi\left(\frac{q}{\langle t\rangle}\right) B(q)\,.$$

The effect of this modification is that the magnetic field B^χ vanishes on growing balls around the origin as t increases, while outside even larger balls B and B^χ coincide, as visualized in Figure 2.2. Now consider the trajectory of some $x \in K$. Because of (2.35)

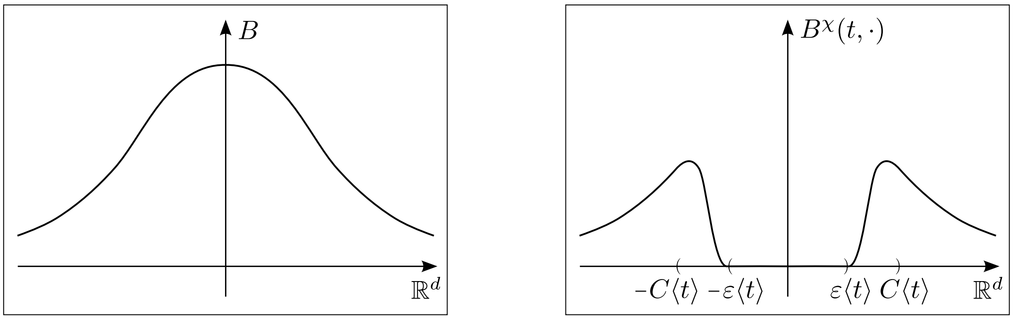


Figure 2.2: Sketches of the graphs of B and $B^\chi(t,\cdot)$

and property (ii) of χ we have

$$B^\chi(t,q^t(x)) = B(q^t(x))$$

for $t \geq T$, and hence, with φ_χ^{t,t_0} denoting the flow induced by B^χ, we obtain the equality

$$\varphi_\chi^{t,T}(\varphi^T(x)) = \varphi^t(x) \quad (t \geq T)\,, \tag{2.36}$$

which is visualized in Figure 2.3. This implies that the magnetic flow $\varphi^t(x)$ solves the

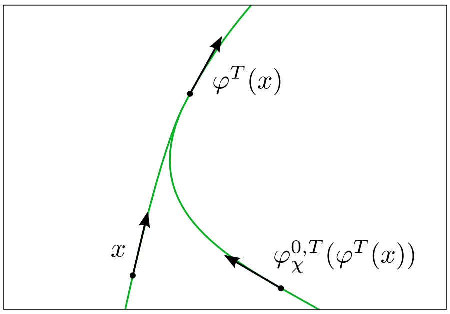


Figure 2.3: Comparison of the flows of B and B^χ

time-dependent magnetic equation (2.6) for $t \geq T$. Hence, for these times any trajectory of the magnetic flow given by B that starts in K is also a trajectory of the flow given by the time-dependent magnetic field B^χ. Finally, note that this also holds for $t < -T$, so we can use the same mechanism for the case $t \to -\infty$. □

The calculations in Section 2.1 assure that the assumptions in Remark 2.3.1 can be met and we will now use the process described above to construct the asymptotic velocity and the asymptotic position for magnetic fields decaying in space.

2.3.1 Asymptotic velocity and asymptotic position

In the case of time decay there were only fixed points (with $\mathcal{E}(x) = 0$) and unbounded states (with $\mathcal{E}(x) > 0$). Now for a time-independent magnetic field non-trivial bounded orbits may occur, for instance circular orbits, and for these the limits of the velocity for $t \to \pm\infty$ do not exist. There are two possibilities to work around this: We can either choose only the scattering orbits as the domain of definition, or we define the asymptotic velocity in a different but consistent way on the whole phase space $\mathbb{P}$. Although only the scattering states $s^{\pm}$ shall be examined further, we give a general definition that is included in the following proposition:

Proposition 2.3.2 *Let the magnetic field satisfy*

$$\int_0^\infty \sup_{|q|\geq r} \|B(q)\| \, dr < \infty \, .$$

Then the following statements hold:

(i) The asymptotic velocity

$$v^+(x) := \lim_{t\to\infty} \frac{q^t(x)}{t}$$

exists for all $x \in \mathbb{P}$.

(ii) For $x \in s^+$ the equation

$$v^+(x) = \lim_{t\to\infty} v^t(x)$$

holds. In particular, v^+ is invariant under the flow.

(iii) The asymptotic velocity is continuous on s^+ and, moreover, on compact subsets of s^+ both limits are uniform.

The analogous results hold for $v^- : \mathbb{P} \to \mathbb{R}^d$ given by

$$v^-(x) := \lim_{t\to-\infty} \frac{q^t(x)}{t} \, .$$

Proof Clearly, for $x \in b^+$ we have $v^+(x) = 0$. Now let $K \subseteq s^+$ be compact. Then by Lemma 2.1.10 there are $C, T > 0$ such that $|q^t(x)| \geq C\langle t\rangle$ for $x \in K$ and $t \geq T$. Now we choose a cut-off function χ with respect to C and define

$$B^\chi(t,q) := \chi\left(\frac{q}{\langle t\rangle}\right) B(q)$$

as in Remark 2.3.1. We want to use Proposition 2.2.4 and to do so we have to compute $\|B^\chi(t,\cdot)\|_\infty$. Because of $\chi(q)=0$ for $|q|<\varepsilon$ as well as $t\le\langle t\rangle$ we have the inequality

$$\|B^\chi(t,q)\| = 0 \quad (|q|\le t\varepsilon)\,.$$

Since $\|B^\chi(t,q)\|\le\|B(q)\|$ holds for all $t\in\mathbb{R}$ and $q\in\mathbb{R}^d$, we obtain the estimate

$$\|B^\chi(t,\cdot)\|_\infty \le \sup_{|q|\ge t\varepsilon}\|B(q)\|$$

and hence B^χ satisfies

$$\int_0^\infty \|B^\chi(t,\cdot)\|_\infty\,dt \le \int_0^\infty \sup_{|q|\ge t\varepsilon}\|B(q)\|\,dt \le \frac{1}{\varepsilon}\int_0^\infty \sup_{|q|\ge r}\|B(q)\|\,dr < \infty\,,$$

i.e. the assumption of Proposition 2.2.4 is met. This means that for all $x\in\mathbb{P}$ the limit

$$v_\chi^+(x) = \lim_{t\to\infty} v_\chi^{t,0}(x) = \lim_{t\to\infty}\frac{q_\chi^{t,0}(x)}{t}$$

exists. Hence, for $x\in K$ we can use identity (2.36) and obtain that the limit

$$\begin{aligned} v^+(x) &\coloneqq \lim_{t\to\infty}\frac{q^t(x)}{t} \\ &= \lim_{t\to\infty}\frac{q_\chi^{t,T}(\varphi^T(x))}{t} \\ &= \lim_{t\to\infty}\frac{q_\chi^{t,0}(\varphi_\chi^{0,T}\circ\varphi^T(x))}{t} \\ &= v_\chi^+(\varphi_\chi^{0,T}\circ\varphi^T(x)) \end{aligned} \tag{2.37}$$

exists. Furthermore, using (2.36) again, v^+ can be expressed as

$$\begin{aligned} v^+(x) &= v_\chi^+(\varphi_\chi^{0,T}\circ\varphi^T(x)) \\ &= \lim_{t\to\infty} v_\chi^{t,0}(\varphi_\chi^{0,T}\circ\varphi^T(x)) \\ &= \lim_{t\to\infty} v_\chi^{t,T}(\varphi^T(x)) \\ &= \lim_{t\to\infty} v^t(x)\,. \end{aligned}$$

Proposition 2.2.4 assures the uniform convergence of $\frac{q_\chi^t}{t}\to v_\chi^+$ and $v_\chi^t\to v_\chi^+$ for $t\to\infty$ on compact subsets of $\mathbb{P}$. Hence, this also holds for the limits

$$\frac{q^t}{t}\to v^+ \quad (t\to\infty)$$

and

$$v^t\to v^+ \quad (t\to\infty)$$

on compact subsets of s^+. ∎

The asymptotic velocity can be used to characterize bounded and unbounded orbits:

Corollary 2.3.3 *For $x \in \mathbb{P}$ the following statements are equivalent:*

(i) $v^{\pm}(x) = 0$.

(ii) $x \in b^{\pm}$.

This is the same as $v^{\pm}(x) \neq 0 \Leftrightarrow x \in s^{\pm}$.

PROOF Since $v^+(x) = 0$ for $x \in b^+$ it remains to show that $x \in s^+$ implies $v^+(x) \neq 0$. For any given $x \in s^+$ we can apply Lemma 2.1.8 and obtain $v^+(x) \geq C > 0$. ∎

After the asymptotic velocity we now introduce the asymptotic position. Since there might be non-trivial bounded orbits, this is only possible for scattering states.

Proposition 2.3.4 *Let the magnetic field satisfy*

$$\int_0^\infty r \sup_{|q| \geq r} \|B(q)\| \, dr < \infty \,. \tag{2.38}$$

Then the following statements hold:

(i) For $x \in s^+$ the asymptotic position

$$q^+(x) := \lim_{t \to \infty} \Big(q^t(x) - t v^+(x) \Big)$$

exists and satisfies

$$q^+(x) = \lim_{t \to \infty} \Big(q^t(x) - t v^t(x) \Big) \,. \tag{2.39}$$

(ii) The mapping $q^+ \colon s^+ \to \mathbb{R}^d$ is continuous and both limits exist uniformly on compact subsets of s^+.

The analogous results hold for $q^- : s^- \to \mathbb{R}^d$ given by

$$q^-(x) := \lim_{t \to -\infty} \Big(q^t(x) - t v^-(x) \Big) \,.$$

PROOF Let $K \subseteq s^+$ be compact. Let $C, T > 0$ be as in Lemma 2.1.10, i.e. such that $|q^t(x)| \geq C\langle t \rangle$ holds for $x \in K$ and $t \geq T$, and let $\chi \in C^\infty(\mathbb{R}^d, \mathbb{R})$ be a cut-off function with respect to C. As in the proof of Proposition 2.3.2 there exists $\varepsilon > 0$ such that the time-dependent magnetic field

$$B^\chi(t, q) := \chi\left(\frac{q}{\langle t \rangle}\right) B(q)$$

vanishes for $|q| < t\varepsilon$. Therefore, the inequality

$$\int_0^\infty t\|B^\chi(t,\cdot)\|_\infty \, dt \leq \int_0^\infty t \sup_{|q|\geq t\varepsilon} \|B(q)\| \, dt < \infty$$

holds and Proposition 2.2.5 yields that the asymptotic position $q_\chi^+ \colon \mathbb{P} \to \mathbb{R}^d$ induced by B^χ exists as a continuous map. Using the equations (2.36) and (2.37) we obtain

$$\begin{aligned} q^+(x) &\coloneqq \lim_{t\to\infty} q^t(x) - tv^+(x) \\ &= \lim_{t\to\infty} q_\chi^{t,T}(\varphi^T(x)) - tv_\chi^+(\varphi_\chi^{0,T} \circ \varphi^T(x)) \\ &= \lim_{t\to\infty} q_\chi^{t,0}(\varphi_\chi^{0,T} \circ \varphi^T(x)) - tv_\chi^+(\varphi_\chi^{0,T} \circ \varphi^T(x)) \\ &= q_\chi^+(\varphi_\chi^{0,T} \circ \varphi^T(x)) \,, \end{aligned} \tag{2.40}$$

i.e. the asymptotic position $q^+(x)$ exists for $x \in K$ and it remains to show property (2.39). For this we make use of the similar statement in Proposition 2.2.5 for the time-dependent case, namely the fact that q_χ^+ satisfies

$$q_\chi^+(x) = \lim_{t\to\infty} q_\chi^{t,0}(x) - tv_\chi^{t,0}(x) \quad (x \in \mathbb{P}) \,.$$

Continuing from (2.40) we obtain

$$\begin{aligned} q^+(x) &= q_\chi^+(\varphi_\chi^{0,T} \circ \varphi^T(x)) \\ &= \lim_{t\to\infty} q_\chi^{t,0}(\varphi_\chi^{0,T} \circ \varphi^T(x)) - tv_\chi^{t,0}(\varphi_\chi^{0,T} \circ \varphi^T(x)) \\ &= \lim_{t\to\infty} q_\chi^{t,T}(\varphi^T(x)) - tv_\chi^{t,T}(\varphi^T(x)) \\ &= \lim_{t\to\infty} q^t(x) - tv^t(x) \,, \end{aligned}$$

and since Proposition 2.2.5 yields that $q_\chi^t - tv_\chi^t \to q_\chi^+$ converges uniformly on compact subsets of $\mathbb{P}$ for $t \to \infty$, this also holds for the limit

$$q^t - tv^t \to q^+ \quad (t \to \infty)$$

on compact subsets of s^+. ∎

2.3.2 Wave transformations

Having obtained the asymptotic velocity and the asymptotic position we can combine them as in the time-dependent case.

Definition 2.3.5 The *(velocity) wave transformations*

$$\Omega^{\pm} \colon s^{\pm} \to \mathbb{P}^0 := T\mathbb{R}^d \setminus \mathbb{R}^d \cong \mathbb{R}^d_q \times \left(\mathbb{R}^d_v \setminus \{0\}\right)$$

given by

$$\Omega^{\pm}(x) = (q^{\pm}(x), v^{\pm}(x))$$

are well defined if the magnetic field satisfies (2.38), namely

$$\int_0^\infty r \sup_{|q|\geq r} \|B(q)\| \, dr < \infty \,.$$

If this condition holds, they coincide with the limit (2.1), i.e.

$$\Omega^{\pm} = \lim_{t\to\pm\infty} \varphi_0^{-t} \circ \varphi^t \,.$$

□

For once we drop assumption (2.38) on the magnetic field and consider $\Omega^{\pm} := (q^{\pm}, v^{\pm})$ as formal limits in order to obtain a similar characterization as in Proposition 2.2.7: If for some point $x \in \mathbb{P}$ the wave transformations $\Omega^{\pm}$ exist, then its magnetic trajectory is asymptotically a straight line. This is described in the following proposition, whose statement and proof are analogous to the time-dependent case.

Proposition 2.3.6 *Let B be any magnetic field. Then, for $x \in \mathbb{P}$ the following statements are equivalent:*

(i) $\Omega^{\pm}(x) = (q^{\pm}(x), v^{\pm}(x)) = y \in \mathbb{P}$ *(i.e. the limit exists and equals y).*

(ii) $\varphi^t(x) - \varphi_0^t(y) \to 0$ *for $t \to \pm\infty$.*

Before proceeding to our main goal in this section (the analysis of the regularity of the wave transformations), we remark that for scattering states $x \in s = s^+ \cap s^-$ we can link the incoming to the outgoing straight line, e.g. in order to consider inverse scattering.

Remark 2.3.7 As for the time-decaying case, we define the scattering transformation

$$\mathcal{S} := \Omega^+ \circ (\Omega^-)^{-1}$$

which assigns to each incoming asymptotic line its corresponding outgoing asymptotic line. The domain of definition of $\mathcal{S}$ is

$$\operatorname{dom}\mathcal{S} = \Omega^-(s)$$

and its image equals

$$\operatorname{im}\mathcal{S} = \Omega^+(s)\,.$$

We can apply a result which is due to C. L. Siegel [43] (for a precise statement and the proof we refer to Proposition 2.1.4 in [12]): The sets s^+ and s^- differ only by a set of measure zero, i.e.

$$(s^+ \setminus s^-) \cup (s^- \setminus s^+)$$

is a nullset in $\mathbb{P}$. This implies that the sets $s^\pm \setminus s$ have measure zero. We will see that with additional assumptions on the magnetic field, the wave transformations $\Omega^\pm\colon s^\pm \to \mathbb{P}^0$ are measure preserving homeomorphisms and therefore, both the domain $\Omega^-(s)$ and the image $\Omega^+(s)$ of $\mathcal{S}$ have full measure in $\mathbb{P}^0 = \Omega^\pm(s^\pm)$ as well as in $\mathbb{P}$. However, since all information about the asymptotic behaviour is coded in the wave transformations, we will focus exclusively on their analysis. $\square$

From now on we shall assume that the magnetic field satisfies at least condition (2.38). In this case, both the asymptotic velocity and the asymptotic position exist, so the wave transformations are well defined continuous maps. In particular, as an immediate consequence of the identities (2.37) and (2.40), we obtain the next corollary, which links the wave transformations of the magnetic field B to the ones of the corresponding time-dependent magnetic fields B^χ. It assures that the asymptotic values of x and $\varphi_\chi^{0,T}(\varphi^T(x))$ coincide, which are computed for the magnetic flows of B and B^χ, respectively. This is visualized in Figure 2.4.

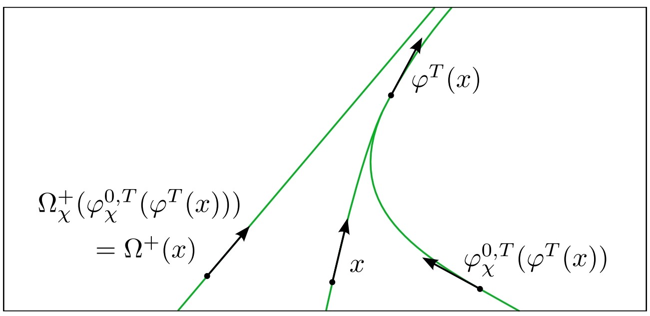


Figure 2.4: Equality of the asymptotic behaviour of x and $\varphi_\chi^{0,T}(\varphi^T(x))$

Corollary 2.3.8 *Let $K \subseteq s^+$ be compact. Then there are constants $C, T > 0$ and a cut-off function χ with respect to C such that*

$$\Omega^+ = \Omega^+_\chi \circ \varphi^{0,T}_\chi \circ \varphi^T$$

holds on K. In particular, Ω^+ is injective. The analogous result holds for Ω^-.

The next lemma shows that the wave transformations intertwine the magnetic flow and the free flow, which is visualized in Figure 2.5.

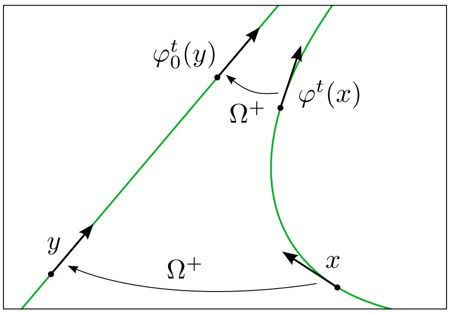


Figure 2.5: The wave transformation Ω^+ intertwines the magnetic and the free flow

Lemma 2.3.9 *The equality*

$$\Omega^\pm \circ \varphi^t = \varphi_0^t \circ \Omega^\pm$$

holds on $s^\pm$.

Proof For $x \in s^+$ we have the equation

$$\begin{aligned}
\Omega^+(\varphi^t(x)) &= \lim_{s\to\infty} (q^s(\varphi^t(x)) - sv^s(\varphi^t(x)), v^s(\varphi^t(x))) \\
&= \lim_{s\to\infty} (q^{s+t}(x) - (s+t)v^{s+t}(x) + tv^{s+t}(x), v^{s+t}(x)) \\
&= (q^+(x) + tv^+(x), v^+(x)) \\
&= \varphi_0^t(\Omega^+(x)) \,.
\end{aligned}$$

∎

We did not write that Ω^+ conjugates the two flows in the sense of Definition 1.4.3 since neither the existence nor the continuity of its inverse are guaranteed. This question is answered by the following theorem, though.

Theorem 2.3.10 *Let the magnetic field $B \in C^1(\mathbb{R}^d, \mathbb{R}^{d\times d})$ be such that the components B_{ij} satisfy*

$$\int_0^\infty r \sup_{|q|\geq r} |B_{ij}(q)|\, dr < \infty$$

and

$$\int_0^\infty r^2 \sup_{|q|\geq r} |\nabla B_{ij}(q)|\, dr < \infty$$

for all $i,j \in \{1,\dots,d\}$. Then the wave transformations $\Omega^\pm\colon s^\pm \to \mathbb{P}^0 = T\mathbb{R}^d \setminus \mathbb{R}^d$ are volume preserving homeomorphisms.

Proof It is to show that Ω^+ is bijective and $(\Omega^+)^{-1}$ is continuous. For both claims we will make use of the corresponding result in the time-decaying case. Let $K \subseteq \mathbb{P}^0$ be compact. Then for any $(q_0,v_0) \in K$ we have the estimate

$$\frac{|q_0 + tv_0|}{t} \geq \min_{(q,v)\in K} |v| - \frac{1}{t} \max_{(q,v)\in K} |q| \geq 2\widetilde{C} > 0 \quad (t \geq \widetilde{T})$$

for some $\widetilde{C} > 0$ and $\widetilde{T} > 0$, which means

$$|q_0 + tv_0| \geq 2\widetilde{C}t \quad (t \geq \widetilde{T})\,.$$

Then, by Lemma 2.1.9 there is a constant $C > 0$ such that

$$|q_0 + tv_0| \geq 2C\langle t\rangle \quad (t \geq \widetilde{T})\,.$$

We now use a cut-off function χ with respect to C and define the time-dependent magnetic field $B^\chi = (B^\chi_{ij})$ by $B^\chi_{ij}(t,q) := \chi(\frac{q}{\langle t\rangle})B_{ij}(q)$. Since $\chi(\frac{q}{\langle t\rangle}) = 0$ holds for $|q| \leq t\varepsilon$ we obtain

$$\int_0^\infty t\|B^\chi(t,\cdot)\|_\infty\, dt \leq \int_0^\infty t \sup_{|q|\geq t\varepsilon} \|B(q)\|\, dt \leq \sum_{i,j=1}^d \int_0^\infty t \sup_{|q|\geq t\varepsilon} |B_{ij}(q)|\, dt < \infty$$

as before. Furthermore, the cut-off function χ satisfies $\nabla\chi(\frac{q}{\langle t\rangle}) = 0$ for $|q| \leq t\varepsilon$ as well as $\chi(q) \equiv 1$ for $|q|$ sufficiently large. In particular, $\nabla\chi$ is bounded by some constant C_1. For fixed $i,j \in \{1,\dots,d\}$ the derivative of B^χ_{ij} with respect to q equals

$$\nabla_q B^\chi_{ij}(t,q) = \frac{1}{\langle t\rangle}\nabla\chi\Big(\tfrac{q}{\langle t\rangle}\Big)B_{ij}(q) + \chi\Big(\tfrac{q}{\langle t\rangle}\Big)\nabla B_{ij}(q)\,,$$

which yields the inequality

$$\|\nabla_q B^\chi_{ij}(t,\cdot)\|_\infty \leq \frac{C_1}{\langle t\rangle} \sup_{|q|\geq t\varepsilon} |B_{ij}(q)| + \sup_{|q|\geq t\varepsilon} |\nabla B_{ij}(q)|\,.$$

Therefore, the function $\ell\colon [0,\infty) \to [0,\infty)$ given by

$$\ell(t) := \sum_{i,j=1}^d \|\nabla_q B^\chi_{ij}(t,\cdot)\|_\infty$$

satisfies

$$\begin{aligned}\|B^\chi(t,q_1)-B^\chi(t,q_2)\| &\le \sum_{i,j=1}^{d} |B^\chi_{ij}(t,q_1)-B^\chi_{ij}(t,q_2)| \\ &\le \sum_{i,j=1}^{d} \|\nabla_q B^\chi_{ij}(t,\cdot)\|_\infty |q_1-q_2| \\ &= \ell(t)|q_1-q_2| \quad (q_1,q_2\in\mathbb{R}^d,\ t\ge 0)\end{aligned}$$

as well as

$$\int_0^\infty t^2\ell(t)\,dt \le \sum_{i,j=1}^{d}\left(C_1\int_0^\infty t\sup_{|q|\ge t\varepsilon}|B_{ij}(q)|\,dt+\int_0^\infty t^2\sup_{|q|\ge t\varepsilon}|\nabla B_{ij}(q)|\,dt\right)<\infty\,.$$

Thus, Theorem 2.2.11 yields that the wave transformation $\Omega^+_\chi\colon\mathbb{P}\to\mathbb{P}$ induced by B^χ exists and is a homeomorphism. Since $q^t_\chi - tv^+_\chi \to q^+_\chi$ converges uniformly on compact sets and $(\Omega^+_\chi)^{-1}(K)$ is compact, there exists some time $T\ge\widetilde{T}$ such that the estimate

$$|q^t_\chi(x)-tv^+_\chi(x)-q^+_\chi(x)|\le C$$

holds for $t\ge T$ and $x\in(\Omega^+_\chi)^{-1}(K)$. These initial values satisfy $(q^+_\chi(x),v^+_\chi(x))\in K$ and therefore we have

$$|q^t_\chi(x)|\ge|q^+_\chi(x)+tv^+_\chi(x)|-|q^t_\chi(x)-tv^+_\chi(x)-q^+_\chi(x)|\ge 2C\langle t\rangle - C\ge C\langle t\rangle$$

for $t\ge T$. This yields $B^\chi(t,q^t(x))=B(q^t(x))$ and therefore

$$\varphi^{t,0}_\chi(x)=\varphi^t(\varphi^{-T}\circ\varphi^{T,0}_\chi(x))$$

for $x\in(\Omega^+_\chi)^{-1}(K)$ and $t\ge T$, which implies

$$\Omega^+_\chi(x)=\Omega^+(\varphi^{-T}\circ\varphi^{T,0}_\chi(x))\,.$$

Hence, K is contained in the image of Ω^+ and due to the arbitrary choice of K this shows that Ω^+ is surjective. Since Ω^+ is injective by Corollary 2.3.8 we obtain that $(\Omega^+)^{-1}$ is given by

$$(\Omega^+)^{-1}(y)=\varphi^{-T}\circ\varphi^{T,0}_\chi\circ(\Omega^+_\chi)^{-1}(y)$$

for $y\in K$. In particular, $(\Omega^+)^{-1}$ is continuous. Also by Corollary 2.3.8, on any compact set we have the identity

$$\Omega^+=\Omega^+_\chi\circ\varphi^{T,0}_\chi\circ\varphi^T\,,$$

and the occurring flows as well as the induced wave transformations Ω^+_χ are volume preserving by Proposition 2.2.13. Thus, this property also holds for Ω^+. ∎

In particular, Ω^+ conjugates the magnetic flow $\varphi^t\big|_{s^+}$ and the free flow $\varphi_0^t\big|_{\mathbb{P}^0}$, each restricted to scattering states s^+ and to states with non-zero energy. The next task is to study if a higher regularity of the magnetic field (and thus, of the magnetic flow) yields a higher regularity of the wave transformations. The conditions involve the second derivative of functions $f \in C^2(\mathbb{R}^d, \mathbb{R})$ and we denote the Hessian by $D^2 f$.

Theorem 2.3.11 *Assume the magnetic field $B \in C^2(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is such that every component $B_{ij}\colon \mathbb{R}^d \to \mathbb{R}$ for $i,j \in \{1,\dots,d\}$ satisfies*

$$\int_0^\infty r \sup_{|q|\geq r} |B_{ij}(q)|\, dr < \infty$$

as well as

$$\int_0^\infty r^2 \sup_{|q|\geq r} |\nabla B_{ij}(q)|\, dr < \infty$$

and

$$\int_0^\infty r^3 \sup_{|q|\geq r} \|D^2 B_{ij}(q)\|\, dr < \infty\,.$$

Then the wave transformations $\Omega^\pm\colon s^\pm \to \mathbb{P}^0$ are C^1-diffeomorphisms.

Proof Similar to the proof of Theorem 2.3.10, for a compact set $K \subseteq s^+$ we consider the induced time-dependent magnetic field B^χ with a suitable cut-off function χ and show that the components B^χ_{ij} of B^χ satisfy the assumptions of Theorem 2.2.15. As in the previous proof, for fixed $i,j \in \{1,\dots,d\}$ the function B^χ_{ij} satisfies both

$$\int_0^\infty t \|B^\chi_{ij}(t,\cdot)\|_\infty\, dt < \infty$$

and

$$\int_0^\infty t^2 \|\nabla_q B^\chi_{ij}(t,\cdot)\|_\infty\, dt < \infty\,,$$

while its second derivative with respect to q equals

$$\begin{aligned} D_q^2 B^\chi_{ij}(t,q) = {} & \frac{1}{\langle t\rangle^2} D^2\chi\Big(\tfrac{q}{\langle t\rangle}\Big) B_{ij}(q) + \frac{1}{\langle t\rangle}\nabla\chi\Big(\tfrac{q}{\langle t\rangle}\Big)\nabla B_{ij}(q)^T \\ & + \frac{1}{\langle t\rangle}\nabla B_{ij}(q)\nabla\chi\Big(\tfrac{q}{\langle t\rangle}\Big)^T + \chi\Big(\tfrac{q}{\langle t\rangle}\Big) D^2 B_{ij}(q)\,. \end{aligned}$$

Since

$$\chi\left(\frac{q}{\langle t\rangle}\right) = \nabla\chi\left(\frac{q}{\langle t\rangle}\right) = D^2\chi\left(\frac{q}{\langle t\rangle}\right) = 0$$

holds for $|q| \leq t\varepsilon$ and $\chi(q) \equiv 1$ for $|q|$ sufficiently large, there are constants $C_1, C_2 > 0$ such that $\|\nabla\chi\|_\infty \leq C_1$ and $\|D^2\chi\|_\infty \leq C_2$, which yields the inequality

$$\|D_q^2 B_{ij}^\chi(t,\cdot)\|_\infty \leq \sup_{|q|\geq t\varepsilon} \left(\frac{C_2}{\langle t\rangle^2}|B_{ij}(q)| + 2\frac{C_1}{\langle t\rangle}|\nabla B_{ij}(q)| + \|D^2 B_{ij}(q)\| \right). \tag{2.41}$$

The function $\ell\colon [0,\infty) \to [0,\infty)$ given by

$$\ell(t) := \|D_q^2 B_{ij}^\chi(t,\cdot)\|_\infty$$

satisfies

$$|\nabla_q B_{ij}^\chi(t,q_1) - \nabla_q B_{ij}^\chi(t,q_2)| \leq \ell(t)|q_1 - q_2| \quad (q_1, q_2 \in \mathbb{R}^d,\ t \geq 0)$$

and, by using (2.41) together with the assumptions about the decay of the magnetic field, we obtain the estimate

$$\int_0^\infty t^3 \ell(t)\, dt \leq \int_0^\infty \sup_{|q|\geq t\varepsilon} \Big(C_2 t|B_{ij}(q)| + 2C_1 t^2 |\nabla B_{ij}(q)| + t^3\|D^2 B_{ij}(q)\| \Big)\, dt < \infty\,.$$

Hence, by applying Theorem 2.2.15 we obtain that Ω_χ^+ is a C^1-diffeomorphism. As the equality $\Omega^+ = \Omega_\chi^+ \circ \varphi^{-T} \circ \varphi_\chi^{0,T}$ holds on K and the compact set $K \subseteq s^+$ was arbitrarily chosen, this also applies for Ω^+. ■

Finally, let us put our results into the context of the ones already existing in the literature:

Remark 2.3.12 To our knowledge, the asymptotic behaviour of a classical particle in a magnetic field has only been treated by M. Loss and B. Thaller [33], who have considered the special case $d = 3$. On $\mathbb{R}^3$ one can describe a magnetic field by a Lipschitz continuous function $\vec{B}\colon \mathbb{R}^3 \to \mathbb{R}^3$ and the magnetic flow is given by the differential equation

$$\ddot{q}(t) = \dot{q}(t) \times \vec{B}(q(t))\,,$$

where $\times$ denotes the vector or cross product (compare Remark 1.2.3). Loss and Thaller considered a magnetic field $\vec{B} \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ such that, using multi-index notation, the estimates

$$|D^\alpha \vec{B}(q)| \leq \mathrm{const}(1+|q|)^{-3/2-\delta-|\alpha|} \quad (\delta > 0,\ |\alpha| = 0,1,2) \tag{2.42}$$

hold. In view of our notation this yields

$$\int_0^\infty r^{1/2} \sup_{|q|\geq r} \|B(q)\|\, dr < \infty$$

as well as

$$\int_0^\infty r^{3/2} \sup_{|q|\geq r} \|DB(q)\| \, dr < \infty$$

and

$$\int_0^\infty r^{5/2} \sup_{|q|\geq r} \|D^2B(q)\| \, dr < \infty \, .$$

For such a magnetic field they have obtained that the wave transformations are volume preserving homeomorphisms. A comparison with our corresponding result given in Theorem 2.3.10 shows that Loss and Thaller allowed a slightly weaker decay of the magnetic field and its derivative, but they required the magnetic field to be a C^2-function instead of a C^1-function. Moreover, they made assumptions on the decay of the second derivative. For the case of the magnetic field being a C^2-function and with slightly faster decay than (2.42) we have even obtained that the wave transformations are diffeomorphisms (see Theorem 2.3.11). However, this result has no analogue in the work of Loss and Thaller, who finish their analysis with the statement of the wave transformations being homeomorphisms. □

Together, Theorem 2.3.10 and Theorem 2.3.11 suggest that a higher regularity of the magnetic field B and an increasing rate of decay for the derivatives yield a higher order of differentiability of the wave transformations $\Omega^\pm$. With this conjecture we conclude the analysis of the wave transformations' regularity and turn to a different aspect.

2.4 Wave transformations on the cotangent bundle

In Section 1.2 we have introduced the Hamiltonian formulation of the magnetic flow on the momentum phase space $T^*\mathbb{R}^d$, which raises the question if there also exist wave transformations on $T^*\mathbb{R}^d$. In contrast to the previous results, for this consideration we have to assume that the magnetic field

$$\beta = \sum_{\substack{i,j=1 \\ i<j}}^{d} B_{ij} dq_i \wedge dq_j$$

is exact. Given a magnetic vector potential α of the magnetic field β and using the Legendre transformation, we have obtained the Hamiltonian flow φ_*^t on $T^*\mathbb{R}^d$ that is generated by the magnetic Hamiltonian $H(q,p) = \frac{1}{2}|p - A(q)|^2$ with the vector field A

corresponding to α. In particular, with the fibre derivative

$$\begin{aligned} \Psi\colon T\mathbb{R}^d &\to T^*\mathbb{R}^d \\ (q,v) &\mapsto (q, v + A(q)) \end{aligned}$$

of the magnetic Lagrangian $L(q,v) = \frac{1}{2}|v| + \langle A(q), v\rangle$, we have the conjugacy

$$\varphi_*^t = \Psi \circ \varphi^t \circ \Psi^{-1}$$

between the Hamiltonian flow φ_*^t on $T^*\mathbb{R}^d$ and the magnetic Euler-Lagrange flow φ^t on $T\mathbb{R}^d$. This construction depends on the magnetic potential α, but since we will later choose one specific α, we omit the reference. Similarly to φ^t, the flow φ_*^t gives rise to the *momentum wave transformations*

$$\Omega_*^\pm := \lim_{t\to\pm\infty} \varphi_{0,*}^{-t} \circ \varphi_*^t\,, \tag{2.43}$$

where $\varphi_{0,*}^t\colon T^*\mathbb{R}^d \to T^*\mathbb{R}^d$ denotes the free flow given by

$$\varphi_{0,*}^t(q,p) = (q+p, p)\,.$$

Note that the momentum wave transformations $\Omega_*^\pm$ are not necessarily conjugated to the velocity wave transformations $\Omega^\pm$. With

$$\begin{aligned} \Psi_0\colon T\mathbb{R}^d &\to T^*\mathbb{R}^d \\ (q,v) &\mapsto (q,v) \end{aligned}$$

denoting the fibre derivative of the Lagrangian $L_0(q,v) := \frac{1}{2}|v|^2$ corresponding to the free flow φ_0^t we have

$$\varphi_{0,*}^t = \Psi_0 \circ \varphi_0^t \circ \Psi_0^{-1}\,,$$

and both of the following diagrams commute. However, due to $\Psi_0 \neq \Psi$, there is no conjugacy between $\varphi_0^{-t} \circ \varphi^t$ and $\varphi_{0,*}^{-t} \circ \varphi_*^t$.

$$\begin{array}{ccc} T^*\mathbb{R}^d & \xrightarrow{\varphi_*^t} & T^*\mathbb{R}^d \\ {\scriptstyle\Psi}\big\uparrow & & \big\uparrow{\scriptstyle\Psi} \\ T\mathbb{R}^d & \xrightarrow[\varphi^t]{} & T\mathbb{R}^d \end{array} \qquad \begin{array}{ccc} T^*\mathbb{R}^d & \xrightarrow{\varphi_{0,*}^t} & T^*\mathbb{R}^d \\ {\scriptstyle\Psi_0}\big\uparrow & & \big\uparrow{\scriptstyle\Psi_0} \\ T\mathbb{R}^d & \xrightarrow[\varphi_0^t]{} & T\mathbb{R}^d \end{array}$$

As of now it is not even obvious if the limits in (2.43) exist. In the following we will show that, with the right choice for the magnetic potential, the limits exist on the sets

$$s_*^\pm := \Psi(s^\pm)\,,$$

and moreover, the momentum wave transformations

$$\Omega^\pm_* \colon s^\pm_* \to \mathbb{P}^0_* := \Psi_0(\mathbb{P}^0) = T^*\mathbb{R}^d \setminus \mathbb{R}^d$$

are symplectic diffeomorphisms. Despite the two different conjugating maps Ψ and Ψ_0, there is a connection between $\Omega^\pm$ and $\Omega^\pm_*$. To establish this, we have to start by choosing a suitable magnetic potential, i.e. we have to find a suitable 1-form α such that

$$d\alpha = \beta = \sum_{\substack{i,j=1 \\ i<j}}^{d} B_{ij} dq_i \wedge dq_j$$

holds. For the magnetic field we assume

$$\int_0^\infty r \sup_{|q|\geq r} |B_{ij}(q)| \, dr < \infty$$

and define

$$\alpha := \sum_{i=1}^{d} A_i(q) dq_i \tag{2.44}$$

with

$$A_i(q) := \sum_{j=1}^{d} \int_1^\infty B_{ij}(qs) q_j s \, ds \,, \tag{2.45}$$

where $B_{ij} := -B_{ji}$ for $i > j$ and $B_{ii} \equiv 0$, as before. This choice for α is adapted from the discussion of the quantum mechanical case by M. Loss and B. Thaller in [33], and with additional assumptions on the magnetic field, α is indeed a magnetic potential.

Lemma 2.4.1 *Assume the magnetic field satisfies*

$$\int_0^\infty r \sup_{|q|\geq r} |B_{ij}(q)| \, dr < \infty$$

and

$$\int_0^\infty r^2 \sup_{|q|\geq r} |\nabla B_{ij}(q)| \, dr < \infty$$

for all $i, j = 1, \ldots, d$. Furthermore, assume that $B_{ij}(\mathbb{R}e_j) \equiv 0$ for all $i, j = 1, \ldots, d$, where e_j denotes the j-th unit vector in $\mathbb{R}^d$. Then α defined as in (2.44) *and* (2.45) *satisfies $d\alpha = \beta$, i.e. α is a magnetic potential for β.*

Proof For any α of the form (2.44) we have

$$d\alpha = \sum_{\substack{i,j=1\\ i<j}}^{d} \left(\frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j} \right) dq_i \wedge dq_j \tag{2.46}$$

as in (1.1). To compute the partial derivatives, we have to check if the order of integration and differentiation in (2.45) can be interchanged. For this, we have to distinguish the two cases $q \neq 0$ and $q = 0$. Consider a bounded open set $U \subseteq \mathbb{R}^d$ such that $0 \notin \overline{U}$, i.e. there are constants $C, \varepsilon > 0$ such that $\varepsilon \leq |q| \leq C$ for $q \in U$. For $i, j, k = 1, \ldots, d$ we have

$$\frac{\partial}{\partial q_k}(B_{ij}(qs)q_j s) = \frac{\partial}{\partial q_k} B_{ij}(qs) q_j s^2 + B_{ij}(qs) s \delta_{jk}$$

and, thus, the inequality

$$\begin{aligned} |\frac{\partial}{\partial q_k}(B_{ij}(qs)q_j s)| &\leq |\frac{\partial}{\partial q_k} B_{ij}(qs) q_j s^2| + |B_{ij}(qs)s| \\ &\leq |q| \sup_{|\tilde{q}|=|q|s} |\nabla B_{ij}(\tilde{q})| s^2 + \sup_{|\tilde{q}|=|q|s} |B_{ij}(\tilde{q})| s \\ &\leq C \sup_{|\tilde{q}|\geq \varepsilon s} |\nabla B_{ij}(\tilde{q})| s^2 + \sup_{|\tilde{q}|\geq \varepsilon s} |B_{ij}(\tilde{q})| s \end{aligned}$$

holds for all $q \in U$ and $s \geq 1$, i.e. there is an integrable majorizing function for the partial derivatives. Therefore, the order of integration and differentiation can be interchanged and we obtain

$$\begin{aligned} \frac{\partial A_i}{\partial q_k}(q) &= \sum_{j=1}^{d} \int_1^\infty \frac{\partial}{\partial q_k} B_{ij}(qs) q_j s^2 + B_{ij}(qs) s \delta_{jk} \, ds \\ &= \int_1^\infty \sum_{j=1}^{d} \frac{\partial}{\partial q_k} B_{ij}(qs) q_j s^2 + B_{ik}(qs) s \, ds \end{aligned} \tag{2.47}$$

for $q \neq 0$. By a straightforward computation we now show that this equation also holds for $q = 0$. For all $h \in \mathbb{R}$ the equality $A_i(he_k) = 0$ holds since every summand in (2.45) vanishes: We have $B_{ij}(he_k) = 0$ for $j = k$ by assumption, and $[e_k]_j = \delta_{jk} = 0$ for $j \neq k$, where $[w]_k = w_k$ denotes the k-th component of $w \in \mathbb{R}^d$. This implies $\frac{\partial A_i}{\partial q_k}(0) = 0$, which coincides with the right hand side of (2.47) for $q = 0$, and therefore this equality holds for all $q \in \mathbb{R}^d$.

Using the identity $B_{ji} - B_{ij} = 2B_{ji}$ this yields

$$\left(\frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j} \right)(q) = \int_1^\infty \sum_{k=1}^{d} \left(\frac{\partial}{\partial q_i} B_{jk}(qs) - \frac{\partial}{\partial q_j} B_{ik}(qs) \right) q_k s^2 + 2s B_{ji}(qs) \, ds \,. \tag{2.48}$$

To simplify this expression, we have to find a relation between the partial derivatives $\frac{\partial}{\partial q_i} B_{jk}$, $\frac{\partial}{\partial q_j} B_{ik}$ and $\frac{\partial}{\partial q_k} B_{ij}$, which we obtain by computing the vanishing coefficient f_{ijk} of $dq_i \wedge d_j \wedge dq_k$ in

$$0 = d\beta =: \sum_{\substack{j_1,j_2,j_3=1 \\ j_1<j_2<j_3}}^{d} f_{j_1 j_2 j_3} dq_{j_1} \wedge d_{j_2} \wedge dq_{j_3} \,.$$

Due to $B_{ij} = -B_{ji}$ we do not need to distinguish between the order of the indices and obtain

$$\begin{aligned} 0 &= f_{ijk} dq_i \wedge dq_j \wedge dq_k \\ &= \left(\frac{\partial}{\partial q_i} B_{jk}\right) dq_i \wedge dq_j \wedge dq_k + \left(\frac{\partial}{\partial q_j} B_{ik}\right) dq_j \wedge dq_i \wedge dq_k + \left(\frac{\partial}{\partial q_k} B_{ij}\right) dq_k \wedge dq_i \wedge dq_j \\ &= \left(\frac{\partial}{\partial q_i} B_{jk} - \frac{\partial}{\partial q_j} B_{ik} + \frac{\partial}{\partial q_k} B_{ij}\right) dq_i \wedge dq_j \wedge dq_k \,, \end{aligned}$$

which yields the equality

$$\frac{\partial}{\partial q_i} B_{jk} - \frac{\partial}{\partial q_j} B_{ik} = -\frac{\partial}{\partial q_k} B_{ij} \,.$$

Following from (2.48) we obtain

$$\begin{aligned} \left(\frac{\partial A_j}{\partial q_i} - \frac{\partial A_i}{\partial q_j}\right)(q) &= \int_1^\infty -\sum_{k=1}^{d} \frac{\partial}{\partial q_k} B_{ij}(qs) q_k s^2 + 2s B_{ji}(qs)\, ds \\ &= -\int_1^\infty \langle \nabla B_{ij}(qs), q\rangle s^2 - 2s B_{ji}(qs)\, ds \\ &= \int_\infty^1 \frac{\partial}{\partial s}\Big(B_{ij}(qs)\Big) s^2 + 2s B_{ij}(qs)\, ds \\ &= \int_\infty^1 \frac{\partial}{\partial s}\left(B_{ij}(qs) s^2\right)\, ds \\ &= B_{ij}(qs) s^2 \Big|_\infty^1 \\ &= B_{ij}(q) \,, \end{aligned}$$

where the last equality holds due to $B_{ij}(qs)s^2 \to 0$ for $s \to \infty$, as in Lemma 2.1.3. Hence, according to (2.46) this yields

$$d\alpha = \sum_{\substack{i,j=1 \\ i<j}}^{d} B_{ij} dq_i \wedge dq_j = \beta$$

and therefore proves the statement. ■

Let us give a description of the additional condition on the magnetic field:

Remark 2.4.2 Recall from Remark 1.2.3 that for the case $d = 3$ the magnetic field can be described by the vector field $(b_1, b_2, b_3)\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by $b_1 = B_{23} = -B_{32}$, $b_2 = -B_{13} = B_{31}$ and $b_3 = B_{12} = -B_{21}$. Then the condition $B_{ij}(\mathbb{R}e_j) \equiv 0$ for all $i, j \in \{1, \dots, d\}$ yields that b_2 and b_3 vanish on the e_1-axis, b_1 and b_3 vanish on the e_2-axis, and b_1 and b_2 vanish on the e_3-axis. This means that on each of the three coordinate axes the magnetic field is parallel to the respective axis.
Note that we could have chosen the magnetic potential given by

$$\sum_{i=1}^{d} \left(\sum_{j=1}^{d} \int_0^1 B_{ji}(qs) q_j s \, ds \right) dq_i$$

instead, where we integrate from zero to one rather than from one to infinity. In this case, the assumptions on the decay as well as the additional condition would not have been necessary. However, with this choice of a potential the following constructions would not be possible. In particular, this potential would not decay to zero along scattering trajectories. □

Before we can proceed to relate $\Omega_*^{\pm}$ to $\Omega^{\pm}$ we have to derive a result on how fast the magnetic potential α decreases along scattering trajectories.

Lemma 2.4.3 *Let the magnetic field satisfy the assumptions of Lemma 2.4.1. Then for any compact set $K \subseteq s_*^+ = \Psi(s^+)$ the limit*

$$\lim_{t \to \infty} t A(q^t(x)) = 0$$

exists uniformly on K. The analogous result holds for $s_*^- = \Psi(s^-)$.

Proof Using Lemma 2.1.8 and inequality (1.5) we obtain constants $C_1, C_2 > 0$ and a time $T > 0$ such that $C_1 t \leq |q^t(x)| \leq C_2 t$ for $t \geq T$. For any indices $i, j \in \{1, \dots, d\}$ and $t \geq T$ we therefore have

$$\begin{aligned} |t \int_1^{\infty} B_{ij}(q^t(x)s)[q^t(x)]_j s \, ds| &\leq \int_1^{\infty} \sup_{|q| \geq C_1 ts} |B_{ij}(q)| C_2 t^2 s \, ds \\ &\overset{u=ts}{=} \frac{C_2}{C_1} \int_t^{\infty} \sup_{|q| \geq C_1 u} |B_{ij}(q)| C_1 u \, du \end{aligned}$$

and together with the assumption $\int_0^\infty \sup_{|q|\geq u} |B_{ij}(q)| u \, du < \infty$ this implies

$$|t \int_1^\infty B_{ij}(q^t(x)s)[q^t(x)]_j s \, ds| \to 0 \quad (t \to \infty) \, .$$

Since the constants C_1, C_2 and T depend on K only, the limit is uniform in $x \in K$. This implies that

$$tA_i(q^t(x)) = \sum_{j=1}^d t \int_1^\infty B_{ij}(q^t(x)s)[q^t(x)]_j s \, ds \to 0 \quad (t \to \infty)$$

converges uniformly on K, which therefore also holds for $tA(q^t(x))$. ■

Having obtained this result on the magnetic potential, we are now able to express the momentum wave transformations $\Omega_*^\pm$ on $T^*\mathbb{R}^d$ in terms of the velocity wave transformations $\Omega^\pm$ on $T\mathbb{R}^d$.

Theorem 2.4.4 *Assume the magnetic field satisfies the conditions of Theorem 2.3.11, namely*

$$\int_0^\infty r \sup_{|q|\geq r} |B_{ij}(q)| \, dr < \infty$$

as well as

$$\int_0^\infty r^2 \sup_{|q|\geq r} |\nabla B_{ij}(q)| \, dr < \infty$$

and

$$\int_0^\infty r^3 \sup_{|q|\geq r} \|D^2 B_{ij}(q)\| \, dr < \infty$$

for all $i, j = 1, \ldots, d$, such that the velocity wave transformations $\Omega^\pm : s^\pm \to \mathbb{P}^0$ on $T\mathbb{R}^d$ are diffeomorphisms. Assume furthermore that $B_{ij}(\mathbb{R}e_j) \equiv 0$ holds for all $i, j = 1, \ldots, d$. Then the momentum wave transformations

$$\Omega_*^\pm : (s_*^\pm, \omega_0) \to (\mathbb{P}_*^0, \omega_0)$$

are symplectomorphisms with $s_*^\pm = \Psi(s^\pm)$ and $\mathbb{P}_*^0 = \Psi_0(\mathbb{P}^0) = T^*\mathbb{R}^d \setminus \mathbb{R}^d$.

PROOF For $x_* \in s_*^+ = \Psi(s^+)$ we have

$$\begin{aligned}
\varphi_{0,*}^{-t} \circ \varphi_*^t(x_*) &= \varphi_{0,*}^{-t} \circ \Psi \circ \varphi^t(\underbrace{\Psi^{-1}(x_*)}_{=:x}) \\
&= \varphi_{0,*}^{-t} \circ \Psi(q^t(x), v^t(x)) \\
&= \varphi_{0,*}^{-t}(q^t(x), v^t(x) + A(q^t(x))) \\
&= (q^t(x) - t(v^t(x) + A(q^t(x))), v^t(x) + A(q^t(x))) \\
&\to (q^+(x), v^+(x)) = \Psi_0 \circ \Omega^+ \circ \Psi^{-1}(x_*) \quad (t \to \infty)\,.
\end{aligned}$$

Hence, the limit

$$\Omega_*^+ = \lim_{t\to\infty} \varphi_{0,*}^{-t} \circ \varphi_*^t$$

exists and satisfies the equality

$$\Omega_*^+ = \Psi_0 \circ \Omega^+ \circ \Psi^{-1}\,.$$

In particular, by using Theorem 2.3.11 we obtain that

$$\Omega_*^+ \colon s_*^+ \to \mathbb{P}_*^0$$

is a diffeomorphism. Furthermore, the convergence is uniform on compact subsets of s_*^+ since this holds for both the components, as shown in Proposition 2.3.2, Proposition 2.3.4 and Lemma 2.4.3. Therefore, the momentum wave transformation Ω_*^+ is the locally uniform limit of the symplectomorphisms $\varphi_{0,*}^{-t} \circ \varphi_*^t$. This allows us to use the result of M. Gromov and Y. Eliashberg described in Theorem 1.3.2: The set of symplectomorphisms $\mathrm{Symp}(s_*^+, \mathbb{P}_*^0; \omega_0, \omega_0)$ is a closed subset of the set of diffeomorphisms $\mathrm{Diff}(s_*^+, \mathbb{P}_*^0)$ with respect to the C^0-topology. Therefore, the diffeomorphism Ω_*^+ is symplectic. ∎

With this result we finish our examination of the scattering orbits and the asymptotic behaviour of the motion. In order to provide a comprehensive study of the dynamics induced by a magnetic field, we shall analyze the behaviour of bounded orbits in the following chapter.

CHAPTER 3

Symbolic dynamics

In Chapter 2 we have obtained that the magnetic flow of a magnetic field of sufficiently fast decay is conjugated to the free flow – when restricted to the scattering states. In this chapter we will see that the motion of the bounded states can become much more complex. To show this, we will restrict ourselves to the case $d = 2$, where a magnetic field is a locally Lipschitz continuous function $B\colon \mathbb{R}^2 \to \mathbb{R}$ and the magnetic flow on the phase space $\mathbb{P} = T\mathbb{R}^2$ is given by

$$\begin{cases} \dot{q} = v \\ \dot{v} = B(q)Jv \end{cases} \tag{3.1}$$

with the skew-symmetric matrix $J := \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

We shall consider multi-bump magnetic fields, i.e. those that consist of several components with disjoint supports. For the corresponding magnetic flows we will obtain a Poincaré (first return) map as well as a coding by symbolic dynamics, which shows that they carry positive topological entropy. Thus, they are chaotic in the sense of Definition 1.4.2. For magnetic fields, the entropy has been studied on compact manifolds, for example by S. Grognet, who exhibited positive topological entropy for high energies in case of negative curvature [18], and by J. Miranda, who showed that certain perturbations of the magnetic field yield positive topological entropy [37]. The closest situation to the one we shall investigate is the case of a multi-bump potential considered in [28], where a non-trivial topological index of the single bumps has been used to obtain the existence of symbolic dynamics. Here, we have non-compact energy surfaces $\mathbb{P}_E$ which are diffeomorphic to $\mathbb{R}^2 \times S^1$ for $E > 0$, so the embedding of symbolic dynamics cannot be based on the non-trivial topology of the energy surface but needs to be established by other methods. One general procedure would be to show that the dynamical system is hyperbolic, but although we will have a circular orbit and derive conditions for its hyperbolicity, the approach we shall take to establish symbolic dynamics will not be based on this. Instead, it will be mostly constructive and involves a detailed analysis of certain trajectories. In particular, our proof also works for the case of non-hyperbolic circular orbits.

In the first part we will consider a magnetic field with rotationally symmetric bumps. To do so, we start by analyzing the motion induced by a single bump. We obtain an additional integral of motion besides the kinetic energy and use it to examine the behaviour of those trajectories that stay outside the largest circular orbit. Afterwards, we use these results to study a magnetic field consisting of n bumps and construct a Poincaré map which is semi-conjugated to the full shift in n symbols. The calculations for the single bumps are based on the existence of the additional integral, which suggests that the rotational symmetry is a necessary condition to obtain symbolic dynamics. In the second part we will show that this is not the case: For a more general setting without an additional integral we exhibit a similar behaviour of the dynamics and, in particular, we also derive a semi-conjugacy to the full shift.

Parts of the analysis for rotationally symmetric bumps are based on the joint work with A. Knauf and K. F. Siburg, see [29].

3.1 Rotationally symmetric magnetic fields

We start by considering the case of a rotationally symmetric magnetic field $B\colon \mathbb{R}^2 \to \mathbb{R}$, i.e. there exists a *profile function* $\widehat{B}\colon [0,\infty) \to \mathbb{R}$ such that $\widehat{B}(|q|) = B(q)$ holds for all $q \in \mathbb{R}^2$. For now, we do not impose any further assumptions on the magnetic field. In particular, it is not necessary that the sign is fixed and, moreover, there may be regions where the magnetic field vanishes, i.e. $\operatorname{supp} B$ does not have be connected.

3.1.1 An additional integral of motion

Due to the additional symmetry one would expect the existence of another integral of motion besides the kinetic energy. In fact, Proposition 3.1.3 will show that the magnetic momentum given in Definition 3.1.2 is an integral of motion. The following remark describes how this formula is found, but we cannot use the procedure for the proof.

Remark 3.1.1 The key idea is to apply Noether's theorem (see e.g. [3]), which states the existence of an integral of motion and, moreover, yields an explicit formula.

Noether's Theorem *Let $L\colon TM \to \mathbb{R}$ be a Lagrangian on some manifold M and let $h^s\colon M \to M$ be a one-parameter family of diffeomorphisms satisfying*

$$L(h^s(q), Dh^s(q)v) = L(q, v)$$

for each $(q, v) \in TM$. Then there exists an integral of motion $I\colon TM \to \mathbb{R}$ for the Euler-Lagrange flow induced by L, which (in coordinates) is given by

$$I(q, v) = \langle \frac{\partial L}{\partial v}(q, v), \left.\frac{d}{ds}\right|_{s=0} h^s(q)\rangle\,,$$

and the value of I is independent of the choice of coordinates.

Inspired by the rotational symmetry we want to apply Noether's theorem to the magnetic Lagrangian

$$L(q, v) = \frac{1}{2}|v|^2 + \alpha_q(v) = \frac{1}{2}|v|^2 + \langle A(q), v\rangle$$

with some magnetic potential α (or A, respectively) and the one-parameter family

$$h^s\colon \mathbb{R}^2 \to \mathbb{R}^2,\ h^s(q) := \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} q =: T_s q$$

of rotations by angle $s \in \mathbb{R}$. For all $q \in \mathbb{R}^2$ we have $Dh^s(q) = T_s$ and hence

$$L(h^s(q), Dh^s(q)v) = \tfrac{1}{2}|T_s v|^2 + \langle A(T_s q), T_s v\rangle\,.$$

If the magnetic potential A satisfies $AT_s = T_s A$, we can use the orthogonality of T_s and obtain

$$L(h^s(q), Dh^s(q)v) = \tfrac{1}{2}|v|^2 + \langle A(q), v\rangle = L(q, v)\,.$$

Then Noether's theorem yields that

$$(q, v) \mapsto \langle \frac{\partial L}{\partial v}(q, v), \left.\frac{d}{ds}\right|_{s=0} h^s(q)\rangle = \langle v + A(q), -Jq\rangle \tag{3.2}$$

is an integral of motion. Hence, in order to be able to apply the theorem, we need to find a suitable magnetic potential such that $AT_s = T_s A$, which we obtain by using polar coordinates r, ϑ. For this we make the additional assumption that the integral

$$\int_0^\infty \widehat{B}(s) s\, ds < \infty$$

exists. In this case, the 1-form

$$\alpha := \left(- \int_r^\infty \widehat{B}(s) s \, ds \right) d\vartheta$$

with the "change in angle" form $d\vartheta$ satisfies

$$d\alpha = \widehat{B}(r) r dr \wedge d\vartheta = B(q) dq_1 \wedge dq_2 = \beta$$

on $\mathbb{R}^2 \setminus \{0\}$. Because of

$$d\vartheta = \frac{q_1 dq_2 - q_2 dq_1}{q_1^2 + q_2^2}$$

we have

$$\alpha = \frac{1}{|q|^2} \int_{|q|}^\infty \widehat{B}(r) r \, dr (q_2 dq_1 - q_1 dq_2) \, ,$$

and thus, the corresponding vector field $A \colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$ is given by

$$A(q) = \frac{1}{|q|^2} \int_{|q|}^\infty \widehat{B}(r) r \, dr \begin{pmatrix} q_2 \\ -q_1 \end{pmatrix} = \left(\frac{1}{|q|^2} \int_{|q|}^\infty \widehat{B}(r) r \, dr \right) Jq \, .$$

In particular, the magnetic potential A satisfies $AT_s = T_s A$ for all $s \in \mathbb{R}$. However, as a downside of using polar coordinates, the 1-form α is no magnetic potential on the whole plane $\mathbb{R}^2$ and the vector field A is not defined in the origin. If one neglected this and applied Noether's theorem anyway, one would obtain from equation (3.2) that

$$(q, v) \mapsto \langle q, Jv \rangle - \int_{|q|}^\infty \widehat{B}(r) r \, dr$$

is an integral of motion – and this term would also be defined for $q = 0$. Unfortunately, we cannot strictly apply Noether's theorem to obtain this result, although it would eventually yield a well defined expression. □

Motivated by this realization we make the following definition.

Definition 3.1.2 For a rotationally symmetric magnetic field $B \colon \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$\int_0^\infty \widehat{B}(r) r \, dr < \infty$$

we define $\mathcal{M}\colon \mathbb{P} \to \mathbb{R}$ by

$$\mathcal{M}(q,v) := \langle q, Jv\rangle - \int_{|q|}^{\infty} \widehat{B}(r) r\, dr\,.$$

We will call this function the *magnetic momentum.* □

Although we could not strictly apply Noether's theorem to obtain an integral, the following proposition shows that the considerations in Remark 3.1.1 gave the correct formula.

Proposition 3.1.3 *For a rotationally symmetric magnetic field $B\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying*

$$\int_{0}^{\infty} \widehat{B}(r) r\, dr < \infty\,,$$

the magnetic momentum $\mathcal{M}\colon \mathbb{P} \to \mathbb{R}$ is an integral of motion of the magnetic flow.

Proof It is to show that the value of $\mathcal{M}$ is constant along trajectories. For a solution curve $(q(t), v(t))$ of the magnetic equation (3.1) we have

$$\begin{aligned}\frac{d}{dt}\int_{|q(t)|}^{\infty} \widehat{B}(r) r\, dr &= -\widehat{B}(|q(t)|)|q(t)| \cdot \langle \frac{q(t)}{|q(t)|}, v(t)\rangle \\ &= -\widehat{B}(|q(t)|)\langle q(t), v(t)\rangle \quad (t \in \mathbb{R})\,.\end{aligned}$$

Note that, although the computation requires $q(t) \neq 0$, the final equation for the derivative also holds if $q(t_0) = 0$: Without loss of generality we assume $t_0 = 0$ and obtain

$$\begin{aligned}|\frac{1}{t}\left(\int_{|q(t)|}^{\infty} \widehat{B}(r) r\, dr - \int_{0}^{\infty} \widehat{B}(r) r\, dr\right)| &\leq \frac{1}{|t|}\int_{0}^{|q(t)|} |\widehat{B}(r)| r\, dr \\ &= \text{const}\frac{|q(t)|^2}{|t|} \\ &\to 0 \quad (t \to 0)\end{aligned}$$

since $|q(0)| = 0$ and $|q(t)| \leq \sqrt{2\mathcal{E}(q(0), v(0))}|t|$ by inequality (1.5). Therefore, the formula for the derivative holds without restrictions and we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{M}(q(t), v(t)) &= \langle q(t), J\dot{v}(t)\rangle + \langle v(t), Jv(t)\rangle + \widehat{B}(|q(t)|)\langle q(t), v(t)\rangle \\ &= \langle q(t), \widehat{B}(|q(t)|) J^2 v(t)\rangle + \widehat{B}(|q(t)|)\langle q(t), v(t)\rangle \\ &= -\widehat{B}(|q(t)|)\langle q(t), v(t)\rangle + \widehat{B}(|q(t)|)\langle q(t), v(t)\rangle \\ &= 0\,,\end{aligned}$$

which proves the statement. ■

Let us point out that we could have defined the magnetic momentum by

$$(q,v) \mapsto \langle q, Jv\rangle + \int_0^{|q|} \widehat{B}(r)r\,dr\,,$$

where no assumption on the decay is required. However, for the calculations in the upcoming sections our choice is more suitable. The magnetic momentum will allow us to gain a deeper insight into the dynamics, but before getting to this, we consider the relation between the magnetic momentum and the first integral, i.e. the kinetic energy $\mathcal{E}$:

Lemma 3.1.4 *The integrals $\mathcal{E}$ and $\mathcal{M}$ are independent (i.e. have linearly independent differentials) on the set*

$$\mathbb{P} \setminus \Big\{(q,v) \in \mathbb{P} \mid v = 0 \text{ or } v = \widehat{B}(|q|)Jq\Big\}\,.$$

In particular, the set

$$\Big\{(q,v) \in \mathbb{P} \mid v = \widehat{B}(|q|)Jq \neq 0\Big\}$$

is the union of all circular orbits.

Proof For the differentials we have

$$\nabla\mathcal{E}(q,v) = \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \nabla\mathcal{M}(q,v) = \begin{pmatrix} Jv + \widehat{B}(|q|)q \\ -Jq \end{pmatrix}.$$

For $v \neq 0$ these vectors are linearly dependent if and only if $Jv = -\widehat{B}(|q|)q$ and $v = -\mu Jq$ for some $\mu \neq 0$. Because of $J^2 = -\mathbb{1}$ this is equivalent to $v = \widehat{B}(|q|)Jq$ and $\mu = -\widehat{B}(|q|)$. It remains to show that initial values $(q_0, v_0) \in \mathbb{P}$ with $v_0 = \widehat{B}(|q_0|)Jq_0 \neq 0$ correspond to circular orbits. Due to the rotational symmetry we can assume $q_0 = (r, 0)$ for some $r > 0$. The only circular trajectory through this point is given by the curve

$$q(t) := r(\cos\omega t, \sin\omega t)$$

with $\omega := -\widehat{B}(r)$: For the derivatives we obtain $\dot{q}(t) = r\omega(-\sin\omega t, \cos\omega t) = \widehat{B}(r)Jq(t)$ and $\ddot{q}(t) = -\widehat{B}(r)^2 q(t)$, which in particular yields the equality $\ddot{q}(t) = \widehat{B}(r)J\dot{q}(t)$ and therefore implies that $(q(t), \dot{q}(t))$ solves the magnetic equation (3.1). Furthermore, we obtain

$$(q(0), \dot{q}(0)) = (q_0, \widehat{B}(r)Jq_0)$$

and thus, the point (q_0, v_0) lies on the circular orbit if and only if $v_0 = \widehat{B}(|q_0|)Jq_0$. ∎

In the following section we shall deepen the analysis of circular orbits, in particular the study of their hyperbolicity.

3.1.2 Circular orbits and hyperbolicity

In a constant magnetic field the trajectories are circles of a fixed radius, the Larmor radius. This raises the question for which energies there are still circular orbits in a rotationally symmetric magnetic field. The first observation is that the curvature of a solution q of the magnetic equation (3.1) at time t equals $\frac{-B(q(t))}{\sqrt{2E}}$ and hence, the existence of a circular orbit in $\mathbb{P}_E = \mathcal{E}^{-1}(E)$ of radius r with respect to the origin of the configuration plane is equivalent to the fact that r satisfies the equation

$$\frac{|\widehat{B}(r)|}{\sqrt{2E}} = \frac{1}{r}\,.$$

In fact, we would have obtained the same condition by using the characterization of the circular orbits in Lemma 3.1.4. With the energy threshold

$$\widetilde{E^\circ} = \max_{r\geq 0} \frac{\widehat{B}(r)^2 r^2}{2} \tag{3.3}$$

defined in Corollary 2.1.7, this yields the following result.

Lemma 3.1.5 *Assume the magnetic field satisfies*

$$\int_0^\infty \widehat{B}(r)\,dr < \infty\,.$$

Then, for every energy $E \in (0, \widetilde{E^\circ})$ there are at least two circular orbits and for $E = \widetilde{E^\circ}$ there is at least one circular orbit.

Proof Note that Lemma 2.1.3 yields $\widehat{B}(r)r \to 0$ for $r \to \infty$ and thus the maximum in (3.3) exists. We fix $E \in (0, \widetilde{E^\circ}]$ and denote the radius where the maximum is attained by $r_{\max} > 0$, i.e.

$$E \leq \max_{r\geq 0} \frac{\widehat{B}(r)^2 r^2}{2} = \frac{\widehat{B}(r_{\max})^2 r_{\max}^2}{2}\,.$$

If equality holds, there is a circular orbit of radius $r_{\max}$. Otherwise, we make use of the facts that $\widehat{B}(r)r = 0$ for $r = 0$ as well as $\widehat{B}(r)r \to 0$ for $r \to \infty$. Then, by continuity, there are radii $r_- < r_{\max} < r_+$ such that

$$\frac{\widehat{B}(r_\pm)^2 r_\pm^2}{2} = E\,,$$

i.e. we have two distinct circular orbits. ∎

Recall from Corollary 2.1.7 that for energies $E > \widetilde{E^\circ}$ only scattering orbits occur. Lemma 3.1.5 now shows that $\widetilde{E^\circ}$ is the optimal energy threshold for this result: For $E \leq \widetilde{E^\circ}$ there are circular orbits, which are, in particular, bounded. This observation leads us to the assumption that the energy threshold $\widetilde{E^\circ}$ coincides with Mañé's critical value, which is an energy threshold at which the behaviour of the dynamics changes (see e.g. [6, 35, 39]). However, we shall not investigate this conjecture further, but turn back to the examination of circular orbits.

After having established the existence of periodic orbits, the next target of their study is to develop a criterion for hyperbolicity or ellipticity. This criterion will be given in Proposition 3.1.8, but before that we present the necessary definitions. Recall that a fixed point $x \in M$ of a diffeomorphism $\Phi\colon M \to M$ of some manifold M is hyperbolic if $D\Phi(x)$ has no eigenvalues λ of absolute value $|\lambda| = 1$ (see e.g. [27] or [40]). This can be generalized to periodic orbits of flows $\varphi^t\colon M \to M$ given by a vector field $f\colon M \to TM$ (compare e.g. Definition 6.2.2 in [27]).

Definition 3.1.6 Let $\varphi^t\colon M \to M$ denote the (local) flow given by a vector field $f\colon M \to TM$ on some n-dimensional manifold M. Furthermore, let $\gamma\colon \mathbb{R} \to M$ be a periodic orbit of period $T > 0$. Assume there is a point $x \in \operatorname{im}\gamma$ such that the linearization $D\varphi^T(x)\colon T_xM \to T_xM$ has the eigenvalues $1, \lambda_2, \ldots, \lambda_n$ with $|\lambda_i| \neq 1$ for $i = 2, \ldots, n$. Then γ is called a *hyperbolic periodic orbit.* □

For a detailed study and, in particular, for the proof that this definition is independent of the choice of $x \in \operatorname{im}\gamma$, we refer to Chapter 5.8 in [40]. In the following we will explain the meaning of this definition for hyperbolicity. First, note that it is no restriction to ask for the eigenvalue $\lambda_1 = 1$: Due to

$$f(x) = f(\varphi^T(x)) = \left.\frac{d}{dt}\right|_{t=T} \varphi^t(x) = \left.\frac{d}{dt}\right|_{t=0} \varphi^T \circ \varphi^t(x) = D\varphi^T(x) f(x)\,,$$

the vector field $f(x)$ is an eigenvector of $D\varphi^T(x)$ with respect to the eigenvalue one. Now assume γ is a periodic orbit and let $x_0 \in \operatorname{im}\gamma$. Let $P \subseteq M$ be a hypersurface with $x_0 \in P$ that is transversal to the orbit γ. For initial values $x \in P$ near x_0 the flow returns to P in time $\tau(x)$, which gives rise to the Poincaré (first return) map $p\colon P \to P$ given by $p(x) \coloneqq \varphi^{\tau(x)}(x)$. We obtain that the eigenvalues of $Dp(x_0)$ are $\lambda_2, \ldots, \lambda_n$ and therefore, the orbit γ is hyperbolic if and only if x_0 is a hyperbolic fixed point of the Poincaré map p.

Similar to the case of hyperbolic fixed points one can generalize the notion of an elliptic fixed point.

Definition 3.1.7 A fixed point $x \in M$ of a diffeomorphism $\Phi\colon M \to M$ is elliptic if all eigenvalues of $D\Phi(x)$ have absolute value one. A periodic orbit γ of a flow $\varphi^t\colon M \to M$ is called *elliptic* if some $x \in \operatorname{im}\gamma$ is an elliptic fixed point of the time T map φ^T, where $T > 0$ is such that $\varphi^T(x) = x$. $\square$

As for hyperbolic periodic orbits, this definition is independent of the choice of $x \in \operatorname{im}\gamma$. With these definitions we are now able to turn to the characterization of circular orbits for the magnetic flow $\varphi^t\colon \mathbb{P} \to \mathbb{P}$. Assume $x_0 \in \mathbb{P}$ belongs to a circular orbit of period $T > 0$. Since the kinetic energy is constant along trajectories, we obtain $\mathcal{E}(\varphi^T(x_0)) = \mathcal{E}(x_0)$ and therefore $D\mathcal{E}(x_0)D\varphi^T(x_0) = D\mathcal{E}(x_0)$ holds, i.e $D\mathcal{E}(x_0)$ is a (left) eigenvector of $D\varphi^T(x_0)$ for the eigenvalue one. Thus, the eigenvalue one occurs twice for $D\varphi^T(x_0)$ and, in particular, the circular orbit cannot be hyperbolic for the flow φ^t. However, if for $E = \mathcal{E}(x_0)$ we consider the restriction

$$\varphi_E \coloneqq \varphi|_{\mathbb{P}_E} : \mathbb{R} \times \mathbb{P}_E \to \mathbb{P}_E$$

of the magnetic flow to the energy surface $\mathbb{P}_E = \mathcal{E}^{-1}(E)$, we eliminate the additional eigenvalue one. In particular, we can consider the question of hyperbolicity with respect to the flow φ_E.

Note that so far we have only examined the kinetic energy $\mathcal{E}$, but the same computations also hold for the magnetic momentum $\mathcal{M}$. However, in Lemma 3.1.4 we have seen that their differentials $D\mathcal{E}$ and $D\mathcal{M}$ are linearly dependent on circular orbits. In particular, the additional integral $\mathcal{M}$ does not increase the multiplicity of the eigenvalue one, so we do not have to restrict the flow further for the consideration of hyperbolicity.

Let us point out that for the following result only the rotational symmetry of the magnetic field is necessary and, since the magnetic momentum is not needed, no assumptions on its decay are required. However, we need the magnetic field to be a C^1-function.

Proposition 3.1.8 *Let $B \in C^1(\mathbb{R}^2, \mathbb{R})$ be any rotationally symmetric magnetic field. Assume there is an energy $E > 0$ such that the corresponding magnetic flow admits a circular orbit $\gamma\colon \mathbb{R} \to \mathbb{P}_E$ of radius $r > 0$ around the origin. Then γ is also a circular orbit for the restricted magnetic flow $\varphi_E = \varphi|_{\mathbb{P}_E}$. If*

$$-r\widehat{B}(r)\widehat{B}'(r) > \widehat{B}(r)^2$$

holds, this orbit is hyperbolic for φ_E, whereas if

$$-r\widehat{B}(r)\widehat{B}'(r) < \widehat{B}(r)^2$$

holds, the orbit is elliptic.

Proof The circular orbit is given by the curve $(q(t), \dot{q}(t))$ defined by

$$q(t) := r(\cos \omega t, \sin \omega t)$$

with $\omega := -\widehat{B}(r)$. By $T := \frac{2\pi}{\omega}$ we denote its period. For the examination whether the periodic orbit is hyperbolic we will choose the point $x_0 := (q(0), \dot{q}(0))$. We have to compute $D\varphi^T(x_0)$, and to do so we shall solve the variational equation

$$\begin{cases} \dot{X}(t) &= Df(\varphi^t(x_0))X(t) \\ X(0) &= \mathbb{1} \end{cases} \tag{3.4}$$

along the circular orbit, where f denotes the magnetic vector field $f(q, v) := (v, B(q)Jv)$. Because of $\nabla(\widehat{B}(|q|)) = \widehat{B}'(|q|)\frac{q}{|q|}$ we have

$$Df(q, v) = \begin{pmatrix} 0 & \mathbb{1} \\ \widehat{B}'(|q|)Jv\frac{q^T}{|q|} & \widehat{B}(|q|)J \end{pmatrix}$$

and together with $J\dot{q}(t) = r\omega(\cos \omega t, \sin \omega t)$ this yields

$$Df(\varphi^t(x_0)) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \widehat{B}'(r)r\omega\cos^2(\omega t) & \widehat{B}'(r)r\omega\cos(\omega t)\sin(\omega t) & 0 & \widehat{B}(r) \\ \widehat{B}'(r)r\omega\cos(\omega t)\sin(\omega t) & \widehat{B}'(r)r\omega\sin^2(\omega t) & -\widehat{B}(r) & 0 \end{pmatrix}$$

along the circular orbit. Unfortunately, this matrix has variable coefficients, so we cannot compute the solution of (3.4) directly via the matrix exponential. Therefore, we transform this problem into an equation with constant coefficients. For this, we introduce a new variable Y defined by the relation

$$X =: \begin{pmatrix} T_{\omega t} & 0 \\ \frac{d}{dt}T_{\omega t} & T_{\omega t} \end{pmatrix} Y =: S_t Y\,, \tag{3.5}$$

where

$$T_{\omega t} := \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}$$

denotes the matrix corresponding to the rotation by angle ωt. The variational equation (3.4) yields $\dot{X} = Df(\varphi^t(x_0))S_t Y$ and from (3.5) we obtain $\dot{X} = \dot{S}_t Y + S_t \dot{Y}$. Therefore, Y satisfies the initial value problem

$$\begin{cases} \dot{Y}(t) &= S_t^{-1}(Df(\varphi^t(x_0))S_t - \dot{S}_t)Y(t) =: A(t)Y(t)\,, \\ Y(0) &= S_0^{-1}X(0) = S_0^{-1}\,. \end{cases} \tag{3.6}$$

The computation of $A(t)$ yields

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -r\widehat{B}(r)\widehat{B}'(r) & 0 & 0 & -\widehat{B}(r) \\ 0 & 0 & \widehat{B}(r) & 0 \end{pmatrix}$$

and, in particular, we obtain that A does not depend on t. Hence, the solution of (3.6) can be described by the matrix exponential and equals $Y(t) = e^{At}S_0^{-1}$. Because of $S_T = S_0$ this yields

$$X(T) = S_T Y(T) = S_0 e^{AT} S_0^{-1}$$

and therefore the eigenvalues of $X(T)$ coincide with those of e^{AT}. To obtain these it is sufficient to compute the eigenvalues of A, which are $\lambda_1 = \lambda_2 = 0$ and

$$\lambda_{3,4} = \pm\sqrt{-\widehat{B}(r)^2 - r\widehat{B}(r)\widehat{B}'(r)}\,.$$

This implies that $X(T) = D\varphi^T(x_0)$ has the eigenvalues $1, 1, e^{\lambda_3 T}, e^{\lambda_4 T}$. The occurrence of one as an eigenvalue of multiplicity two is as expected, so only the other eigenvalues are of further interest. For

$$-r\widehat{B}(r)\widehat{B}'(r) > \widehat{B}(r)^2$$

one obtains a positive and a negative real eigenvalue of A. Consequently, the eigenvalues $e^{\lambda_{3,4}T}$ of $D\varphi^T(x_0)$ satisfy $|e^{\lambda_{3,4}T}| \neq 1$, which implies the hyperbolicity of the periodic orbit. In case of

$$-r\widehat{B}(r)\widehat{B}'(r) < \widehat{B}(r)^2$$

the eigenvalues $\lambda_{3,4}$ are imaginary, so the orbit is elliptic. ∎

In the following we will explore this condition further. Firstly, in Remark 3.1.9 we provide a different approach to the previous proof that might grant a better motivation for the substitution (3.5). Secondly, we give an example where one can recognize the hyperbolic and elliptic behaviour in the phase portrait. Finally, we examine the geometric meaning of the hyperbolicity condition.

Remark 3.1.9 By the canonical correspondence $J = -i$ we can consider the real two-dimensional equation $\ddot{q} = \widehat{B}(|q|)J\dot{q}$ as the complex equation $\ddot{z} = -i\widehat{B}(|z|)\dot{z}$, or equivalently

$$\ddot{z} + i\widehat{B}(|z|)\dot{z} = 0\,.$$

Instead of working with the total derivative of the flow, we study the directional derivatives of the solution $z(t,x)$ with respect to the initial value $x \in \mathbb{C}^2$. The circular solution is given by $z_0(t) \coloneqq re^{i\omega t}$ with $\omega \coloneqq -\widehat{B}(r)$, and by $x_0 \coloneqq (z_0(0), \dot{z}_0(0)) \in \mathbb{C}^2$ we denote the corresponding initial value. We fix a direction $\hat{\xi} \in \mathbb{C}^2 \setminus \{0\}$ and define $z_\varepsilon(t) \coloneqq z(t, x_0 + \varepsilon\hat{\xi})$ for $\varepsilon \in \mathbb{R}$ as well as the directional derivative

$$\xi \coloneqq \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0} z_\varepsilon \,.$$

Then, we obtain

$$\frac{\partial}{\partial \varepsilon}\Big(\ddot{z}_\varepsilon + i\widehat{B}(|z_\varepsilon|)\dot{z}_\varepsilon\Big) = 0$$

and hence the equation

$$\ddot{\xi} + i\,\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0}\Big(\widehat{B}(|z_\varepsilon|)\Big)\dot{z}_0 + i\widehat{B}(|z_0|)\dot{\xi} = 0.$$

Because of

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0}|z_\varepsilon| = \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0}\sqrt{z_\varepsilon\overline{z_\varepsilon}} = \frac{1}{2\sqrt{z_0\overline{z_0}}}(\xi\overline{z_0} + z_0\overline{\xi}) = \frac{\operatorname{Re}(\xi\overline{z_0})}{|z_0|}$$

we can deduce

$$\frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0}\Big(\widehat{B}(|z_\varepsilon|)\Big) = \widehat{B}'(|z_0|)\frac{\operatorname{Re}(\xi\overline{z_0})}{|z_0|} = \widehat{B}'(r)\frac{\operatorname{Re}(\xi re^{-i\omega t})}{r} = \widehat{B}'(r)\operatorname{Re}(\xi e^{-i\omega t})$$

and altogether we obtain

$$\begin{aligned} 0 &= \frac{\partial}{\partial \varepsilon}\bigg|_{\varepsilon=0}\Big(\ddot{z}_\varepsilon + i\widehat{B}(|z_\varepsilon|)\dot{z}_\varepsilon\Big) \\ &= \ddot{\xi} + i\widehat{B}'(r)\operatorname{Re}(\xi e^{-i\omega t})\dot{z}_0 + i\widehat{B}(r)\dot{\xi} \\ &= \ddot{\xi} - \widehat{B}'(r)\operatorname{Re}(\xi e^{-i\omega t})r\omega e^{i\omega t} + i\widehat{B}(r)\dot{\xi}. \end{aligned} \tag{3.7}$$

In order to derive an equation with constant coefficients we change the coordinates to the rotating frame and consider the equation in the new variable η given by

$$\xi = e^{i\omega t}\eta\,.$$

For the derivatives we have

$$\begin{aligned} \dot{\xi} &= i\omega e^{i\omega t}\eta + e^{i\omega t}\dot{\eta} \\ &= e^{i\omega t}(i\omega\eta + \dot{\eta}) \end{aligned}$$

as well as

$$\begin{aligned} \ddot{\xi} &= i\omega e^{i\omega t}(i\omega\eta + \dot{\eta}) + e^{i\omega t}(i\omega\dot{\eta} + \ddot{\eta}) \\ &= e^{i\omega t}(-\omega^2\eta + 2i\omega\dot{\eta} + \ddot{\eta})\,, \end{aligned}$$

so in the new variable η equation (3.7) has the form

$$e^{i\omega t}(-\omega^2\eta + 2i\omega\dot\eta + \ddot\eta) - \widehat{B}'(r)\operatorname{Re}(\eta)r\omega e^{i\omega t} + i\widehat{B}(r)e^{i\omega t}(i\omega\eta + \dot\eta) = 0.$$

Multiplying this equation by $e^{-i\omega t}$ and sorting by derivatives of η yields

$$\ddot\eta + (2i\omega + i\widehat{B}(r))\dot\eta + (-\omega^2 - \widehat{B}(r)\omega)\eta - r\widehat{B}'(r)\omega\operatorname{Re}(\eta) = 0\,,$$

where the occurring real part of η makes it necessary to return to the real two-dimensional view. Note that $\widehat{B}(r) = -\omega$ implies $2i\omega + i\widehat{B}(r) = -i\widehat{B}(r)$ as well as $-\omega^2 - \widehat{B}(r)\omega = 0$ and thus, we obtain

$$\ddot\eta + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\widehat{B}(r)\dot\eta + \begin{pmatrix} -r\widehat{B}(r)\widehat{B}'(r) & 0 \\ 0 & 0 \end{pmatrix}\eta = 0.$$

With $\zeta \coloneqq \dot\eta$ this gives rise to the system

$$\begin{pmatrix} \dot\eta \\ \dot\zeta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -r\widehat{B}(r)\widehat{B}'(r) & 0 & 0 & -\widehat{B}(r) \\ 0 & 0 & \widehat{B}(r) & 0 \end{pmatrix} \cdot \begin{pmatrix} \eta \\ \zeta \end{pmatrix}$$

as in the original proof of Proposition 3.1.8. □

As an example we consider the magnetic field given by

$$\widehat{B}(r) = \begin{cases} 10(1-r) & \text{for } r \le 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For a visualization it is best to consider the radial phase portrait $(r, \dot r)$ and for this we need to express the values of the integrals in polar coordinates. For $x \in \mathbb{P}_E$ let $(q(t), v(t)) \coloneqq \varphi^t(x)$ denote its trajectory and by introducing polar coordinates we obtain

$$q(t) = (r(t)\cos\vartheta(t), r(t)\sin\vartheta(t))$$

as well as

$$v(t) = (\dot r(t)\cos\vartheta(t) - r(t)\dot\vartheta(t)\sin\vartheta(t), \dot r(t)\sin\vartheta(t) + r(t)\dot\vartheta(t)\cos\vartheta(t))\,.$$

Hence, $2E = |v(t)|^2 = \dot r(t)^2 + r(t)^2\dot\vartheta(t)^2$ is constant along the trajectory, which yields

$$|\dot\vartheta(t)| = \frac{\sqrt{2E - \dot r(t)^2}}{r(t)}\,. \tag{3.8}$$

Furthermore, we have the identity

$$\begin{aligned}\langle q(t), Jv(t)\rangle &= \langle \begin{pmatrix} r(t)\cos\vartheta(t) \\ r(t)\sin\vartheta(t)\end{pmatrix}, \begin{pmatrix} \dot r(t)\cos\vartheta(t) + r(t)\dot\vartheta(t)\cos\vartheta(t) \\ -\dot r(t)\cos\vartheta(t) + r(t)\dot\vartheta(t)\sin\vartheta(t)\end{pmatrix}\rangle \\ &= r(t)^2\dot\vartheta(t)\end{aligned} \tag{3.9}$$

and derive from the magnetic momentum that the value of $r^2\dot\vartheta - \int_r^\infty \widehat{B}(s)s\,ds$ is constant. Since the magnetic field is positive, the circular orbits rotate counter-clockwise, i.e. the value of $\dot\vartheta$ is negative. Therefore, according to (3.8) we obtain that the magnetic momentum equals

$$-r\sqrt{2E - \dot r^2} - \int_r^\infty \widehat{B}(s)s\,ds\,.$$

For the kinetic energy $E = \frac{1}{2}$ the level sets are depicted in Figure 3.1 and one recognizes the two circular orbits with radii ≈ 0.89 and ≈ 0.11 as fixed points: the outer one being hyperbolic, the inner one elliptic for the flow $\varphi_E = \varphi|_{\mathbb{P}_E}$. This concludes the example.

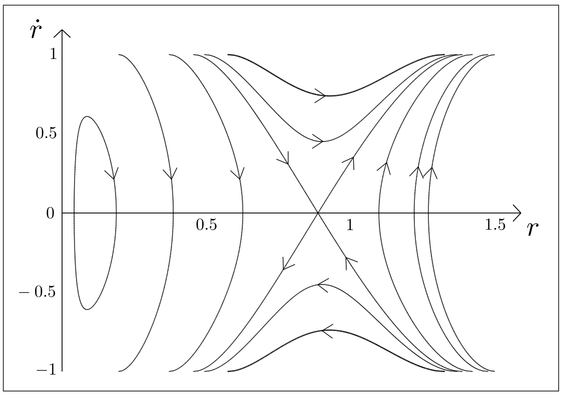


Figure 3.1: A hyperbolic and an elliptic orbit

Now we want to explore the meaning of the condition

$$-r\widehat{B}(r)\widehat{B}'(r) > \widehat{B}(r)^2\,.$$

If the magnetic field is positive along the circular orbit, i.e. $\widehat{B}(r) > 0$, this yields the inequality

$$-r\widehat{B}'(r) > \widehat{B}(r).$$

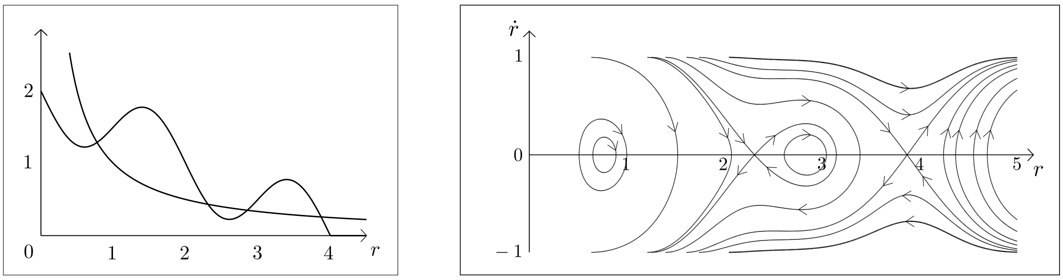
Since for circular orbits the equality $\widehat{B}(r) = \frac{\sqrt{2E}}{r}$ holds, this is equivalent to

$$\frac{d}{dr}\left(\frac{\widehat{B}(r)}{\sqrt{2E}} - \frac{1}{r}\right) = \frac{\widehat{B}'(r)}{\sqrt{2E}} + \frac{1}{r^2} = \frac{1}{r\sqrt{2E}}\left(r\widehat{B}'(r) + \widehat{B}(r)\right) < 0\,.$$

Therefore, a hyperbolic circular orbit corresponds to a transversal intersection of the two graphs of $r \mapsto \frac{1}{r}$ and $\frac{\widehat{B}}{\sqrt{2E}}$, where $\frac{\widehat{B}}{\sqrt{2E}}$ intersects $\frac{1}{r}$ "from top to bottom". Accordingly, a transversal intersection of the type "from bottom to top" corresponds to an elliptic orbit. Let us point out that "top" and "bottom" are meant with respect to the graph of $\frac{1}{r}$. If $\widehat{B}$ has compact support, this implies that hyperbolic and elliptic circular orbits occur in pairs, provided that all intersections are transversal. In particular, under this condition the innermost circular orbit is always elliptic and the outermost one always hyperbolic. For the magnetic field given by

$$\widehat{B}(r) = \begin{cases} \frac{1}{2}(4 - r + \sin(r\pi + \pi)) & \text{for } r \leq 4\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

this alternating occurrence can be seen in Figure 3.2(b) for $E = \frac{1}{2}$. The circular orbits have the properties which one expects from the graphs in Figure 3.2(a).



(a) Graphs of $\widehat{B}$ and $\frac{1}{r}$ (b) Phase portrait of φ_E for $E = \frac{1}{2}$

Figure 3.2: Hyperbolic and elliptic orbits occurring in pairs

3.1.3 The motion outside the largest circular orbit

From this point on we assume the magnetic field B to have compact support and denote the radius of the *supporting disc*

$$D \coloneqq \left\{ q \in \mathbb{R}^2 \mid |q| \leq R \right\}$$

by

$$R \coloneqq \sup(\operatorname{supp} \widehat{B}) > 0\,.$$

In Lemma 3.1.5 we have shown the existence of circular orbits for energies below a certain threshold $\widetilde{E^\circ}$. If we denote the largest radius of all circular orbits with a fixed energy

$E \leq \widetilde{E^\circ}$ by $R^+ = R^+(E)$, then we have

$$\frac{|\widehat{B}(r)|}{\sqrt{2E}} = \frac{1}{r} \quad \text{for } r = R^+$$

and

$$\frac{|\widehat{B}(r)|}{\sqrt{2E}} < \frac{1}{r} \quad \text{for } r > R^+. \tag{3.10}$$

Note that the sign of $\widehat{B}(R^+) \neq 0$ determines the orientation of the outermost circular orbit. The orbit winds clockwise around the origin if $\widehat{B}(R^+) > 0$, and counter-clockwise if $\widehat{B}(R^+) < 0$.

This section shall be devoted to the study of orbits that enter the supporting disc D but stay outside the largest circular orbit. Let us point out that this outermost circular orbit plays an important role for the dynamics. In fact, since

$$\|B(q)J\| = |\widehat{B}(|q|)|\,,$$

its radius $R^+(E)$ coincides with the virial radius $R_{\mathrm{vir}}(E)$ given in Definition 2.1.4. Therefore, Proposition 2.1.5 now reads:

Proposition 3.1.10 *Let $E \in (0, \widetilde{E^\circ}]$ and $x_0 = (q_0, v_0) \in \mathbb{P}_E$ with $|q_0| > R^+(E)$ as well as $\langle q_0, v_0 \rangle \geq 0$. Then there exists $\delta > 0$ such that*

$$|q^t(x_0)|^2 \geq |q_0|^2 + \delta t^2$$

holds for all $t \geq 0$. In particular, $x_0 \in s_E^+$ is scattering in the future and s_E^+ is open. The analogous result applies for $t \leq 0$ and s_E^- if we assume $\langle q_0, v_0 \rangle \leq 0$.

Note that the rotational symmetry is not necessary for the proof of Proposition 2.1.5, only the estimate (3.10) in the outer annulus is needed. Furthermore, the calculation in this proof, namely equation (2.4), shows that $t \mapsto |q^t(x)|^2$ is convex while $q^t(x)$ is outside the disc $\{|q| \leq R^+\}$ of radius R^+. Hence, for an orbit staying outside this disc, $|q^t(x)|$ either attains its minimum or converges to its infimum as $t \to \infty$. The next result shows that in the second case only R^+ can occur as infimum.

Proposition 3.1.11 *Let $E \in (0, \widetilde{E^\circ}]$ and $x \in \mathbb{P}_E$ such that $|q^t(x)| > R^+$ holds for $t \geq 0$. Then either*

$$\min_{t \geq 0} |q^t(x)| > R^+$$

or

$$\lim_{t \to \infty} |q^t(x)| = R^+$$

applies.

PROOF We set $(q(t), v(t)) := \varphi^t(x)$. If $\langle q(t), v(t)\rangle > 0$ holds for all $t \geq 0$, then $|q(t)|^2$ is strictly increasing and we have $\min_{t\geq 0}|q(t)| = |q(0)| > R^+$. If there is a time $T \geq 0$ such that $\langle q(T), v(T)\rangle = 0$, then by Proposition 3.1.10 the minimum is attained at time T. Hence, in the following we assume

$$\langle q(t), v(t)\rangle < 0 \quad (t \geq 0)\,.$$

This implies that $|q(t)|^2$ is strictly decreasing and therefore the same holds for $|q(t)|$, from which we deduce the convergence

$$\lim_{t\to\infty}|q(t)| = \inf_{t\geq 0}|q(t)| =: \widetilde{R} \geq R^+\,. \tag{3.11}$$

Furthermore, equation (2.4) yields that $\langle q(t), v(t)\rangle$ is increasing as long as $|q(t)| > R^+$ holds. Therefore we obtain

$$\lim_{t\to\infty}\langle q(t), v(t)\rangle = 0\,,$$

since otherwise $\langle q(t), v(t)\rangle < 0$ would be bounded away from 0, resulting in $|q(t)| \leq R^+$ for some $t \geq 0$ and hence contradicting (3.11). By assumption we have

$$\varphi^t(x) \in \left\{(q, v) \in \mathbb{P} \mid \widetilde{R} \leq |q| \leq |q_0|,\, |v| = \sqrt{2E}\right\} \quad (t \geq 0)$$

and the compactness of this set yields an increasing sequence of times $t_n \to \infty$ as well as a point $x_\infty \in \mathbb{P}_E$ such that

$$(q(t_n), v(t_n)) = \varphi^{t_n}(x) \to x_\infty =: (q_\infty, v_\infty) \quad (n \to \infty)\,.$$

This implies $|q_\infty| = \widetilde{R} > R^+$ as well as $\langle q_\infty, v_\infty\rangle = 0$ and Proposition 3.1.10 yields $x_\infty \in s_E^+$. But since $|q(t)| \to \widetilde{R}$ for $t \to \infty$ we have $\varphi^{t_n}(x) \notin s_E^+$ for any $n \in \mathbb{N}$. Because of $\varphi^{t_n}(x) \to x_\infty$ for $n \to \infty$, this is a contradiction to s_E^+ being open. Hence, the statement

$$\lim_{t\to\infty}|q(t)| = \inf_{t\geq 0}|q(t)| = R^+$$

holds. ∎

In order to understand the motion of orbits entering the support of B, we consider the set of points

$$U_E := \left\{(q, v) \in \mathbb{P}_E \mid |q| = R,\, \langle q, v\rangle \leq 0\right\} \tag{3.12}$$

through which orbits (in the configuration space $\mathbb{R}^2$) enter the supporting disc D. We want to examine the orbits staying outside the R^+-disc given by the largest circular orbit and for this we make use of the magnetic momentum.

Definition 3.1.12 For the energy $E \in (0, \widetilde{E^\circ}]$ we define

$$\mathcal{M}^+ = \mathcal{M}^+(E) := -\big(\operatorname{sign} \widehat{B}(R^+)\big) R^+ \sqrt{2E} - \int_{R^+}^{R} \widehat{B}(r) r \, dr$$

as the value of $\mathcal{M}$ on the circular orbit of radius $R^+ = R^+(E)$ and call this quantity the *critical magnetic momentum (with respect to the energy E).* □

For the following considerations we assume $\widehat{B}(R^+) > 0$, i.e.

$$\mathcal{M}^+ = -R^+ \sqrt{2E} - \int_{R^+}^{R} \widehat{B}(r) r \, dr \,,$$

and refer to Remark 3.1.19 for the case $\widehat{B}(R^+) < 0$. In this setting, the next result assures that trajectories with magnetic momentum less than $\mathcal{M}^+$ cannot enter the disc of radius R^+.

Lemma 3.1.13 *If $\mathcal{M}(x) \leq \mathcal{M}^+$ for $x \in U_E$, then $|q^t(x)| > R^+$ holds for all $t \in \mathbb{R}$.*

Proof Outside of $\operatorname{supp} B$ the motion coincides with the free motion, hence the statement $|q^t(x)| \geq R > R^+$ holds for all $t \leq 0$. Now let us assume there is some starting point $x \in U_E$ with $\mathcal{M}(x) \leq \mathcal{M}^+$, whose trajectory intersects the circle of radius R^+ at some time $T > 0$, i.e. $|q^T(x)| = R^+$. Due to the uniqueness of the solutions we have

$$\langle q^T(x), Jv^T(x) \rangle \neq -R^+ \sqrt{2E} \,,$$

since otherwise the trajectory would coincide with the circular orbit, contradicting the assumption of $x \in U_E$. Hence, we have

$$\langle q^T(x), Jv^T(x) \rangle > -R^+ \sqrt{2E}$$

and therefore

$$\mathcal{M}(x) = \langle q^T(x), Jv^T(x) \rangle - \int_{R^+}^{R} \widehat{B}(r) r \, dr > -R^+ \sqrt{2E} - \int_{R^+}^{R} \widehat{B}(r) r \, dr = \mathcal{M}^+ \,,$$

contradicting the initial assumption $\mathcal{M}(x) \leq \mathcal{M}^+$. ■

This result gives insight to the evolution of the radius $|q^t(x)|$ for an initial value $x \in U_E$ with $\mathcal{M}(x) \leq \mathcal{M}^+$. Our next aim is to investigate how such trajectories rotate around the origin. For this it is convenient to describe the motion $q(t) := q^t(x)$ by polar coordinates $q(t) = (r(t)\cos\vartheta(t), r(t)\sin\vartheta(t))$, which is possible due to $r(t) = |q(t)| > R^+$ for $t \in \mathbb{R}$. As in (3.9) the equation

$$\langle q(t), Jv(t)\rangle = r(t)^2\dot{\vartheta}(t) \quad (t \in \mathbb{R})$$

holds and hence, we have to consider the sign of $\langle q(t), Jv(t)\rangle$ to study the rotation. We expect the orbits to rotate in the same orientation around the origin as the circular orbit of radius R^+, but depending on the magnetic field we might have to decrease the energy threshold $\widetilde{E^\circ}$ for this to hold.

Definition 3.1.14 If $\widehat{B} \geq 0$ holds on $[R^+, R]$, then we define

$$E^\circ := \widetilde{E^\circ}\,.$$

If $\widehat{B}$ attains positive as well as negative values on this interval, then we set

$$E' := \frac{1}{2}\Big(\widehat{B}(\tfrac{R}{2})\tfrac{R}{2}\Big)^2$$

and define

$$E^\circ := \min\Big\{\widetilde{E^\circ}, E'\Big\}\,.$$

□

In the second case, any energy $E \leq E^\circ$ satisfies the inequality

$$\frac{1}{2}\Big(\widehat{B}(\tfrac{R}{2})\tfrac{R}{2}\Big)^2 \geq E\,,$$

and due to $(\widehat{B}(R)R)^2 = 0$, there is some radius $\tilde{r} \geq \frac{R}{2}$ such that

$$\frac{1}{2}\Big(\widehat{B}(\tilde{r})\tilde{r}\Big)^2 = E\,.$$

This yields a circular orbit of radius $\tilde{r}$ and, in particular, for the largest radius of the circular orbits we have the estimate

$$R^+ \geq \frac{R}{2}\,. \tag{3.13}$$

Considering the energy $E \in (0, E^\circ]$ fixed, this now allows us to examine how the orbits rotate around the origin:

Proposition 3.1.15 *Let $x \in U_E$ with $\mathcal{M}(x) \leq \mathcal{M}^+$. Then, there is a constant $c < 0$ such that $\langle q^t(x), Jv^t(x)\rangle \leq c$ holds for all $t \in \mathbb{R}$.*

Proof With $(q(t), v(t)) := \varphi^t(x)$ the inequality

$$\begin{aligned}\langle q(t), Jv(t)\rangle &= \mathcal{M}(x) + \int_{|q(t)|}^{R} \widehat{B}(r)r\,dr \\ &\leq -R^+\sqrt{2E} - \int_{R^+}^{R} \widehat{B}(r)r\,dr + \int_{|q(t)|}^{R} \widehat{B}(r)r\,dr \\ &= -R^+\sqrt{2E} - \int_{R^+}^{|q(t)|} \widehat{B}(r)r\,dr\end{aligned}$$

holds for all $t \in \mathbb{R}$. If $\widehat{B}(r) \geq 0$ applies for all $r \geq R^+$, then we have

$$\langle q(t), Jv(t)\rangle \leq -R^+\sqrt{2E} =: c < 0$$

for any $t \in \mathbb{R}$ since Lemma 3.1.13 assures $|q(t)| > R^+$. If there is no fixed sign of the magnetic field, estimate (3.10) yields

$$\left|\int_{R^+}^{|q(t)|} \widehat{B}(r)r\,dr\right| \leq \int_{R^+}^{R} |\widehat{B}(r)r|\,dr \leq \int_{R^+}^{R} \sqrt{2E}\,dr - \varepsilon = (R - R^+)\sqrt{2E} - \varepsilon$$

for some $\varepsilon > 0$. Using $R - 2R^+ \leq 0$ as given by (3.13), this implies

$$\langle q(t), Jv(t)\rangle \leq -R^+\sqrt{2E} + \left|\int_{R^+}^{|q(t)|} \widehat{B}(r)r\,dr\right| \leq (R - 2R^+)\sqrt{2E} - \varepsilon =: c < 0$$

for all $t \in \mathbb{R}$, which completes the proof. ∎

Note that in view of Lemma 3.1.13, trajectories of points $x \in U_E$ with $\mathcal{M}(x) \leq \mathcal{M}^+$ cannot enter the disc of radius R^+. Furthermore, for these trajectories we already know that $|q(t)|$ assumes a global minimum or converges to its infimum for $t \to \infty$. Which behaviour occurs can now be precisely described by the magnetic momentum.

Proposition 3.1.16 *Let $x \in U_E$. Then the following statements hold:*

(i) $\mathcal{M}(x) = \mathcal{M}^+ \iff \lim_{t\to\infty} |q^t(x)| = R^+$.

(ii) $\mathcal{M}(x) < \mathcal{M}^+ \implies \min_{t\in\mathbb{R}} |q^t(x)| > R^+$.

Proof *(i)* We show the implication "$\Longrightarrow$" by contradiction and set $(q(t), v(t)) := \varphi^t(x)$. If $\mathcal{M}(x) = \mathcal{M}^+$ holds and $|q(t)|$ does not converge to R^+, then Proposition 3.1.11 yields that $|q(t)|$ attains its minimum $R_{\min} > R^+$ for some time $T \geq 0$. This implies $\langle q(T), v(T)\rangle = 0$ and therefore the equality

$$\langle q(T), Jv(T)\rangle = \pm R_{\min}\sqrt{2E}$$

holds, where Proposition 3.1.15 assures

$$\langle q(T), Jv(T)\rangle = -R_{\min}\sqrt{2E}\,.$$

Thus, since the value of $\mathcal{M}$ is constant along trajectories, we have

$$\begin{aligned}
\mathcal{M}(x) &= -R_{\min}\sqrt{2E} - \int\limits_{R_{\min}}^{R} \widehat{B}(r)r\,dr \\
&= -R_{\min}\sqrt{2E} + \int\limits_{R^+}^{R_{\min}} \widehat{B}(r)r\,dr - \int\limits_{R^+}^{R} \widehat{B}(r)r\,dr \\
&< -R_{\min}\sqrt{2E} + (R_{\min} - R^+)\sqrt{2E} - \int\limits_{R^+}^{R} \widehat{B}(r)r\,dr \\
&= -R^+\sqrt{2E} - \int\limits_{R^+}^{R} \widehat{B}(r)r\,dr \\
&= \mathcal{M}^+\,,
\end{aligned}$$

which contradicts the premise.

We now show the second implication "$\Longleftarrow$": Due to $|q(t)| \to R^+$ we have the convergence $\langle q(t), v(t)\rangle \to 0$ for $t \to \infty$ as in the proof of Proposition 3.1.11, and therefore $\langle q(t), Jv(t)\rangle \to \pm R^+\sqrt{2E}$ for $t \to \infty$. If the limit was positive, we would have

$$\frac{d}{dt}\langle q(t), v(t)\rangle = 2E + B(q(t))\langle q(t), Jv(t)\rangle \to 2E + \frac{\sqrt{2E}}{R^+}R^+\sqrt{2E} = 4E \quad (t \to \infty)\,,$$

which contradicts the fact $\langle q(t), v(t)\rangle \to 0$. This implies

$$\langle q(t), Jv(t)\rangle \to -R^+\sqrt{2E} \quad (t \to \infty)$$

and since the value of $\mathcal{M}$ is constant along the trajectory of x, we obtain

$$\mathcal{M}(x) = -R^+\sqrt{2E} - \int\limits_{R^+}^{R} \widehat{B}(r)r\,dr = \mathcal{M}^+\,.$$

(ii) Let $x \in U_E$ with $\mathcal{M}(x) < \mathcal{M}^+$. For $t \leq 0$ the inequality $|q^t(x)| \geq R > R^+$ holds for any $x \in U_E$. Furthermore, since $\lim_{t\to\infty} |q(t)| = R^+$ is not possible by the first part, Proposition 3.1.11 guarantees

$$\min_{t\in\mathbb{R}} |q(t)| = \min_{t\geq 0} |q(t)| > R^+ \,.$$

■

Let us point out that the converse in assertion *(ii)* is false: If $x \in U_E$ belongs to a trajectory passing $\operatorname{supp} B$ tangentially to the right, then we have $|q^t(x)| \geq R > R^+$ for all $t \in \mathbb{R}$, but $\mathcal{M}(x) = R\sqrt{2E} > \mathcal{M}^+$.

It is our aim to examine how the angle evolves as an orbit rotates around the origin. In order to keep track of how it varies after the trajectory has entered the support, we define the rotated angle. For a trajectory that eventually leaves the support, we can describe its total change by the exit angle.

Definition 3.1.17 For a point $x \in U_E$ with $\mathcal{M}(x) \leq \mathcal{M}^+$ we define the *rotated angle* as

$$\theta(t,x) := \int_0^t \frac{\langle q^s(x), Jv^s(x)\rangle}{|q^s(x)|^2}\, ds = \int_0^t \frac{d}{ds}\vartheta^s(x)\, ds \,.$$

Furthermore, for $\mathcal{M}(x) < \mathcal{M}^+$ we consider the *exit angle*

$$\theta^e(x) := \theta(T^e(x), x)\,,$$

where by $T^e(x)$ we denote the *exit time* of x with respect to the supporting disc D, i.e. the unique time $T^e(x) \geq 0$ such that $|q^t(x)| \geq R$ holds for $t \geq T^e(x)$. □

In view of this definition, our observations in Proposition 3.1.15 and Proposition 3.1.16 immediately yield the following result.

Corollary 3.1.18 *Let $x \in U_E$. Then the following statements hold:*

(i) If $\mathcal{M}(x) \leq \mathcal{M}^+$, then the function $\theta(t,x)$ is strictly decreasing with respect to t.

(ii) If $\mathcal{M}(x) = \mathcal{M}^+$, then we have $\theta(t,x) \to -\infty$ as $t \to \infty$.

Most importantly, the equality

$$\lim_{n\to\infty} \theta^e(x_n) = -\infty$$

holds for any sequence $(x_n)_{n\in\mathbb{N}} \subseteq U_E$ such that $\mathcal{M}(x_n) < \mathcal{M}^+$ for all $n \in \mathbb{N}$ and $\mathcal{M}(x_n) \to \mathcal{M}^+$ as $n \to \infty$.

Remark 3.1.19 The previous observations apply for $\widehat{B}(R^+) > 0$. In fact, these calculations work in a similar way for the case $\widehat{B}(R^+) < 0$, where the sign of $\dot{\vartheta}$ is switched since the circular orbit turns in the opposite direction. In particular, along the circular orbit the equality

$$(R^+)^2\dot{\vartheta}(t) = \langle q(t), Jv(t)\rangle = +R^+\sqrt{2E}$$

holds. Thus, for $\widehat{B}(R^+) < 0$ we obtain

$$\mathcal{M}^+ = R^+\sqrt{2E} - \int\limits_{R^+}^{R} \widehat{B}(r)r\,dr$$

as the critical magnetic momentum, and given $x \in U_E$, the analogue of Proposition 3.1.16 now reads:

(*i*) $\mathcal{M}(x) = \mathcal{M}^+ \Longleftrightarrow \lim\limits_{t\to\infty} |q(t)| = R^+$.

(*ii*) $\mathcal{M}(x) > \mathcal{M}^+ \Longrightarrow \min\limits_{t\in\mathbb{R}} |q(t)| > R^+$.

Note that the first assertion has not changed. Furthermore, we obtain that the rotated angle $\theta(t,x)$ given in Definition 3.1.17 is strictly increasing with respect to t for any $x \in U_E$ with $\mathcal{M}(x) \geq \mathcal{M}^+$. If $\mathcal{M}(x) = \mathcal{M}^+$, it satisfies $\theta(t,x) \to +\infty$ as $t \to \infty$. Consequently, for any sequence $(x_n)_{n\in\mathbb{N}} \subseteq U_E$ satisfying $\mathcal{M}(x_n) > \mathcal{M}^+$ for all $n \in \mathbb{N}$ and $\mathcal{M}(x_n) \to \mathcal{M}^+$ as $n \to \infty$, this yields

$$\lim_{n\to\infty} \theta^e(x_n) = +\infty$$

for the exit angle θ^e. □

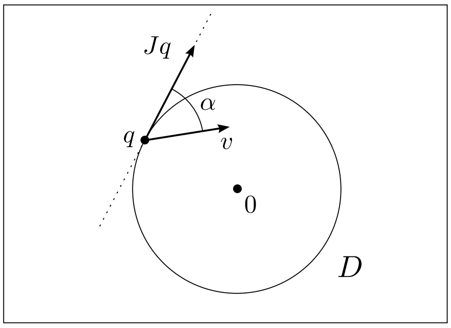
Finally, we give a geometric interpretation for the magnetic momentum on U_E:

Remark 3.1.20 Instead of regarding the value $\mathcal{M}(x)$ of the magnetic momentum for a point $x = (q,v) \in U_E$, we can consider the angle

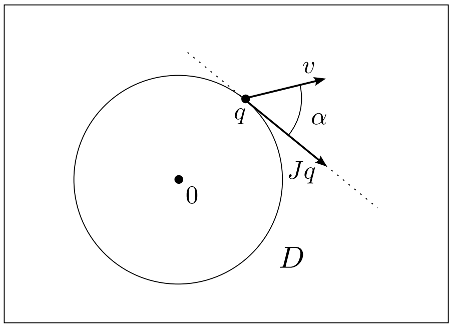
$$\alpha(x) := \arccos \frac{\langle Jq, v\rangle}{R\sqrt{2E}}$$

between Jq and v, as shown in Figure 3.3(a). Since we are interested in certain entrance directions, it is more convenient for the following to think of a critical entrance angle α^+ instead of the critical momentum $\mathcal{M}^+$. This is possible due to the orientation preserving homeomorphism between the magnetic momentum $\mathcal{M} \in [-R\sqrt{2E}, R\sqrt{2E}]$ and the angle $\alpha \in [0,\pi]$. In the following we will also consider the rotated angle θ given above and therefore, to avoid confusions, we shall refer to α as the *entrance direction* and to α^+ as the *critical entrance direction.* □

Note that we can consider the angle α between Jq and v also for points $(q, v) \in \mathbb{P}$ with $(q, -v) \in U_E$. These correspond to trajectories leaving the supporting disc, and the value of α describes the exit direction as in Figure 3.3(b). Due to the correspondence



(a) The entrance direction



(b) The exit direction

Figure 3.3: The angle α between Jq and v

between the entrance direction and the magnetic momentum we obtain the following relation.

Corollary 3.1.21 *Let $x_0 = (q_0, v_0) \in U_E$ and assume there is a time $T \geq 0$ such that $(q^T(x_0), -v^T(x_0)) \in U_E$, i.e. the trajectory of x_0 exits the supporting disc D at time T. Then the equation*

$$\alpha(x_0) = \alpha(\varphi^T(x_0))$$

holds, i.e. the entrance direction and the exit direction coincide.

PROOF Due to $|q_0| = |q^T(x_0)| = R$ the magnetic momentum equals

$$\mathcal{M}(x_0) = \langle q_0, Jv_0 \rangle$$

and

$$\mathcal{M}(\varphi^T(x_0)) = \langle q^T(x_0), Jv^T(x_0) \rangle \,.$$

As an integral of motion, $\mathcal{M}$ is constant along the trajectory of x_0, and thus we obtain the equation

$$\alpha(x_0) = \frac{\langle Jq_0, v_0 \rangle}{R\sqrt{2E}} = \frac{\langle Jq^T(x_0), v^T(x_0) \rangle}{R\sqrt{2E}} = \alpha(\varphi^T(x_0)) \,.$$

■

With this we conclude the analysis of rotationally symmetric magnetic fields, which allows us to conduct the examination of symbolic dynamics.

3.2 Symbolic dynamics for rotationally symmetric components

In this section, we turn to magnetic fields consisting of a finite number of rotationally symmetric components and study the complexity of the bounded orbits. We shall do this by the technique of symbolic dynamics, i.e. by coding the dynamics on a subset of $\mathbb{P}$ by a symbol space and the corresponding shift map. We will choose the coding in such a way that it mirrors the consecutive intersections of a trajectory with the supports of the magnetic field's components as well as the occurring rotations. Therefore, the coding can be regarded as the itinerary of the trajectory. The magnetic fields we shall consider in this section will be of the following kind: We assume that the magnetic field

$$B \coloneqq \sum_{k=1}^{n} B_k$$

consists of n components, where each component B_k is rotationally symmetric with respect to some centre $q_k \in \mathbb{R}^2$, i.e. $B_k(q) = \widehat{B}_k(|q - q_k|)$. Furthermore, with

$$R_k \coloneqq \sup(\operatorname{supp} \widehat{B}_k)$$

we consider the supporting discs

$$\operatorname{supp} B_k \subseteq D_k \coloneqq \left\{q \in \mathbb{R}^2 \mid |q - q_k| \leq R_k\right\}$$

and assume they are disjoint, i.e.

$$D_k \cap D_l = \emptyset \quad (k \neq l)\,.$$

Let us point out that the components B_k do not need to have a fixed sign. Furthermore, there may also be parts of the disc $\{|q - q_k| \leq R_k\}$ where B_k vanishes.

For a rotationally symmetric magnetic field B we have seen in Section 3.1.2 that every energy below some threshold $E^\circ = E^\circ(B) > 0$ yields a circular orbit. In the case of several components we require the existence of circular orbits for each component B_k and hence define

$$E^\circ \coloneqq \min_{1 \leq k \leq n} E^\circ(B_k) > 0\,,$$

where $E^\circ(B_k)$ is computed with respect to the centre q_k of the symmetry. We point out that the magnetic momentum introduced in Proposition 3.1.3 is no longer an integral of motion since B is not rotationally symmetric anymore. However, for each B_k we may compute its local magnetic momentum $\mathcal{M}_k$ (with q_k taking the part of the origin) and obtain a local integral in the sense that $\mathcal{M}_k$ is constant along trajectories as long as they stay outside the other supports. For the following considerations we fix an energy $E \in (0, E^\circ]$ and let $R_k^+ = R_k^+(E)$ denote the radius of the outermost circular orbit in D_k. We write $\mathcal{M}_k^+ = \mathcal{M}_k^+(E)$ for the critical magnetic momentum of B_k given by Definition 3.1.12 and $\alpha_k^+ = \alpha_k^+(E)$ for the critical entrance direction described by Remark 3.1.20.

In the next part we will need two geometric conditions on the configuration of the components $\operatorname{supp} B_k$ of the magnetic field's support. Magnetic fields satisfying these conditions are said to be "in general position", which we will define precisely in the following (see Definition 3.2.1). The first condition will allow transitions from one support to any other. If the supports were placed in a row, for example, then the transition from one support to an arbitrary other one would not be possible. The easiest way to guarantee this is to demand for the convex hull of two supports to have an empty intersection with the other supports. However, this is a rather restrictive condition and we shall use a weaker one, only demanding for certain parts of the convex hull (depending on the chosen energy) to have an empty intersection. The weaker condition on the areas of empty intersection allows a configuration as in Figure 3.4. Here, the areas where even

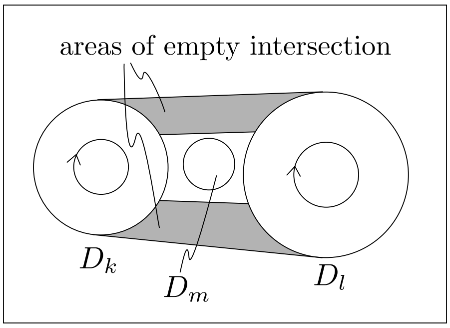
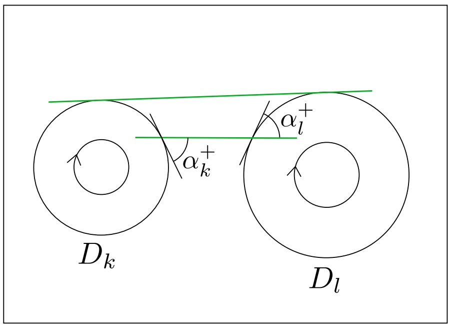


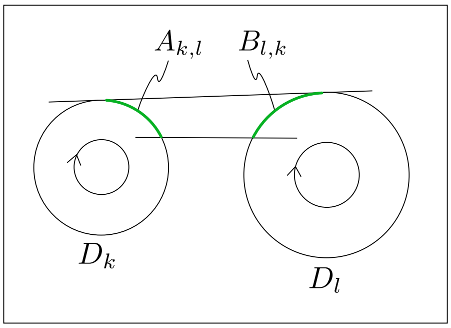
Figure 3.4: Areas of empty intersection

the weaker assumption requires an empty intersection are shaded in grey. Before we will describe how these areas are obtained, we introduce the second geometric condition. This one assures the possibility to choose an appropriate Poincaré section which counts the revolutions around the given centre q_k in the right way. We want to avoid that nearby orbits going from D_k to D_l have different numbers of intersections and therefore assume that the areas we will use for the transition from D_k to the other components do not cover the whole boundary ∂D_k of the supporting disc.

The areas we require for the transitions between two supports, as depicted in Figure 3.4, are each given by the space between two straight lines. They depend on the orientation of their circular orbits, or equivalently, on the sign of the magnetic field B along these orbits. For this we define $b_k := \widehat{B}_k(R_k^+)$ and consider all four possible sign combinations. If for fixed $k \neq l$ both signs are positive, i.e. $(b_k, b_l) = (+1, +1)$, we consider the tangent line to ∂D_k and ∂D_l passing both supports to the left as well as the line which hits ∂D_k with direction α_k^+ and ∂D_l with direction α_l^+; see Figure 3.5(a). By $A_{k,l} = A_{k,l}(E)$ and $B_{l,k} = B_{l,k}(E)$ we denote the closed sets of points on ∂D_k and ∂D_l which lie in between these two lines as depicted in Figure 3.5(b).



(a) The defining lines



(b) The sets $A_{k,l}$ and $B_{l,k}$

Figure 3.5: Transition from D_k to D_l for $(b_k, b_l) = (+1, +1)$

If both signs are negative we basically consider the same lines as in the previous case, but replace the left tangent by the right tangent as shown in Figure 3.6.

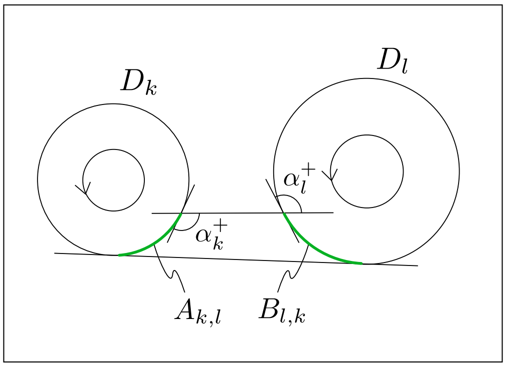
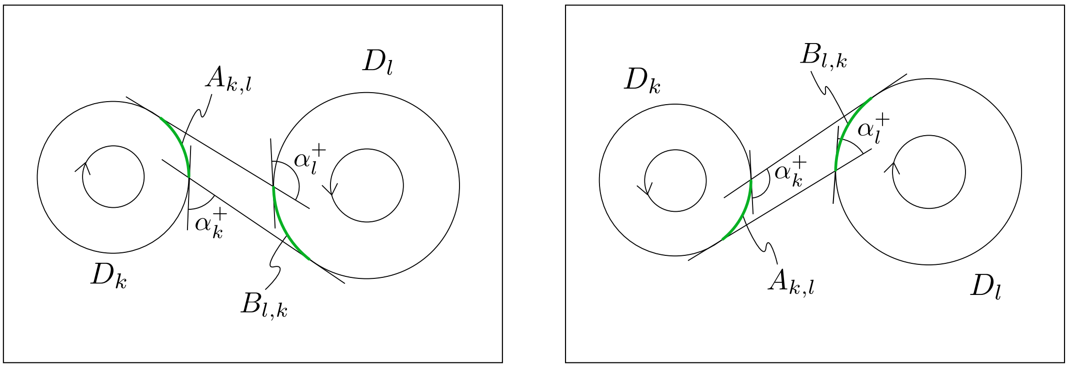


Figure 3.6: The sets $A_{k,l}$ and $B_{l,k}$ for $(b_k, b_l) = (-1, -1)$

For $(b_k, b_l) = (+1, -1)$ we consider the left tangent to ∂D_k that hits ∂D_l with direction α_l^+ as well as the line which hits ∂D_k with direction α_k^+ and is tangent to D_l on the right side; see Figure 3.7(a) for a visualization and the choice of $A_{k,l}$ and $B_{l,k}$ in this case. If $(b_k, b_l) = (-1, +1)$ we again consider the lines as in the case $(+1, -1)$ with left and right tangent interchanged; see Figure 3.7(b).



(a) $(b_k, b_l) = (+1, -1)$ (b) $(b_k, b_l) = (-1, +1)$

Figure 3.7: The sets $A_{k,l}$ and $B_{l,k}$ in case of opposite signs

The supports are supposed to be placed in such a way that for $k \neq l$ the convex hull of $A_{k,l}$ and $B_{l,k}$ intersects no other support. Note that it is possible that one small support lies in between two larger ones as shown in Figure 3.4. Let us point out that the sets $A_{k,l}$ and $B_{l,k}$ as well as the critical directions α_k^+ and α_l^+ depend on the energy value $E \in (0, E^\circ]$ that we have fixed at the beginning. Thus, all the following constructions depend on E as well, but for the sake of readability we occasionally omit the explicit reference when there are no ambiguities. We now give the precise definition of the geometric conditions we impose on the configuration, where the convex hull of a set M is denoted by $\operatorname{conv} M$.

Definition 3.2.1 Let $B = \sum_{k=1}^{n} B_k$ be a magnetic field on $\mathbb{R}^2$ whose components B_k are rotationally symmetric (with respect to their centres q_k) and have pairwise disjoint supporting discs $D_k = \{q \in \mathbb{R}^2 \mid |q - q_k| \leq R_k\}$. We say that B is *in general position with respect to the energy* $E \in (0, E^\circ]$ if in addition the following two conditions hold:

(*i*) $\operatorname{conv}\left(A_{k,l} \cup B_{l,k}\right) \cap D_m = \emptyset$ for distinct $k, l, m \in \{1, \ldots, n\}$.

(*ii*) $\partial D_k \Big\backslash \Big(\bigcup_{l \neq k} A_{k,l} \cup \bigcup_{l \neq k} B_{k,l} \Big) \neq \emptyset$ for all $k \in \{1, \ldots, n\}$. □

Note that the first condition guarantees that all the transitions from D_k to any other D_l are possible. The second condition allows us to put a Poincaré section at an appropriate place; it will be a radial segment ending in the non-empty set considered there. In particular, (ii) assures the existence of points

$$q_k^* = q_k^*(E) \in \partial D_k \Big\backslash \Big(\bigcup_{l \neq k} A_{k,l} \cup \bigcup_{l \neq k} B_{k,l} \Big) \,;$$

we pick such points q_k^* and consider them fixed in the following. With these points we define the Poincaré sections

$$P_k = P_{k,E} := \Big\{ (q,v) \in \mathbb{P}_E \mid q = q_k + \lambda(q_k^* - q_k) \text{ for some } \lambda \in (0,1),\ b_k \langle q - q_k, Jv \rangle < 0 \Big\}$$

as well as the Poincaré map

$$p = p_E \colon P_E := \bigcup_{k=1}^{n} P_{k,E} \ \longrightarrow\ \overline{P_E} \cup \{\infty\}$$

by setting $p(x)$ as the first point $\varphi^t(x) \in \overline{P_E}$ for $t > 0$, if such a point exists. If this is not the case, we define $p(x) := \infty$. Both cases are illustrated in Figure 3.8.

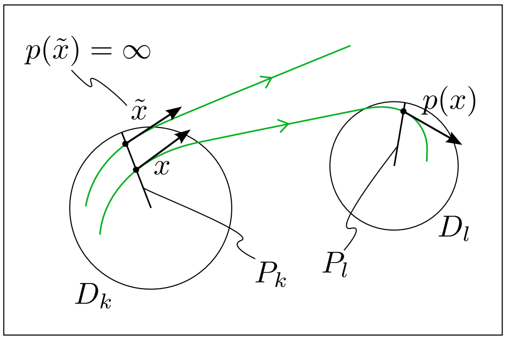


Figure 3.8: Definition of the Poincaré map p

Remark 3.2.2 Let us point out that the Poincaré map p is continuous at $x \in P_E$ if $p(x) \in P_E$ holds. We chose $p(x)$ as the first point in $\overline{P_E}$ instead of the first point in P_E to get this useful criterion for continuity. In particular, if $x_m \to x_\infty \in P_E$ and $p(x_m) \to y_\infty \in P_E$, then the continuity of the flow yields $p(x_\infty) = y_\infty$ and p is continuous at x_∞. $\square$

We want to use the Poincaré section to construct the itinerary of a trajectory, and to do so we have to restrict ourselves to those trajectories which admit such a coding. Therefore, we consider the set

$$\Lambda_E := \Big\{ x \in P_E \mid p^i(x) \in P_E \text{ for all } i \in \mathbb{Z} \Big\}$$

which is invariant under p. Note that at this point, Λ_E might contain only the points $x \in P_E$ lying on the circular orbits. Let

$$h = h_E \colon \Lambda_E \to \Sigma_n = \{1, \ldots, n\}^{\mathbb{Z}}$$

denote the canonical coding map where $h(x) = (s_i)_{i \in \mathbb{Z}}$ is defined by the condition that $p^i(x) \in P_{s_i,E}$ holds for all $i \in \mathbb{Z}$. Furthermore, let $\sigma \colon \Sigma_n \to \Sigma_n$ be the left shift map, shifting a sequence $(s_i)_{i \in \mathbb{Z}}$ one position to the left, as given in Section 1.4. By construction we obtain the identity

$$h \circ p = \sigma \circ h \,.$$

With these definitions we can formulate the first main theorem of this chapter.

Theorem 3.2.3 *Assume the magnetic field B is in general position with respect to the energy $E \in (0, E^\circ]$. Then there is a compact p_E-invariant subset $\Lambda_E' \subseteq \Lambda_E$ such that the coding map $h_E \colon \Lambda_E' \to \Sigma_n$ is continuous and surjective. In other words, the Poincaré map $p_E|_{\Lambda_E'} \colon \Lambda_E' \to \Lambda_E'$ is semi-conjugated to the full shift $\sigma \colon \Sigma_n \to \Sigma_n$.*

Proposition 1.4.4 assures that a dynamical system has at least the topological entropy of a second one it is semi-conjugated to. Thus, we immediately get the following corollary.

Corollary 3.2.4 *The Poincaré map $p_E|_{\Lambda_E'} \colon \Lambda_E' \to \Lambda_E'$ has positive topological entropy*

$$h_{\mathrm{top}}(p_E|_{\Lambda_E'}) \geq \log n \,.$$

In particular, this discrete dynamical system is chaotic.

For the following considerations and consequently for the proof of Theorem 3.2.3 we need to define various sets and maps. Recall from (3.12) in Section 3.1.3 the definition of the set U_E of points in the phase space through which orbits can enter the support. Due to the different centres of the supports, this will now be replaced by

$$U_E \coloneqq \bigcup_{k=1}^{n} U_{k,E}$$

with

$$U_k = U_{k,E} \coloneqq \Big\{ (q, v) \in \mathbb{P}_E \mid |q - q_k| = R_k,\, \langle q - q_k, v \rangle \leq 0 \Big\} \,.$$

Furthermore, we need two additional maps describing the transitions between P_E and U_E. For $x \in U_E$, i.e. for an orbit entering the support, let

$$u \colon U_E \to P_E \cup \{\infty\}$$

denote the first point $\varphi^t(x) \in P_E$ for $t \geq 0$ where the trajectory hits the Poincaré section. For a point $x \in P_E$ let

$$w\colon P_E \to U_E \cup P_E \cup \{\infty\}$$

denote the first point $\varphi^t(x) \in U_E \cup P_E$ for $t \geq 0$ where the orbit intersects the Poincaré section again or enters the support of some other component. In case such points do not exist, we set $u(x) := \infty$ and $w(x) := \infty$, respectively.

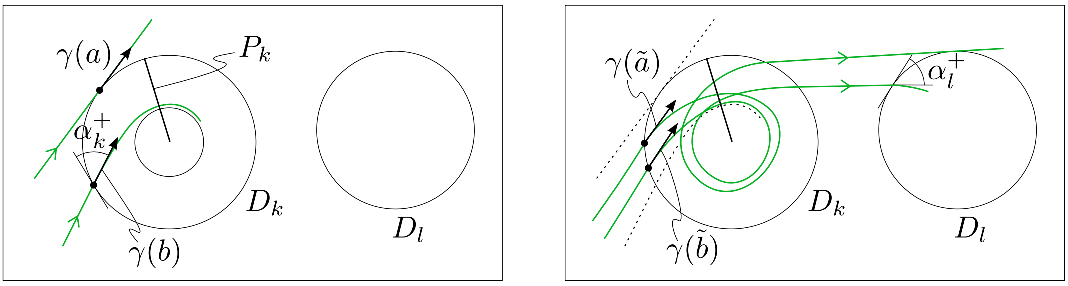
We start by describing the basic mechanism of how to find a point that realizes a prescribed itinerary. Let $I \subseteq U_k = U_{k,E}$ be a segment consisting of phase space points entering D_k, i.e.

$$I := \gamma([a, b])$$

is the image of a curve $\gamma\colon [a, b] \to U_k$. Assume that the trajectory of one endpoint of I passes D_k tangentially to the left and the other endpoint has the critical entrance direction α_k^+, i.e. its trajectory converges to the outermost circular orbit inside D_k; see Figure 3.9(a). Then, in I we find a subsegment

$$\widetilde{I} := \gamma([\tilde{a}, \tilde{b}]) \subseteq \gamma([a, b])$$

of points whose trajectories hit P_k exactly j times before leaving D_k in direction of D_l. In particular, the resulting segment $\widetilde{I}$ of points in U_l has the same configuration as the original segment entering U_k: The trajectory of one endpoint of $\widetilde{I}$ passes D_l tangentially to the left and the other endpoint has the critical entrance direction α_l^+; see Figure 3.9(b).



(a) Endpoints of original segment $\gamma([a, b])$ (b) Endpoints of resulting segment $\gamma([\tilde{a}, \tilde{b}])$

Figure 3.9: Aiming mechanism for $j = 2$

This procedure, which we will refer to as *aiming mechanism* (since one has to aim precisely into the right direction), is stated precisely in the following lemma. Note that the previous description is valid for the case of the magnetic fields being positive along the circular orbits in D_k and D_l. The statement as well as the proof are also given for this case while the general situation is treated afterwards in Remark 3.2.6.

Lemma 3.2.5 *Assume that the magnetic field B is in general position with respect to the energy $E \in (0, E^\circ]$. Let $k \neq l \in \{1, \dots, n\}$ be different indices of supports with $(b_k, b_l) = (+1, +1)$ and let $j \in \mathbb{N}$ denote the desired number of rotations. Furthermore, let $\gamma\colon [a, b] \to U_{k,E}$ be a curve with*

(a) $\alpha_k(\gamma(a)) = 0\,,$

(b) $\alpha_k(\gamma([a, b))) \subseteq [0, \alpha_k^+)\,,$

(c) $\alpha_k(\gamma(b)) = \alpha_k^+$ *and*

(d) $q_k^* \notin \operatorname{im} \gamma\,.$

Then there exists a subinterval $[\tilde{a}, \tilde{b}] \subseteq [a, b]$ such that for all points $x \in \operatorname{im} \gamma|_{[\tilde{a},\tilde{b}]}$ in the image of $\gamma|_{[\tilde{a},\tilde{b}]}$ we have

(i) $p^i(u(x)) \in P_{k,E}$ *for* $i \in \{0, \dots, j-1\}$,

(ii) $w(p^{j-1}(u(x))) \in U_{l,E}$,

and the curve $\gamma_1 := (w \circ p^{j-1} \circ u \circ \gamma)|_{[\tilde{a},\tilde{b}]} : [\tilde{a}, \tilde{b}] \to U_{l,E}$ *satisfies*

(a′) $\alpha_l(\gamma_1(\tilde{a})) = 0$,

(b′) $\alpha_l(\gamma_1([\tilde{a}, \tilde{b}))) \subseteq [0, \alpha_l^+)$,

(c′) $\alpha_l(\gamma_1(\tilde{b})) = \alpha_l^+$ *and*

(d′) $q_l^* \notin \operatorname{im} \gamma_1$.

Proof Let $q_a, q_b \in \partial D_k$ denote the two points of $\partial A_{k,l}$ as shown in Figure 3.10. The

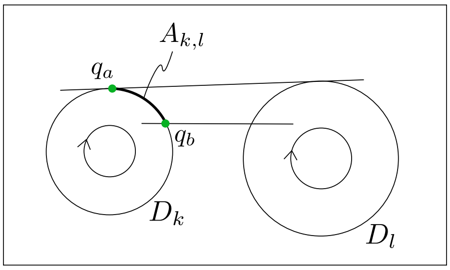


Figure 3.10: Definition of q_a and q_b

conditions (b) and (c) imply $\mathcal{M}_k(\gamma(s)) < \mathcal{M}_k^+$ for $s < b$ as well as $\mathcal{M}_k(\gamma(s)) \to \mathcal{M}_k^+$ for $s \to b$, and thus, we can apply Corollary 3.1.18 to obtain

$$\theta_k^e(\gamma(s)) \to -\infty \quad (s \to b)\,.$$

Hence, there are parameters $s_a < s_b$ in $[a, b]$ such that the trajectory of $\gamma(s_a)$ exits through q_a, the one of $\gamma(s_b)$ exits through q_b, and for $s \in [s_a, s_b]$ the forward trajectory hits $P_{k,E}$ exactly j times before leaving D_k. The trajectory of $\gamma(s_a)$ passes D_l on the left hand side and, since by Corollary 3.1.21 the entering and exiting directions coincide, the trajectory of $\gamma(s_b)$ leaves D_k with an angle less than α_k^+; see Figure 3.11. This implies

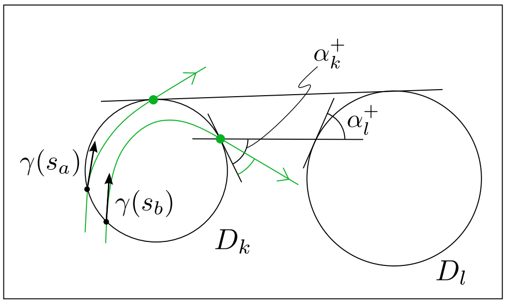


Figure 3.11: Trajectories of $\gamma(s_a)$ and $\gamma(s_b)$

that the trajectory of $\gamma(s_b)$ passes to the right of the line hitting D_l with direction α_l^+ and hence, it hits D_l with an angle strictly larger than α_l^+. Therefore, there are new parameters $\tilde{a} < \tilde{b}$ in $[s_a, s_b]$ such that the trajectory of $\gamma(\tilde{a})$ is the last to pass tangentially to the left of D_l and the trajectory of $\gamma(\tilde{b})$ is the first to hit D_l with direction α_l^+. Furthermore, this implies that for $s \in [\tilde{a}, \tilde{b}]$ the trajectory of $\gamma(s)$ hits D_l with an angle strictly less than α_l^+. Then $\gamma_1 \colon [\tilde{a}, \tilde{b}] \to U_{l,E}$ with $\gamma_1(s) := (w \circ p^j \circ u \circ \gamma)(s)$ is well defined and satisfies the stated properties. ■

Remark 3.2.6 The mechanism in the proof of Lemma 3.2.5 is manufactured for the situation where both magnetic fields have positive values on their outermost circular orbits. We argue that an analogous result holds for all other cases as well.

In the case of $(b_k, b_l) = (+1, -1)$, i.e. when for B_k the outermost circular orbit rotates clockwise and the one for B_l counter-clockwise, we can choose the curve γ_1 in such a way that the trajectory of $\gamma(\tilde{a})$ hits D_l with direction α_l^+, and the trajectory of $\gamma(\tilde{b})$ passes tangentially to the right of D_l.

Finally, a similar mechanism still works in the case of $b_k = -1$, i.e. for the circular orbit rotating counter-clockwise: If we start with a curve $\gamma \colon [a, b] \to U_{k,E}$ such that $\alpha_k(\gamma(a)) = \pi$ and $\alpha_k(\gamma(b)) = \alpha_k^+$, we have $\theta_k^e(\gamma(s)) \to +\infty$ as $s \to b$. Then again, there is a subinterval of points whose trajectories intersect $P_{k,E}$ the prescribed number of times and then leave D_k in direction of D_l. Depending on the sign of B_l, we can choose the segment of the intersection with D_l, as we did for $b_k = +1$. □

The mechanism described above allows us to show that for every half-infinite sequence there is a point realizing the prescribed itinerary.

Proposition 3.2.7 *For any half-infinite sequence $(k_i)_{i\in\mathbb{N}_0} \in \{1,\dots,n\}^{\mathbb{N}_0}$ there exists a point $x_0 \in P_{k_0,E}$ such that $p_E^i(x_0) \in P_{k_i,E}$ holds for every $i \in \mathbb{N}_0$.*

Proof We shall start with a curve $\gamma\colon [0,1] \to U_{k_0} = U_{k_0,E}$ such that $\alpha_{k_0}(\gamma(0)) = 0$ (or $\alpha_{k_0}(\gamma(0)) = \pi$, if B_{k_0} is negative along the circular orbit) and $\alpha_{k_0}(\gamma(1)) = \alpha_{k_0}^+$. Let us assume first that the sequence $(k_i)_{i\in\mathbb{N}_0}$ does not become constant eventually. Let $j_1 \geq 1$ denote the number of consecutive values of k_0, i.e. $k_i = k_0$ for $i \leq j_1 - 1$ and $k_{j_1} \neq k_0$. Then, by Lemma 3.2.5 there is a parameter interval $[a^{(1)}, b^{(1)}] \subseteq [0,1]$ such that

$$p^i(u(\gamma(s))) \in P_{k_0} = P_{k_i} \quad (i \in \{0,\dots,j_1-1\})$$

holds for all $s \in [a^{(1)}, b^{(1)}]$ and we obtain a corresponding curve $\gamma_1\colon [a^{(1)}, b^{(1)}] \to U_{k_{j_1}}$. Applying Lemma 3.2.5 to the curve γ_1 and to the number $j_2 \geq 1$ of consecutive values of k_{j_1} yields a subset $[a^{(2)}, b^{(2)}] \subseteq [a^{(1)}, b^{(1)}]$ such that the condition

$$p^i(u(\gamma(s))) \in P_{k_{j_1}} = P_{k_i} \quad (i \in \{j_1,\dots,j_1+j_2-1\})$$

holds for all $s \in [a^{(2)}, b^{(2)}]$. Hence, by iteration, we obtain a sequence of subintervals $[a^{(m)}, b^{(m)}] \subseteq [a^{(m-1)}, b^{(m-1)}]$ such that

$$p^i(u(\gamma(s))) \in P_{k_i} \quad (i \in \{0,\dots,j_1+\cdots+j_m-1\})$$

holds for all $s \in [a^{(m)}, b^{(m)}]$ and for i (at least) up to $m-1$. Then, there is a point

$$\bar{x} \in \bigcap_{m\in\mathbb{N}} \gamma([a^{(m)}, b^{(m)}]) \neq \emptyset$$

and hence

$$x_0 \coloneqq u(\bar{x}) \in P_{k_0}$$

satisfies $p^i(x_0) \in P_{k_i}$ for $i \geq 0$, which proves the claim if $(k_i)_{i\in\mathbb{N}_0}$ does not become constant eventually. However, if this is the case we follow the same procedure up to the index where $(k_i)_{i\in\mathbb{N}_0}$ becomes constant. At this index we simply choose the point with critical entrance direction and do not need to iterate further. The trajectory will converge to the corresponding circular orbit with an infinite number of intersections of the Poincaré segment. ∎

With this result for half-infinite sequences we can now turn to the proof of Theorem 3.2.3. In order to show that every bi-infinite sequence can be realized by some trajectory, we take increasing, half-infinite parts of this sequence and show that the corresponding points obtained by Proposition 3.2.7 converge to a point that has the prescribed itinerary.

Proof (of Theorem 3.2.3) Let us define

$$\Lambda_E' := \Big\{ x \in \Lambda_E \mid |q^t(x) - q_k| \geq R_k^+ \ \text{ for all } t \in \mathbb{R},\ k = 1, \dots, n \Big\}$$

as the subset of points whose trajectories stay outside the open discs $\{|q - q_k| < R_k^+\}$ for all times. To show that $h|_{\Lambda_E'}$ is surjective, let $(k_i)_{i\in\mathbb{Z}} \in \{1, \dots, n\}^{\mathbb{Z}}$. For each $m \in \mathbb{N}$, Proposition 3.2.7 yields a point $y_m \in P_{k_{-m}} = P_{k_{-m},E}$ such that $p^i(y_m) \in P_{k_{i-m}}$ holds for all $i \geq 0$, and we define

$$x_m := p^m(y_m) \in P_{k_0} \,.$$

Since $\overline{P_{k_0}}$ is compact we have a convergent subsequence which we denote by x_m again, i.e. we have $x_m \to x_\infty \in \overline{P_{k_0}}$ as $m \to \infty$. We claim that x_∞ lies in Λ_E' and satisfies $h(x_\infty) = (k_i)_{i\in\mathbb{Z}}$, i.e.

$$p^i(x_\infty) \in P_{k_i} \quad (i \in \mathbb{Z})\,.$$

In order to prove this, it suffices to show that the trajectory of x_∞ does not intersect the boundary ∂P_E of the Poincaré section: As described in Remark 3.2.2, the continuity of the flow then yields

$$p^i(x_\infty) = \lim_{m\to\infty} p^i(x_m) \in P_{k_i}$$

for all $i \in \mathbb{Z}$, which means $h(x_\infty) = (k_i)_{i\in\mathbb{Z}}$. We argue by contradiction and assume that the trajectory intersects ∂P_E at the index $i = 0$, i.e.

$$x_\infty =: (q_\infty, v_\infty) \in \partial P_{k_0} = \Big\{ (q, v) \in \mathbb{P}_E \mid q = q_{k_0} \text{ or } q = q_{k_0}^* \text{ or } \langle q - q_{k_0}, Jv\rangle = 0 \Big\}\,.$$

Now, $\langle q_\infty - q_{k_0}, Jv_\infty\rangle = 0$ is not possible since $\langle q - q_{k_0}, Jv\rangle < 0$ by Proposition 3.1.15, which also implies $q \neq q_{k_0}$. Therefore we have $q_\infty = q_{k_0}^*$, where three cases can occur. However, each of them will lead to a contradiction.

Case 1: $k_1 \neq k_0$. Since for $m \in \mathbb{N}$ the trajectory of x_m exits the supporting disc D_{k_0} in the direction of D_{k_1}, it passes the subset A_{k_0,k_1}. As this set is closed we obtain $q_\infty \in A_{k_0,k_1}$, but since $q_{k_0}^* = q_\infty$ this contradicts the choice of $q_{k_0}^* \notin A_{k_0,k_1}$.

Case 2: $k_{-1} \neq k_0$. Similar to Case 1, the backward trajectories of the points x_m pass the closed subset B_{k_0,k_1}, contradicting the choice of $q_{k_0}^* \notin B_{k_0,k_{-1}}$.

Case 3: $k_{-1} = k_0 = k_1$. We have $|q^t(x_m) - q_{k_0}| < R_{k_0}$ for $|t| < T$ with $T > 0$ independent of $m \in \mathbb{N}$, and therefore we obtain

$$|q^t(x_\infty) - q_{k_0}| \leq R_{k_0} = |q_\infty - q_{k_0}| \quad (|t| < T)\,. \tag{3.14}$$

This yields

$$\langle q_\infty - q_{k_0}, v_\infty\rangle = \frac{d}{dt}\Big|_{t=0} \frac{1}{2}|q^t(x_\infty) - q_{k_0}|^2 = 0\,,$$

i.e. the trajectory passes tangentially to D_{k_0} and therefore satisfies

$$|q^t(x_\infty) - q_{k_0}| > R_{k_0}$$

for small values of $|t|$, which contradicts inequality (3.14).

This shows that all three cases lead to contradictions and therefore, the assumption $x_\infty \in \partial P_{k_0}$ is false. The trajectory of x_∞ does not intersect ∂P_E and thus, $h|_{\Lambda'_E}$ is surjective.

It remains to show the continuity of h on Λ'_E. Due to the continuity of the flow φ^t, it is sufficient to show that the time between two consecutive intersections of the Poincaré section is uniformly bounded. Two cases are possible: The intersections can occur in the same support or in different supports. Proposition 3.1.15 gives an upper bound for the time that a trajectory can stay inside the outer annulus in one support without intersecting the Poincaré section. Furthermore, outside the support the trajectories are straight lines and, since $|\dot{q}| \equiv \sqrt{2E}$, the time is proportional to the length of the trajectory. Therefore, we have a uniform bound for the length of the segments between two consecutive intersections of the orbit with the Poincaré section and thus, a uniform bound for the time, which consequently yields the continuity of h on Λ'_E. ∎

With this we have shown that the Poincaré map $p_E|_{\Lambda'_E}$ is semi-conjugated to the full shift in n symbols, i.e. for the bounded states the motion exhibits chaotic behaviour. Let us point out that the required rotational symmetry of the magnetic field is a rather restrictive constraint, but in the following section we shall see that the characteristics of the motion do not depend on this condition.

3.3 Non-rotationally symmetric magnetic fields

In this section we will show that the result of Theorem 3.2.3 does not depend on the existence of the additional integral, the magnetic momentum $\mathcal{M}$, although its frequent use might suggest this at first. Its existence was helpful for the proofs, but we will now show that it is not necessary to obtain a semi-conjugacy to the full shift. Similarly to the rotationally symmetric case we start by studying the motion inside a single bump, which will later be one component of the magnetic field.

3.3.1 The motion outside the largest circular orbit

We consider a magnetic field $B\colon \mathbb{R}^2 \to \mathbb{R}$ with compact support

$$\operatorname{supp} B \subseteq D := \Big\{ q \in \mathbb{R}^2 \mid |q| \leq R \Big\}$$

for some $R > 0$. For a fixed energy $E > 0$ we assume that there is a largest circular orbit of radius $R^+ = R^+(E)$, i.e.

$$\frac{|B(q)|}{\sqrt{2E}} = \frac{1}{|q|} \quad \text{for } |q| = R^+ \,.$$

Along this circular orbit the magnetic field has to be constant, but apart from this we do not require rotational symmetry. We do require, though, that the strength of the magnetic field is sufficiently weak outside of this circular orbit, as it was the case for rotationally symmetric magnetic fields. In particular, we assume

$$\frac{|B(q)|}{\sqrt{2E}} < \frac{1}{|q|} \quad \text{for } |q| > R^+ \,. \tag{3.15}$$

The final assumption is that the magnetic field does not change its sign outside the circular orbit, i.e. either $B(q) \geq 0$ for $|q| \geq R^+$ or $B(q) \leq 0$ for $|q| \geq R^+$. Before we gather these conditions in Definition 3.3.1, let us point out that for the rotationally symmetric case they describe the energies $E \in (0, E^\circ]$. Furthermore, we observe that Proposition 3.1.10 and Proposition 3.1.11 both hold for these magnetic fields as well since their proofs only make use of estimate (3.15) and do not depend on the rotational symmetry of the magnetic field. In particular, this implies that there is no other periodic orbit with $|q^t(x)| > R^+$ for some $t \in \mathbb{R}$.

Definition 3.3.1 Assume that for an energy $E > 0$ there is a radius $R^+ = R^+(E)$ such that the following conditions are satisfied:

(i) $\dfrac{|B(q)|}{\sqrt{2E}} = \dfrac{1}{|q|}$ holds for $|q| = R^+$.

(ii) $\dfrac{|B(q)|}{\sqrt{2E}} < \dfrac{1}{|q|}$ holds for $|q| > R^+$.

(iii) Either $B(q) \geq 0$ or $B(q) \leq 0$ holds for $|q| \geq R^+$.

Then we call E a *circular energy.* The union of all circular energies we denote by $\mathcal{C}$. $\square$

For the following considerations we fix a circular energy $E \in \mathcal{C}$. To avoid having to distinguish between positive and negative magnetic fields, we consider the first case explicitly and treat the second one later in Remark 3.3.8. Since we want to prescribe the order in which a trajectory hits the supports of the components, we now take a look at the set of points

$$U_E := \Big\{(q,v) \in \mathbb{P}_E \mid |q| = R,\ \langle q, v\rangle \le 0\Big\}$$

through which orbits (in the configuration space $\mathbb{R}^2$) enter the supporting disc D. Using the angle

$$\alpha\colon U_E \to [0,\pi],\ \alpha(q,v) := \arccos \frac{\langle Jq, v\rangle}{R\sqrt{2E}}$$

between Jq and v, we define the sets

$$U_E^\beta := \Big\{x \in U_E \mid \alpha(x) = \beta\Big\}$$

of points $x \in U_E$ whose trajectories enter the supporting disc with angle $\alpha(x) = \beta$. Initial values $x \in U_E^0$ correspond to trajectories that pass D tangentially to the left, and for $x \in U_E^\pi$ the trajectory passes D tangentially to the right. As in Remark 3.1.20 we will refer to α as the *entrance direction*. Furthermore, by

$$\widehat{U}_E := \Big\{x \in U_E \mid \inf_{t\ge 0} |q^t(x)| > R^+\Big\}$$

we denote the subset of initial values whose trajectories stay away from the closed disc of radius R^+. As a consequence of Proposition 3.1.11, these trajectories are scattering:

Lemma 3.3.2 *The set $\widehat{U}_E$ consists of scattering states, i.e. $\widehat{U}_E \subseteq s_E^+$.*

PROOF Let $x \in \widehat{U}_E$. By Proposition 3.1.11 there is a time $T \ge 0$ such that

$$|q^T(x)| = \min_{t\ge 0} |q^t(x)| > R^+$$

holds. This implies $\langle q^T(x), v^T(x)\rangle = 0$ and thus, Proposition 3.1.10 yields $x \in s_E^+$. ∎

From this lemma we obtain the next result.

Lemma 3.3.3 *The set $\widehat{U}_E$ is open in U_E.*

PROOF Let $x_0 \in \widehat{U}_E$. Since $x_0 \in s_E^+$, there is a time $T > 0$ with $|q^t(x_0)| > R$ for $t \ge T$, and $\langle q^T(x_0), v^T(x_0)\rangle > 0$. Furthermore, we have the identity

$$\inf_{t\ge 0} |q^t(x_0)| = \min_{t\in[0,T]} |q^t(x_0)| > R^+\,.$$

Thus, there is a neighbourhood $N \subseteq U_E$ of x_0 such that $|q^T(x)| > R$, $\langle q^T(x), v^T(x)\rangle > 0$ and

$$\min_{t\in[0,T]} |q^t(x)| > R^+$$

hold for all $x \in N$. In particular, Proposition 3.1.10 yields $|q^t(x)| > R > R^+$ for $t \geq T$ and $x \in N$. This implies

$$\inf_{t \geq 0} |q^t(x)| = \min_{t \in [0,T]} |q^t(x)| > R^+ \quad (x \in N)$$

and therefore $N \subseteq \widehat{U}_E$, i.e. $\widehat{U}_E$ is open. ∎

In order to analyze the motion, it is our aim to imitate the proofs from the rotationally symmetric case. Since we cannot make use of another integral besides the kinetic energy, we need an adequate replacement for the critical magnetic momentum $\mathcal{M}^+$. This replacement will be one of the connected components of $\widehat{U}_E$; for the following arguments we need to make sure that there are at least two of them.

Lemma 3.3.4 *The sets U_E^0 and U_E^π are in different connected components of $\widehat{U}_E$.*

Proof Let us assume otherwise. Then, since $\widehat{U}_E$ is open, U_E^0 and U_E^π are in the same path-connected component and there is a curve $\gamma\colon [0,1] \to \widehat{U}_E$ such that $\gamma(0) \in U_E^0$ and $\gamma(1) \in U_E^\pi$. On the compact set $\operatorname{im}\gamma \subseteq s_E^+$ the continuous function $T^e\colon U_E \cap s_E^+ \to \mathbb{R}$, which denotes the exit time with respect to D, is bounded from above by some $T_{\max} > 0$. We now consider the function $f\colon [0, T_{\max}] \times [0,1] \to \mathbb{R}$ given by

$$f(t,s) := \langle q^t(\gamma(s)), Jv^t(\gamma(s))\rangle\,.$$

This function is continuous and, in particular, gives rise to the continuous function $\eta\colon [0,1] \to \mathbb{R}$ by

$$\eta(s) := \max_{t \in [0, T_{\max}]} f(t,s)\,.$$

For any initial value $x \in \mathbb{P}_E$ we have

$$\begin{aligned}\frac{d}{dt}\langle q^t(x), Jv^t(x)\rangle &= \langle v^t(x), Jv^t(x)\rangle + B(q^t(x))\langle q^t(x), J^2 v^t(x)\rangle \\ &= -B(q^t(x))\langle q^t(x), v^t(x)\rangle\,,\end{aligned} \tag{3.16}$$

and therefore the value of $\langle q, Jv\rangle$ is constant while the trajectory is outside $\operatorname{supp} B$. This implies $\eta(0) = -R\sqrt{2E}$ as well as $\eta(1) = R\sqrt{2E}$ and hence, by the continuity of η, there is some parameter $s_0 \in (0,1)$ such that

$$\eta(s_0) = 0\,.$$

In particular, there is a time $t_0 \in [0, T_{\max}]$ such that $f(t_0, s_0) = \eta(s_0) = 0$, i.e.

$$\langle q^{t_0}(\gamma(s_0)), Jv^{t_0}(\gamma(s_0))\rangle = 0\,.$$

This means that q and v either point in the same direction or in opposite directions, i.e. $v = \pm\frac{\sqrt{2E}}{|q|}q$, which implies

$$\langle q^{t_0}(\gamma(s_0)), v^{t_0}(\gamma(s_0))\rangle = \pm|q^{t_0}(\gamma(s_0))|\sqrt{2E}\,.$$

We first consider the case of opposite directions, i.e. a negative sign; the other case works in the same way. Using (3.16) we obtain

$$\frac{d}{dt}\bigg|_{t=t_0} f(t, s_0) = B(q^{t_0}(\gamma(s_0)))|q^{t_0}(\gamma(s_0))|\sqrt{2E} \geq 0\,, \tag{3.17}$$

where strict inequality or equality depends on whether the magnetic field satisfies either $B(q^{t_0}(\gamma(s_0))) > 0$ or $B(q^{t_0}(\gamma(s_0))) = 0$. In the case of strict inequality in (3.17) we have

$$f(t, s_0) > f(t_0, s_0) = 0$$

for $t > t_0$ with $t - t_0$ small enough, and hence

$$\eta(s_0) = \max_{t\in[0,T_{\max}]} f(t, s_0) > 0\,. \tag{3.18}$$

This contradicts the choice of s_0, which asserts $\eta(s_0) = 0$. If, on the other hand, equality holds in (3.17) for the time t_0, then we claim that there is a time $t_1 > t_0$ with

$$\frac{d}{dt}\bigg|_{t=t_1} f(t, s_0) > 0\,,$$

which again results in contradiction (3.18). We show this claim itself by contradiction: Assume that

$$\frac{d}{dt} f(t, s_0) = -B(q^t(\gamma(s_0)))\langle q^t(\gamma(s_0)), v^t(\gamma(s_0))\rangle = 0$$

holds for all $t \geq t_0$. This implies that B equals zero along the trajectory as long as $\langle q^t(\gamma(s_0)), v^t(\gamma(s_0))\rangle \neq 0$ and hence, the trajectory coincides with the straight line

$$q^{t+t_0}(\gamma(s_0)) = q^{t_0}(\gamma(s_0)) - t\frac{\sqrt{2E}}{|q^{t_0}(\gamma(s_0))|}q^{t_0}(\gamma(s_0))$$

for $t \geq 0$ until $q^{t+t_0}(\gamma(s_0)) = 0$. In particular, there is a time $T > 0$ such that

$$|q^{T+t_0}(\gamma(s_0))| = R^+\,,$$

contradicting the assumption $\gamma(s_0) \in \widehat{U}_E$, which assures

$$\inf_{t\geq 0} |q^t(\gamma(s_0))| > R^+\,.$$

Thus, strict inequality and equality in (3.17) both lead to a contradiction. This shows that the initial assumption of U_E^0 and U_E^π being in the same connected component is false and hence proves the statement. ■

This result shows that there are at least two connected components of $\widehat{U}_E$. In the following, we will only use the connected component containing U_E^0 which we denote by U_E^+. A visualization of these subsets of U_E is given in Figure 3.12. Recall the special

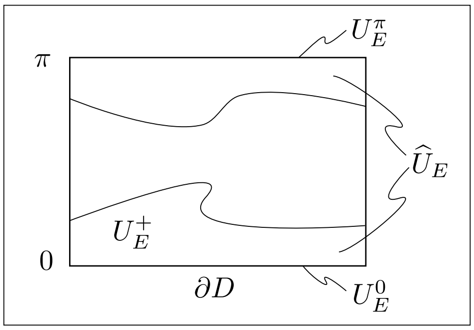


Figure 3.12: Visualization of the subsets of $U_E \cong \partial D \times [0, \pi]$

case of a rotationally symmetric magnetic field, in which we have

$$U_E^+ = \left\{ x \in U_E \mid \alpha(x) \in [0, \alpha^+) \right\},$$

or equivalently, $x \in U_E^+$ if and only if $\mathcal{M}(x) < \mathcal{M}^+$. The following two lemmas show that trajectories with magnetic momentum $\mathcal{M}(x) \leq \mathcal{M}^+$ in the rotationally symmetric case have a similar behaviour as those in the general case which start at $\overline{U_E^+}$.

Lemma 3.3.5 *Let $x \in U_E^+$. Then $\langle q^t(x), Jv^t(x) \rangle < 0$ holds for all $t \geq 0$.*

Proof For $x \in U_E^+$ there exists a curve $\gamma\colon [0,1] \to U_E^+$ such that $\gamma(0) \in U_E^0$ and $\gamma(1) = x$. If we assume that there is a time $T \geq 0$ such that $\langle q^T(x), Jv^T(x) \rangle \geq 0$ holds, then we can apply the arguments from the proof of Lemma 3.3.4 and obtain the same contradiction. This proves the statement. ∎

This means that these orbits rotate around the origin in the same orientation as the circular orbit, which has also been the case in a rotationally symmetric magnetic field (see Proposition 3.1.15). Furthermore, we can transfer the important properties of points $x \in \mathbb{P}$ with critical momentum $\mathcal{M}(x) = \mathcal{M}^+$ to the non-rotationally symmetric case.

Lemma 3.3.6 *Let $x \in \partial U_E^+$. Then the following two conditions hold:*

(i) $|q^t(x)| \to R^+$ *for* $t \to \infty$.

(ii) $\langle q^t(x), Jv^t(x) \rangle \to -R^+\sqrt{2E}$ *for* $t \to \infty$.

Proof *(i)* For $x \in \partial U_E^+$ there is an increasing sequence of times $t_n \geq 0$ such that $|q^{t_n}(x)| \to R^+$ converges for $n \to \infty$. If we assume

$$t_\infty := \sup_{n \in \mathbb{N}} t_n < \infty\,,$$

then the minimum R^+ of the radius $|q^t(x)|$ is attained at time $t_\infty \in \mathbb{R}$, which implies the equation $\langle q^{t_\infty}(x), v^{t_\infty}(x)\rangle = 0$. This yields that either

$$\langle q^{t_\infty}(x), Jv^{t_\infty}(x)\rangle = -R^+\sqrt{2E}$$

or

$$\langle q^{t_\infty}(x), Jv^{t_\infty}(x)\rangle = R^+\sqrt{2E}$$

holds. The first option is not possible because the trajectory would coincide with the circular orbit of radius R^+. The second option is also not possible since the mapping

$$\mathbb{R} \times U_E \to \mathbb{R},\ (t, x) \mapsto \langle q^t(x), Jv^t(x)\rangle$$

is continuous and, by Lemma 3.3.5, we have $\langle q^t(x), Jv^t(x)\rangle < 0$ for all $x \in U_E^+$ and $t \geq 0$. Hence, $t_\infty < \infty$ is not possible, i.e. $t_\infty = \infty$. Proposition 3.1.10 guarantees that

$$\langle q^t(x), v^t(x)\rangle < 0$$

holds for all $t \geq 0$, which means that $|q^t(x)|^2$ is strictly decreasing, and therefore we have $|q^t(x)| \to R^+$ for $t \to \infty$.

(ii) We use that the conditions $|q^t(x)| \to R^+$ for $t \to \infty$ and $\langle q^t(x), v^t(x)\rangle < 0$ for $t \geq 0$ together imply the convergence $\langle q^t(x), v^t(x)\rangle \to 0$ for $t \to \infty$, as previously in the proof of Proposition 3.1.11. This yields

$$\langle q^t(x), Jv^t(x)\rangle \to \pm R^+\sqrt{2E} \quad (t \to \infty)\,,$$

where the positive value is not possible, as in the proof of the first part. This shows the convergence

$$\langle q^t(x), Jv^t(x)\rangle \to -R^+\sqrt{2E} \quad (t \to \infty)\,.$$

∎

These two results allow us to proceed similarly to the rotationally symmetric case. For a point $x \in \overline{U_E^+}$ we define the rotated angle

$$\theta(t, x) := \int_0^t \frac{\langle q^s(x), Jv^s(x)\rangle}{|q^s(x)|^2}\, ds$$

and for $x \in U_E^+$ we consider the exit angle

$$\theta^e(x) := \theta(T^e(x), x)\,,$$

where $T^e(x)$ again denotes the exit time of x with respect to the supporting disc D. Lemma 3.3.5 and Lemma 3.3.6 immediately yield the next result.

Corollary 3.3.7 *For fixed $x \in \overline{U_E^+}$ the function $\theta(t, x)$ is strictly decreasing with respect to the time t and satisfies $\theta(t, x) \to -\infty$ as $t \to \infty$ for $x \in \partial U_E^+$. Furthermore, we have*

$$\theta^e(x) \to -\infty \quad (x \to \partial U_E^+)\,.$$

This result will be the main tool for the analysis of symbolic dynamics. As stated at the beginning, it applies for the case that $B(q) \geq 0$ holds for $|q| \geq R^+$. The other case is treated in the following remark.

Remark 3.3.8 The previous calculations work in a similar way for magnetic fields with $B(q) \leq 0$ for $|q| \geq R^+$. In this case we define $U_E^+ \subseteq \widehat{U}_E$ to be the connected component of U_E^π and, given an initial value $x \in U_E^+$, we obtain that the inequality

$$\langle q^t(x), Jv^t(x)\rangle > 0$$

holds for all $t \geq 0$. For $x \in \partial U_E^+$ we still have $|q^t(x)| \to R^+$ as $t \to \infty$, but now the convergence

$$\langle q^t(x), Jv^t(x)\rangle \to +R^+\sqrt{2E}$$

applies for $t \to \infty$. For the angle θ and the exit angle θ^e defined as above this yields

$$\theta^e(x) \to +\infty \quad (x \to \partial U_E^+)\,. \qquad \square$$

With this we finish the study of how the motion behaves inside one component and we now turn to the situation of a magnetic field with multiple components of this type.

3.3.2 Symbolic dynamics for non-rotationally symmetric components

We consider a magnetic field

$$B := \sum_{k=1}^{n} B_k\,,$$

that satisfies the following conditions:

(*i*) Each component B_k has compact support

$$\operatorname{supp} B_k \subseteq D_k \coloneqq \{q \in \mathbb{R}^2 \mid |q - q_k| \leq R_k\}$$

for some $q_k \in \mathbb{R}^2$ and $R_k > 0$.

(*ii*) The supporting discs D_k are disjoint.

(*iii*) The intersection of the circular energies

$$\mathcal{C} \coloneqq \bigcap_{k=1}^{n} \mathcal{C}(B_k)$$

is non-empty.

Note that the circular energies $\mathcal{C}(B_k)$ as given in Definition 3.3.1 are now obtained with respect to the centres q_k, i.e. these points now take the part of the origin regarding the results in the previous section. For an energy $E \in \mathcal{C}$ we denote the sign of B_k along the circular orbit of radius $R_k^+(E)$ by $b_k = b_k(E)$. In the following we fix some energy $E \in \mathcal{C}$ and omit the reference to E where appropriate.

Due to the multiple components of the support, for $k \in \{1, \dots, n\}$ we consider the set $U_{k,E}$ of points entering the supporting disc D_k and denote their union by U_E. In particular, instead of the angle $\alpha \colon U_E \to [0, \pi]$ we now have

$$\alpha_k \colon U_{k,E} \to [0, \pi],\ \alpha_k(q, v) \coloneqq \arccos \frac{\langle J(q - q_k), v\rangle}{R_k \sqrt{2E}}\,.$$

As in the rotationally symmetric case, there are some obstructions on the configuration of the supports and on where to place the Poincaré section. Due to the lack of the additional integral we have less control about the trajectories, so we have to make some adjustments to the previous definition. For $k \in \{1, \dots, n\}$ we define the *critical entrance direction*

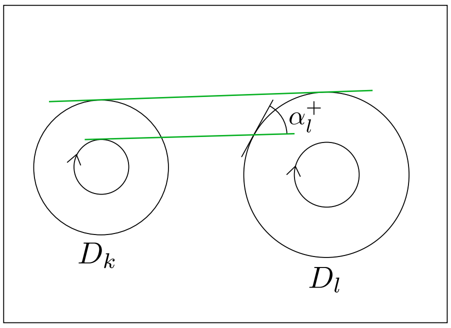
$$\alpha_k^+ = \alpha_k^+(E) \coloneqq \max_{x \in \overline{U_E^+}} \alpha_k(x)$$

if $b_k > 0$, and

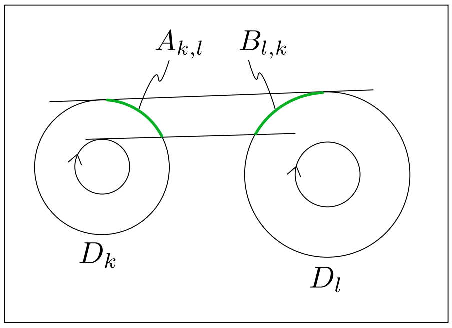
$$\alpha_k^+ = \alpha_k^+(E) \coloneqq \min_{x \in \overline{U_E^+}} \alpha_k(x)$$

if $b_k < 0$. As previously, for fixed $k \neq l$ there are four different sign combinations of (b_k, b_l). If both signs are positive, i.e. $(b_k, b_l) = (+1, +1)$, we consider the tangent line to ∂D_k and ∂D_l passing both supporting discs D_k and D_l tangentially to the left. In addition, we consider the left tangent line to the circular orbit of B_k which hits ∂D_l

with direction α_l^+; see Figure 3.13(a). By $A_{k,l} = A_{k,l}(E)$ and $B_{l,k} = B_{l,k}(E)$ we again denote the closed sets of points on ∂D_k and ∂D_l between these two lines as shown in Figure 3.13(b).



(a) The defining lines



(b) The sets $A_{k,l}$ and $B_{l,k}$

Figure 3.13: Transition from D_k to D_l for $(b_k, b_l) = (+1, +1)$

If both signs are negative, we basically consider the same lines as in the previous case, but with the right tangent replacing the left tangent.

If $(b_k, b_l) = (+1, -1)$, we consider the left tangent to ∂D_k which hits ∂D_l with direction α_l^+ as well as the line which is tangent to the circular orbit of radius R_k^+ in D_k on the left hand side, and tangent to ∂D_l on the right hand side. If $(b_k, b_l) = (-1, +1)$, we again consider the lines as in the case $(+1, -1)$ with left and right tangent interchanged.

With these sets $A_{k,l}$ and $B_{k,l}$ we are able to define the constraints on the configuration of the supports which the magnetic field has to satisfy.

Definition 3.3.9 Let $B = \sum_{k=1}^n B_k$ be a magnetic field on $\mathbb{R}^2$ and assume that the components B_k have pairwise disjoint supporting discs $D_k = \{q \in \mathbb{R}^2 \mid |q - q_k| \leq R_k\}$. We say that B is *in suitable position with respect to the energy* $E \in \mathcal{C}$ if the two placement conditions

(i) $\operatorname{conv}\left(A_{k,l} \cup B_{l,k}\right) \cap D_m = \emptyset$ for distinct $k, l, m \in \{1, \ldots, n\}$ and

(ii) $\partial D_k \Big\backslash \Big(\bigcup_{l \neq k} A_{k,l} \cup \bigcup_{l \neq k} B_{k,l} \Big) \neq \emptyset$ for all $k \in \{1, \ldots, n\}$

are satisfied. $\square$

Now, the following constructions work analogously to the rotationally symmetric case: The second condition yields the existence of some points

$$q_k^* = q_k^*(E) \in \partial D_k \Big\backslash \Big(\bigcup_{l \neq k} A_{k,l} \cup \bigcup_{l \neq k} B_{k,l} \Big) \quad (k = 1, \dots, n)\,,$$

which we consider fixed in the following. With these we define the Poincaré sections

$$P_k = P_{k,E} := \Big\{ (q,v) \in \mathbb{P}_E \mid q = q_k + \lambda(q_k^* - q_k) \text{ for some } \lambda \in (0,1),\ b_k \langle q - q_k, Jv \rangle < 0 \Big\}$$

and the Poincaré map

$$p = p_E \colon P_E := \bigcup_{k=1}^{n} P_{k,E} \ \longrightarrow\ \overline{P_E} \cup \{\infty\}\,,$$

which admits the p-invariant set

$$\Lambda_E := \Big\{ x \in P_E \mid p^i(x) \in P_E \text{ for all } i \in \mathbb{Z} \Big\}\,.$$

By construction, the canonical coding map $h = h_E \colon \Lambda_E \to \Sigma_n = \{1, \dots, n\}^{\mathbb{Z}}$, which is given by $h(x) = (s_i)_{i \in \mathbb{Z}}$ such that $p^i(x) \in P_{s_i,E}$ holds for all $i \in \mathbb{Z}$, satisfies the identity

$$h \circ p = \sigma \circ h$$

with the left shift map $\sigma \colon \Sigma_n \to \Sigma_n$. Finally, we have the maps $u \colon U_E \to P_E \cup \{\infty\}$ and $w \colon P_E \to U_E \cup P_E \cup \{\infty\}$, which describe the first intersections of a trajectory with the corresponding sets.

These definitions now allow us to formulate the main result for magnetic fields with non-rotationally symmetric components.

Theorem 3.3.10 *Assume the magnetic field B is in suitable position with respect to the energy $E \in \mathcal{C}$. Then there is a compact p_E-invariant subset $\Lambda_E' \subseteq \Lambda_E$ such that the coding map $h_E \colon \Lambda_E' \to \Sigma_n$ is continuous and surjective. In other words, $p_E|_{\Lambda_E'} \colon \Lambda_E' \to \Lambda_E'$ is semi-conjugated to the full shift $\sigma \colon \Sigma_n \to \Sigma_n$.*

Again, this yields the following corollary for the topological entropy of the Poincaré map.

Corollary 3.3.11 *The Poincaré map $p_E|_{\Lambda_E'} \colon \Lambda_E' \to \Lambda_E'$ has positive topological entropy*

$$h_{\text{top}}(p_E|_{\Lambda_E'}) \geq \log n\,.$$

In particular, this discrete dynamical system is chaotic.

The proof of Theorem 3.3.10 parallels the one of the corresponding result in the rotationally symmetric case given in Theorem 3.2.3: We can adopt Proposition 3.2.7 and its proof, which assures that every half-infinite sequence can be realized. This allows us to copy the proof of Theorem 3.2.3, where it is shown that every bi-infinite sequence can be realized using initial values that are provided by increasing half-infinite parts of this sequence, and applying a compactness argument. The only result we have to adjust to the present situation is the aiming mechanism specified in Lemma 3.2.5, which describes how to get from one support to the next with the prescribed number of intersections of the Poincaré section. This lemma has to be replaced by the following one, which is formulated for the case $(b_k, b_l) = (+1, +1)$. The other cases work similarly, as it was the case for rotationally symmetric magnetic fields. To avoid confusion by only pointing out which parts of the proof of Lemma 3.2.5 change and which do not, we give the full modified proof instead.

Lemma 3.3.12 *Assume the magnetic field B is in suitable position with respect to the circular energy $E \in \mathcal{C}$. Let $k \neq l \in \{1, \ldots, n\}$ be different indices of supports such that $(b_k, b_l) = (+1, +1)$ and let $j \in \mathbb{N}$ denote the desired number of rotations. Furthermore, let $\gamma\colon [a, b] \to U_{k,E}$ be a curve with*

(a) $\gamma(a) \in U^0_{k,E}$,

(b) $\gamma([a, b)) \subseteq U^+_{k,E}$,

(c) $\gamma(b) \in \partial U^+_{k,E}$ *and*

(d) $q^*_k \notin \operatorname{im}\gamma$.

Then there exists a subinterval $[\tilde{a}, \tilde{b}] \subseteq [a, b]$ such that for all points $x \in \operatorname{im}\gamma|_{[\tilde{a},\tilde{b}]}$ in the image of $\gamma|_{[\tilde{a},\tilde{b}]}$ we have

(i) $p^i(u(x)) \in P_{k,E}$ *for* $i \in \{0, \ldots, j-1\}$,

(ii) $w(p^{j-1}(u(x))) \in U_{l,E}$,

and the curve $\gamma_1 \coloneqq (w \circ p^{j-1} \circ u \circ \gamma)\big|_{[\tilde{a},\tilde{b}]} : [\tilde{a}, \tilde{b}] \to U_{l,E}$ *satisfies*

(a′) $\gamma_1(\tilde{a}) \in U^0_{l,E}$,

(b′) $\gamma_1([\tilde{a}, \tilde{b})) \subseteq U^+_{l,E}$,

(c′) $\gamma_1(\tilde{b}) \in \partial U^+_{l,E}$ *and*

(d′) $q^*_l \notin \operatorname{im}\gamma_1$.

Proof Let $q_a, q_b \in \partial D_k$ denote the two points of $\partial A_{k,l}$ as previously shown in Figure 3.10. Since $\gamma(s) \to \partial U_k^+ = \partial U_{k,E}^+$ we have

$$\theta_k^e(\gamma(s)) \to -\infty \quad (s \to b)$$

by Corollary 3.3.7. Hence, there are parameters $s_a < s_b \in [a, b]$ such that

$$q^{T^e(\gamma(s_a))}(\gamma(s_a)) = q_a \, ,$$

i.e. the trajectory of $\gamma(s_a)$ exits D_k through q_a;

$$q^{T^e(\gamma(s_b))}(\gamma(s_b)) = q_b \, ,$$

i.e. the trajectory of $\gamma(s_b)$ exits D_k through q_b; and

$$p^i(u(\gamma(s))) \in P_k \quad (i \in \{0, \dots, j-1\},\, s \in [s_a, s_b])$$

as well as

$$w(p^{j-1}(u(\gamma(s)))) \notin P_k \quad (s \in [s_a, s_b]) \, ,$$

i.e. the trajectories hit P_k exactly j times before leaving the support. The trajectory of $\gamma(s_a)$ passes D_l on the left hand side. Since the curvature of any solution curve is negative while in the outer annulus

$$\left\{q \in \mathbb{R}^2 \mid R_k^+ \le |q - q_k| \le R_k \right\} ,$$

the trajectory of $\gamma(s_b)$ moves towards D_l on the right hand side of the line we used to define $A_{k,l}$ and $B_{l,k}$ – the one being tangent to the circle of radius R_k^+ and intersecting D_l with angle α_l^+. Hence, it intersects ∂D_l with an angle $\beta > \alpha_l^+$. If the trajectory does not intersect D_l, then it passes somewhere to the right and there is a parameter $\tilde{s_b} < s_b$ such that the trajectory of $\gamma(\tilde{s_b})$ passes tangentially to the right of D_l, i.e. intersects ∂D_l with angle $\beta = \pi$. For simplicity we then denote this parameter by s_b. Since the trajectory of $\gamma(s_a)$ passes somewhere to the left of D_l, there is a parameter $\tilde{a} \in [s_a, s_b]$ such that the trajectory of $\gamma(\tilde{a})$ passes tangentially to the left of ∂D_l. Furthermore, we choose $\tilde{a}$ such that

$$w \circ p^{j-1} \circ u \circ \gamma(s) \in U_l \quad (s \in [\tilde{a}, s_b])$$

holds, i.e. for $s \in [\tilde{a}, s_b]$ the trajectory of $\gamma(s)$ intersects ∂D_l. This means we have a curve $\widetilde{\gamma}\colon [\tilde{a}, s_b] \to U_l$ given by

$$\widetilde{\gamma}(s) = (w \circ p^j \circ u \circ \gamma)(s) \, ,$$

such that

$$\alpha_l(\widetilde{\gamma}(\tilde{a})) = 0$$

and

$$\alpha_l(\widetilde{\gamma}(s_b)) = \beta > \alpha_l^+ = \max_{x \in \overline{U_l^+}} \alpha_l(x)$$

hold. This implies $\widetilde{\gamma}(\tilde{a}) \in U_l^+$ as well as $\widetilde{\gamma}(s_b) \notin U_l^+$ and hence, there exists a minimal parameter $\tilde{b} \in [\tilde{a}, s_b]$ with

$$\widetilde{\gamma}(\tilde{b}) \in \partial U_l^+ .$$

Therefore, the curve

$$\gamma_1 \coloneqq \widetilde{\gamma}|_{[\tilde{a},\tilde{b}]} : [\tilde{a}, \tilde{b}] \to U_l$$

satisfies the stated properties. ■

This describes how the aiming mechanism works for the signature $(b_k, b_l) = (+1, +1)$. The other cases rely on the same procedure with only slight modifications regarding the starting and the resulting curves. This parallels the rotationally symmetric case and works just as described in Remark 3.2.6. Having obtained the aiming mechanism now allows us to proceed as in the previous section, and copying the proof of Theorem 3.2.3 then proves Theorem 3.3.10. In particular, this shows that the semi-conjugacy to the full shift does not depend on the rotational symmetry of the components and, moreover, not on the existence of an additional integral of motion.

Let us point out that the magnetic fields considered in this chapter have compact support and therefore all the results from Chapter 2 apply. This demonstrates a fundamental difference between scattering and bounded states: For the scattering states the motion is conjugated to the elementary free flow, while for the bounded states we even exhibit chaotic behaviour.

List of Symbols

B	magnetic field	p. 10
$\widehat{B}$	profile function of B	p. 75
$\mathbb{P} \coloneqq T\mathbb{R}^d$	phase space	p. 10
$\mathbb{P}_E \coloneqq \mathcal{E}^{-1}(E)$	energy surface of the kinetic energy	p. 11
$\mathbb{P}^0 \coloneqq \mathbb{P} \setminus \mathbb{P}_0$	phase space without fixed points	p. 59
$\varphi^t(x) = (q^t(x), v^t(x))$	magnetic flow on $\mathbb{P}$	p. 10
$\varphi^t_*(x)$	magnetic flow on $T^*\mathbb{R}^d$	p. 66
$\varphi^t_0(x)$	free flow on $\mathbb{P}$	p. 19
$\varphi^t_{0,*}(x)$	free flow on $T^*\mathbb{R}^d$	p. 67
b, $b^\pm$	set of bounded states	p. 21
s, $s^\pm$	set of scattering states	p. 21
$s^\pm_*$	set of scattering states on $T^*\mathbb{R}^d$	p. 67
$q^\pm(x)$	asymptotic positions	pp. 31, 57
$v^\pm(x)$	asymptotic velocities	pp. 28, 55
$\Omega^\pm \coloneqq (q^\pm, v^\pm)$	(velocity) wave transformations on $\mathbb{P}$	pp. 32, 59
$\Omega^\pm_*$	(momentum) wave transformations on $T^*\mathbb{R}^d$	p. 67
$\mathcal{E}$	kinetic energy	p. 11
$\mathcal{M}$	magnetic momentum	p. 78
$\mathcal{M}^+$	critical magnetic momentum	p. 91
α^+	critical entrance direction	pp. 96, 117
D	supporting disc of B	pp. 88, 110
$R^+ = R^+(E)$	radius of largest circular orbit	pp. 88, 110
$R_{\mathrm{vir}}(E)$	virial radius	p. 22
U_E	set of points with energy E entering the support	pp. 90, 111
$\theta^e(x)$	exit angle	pp. 95, 115
$T^e(x)$	exit time with respect to D	p. 95
$\widetilde{E^\circ}$, E°	energy thresholds	pp. 24, 92
$\mathcal{C}$	set of circular energies	pp. 110, 117
P_E	Poincaré section	pp. 102, 119
$p = p_E$	Poincaré map	pp. 102, 119
Σ_N, σ	shift space in N symbols, shift map	p. 17

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