Uniqueness and Regularity for Porous Media Equations with x-dependent Coefficients

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Dipl.-Math. Jan-Christopher Koch

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Promotionsausschuss:

Vorsitzender:	Prof. Dr. Stefan Turek
Erster Gutachter:	Prof. Dr. Ben Schweizer
Zweiter Gutachter:	Prof. Dr. Matthias Röger
Zusätzlicher Prüfer:	Prof. Dr. Christian Meyer

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1 Introduction

The scope of this thesis is the investigation of two-phase immiscible flow. More precisely, inside a porous medium, like soil, rock or a fuel cell, we examine equations that describe the flow of two immiscible fluids, like water and air, water and oil or water and water vapor. Such flow processes may occur e.g. in hydrology, oil recovery or power generation.

To describe the flow of two immiscible fluids with (reduced) saturations s_1 and s_2, and pressures p_1 and p_2, we use the so called two-phase flow equations that have essentially the form

$$
\begin{aligned}
\partial_t s_1 &= \nabla \cdot (\lambda_1(s_1)(\nabla p_1 + g)), \\
\partial_t s_2 &= \nabla \cdot (\lambda_2(s_2)(\nabla p_2 + g)), \\
p_c(s_1) &= p_1 - p_2, \\
1 &= s_1 + s_2,
\end{aligned}
\tag{1.1}
$$

inside a cylinder $Q := \Omega \times (0, T)$. Typically, we find $\Omega \subset \mathbb{R}^2$ or $\mathbb{R}^3$ and we consider $T > 0$. We only consider the case where s_1 and the pressure difference $p_1 - p_2$ are linked by a functional dependence p_c. Typical shapes of these functions we have in mind are depicted in Figure 1.1. Particularly, our main concern are the degenerate cases shown there, i.e. $\lambda_1(0) = \lambda_2(0) = 0$ and $\lim_{s_1 \to 1} p_c(s_1) = -\lim_{s_1 \to 0} p_c(s_1) = \infty$.

In many physical situations, for example in the case that one of the fluid phases is gas and the other water, the variations in the pressure of one phase can be neglected in comparison to the pressure variations in the other phase. In this situation it is reasonable to assume that one pressure is constant, e.g. $p_2 = 0$. With $s = s_1$, $p = p_1$ and $\lambda_1 = \lambda$, we reduce (1.1) to

$$
\begin{aligned}
\partial_t s &= \nabla \cdot (\lambda(s)(\nabla p + g)), \\
p_c(s) &= p
\end{aligned}
\tag{1.2}
$$

on Q. Equation (1.2) is called the Richards equation for unsaturated flow.

To investigate these equations, we use certain transformations and realize that the transformed problems of (1.1) and (1.2) are related to the generalized porous medium equation:

$$
\partial_t s = \Delta \Phi(s) \text{ on } Q
\tag{1.3}
$$

Here, Φ has one of the forms depicted in Figure 4.1. The degeneracies of λ_1, λ_2 and p_c may lead to a so-called doubly degenerate Φ. Particularly, we think of the cases $\Phi'(0) = 0$ and $\lim_{s \to 1} \Phi'(s) \in \{0, \infty\}$.

So far, we did not comment on the x-dependence of the coefficients. Naturally, properties of a porous medium may change from one space position to another. Hence, the coefficient functions, i.e. λ_1, λ_2 and Φ in (1.1), (1.2) and (1.3) may depend on the spatial position $x \in \Omega$. Essentially, we consider two different types of x-dependencies. For (1.1), we assume that the variations in x-directions of the coefficients are smooth; we provide a local Hölder continuity result for s_1. The precise problem is stated in chapter 3 and the regularity is shown in chapter 6 using the method of intrinsic scaling. The x-dependence for the unsaturated flow problem is as follows. The porous medium Ω is separated by an interface Γ into two subdomains Ω_l and Ω_r (see Figure 1.2). On each of these subdomains the coefficients are assumed to be constant with potentially different values. The flow of the fluid phase is given by (1.2). To pose a well-defined problem we require transmission conditions at the interface. Under the assumption that the pressure and the flux across the interface are continuous, we show uniqueness of the saturation. The problem is formulated in chapter 2 and the uniqueness, particularly a generalized L^1-contraction result, is proved in chapter 5 using the method of doubling the variables. To tackle both problems, we provide technical tools in chapter 4 motivated and demonstrated by means of equation (1.3).

Before we turn our attention to the problems presented above, we state the main results more precisely in the next section. We briefly derive variants of equations (1.1) and (1.2) by means of physical principles in section 1.2 and comment on the literature in section 1.3. Basic assumptions, results and notation are provided in section 1.4.

1.1 Main Results

In this section, we specify the equations under consideration and state our main results more precisely. As for the unsaturated flow problem, we assume that Ω is divided by an interface Γ into two domains Ω_l and Ω_r (see Figure 1.2). We write $Q_j = \Omega_j \times (0, T)$ for $j \in \{l, r\}$. We attach an index to the functions that corresponds to the domains. For $j \in \{l, r\}$, the functions λ_j and $p_{c,j}$ have the shapes of λ_1 and p_c from Figure 1.1, respectively. We use the Kirchhoff transform

$$\Phi_j(s) := \int_0^s \lambda_j(\sigma) p'_{c,j}(\sigma) \, \mathrm{d}\sigma$$

and assume that Θ_j is an increasing function of s that that handles the continuity of the pressure across the interface. As we see in section 2.1, we obtain

$$\partial_t s = \nabla \cdot (\nabla[\Phi_j(s)] + \lambda_j(s)g_j) + f_j \quad \text{on } Q_j \text{ for } j \in \{l, r\},$$
$$0 = (\nabla[\Phi_l(s)] + \lambda_l(s)g_l]) \cdot \nu_l + (\nabla[\Phi_r(s)] + \lambda_r(s)g_r]) \cdot \nu_r \quad \text{on } \Gamma \times (0, T), \quad (1.4)$$
$$\Theta_l(s) = \Theta_r(s) \quad \text{on } \Gamma \times (0, T),$$

where f_j are source terms and ν_j are the outer normal vectors of Ω_j on Γ for $j \in \{l, r\}$. Assuming essentially that $(\Phi_j \circ \Theta_j^{-1})'$ is Lipschitz continuous, Theorem 2.4 states that solutions of (1.4) possess a generalized L^1-contraction property. That is, for two solutions s_1 and s_2 with initial data $s_{0,1}$ and $s_{0,2}$ there holds

$$\|s_1(t) - s_2(t)\|_{L^1(\Omega)} \leq e^{Lt} \|s_{0,1} - s_{0,2}\|_{L^1(\Omega)},$$

for a.e. $t \in (0, T)$. The constant L is determined by the Lipschitz constants of the source terms f_l and f_r. Without the source terms, we find $L = 0$ and thus the classical, non-generalized L^1-contraction property. We emphasize that the L^1-contraction implies uniqueness.

Concerning the two-phase flow problem, we introduce the global pressure

$$p = p_1 - \int_{p_c(0)}^{p_c(s)} \frac{\lambda_2(p_c^{-1}(u))}{\lambda(p_c^{-1}(u))} \, \mathrm{d}u$$

and the transformation

$$\Phi(s) := \int_0^s \frac{\lambda_1(\xi)\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \, \mathrm{d}\xi.$$

Here, λ_1, λ_2 and p_c are as in Figure 1.1. As it is pointed out in section 3.1, we obtain

$$\phi \partial_t s = \nabla \cdot (\kappa \, (\nabla[\Phi(s)] - \nabla_x \Phi(s) + B(s)) + D(s)u) \text{ on } Q,$$
$$0 = \nabla \cdot (\kappa \, [\lambda(s)\nabla p + E(s)]) + f \text{ on } Q, \quad (1.5)$$

with

$$B(s) = \frac{\lambda_1(s)\lambda_2(s)}{\lambda(s)} \left(\nabla_x[p_c(s)] + g_1 - g_2 \right),$$

$$D(s) = -\frac{\lambda_1(s)}{\lambda(s)} \text{ or } \frac{\lambda_2(s)}{\lambda(s)},$$

$$E(s) = -\lambda_2(s)\nabla_x p_c(s) + \lambda \int_0^s \nabla_x \left(\frac{\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \right) \mathrm{d}\xi + \lambda_1(s)g_1 + \lambda_2(s)g_2.$$

Our main result for (1.5) is stated in Theorem 3.12. Assuming that Φ' is smooth and behaves like a power near zero and one with the same order, we prove that the saturation s is locally Hölder continuous. As far as we know, there is no proof for the local Hölder continuity of s for x-dependent Φ available in the literature.

As it is sketched in [Che02] for x-independent Φ, the method to obtain local Hölder continuity of s leads may also be used to obtain Hölder continuity at the boundary. From there on, also global Hölder continuity can be obtained, which, in turn, is used to obtain an uniqueness result. Thus, the local Hölder continuity result we provide could be the first step to obtain an uniqueness result for (1.5) in the case of x-dependent Φ.

1.2 Modelling of Flow in Porous Media

In this section, we derive the equations describing the macroscopic flow of two immiscible fluids in a porous medium. For further details, we refer to the books of Bear [Bea88, ch. 9], Bear and Bachmat [BB90, ch. 5] and Chavent and Jaffré [CJ86, ch. I.IV and III.II]. Additionally, for the consideration of interfaces between porous media for unsaturated flow processes we refer to [OS06] and the references therein.

Two-Phase Flow

The porous medium is denoted by $\Omega \subset \mathbb{R}^d$. The function $\phi : \Omega \to [0,1]$ describes the porosity of the medium, i.e. the amount of pore space relative to the bulk volume. We assume that Ω is not deformable; particularly, ϕ is independent of time and pressure. For each of the two fluid phases, i.e. for the α-phase with $\alpha = 1, 2$, the mass balance (or continuity equation) is given by

$$\phi \partial_t (\rho_\alpha s_\alpha) + \nabla \cdot (\rho_\alpha u_\alpha) = \rho_\alpha f_\alpha, \tag{1.6}$$

where $\rho_\alpha, u_\alpha, \rho_\alpha u_\alpha, s_\alpha$ and f_α are, respectively, the density, volumetric flow rate, mass flux, (reduced) saturation, and external volumetric flow rate of the α-phase. In the literature one often finds the notation $\alpha = w, n$, where w and n denotes the wetting and non-wetting phase, respectively.

Assuming that the fluids are homogeneous and incompressible, i.e. $\rho_\alpha = const$, equation (1.6) becomes

$$\phi \partial_t (s_\alpha) + \nabla \cdot u_\alpha = f_\alpha. \tag{1.7}$$

The fluxes u_α obey Darcy's law

$$u_\alpha = -\kappa \frac{k_{r\alpha}(s_\alpha)}{\mu_\alpha} \left(\nabla p_\alpha + \rho_\alpha g \right), \tag{1.8}$$

where κ is the absolute permeability of the porous medium, $-g$ is the gravity force, and $k_{r\alpha}, \mu_\alpha$ and p_α are, respectively, the relative permeability, viscosity and pressure of the α-phase. The permeability κ is a symmetric, positive definite matrix. Thus, κ allows to describe anisotropic porous media. We define the phase mobilities

$$\lambda_\alpha(x, s_\alpha) = \frac{k_{r\alpha}(x, s_\alpha)}{\mu_\alpha(x)}.$$
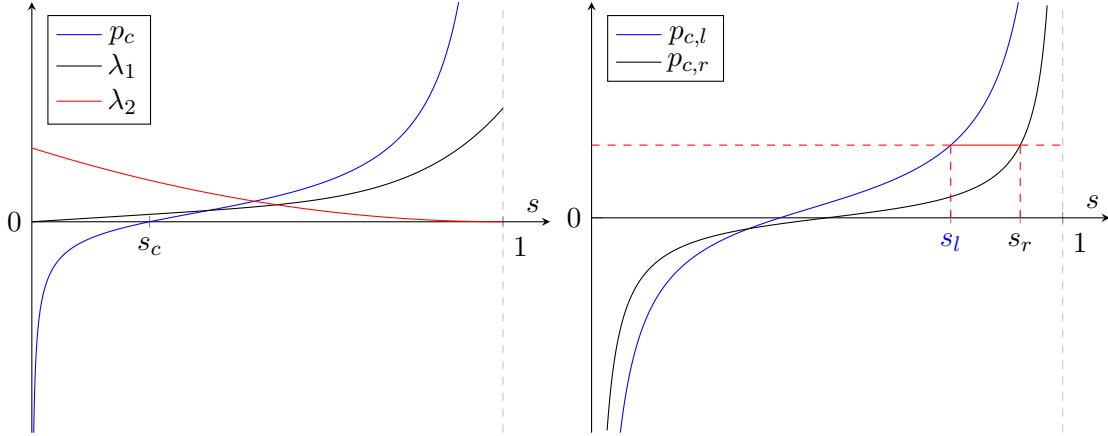
Figure 1.1: *In the left picture, we see typical curves of the $\lambda_1(s), \lambda_2(s)$ and $p_c(s)$. Also the critical saturation s_c such that $p_c(s_c) = 0$ is shown. The picture on the right shows that the continuity of the pressure across an interface may yield a discontinuity of the saturation.*

The pores of the medium are completely filled by the two phases, i.e.

$$s_1 + s_2 = 1 \text{ and we choose } s := s_1. \tag{1.9}$$

Let us assume for the moment that the 1-phase is water, i.e. s describes the water saturation inside the porous medium. A decrease in saturation of water may lead to a disconnectedness in the water phase. Particularly, water pools inside the porous medium may loose their connection through pores to each other. In such a situation the water ceases to flow, causing the permeability to vanish. For $0 \le s_m \le s_M \le 1$, this occurs at the residual saturations $s = s_m, s_M$ with $k_{r1}(s_m) = 0$ and $k_{r2}(s_M) = 0$, respectively. With the transformation $\tilde{s} = (s_M - s_m)s + s_m \in [0,1]$, we may assume $s \in [0,1]$, $s_m = 0$ and $s_M = 1$ in the following. Thus, we only speak of saturation instead of reduced saturation in the remainder of this thesis.

The bulk model is completed by a functional dependence of the pressure differences and the saturation, namely by the capillary pressure relation

$$p_c(s) = p_1 - p_2. \tag{1.10}$$

This relation may also depend on the spatial position $x \in \Omega$. The capillary pressure law (1.10) is motivated by Laplace's law of surface tension. The pressure difference across the fluid interface depends linearly on the mean curvature of the interfaces between the two fluids. Furthermore, the curvature of these interface depends on the typical size of the pores where these interfaces are located. Since the typical pore size depends on the saturation, we obtain a relation between saturation and pressure.

We emphasize that the definition of capillary pressure presented here is different from the usual choice in the literature. In particular, our definition leads to an increasing p_c function. Summarizing the previous deduction and redefining the functions λ_α in terms of s and using $s_1 + s_2 = 1$ as well as $g_\alpha = \rho_\alpha g$ for $\alpha \in \{1, 2\}$ we obtain the two-phase flow system

$$
\begin{aligned}
\phi \partial_t s &= \nabla \cdot (\kappa \lambda_1(s)[\nabla p_1 + g_1]) + f_1, \\
-\phi \partial_t s &= \nabla \cdot (\kappa \lambda_2(s)[\nabla p_2 + g_2]) + f_2, \\
p_c(s) &= p_1 - p_2.
\end{aligned}
\tag{TP}
$$
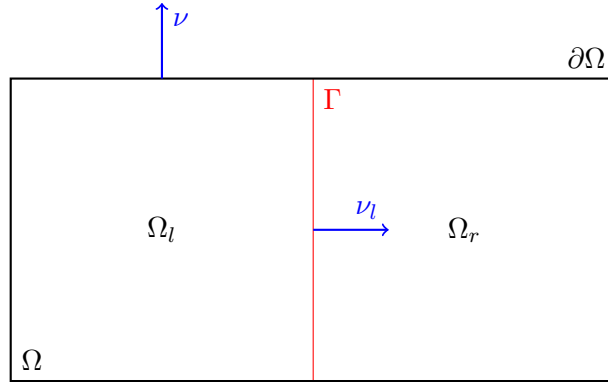
Figure 1.2: *This picture shows the situation where the domain Ω is divided by the interface Γ into two subdomains Ω_l and Ω_r*

For completeness, we also consider a porous medium with a sharp interface in the case of two-phase flow. Assume that Ω is divided by an interface Γ into two subdomains which are denoted by Ω_l and Ω_r. This situation is shown in Figure 1.2. We consider (TP) on Ω_j and attach an index $j \in \{l, r\}$ to the functions. Mass conservation yields that the fluxes of each of the phases across the interfaces are continuous, i.e. for $\alpha \in \{1, 2\}$ there holds

$$
(\kappa \lambda_{\alpha,l}(s_l)[\nabla p_{\alpha,l} + g_{\alpha,l}]) \cdot \nu_l = -(\kappa \lambda_{\alpha,r}(s_r)[\nabla p_{\alpha,r} + g_{\alpha,r}]) \cdot \nu_r
\tag{1.11}
$$

on Γ. Here $\nu_l = -\nu_r$ are the outward pointing unit normal vectors of Ω_l and Ω_r on Γ. Balance of forces provides the continuity of the pressures across the interface, which is

$$
p_{c,l}(s_l) = p_{1,l} - p_{2,l} = p_{1,r} - p_{2,r} = p_{c,r}(s_r)
\tag{1.12}
$$

on Γ. The last assumption may yield a discontinuous saturation across the interface, i.e. we find $s_l \neq s_r$ on Γ. This situation is shown in Figure 1.1.

Unsaturated Flow

As we stated in the introduction, it is often reasonable to assume that the pressure variations in the second phase are negligible when compared to the pressure variations

in the first phase. For example, in groundwater flow, one considers water and air inside soil. Assuming that the gas inside the soil is connected to the surrounding yields a constant atmospheric pressure $p_2 = p_{atm}$ inside the gas phase. Usually, the atmospheric pressure is normalized to $p_{atm} = 0$.

With this normalization, dropping the index 1 for the first phase, and the choice $\phi = \kappa = 1$ we obtain from (TP) the Richards equation for unsaturated flow:

$$\partial_t s = \nabla \cdot [(\lambda(s)(\nabla p + g))] + f,$$
$$p = p_c(s). \tag{R}$$

In the case of a domain Ω that is divided by an interface Γ, we assume, as before, that (R) holds on Ω_l and Ω_r, respectively, attach an index $j \in \{l, r\}$ to the coefficient functions and obtain from (1.11) and (1.12) the equations

$$(\lambda_l(s_l)[\nabla p_l + g_l]) \cdot \nu_l = - (\lambda_r(s_r)[\nabla p_r + g_r]) \cdot \nu_r$$
$$p_{c,l}(s_l) = p_l = p_r = p_{c,r}(s_r) \tag{1.13}$$

on Γ. In general, the coefficient functions $p_c(x,s) := p_{c,l}(s)\, \mathbb{1}_{\Omega_l}(x) + p_{c,r}(s)\, \mathbb{1}_{\Omega_r}(x)$ and $\lambda(x,s) := \lambda_l(s)\, \mathbb{1}_{\Omega_l}(x) + \lambda_r(s)\, \mathbb{1}_{\Omega_r}(x)$ are discontinuous in x across Γ. Hence, we call (R) on Ω_l and Ω_r linked by the transmission conditions (1.13) the discontinuous Richards equation.

1.3 Survey of Literature

Considering (R) and having the shape of λ_1 and p_c from Figure 1.1 in mind, we see that we can only expect estimates of ∇p or ∇s with a weight. More precisely, with the test function p we expect that

$$\int_\Omega \lambda(s)\nabla p \cdot \nabla p = \int_\Omega \lambda(s)(p_c'(s))^2\, |\nabla s|^2$$

is bounded and with the test function s we expect

$$\int_\Omega \lambda(s)\nabla p \cdot \nabla s = \int_\Omega \lambda(s)p_c'(s)\, |\nabla s|^2$$

is bounded. This implies a lack of regularity of p and s (in comparison to the heat equation). Such a lack of regularity may result in a lack of compactness of sequences p_k or s_k, which is needed to prove existence. One possible trick to recover compactness is to use the so called Kirchhoff transformation

$$\Phi(s) = \int_0^s \lambda(\sigma)p_c'(\sigma)\, \mathrm{d}\sigma. \tag{K}$$

We can expect to find estimates for

$$\int_\Omega |\nabla[\Phi(s)]|^2$$

without a weight and compactness for sequences $u_k = \Phi(s_k)$ can be inferred. With (K), the Richards equation (R) is transformed into

$$\partial_t s = \nabla \cdot (\nabla[\Phi(s)] + \lambda(s)g) + f. \tag{1.14}$$

Equation (1.14) is a generalization of equation (1.3). In [Váz06] the generalized porous medium equation (1.3) has been investigated extensively. With $u = \Phi(s)$ and $b = \Phi^{-1}$, we cast (1.14) into

$$\partial_t b(u) = \nabla \cdot (\nabla u + \lambda(b(u))g) + f \tag{1.15}$$

and obtain a quasilinear elliptic-parabolic equation for which vast amounts of literature is available. For example, existence is provided in the fundamental work of [AL83] under standard boundary conditions. A uniqueness result for time-independent boundary data is provided in [Ott95].

Equation (1.15) is also suited to describe so called unsaturated-saturated flow processes. For x-independent coefficients, existence for standard and outflow boundary data is shown in [ALV84]. A uniqueness result has been provided in [Ott97]. For unsaturated-saturated flow processes, one may assign for each pressure value p a unique saturation value s, but not the other way round. This corresponds to the situation of a multivalued capillary pressure relation, i.e. $p \in p_c(s)$. In comparison to Figure 1.1, we find for some p^* the relation $\lim_{s \to 1} p_c(s) = p^*$ and $p_c(1) = [p^*, \infty]$ instead of merely $\lim_{s \to 1} p_c(s) = \infty$. A similar behaviour may occur for $\lim_{s \to 0} p_c(s)$. For such a multivalued relation, one defines a suitable analogue of (K) and obtains, instead of (1.14), under the assumption $g = 0$, the equation

$$\partial_t s = \Delta u + f, \quad u \in \Phi(s).$$

This equation, supplemented with an outflow boundary condition, was investigated in [Sch07] and an existence result in the case of x-dependent coefficients was obtained via a regularization argument.

For nondegenerate two-phase flow equations (TP), existence is shown in [KS77]. For degenerate two-phase flow equations under standard and outflow boundary conditions, existence results are provided in [KL84; AD85a; Arb92; Che01] and [LS10]. All these references have in common that at most a smooth x-dependence of the coefficients is considered. Regularity of the saturation has been investigated in [AD85a; Che01; Che02] and recently in [DGV10]. General uniqueness results are not available in the literature. Under restrictive assumptions, essentially loosing the structure of the problem, a uniqueness result in the case of x-dependent coefficients is stated in [Che01]. For the x-independent case, a uniqueness result is stated in [Che02].

Multivalued capillary pressure relations are also investigated for two-phase flow equations. Usually, relations of the form $p_1 - p_2 \in p_c(s)$ are considered in the literature. In the one-dimensional situation, existence is shown in [BLS09] considering an interface and in [Koc09] considering outflow boundary conditions. For higher-dimensional problems and under consideration of interfaces, an existence result has been provided in [CGP09]. The continuity of the pressures across the interfaces is translated into $p_{\alpha,l} \cap p_{\alpha,r} \neq \emptyset$ for $\alpha \in \{1, 2\}$.

In the case of nondegenerate capillary pressures, i.e. in the case that p_c is a bounded function, existence for the two-phase flow equation with interfaces is shown in [EEM06]. For nondegenerate capillary pressures, existence and uniqueness for the discontinuous Richards equation is provided in [Can08].

Recently, progress on existence results for capillary pressures with hysteresis for two-phase and unsaturated flows were obtained. We refer to [KRS13; Sch12b; Sch12a] and [LRS11] and the references therein.

1.4 Notation and Function Spaces

For $d \geq 1$, let $E, U, V \subset \mathbb{R}^d$ such that U, V open, $a, b \in \mathbb{R}$ and a function $f : E \to \mathbb{R}$ be given. We use the following definitions and notations.

- $/a, b/ := [\min\{a, b\}, \max\{a, b\}]$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ denotes the extended real line

- For $\varepsilon > 0$, each of the inequalities $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$ and $ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}$ is called Cauchy's inequality (see [Eva98, B.2]).

- We write $V \subset\subset U$ and say V is compactly contained in U, if $V \subset \overline{V} \subset U$ and $\overline{V}$ is compact

- $|E|$ and $\mathcal{H}^d(E)$ denote the d-dimensional Lebesgue and Hausdorff measure of E, respectively, and the characteristic function of E is denoted by $\mathbb{1}_E$

- For $l < k \in \mathbb{R}$, we write

$$\{l < f < k\} = \{x \in E \mid l < f(x) < k\}$$

and the obvious variants with, for example, only one lower or upper bound for $f(x)$ and relations "$\leq$" and "$=$"

- If E is measurable and bounded and $f \in L^1(E)$, we write

$$\fint_E f := \frac{1}{|E|} \int_E f$$

- $f_+ := \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$. This implies $f = f_+ - f_-$ and $|f| = f_+ + f_-$

- For $u \in \mathbb{R}$ and $\varepsilon > 0$, we define the sign-function and the approximation sign_ε in virtue of

$$\text{sign}(u) := \begin{cases} 1 & u > 0, \\ 0 & u = 0, \\ -1 & u < 0, \end{cases} \quad \text{and} \quad \text{sign}_\varepsilon(u) := \begin{cases} 1 & u > \varepsilon, \\ \frac{u}{\varepsilon} & u \in [-\varepsilon, \varepsilon], \\ -1 & u < -\varepsilon \end{cases} \quad (1.16)$$

- For $\rho > 0$ and $x_0 \in \mathbb{R}^d$, the Euclidean norm of x_0 is denoted by $|x_0| = \|x_0\|_2$ and the ball of radius ρ centered at x_0 by

$$B_\rho(x_0) := \{x \in \mathbb{R}^d \mid |x - x_0| < \rho\} \text{ and abbreviate } B_\rho = B_\rho(0)$$

- We use the standard notations $C_c^\infty(U) = \mathcal{D}(U)$ for the space of functions that are compactly supported in U and arbitrarily often differentiable. The space of distributions is denoted by $\mathcal{D}'(U)$

- For $\varphi \in C_c^\infty(B_1)$ with $\varphi \geq 0$ and $\int_{\mathbb{R}^d} \varphi = 1$, and $\varepsilon > 0$, we call the sequence $(\varphi_\varepsilon)_{\varepsilon > 0}$ defined in virtue of

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^d} \varphi\left(\frac{x}{\varepsilon}\right)$$

for $x \in \mathbb{R}^d$ a (standard) Dirac sequence; see for example [Alt06, 2.13]

Assume now that E is measurable. The support of f is defined as

$$\text{supp}(f) := \overline{\{x : f(x) \neq 0\}}.$$

This definition is suitable for continuous functions. When working with equivalence classes of functions, such as in the L^p-spaces, this definition is not adequate. A suitable definition of the support should be independent of the representative element of the equivalence class, but since $\mathbb{1}_\mathbb{Q} = 0$ a.e. in $\mathbb{R}$ and $\mathbb{R} = \text{supp}(\mathbb{1}_\mathbb{Q}) \neq \text{supp}(0) = \emptyset$ this is not the case.

Proposition 1.1 (and definition of support [Bre10, Proposition 4.17]). *Let $f : \mathbb{R}^d \to \mathbb{R}$ be any function. Consider the family $(\omega_i)_{i \in I}$, of all open sets of $\mathbb{R}^d$, for an appropriate index set I, such that for each $i \in I$, $f = 0$ a.e. on ω_i. Set $\omega = \bigcup_{i \in I} \omega_i$. Then $f = 0$ a.e. on ω and define $\text{supp}(f) := \mathbb{R}^d \setminus \omega$.*

Function spaces For $d \geq 1$, let $\Omega \subset \mathbb{R}^d$ be a given domain. The Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}) = W^{k,p}(\Omega)$ are defined by

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega) \text{ for every } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k\}$$

with norm

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \le k} \|D^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p} & (1 \le p < \infty) \\ \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty(\Omega)} & (p = \infty) \end{cases}$$

for $p \in [1, \infty]$. Equivalently, by the Theorem of Meyers and Serrin [AF03, Theorem 3.18], the space $W^{k,p}(\Omega)$ can be characterized as the closure of $C^\infty(\Omega)$ under $\|\cdot\|_{W^{k,p}(\Omega)}$. The space $W_0^{k,p}(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{k,p}(\Omega)}$. If Ω has a Lipschitz boundary, we find the characterization

$$W_0^{k,p}(\Omega) = \{u \in W^{k,p}(\Omega) \mid u|_{\partial\Omega} = 0\},$$

where $u|_{\partial\Omega}$ denotes the trace of u on $\partial\Omega$. We use the abbreviations $H^k(\Omega) = W^{k,2}(\Omega)$ and $H_0^1(\Omega) = W_0^{1,2}(\Omega)$. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. For further reference on Sobolev spaces, compare to [Alt06], [AF03] or [Bre10].

Concerning the local Hölder regularity for two-phase flow, we exploit the following Poincaré type inequality.

Proposition 1.2 ([DiB93, chapter I, Proposition 2.1])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded convex set and let $\varphi \in C(\overline{\Omega})$ be such that $0 \le \varphi(x) \le 1$ for every $x \in \Omega$ and such that the sets $\{\varphi > k\}$ are convex for every $k \in (0,1)$. Let $u \in W^{1,p}(\Omega)$, $1 \le p < \infty$, and assume that the set $\mathcal{E} := \{u = 0\} \cap \{\varphi = 1\}$ has positive measure.*

Then there exists a constant C depending only upon d and p, not depending on u and φ, such that

$$\left(\int_\Omega \varphi \, |u|^p\right)^{\frac{1}{p}} \le C \frac{\mathrm{diam}(\Omega)^d}{|\mathcal{E}|^{\frac{d-1}{d}}} \left(\int_\Omega \varphi \, |Du|^p\right)^{\frac{1}{p}}. \tag{1.17}$$

We also use Bochner spaces. Let $T > 0$, $p \in [1, \infty]$ and a Banach space X be be given. We define the space $L^p(0, T; X)$ as the space of strongly measurable functions $u : [0, T] \to X$, such that the Bochner norm

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\right)^{1/p}$$

is bounded. For $p = \infty$, we define $L^\infty(0, T; X)$ as the space of strongly measurable functions $u : [0, T] \to X$ such that

$$\|u\|_{L^\infty(0,T;X)} = \operatorname*{ess\,sup}_{t \in [0,T]} \|u(t)\|_X < \infty.$$

We remark that $L^\infty(0, T; L^\infty(\Omega)) \subsetneq L^\infty(\Omega \times [0, T])$ as the example $f(x, y) = \mathbb{1}_{\{x \le y\}} = \mathbb{1}_{[0,y]}(x)$ on $[0,1]^2$ shows. Clearly, we find that $f \in L^\infty([0,1]^2)$. However, the induced map $F : [0,1] \to L^\infty([0,1])$ given by $y \mapsto \mathbb{1}_{[0,y]}$ is not strongly measurable. For further

results on the Bochner integral, we refer to [DU77], [Boc33], [AB07, ch. 11.1] and [Sch13, ch. 10.1].

Additionally, we define certain parabolic spaces on $Q := \Omega \times (0, T)$ in virtue of

$$V^p(Q) = L^\infty(0, T, L^p(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega))$$
$$V_0^p(Q) = L^\infty(0, T, L^p(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega))$$

both equipped with the norm

$$\|v\|_{V^p(Q)} = \operatorname*{ess\,sup}_{0 < t < T} \|v(\cdot, t)\|_{L^p(\Omega)} + \|\nabla v\|_{L^p(0,T;L^p(Q))}.$$

In chapter 6, we exploit the following embedding theorem.

Theorem 1.3. *Let $p > 1$ and let $\Omega \subset \mathbb{R}^d$ be bounded. There exists a constant C depending only on d and p such that for every $v \in V_0^p(Q)$ holds*

$$\|v\|_{L^p(Q)}^p \le C \, |\{|v| > 0\}|^{\frac{p}{d+p}} \, \|v\|_{V^p(Q)}^p. \tag{1.18}$$

The theorem is stated as a corollary in [DPV11, I.Corollary 4.1].

The duality pairing between a Banach space X and its dual X' is denoted by $\langle x', x \rangle_{X',X}$. As long as the domains are clear, we write $\|u\|_p$ or $\|u\|_{L^p}$ instead of $\|u\|_{L^p(Q)}$ and likewise for other norms.

Notation for spatial derivatives The spatial gradient of a real-valued function is denoted by ∇. Let $f : \mathbb{R}^{d+1} \to \mathbb{R}$ be given. For $u : \mathbb{R}^d \to \mathbb{R}$, we write

$$\nabla[f(u)] = \nabla[f(x, u(x))] = \nabla_x f(x, u(x)) + f'(x, u(x))\nabla u(x) = \nabla_x f(u) + f'(u)\nabla u.$$

Particularly, $\nabla_x f(x, u(x))$ denotes the evaluation of the d partial x-derivatives of f in the point $(x, u(x))$ and $f'(x, u(x))$ denotes the evaluation of the $d + 1$-st partial derivative of f at $(x, u(x))$. Occasionally, for the column vector $\nabla_x f(x, u(x))$ we use the notation

$$\nabla_x f(x, u(x)) = (f_1(x, u(x)), f_2(x, u(x)), \ldots, f_d(x, u(x)))^\top.$$

In addition, we use the standard symbol D to denote the vector of the derivatives of a function, see [Eva98, Appenidx A]. Consequently, we have $\nabla u = (Du)^\top$ but $\nabla f = \pi(Df)^\top$, where π is the projection onto the first d-coordinates. The matrix of the second order derivatives of u is denoted by $D^2 u$.

Definition 1.4. *For $A, B \in \mathbb{R}^{d \times d}$ with $A = (a_{jk})$ and $B = (b_{jk})$, we define*

$$A : B := \sum_{j,k=1}^{d} a_{jk} b_{jk}.$$

For $M \in C^1(\Omega, \mathbb{R}^{d \times d})$, let $M_k(x)$ be the $k-$th column of M. We define the column-wise divergence of M as

$$\nabla \cdot M(x) := (\nabla \cdot M_1(x), \dots, \nabla \cdot M_d(x)).$$

For $A \in C^1(\Omega, \mathbb{R}^{d \times d})$, $f \in C^1(\Omega, \mathbb{R})$ and $g \in C^1(\Omega, \mathbb{R}^d)$, we find the product rule

$$\nabla \cdot (fAg) = Ag \cdot \nabla f + f \nabla \cdot (Ag) = Ag \cdot \nabla f + f(A : (Dg)^\top) + f(\nabla \cdot A)g. \qquad (1.19)$$

Further Notation   Let $d \geq 1$ be given. For domains in $\mathbb{R}^d$, space-time domains in $\mathbb{R}^{d+1}$ and their boundaries, we use the following notation. Let

$$\Omega \subset \mathbb{R}^d \text{ be a bounded domain with boundary } \partial\Omega. \qquad (1.20)$$

The outward normal vector of Ω on $\partial\Omega$ is denoted by ν. We denote the Dirichlet and Neumann parts of $\partial\Omega$ as Γ_D and Γ_N, respectively. The decomposition is such that

$$\partial\Omega = \Gamma_D \cup \Gamma_N \text{ with } \Gamma_D \cap \Gamma_N = \emptyset \qquad (1.21)$$

holds. Let $T > 0$ be given. For $0 < t \leq T$ we define the space-time cylinders

$$Q_t := \Omega \times (0, t) \text{ denote its boundary by } \partial_p Q_t := \overline{Q}_t \setminus \Omega \times (0, t] \qquad (1.22)$$

and abbreviate $Q = Q_T$. The boundary $\partial_p Q_t$ is called the parabolic boundary of Q_t. For Ω with Lipschitz boundary and given $\Gamma_D \subset \partial\Omega$, we define

$$V := \left\{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \right\}. \qquad (1.23)$$

We are often concerned with Carathéodory functions, e.g. functions $g : \Omega \times [0, 1] \to \overline{\mathbb{R}}$, such that $g(x, \cdot) : [0, 1] \to \overline{\mathbb{R}}$ is continuous for a.e. $x \in \Omega$ and $g(\cdot, s)$ is measurable for every $s \in [0, 1]$. Such functions are jointly measurable. Furthermore, for a measurable function $u : \Omega \to [0, 1]$ the mapping $x \mapsto g(x, u(x))$, or as we write $g(\cdot, u)$, is measurable (see [AB07, 2.75 and 4.49-4.51]). With slight abuse of notation, we denote for measurable $u : Q \to [0, 1]$ the map $(x, t) \mapsto g(x, u(x, t))$ also by $g(\cdot, u)$.

Remark 1.5 (Continuity in $\overline{\mathbb{R}}$)**.** Continuity with values in $\overline{\mathbb{R}}$ is understood with respect to the topology generated by the metric $d(x,y) = |g(x) - g(y)|$ for $x, y \in \overline{\mathbb{R}}$, where

$$
g(x) = \begin{cases} -1 & \text{for } x = -\infty, \\ \frac{x}{1+|x|} & \text{for } x \in \mathbb{R}, \\ 1 & \text{for } x = \infty. \end{cases}
$$

This is the topology generated by the intervals (a, b), $(a, \infty]$ and $[-\infty, a)$ for $a, b \in \mathbb{R}$. See also [Alt06, 0.8 and 0.11] and [AB07, 2.75] ◁

Definition 1.6 (Inverse of a function in one direction)**.** *Let $\Phi \in C(\overline{\Omega} \times [0,1])$ be such that the map $\Phi_{(x)} : s \mapsto \Phi(x, s)$ is increasing for every $x \in \Omega$ and denote the inverse by $\Phi_{(x)}^{-1}$. For $\sigma \in (0, \frac{1}{2})$, we define the sets*

$$
K_\Phi := \{(x, u) \mid x \in \overline{\Omega}, \Phi(x, 0) \leq u \leq \Phi(x, 1)\} \text{ and}
$$
$$
K_\Phi^\sigma := \{(x, u) \mid x \in \overline{\Omega}, \Phi(x, \sigma) \leq u \leq \Phi(x, 1 - \sigma)\}.
$$

Furthermore, with abuse of notation, we define the map

$$
\Phi^{-1} : \quad K_\Phi \to [0, 1]
$$
$$
(x, u) \mapsto \Phi_{(x)}^{-1}(u).
$$

Remark 1.7. We emphasize that $\Phi_{(x)}([0,1]) = [\Phi(x, 0), \Phi(x, 1)]$ and that K_Φ and K_Φ^σ are closed. Additionally, by definition of Φ^{-1} we find $\Phi^{-1}(x, \Phi(x, s)) = s$ and $\Phi(x, \Phi^{-1}(x, u)) = u$ for fixed $x \in \overline{\Omega}$. The domain K_Φ and K_Ψ^σ are depicted in Figure 4.1. ◁

2 The Unsaturated Flow Problem

In this chapter, we state the main result concerning the problem for unsaturated flows. We consider a Lipschitz domain Ω that is separated by an interface Γ into two Lipschitz subdomains Ω_l and Ω_r as depicted in Figure 1.2. We assume that $\Omega_l \cap \Omega_r = \emptyset$ and the interface Γ is such that $\overline{\Gamma} = \overline{\Omega}_l \cap \overline{\Omega}_r$ holds. The outward unit normal vectors of Ω, Ω_l and Ω_r on $\partial\Omega, \partial\Omega_l$ and $\partial\Omega_r$ are denoted by ν, ν_l and ν_r, respectively. On $\Gamma, \partial\Omega_l \cap \partial\Omega$ and $\partial\Omega_r \cap \partial\Omega$, we find $\nu_l = -\nu_r, \nu = \nu_l$ and $\nu = \nu_r$, respectively. We consider the discontinuous Richards equation such a domain Ω. Particularly, for $j \in \{l, r\}$, we assume that functions λ_j and $p_{c,j}$ as well as vectors g_j are given and postulate that the flow on Ω_j is described by (R) from page 15. On the interface Γ we prescribe the transmission conditions (1.13).

To handle the discontinuous Richards equation, we require two transformations, both similar to the Kirchhoff transform (K) from page 15. This is executed in section 2.1. For the transformed problem, we provide a weak solution concept, and state the main result for the discontinuous Richards equation, an L^1-contraction in section 2.2. The proof of the main result is presented in chapter 5.

Before we investigate the problem stated above, we make some notational remarks. For $j \in \{l, r\}$, we abbreviate $Q_j := \Omega_j \times (0, T)$ and use for $h_j : \Omega_j \times [0, 1] \to \mathbb{R}$ and $u_j : \Omega_j \to [0, 1]$ the notation

$$u := u_l \, \mathbb{1}_{\Omega_l} + u_r \, \mathbb{1}_{\Omega_r} \text{ and } h(u) := h(x, u) := h_l(u) \, \mathbb{1}_{\Omega_l}(x) + h_r(u) \, \mathbb{1}_{\Omega_r}(x). \qquad (2.1)$$

2.1 Transformation of the Equations

We recall the equations we intend to consider. For $j \in \{l, r\}$, we use the notation (2.1) for s and obtain in virtue of (R) from page 15 the equations

$$
\begin{aligned}
\partial_t s &= \nabla \cdot (\lambda_j(s)(\nabla p_j + g_j)) + f_j \\
p_j &= p_{c,j}(s)
\end{aligned}
\qquad (2.2)
$$

on Q_j and from (1.13), again with abuse of notation, the equations

$$
\begin{aligned}
(\lambda_l(s)[\nabla p_l + g_l]) \cdot \nu_l &= -\,(\lambda_r(s)[\nabla p_r + g_r]) \cdot \nu_r \\
p_{c,l}(s) = p_l &= p_r = p_{c,r}(s)
\end{aligned}
\qquad (2.3)
$$

on $\Gamma \times (0, T)$. As in (K), we define for $j \in \{l, r\}$ and $s \in [0, 1]$ the transformation

$$\Phi_j(s) := \int_0^s \lambda_j(\sigma) p'_{c,j}(\sigma) \, \mathrm{d}\sigma = \int_{p_{c,j}(0)}^{p_{c,j}(s)} \lambda_j(p_{c,j}^{-1}(\xi)) \, \mathrm{d}\xi \,. \tag{2.4}$$

Consequently, (2.2) is transformed into

$$\partial_t s = \nabla \cdot (\nabla[\Phi_j(s)] + \lambda_j(s) g_j) + f_j \tag{2.5}$$

and the continuity of the flux from (2.3) reads

$$(\nabla[\Phi_l(s)] + \lambda_l(s) g_l]) \cdot \nu_l = -(\nabla[\Phi_r(s)] + \lambda_r(s) g_r]) \cdot \nu_r. \tag{2.6}$$

Concerning the continuity of the pressure, we impose as a compatibility condition that the ranges of $p_{c,l}$ and $p_{c,r}$ coincide. More precisely, we assume that $p_{c,l}$ and $p_{c,r}$ are increasing and such that $p_{c,l}(0) = p_{c,r}(0) \in [-\infty, \infty)$ and $p_{c,l}(1) = p_{c,r}(1) \in (-\infty, \infty]$ holds.

From the second equality in (2.4) we see that continuity of the pressures across Γ does not lead to a continuity of Φ across Γ, in general. Particularly, let s_l and $s_r \in [0, 1]$ be arbitrary, then

$$\Phi_l(s_l) = \Phi_r(s_r) \in [0, \infty] \quad \not\Longleftrightarrow \quad p_{c,l}(s_l) = p_{c,r}(s_r) \in [-\infty, \infty]$$

except if $\lambda_l(p_{c,l}^{-1}(u)) = \lambda_r(p_{c,r}^{-1}(u))$ for any $u \in [p_{c,l}(0), p_{c,r}(1)]$, which is not the case we want to consider.

Following [Can08], we define a transformation similar to (2.4) that contains the continuity information of the pressure across Γ. For $j \in \{l, r\}$, we define

$$\begin{aligned} \Theta_j(s) &:= \int_0^s \min_{k \in \{l, r\}} \left\{ \sqrt{\lambda_k(p_{c,k}^{-1}(p_{c,j}(\sigma)))} \right\} p'_{c,j}(\sigma) \, d\sigma \\ &= \int_{p_{c,j}(0)}^{p_{c,j}(s)} \min_{k \in \{l, r\}} \left\{ \sqrt{\lambda_k(p_{c,k}^{-1}(\xi))} \right\} d\xi. \end{aligned} \tag{2.7}$$

As the second equality in (2.7) shows, for arbitrary $s_l, s_r \in [0, 1]$ we obtain

$$\Theta_l(s_l) = \Theta_r(s_r) \in [0, \infty] \iff p_{c,l}(s_l) = p_{c,r}(s_r) \in [-\infty, \infty] \tag{2.8}$$

as long as $p_{c,l}(0) = p_{c,r}(0)$.

Summarizing, we obtain the transformed discontinuous Richards equation

$$\begin{aligned} \partial_t s &= \nabla \cdot (\nabla[\Phi_j(s)] + \lambda_j(s) g_j) + f_j \quad \text{on } Q_j \text{ for } j \in \{l, r\}, \\ 0 &= (\nabla[\Phi_l(s)] + \lambda_l(s) g_l]) \cdot \nu_l + (\nabla[\Phi_r(s)] + \lambda_r(s) g_r]) \cdot \nu_r \quad \text{on } \Gamma \times (0, T), \quad \text{(TDR)} \\ \Theta_l(s) &= \Theta_r(s) \quad \text{on } \Gamma \times (0, T). \end{aligned}$$

For $j \in \{l, r\}$, we consider the disjoint decomposition $\partial\Omega_j = \Gamma \cup \Gamma_{D,j} \cup \Gamma_{N,j}$, use $\nu_j = \nu$ on $\partial\Omega_j \setminus \Gamma$ to supplement (TDR) with the boundary conditions

$$
\begin{aligned}
0 &= \nu \cdot (\nabla\left[\Phi_j(s)\right] + \lambda_j(s)g_j) \quad \text{on } \Gamma_{N,j} \times (0, T), \\
\Phi_j(s) &= \Phi_{D,j} \quad \text{on } \Gamma_{D,j} \times (0, T)
\end{aligned}
\tag{2.9}
$$

and with the initial condition

$$
s(x, 0) = s_0(x) \text{ for } x \in \Omega
\tag{2.10}
$$

for appropriate functions $\Phi_{D,j}$ and s_0.

2.2 Weak Solutions and Main Result

We start with the assumptions on the domain and recall the introduction of the interface Γ at the beginning of this chapter.

Assumption A2.1. *Let $d \geq 1$ and $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary. Additionally, there are Lipschitz domains $\Omega_l, \Omega_r \subset \Omega$ such that $\Omega_l \cap \Omega_r = \emptyset$ and $\overline{\Omega}_l \cup \overline{\Omega}_r = \overline{\Omega}$. Γ is such that $\overline{\Gamma} = \overline{\Omega}_l \cap \overline{\Omega}_r$. For $j \in \{l, r\}$, we assume that the disjoint decompositions $\partial\Omega_j = \Gamma_{N,j} \cup \Gamma_{D,j} \cup \Gamma$ and $\partial\Omega = \Gamma_D \cup \Gamma_N$ hold, where $\Gamma_D = \Gamma_{D,l} \cup \Gamma_{D,r}$ and $\Gamma_N = \Gamma_{N,l} \cup \Gamma_{N,r}$.*

We only use the following assumptions on the coefficients.

Assumption A2.2. *For $j \in \{l, r\}$, we assume that $f_j \in C^{0,1}([0,1])$, $\lambda_j \in C([0,1])$ and $g_j \in \mathbb{R}^d$ and that there are measurable functions $s_{D,j} : \Omega \to [0,1]$ such that $\Phi_{D,j} = \Phi_j(s_{D,j}) \in H^1(\Omega_j)$. The Lipschitz constant of f_j is denoted by L_j.*

Assumption A2.3. *Let $j \in \{l, r\}$. Assume that there exist functions $\Theta_j, \Phi_j : [0,1] \to [0,\infty]$, that are increasing, continuous in the sense of Remark 1.5 and that the compatibility conditions $\Theta_l(0) = \Theta_r(0) = 0$ and $\Theta_l(1) = \Theta_r(1) \in (0,\infty]$ hold. If $\Theta_l(1) < \infty$, we impose $\Theta_j, \Phi_j \in C^1([0,1])$. Otherwise, we impose $\Theta_j, \Phi_j \in C^1([0,1))$.*

Remark 2.1. In the following, we only work with the regularity of Φ_j and Θ_j from Assumption A2.3. Thus, it is not necessary to impose assumptions on $p'_{c,j}$ or further assumptions on λ_j.

However, assuming integrability near zero, additional conditions like $p_{c,j} \in C^1((0,1))$, $\lambda_j \in C^{0,1}([0,1])$ and $p'_{c,j}(s), \lambda_j(s) > 0$ for $s \in (0,1)$ and $j \in \{l, r\}$ provide the regularity of Φ_j and Θ_j stated in Assumption A2.3. In particular, the conditions of [Can08] are allowed, i.e. $\lambda_j(0) = \lambda_j(1) = 0$ and $p_{c,j} \in C^1([0,1])$ for $j \in \{l, r\}$. Moreover, choices of λ_j and $p_{c,j}$ as in Figure 1.1 are possible and Φ_j and Θ_j can have both shapes depicted in Figure 4.1. ◁

We recall the definition and V from (1.23) and apply the notation to Ω_j, i.e. we define

$$V_j := \left\{ v \in H^1(\Omega_j) \mid v|_{\Gamma_{D,j}} = 0 \right\} \tag{2.11}$$

for $j \in \{l, r\}$. With the notation of (2.1), the abbreviations Q from (1.22) and Q_j from the beginning of this chapter, we define weak solutions for the discontinuous Richards equation.

Definition 2.2. *We call $s \in L^\infty(Q, [0,1])$ a weak solution of* (TDR) *with initial data $s_0 \in L^\infty(\Omega, [0,1])$, if the following properties hold:*

1. $\partial_t s \in L^2(0, T; V')$ *and*

$$\int_0^T \langle \partial_t s, \xi \rangle_{V',V} + \int_Q s \partial_t \xi = - \int_\Omega s_0 \xi(\cdot, 0) \tag{2.12}$$

for every $\xi \in L^2(0, T; V) \cap W^{1,1}(0, T; L^1(\Omega))$ such that $\xi(\cdot, T) = 0$

2. $\Phi_j(s) \in L^2(0, T; V_j) + \Phi_{D,j}$ *for $j \in \{l, r\}$ and*

$$\int_Q \langle \partial_t s, \xi \rangle_{V',V} + \sum_{j=l}^r \int_{Q_j} [\nabla \Phi_j(s) + \lambda_j(s) g_j] \cdot \nabla \xi = \int_Q f(s) \xi \tag{2.13}$$

for every $\xi \in L^2(0, T; V)$

3. $\Theta(s) \in L^2(0, T; V)$.

Remark 2.3. In Definition 2.2 item 1 states $s(0) = s_0$, item 2 covers the continuity of the flux and item 3 covers the continuity of the pressure on Γ. We emphasize that item 3 needs to be read with the notation from (2.1). The only assumptions on Θ_j are those imposed in Assumption A2.3. In particular, different definitions of Θ_j than that of (2.7) may be used.

In the case without an interface we consider $\Gamma = \emptyset$, $\Omega_l = \Omega_r$, $\lambda_l(s) = \lambda_r(s)$ and $p_{c,l}(s) = p_{c,r}(s)$. Consequently, item 3 in Definition 2.2 is not required and the sum in (2.13) is replaced by a single integral over Q.

In [Can08] an existence result for nondegenerate capillary pressures, $\lambda_j(0) = \lambda_j(1) = 0$ and the same solution concept is shown. We are not going to provide existence for our more general choice of Θ. $\triangleleft$

To prove the L^1-contraction, we have to impose the following assumption.

Assumption A2.4. *For $j \in \{l, r\}$, we assume that $\lambda_j \circ \Theta_j^{-1}$ is Lipschitz continuous on $[0, \Theta_j(1))$ and that $\Phi_j \circ \Theta_j^{-1}$ is differentiable on $(0, \Theta_j(1))$. We define $\Lambda_j := (\Phi_j \circ \Theta_j^{-1})'$ and assume additionally that Λ_j is Lipschitz continuous and bounded on $(0, \Theta_j(1))$.*

The Lipschitz continuity of Λ_j is also required in [Can08] and is crucial to infer the L^1-contraction. In addition, we require bounds on Λ_j since we consider potentially unbounded functions Θ_j. Our main result on the discontinuous Richards equation (TDR) from page 24 is the following:

Theorem 2.4 (L^1-contraction and uniqueness)**.** *Let Assumptions A2.1, A2.2, A2.3 and A2.4 hold. Let s_1, s_2 be weak solutions of* (TDR) *in the sense of Definition 2.2 with initial data $s_{0,1}$ and $s_{0,2}$, respectively. Then there holds*

$$
\begin{aligned}
&\int_Q (|s_{0,1} - s_{0,2}| - |s_1 - s_2|)\partial_t\gamma + \\
&+ \sum_{j\in\{l,r\}} \int_{Q_j} \operatorname{sign}(\Phi_j(s_1) - \Phi_j(s_2))\nabla[\Phi_j(s_1) - \Phi_j(s_2)]\cdot\nabla\gamma \\
&+ \sum_{j\in\{l,r\}} \int_{Q_j} \operatorname{sign}(\Phi_j(s_1) - \Phi_j(s_2))[\lambda_j(s_1) - \lambda_j(s_2)]g_j\cdot\nabla\gamma \\
&\leq \int_Q \operatorname{sign}(s_1 - s_2)\left[f(s_1) - f(s_2)\right]\gamma
\end{aligned}
\tag{2.14}
$$

for every non-negative $\gamma \in C_c^\infty((-\infty, T)\times\mathbb{R}^n)$. Moreover, for $L = \max\{L_1, L_2\}$, there holds the following generalized L^1-contraction property

$$
\|s_1(t) - s_2(t)\|_{L^1(\Omega)} \leq e^{Lt}\|s_{0,1} - s_{0,2}\|_{L^1(\Omega)},
\tag{2.15}
$$

for almost every $t \in (0, T)$. Consequently, there is at most one solution to the discontinuous Richards equation (TDR)*.*

The proof of the theorem is presented in chapter 5 and uses the method of doubling the variables. This method was introduced by [Kru70] and is presented, for example, in [Ott95] or [Can08]. An important tool required to perform the method of doubling the variables is the integration by parts formula from Lemma 4.36.

We provide two examples of coefficient functions, such that Λ_j is Lipschitz continuous though $p_{c,j}$ is unbounded. In [Can08] only bounded capillary pressure functions are considered. Consequently, Theorem 2.4 generalizes Theorem 3.1 from [Can08] in a substantial way.

For simplicity, both examples only show that Λ_j is Lipschitz continuous sufficiently close to zero. We use the inverse function theorem to obtain the identity

$$
\Lambda_j(u) = \frac{\lambda_j(\Theta_j^{-1}(u))}{\min\limits_{k\in\{l,r\}}\sqrt{\lambda_k(p_{c,k}^{-1}(p_{c,j}(\Theta_j^{-1}(u))))}},
\tag{2.16}
$$

where $0 < u$ is sufficiently small and $j \in \{l, r\}$.

Example 2.5 (Logarithmic Pressures). *For $j \in \{l, r\}$, let $\alpha_j, A_j, B_j > 0$, and $C_j \in \mathbb{R}$ be given. On a small interval near zero, we assume*

$$\lambda_j(s) = A_j s^{\alpha_j} \text{ for } s \geq 0 \text{ and } p_{c,j}(s) = B_j \ln(s) + C_j \text{ for } s > 0.$$

The coefficients are chosen such that

$$\alpha_j B_k = \alpha_k B_j, \tag{2.17}$$

where $k \in \{l, r\}$ with $k \neq j$.

We find

$$p_{c,k}^{-1}(p_{c,j}(s)) = \begin{cases} s, & k = j \\ s^{\frac{B_j}{B_k}} \exp(\frac{C_j - C_k}{B_k}), & j \neq k \end{cases}$$

and with (2.17) also

$$\min_{k \in \{l,r\}} \sqrt{\lambda_k(p_{c,k}^{-1}(p_{c,j}(s)))} = \min_{k \in \{l,r\}} \left\{ \sqrt{A_k} \exp\left(\alpha_k \frac{C_j - C_k}{2B_k} \right) \right\} s^{\frac{\alpha_j}{2}} =: D_j s^{\frac{\alpha_j}{2}}.$$

For $j \in \{l, r\}$, this leads to

$$\Theta_j(s) = \frac{2}{\alpha_j} B_j D_j s^{\frac{\alpha_j}{2}}.$$

Inversion of Θ_j and application to (2.16) yields

$$\Theta_j^{-1}(u) = \left(\frac{\alpha_j u}{2B_j D_j} \right)^{\frac{2}{\alpha_j}} \text{ and } \Lambda_j(u) = \frac{A_j}{D_j} \left(\frac{\alpha_j}{2B_j D_j} \right)^2 u$$

for u close to zero; thus, Λ_j is Lipschitz continuous near zero. Particularly, the case where $p_{c,l}$ and $p_{c,r}$ differ only by a vertical shift is allowed.

Example 2.6 (Higher Order Pressures). *For $j \in \{l, r\}$, let α_j, β_j, A_j and $B_j > 0$ be given. For s sufficiently small, we assume*

$$\lambda_j(s) = A_j s^{\alpha_j} \text{ for } s \geq 0 \quad \text{and} \quad p_{c,j}(s) = -\frac{B_j}{s^{\beta_j}} \text{ for } s > 0.$$

The coefficients are chosen such that

$$\alpha_l \beta_r = \alpha_r \beta_l \text{ and } \alpha_j > 2\beta_j. \tag{2.18}$$

We find

$$p_{c,k}^{-1}(p_{c,j}(s)) = \begin{cases} s, & j = k \\ \left(\frac{B_k}{B_j} \right)^{\frac{1}{\beta_k}} s^{\frac{\beta_j}{\beta_k}}, & j \neq k. \end{cases}$$

Due to (2.18), we infer for $j \in \{l, r\}$ also

$$\min_{k \in \{l,r\}} \sqrt{\lambda_k(p_{c,k}^{-1}(p_{c,j}(s)))} = \min_{k \in \{l,r\}} \left\{ \sqrt{A_k} \left(\frac{B_k}{B_j} \right)^{\frac{\alpha_k}{2\beta_k}} \right\} s^{\frac{\alpha_j}{2}} =: D_j s^{\frac{\alpha_j}{2}}.$$

Under consideration of (2.18), this leads to

$$\Theta_j(s) = \frac{2D_j B_j \beta_j}{\alpha_j - 2\beta_j} s^{\frac{\alpha_j - 2\beta_j}{2}}.$$

Inversion of Θ_j and application to (2.16) yields

$$\Theta_j^{-1}(u) = \left(\frac{\alpha_j - 2\beta_j}{2D_j B_j \beta_j} \right)^{\frac{2}{\alpha_j - 2\beta_j}} u^{\frac{2}{\alpha_j - 2\beta_j}} \;\; and \;\; \Lambda_j(u) = \frac{A_j}{D_j} \left(\frac{\alpha_j - 2\beta_j}{2D_j B_j \beta_j} \right)^{\frac{\alpha_j}{\alpha_j - 2\beta_j}} u^{\frac{\alpha_j}{\alpha_j - 2\beta_j}}$$

for u close to zero; thus, due to (2.18), Λ_j is Lipschitz continuous near zero.

In both examples, the assumptions on the coefficients are made to reduce complexity and to obtain straight forward calculations.

3 The Two-Phase Flow Problem

In this chapter, we state the main result concerning the two-phase flow problem (TP) from page 14. Prior to that, we provide two ways to transform the system into a coupled system of an elliptic and a parabolic partial differential equation (section 3.1). In section 3.2, we provide assumptions required to derive existence for the transformed problems (TP1) and (TP2). We specify assumptions needed to derive local Hölder continuity of the saturation for weak solutions of (TP1) in section 3.3. Throughout this chapter, we usually suppress the x-dependence of the occurring functions.

3.1 Transformation of the Equations

In this section, we transform the system (TP) into a system consisting of one parabolic and one elliptic equation. To this end, we introduce the so called global pressure and a transformation that resembles the Kirchhoff transformation (K). Having done that, we complete the problem by adding boundary and initial data.

We add the two elliptic-parabolic equations in (TP) from page 14, neglect the source terms, i.e. we assume $f_1 = f_2 = 0$, and use the capillary pressure relation to obtain

$$
\begin{aligned}
0 &= \nabla \cdot (\kappa[\lambda_1(s)\nabla p_1 + \lambda_2(s)\nabla p_2 + \lambda_1(s)g_1 + \lambda_2(s)g_2]) \\
&= \nabla \cdot \left(\kappa \left[\lambda(s) \left(\nabla p_1 - \frac{\lambda_2(s)}{\lambda(s)} \nabla[p_c(s)] \right) + \lambda_1(s)g_1 + \lambda_2(s)g_2 \right] \right),
\end{aligned}
\tag{3.1}
$$

with the definitions $\lambda(s) := \lambda_1(s) + \lambda_2(s)$ and $g_j := \rho_j g$. Equation (3.1) can be expressed with ∇p_2 instead of ∇p_1, in which case $-\lambda_2/\lambda$ changes to λ_1/λ.

We define a global pressure

$$
p := p_1 - \int_0^s \frac{\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \, \mathrm{d}\xi = p_2 + \int_0^s \frac{\lambda_1(\xi)}{\lambda(\xi)} p_c'(\xi) \, \mathrm{d}\xi
\tag{3.2}
$$

and find with a substitution

$$
p = p_1 - \int_{p_c(0)}^{p_c(s)} \frac{\lambda_2(p_c^{-1}(u))}{\lambda(p_c^{-1}(u))} \, \mathrm{d}u \,.
\tag{3.3}
$$

The global pressure p is often regarded as a mean pressure; particularly, in the case $\lambda_1 = \lambda_2$ we obtain $p = \frac{p_1+p_2}{2}$. The gradients of the global and capillary pressure are

$$\nabla p = \nabla p_1 - \frac{\lambda_2(s)}{\lambda(s)} p_c'(s) \nabla s - \int_0^s \nabla_x \left(\frac{\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \right) \mathrm{d}\xi,$$

$$\nabla[p_c(s)] = p_c'(s) \nabla s + \nabla_x p_c(s) \tag{3.4}$$

and we refer to the notation from section 1.4. Definitions (3.2) and (3.4) applied to (3.1), yield

$$0 \overset{(3.1)}{=} \nabla \cdot \left(\kappa \left[\lambda(s) \left(\nabla p_1 - \frac{\lambda_2(s)}{\lambda(s)} \nabla[p_c(s)] \right) + \lambda_1(s) g_1 + \lambda_2(s) g_2 \right] \right)$$

$$= \nabla \cdot (\kappa [\lambda(s) \nabla p + E(s)]) \tag{3.5}$$

with

$$E(s) := -\lambda_2(s) \nabla_x p_c(s) + \lambda \int_0^s \nabla_x \left(\frac{\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \right) \mathrm{d}\xi + \lambda_1(s) g_1 + \lambda_2(s) g_2.$$

For $\alpha \in \{1, 2\}$, the fluxes u_α specified in (1.8) read

$$u_\alpha = -\kappa \lambda_\alpha(s)(\nabla p_\alpha + g_\alpha). \tag{3.6}$$

We define the global flux $u := u_1 + u_2$ and find

$$u = u_1 + u_2 = -\kappa(\lambda(s) \nabla p + E(s)). \tag{3.7}$$

To reformulate one of the parabolic equations in (TP) in a way that it contains the information of u, we use the identity

$$\lambda u_1 = (\lambda_1 + \lambda_2) u_1 = \lambda_2 u_1 - \lambda_1 u_2 + \lambda_1 (u_1 + u_2) = \lambda_2 u_1 - \lambda_1 u_2 + \lambda_1 u. \tag{3.8}$$

For the equation of p_1 in (TP), we obtain

$$\phi \partial_t s = \nabla \cdot (\kappa \lambda_1(s)[\nabla p_1 + g_1]) \overset{(3.6)}{=} \nabla \cdot (-u_1)$$

$$\overset{(3.8)}{=} \nabla \cdot \left(\frac{1}{\lambda(s)} (\lambda_1(s) u_2 - \lambda_2(s) u_1) - \frac{\lambda_1(s)}{\lambda(s)} u \right) \tag{3.9}$$

$$\overset{(3.6)}{=} \nabla \cdot \left(\kappa \frac{1}{\lambda(s)} [\lambda_1(s)\lambda_2(s) \nabla[p_1 - p_2] + \lambda_1(s)\lambda_2(s)(g_1 - g_2)] - \frac{\lambda_1(s)}{\lambda(s)} u \right).$$

Using the equation for p_2 and a similar reasoning, we obtain (3.9) with $\lambda_2(s)/\lambda(s)\, u$ instead of $-\lambda_1(s)/\lambda(s)\, u$. For further manipulations, we define a pseudo pressure in the spirit of the the Kirchhoff transformation (K) and emphasize that the functions are x-dependent

$$\Phi(s) := \int_0^s \frac{\lambda_1(\xi)\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \, \mathrm{d}\xi. \tag{3.10}$$

We remark the identity

$$\nabla_x \Phi(s) = \int_0^s \nabla_x \left(\frac{\lambda_1(\xi)\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \right) \mathrm{d}\xi = \int_0^s \nabla_x \Phi'(s). \tag{3.11}$$

Using the definition of Φ and the gradient of the capillary pressure p_c from (3.4) in (3.9), we derive

$$\phi \partial_t s = \nabla \cdot \left(\kappa \left(\nabla[\Phi(s)] - \nabla_x \Phi(s) + B(s) \right) + D(s)u \right) \tag{3.12}$$

with

$$B(s) = \frac{\lambda_1(s)\lambda_2(s)}{\lambda(s)} \left(\nabla_x[p_c(s)] + g_1 - g_2 \right) \text{ and } D(s) = -\frac{\lambda_1(s)}{\lambda(s)} \text{ or } \frac{\lambda_2(s)}{\lambda(s)}.$$

With $D(s) = -\lambda_1(s)/\lambda(s)$ and the definition of u, we can express (3.12) as

$$\phi \partial_t s = \nabla \cdot \left(\kappa \left(\nabla[\Phi(s)] + \lambda_1(s)\nabla p + \gamma_1(s) \right) \right), \tag{3.13}$$

where

$$\begin{aligned}
\gamma_1(s) &:= -\nabla_x \Phi(s) + B(s) + \frac{\lambda_1(s)}{\lambda(s)} E(s) \\
&= \lambda_1(s) \int_0^s \nabla_x \left(\frac{\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \right) \mathrm{d}\xi - \int_0^s \nabla_x \left(\frac{\lambda_1(\xi)\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \right) \mathrm{d}\xi + \lambda_1(s)g_1.
\end{aligned} \tag{3.14}$$

Likewise, with $D(s) = \lambda_2(s)/\lambda(s)$ we may write (3.12) as

$$\phi \partial_t s = \nabla \cdot \left(\kappa \left(\nabla[\Phi(s)] - \lambda_2(s)\nabla p + \gamma_2(s) \right) \right), \tag{3.15}$$

where

$$\begin{aligned}
\gamma_2(s) &:= \lambda_2(s)\nabla_x[p_c(s)] - \lambda_2(s) \int_0^s \nabla_x \left(\frac{\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \right) \mathrm{d}\xi \\
&\quad - \int_0^s \nabla_x \left(\frac{\lambda_1(\xi)\lambda_2(\xi)}{\lambda(\xi)} p_c'(\xi) \right) \mathrm{d}\xi - \lambda_2(s)g_2.
\end{aligned} \tag{3.16}$$

On the other hand, under consideration of the elliptic equation (3.5), we can deduce (3.15) also from (3.13), and vice verca.

We recall $\Omega, Q, \partial\Omega, \Gamma_N$ and Γ_D from (1.20)–(1.22). Summarizing the previous considerations, we may write (TP), formally equivalent, in one of the two forms below:

$$\phi \partial_t s = \nabla \cdot \left(\kappa \left(\nabla[\Phi(s)] - \nabla_x \Phi(s) + B(s) \right) + D(s)u \right) \text{ on } Q, \tag{TP1$_1$}$$
$$0 = \nabla \cdot \left(\kappa \left[\lambda(s)\nabla p + E(s) \right] \right) \text{ on } Q \tag{TP1$_2$}$$

or

$$\phi \partial_t s = \nabla \cdot (\kappa (\nabla[\Phi(s)] + \lambda_1(s)\nabla p + \gamma_1(s))) \text{ on } Q, \qquad (\text{TP2}_1)$$

$$0 = \nabla \cdot (\kappa [\lambda(s)\nabla p + E(s)]) \text{ on } Q. \qquad (\text{TP2}_2)$$

To complete these problems, we prescribe initial and Dirichlet boundary data. Particularly, we assume $\Gamma_N = \emptyset$. On $\Gamma_D \times (0, T] = \partial\Omega \times (0, T]$ we prescribe

$$p = p_D \text{ and } \Phi(s) = \Phi_D \text{ on } \Gamma_D \times (0, T]. \qquad (3.17)$$

and complete the systems by prescribing initial data

$$s(0) = s_0 \text{ on } \Omega \times \{0\}. \qquad (3.18)$$

Remark 3.1 (Comparison to [AD85a]). The article of Alt and DiBenedetto considers existence of the two-phase flow problem in the form of (TP) with various boundary conditions. As is shown there, solutions to this problem provide, via approximation, a solution to the bulk two-phase flow problem in a form similar to (TP1). In their transformed setting local uniform continuity, i.e. continuity on sets $\mathcal{K} \subset\subset Q$, of the saturation is shown. Particularly, continuity of the saturation at the parabolic boundary of Q is not provided there.

In the remainder of this remark, we compare our notation to the notation of Alt and DiBenedetto and assume for simplicity that $\kappa = 1$. In their notation the saturations s_1 and s_2 are functions of the pressure difference $p_1 - p_2$ and of x. Comparing this to our notation, we realize that the capillary pressure function $p_c(x, s)$ can be inverted for every $x \in \Omega$, as long as $p_c(x, \cdot)$ has the shape as shown in Figure 1.1. This leads to $s_1(x, p_1 - p_2) = s = p_c^{-1}(x, p_1 - p_2)$ with the notation from Definition 1.6.

The definition of global pressure Alt and DiBenedetto use is again slightly different and reads $p = p_1 - \int_0^{p_c(s)} \frac{\lambda_2}{\lambda}$. If we introduce a critical saturation s_c, i.e. $s_c \in [0, 1]$ such that $p_1 - p_2 = p_c(x, s) = 0 \iff s = s_c$ (see [CJ86, chapters I, III.3.2 and equation III.(2.13)]), we deduce with a change of variables $p = p_1 - \int_{s_0}^{s} \frac{\lambda_2}{\lambda} p_c'$. This coincides with (3.2) as long as $s_c \equiv 0$. We infer that the global pressures in [AD85a] and the one we use differ only by an x-dependent function. In particular, if the capillary pressure curves are independent of x, then the pseudo pressures differ only by a constant.

Starting from (3.9) Alt and DiBenedetto arrive at

$$\phi \partial_t s = \nabla \cdot \left(\frac{\lambda_1(s)\lambda_2(s)}{\lambda(s)} p_c'(s)\nabla s + B(s) + D(s)u \right). \qquad (3.19)$$

With Φ from (3.10) and realizing that

$$\frac{\lambda_1(s)\lambda_2(s)}{\lambda(s)} p_c'(s)\nabla s = \Phi'(s)\nabla s = \nabla[\Phi(s)] - \nabla_x[\Phi(s)],$$

we might cast (3.19) into the form of (3.12). However, this is not done in [AD85a]. In the derivation of the local uniform continuity of the saturation equation (3.19) appears, but as a limit problem.

We remark that $\partial_z s_1^{-1}(x, \cdot)$ in their notation coincides with $p_c'(x, \cdot)$ in our notation. At first glance, a difference occurs in the terms b in their notation and B in our notation. However, this difference is based on a typing error in [AD85a, (1.19)] where instead of $\nabla_z(\frac{s_1}{s_0})$ it should be $\nabla_x(\frac{s_1}{s_0})$. Assuming $s_0 = 1$, this can be seen in the following way. For $x \in \Omega$ and $z \in [0, 1]$, we find the identity $p_c(x, p_c^{-1}(x, z)) = z$. The chain rule yields

$$0 = \nabla[p_c(x, p_c^{-1}(x, z))] = \nabla_x p_c(x, p_c^{-1}(x, z)) + p_c'(x, p_c^{-1}(x, z))\nabla_x p_c^{-1}(x, z)$$

and hence

$$\nabla_x p_c^{-1}(x, z) = -\frac{1}{p_c'(x, p_c^{-1}(x, z))}\nabla_x p_c(x, p_c^{-1}(x, z)). \tag{3.20}$$

With the previous considerations, we see that (3.20) corresponds to

$$\nabla_x s_1(x, z) = -\frac{1}{\partial_z s_1^{-1}(x, s_1(x, z))}\nabla_x s_1^{-1}(x, s_1(x, z)).$$

Using this in [AD85a, (1.19)], we find that b and B coincide. ◁

Remark 3.2 (Comparison to [Che01] and [Arb92])**.** We mention that the articles [Che01] and [Arb92] are quite similar. The main differences are a slightly different definition of the pseudo pressure and a more general right-hand side of the equation in Arbogast's article. However, since we use the same pseudo pressure as Chen does, we comment mainly on his article.

The article of Chen uses the second form (TP2) of the two-phase flow system. Implicitly, it is assumed in [Che01, (A4)] that p_c is nonincreasing. This is explained more detailed in [Arb92, (A5b*)]. Compared to our deduction, Chen uses the negative of the pseudo pressure Φ and $s = s_2$ instead of s_1. Essentially, our definitions of γ_1 and γ_2 coincide with γ_3 and γ_2 in Chen's notation, except for signs due to the different choices of s, p_c and Φ and the orientation of g_j. ◁

3.2 Weak Solutions

Following [Che01], we briefly state assumptions required to derive an existence result for (TP2) modified due to our choices of Φ, s and boundary conditions.

Assumption A3.1. *Let $d \geq 1$ and $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain.*

Assumption A3.2. *We assume that $\phi \in L^\infty(\Omega)$ is such that $0 < \phi_* \leq \phi(x) \leq \phi^* < \infty$ and that $\kappa(x)$ is a bounded, symmetric and uniformly positive definite matrix, i.e.*

$$0 < \kappa_* \leq |\xi|^{-2} \sum_{i,j=1}^{n} \kappa_{ij}(x)\xi_i\xi_j \leq \kappa^* < \infty, \quad x \in \Omega, \quad 0 \neq \xi \in \mathbb{R}^d$$

Assumption A3.3. *For $\alpha \in \{1, 2\}$, let $\lambda_\alpha(x, s)$ be bounded Carathéodory functions as introduced in section 1.4. Additionally, assume that $\lambda_1(0) = 0$ and $\lambda_1(s) > 0$ for $s > 0$, $\lambda_2(1) = 0$ and $\lambda_2(s) > 0$ for $s < 1$, and*

$$0 < \lambda_* \leq \lambda(x, s) \leq \lambda^* < \infty \text{ for } x \in \Omega \text{ and } s \in [0, 1].$$

Assumption A3.4. *$\Phi : \Omega \times [0, 1] \to \overline{\mathbb{R}}$ is a Carathéodory function, such that $\Phi(x, s)$ is strictly increasing in s for every $x \in \Omega$. In addition, assume that $\Phi(x, 0) = 0$ and $0 < \Phi(x, 1)$ for every $x \in \Omega$ and $\Phi(\cdot, 1) \in H^1(\Omega)$.*

As in [Che01] and [Arb92], we introduce the following notation. For $v = v(x, s)$ and any norm $\|\cdot\|$ for x-dependent functions, we define the norm $\|\|\cdot\|\|$ by

$$\|v\| = \left\| \sup_{s \in [0,1]} |v(\cdot, s)| \right\|. \tag{3.21}$$

Assumption A3.5. *E and γ_1 are Carathéodory functions and the norms*

$$\|E\|_{L^\infty(0,T;L^2(\Omega))} \text{ and } \|\gamma_1\|_{L^2(Q)}$$

are bounded. Furthermore, $p_D \in L^\infty(0, T; H^1(\Omega))$, $\Phi_D \in L^2(0, T; H^1(\Omega))$, $\partial_t \Phi_D \in L^1(Q)$ and $0 \leq \Phi_D(x, t) \leq \Phi(x, 1)$ almost everywhere on Q. The initial data fulfill

$$0 \leq \Phi(s_0) \leq \Phi(1) \text{ a.e. in } \Omega \text{ and } \Phi(s_0) \in L^2(\Omega).$$

Since we only consider the Dirichlet problem here, we find $\Gamma_D = \partial\Omega$. Thus, considering (1.23) we have $V = H_0^1(\Omega)$ and $V' = H^{-1}(\Omega)$.

Definition 3.3 (Weak solutions for (TP2))**.** *A weak solution of system* (TP2) *with boundary and initial data* (3.17)–(3.18) *is a pair of functions (s, p) with $p \in L^\infty(0, T; V) + p_D$, $\Phi(s) \in L^2(0, T; V) + \Phi_D$, $\phi \partial_t s \in L^2(0, T; V)$, $0 \leq s(x, t) \leq 1$ a.e. $(x, t) \in Q$ and such that the following identities hold:*

$$\int_\Omega \kappa(\lambda(s)\nabla p + E(s)) \cdot \nabla w = 0 \quad \forall w \in L^\infty(0, T; V),$$

$$\int_0^T \langle \phi \partial_t s, v \rangle_{V',V} \, dt + \int_Q \kappa(\nabla[\Phi(s)] + \lambda_1(s)\nabla p + \gamma_1(s)) \cdot \nabla v = 0 \quad \forall v \in L^2(0, T; V),$$

$$\int_0^T \langle \phi \partial_t s, v \rangle_{V',V} \, dt + \int_0^T \int_\Omega \phi(s - s_0) \partial_t v \, dt = 0$$
$$\forall v \in L^2(0, T; V) \cap W^{1,1}(0, T; L^1(\Omega)), \ v(x, T) = 0.$$

An adaption of [Che01, Theorem 2.1] yields existence of weak solutions.

Theorem 3.4. *Under Assumptions A3.1–A3.5, the system* (TP2) *has a weak solution in the sense of Definition 3.3.*

We define weak solutions of (TP1) in the following sense.

Definition 3.5 (Weak solutions of (TP1))**.** *A weak solution of system* (TP1) *with boundary and initial data* (3.17)–(3.18) *is a pair of functions* (s, p) *with* $p \in L^\infty(0, T; V) + p_D$, $\Phi(s) \in L^2(0, T; V) + \Phi_D$, $\phi \partial_t s \in L^2(0, T; V')$, $0 \le s(x, t) \le 1$ *a.e.* $(x, t) \in Q$ *and such that the following identities hold:*

$$\int_\Omega \kappa(\lambda(s)\nabla p + E(s)) \cdot \nabla w = 0 \quad \forall w \in L^\infty(0, T; V), \tag{3.22}$$

$$\int_0^T \langle \phi \partial_t s, v \rangle_{V', V} \, \mathrm{d}t + \int_Q (\kappa[\nabla[\Phi(s)] - \nabla_x \Phi(s) + B(s)] + D(s)u) \cdot \nabla v = 0$$
$$\forall v \in L^2(0, T; V), \tag{3.23}$$

$$\int_0^T \langle \phi \partial_t s, v \rangle_{V', V} \, \mathrm{d}t + \int_0^T \phi(s - s_0) \partial_t v \, \mathrm{d}t = 0 \tag{3.24}$$
$$\forall v \in L^2(0, T; V) \cap W^{1,1}(0, T; L^1(\Omega)), \ v(x, T) = 0.$$

To obtain weak solutions of (TP1) from weak solutions of (TP2) provided by Theorem 3.4, we need to ensure that

$$\int_Q \kappa(\lambda_1(s)\nabla p + \gamma_1(s)) \cdot \nabla v = \int_Q [\kappa(-\nabla_x \Phi(s) + B(s)) + D(s)u] \cdot \nabla v \tag{3.25}$$

holds for every $v \in L^2(0, T; V)$. Particularly, we have to take the alternatives in the definition of $D(s)$ into account.

Lemma 3.6 (Equivalence of weak solutions)**.** *Let Assumptions A3.1 – A3.5 hold, and assume that the functions* $\nabla_x \Phi(s)$ *and* B *are bounded Carathéodory functions on* $\Omega \times [0, 1]$. *Then any weak solution* (s, p) *of* (TP2) *in the sense of Definition 3.3 is a weak solution of* (TP1) *in the sense of Definition 3.5 and vice versa. In particular, there exists a weak solution of* (TP1) *in the sense of Definition 3.5.*

Proof. To show that the solution concepts are equivalent, it suffices to show that (3.25) holds for every $v \in L^2(0, T; V)$. Due to the assumptions, we obtain that the integrals in (3.25) are well-defined.

Let (s, p) be a weak solution of in the sense of Definition 3.5. We choose the alternative $D(s) = -\lambda_1(s)/\lambda(s)$. Using the definitions of $\gamma_1(s)$ and u from (3.14) and (3.7), respectively, we obtain the pointwise identity

$$\kappa\left(-\nabla_x \Phi(s) + B(s)\right) - \frac{\lambda_1(s)}{\lambda(s)} u = \kappa\left(\lambda_1(s)\nabla p + \gamma_1(s)\right)$$

a.e. on Q. Consequently, for our choice of $D(s)$, we infer that (3.25) holds for every $v \in L^2(0, T; V)$.

Concerning the alternative $D(s) = {}^{\lambda_2(s)}/{}_{\lambda(s)}$, we need to take the elliptic equation into account. For $v \in C_c^\infty(0, T; V)$, we use (3.22) to infer

$$\int_Q \kappa(\nabla \Phi(s) + \lambda_1(s)\nabla p + \gamma_1(s)) \cdot \nabla v$$

$$= \int_Q \kappa(\nabla \Phi(s) + \lambda \nabla p - \lambda_2(s)\nabla p + \gamma_1(s)) \cdot \nabla v$$

$$\int_Q \kappa(\nabla \Phi(s) - E(s) - \lambda_2(s)\nabla p + \gamma_1(s)) \cdot \nabla v.$$

From

$$\frac{\lambda_1(s)}{\lambda(s)} E(s) - E(s) = -\frac{\lambda_2(s)}{\lambda(s)} E(s),$$

we derive the identity

$$\kappa(\gamma_1(s) - E(s) - \lambda_2(s)\nabla p) = \kappa(-\nabla_x \Phi(s) + B(s)) + \frac{\lambda_2(s)}{\lambda(s)} u$$

a.e. on Q as above. Consequently, (3.25) holds for every $v \in C_c^\infty(0, T; V)$ and the choice $D(s) = {}^{\lambda_2(s)}/{}_{\lambda(s)}$. With a density argument, we infer that the argument also holds for $v \in L^2(0, T; V)$. Thus, (s, p) is a weak solution of (TP2) in the sense of Definition 3.3.

Considering the previous arguments, we realize that starting from weak solutions of (TP2) in the sense of Definition 3.3 yields a weak solution to (TP1) in the sense of Definition 3.5. This concludes the proof. $\qquad\square$

3.3 Main Result

We specify assumptions on the coefficients needed to derive the local Hölder continuity of the saturation.

Assumption A3.6. *We assume that p_c is differentiable in x for every (x, s) in $\overline{\Omega} \times [0, 1]$ and in s for $(x, s) \in \overline{\Omega} \times (0, 1)$. We assume that*

$$0 < p_* := \inf_{\substack{x \in \overline{\Omega} \\ s \in (0,1)}} p_c'(x, s) < \infty \ \text{and} \ 0 \le p^* := \max_{\substack{x \in \overline{\Omega} \\ s \in [0,1]}} |\nabla_x p_c(x, s)| < \infty. \tag{3.26}$$

For a typical shape of the function Φ, we refer to Figure 4.1. The assumptions imposed on Φ from Assumption A4.1 (page 51) are also contained in the following assumptions.

Assumption A3.7. *Let* $\Phi \in C^1(\overline{\Omega} \times [0,1])$ *be such that* $\Phi'(x,s) > 0$ *for* $s \in (0,1)$ *and every* $x \in \overline{\Omega}$, *and* $\Phi'(x,0) = \Phi'(x,1) = \Phi(x,0) = 0$ *for every* $x \in \overline{\Omega}$.

Let $\delta_0 \in (0, \frac{1}{8})$ *be given. We define*

$$
\begin{aligned}
\lambda_{2,*} &:= \min_{\substack{x \in \overline{\Omega} \\ s \in [0,4\delta_0]}} \lambda_2(x,s), \quad \lambda_{1,*} := \min_{\substack{x \in \overline{\Omega}, \\ s \in [1-4\delta_0,1]}} \lambda_1(x,s) \\
\text{and} \quad \lambda_{1,2,*} &:= \min_{\substack{x \in \overline{\Omega}, \\ s \in [\delta_0, 1-\delta_0]}} \min\{\lambda_1(x,s), \lambda_2(x,s)\}
\end{aligned}
\tag{3.27}
$$

and assume that there are constants $0 < C_* < C^*$ *such that*

$$
\begin{aligned}
&C_* \leq \min\left\{\lambda_*, \phi_*, \kappa_*, p_*, \lambda_{2,*}, \lambda_{1,*}, \lambda_{1,2,*}\right\} \\
&\text{and } \max\left\{\frac{1}{C_*^2}, \lambda^*, \kappa^*, \phi^*, 3\frac{p^*}{p_*}, 3\frac{|g_1|}{p_*}, 3\frac{|g_2|}{p_*}\right\} \leq C^*.
\end{aligned}
\tag{3.28}
$$

In addition, we demand $\kappa \in W^{1,\infty}(\Omega; \mathbb{R}^{d \times d})$,

$$
\begin{aligned}
&\|\Phi\|_{C^1(\overline{\Omega} \times [0,1])} + \|\kappa\|_{W^{1,\infty}} + \|\!|E|\!\|_{L^\infty(0,T;L^2(\Omega))} + \|p_D\|_{L^\infty(0,T;H^1(\Omega))} \leq C^* \text{ and} \\
&|B(x,s)| + |D(x,s)| + |E(x,s)| \leq C^* \text{ for every } x \in \overline{\Omega} \text{ and } s \in [0,1].
\end{aligned}
\tag{3.29}
$$

Furthermore, we impose that $\Phi'(x,s)$ *and* $D(x,s)$ *are differentiable with respect to* x, *and that for every* $x \in \overline{\Omega}$ *and* $s \in [0,1]$ *there holds*

$$
\nabla_x \Phi(x,s) = \int_0^s \nabla_x \Phi'(x,\sigma) \, \mathrm{d}\sigma \quad \text{and} \quad |\nabla_x \Phi'(x,s)| + |\nabla_x D(x,s)| \leq C^*.
\tag{3.30}
$$

Concerning the structure of Φ', *we assume that for every* $x \in \overline{\Omega}$ *there holds*

$$
\begin{aligned}
\Phi_{0,l}(s) &\leq \Phi'(x,s) \leq \Phi_{0,u}(s), \quad \text{for } s \in [0,4\delta_0] \\
\Phi_{1,l}(1-s) &\leq \Phi'(x,s) \leq \Phi_{1,u}(1-s), \quad \text{for } s \in [1-4\delta_0, 1].
\end{aligned}
\tag{3.31}
$$

For $j \in \{0,1\}$ *and* $k \in \{l, u\}$, *we consider positive constants* $c_{j,k}$ *and* α_j *such that* $\Phi_{j,k}(v) := c_{j,k} v^{\alpha_j}$ *holds for* $v \in [0, 4\delta_0]$. *In addition, we assume* $C_* \leq \Phi'(x,s) \leq C^*$ *for* $s \in [\delta_0, 1-\delta_0]$ *and every* $x \in \overline{\Omega}$. *For the powers* α_j, *we assume that*

$$
\alpha_0 = \alpha_1.
\tag{3.32}
$$

Remark 3.7. The assumption $\alpha_0 = \alpha_1$ is crucial to obtain Hölder continuity. In the x-indepedent case, Hölder continuity for the cases $\alpha_1 \neq \alpha_0$ is stated in the literature. However, there seem to be flaws in the given proofs in that case. We comment on these issues more detailed in section 6.6.1. ◁

Remark 3.8. Some of the assumptions in (3.28) are only made to simplify calculations later. Particularly, we derive the following estimates. Due to the choices of p_* and p^*, we infer

$$|B(s)| \leq \left|\frac{\lambda_1(s)\lambda_2(s)}{\lambda(s)}\left(\nabla_x p_c + g_1 - g_2\right)\right| \leq \frac{\lambda_1(s)\lambda_2(s)}{\lambda(s)p_*}p_c'(s)\left(|\nabla_x p_c| + |g_1| + |g_2|\right)$$

$$\leq \Phi'(s)\frac{p^* + |g_1| + |g_2|}{p_*} \leq C^*\Phi'(s).$$

The assumption on $\lambda_{1,*}$, $\lambda_{2,*}$ and $\lambda_{1,2,*}$ are connected to the alternative in the definition of $D(s)$. Assume for the moment that $s \in [0, 4\delta_0]$. Thus, we obtain

$$|D(x,s)| \leq \left|\frac{\lambda_1(s)\lambda_2(s)}{\lambda(s)\lambda_2(s)p_*}\right|p_c'(s) \leq \frac{1}{C_*^2}\Phi'(s) \leq C^*\Phi'(s). \tag{3.33}$$

With the same argument, we obtain the estimate of equation (3.33) also on $[1 - 4\delta_0, 1]$ and on $[\delta_0, 1 - \delta_0]$. This is used as follows. In the analysis in chapter 6, we use test functions that are only supported in either $[0, 4\delta_0]$, $[1 - 4\delta_0, 1]$ or $[\delta_0, 1 - \delta_0]$. Depending on the test function, we then choose D such that (3.33) holds.

We remark that the assumption on p^* is not needed in the case without x-dependence. ◁

Definition 3.9 (Parabolic Metric and Distance)**.** *On $\mathbb{R}^{d+1}$ we define the parabolic metric d_2 as*

$$d_2((x_1, t_1), (x_2, t_2)) = |x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}}$$

for $x_1, x_2 \in \mathbb{R}^d$ and $t_1, t_2 \in \mathbb{R}$. According to this metric for $A, B \subset \mathbb{R}^{d+1}$ we define the parabolic distance

$$\text{dist}_2(A, B) := \inf_{z_1 \in A,\, z_2 \in B} d_2(z_1, z_2). \tag{3.34}$$

Definition 3.10. *The constants $C_*, C^*, c_{j,k}$ and α_j, for $j \in \{0, 1\}$ and $k \in \{u, l\}$ from Assumption A3.7 and the dimension d from Assumption A3.1 are called the data. We say that a constant γ depends only on the data, if γ can be determined only in terms of these quantities.*

Furthermore, let $\mathcal{K} \subset Q$ be given. We say that a constant $\gamma \in \mathbb{R}$ depends only on the data and on $\mathcal{K}$, if γ can be determined only in terms of the data and in terms of the distance from $\mathcal{K}$ to $\partial_p Q$, i.e. of $d_2(\mathcal{K}, \partial_p Q)$. In this case, we write $\gamma = \gamma(data, \mathcal{K})$. Additionally, if we write $\gamma = \gamma(data, \mathcal{K}, l)$, we mean that γ can be determined only in terms of the data, $\mathcal{K}$ and some quantity l.

Remark 3.11. Since we only intend to provide local Hölder continuity of s on sets compactly contained in Q, initial and boundary data for s do net enter in Definition 3.10. Due to (3.27), the constants C_* and C^* also depend on the choice of δ_0. Thus, when speaking of a quantity depending on the data, the dependence on δ_0 is implied.

We emphasize that, if for the saturation s of (TP1) on a domain $\mathcal{K}$ a constant $\gamma = \gamma(data, \mathcal{K})$ is determined, then we emphasize that γ is particularly independent of s. ◁

Remark 3.11 also applies to the previous definition. Our main result for two-phase flows reads as follows.

Theorem 3.12 (Local Hölder continuity of *s*)**.** *We assume that Assumptions A3.1 – A3.7 hold. Let (s, p) be any weak solution of problem* (TP1) *in the sense of Definition 3.5. Then s is locally Hölder continuous on Q. For any compact set $\mathcal{K} \subset Q$, there are constants $\gamma \in \mathbb{R}$ and $\alpha \in (0, 1)$, depending only on the data and on $\mathcal{K}$, such that*

$$|s(x_1, t_1) - s(x_2, t_2)| \leq \gamma \left(d_2((x_1, t_1), (x_2, t_2)) \right)^{\alpha} \tag{3.35}$$

for every $(x_1, t_1), (x_2, t_2) \in \mathcal{K}$.

We emphasize that γ and α do not depend on the solution. The proof is presented in chapter 6 and uses the method of intrinsic scaling. To apply the technique, it is necessary to show that truncations of *s* are regular. This is shown in the next chapter.

Remark 3.13. We emphasize again that the proof of the local Hölder continuity we present in chapter 6 only works in the case $\alpha_0 = \alpha_1$. For an easier comparison to the literature as well as a potential extension, we chose the given presentation. Particularly, we do not simplify the notation from (3.31).

In view of Theorem 6.3 and Remark 6.4, we see that dropping the dependence of C^* on p_D in (3.29) yields a constant γ which also depends on the $L^{\infty}(0, T; L^2(\Omega))$-norm of p. We emphasize that C^* is independent of Φ_D and that Assumption A3.4 is contained completely in Assumption A3.7.

In the literature, statements similar to that of Theorem 3.12 can be found for so-called local weak solutions. Essentially, these are solutions as in Definition 3.5 but the integrals are restricted to sets $E \subset\subset Q$. Particularly, boundary and initial data are not specified there. Since the result we provide is a local result, the theorem could also be stated with minor changes in the setting of local weak solutions. ◁

4 Chain Rules and Integration by Parts

Due to the similarities of the Richards equation (R) from page 15 and the parabolic equation of the two-phase flow equations (TP1) from page 33, we consider the model problem

$$\begin{aligned}
\partial_t s &= \Delta[\Phi(x,s)] \text{ on } Q, \\
\Phi(x,s) &= \Phi_D \text{ on } \Gamma_D \times (0,T), \\
\partial_\nu \Phi(x,s) &= 0 \text{ on } \Gamma_N \times (0,T), \\
s(x,0) &= s_0(x) \text{ on } \Omega,
\end{aligned} \tag{MP}$$

where $s(x,t) \in [0,1]$ for $(x,t) \in Q$. We recall $\Omega, \Gamma_D, \Gamma_N, Q$ and V from (1.20)–(1.23).

Equation (MP) is the so-called *generalized porous medium equation* [Váz06]. The function $\Phi : \Omega \times [0,1] \to [0,\infty]$ is smooth in $\Omega \times [0,1]$ and increasing in s. When considering the Richards equation one usually finds Φ such that $\Phi(x,0) = \Phi'(x,0) = 0$ and $\Phi(x,s) \to c > 0$ as $s \to 1$, where c is usually infinity.

For the transformed two-phase flow equations (TP1), the function Φ does in general not tend to infinity when s approaches one. Typically, we find that Φ is flat near one, i.e. $\Phi'(x,1) = 0$. Common shapes of Φ are depicted in figure 4.1.

Though we are not going to show the existence of weak solutions, let us fix a setting where we expect to find solutions in.

Definition 4.1 (Weak solutions of (MP)). *A function $s \in L^\infty(Q; [0,1])$ with $\partial_t s \in L^2(0,T;V')$ and $\Phi(\cdot, s) \in L^2(0,T;V) + \Phi_D$ is called a weak solution of* (MP) *provided the following two properties are fulfilled:*

1. *For every $\xi \in L^2(0,T;V)$, there holds*

$$\int_0^T \langle \partial_t s, \xi \rangle_{V',V} + \int_Q \nabla \Phi(\cdot, s) \nabla \xi = 0. \tag{4.1}$$

2. *The initial data $s(0) = s_0$ are assumed in the sense of traces, i.e. there holds*

$$\int_0^T \langle \partial_t s(t), \xi(.,t) \rangle_{V',V} \, \mathrm{d}t + \int_Q s \partial_t \xi = - \int_\Omega s_0 \xi(.,0) \tag{4.2}$$

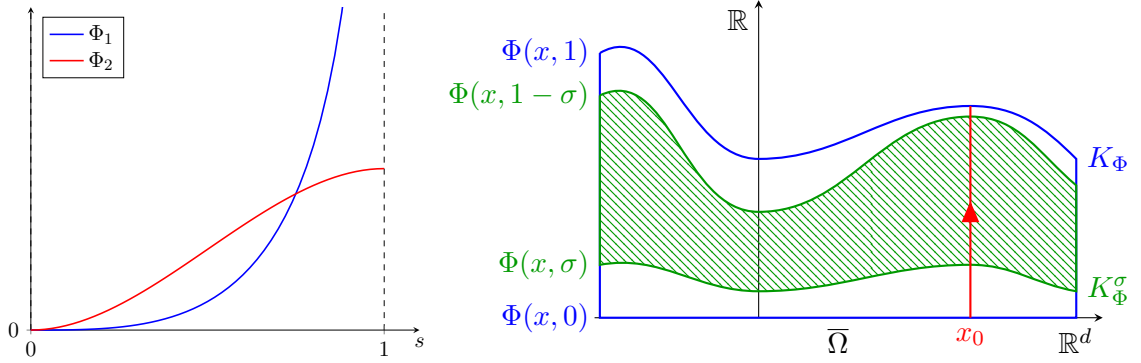*for every $\xi \in L^2(0,T;V) \cap W^{1,1}(0,T;L^1(\Omega))$ with $\xi(\cdot, T) = 0$.*

Figure 4.1: *The left picture shows typical shapes of Φ for fixed $x \in \Omega$. For Richards equations, Φ typically looks like Φ_1. In the case of the transformed two-phase flow equations* (TP1) *the shape of Φ is typically as that of Φ_2. The picture on the right shows the domain of the function Φ^{-1} in the sense of Definition 1.6. Particularly, the sets K_Φ and K_Φ^σ are depicted modulo a cross section. For fixed $x_0 \in \Omega$, the function $s \mapsto \Phi(x_0, s)$ is increasing and we consider it to have the shape of Φ_2 depicted in the left picture.*

For the sake of simplicity, let us assume for the rest of this section that $\Phi_D = 0$. An important subclass of weak solutions in the sense of definition 4.1 are so called weak energy solutions; existence of such solutions is shown in [Váz06, chapter 5.4] and [AL83]. Weak energy solutions rely on the energy estimate

$$\int_\Omega \Psi(\cdot, s(t)) + \int_{Q_t} |\nabla \Phi(\cdot, s)|^2 \leq \int_\Omega \Psi(\cdot, s_0), \tag{4.3}$$

where Ψ is a primitive of Φ with respect to s and such that $\Psi(x, 0) = 0$. Formally, this estimate is obtained by testing (MP) with $\Phi(x, s(x))$. For a rigorous proof of (4.3), the crucial point is to show the chain rule

$$\int_0^t \langle \partial_t s, \Phi(s) \rangle_{V', V} = \int_{Q_t} \partial_t \Psi(s) = \int_\Omega \Psi(s(t)) - \int_\Omega \Psi(s_0). \tag{4.4}$$

Essentially, the literature contains two methods to prove this chain rule. For the first approach, e.g. found in [Váz06, chapter 5], one constructs a sequence of smooth functions s_n that converges to s. For such smooth functions, (4.4) is nothing but the classical chain rule; one obtains (4.3) for the approximations s_n. Passing to the limit $n \to \infty$, the estimate for s is maintained. The downside of this procedure is, that the energy estimate (4.3) is only deduced for functions s that are the limits of an appropriate sequence s_n.

The second approach is due to Alt and Luckhaus [AL83]; there, the chain rule (4.4) is shown for any function in the function space of interest. In particular, the energy estimate (4.3) is obtained without explicitly constructing an approximating sequence.

We follow the method of Alt and Luckhaus and show in section 4.4 chain rules and related integration by parts formulae similar to (4.4), with $\Phi(\cdot, s)$ replaced by more general functions. Preliminary to that, we collect properties of the Steklov average that are presented in section 4.3. Additionally and more importantly for our objectives is that these more general chain rules can be used to derive an L^1-contraction property for weak solutions. We provide a formal proof in section 4.5.3 that also uses the so called method of *doubling the variables*. To derive a uniqueness result from an L^1-contraction property, it is mandatory to infer the L^1-contraction property for all solutions in the considered functions spaces. If one seeks uniqueness, it does not suffice to show an L^1-contraction only for solutions that are merely the limits of certain approximations.

Further typical choices of test functions for (MP) are s and $s\xi^2$, where ξ is a smooth and compactly supported function. The latter test function leads to so called Caccioppoli estimates [GM12, chapter 4]. However, for the cases of Φ we want to consider s does in general no possess a weak gradient. As the formal calculation

$$\nabla[\Phi(x, s(x))] = \nabla_x \Phi(x, s(x)) + \Phi'(x, s(x))\nabla s(x), \tag{4.5}$$

indicates and since $\Phi'(x, 0) = \Phi'(x, 1) = 0$, we can only expect control of ∇s with a weight. We emphasize that for smooth s, e.g. $s \in H^1(\Omega)$, Stampacchia's Lemma yields that $\nabla s = 0$ a.e. on the level sets $\{s = 0\}$ and $\{s = 1\}$. Due to the lack of regularity of s this information cannot be used here.

However, if s was bounded away from zero and one, then we could bound Φ' away from zero and expect s to have a weak gradient. To this end, consider $0 < \varepsilon < \frac{1}{2}$, use $\Phi^{-1}(x, \Phi(x, s)) = s$, the chain rule and the same reasoning as in (3.20), to infer

$$
\begin{aligned}
\nabla s \, \mathbb{1}_{\{\varepsilon < s < 1-\varepsilon\}} &= \nabla[\Phi^{-1}(\cdot, (\Phi(\cdot, s))] \, \mathbb{1}_{\{\varepsilon < s < 1-\varepsilon\}} \\
&= \left(\nabla_x \Phi^{-1}(\cdot, \Phi(\cdot, s)) + \frac{1}{\Phi'(\cdot, s)} \nabla[\Phi(\cdot, s)] \right) \mathbb{1}_{\{\varepsilon < s < 1-\varepsilon\}} \\
&= \frac{1}{\Phi'(\cdot, s)} \left(-\nabla_x \Phi(\cdot, s) + \nabla[\Phi(\cdot, s)] \right) \mathbb{1}_{\{\varepsilon < s < 1-\varepsilon\}}.
\end{aligned}
\tag{4.6}
$$

In the situation that Φ does not depend explicitly on x Stampacchia's Lemma implies that

$$\lim_{\varepsilon \to 0} \nabla[\Phi(s)] \, \mathbb{1}_{\{\varepsilon < s < 1-\varepsilon\}} = \nabla[\Phi(s)].$$

This raises the question whether we can perform the limit also in the case where Φ depends on x. We ask, whether

$$\nabla[\Phi(\cdot, s)] = \lim_{\varepsilon \to 0} [\Phi'(\cdot, s)\nabla s + \nabla_x \Phi(\cdot, s)] \, \mathbb{1}_{\{\varepsilon < s < 1-\varepsilon\}} \tag{4.7}$$

holds in an appropriate sense or not. To investigate this question, we present well known results in the environment of Stampacchia's lemma in section 4.1. Starting from

Stampacchia's Lemma, we show in section 4.2 that truncations of s are regular and prove in section 4.5.1 the validity of (4.7).

In section 4.4, we show integration by parts formulae as an replacement for the formal chain rule $\partial_t s g(s) = G(s)$, where G is a primitive of g. These formulae are tailored to handle truncations of s as test functions, but are not suited to treat time dependent Dirichlet data Φ_D.

4.1 Chain Rules and Stampacchia's Lemma

In this section, we provide a brief overview on chain rules for Sobolev functions and Stampacchia type lemmas. Consider the classical chain rule

$$\nabla[g \circ u](x) = g'(u(x))\nabla u(x) \tag{4.8}$$

for $g \in C^1(\mathbb{R})$, $u \in C^1(\Omega)$ and $x \in \Omega$. With an approximation argument one infers that (4.8) holds for a.e. $x \in \Omega$ if $u \in W^{1,p}(\Omega)$ and $g \in C^1(\mathbb{R})$.

Proposition 4.2 ([GM12, Proposition 3.22])**.** *Let a bounded domain $\Omega \subset \mathbb{R}^d$, $g \in C^1(\mathbb{R})$ with $g' \in L^\infty(\mathbb{R})$, and $u \in W^{1,p}(\Omega)$ for some $p \in [1, \infty]$ be given. Then $g \circ u \in W^{1,p}(\Omega)$ and (4.8) holds for almost every $x \in \Omega$.*

Remark 4.3. Similar results can be found in [KS00, Lemma A.3; DiB02, VII.Proposition 20.1] and [GT98, Lemma 7.5 and p. 154]. For unbounded domains one requires that $g(0) = 0$. Under that additional assumption the chain rule is proved in [Bre10, Proposition 9.5]. ◁

With an approximation argument, the chain rule can be extended to the case where g is piecewise C^1 with finitely many discontinuities of g'. In particular, for the choice $g(u) = (u)_+$ the chain rule is often referred to as Stampacchia's lemma.

Lemma 4.4 (Stampacchia's lemma [see GM12, Proposition 3.23])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $u \in W^{1,p}(\Omega)$ and $p \in [1, \infty]$. Then u_+, u_- and $|u|$ belong to $W^{1,p}(\Omega)$ with*

$$\nabla(u_+) = \nabla u \; \mathbb{1}_{\{u>0\}}, \;\; \nabla(u_-) = -\nabla u \; \mathbb{1}_{\{u<0\}}, \quad and \;\; \nabla|u| = \nabla(u_+) + \nabla(u_-). \tag{4.9}$$

Furthermore, given any $y \in \mathbb{R}$, we find $\nabla u = 0$ a.e. on every level set $\{u = y\}$.

With induction one obtains

Corollary 4.5 ([GM12, Proposition 3.24])**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $g \in C(\mathbb{R})$ be piecewise $C^1(\mathbb{R})$, i.e. for $l \in \mathbb{N}$ there are points $t_1, \ldots, t_l$ such that $g \in C^1(-\infty, t_1])$, $g \in C^1([t_1, t_2])$, ..., $g \in C^1([t_l, \infty))$. Additionally, assume that $g' \in L^\infty(\mathbb{R})$. For $p \in [1, \infty]$ and every $u \in W^{1,p}(\Omega)$, there holds $g \circ u \in W^{1,p}(\Omega)$ and*

$$\nabla(g \circ u) = g'(u)\nabla u \; \mathbb{1}_{\{u \neq \{t_1, \ldots, t_l\}\}}. \tag{4.10}$$

Further references for either Stampacchia's lemma or the corollary are [Sch13, Lemma 7.4; KS00, Appendix A; DiB02, chapter 7, Proposition 20.2] and [GT98, Lemma 7.6, Theorem 7.8].

Corollary 4.6. *For $p \in [1, \infty]$, let $u, v \in W^{1,p}(\Omega)$ be given. There holds $\max\{u, v\} \in W^{1,p}(\Omega)$ and $\min\{u, v\} \in W^{1,p}(\Omega)$ with*

$$\nabla \max\{u, v\} = \nabla u \, \mathbb{1}_{\{u \geq v\}} + \nabla v \, \mathbb{1}_{\{u < v\}} \text{ and } \nabla \max\{u, v\} = \nabla u \, \mathbb{1}_{\{u \leq v\}} + \nabla v \, \mathbb{1}_{\{u > v\}}.$$

Proof. We recall

$$\max\{u, v\} = \frac{u + v + |u - v|}{2} \text{ and } \min\{u, v\} = \frac{u + v - |u - v|}{2},$$

and apply Lemma 4.4. □

Remark 4.7 (Coordinate wise chain rules)**.** The previous results admit also coordinate-wise chain rules. For example, Proposition 4.2 implies

$$\partial_j (g \circ u)(x) = g'(u(x)) \partial_j u(x) \tag{4.11}$$

for a.e. $x \in \Omega$ and every $j \in \{1, \ldots n\}$. In each coordinate this equation is well defined for $u \in L^p(\Omega)$ and $\partial_j u \in L^p(\Omega)$. This raises the question whether (4.11) also holds if one merely assumes $u \in L^p(\Omega)$ and $\partial_j u \in L^p(\Omega)$. Thanks to Proposition [GT98, Theorem 7.4] this question can be answered positively; u and its partial derivative u_j, for $j \in \{1, \ldots, d\}$ can always be approximated by a sequence of smooth u_k with $\partial_j u_k \to \partial_j u$ and $u_k \to u$ in $L^1_{\text{loc}}(\Omega)$. In particular, the previous chain rules are applicable if one has e.g. a time dependent function $u \in L^p(Q)$ that lacks spatial regularity but has a regular time derivative $u_t \in L^p(Q)$. ◁

Formula (4.8) extends to the case where g is merely a Lipschitz continuous function. The proof of this result requires a different technique; one exploits that $W^{1,p}(\Omega)$ functions are absolutely continuous on a.e. line-segments parallel to the coordinate axes [see Leo09, Theorem 10.37 and Exercise 10.37; LM07; Zie89, Theorem 2.1.11] and [KS00, Appendix A]. In this case the right-hand side of (4.8) is interpreted to be zero whenever $\nabla u(x)$ is zero irrespective of whether $g'(u(x))$ is defined or not.

Let us turn our attention to chain rules for functions that depend also on $x \in \Omega$. For $g \in C^1(\Omega \times \mathbb{R})$ and $u \in C^1(\Omega)$, the classical chain rule reads

$$\nabla [g(x, u(x))] = \nabla_x g(x, u(x)) + g'(x, u(x)) \nabla u(x) \tag{4.12}$$

for every $x \in \Omega$. If g is smooth on $\overline{\Omega}$, has bounded derivatives and $u \in W^{1,p}(\Omega)$, we can easily extend the proof of Proposition 4.2 to the case here. We present the proof under the weaker assumption of regularity in a single coordinate direction: $u, \partial_j u \in L^p(\Omega)$.

Proposition 4.8. *Let a bounded domain $\Omega \subset \mathbb{R}^d$ and $p \in [1, \infty]$ be given. For $g \in C^1(\overline{\Omega} \times \mathbb{R})$ with bounded partial derivatives and $u \in L^p(\Omega)$ with $\partial_j u \in L^p(\Omega)$, there holds $g(\cdot, u), \partial_j g(\cdot, u) \in L^p(\Omega)$ and*

$$\partial_j[g(x, u(x))] = g_j(x, u(x)) + g'(x, u(x))\partial_j u(x) \text{ for a.e. } x \in \Omega. \qquad (4.13)$$

Proof. It suffices to prove the proposition in the case $p = 1$. If $u, \partial_j u \in L^p(\Omega)$, then, due to the boundedness of Ω, we find $u, \partial_j u \in L^1(\Omega)$. Application of the proposition in the case $p = 1$ yields that the jth weak partial $g(\cdot, u)$ is determined by the right-hand side of (4.13) which belongs to $L^p(\Omega)$. Hence, let us assume $p = 1$. Since $\overline{\Omega}$ is compact and g is continuous, we deduce $|g(x, 0)| < C$ for an appropriate constant $C > 0$. Potentially increasing C and using that the partial derivatives of g are bounded, we infer $|g(x, s)| \leq \sup_{\sigma \in \mathbb{R}} |g'(x, \sigma)| \, |s| + |g(x, 0)| \leq C(1 + |s|)$ and obtain $g(\cdot, u) \in L^1(\Omega)$.

Thanks to [GT98, Theorem 7.4] there is a sequence $u_k \in C^\infty(\Omega)$ such that $u_k \to u$ and $\partial_j u_k \to \partial_j u$ in $L^1_{\text{loc}}(\Omega)$ as $k \to \infty$. Since (4.13) holds for smooth functions u, we obtain for $\varphi \in C_c^\infty(\Omega)$ using integration by parts the identity

$$\int g(\cdot, u_k)\partial_j\varphi = -\int_\Omega \partial_j[g(\cdot, u_k)]\varphi = -\int_\Omega \left[g_j(\cdot, u_k) + g'(\cdot, u_k)\partial_j u_k\right]. \qquad (4.14)$$

From

$$|g(x, u_k(x)) - g(x, u(x))| \leq \|g'\|_\infty \, |u_k(x) - u(x)| \, , \text{ for a.e. } x \in \Omega,$$

we deduce $g(\cdot, u_k) \to g(\cdot, u)$ in $L^1_{\text{loc}}(\Omega)$. Consequently the left-hand side in (4.14) has the desired limit. Moreover, up to extracting a subsequence, $u_k \to u$ a.e. in Ω. Hence, we also find $g_j(x, u_k(x)) \to g_j(x, u(x))$ and $g'(x, u_k(x)) \to g'(x, u(x))$ a.e. in Ω. Since g has bounded derivatives, Lebesgue's dominated convergence theorem implies $g_j(\cdot, u_k) \to g_j(\cdot, u) \in L^1(\Omega)$ and

$$\int_\Omega |g'(\cdot, u_k)\partial_j u_k - g'(\cdot, u)\partial_j u| \, |\varphi|$$
$$\leq \|g'\|_\infty \int_\Omega |\partial_j(u - u_k)| \, |\varphi| + \int_\Omega |g'(\cdot, u_k) - g'(\cdot, u)| \, |\partial_j u| \, |\varphi| \xrightarrow[k \to \infty]{} 0.$$

The limit $k \to \infty$ of (4.14) yields that $g_j(\cdot, u) + g'(\cdot, u)\partial_j u \in L^1(\Omega)$ is the jth weak partial derivative of $g(\cdot, u)$ and we conclude. $\qquad \square$

Corollary 4.9. *Let $p \in [1, \infty]$, a bounded domain $\Omega \subset \mathbb{R}^d$ and $g \in C^1(\overline{\Omega} \times \mathbb{R})$ with bounded partial derivatives be given. For $u \in W^{1,p}(\Omega)$, there holds $g(\cdot, u) \in W^{1,p}(\Omega)$ and*

$$\nabla[g(\cdot, u)] = \nabla_x g(\cdot, u) + g'(\cdot, u)\nabla u. \qquad (4.15)$$

On the case of an (x, u)**-dependent Lipschitz function** g In the case where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is merely Lipschitz and $u \in H^1(\Omega)$ one obtains that $g(\cdot, u) \in H^1(\Omega)$. But, as the following example shows, a chain rule, as in (4.12), can not be expected in general.

Example 4.10. *Let $\Omega \subset \mathbb{R}$ be an open interval, consider $g(x, v) := \max\{x, v\}$ and choose $u(x) = x \in C^1(\overline{I})$. Then g is not differentiable on the set $B := \{(x, v) \subset \mathbb{R}^2 | x = v\}$ and B has measure zero by Rademacher's theorem [EG92, chapter 3.1.2]. We find $g(x, u(x)) = x \in C^1(\overline{\Omega})$ and consequently $\nabla[g(x, u(x))] = 1$. We find that the left-hand side of (4.12) equals one. However, the right-hand side of (4.12) is nowhere defined, since $(x, u(x)) = (x, x) \in B$ for any $x \in \Omega$.*

The previous example and further results on general chain rules can be found in [LM07], [MM72] and the references therein.

4.2 Regularity of Truncations

The preliminary results of the previous section are used to investigate the regularity of truncations of s.

Definition 4.11 (Truncations)**.** *Let $a, b \in \mathbb{R}$, $a \leq b$ be given. For $s \in \mathbb{R}$, we define truncations of s at levels a and b as*

$$T_a(s) := \max\{s, a\}, \quad T^b(s) = \min\{s, b\} \quad and \quad T_a^b(s) = \max\{\min\{s, b\}, a\}.$$

We mention the identities $T_a \circ T^b = T^b \circ T_a = T_a^b$. In the following we investigate the cases where Φ is x-independent and x-dependent separately. We refer to them simply as the x-independent case and the x-dependent case, respectively. We also recall Figure 4.1 where a typical shape of Φ and, for the x-dependent case, the sets K_Φ and K_Φ^σ are shown.

4.2.1 The x-independent Case

Lemma 4.12 (Characterization of $\Phi(T_a^b(s))$)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $\Phi : [0, 1] \to \mathbb{R}$ be continuous and increasing and let $s \in L^\infty(\Omega, [0, 1])$ be such that $\Phi(s) \in H^1(\Omega)$. For $0 \leq a \leq b \leq 1$ and a.e. $x \in \Omega$, the identities*

$$\begin{aligned}
\Phi(s(x)) &= (\Phi(s(x)) - \Phi(b))_+ + \Phi(T_a^b(s(x))) - (\Phi(s(x)) - \Phi(a))_-, \\
\Phi(s(x)) &= (\Phi(s(x)) - \Phi(b))_+ + \Phi(T^b(s)), \\
\Phi(s(x)) &= \Phi(T_a(s(x))) - (\Phi(s(x)) - \Phi(a))_-
\end{aligned} \qquad (4.16)$$

hold. Additionally, $\Phi(T_a^b(s)), \Phi(T^b(s))$ *and* $\Phi(T_a(s)) \in H^1(\Omega)$ *with*

$$
\begin{aligned}
\nabla[\Phi(T_a^b(s))] &= \nabla[\Phi(s)] \, \mathbb{1}_{\{a<s<b\}} = \nabla[\Phi(s)] \, \mathbb{1}_{\{a\le s<b\}} = \nabla[\Phi(s)] \, \mathbb{1}_{\{a\le s\le b\}}, \\
\nabla[\Phi(T^b(s))] &= \nabla[\Phi(s)] \, \mathbb{1}_{\{s<b\}} = \nabla[\Phi(s)] \, \mathbb{1}_{\{s\le b\}}, \\
\nabla[\Phi(T_a(s))] &= \nabla[\Phi(s)] \, \mathbb{1}_{\{a<s\}} = \nabla[\Phi(s)] \, \mathbb{1}_{\{a\le s\}}.
\end{aligned}
\tag{4.17}
$$

Proof. The identities in (4.16) follow directly from the definition of positive and negative part of a function. Since $\Phi(a)$ and $\Phi(b)$ are constants and $\Phi(s) \in H^1(\Omega)$, we obtain from Lemma 4.4 that $(\Phi(s)-\Phi(b))_+$ and $(\Phi(s)-\Phi(a))_+$ belong both to $H^1(\Omega)$. Consequently, $\Phi(T_a^b(s)), \Phi(T^b(s))$ and $\Phi(T_a(s)) \in H^1(\Omega)$. Furthermore, with repeated application of Lemma 4.4 and since Φ is increasing we infer that

$$
\nabla(\Phi(s) - \Phi(b))_+ = \nabla(\Phi(s)) \, \mathbb{1}_{\{\Phi(b)<\Phi(s)\}} = \nabla\Phi(s) \, \mathbb{1}_{\{b<s\}} = \nabla\Phi(s) \, \mathbb{1}_{\{b\le s\}}.
$$

Analogous formulae hold for $(\Phi(s) - \Phi(a))_-$. Combination of these results yields (4.17). $\qquad\square$

With the previous lemma, the regularity of $T_a^b(s)$ is a direct consequence, provided Φ is smooth.

Proposition 4.13 (Regularity of truncations I)**.** *Let $a, b \in (0,1)$ and a bounded domain $\Omega \subset \mathbb{R}^d$ be given. Furthermore, let $\Phi \in C^1([0,1])$ be such that $\Phi'(s) > 0$ for $s \in (0,1)$, $\Phi'(0) = 0$, and $\Phi'(1) = 0$. For $s \in L^\infty(\Omega,[0,1])$ with $\Phi(s) \in H^1(\Omega)$, there holds $T_a^b(s) \in H^1(\Omega)$ and*

$$
\Phi'[T_a^b(s)]\nabla[T_a^b(s)] = \nabla[\Phi(T_a^b(s))]. \tag{4.18}
$$

Proof. With $(\Phi^{-1})'(u) > 0$ for $u \in [\Phi(a), \Phi(b)]$, the inverse function theorem implies that $\Phi^{-1} \in C^1([\Phi(a), \Phi(b)])$. We extend Φ^{-1} linearly and continuously differentiable onto $\mathbb{R}$ and denote the extension by $\tilde{\Phi}^{-1}$. Application of Proposition 4.2, i.e. of the chain rule, and Lemma 4.12 yields $T_a^b(s) = \tilde{\Phi}^{-1}(\Phi(T_a^b(s))) \in H^1(\Omega)$. With this information and realizing that, due to the assumptions, the continuous and constant extension of Φ is in $C^1(\mathbb{R})$ with bounded derivative, we apply Proposition 4.2 and deduce

$$
\nabla[\Phi(T_a^b(s))] = \Phi'(T_a^b(s))\nabla[T_a^b(s)].
$$

$\qquad\square$

Remark 4.14. The statement of Proposition 4.13 may be extended to the case where $a = 0$ or $b = 1$, provided $\Phi'(0) > 0$ or $\Phi'(1) > 0$, respectively. In that case the argument presented above remains unchanged. $\qquad\triangleleft$

Lemma 4.12 and Proposition 4.13 imply the following identities for $\nabla[T_a^b(s)]$.

Corollary 4.15. *Let the assumptions of Proposition 4.13 be fulfilled. Then, there holds*

$$\nabla[T_a^b(s)] = \frac{1}{\Phi'(s)} \nabla[\Phi(s)] \mathbb{1}_{\{a \le s < b\}} = \frac{1}{\Phi'(s)} \nabla[\Phi(s)] \mathbb{1}_{\{a < s \le b\}}$$
$$= \frac{1}{\Phi'(s)} \nabla[\Phi(s)] \mathbb{1}_{\{a \le s \le b\}}.$$

(4.19)

Furthermore, for every $y \in [0, 1]$ there holds $\nabla[T_a^b(s)] = 0$ a.e. on $\{s = y\}$.

4.2.2 The x-dependent Case

We recall the main steps to obtain the result of Proposition 4.13. Firstly, we used the inverse function theorem to derive that Φ^{-1} is smooth on an interval bounded away from zero and one. Secondly, we extended the function smoothly with bounded derivatives. Thirdly, the characterization of $\Phi(T_a^b(s))$ together with the chain rule provided that $T_a^b(s) \in H^1(\Omega)$. Lastly, we applied the chain rule again to obtain the identity in (4.18).

In the x-dependent case, we want to implement the same program. With Corollary 4.9 the chain rule is already proven. On Φ we impose assumptions that resemble those of Proposition 4.13 adapted to the x-dependent problem. A typical shape of such a function Φ is depicted in Figure 4.1 and labeled Φ_2.

Assumption A4.1. *Let $\Phi \in C^1(\overline{\Omega} \times [0, 1])$, be such that $\Phi'(x, s) > 0$ for $s \in (0, 1)$ and every $x \in \overline{\Omega}$, and $\Phi'(x, 0) = \Phi'(x, 1) = \Phi(x, 0) = 0$ for every $x \in \overline{\Omega}$.*

The inverse of Φ needs to be understood in the sense of Definition 1.6 and we denote it by Φ^{-1}. As in the x-independent case, regularity properties of Φ provide regularity of Φ^{-1}.

Lemma 4.16 (Properties of Φ^{-1}). *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, let Φ be as in Assumption A4.1 and let $\Phi^{-1}, \sigma, K_\Phi$ and K_Φ^σ be as in Definition 1.6. There holds $\Phi^{-1} \in C(K_\Phi)$ and $\Phi^{-1} \in C^1(K_\Phi^\sigma)$. On K_Φ^σ we find*

$$\partial_u \Phi^{-1}(x, u) = \frac{1}{\Phi'(x, \Phi^{-1}(x, u))}$$

(4.20)

and

$$\nabla_x[\Phi^{-1}(x, u)] = -\frac{1}{\Phi'(x, \Phi^{-1}(x, u))} \nabla_x \Phi(x, \Phi^{-1}(x, u)).$$

(4.21)

We recall that the sets K_Φ and K_Φ^σ are depicted in Figure 4.1.

Proof. **Step 1: Uniform continuity of Φ^{-1}.** We start with the uniform continuity of Φ^{-1}. Let $(x, u) \in K_\Phi$ be given and $(x_k, u_k) \in K_\Phi$ be a sequence with $(x_k, u_k) \to (x, u)$. Thanks to the properties of Φ, we find for each pair $(x, u), (x_k, u_k)$, for $k \in \mathbb{N}$, a unique $s, s_k \in [0, 1]$ such that $\Phi(x, s) = u$ and $\Phi(x_k, s_k) = u_k$, respectively. The corresponding sequence s_k is bounded and thus admits a convergent subsequence $s_{k'} \to \tilde{s}$. Thanks to the continuity of Φ and the convergence of $(x_k, u_k) \to (x, u)$ we obtain

$$\Phi(x, \tilde{s}) \leftarrow \Phi(x_{k'}, s_{k'}) = u_{k'} \to u = \Phi(x, s).$$

Consequently, due to the properties of Φ, we conclude $s = \tilde{s}$ and, by a standard argument [DiB02, I.1.1], that the whole sequence s_k converges to s. Furthermore, we obtain

$$\Phi^{-1}(x_k, u_k) = s_k \to s = \Phi^{-1}(x, u)$$

and we conclude, due to the compactness of K_Φ, that Φ^{-1} is uniformly continuous.

Step 2: Uniform continuity of $\partial_u \Phi^{-1}(x, u)$. Due to the first step of the proof and the compactness of K_Φ^σ, it suffices to show that the partial derivatives of Φ^{-1} are uniformly continuous in $\text{int}(K_\Phi^\sigma)$ to infer $\Phi^{-1} \in C^1(K_\Phi^\sigma)$. From the properties of Φ and the inverse function theorem in one dimension we infer equation (4.20) on $\text{int}(K_\Phi)$. Since K_Φ^σ is compact and due to the properties of Φ, there exists $\varepsilon > 0$ such that $(x, \Phi^{-1}(x, u)) \in \overline{\Omega} \times [\varepsilon, 1 - \varepsilon]$ for $(x, u) \in K_\Phi^\sigma$. Hence, there exists a positive c such that $\Phi'(x, s) > c > 0$ on $\overline{\Omega} \times [\varepsilon, 1 - \varepsilon]$ and together with the uniform continuity of Φ' on $\overline{\Omega} \times [\varepsilon, 1 - \varepsilon]$, the uniform continuity of Φ^{-1} on K_Φ and equation (4.20) on $\text{int}(K_\Phi^\sigma)$, we infer that $\partial_u \Phi^{-1}$ is uniformly continuous on $\text{int}(K_\Phi^\sigma)$.

Step 3: Uniform continuity of $\partial_j \Phi^{-1}(x, u)$. To consider the x-derivatives, we are going to exploit the implicit function theorem. Let $(x_0, u_0) \in \text{int}(K_\Phi^\sigma)$ be given and let $s_0 \in (0, 1)$ be the unique solution of $\Phi(x_0, s_0) = u_0$. Since $\Phi'(x, s) > 0$ for every $x \in \Omega$ and $s \in (0, 1)$, the implicit function theorem states that there is locally a unique, continuously differentiable function $g^{u_0}(x)$ with $g^{u_0}(x_0) = s_0$ and $\Phi(x, g^{u_0}(x)) = u_0$. Since $\Phi^{-1}(x, u_0)$ also has these properties we infer $\Phi^{-1}(x, u_0) = g^{u_0}(x)$. For the partial derivatives, we apply again the implicit function theorem and compute

$$\partial_j [\Phi^{-1}(x, u_0)] = -\frac{1}{\Phi'(x, \Phi^{-1}(x, u_0))} \Phi_j(x, \Phi^{-1}(x, u_0)) \qquad (4.22)$$

for $(x, u_0) \in \text{int}(K_\Phi)$ and $j \in \{1, \dots, n\}$. Due to the assumptions on Φ and using the results of the first and second step we infer that $\partial_j \Phi^{-1}$ is uniformly continuous on $\text{int}(K_\Phi^\sigma)$ for every $j \in \{1, \dots, n\}$. From (4.22) we infer (4.21) which concludes the proof. $\qquad\square$

Using a reflection, we next show an abstract extension lemma. Later, we apply this lemma to Φ^{-1}.

Lemma 4.17 (Extension lemma)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let $g_0, g_1 \in C^1(\overline{\Omega})$ be such that $g_0(x) < g_1(x)$ for every $x \in \overline{\Omega}$. For $G_0^1 := \{(x,t) \mid x \in \overline{\Omega}, g_0(x) \le t \le g_1(x)\}$ let $f \in C^1(G_0^1)$ be given. Then, there is an constant $C > 0$ and an extension $\tilde{f}$ of f such that*

$$\tilde{f} \in C^1(\overline{\Omega} \times \mathbb{R}), \quad \tilde{f}|_{G_0^1} = f \ and \ \|f\|_{C^1(\overline{\Omega} \times \mathbb{R})} \le C.$$

Proof. **Step 1:** $G_0^1 = \overline{\Omega} \times [0,1]$**.** We assume that $g_0 \equiv 0$ and $g_1 \equiv 1$. Consequently, $G_0^1 = \overline{\Omega} \times [0,1]$. We extend f smoothly on $\overline{\Omega} \times [0,\infty)$ with a reflection. The function

$$f_1(x,t) = \begin{cases} f(x,t) & \text{for } (x,t) \in \overline{\Omega} \times [0,1], \\ 2f(x,1) - f\left(x, \frac{1}{t}\right) & \text{for } (x,t) \in \overline{\Omega} \times (1,\infty) \end{cases}$$

has the desired properties and, particularly, $\|f\|_{C^1(\overline{\Omega} \times [0,\infty)}$ is bounded. With a reflection at the set where $t = 0$, we obtain the extension

$$\tilde{f}(x,t) = \begin{cases} f_1(x,t) & \text{for } (x,t) \in \overline{\Omega} \times [0,\infty) \\ 2f_1(x,0) - f_1(x,-t) & \text{for } (x,t) \in \overline{\Omega} \times (-\infty, 0). \end{cases}$$

with the desired properties.

Step 2: General case. We define the transformation

$$F : \overline{\Omega} \times \mathbb{R} \to \overline{\Omega} \times \mathbb{R}$$
$$(x,\tau) \mapsto (x, \tau g_1(x) + (1-\tau)g_0(x))$$

with inverse

$$F^{-1}(x,t) := \left(x, \frac{t - g_0(x)}{g_1(x) - g_0(x)}\right)$$

in t for fixed $x \in \overline{\Omega}$. By the properties of g_0 and g_1 we see that $F, F^{-1} \in C^1(\overline{\Omega} \times \mathbb{R})$. Furthermore, we obtain

$$F|_{\overline{\Omega} \times [0,1]}(\overline{\Omega} \times \mathbb{R}) = G_0^1 \text{ and } F^{-1}(G_0^1) = \overline{\Omega} \times [0,1].$$

Hence, $h := f \circ F \in C^1(\overline{\Omega} \times [0,1])$. Extend h by the first step and obtain $\tilde{h} \in C^1(\overline{\Omega} \times \mathbb{R})$. Choosing $\tilde{f} = \tilde{h} \circ F^{-1}$ completes the proof. $\qquad\square$

Remark 4.18. We emphasize that the proof of the extension lemma does not rely on boundary regularity of Ω since we only extended perpendicular to Ω. In typical situations, Ω is assumed to have Lipschitz boundary and an alternative proof can be obtained with the remarkable Whitney extension theorem. It provides a C^1 extension of f to an open domain in $\mathbb{R}^{d+1}$ if G_0^1 is quasiconvex; see e.g. [BB11, Theorem 2.64], the classical articles by Whitney [Whi34a], [Whi34b], or the book [EG92, chapter 6.5]. For a Lipschitz domain Ω, we find that G_0^1 is also Lipschitz and thus quasiconvex; this is essentially the content of [Alt06, section 8.4]. $\qquad\triangleleft$

It remains to show the characterization of $\Phi(\cdot, T_a^b(s))$ to have the tools required to perform the program presented at the beginning of this section.

Lemma 4.19 (Characterization of $\Phi(\cdot, T_a^b(s))$)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, let Φ be as in Assumption A4.1 and let $s \in L^\infty(\Omega, [0,1])$ be such that $\Phi(\cdot, s) \in H^1(\Omega)$. For $0 < a < b < 1$, there holds*

$$
\begin{aligned}
\Phi(x, s(x)) &= (\Phi(x, s(x)) - \Phi(x, b))_+ + \Phi(x, T_a^b(s(x))) - (\Phi(x, s(x)) - \Phi(x, a))_- \\
\Phi(x, s(x)) &= (\Phi(x, s(x)) - \Phi(x, b))_+ + \Phi(x, T^b(s(x))) \\
\Phi(x, s(x)) &= \Phi(x, T_a(s(x))) - (\Phi(x, s(x)) - \Phi(x, a))_-.
\end{aligned}
\tag{4.23}
$$

Additionally, there holds $\Phi(\cdot, T_a^b(s)), \Phi(\cdot, T_b(s))$ and $\Phi(\cdot, T_a(s)) \in H^1(\Omega)$ with

$$
\begin{aligned}
\nabla[\Phi(\cdot, T_a^b(s))] &= \nabla[\Phi(\cdot, s)]\,\mathbb{1}_{\{a < s < b\}} + \nabla[\Phi(\cdot, a)]\,\mathbb{1}_{\{s \le a\}} + \nabla[\Phi(\cdot, b)]\,\mathbb{1}_{\{b \le s\}} \\
&= \nabla[\Phi(\cdot, s)]\,\mathbb{1}_{\{a \le s < b\}} + \nabla[\Phi(\cdot, a)]\,\mathbb{1}_{\{s < a\}} + \nabla[\Phi(\cdot, b)]\,\mathbb{1}_{\{b \le s\}} \\
&= \nabla[\Phi(\cdot, s)]\,\mathbb{1}_{\{a \le s \le b\}} + \nabla[\Phi(\cdot, a)]\,\mathbb{1}_{\{s < a\}} + \nabla[\Phi(\cdot, b)]\,\mathbb{1}_{\{b < s\}}
\end{aligned}
\tag{4.24}
$$

and

$$
\begin{aligned}
\nabla[\Phi(\cdot, T^b(s))] &= \nabla[\Phi(\cdot, s)]\,\mathbb{1}_{\{s < b\}} + \nabla[\Phi(\cdot, b)]\,\mathbb{1}_{\{b \le s\}} \\
&= \nabla[\Phi(\cdot, s)]\,\mathbb{1}_{\{s \le b\}} + \nabla[\Phi(\cdot, b)]\,\mathbb{1}_{\{b < s\}} \\
\nabla[\Phi(\cdot, T_a(s))] &= \nabla[\Phi(\cdot, s)]\,\mathbb{1}_{\{a < s\}} + \nabla[\Phi(\cdot, a)]\,\mathbb{1}_{\{s \le a\}} \\
&= \nabla[\Phi(\cdot, s)]\,\mathbb{1}_{\{a \le s\}} + \nabla[\Phi(\cdot, a)]\,\mathbb{1}_{\{s < a\}}.
\end{aligned}
\tag{4.25}
$$

Proof. The proof is performed along the lines of the proof of Lemma 4.12; it suffices to take the x-dependence of $\Phi(x, b)$ and $\Phi(x, a)$ into account in each step of the proof. $\square$

Remark 4.20. In equations (4.24) and (4.25) we may use the identities $\nabla[\Phi(x, a)] = \nabla_x \Phi(x, a)$ and $\nabla[\Phi(x, b)] = \nabla_x \Phi(x, b)$ for a.e. $x \in \Omega$. $\triangleleft$

Gathering the previous statements permits to obtain regularity of truncations of s also in the x-dependent case.

Proposition 4.21 (Regularity of truncations II)**.** *For a bounded domain $\Omega \subset \mathbb{R}^d$, Φ as in Assumption A4.1, $s \in L^\infty(\Omega, [0,1])$ with $\Phi(\cdot, s) \in H^1(\Omega)$ and $0 < a < b < 1$, there holds $T_a^b(s) \in H^1(\Omega)$. Furthermore, we find*

$$
\nabla[\Phi(\cdot, T_a^b(s))] = \Phi'(\cdot, T_a^b(s))\nabla[T_a^b(s)] + \nabla_x \Phi(\cdot, T_a^b(s)).
\tag{4.26}
$$

Proof. For $\sigma < \min\{a, b\}$, we infer from Lemma 4.16 that $\Phi^{-1} \in C^1(K_\Phi^\sigma)$; thus, with $g_0(x) = \Phi(x, a)$ and $g_1(x) = \Phi(x, b)$ and G_0^1 defined in Lemma 4.17, $\Phi^{-1} \in C^1(G_0^1)$. Application of the extension lemma yields a C^1 extension of Φ^{-1} from G_0^1 onto $\Omega \times \mathbb{R}$ that is denoted by $\tilde{\Phi}^{-1}$.

From Lemma 4.19 we infer that $\Phi(\cdot, T_a^b(s)) \in H^1(\Omega)$. Consequently, Corollary 4.9 applied with $g = \tilde{\Phi}^{-1}$ and $u = \Phi(x, T_a^b(s(x)))$ yields $T_a^b(s) \in H^1(\Omega)$. Due to Assumption A4.1, the extension

$$\tilde{\Phi}(x, s) := \begin{cases} 0 & s < 0, \\ \Phi(x, s) & s \in [0, 1], \\ \Phi(x, 1) & s > 1 \end{cases}$$

of Φ is in $C^1(\overline{\Omega} \times \mathbb{R})$ with bounded derivatives. Thus, we can apply the chain rule from Corollary 4.9 again and obtain (4.26). $\qquad \square$

Remark 4.22. A remark similar to Remark 4.14 applies to Proposition 4.21. The proposition holds for $a = 0$ and $b = 1$ provided $\Phi'(x, 0) > 0$ and $\Phi'(x, 1) > 0$ for every $x \in \overline{\Omega}$, respectively. Furthermore, the assumption $\Phi(x, 0) = 0$ is not essential; we only require that $\Phi(x, 0)$ is a smooth curve. $\qquad \triangleleft$

Analogously to section 4.2.1, we derive from Lemma 4.19 and Proposition 4.21 the following identities for $\nabla[T_a^b(s)]$.

Corollary 4.23. *Let the assumptions of Proposition 4.21 be fulfilled. Then there holds*

$$\begin{aligned} \nabla[T_a^b(s)] &= \frac{1}{\Phi'(\cdot, s)}[\nabla[\Phi(\cdot, s)] - \nabla_x \Phi(\cdot, s)]\, \mathbb{1}_{\{a < s < b\}} \\ &= \frac{1}{\Phi'(\cdot, s)}[\nabla[\Phi(\cdot, s)] - \nabla_x \Phi(\cdot, s)]\, \mathbb{1}_{\{a \leq s < b\}} \\ &= \frac{1}{\Phi'(\cdot, s)}[\nabla[\Phi(\cdot, s)] - \nabla_x \Phi(\cdot, s)]\, \mathbb{1}_{\{a \leq s \leq b\}} \end{aligned} \qquad (4.27)$$

Furthermore, for every $y \in [0, 1]$ there holds $\nabla[T_a^b(s)] = 0$ a.e. on $\{s = y\}$.

4.3 The Steklov Average

Throughout this section, let $I = (a, b) \subset \mathbb{R}$ with $-\infty \leq a < b \leq \infty$ be an open interval and X be a Banach space. We use the Bochner-integral to define Banach-space valued integrals, see e.g. [Boc33], [DU77] or [Sch13].

Definition 4.24. *For $p \in [1, \infty]$, let $u \in L^p(I; X)$ be a function extended by zero on $\mathbb{R} \setminus I$. For $h \neq 0$, the* Steklov average *of u is defined as*

$$u_h(t) := \frac{1}{h} \int_t^{t+h} u(s)\, \mathrm{d}s \in X.$$

Remark 4.25. The Steklov average of u can be written as a convolution

$$u_h(t) = \left(u * \frac{\mathbb{1}_{/-h,0/}}{|h|} \right)(t),$$

where we used the notation for $/a, b/$ introduced in section 1.4. Consequently, we obtain the estimate $\|u_h\|_p \leq \|u\|_p$ for any $u \in L^p(I; X)$ with $p \in [1, \infty]$; see [Alt06, chapter 2.12]. ◁

Lemma 4.26. *Let* $p, p' \in [1, \infty]$ *with* $\frac{1}{p} + \frac{1}{p'} = 1$, $h \neq 0$, $f \in L^p(\mathbb{R}; X)$ *and* $g \in L^{p'}(\mathbb{R})$ *be given. Then there holds*

$$\int_a^b \int_t^{t+h} f(s) \, \mathrm{d}s \, g(t) \, \mathrm{d}t = \int_a^b \int_{t-h}^t g(s) \, \mathrm{d}s \, f(t) \, \mathrm{d}t$$
$$+ \int_b^{b+h} \int_{t-h}^b g(s) \, \mathrm{d}s \, f(t) \, \mathrm{d}t - \int_a^{a+h} \int_{t-h}^a g(s) \, \mathrm{d}s \, f(t) \, \mathrm{d}t \in X. \tag{4.28}$$

Remark 4.27. For unbounded intervals, we use the convention $\infty + h = \infty$ and analogous definitions. Then, for $a = -\infty$ or $b = \infty$, the last two integrals in equation (4.28) may vanish depending on the form of I. For negative h, we exploit $\int_t^{t+h} = -\int_{t+h}^t$. ◁

Proof. We only consider the case of a bounded interval I. The proof for unbounded intervals works along the lines of this proof with the obvious modifications.

From Remark 4.25, we infer

$$\left\| \int_t^{t+h} f(s) \right\|_p = |h| \, \|f_h\|_p \leq |h| \, \|f\|_p \,.$$

An analogous result also holds for g. Hence, the integrals in (4.28) are well-defined. Using Fubini's theorem and several substitutions we deduce

$$\int_a^b \int_t^{t+h} f(s) \, \mathrm{d}s \, g(t) \, \mathrm{d}t = \int_0^h \int_a^b f(t+s)g(t) \, \mathrm{d}t \, \mathrm{d}s = \int_0^h \int_{a+s}^{b+s} f(t)g(t-s) \, \mathrm{d}t \, \mathrm{d}s$$
$$= \int_0^h \int_a^b f(t)g(t-s) \, \mathrm{d}t \, \mathrm{d}s + \int_0^h \left[\int_b^{b+s} f(t)g(t-s) \, \mathrm{d}t - \int_a^{a+s} f(t)g(t-s) \, \mathrm{d}t \right] \mathrm{d}s$$
$$= \int_a^b \int_{t-h}^t g(s) \, \mathrm{d}s \, f(t) \, \mathrm{d}t + \int_b^{b+h} \int_{t-h}^b g(s) \, \mathrm{d}s \, f(t) \, \mathrm{d}t - \int_a^{a+h} \int_{t-h}^a g(s) \, \mathrm{d}s \, f(t) \, \mathrm{d}t \,.$$

$\square$

Lemma 4.28 (Properties of the Steklov average)**.** *The Steklov average, as given by Definition 4.24, of a function* $u \in L^p(I; X)$ *has the properties*

   1. $u_h \in C(\bar{I}; X)$

2. *For $1 \leq p < \infty$, we find the convergence $u_h \to u$ in $L^p(I; X)$ as $h \to 0$*

3. *$u_h \in W^{1,p}(I; X)$ with weak derivative $\partial_t[u_h(t)] = \frac{u(t+h)-u(t)}{h} \in X$ for almost every $t \in I$*

4. *Let I be a bounded interval and $u \in W^{1,1}(I; X)$. Then $\partial_t[u_h(t)] = [\partial_t u]_h(t)$ in X for a.e. $t \in (\max\{a, a-h\}, \min\{b, b-h\})$.*

Proof. **Ad 1.** Let $a < s < t < b$ be given and estimate

$$
\|u_h(t) - u_h(s)\|_X = \frac{1}{h} \left\| \int_t^{t+h} u(\tau)\,\mathrm{d}\tau - \int_s^{s+h} u(\tau)\,\mathrm{d}\tau \right\|_X
$$

$$
= \frac{1}{h} \left\| \int_{s+h}^{t+h} u(\tau)\,\mathrm{d}\tau - \int_s^t u(\tau)\,\mathrm{d}\tau \right\|_X \leq \frac{1}{h} \int_s^t \|u(\tau)\|_X + \|u(\tau + h)\|_X \,\mathrm{d}\tau .
$$

Due to the absolute continuity of the Lebesgue integral, see e.g. [DiB02, III.Theorem 11.1], we conclude that u_h is uniformly continuous in t. We emphasize that due to the definition of the Steklov average, $u = 0$ on $\mathbb{R} \setminus I$.

Ad 2. Since the Steklov average can be regarded as a convolution, the convergence in $L^p(I; X)$ follows from standard L^p-theory [see Alt06, section 2.14].

Ad 3. Let $\varphi \in C_c^\infty(I)$ and $h \neq 0$ be given. We extend u on $\mathbb{R} \setminus I$ by zero. We apply Lemma 4.26, take into account that $\operatorname{supp}(u), \operatorname{supp}(\varphi) \subset \overline{I}$, and exploit the summation by parts formula to obtain

$$
\int_I u_h(t)\varphi'(t)\,\mathrm{d}t = \int_I \frac{1}{h} \int_{t-h}^t \varphi'(s)\,\mathrm{d}s\, u(t)\,\mathrm{d}t
$$

$$
= \int_I \frac{\varphi(t) - \varphi(t-h)}{h} u(t)\,\mathrm{d}t = - \int_I \frac{u(t+h) - u(t)}{h} \varphi(t)\,\mathrm{d}t .
$$

Ad 4. Since $u \in W^{1,1}(I; X)$, it possesses an absolutely continuous representative, see e.g [Sch13, Proposition 10.8] and [Eva98, section 5.9, Theorem 2]. Hence, we find for a.e. $t \in (\max\{a, a-h\}, \min\{b, b-h\})$ the identity

$$
u(t+h) - u(t) = \int_t^{t+h} \partial_t u(\tau)\,\mathrm{d}\tau \in X.
$$

Division by h proves *4*. $\qquad\square$

Remark 4.29. Occasionally, we require pointwise properties of integrable functions f. Such properties are often deduced in Lebesgue points, i.e. points t such that

$$
\fint_t^{t+h} \|f(s) - f(t)\|_X \to 0 \text{ as } h \to 0 \tag{4.29}
$$

holds. For real-valued functions we refer to [Rud99, 7] and for vector valued functions we refer to [DU77, II.Theorem 9]. We emphasize that the Lebesgue differentiation theorem holds in both cases, i.e. (4.29) holds for a.e. t in the domain of f. Consequently, with the properties of the Steklov average one derives from (4.28) for $u \in W^{1,p}(I; X)$ and $\varphi \in C_c^\infty(\mathbb{R})$ the classical integration by parts formula

$$\int_{t_1}^{t_2} \partial_t u \varphi = u(t_2)\varphi(t_2) - u(t_1)\varphi(t_1) - \int_{t_1}^{t_2} u \partial_t \varphi \in X \tag{4.30}$$

for a.e. $t_1 < t_2 \in I$. With a density argument and properties of the Bochner integral, this formula can be extended to the case $\varphi \in W^{1,p'}(I; X')$ for suitable p, p' and spaces X, X'. In this case the product needs to be replaced by an appropriate duality pairing. ◁

When considering the product of an integrable function f with a Lipschitz function g, we find Lebesgue points of the product in zeroes of g.

Lemma 4.30 (Lebesgue points of products). *Let $f \in L^1(I; X)$ and $g : I \to \mathbb{R}$ be Lipschitz continuous. Let $t_0 \in I$ be such that $g(t_0) = 0$. Then t_0 is a Lebesgue point of the product fg, i.e.*

$$\fint_{t_0}^{t_0+h} \|f(t)g(t)\|_X \xrightarrow[h \to 0]{} 0.$$

Proof. For $|h| < h_0$ small enough, we find $B_h(t_0) \subset B_{h_0}(t_0) \subset\subset I$ and, with the Lipschitz constant L of g on $B_{h_0}(t_0)$ and $g(t_0) = 0$, we compute

$$\frac{1}{h} \int_{t_0}^{t_0+h} \|f(t)g(t)\|_X \, \mathrm{d}t \leq \frac{1}{h} \int_{t_0}^{t_0+h} \|f(t)[g(t) - g(t_0)]\|_X \, \mathrm{d}t$$

$$\leq L \int_{t_0}^{t_0+h} \|f(t)\|_X \frac{|t - t_0|}{h} \, \mathrm{d}t \leq L \int_{t_0}^{t_0+h} \|f(t)\|_X \, \mathrm{d}t \xrightarrow[h \to 0]{} 0$$

due to the absolute continuity of the Lebesgue integral. □

Lemma 4.31 (Lebesgue points of a composition). *Let $g : X \to X$ be Lipschitz continuous, let $f \in L^1(I; X)$ and let $t_0 \in I$ be a Lebesgue point of f. Then t_0 is a Lebesgue point of $g \circ f$.*

Proof. The lemma follows from (4.29) and the Lipschitz continuity of g with respect to the X-norm. More precisely, we find i.e. $\|g(f(t)) - g(f(s))\|_X \leq L \|f(t) - f(s)\|_X$, where L is the Lipschitz constant of g and $t, s \in I$ are arbitrary. □

4.4 Integration by Parts

With the properties of the Steklov average, we are in the position to derive integration by parts formulas related to (4.4).

Lemma 4.32 (Integration by parts I)**.** *Let Ω, Q and V be as in (1.20)–(1.23). Let $s \in L^\infty(Q; [0,1])$ be such that $\partial_t s \in L^2(0, T; V')$ and assume there exists $\varepsilon > 0$ such that $T_\varepsilon^{1-\varepsilon}(s) \in L^2(0, T; H^1(\Omega))$. Let a Lipschitz continuous function $g : [0,1] \to \mathbb{R}$ be given such that g is constant on $[0, \varepsilon] \cup [1 - \varepsilon, 1]$. For any antiderivative G of g, there holds*

$$\int_{t_1}^{t_2} \langle \partial_t s, g(s)\xi \rangle_{V', V} = \int_\Omega G(s(t))\xi(t) \Big|_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2} \int_\Omega G(s) \partial_t \xi \tag{4.31}$$

for every $\xi \in C_c^\infty(\Omega \times \mathbb{R})$ and a.e. $0 < t_1 < t_2 < T$.

Proof. It suffices to show the lemma for nondecreasing g. Since Lipschitz continuous functions are of bounded variation, we may decompose $g = g_1 - g_2$, where g_1, g_2 are nondecreasing functions. Particularly, choosing g_1 and g_2 via the Jordan decomposition, we obtain that both functions are Lipschitz continuous and constant on $[0, \varepsilon] \cup [1 - \varepsilon, 1]$, see [DiB02, chapter IV.1]. Furthermore, thanks to linearity, it suffices to consider nonnegative $\xi \in C_c^\infty(\Omega \times \mathbb{R})$. Hence, let us assume that g is a nondecreasing, Lipschitz continuous function and that $\xi \in C_c^\infty(\Omega \times \mathbb{R})$ is nonnegative.

From $s \in L^\infty(Q)$ we deduce $g(s), G(s) \in L^\infty(Q)$. Furthermore, $g(s) = g(T_\varepsilon^{1-\varepsilon}(s)) \in L^2(0, T; H^1(\Omega))$ and hence $g(s)\xi \in L^2(0, T; H_0^1(\Omega)) \subset L^2(0, T; V)$. Consequently, for a.e. $t \in (0, T)$ the duality pairing $\langle \partial_t s(t), g(s(t))\xi(t) \rangle_{V', V}$ and the right hand side of (4.31) are well-defined. Using the properties of the Steklov average from lemma 4.28 and integration with respect to t over (t_1, t_2), we obtain

$$\int_{t_1}^{t_2} \langle \partial_t s, g(s)\xi \rangle_{V', V} \xleftarrow[h \to 0]{} \int_{t_1}^{t_2} \int_\Omega [\partial_t s]_h g(s)\xi.$$

The monotony of g implies the convexity of G. For $0 < h < T - t_2$ we use again Lemma 4.28, exploit the boundedness and the convexity of G, i.e. $G(b) - G(a) \geq g(a)(b - a)$, and compute with summation by parts

$$\begin{aligned}
\int_{t_1}^{t_2} \int_\Omega [\partial_t s]_h g(s)\xi &= \int_{t_1}^{t_2} \int_\Omega \frac{s(t+h) - s(t)}{h} g(s(t))\xi(t) \, \mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \int_\Omega \frac{G(s(t+h)) - G(s(t))}{h} \xi(t) \, \mathrm{d}t = \int_{t_2}^{t_2+h} \int_\Omega \frac{G(s(t))\xi(t-h)}{h} \, \mathrm{d}t \\
&\quad - \int_{t_1}^{t_1+h} \int_\Omega \frac{G(s(t))\xi(t-h)}{h} - \int_{t_1}^{t_2} \int_\Omega G(s(t)) \frac{\xi(t) - \xi(t-h)}{h} \, \mathrm{d}t \\
&\xrightarrow{h \to 0} \int_\Omega G(s(t))\xi(t) \Big|_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2} \int_\Omega G(s(t)) \partial_t \xi \, \mathrm{d}t,
\end{aligned} \tag{4.32}$$

for a.e. $0 < t_1 < t_2 < T$. The convergence on the right hand side of (4.32) follows from the continuity of ξ, the uniform boundedness of difference quotients of ξ and their uniform convergence towards $\partial_t \xi$, and from Lebesgue's differentiation theorem [DiB02, chapter IV.11]. For negative h with $0 < -h =: \eta < t_1$, we see that $[\partial_t s]_h$ is a backward difference quotient and, similarly to (4.32), we obtain

$$
\begin{aligned}
\int_{t_1}^{t_2} \int_\Omega [\partial_t s]_h g(s)\xi &= \int_{t_1}^{t_2} \int_\Omega [\partial_t s]_{-\eta} g(s)\xi = \int_{t_1}^{t_2} \int_\Omega \frac{s(t) - s(t-\eta)}{\eta} g(s(t))\xi(t) \, \mathrm{d}t \\
&\geq \int_{t_1}^{t_2} \int_\Omega \frac{G(s(t)) - G(s(t-\eta))}{\eta} \xi(t) \, \mathrm{d}t = \int_{t_2-\eta}^{t_2} \int_\Omega \frac{G(s(t))\xi(t+\eta)}{\eta} \, \mathrm{d}t \\
&\quad - \int_{t_1-\eta}^{t_1} \int_\Omega \frac{G(s(t))\xi(t+\eta)}{\eta} \, \mathrm{d}t - \int_{t_1}^{t_2} \int_\Omega G(s(t)) \frac{\xi(t+\eta) - \xi(t)}{\eta} \, \mathrm{d}t \\
&\xrightarrow{h \to 0} \int_\Omega G(s(t))\xi(t) \Big|_{t=t_1}^{t=t_2} - \int_{t_1}^{t_2} \int_\Omega G(s(t))\partial_t\xi \, \mathrm{d}t,
\end{aligned}
\tag{4.33}
$$

for a.e. $0 < t_1 < t_2 < T$. With reasoning as in (4.32) we obtain convergence to the same limit and we conclude. $\qquad\square$

Remark 4.33. Without a proof, the previous lemma can be found in [AD85a, p. 366] for more general ξ lying in an appropriate Sobolev space. Furthermore, the proof of Lemma 4.32 remains unchanged if $\xi \in C_c^\infty(\Omega \cup \Gamma_N \times \mathbb{R})$ or if $g(s) \in L^2(0, T; V)$. In the latter case $\xi \in C^\infty(\mathbb{R}^{d+1})$ is also allowed. $\qquad\triangleleft$

Lemma 4.32 provides, at least locally, the admissibility of certain truncations of s as test functions in (MP). We emphasize that, in general, the lemma is not suited to take either Dirichlet boundary or initial data into account.

We turn our attention to the justification of e.g. $\Phi(\cdot, s)$ as a test function in (MP) from page 43. We are interested in shapes of $\Phi(x, \cdot)$ as depicted in Figure 4.1 for $x \in \Omega$. In these cases Φ is not necessarily Lipschitz continuous, flat near zero and one, and it is x-dependent. Hence, Lemma 4.32 does not suffice to justify the use of $\Phi(\cdot, s)$ as a test-function. Imitating the previous proof, while using the definition of initial data in the sense of traces (4.2), we obtain a formula similar to (4.31) that is suited to treat initial and time-independent boundary data. Before proving such an improved integration by parts formula, we define a primitive of a function g subject to a function h with $g(x, h(x)) = 0$.

Definition 4.34. *Let $h(x) \in L^\infty(\Omega; [0, 1])$ and a Carathéodory function $g : \Omega \times [0, 1] \to \overline{\mathbb{R}}$ be given. We assume that for every $x \in \Omega$ there holds that $g(x, \cdot)$ is nondecreasing and $g(x, h(x)) = 0$. Furthermore, we assume that $-\infty < g(x, s) < \infty$ holds on $\Omega \times (0, 1)$. We define $G : \Omega \times [0, 1] \to \overline{\mathbb{R}}$ as*

$$
G(x, s) = \int_{h(x)}^s g(x, \sigma) \, \mathrm{d}\sigma
\tag{4.34}
$$

for $(x, s) \in \Omega \times [0, 1]$ and say that G is the (nonnegative) primitive of g with respect to h.

From the previous definition we infer that $G(x, h(x)) = 0$ for $x \in \Omega$. Since $g(x, \cdot)$ is non-decreasing we obtain that $G(x, \cdot)$ is nonnegative and convex for every $x \in \Omega$.

Remark 4.35 (On the subdifferential of G). Assume $g \in C(\Omega \times [0, 1])$ in Definition 4.34. Then, the fundamental theorem of calculus yields that $G'(x, v) = g(x, v)$ for every $v \in [0, 1]$ and $x \in \Omega$. Additionally, for given $v \in [0, 1]$ the subdifferential relation $G(x, u) \geq G(x, v) + g(v)(u - v)$ holds for every $u \in [0, 1]$. With the subdifferential notation, this is written as $\partial G(x, v) = g(x, v)$.

We remark that neither of the mentioned identities needs to hold in points that satisfy $g(x, v) = \infty$. Exemplary, assume $h(x) \equiv 0$, $g(x, 1) = \infty$ and $0 \leq g(x, v) < \infty$ for $x \in \Omega$ and $v \in [0, 1)$. In that situation we obtain $G(x, 1) = \infty$, but the differential of $G(x, v)$ is undefined for $v = 1$. By definition of the subdifferential, we obtain $\partial G(x, 1) = \emptyset$ but $\infty = g(x, 1)$. Particularly, the subdifferential relation is not satisfied for $v = 1$. ◁

Lemma 4.36 (Integration by parts II). *Let Ω, Q and V be as in (1.20)–(1.23) and let h, g and G be as in Definition 4.34. Let $s_0 \in L^\infty(\Omega, [0, 1])$ and $s \in L^\infty(Q, [0, 1])$ be given. Furthermore, we assume that $\partial_t s \in L^2(0, T; V')$, $g(\cdot, s) \in L^2(0, T; V)$ and $s(\cdot, 0) = s_0$ in the sense of traces, i.e. such that (4.2) holds. If $G(\cdot, s_0) \in L^1(\Omega)$, then $G(\cdot, s) \in L^\infty(0, T; L^1(\Omega))$ and*

$$\int_0^{t_0} \langle \partial_t s, g(s)\xi \rangle_{V', V} \geq \int_\Omega G(s(t_0))\xi(t_0) - \int_\Omega G(s_0)\xi(0) - \int_0^{t_0} \int_\Omega G(s(t))\partial_t \xi(t) \quad (4.35)$$

for every nonnegative $\xi \in C_c^\infty(\mathbb{R}^d \times [0, T))$ and a.e. $t_0 \in (0, T)$. Furthermore, if, for some $p \in [1, \infty)$, there exists a set $E \subset (0, T)$ of measure zero such that $s(t) \rightharpoonup s_0$ in $L^p(\Omega)$ for $t \to 0$, $t \in (0, T) \setminus E$, then equality holds in (4.35).

In Lemma 4.43 we observe that the convergence $s(t) \rightharpoonup s_0$ can often be obtained by means of the differential equation.

Proof of Lemma 4.36. Let $h > 0$ and $t_0 \in (0, T)$ be given $(0 < h < T - t_0)$. We extend s on $-h < t < 0$ via $s(t) := s_0$; since (4.2) holds, i.e. $s_0 = s(0)$ in the sense of traces, this extension is in $W^{1,1}(-h, T; V')$, see [Sch13, Lemma 10.10]. Due to the assumptions, the left-hand side in (4.35) is well-defined.

To show that the right-hand side of (4.35) is well-defined, we need to exploit the convexity of G and the subdifferential propberties of G under consideration of Remark 4.35. Since $g(\cdot, s) \in L^2(0, T; V)$ we infer that $g(\cdot, s(t)) \in L^2(\Omega)$ for a.e. $t \in (0, T)$ and, hence, for a.e. $t \in (0, T)$ there holds $-\infty < g(x, s(x, t)) < \infty$ for a.e. $x \in \Omega$. In particular, we find

for a.e. t that $g(x, s(x,t)) \in \partial G(x, s(x,t))$ for a.e. $x \in \Omega$. Hence, the convexity of G in s implies for a.e. $t > 0$ that

$$(s(t) - s(t-h))g(\cdot, s(t)) \geq G(\cdot, s(t)) - G(\cdot, s(t-h)) \qquad (4.36)$$

pointwise almost everywhere in Ω. Since G is nonnegative, we infer

$$0 \leq G(\cdot, s(t)) \leq g(\cdot, s(t))(s(t) - s(t-h)) + G(\cdot, s(t-h)).$$

Due to the assumptions and particularly since $G(\cdot, s_0) \in L^1(\Omega)$ we deduce with an inductive argument that $G(\cdot, s) \in L^1(\Omega \times (-h, t_0 + h))$.

With the integrability of $G(\cdot, s)$, Lemma 4.28, particularly with the identity $[\partial_t s]_{-h} = \partial_t[s_{-h}] \in V'$ a.e. on $(0, t_0)$ for the backward difference quotient, and equation (4.36), we proceed as in (4.33) with $t_1 = 0$ and $t_2 = t_0$ to obtain equation (4.35) for a.e. $t_0 \in (0, T)$. We want to point out the difference in comparison to the proof of Lemma 4.32. The integral regarding $t_1 = 0$ is treated with the extension of s; particularly, we obtain

$$\int_{-h}^{0} \int_{\Omega} \frac{s(t)\xi(t+h)}{h} = \int_{-h}^{0} \int_{\Omega} \frac{s_0 \xi(t+h)}{h} \xrightarrow[h \to 0]{} \int_{\Omega} s_0 \xi(0).$$

Furthermore, with $\xi \equiv 1$ on $[0, t_0]$ we infer from (4.35) that $G(\cdot, s) \in L^\infty(0, T; L^1(\Omega))$.

With reasoning as above and as in equation (4.32) but taking the inferior limit, we infer the upper estimate

$$
\begin{aligned}
\int_0^{t_0} \langle \partial_t s, g(s)\xi \rangle_{V',V} \leq {} & \int_{\Omega} G(\cdot, s(t_0))\xi(t_0) \\
& - \limsup_{h \to 0} \frac{1}{h} \int_0^h \int_{\Omega} G(\cdot, s(t))\xi((t)) - \int_0^{t_0} \int_{\Omega} G(\cdot, s(t))\partial_t \xi(t)
\end{aligned}
\qquad (4.37)
$$

for a.e. $t_0 \in (0, T)$. Comparing (4.35) and (4.37), we conclude

$$\limsup_{h \to 0} \frac{1}{h} \int_0^h \int_{\Omega} G(s(t))\xi(t) \leq \int_{\Omega} G(s_0)\xi(0).$$

Consequently, for the proof of the reverse inequality (4.37) it suffices to show that

$$\liminf_{h \to 0} \frac{1}{h} \int_0^h \int_{\Omega} G(\cdot, s(t))\xi(t) \geq \int_{\Omega} G(\cdot, s_0)\xi(0). \qquad (4.38)$$

Lemma 4.37 is applicable to $f = G\xi(0)$ and we find $s_0 \in M_f$. We conclude that $M_{G\xi(0)}$ is a convex set and that the corresponding functional is weakly lower semicontinuous. Since $s(t) \rightharpoonup s_0$ in $L^p(\Omega)$ for some $p \in [1, \infty)$ and $t \in (0, T) \setminus E$, we infer that

$$\liminf_{t \to 0, t \notin E} \int_{\Omega} G(\cdot, s(t))\xi(0) \geq \int_{\Omega} G(\cdot, s_0)\xi(0).$$

We denote the limit inferior by c and define $\eta(t) = \int_\Omega G(\cdot, s(t))\xi(0)$. By definition, we obtain for every $\varepsilon > 0$ a $\delta > 0$ such that $\eta(t) \geq c - \varepsilon$ for every $t \in (0, \delta) \setminus E$. For $h_0 = \delta$ we deduce $\fint_0^h \eta(t) \geq c - \varepsilon$ for every $0 < h < h_0$. Hence, we infer

$$\liminf_{h \to 0} \fint_0^h \eta(t) \geq \liminf_{t \to 0, t \notin E} \eta(t) \geq \int_\Omega G(\cdot, s_0)\xi(0), \tag{4.39}$$

which is (4.38), and we conclude. $\qquad\square$

Lemma 4.37. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $f : \Omega \times [0,1] \to [0, \infty]$ be a Carathéodory function such that $s \mapsto f(x, s)$ is convex for a.e. $x \in \Omega$. Define*

$$M_f := \{v \in L^\infty(\Omega, [0,1]) \mid f(\cdot, v) \in L^1(\Omega)\}$$

and let $p \in [1, \infty)$. If $M_f \neq \emptyset$, then $M_f \subset L^p(\Omega)$ is a convex set and $F : M_f \to [0, \infty]$ defined by

$$F(v) = \int_\Omega f(x, v(x)) \,\mathrm{d}x \tag{4.40}$$

is a convex functional. Furthermore, F is lower semicontinuous with respect to the weak convergence on $L^p(\Omega)$.

Proof. Since Ω is bounded, any $L^\infty(\Omega)$ function is in $L^p(\Omega)$ and consequently $M_f \subset L^p(\Omega)$. The convexity of f in s implies that M_f is a convex set and that $F : M_f \to [0, \infty]$ is a convex functional.

Let $v_k, v \in M_f$ with $v_k \to v \in L^p(\Omega)$. By definition of the inferior limit, there exists a subsequence k_l such that $\lim_{l \to \infty} F(v_{k_l}) = \liminf_{k \to \infty} F(v_k)$. The subsequence can be chosen such that $v_{k_l} \to v$ pointwise a.e. in Ω. The continuity of f implies that $f(x, v_{k_l}(x)) \to f(x, v(x))$ for $l \to \infty$ for a.e. $x \in \Omega$. Since $f(x, v_{k_l}(x)) \geq 0$, we apply Fatou's lemma to deduce

$$\liminf_{k \to \infty} F(v_k) = \lim_{l \to \infty} \int_\Omega f(x, v_{k_l}(x)) \,\mathrm{d}x \geq \int_\Omega f(x, v(x)) \,\mathrm{d}x = F(v),$$

i.e. the lower semi-continuity of F with respect to the strong convergence in $L^p(\Omega)$. From the separation theorem for convex sets we derive that F is weakly lower semi-continuous in $L^p(\Omega)$, see [Sch13, Theorem 13.8]. $\qquad\square$

Remark 4.38 (Extensions of the integration by parts formulae)**.** We consider the proofs of the integration by parts formulae again and take Lemma 4.30 into accounts. After a suitable extension of s for $t > T$, e.g. extension by its trace value at $t = T$, we realize that 4.32 and 4.36 also hold for $t_2 = T$ and $t_0 = T$, respectively. We find also that Lemma 4.32 holds for x-dependent g as in Definition 4.34.

We also intend to apply both integration by parts formulae to equation (3.23) of the two-phase problem. Hence, we require integration by parts formulae that take the porosity ϕ into account. Since the porosity ϕ is time-independent, we find in $\phi \partial_t s = \partial_t(\phi s) \in L^2(0, T; V')$. Using this in the proofs of Lemmas 4.32 and 4.36, we obtain equations (4.31) and (4.35) with a factor ϕ inside every duality pairing or integral over Ω. $\qquad\triangleleft$

Boundary Data in Lemma 4.36 and Comparison to the Literature   We emphasize that the proof of the integration by parts formulae relies on the convexity estimate (4.36). Particularly, we use that g and G do not depend explicitly on the time t. Hence, the previous integration by parts formulae are only appropriate to treat time-independent boundary data $\Phi_D \in H^1(\Omega)$. As a consequence the test function $g(x,s) = \Phi(x,s) - \Phi_D(x)$ can be applied to (MP).

Integration by parts formulae, similar to the ones stated above, can also be found in the literature. With the notation $b = \Phi^{-1}$, i.e. $\Phi(s) = u$ we cast (MP) into the form

$$
\begin{aligned}
&\partial_t[b(u)] = \Delta u \text{ on } Q, \\
&u = u_D \text{ on } \Gamma_D \times (0,T), \\
&\partial_\nu u = 0 \text{ on } \Gamma_N \times (0,T), \\
&b(u(x,0)) = b_0(x) \text{ on } \Omega,
\end{aligned}
\tag{4.41}
$$

and explain the use of integration by parts formulae with respect to (4.41). Typically, only results for x-independent b are proven in the literature, but generalizations for the x-dependent case are stated. The following references have in common, that b needs not to be strictly increasing. This implies that multivalued pressures and, in contrast to the work at hand, that the unsaturated-saturated flow problem can be handled.

In [AL83, Lemma 1.5] time-dependent Dirichlet boundary data are treated. Particularly, an integration by parts formulae for the function $u - u_D$ applied to (4.41) is used. The formulae is shown in the case that b is independent of x and is used to obtain an existence result. Without a proof it is stated that the formula can be extended to the case where b depends on x. As we saw above, our argument confirms this claim in the case that u_D is time-independent and Φ is as in Assumption A4.1.

In [Car99, Lemma 4, p.324], [CW99, Lemma 4.3] and [Ott95, Lemma 1] integration by parts formulae are shown to derive an L^1-contraction result. In view of (4.41), the use of the test function $\psi(u - f)$ is justified, where ψ is a sufficiently smooth function that is used to approximate the sign-function and f depends on x or is constant. As in [AL83, Lemma 1.5], the integration by parts formulae are stated in the case that b is x-independent, but extensions are possible. To apply the method of doubling the variables assumptions on f must be made. In [Car99] and [CW99] the space and time variables are being doubled, which necessitates that f is constant. In [Ott95] only the time variable is being doubled, whence f is allowed to depend on x. Particularly, f is chosen such that $\psi(u - f)$ has the right boundary data. In [Car99] and [CW99] only homogeneous Dirichlet data are considered, whereas in [Ott95] time-independent Dirichlet data u_D and also Neumann boundary data are treated. Translating this to our notation, we find that $g(x,s) = \mathrm{sign}_\varepsilon(\Phi(x,s) - \Phi_D(x))$ is a valid test-function for the model problem (MP). However, we need to use the test function $g(x,s) = \mathrm{sign}_\varepsilon(\Theta(x,s) - \Theta(x,f(x))$

to perform the method of doubling the variables applied to the discontinuous Richards equation (TDR) from page 24. Since Θ is nonlinear, this case is not covered by the references mentioned above.

In the succeeding article [Ott97], Otto was able to obtain L^1-contraction in the setting of outflow boundary problems; particularly, time dependent boundary data u_D are involved and, again, the test function $\psi(u-u_D)$ is applied to a variant of (4.41).

4.5 The Model Problem Revisited

In this section, we apply the results of sections 4.1 – 4.4 to the model problem (MP) from page 43. Particularly, we show that the formal calculation (4.7) can be made rigorous. We show that $s(t) \rightharpoonup s_0$ as $t \to 0$ and sketch the proof of an L^1-contraction result for weak solutions of (MP).

4.5.1 Justification of (4.7)

As before, we consider the x-independent and x-dependent case separately.

Lemma 4.39. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Let $\Phi \in C^1([0,1])$ be as in Proposition 4.13 and let $s \in L^\infty(\Omega, [0,1])$ be such that $\Phi(s) \in H^1(\Omega)$. For given $0 < \varepsilon < \frac{1}{2}$, we obtain with $T_\varepsilon := T_\varepsilon^{1-\varepsilon}$ the identity*

$$\nabla[\Phi(s)] = \lim_{\varepsilon \to 0} \Phi'(s)\nabla[T_\varepsilon(s)]. \tag{4.42}$$

Proof. Since $T_\varepsilon(s) \to s$ uniformly and since Φ is continuous, we obtain

$$\Phi'(s)\nabla[T_\varepsilon(s)] = \nabla[\Phi(T_\varepsilon(s))] \xrightarrow[\varepsilon \to 0]{} \nabla[\Phi(s)] \text{ in } \mathcal{D}'(\Omega), \tag{4.43}$$

where we applied the Proposition 4.13 and Corollary 4.15 to obtain the first equality. The corollary is used to justify the replacement of $\Phi'(T_\varepsilon(s))$ by $\Phi'(s)$. Hence, a potential L^2-limit is identified. To show the L^2-convergence we use the same corollary to find almost everywhere on Ω the identity

$$\Phi'(s)\nabla[T_\varepsilon(s)]\,\mathbb{1}_{\{\varepsilon<s<1-\varepsilon\}} = \nabla[\Phi(s)]\,\mathbb{1}_{\{\varepsilon<s<1-\varepsilon\}}. \tag{4.44}$$

Determining the pointwise limit on the right-hand side of (4.44) and using $\Phi(s) \in H^1(\Omega)$, we infer

$$\Phi'(s)\nabla[T_\varepsilon(s)]\,\mathbb{1}_{\{\varepsilon<s<1-\varepsilon\}} \xrightarrow[\varepsilon \to 0]{} \nabla[\Phi(s)]\,\mathbb{1}_{\{0<s<1\}} \text{ in } L^2(\Omega) \tag{4.45}$$

from Lebesgue's theorem. Since the distributional limit is unique and since L^2-convergence implies $\mathcal{D}'$-convergence, we obtain (4.42). $\qquad\square$

Lemma 4.40. *Let Φ fulfill Assumption A4.1 and let $s \in L^\infty(\Omega, [0, 1])$ be such that $\Phi(\cdot, s) \in H^1(\Omega)$. For $0 < \varepsilon < \frac{1}{2}$, let $T_\varepsilon := T_\varepsilon^{1-\varepsilon}$ be as in Definition 4.11. Then there holds*

$$\nabla[\Phi(\cdot, s)] = \lim_{\varepsilon \to 0} \left(\nabla_x \Phi(\cdot, T_\varepsilon(s)) + \Phi'(\cdot, s)\nabla[T_\varepsilon(s)] \right)$$

$$= \nabla_x \Phi(\cdot, s) + \lim_{\varepsilon \to 0} \left(\Phi'(\cdot, s)\nabla[T_\varepsilon(s)] \right) \tag{4.46}$$

in $L^2(\Omega)$ and the identity

$$\nabla[\Phi(\cdot, s)] \mathbb{1}_{\{0 < s < 1\}} + \nabla_x \Phi(\cdot, s) \mathbb{1}_{\{\{s=0\} \cup \{s=1\}\}} = \nabla[\Phi(\cdot, s)].$$

Proof. The proof is similar to that of Lemma 4.39. Since Φ is continuous and $T_\varepsilon(s) \to s$ uniformly ad $\varepsilon \to 0$, we deduce, using Proposition 4.21 and Corollary 4.23, the convergence

$$\nabla_x \Phi(\cdot, T_\varepsilon(s)) + \Phi'(\cdot, s)\nabla[T_\varepsilon(s)] = \nabla[\Phi(\cdot, T_\varepsilon(s))] \xrightarrow[\varepsilon \to 0]{} \nabla\Phi(\cdot, s) \text{ in } \mathcal{D}'(\Omega). \tag{4.47}$$

As in the proof of Lemma 4.39, Corollary 4.23 is only needed to justify the replacement of $\Phi'(T_\varepsilon(s))$ by $\Phi'(s)$. With (4.47) a potential L^2-limit is identified. Due to the assumptions on Φ and the uniform convergence of $T_\varepsilon(s) \to s$ we infer that $\nabla_x \Phi(\cdot, T_\varepsilon(s)) \to \nabla_x(\Phi(\cdot, s))$ in $L^2(\Omega)$. With Corollary 4.23, we derive

$$\Phi'(\cdot, s)\nabla[T_\varepsilon(s)] = [\nabla[\Phi(\cdot, s)] - \nabla_x \Phi(\cdot, s)] \mathbb{1}_{\{\varepsilon < s < 1-\varepsilon\}} \tag{4.48}$$

for a.e. in Ω. Determining the pointwise limit in (4.48) and using that $\nabla\Phi(\cdot, s)$, $\nabla_x \Phi(\cdot, s) \in L^2(\Omega)$ we obtain by Lebesgue's theorem that

$$\Phi'(\cdot, s)\nabla[T_\varepsilon(s)] \xrightarrow[\varepsilon \to 0]{} [\nabla[\Phi(\cdot, s)] - \nabla_x \Phi(\cdot, s)] \mathbb{1}_{\{0 < s < 1\}} \text{ in } L^2(\Omega). \tag{4.49}$$

Since L^2-convergence implies $\mathcal{D}'$-convergence and since the distributional limit is unique we obtain (4.46). Collecting the previous identities also

$$\nabla[\Phi(\cdot, s)] \mathbb{1}_{\{0 < s < 1\}} + \nabla_x \Phi(\cdot, s) \mathbb{1}_{\{\{s=0\} \cup \{s=1\}\}} = \nabla[\Phi(\cdot, s)]$$

and conclude. $\square$

Remark 4.41. Taking in Lemmas 4.39 and 4.40 the "simultaeneous" limit $T_\varepsilon^{1-\varepsilon}$ was only done to simplify notation. One obtains similar results if one merely considers the upper or lower limits. ◁

With the results from this section, we see that the formal identity (4.7) can be justified by replacing $\nabla s \, \mathbb{1}_{\{\varepsilon < s < 1-\varepsilon\}}$ essentially by $\nabla[T_\varepsilon(s)]$.

Remark 4.42. We want to continue Remarks 3.1 and 3.2. In [AD85a, Lemma 3.3 and p. 389] a solution is constructed that fulfills (3.19) only in the sense that

$$\phi \partial_t s = \nabla \cdot \left(\lim_{\varepsilon \to 0} \frac{\lambda_1(s)\lambda_2(s)}{\lambda(s)} p_c'(s) \nabla T_\varepsilon(s) + B(s) + D(s)u \right). \tag{4.50}$$

Under certain assumptions on Φ, the previous two lemmas show that (4.50) is equivalent to

$$\phi \partial_t s = \nabla \cdot (\Delta[\Phi(s)] - \nabla_x \Phi(s) + B(s) + D(s)u) \tag{4.51}$$

and essentially (3.23) holds. ◁

4.5.2 Weak Convergence Towards Initial Data

Lemma 4.43 (Weak convergence towards initial data)**.** *Let s be a weak solution of the model problem* (MP) *in sense of Definition 4.1. Then, there is a set $E \subset (0, T)$ with $|E| = 0$ and such that*

$$s(t) \rightharpoonup s_0 \in L^2(\Omega)$$

as $t \to 0$ for $t \in (0, T) \setminus E$.

Proof. Let $\beta \in C_c^\infty(\Omega)$ be given and ,for $\delta > 0$, let $\alpha_{\delta,t^*} \in W^{1,1}(0, T)$ be a sequence of continuous piecewise linear cut off functions of the interval $[0, t^*]$, i.e.

$$\alpha_{\delta,t^*}(t) = \begin{cases} 1 & 0 \le t \le t^*, \\ 0 & t \ge t^* + \delta, \\ \text{linear} & t^* < t < t^* + \delta. \end{cases}$$

Particularly, there holds $\left| \alpha_{\delta,t^*}' \right| \le 1/\delta$. For $0 < t^* < T$ and δ small enough, we may choose $\xi = \alpha_{\delta,t^*}\beta$ in (4.1) and (4.2), and we obtain

$$\fint_{t^*}^{t^*+\delta} \int_\Omega s\beta + \int_0^{t^*+\delta} \int_\Omega \alpha_{\delta,t^*} \nabla[\Phi(\cdot, s)] \nabla\beta = \int_\Omega s_0 \beta. \tag{4.52}$$

To pass with $\delta \to 0$, we apply Lebesgue's differentiation theorem. This yields a set $E \subset (0, T)$ with $|E| = 0$ such that

$$\int_\Omega s(t^*)\beta + \int_0^{t^*} \int_\Omega \nabla[\Phi(\cdot, s)] \nabla\beta = \int_\Omega s_0 \beta \tag{4.53}$$

for any $t^* \in (0, T) \setminus E$. Due to the absolute continuity of the Lebesgue integral, we obtain passing with $t^* \to 0$ the convergence

$$s(t) \xrightarrow[t \in (0,T) \setminus E]{t \to 0} s_0 \text{ in } \mathcal{D}'(\Omega).$$

Furthermore, since $s(t) \in L^\infty(0, T; L^2(\Omega))$, we find that $\{s(t)\}_{t \in (0,T) \setminus E} \subset L^2(\Omega)$ is a bounded set. Hence, up to a subsequence and since the distributional limit is unique, we obtain $s(t) \rightharpoonup s_0$ in $L^2(\Omega)$ for any sequence $t \to 0$ subject to $t \in (0, T) \setminus E$. We conclude with a standard argument, see [DiB02, I.1.1]. $\qquad\square$

Remark 4.44. The previous proof exploits that $s \in L^\infty(0, T; L^p(\Omega))$ for $p = 2$. The same reasoning can be used for any $p \in [1, \infty)$. For $p = 1$, we require additionally that $\{s(t)\}$ is uniformly or equi-integrable to apply the Dunford-Pettis theorem [Bre10, Theorem 4.30]; this hint is also given in [Ott95, p. 36]. $\qquad\triangleleft$

Application of Lemma 4.43 and Lemma 4.36, to the model problem (MP), yields with the test function $\Phi(x, s)\alpha_{\delta,t}\beta$, where $\beta \equiv 1$ on Ω, under the assumption $\Phi_D = h(x) = 0$ the a priori estimate (4.3) for a.e. $t \in (0, T)$. For more general Dirichlet data, we require $\Phi_D = \Phi(x, h(x)) \in H^1(\Omega)$ for some $h \in L^\infty(\Omega, [0, 1])$.

4.5.3 Formal L^1-contraction and Doubling the Variables

In this section, we assume that Φ does not depend on x. The basic idea to show L^1-contraction for the model problem (MP) is to multiply the equations for any two solutions s_1 and s_2 by $\text{sign}(\Phi(s_1) - \Phi(s_2)) = \text{sign}(s_2 - s_2)$. To obtain the latter identity, the monotony of $\Phi(s)$ is used. For $j \in \{1, 2\}$, we obtain

$$\int_\Omega \partial_t s_j \, \text{sign}(s_1 - s_2) + \nabla(\Phi(s_j))\nabla[\text{sign}(\Phi(s_1) - \Phi(s_2))]. = 0 \qquad (4.54)$$

Subtracting the equations for s_1 and s_2, and using the total time derivative $\partial_t u \, \text{sign}(u) = \partial_t |u|$ yields

$$\int_\Omega \partial_t |s_1 - s_2| + |\nabla(\Phi(s_1) - \Phi(s_2))|^2 \, \text{sign}'(\Phi(s_1) - \Phi(s_2)) = 0. \qquad (4.55)$$

We interpret sign' to be nonnegative and obtain, after integration over $\tau \in (0, t)$ with the initial data $s_{0,1}$ and $s_{0,2}$ of s_1 and s_2, respectively, the estimate

$$\int_\Omega |s_1 - s_2|(t) \le \int_\Omega |s_{0,1} - s_{0,2}|.$$

This estimate implies uniqueness of solutions and completes a first formal proof.

Remark 4.45. In the case of x-dependent Φ an additional term of the form

$$\int_\Omega \nabla(\Phi(s_1) - \Phi(s_2)) \, \text{sign}'(\Phi(s_1) - \Phi(s_2))(\nabla_x(\Phi(s_1) - \Phi(s_2)))$$

would appear in (4.55). Such a term does, in general, not possess a sign and the argument fails. $\qquad\triangleleft$

There are several issues in this formal proof which all are related to the validity of the test function. The most obvious problem is the lack of regularity; in general $\text{sign}(\Phi(s_1) - \Phi(s_2)) \notin L^2(0,T;V)$. To tackle this issue, one approximates the sign function by a regularized function sign_ε. The downside of this approximation is that the equality $\text{sign}(s_1 - s_2) = \text{sign}(\Phi(s_1) - \Phi(s_2))$ is not maintained under this approximation. The use of this equality was essential in the in the step from (4.54) to (4.55). Furthermore, due to the non-linearity of Φ we cannot expect that $\partial_t(s_1 - s_2)\text{sign}(\Phi(s_1) - \Phi(s_2))$ is a total time derivative of any function.

At this point the chain rule and *doubled time variables* come into play. However, the chain rule is not directly applicable since

$$\partial_t s_1 \text{sign}_\varepsilon(\Phi(s_1) - \Phi(s_2))$$

is not a total time derivative unless s_2 is constant in t. Assuming that for the moment, we obtain

$$
\begin{aligned}
\partial_t s_1 g_\varepsilon(s_1, s_2) &:= \partial_t s_1 \text{sign}_\varepsilon(\Phi(s_1) - \Phi(s_2)) \\
&= \partial_t \left[\int_{s_2}^{s_1} \text{sign}_\varepsilon(\Phi(u) - \Phi(s_2)) \right] =: \partial_t G_\varepsilon(s_1, s_2).
\end{aligned}
\tag{4.56}
$$

Hence, let us assume that s_1 is a solution for the time variable t and s_2 is a solution for the time variable τ. We repeat the procedure obtained to deduce (4.55) with the following changes. We use the total time deriative (4.56) for s_1 and s_2, respectively, exploit the symmetry of the sign function to obtain $-\partial_\tau s_2 \text{sign}_\varepsilon(\Phi(s_1) - \Phi(s_2)) = \partial_\tau G_\varepsilon(s_2, s_1)$ and infer, similarly to (4.55), the identity

$$
\begin{aligned}
&\int_\Omega \partial_t G_\varepsilon(s_1(t), s_2(\tau)) + \partial_\tau G_\varepsilon(s_2(t), s_1(\tau)) \\
&+ \int_\Omega |\nabla(\Phi(s_1(t)) - \Phi(s_2(\tau)))|^2 \text{sign}'_\varepsilon(\Phi(s_1(t)) - \Phi'(s_2(\tau))) = 0.
\end{aligned}
\tag{4.57}
$$

The last integral has a sign and we pass formally to the limit $\varepsilon \to 0$. This yields

$$\int_\Omega \partial_t G(s_1(t), s_2(\tau)) + \partial_\tau G(s_2(t), s_1(\tau)) \leq 0$$

Linking the times again by choosing $\tau = t$, using that $G(s_1, s_2) = |s_1 - s_2|$ and integration over $(0, t_0)$ yields the L^1-contraction

$$\int_\Omega |s_1(t_0) - s_2(t_0)| \leq \int_\Omega |s_{0,1} - s_{0,2}|$$

and we finish the second formal proof.

The lack of rigor in this second proof on L^1-contraction essentially affects the limiting process $\varepsilon \to 0$. Particularly, the convergence of e.g. $\partial_t G_\varepsilon \to \partial_t G$ and choosing

$t = \tau$ afterwards is unclear. As we are going to see, the tool to cope with these issues is the integration by parts formula provided by Lemma 4.36. Using this lemma, the time derivative acts on a smooth function ξ instead of $G(s_1, s_2)$ and $G(s_2, s_1)$, respectively. Consequently, we only need to show convergence $G_\varepsilon(s_1, s_2) \to G_\varepsilon(s_1, s_2)$ instead of $\partial_t G_\varepsilon(s_1, s_2) \to \partial_t G_\varepsilon(s_1, s_2)$. There are two terms coming into play due to the occurrence of ξ. The $\nabla \xi$-term vanishes by simply choosing ξ constant in space and the term containing $\partial_t \xi + \partial_\tau \xi$ is used to approximate a Dirac measure in $t = \tau$.

Remark 4.46. The method of doubling the variables was introduced by Kruzhkov [Kru70] in the context of conservation laws. Otto [Ott95; Ott97] and Carrillo[Car99] extended the method to degenerate parabolic equations. In Otto's works it suffices to double on the time variable, whereas in Kruzhkov's and Carrillo's articles it is required to double the space and time variable. For a survey on recent results in the environment of the method of doubling the variables, we refer to [AI12]. ◁

5 L^1-contraction for Equations of Richards type with x-dependence

In this chapter, we provide a proof of Theorem 2.4, following closely the ideas presented in the formal L^1-contraction proof of section 4.5.3. There it was used that Φ is independent of x. The x-dependence of Φ in our problem makes adaptions necessary. In section 5.1, we provide notation and results to treat the problem with an x-dependent Φ. In section 5.2, we state the main proposition, which is extended in section 5.3. The extension is used to prove the main theorem, Theorem 2.4.

In the following, we use the notation introduced at the beginning of chapter 2 in equation (2.1) and usually suppress the x-dependence of the functions.

5.1 Preliminaries

In contrast to the formal L^1-contraction proof in section 4.5.3, we choose $\text{sign}_\varepsilon(\Theta(s_1) - \Theta(s_2))$ instead of $\text{sign}_\varepsilon(\Phi(s_1) - \Phi(s_2))$. The first function is weakly differentiable on Ω. The second function is only weakly differentiable on Ω_l and Ω_r separately. As seen in (4.56) we need to define a transformation similar to G_ε adapted to our problem.

Definition 5.1. *Let $j \in \{l, r\}$, $v \in [0, 1]$ and $\varepsilon > 0$ be given and let Θ_j be as in Assumption A2.3. We define the mappings $q_{\varepsilon,j}$ and $q_j : [0, 1]^2 \to \mathbb{R}$ as*

$$q_{\varepsilon,j}(u, v) = \int_v^u \text{sign}_\varepsilon(\Theta_j(\xi) - \Theta_j(v)) \, \mathrm{d}\xi$$

and

$$q_j(u, v) = |u - v| \,.$$

Lemma 5.2 (Properties of $q_{\varepsilon,j}$)**.** *Let $j \in \{l, r\}$ and $\varepsilon > 0$ be given. Let $q_{\varepsilon,j}$ and q_j be as in Definition 5.1. For $u, v \in [0, 1]$, the following properties hold*

1. *The mapping $q_{\varepsilon,j}(\cdot, v) : [0, 1] \to \mathbb{R}$ is convex for every $v \in [0, 1]$*

2. *$0 \leq q_{\varepsilon,j}(u, v) \nearrow q_j(u, v)$ as $\varepsilon \to 0$.*

3. *$q_{\varepsilon,j} \in C([0, 1]^2)$*

4. $q_{\varepsilon,j} \to q_j$ uniformly as $\varepsilon \to 0$.

Proof. **Ad 1.** We consider two cases. If Θ_j is bounded or if $v < 1$, then the integrand of q_ε is nondecreasing in ξ. Consequently, $q_\varepsilon(\cdot, v)$ is convex. In the remaining case, i.e. in the case$\Theta_j(1) = \infty$ and $v = 1$, we obtain that the integrand in $q_{\varepsilon,j}$ equals -1 for $u \in [0, 1)$ and consequently we obtain for $u \in [0, 1]$ the relation $q_{\varepsilon,j}(u, 1) = q_j(u, 1) = |u - 1|$ which is clearly convex.

Ad 2. Let Θ_j be bounded or $v < 1$. With $0 \le \operatorname{sign}(u - v) \operatorname{sign}_\varepsilon(\Theta_j(u) - \Theta_j(v)) \le 1$, we find
$$0 \le q_{\varepsilon,j}(u, v) \le |u - v| = q_j(u, v)$$
Since Θ_j is an increasing function we obtain
$$\operatorname{sign}(u - v) \operatorname{sign}_\varepsilon(\Theta_j(u) - \Theta_j(v)) \nearrow 1 - \delta_{uv}$$
as $\varepsilon \to 0$ and δ denotes the Kronecker delta. Thus, from the monotone convergence theorem [Alt06, A1.12], we infer $q_{\varepsilon,j}(u, v) \nearrow |u - v| = q_j(u, v)$ for $u \in [0, 1]$ as $\varepsilon \to 0$. The remaining case, i.e. $\Theta_j(1) = \infty$ and $v = 1$, is immediate since $\operatorname{sign}_\varepsilon(\Theta_j(u) - \Theta_j(1)) = -1$ for $u \in [0, 1)$. Hence $q_\varepsilon(u, 1) = 1 - u = |u - 1|$ for every $u \in [0, 1]$.

Ad 3. Let $u_k, v_k \in [0, 1]$ for $k \in \mathbb{N}$ be such that $u_k \to u$ and $v_k \to v$. With Lebesgue's dominated convergence theorem, we conclude
$$\begin{aligned}
&|q_{\varepsilon,j}(u, v) - q_{\varepsilon,j}(u_k, v_k)| \\
&= \left| \int_0^1 \mathbb{1}_{/u,v/}(\xi) \operatorname{sign}_\varepsilon(\Theta_j(\xi) - \Theta_j(v)) - \mathbb{1}_{/u_k,v_k/}(\xi) \operatorname{sign}_\varepsilon(\Theta_j(\xi) - \Theta_j(v_k)) \, \mathrm{d}\xi \right| \\
&\le \int_0^1 \left| \mathbb{1}_{/u,v/} - \mathbb{1}_{/u_k,v_k/} \right|(\xi) + |\operatorname{sign}_\varepsilon(\Theta_j(\xi) - \Theta_j(v)) - \operatorname{sign}_\varepsilon(\Theta_j(\xi) - \Theta_j(v_k))| \, \mathrm{d}\xi \\
&\xrightarrow[k \to \infty]{} 0.
\end{aligned}$$

Ad 4. Thanks to items 2 and 3, we can apply Dini's theorem [DiB02, I.Theorem 7.3]. This proves the claim. $\qquad\square$

Lemma 5.3. *We assume that Assumptions A2.1–A2.3 hold. Let s be a weak solution of* (TDR) *in the sense of Definition 2.2. Let $v \in L^\infty(\Omega; [0, 1])$ be such that $\Theta(\cdot, v) \in H^1(\Omega)$ and $\Theta(\cdot, s) - \Theta(\cdot, v) \in L^2(0, T; V)$. Then there holds*
$$\begin{aligned}
\int_Q &[q_\varepsilon(s_0, v) - q_\varepsilon(s, v)] \partial_t \gamma \\
&+ \sum_{j \in \{l,r\}} \int_{Q_j} [\nabla \Phi_j(s) + \lambda_j(s) g_j] \cdot \nabla[\operatorname{sign}_\varepsilon(\Theta_j(s) - \Theta_j(v))\gamma] \\
&\le \int_Q f(s) \left[\operatorname{sign}_\varepsilon(\Theta(s) - \Theta(v))\gamma \right]
\end{aligned} \tag{5.1}$$

for every nonnegative $\gamma \in C_c^\infty([0, T] \times \mathbb{R}^d)$ and $\varepsilon > 0$.

Proof. Since $\Theta(\cdot, s) - \Theta(\cdot, v) \in L^2(0, T; V)$ and since sign_ε is Lipschitz continuous, we infer from Corollary 4.5 that $g(\cdot, s) = \text{sign}_\varepsilon(\Theta(\cdot, s) - \Theta(\cdot, v)) \in L^2(0, T; V)$. Hence, it is allowed to choose $\xi = g(\cdot, s)\gamma$ in (2.13) and we obtain (5.1) modulo the inequality and the term containing the time derivative.

To treat the term containing the time derivative, let G be the primitive of g with respect to v in the sense of Definition 4.34. Since g is bounded on $\Omega \times (0, 1)$, we find $G(x, s) = q_\varepsilon(x, s, v)$ for $x \in \Omega$ and $s \in [0, 1]$. Thus, under consideration of Remark 4.38 with $t_0 = T$, we apply Lemma 4.36 and obtain

$$\int_Q \langle \partial_t s, g(\cdot, s)\gamma \rangle_{V', V} \geq - \int_\Omega q_\varepsilon(s_0, v)\gamma(0) - \int_Q q_\varepsilon(s, v)\partial_t \gamma$$
$$= \int_Q [q_\varepsilon(s_0, v) - q_\varepsilon(s, v)]\partial_t \gamma.$$

Summarizing, we obtain (5.1) and conclude the proof. $\qquad\square$

5.2 The Kato Inequality

The next steps in the formal proof of Section 4.5.3 were to *double the time variable*, send the regularization parameter of sign_ε to zero and finally reduce the doubling of the time variable. These steps are performed in Proposition 5.4 and yield the so called Kato-inequality.

As in [Ott95], the Kato-inequality presented here requires test functions that vanish in $t = 0$. In section 5.3, the Kato-inequality is extended to test functions with arbitrary values at $t = 0$. It is also possible to prove this extended Kato-inequality directly, which requires, in comparison to our proof, a different choice of test functions.

Proposition 5.4 (Kato inequality)**.** *Let s_1, s_2 be weak solutions of* (TDR) *in the sense of Definition 2.2. Under Assumptions A2.1, A2.2, A2.3 and A2.4 there holds*

$$\int_Q (-|s_1 - s_2|)\partial_t \gamma + \sum_{j \in \{l, r\}} \int_{Q_j} \text{sign}(\Phi_j(s_1) - \Phi_j(s_2))\nabla[\Phi_j(s_1) - \Phi_j(s_2)] \cdot \nabla\gamma$$
$$+ \sum_{j \in \{l, r\}} \int_{Q_j} \text{sign}(s_1 - s_2)[\lambda_j(s_1) - \lambda_j(s_2)]g_j \cdot \nabla\gamma \tag{5.2}$$
$$\leq \int_Q \text{sign}(s_1 - s_2)[f(s_1) - f(s_2)]\gamma$$

for every $\gamma \in C_c^\infty((0, T) \times \mathbb{R}^d)$ with $\gamma \geq 0$.

Proof of the Kato Inequality

As indicated in section 4.5.3, the proof of the Kato inequality consists of three steps. Firstly, we double the time variable. Secondly, we pass with $\varepsilon \to 0$, i.e. we consider $\text{sign}_\varepsilon \to \text{sign}$. Lastly, we reduce the doubling of the time variable.

Step 1: Doubling the time variable

Let $(t_1, t_2, x) \in (0, T)^2 \times \Omega := \tilde{Q}$ and define $\tilde{Q}_j := (0, T)^2 \times \Omega_j$ for $j \in \{l, r\}$. Consider $s_1(x, t_1, t_2) := s_1(x, t_1)$ as a function on $\tilde{Q}$ independent of t_2 and $s_2(x, t_1, t_2) := s_2(x, t_2)$ as a function on $\tilde{Q}$ independent of t_1.

Let $\tilde{\gamma} \in C_c^\infty((0, T)^2 \times \mathbb{R}^d)$ be such that $\tilde{\gamma} \geq 0$. For $\varepsilon > 0$ and a.e. $t_2 \in (0, T)$, we infer from Lemma 5.3 with $v = s_2(t_2)$, $t = t_1$ and with the choice $\gamma(t) = \tilde{\gamma}(t, t_2)$ the estimate

$$
\begin{aligned}
&- \int_Q q_\varepsilon(s_1, s_2(t_2)) \partial_{t_1} \tilde{\gamma}(t_2) \\
&+ \sum_{j \in \{l, r\}} \int_{Q_j} [\nabla \Phi_j(s_1) + \lambda_j(s_1) g_j] \cdot \nabla [\text{sign}_\varepsilon [\Theta_j(s_1) - \Theta_j(s_2(t_2))] \tilde{\gamma}(t_2))] \\
&\leq \int_Q f(s_1) [\text{sign}_\varepsilon [\Theta(\cdot, s_1) - \Theta(\cdot, s_2(t_2))] \tilde{\gamma}(t_2)] .
\end{aligned}
\tag{5.3}
$$

Exchanging the roles of s_1 and s_2, we obtain for a.e. $t_1 \in (0, T)$

$$
\begin{aligned}
&- \int_Q q_\varepsilon(s_2, s_1(t_1)) \partial_{t_2} \tilde{\gamma}(t_1) \\
&+ \sum_{j \in \{l, r\}} \int_{Q_j} [\nabla \Phi_j(s_2) + \lambda_j(s_2) g_j] \cdot \nabla [\text{sign}_\varepsilon [\Theta_j(s_2) - \Theta_j(s_1(t_1))] \tilde{\gamma}(t_1))] \\
&\leq \int_Q f(s_2) [\text{sign}_\varepsilon [\Theta(\cdot, s_2) - \Theta(\cdot, s_1(t_1))] \tilde{\gamma}(t_1)] .
\end{aligned}
\tag{5.4}
$$

Integrating (5.3) and (5.4) with respect to $t_2 \in (0, T)$ and $t_1 \in (0, T)$, respectively, adding the resulting equations and exploiting the symmetry of sign_ε, i.e. $\text{sign}_\varepsilon(\sigma) = -\text{sign}_\varepsilon(-\sigma)$ yields

$$
\begin{aligned}
&- \int_{\tilde{Q}} q_\varepsilon(s_1, s_2) \partial_{t_1} \tilde{\gamma} + q_\varepsilon(s_2, s_1) \partial_{t_2} \tilde{\gamma} \\
&\quad + \sum_{j \in \{l, r\}} \int_{\tilde{Q}_j} [\nabla \Phi_j(s_1) - \nabla \Phi_j(s_2)] \cdot \nabla [\text{sign}_\varepsilon [\Theta_j(s_1) - \Theta_j(s_2)] \tilde{\gamma}] \\
&\quad + \sum_{j \in \{l, r\}} \int_{\tilde{Q}_j} [\lambda_j(s_1) - \lambda_j(s_2)] g_j \cdot \nabla [\text{sign}_\varepsilon [\Theta_j(s_1) - \Theta_j(s_2)] \tilde{\gamma}] \\
&\leq \int_{\tilde{Q}} [f(s_1) - f(s_2)] [\text{sign}_\varepsilon [\Theta(\cdot, s_1) - \Theta(\cdot, s_2)] \tilde{\gamma}] .
\end{aligned}
\tag{5.5}
$$

Step 2: The Limit $\varepsilon \to 0$

To treat the first integral on the left-hand side of (5.5), we use the properties of q_ε from Lemma 5.2 to obtain $q_\varepsilon(s_1, s_2), q_\varepsilon(s_2, s_1) \nearrow |s_1 - s_2|$ a.e. on $\tilde{Q}$ as $\varepsilon \to 0$. The boundedness of $|s_1 - s_2|$ and the regularity of $\tilde{\gamma}$ allow to apply Lebesgue's theorem. This yields

$$-\int_{\tilde{Q}} q_\varepsilon(s_1, s_2) \partial_{t_1} \tilde{\gamma} + q_\varepsilon(s_2, s_1) \partial_{t_2} \tilde{\gamma} \xrightarrow[\varepsilon \to 0]{} -\int_{\tilde{Q}} |s_1 - s_2| \left(\partial_{t_1} + \partial_{t_2} \right) \tilde{\gamma}. \qquad (5.6)$$

Concerning the second integral on the left-hand side of (5.5), we compute

$$\begin{aligned}
\int_{\tilde{Q}_j} & [\nabla \Phi_j(s_1) - \nabla \Phi_j(s_2)] \cdot \nabla[\text{sign}_\varepsilon[\Theta_j(s_1) - \Theta_j(s_2)]\tilde{\gamma})] \\
&= \int_{\tilde{Q}_j} \text{sign}_\varepsilon[\Theta_j(s_1) - \Theta_j(s_2)] \left[\nabla \Phi_j(s_1) - \nabla \Phi_j(s_2) \right] \cdot \nabla \tilde{\gamma} \\
&\quad + \int_{\tilde{Q}_j} \text{sign}'_\varepsilon[\Theta_j(s_1) - \Theta_j(s_2)] \left[\nabla \Phi_j(s_1) - \nabla \Phi_j(s_2) \right] \cdot \left[\nabla \Theta_j(s_1) - \nabla \Theta_j(s_2) \right] \tilde{\gamma} \\
&= I_1^\varepsilon + I_2^\varepsilon,
\end{aligned} \qquad (5.7)$$

where we use Corollary 4.5 from Stampacchia's lemma to understand the right-hand side, i.e. $\text{sign}'_\varepsilon(\Theta_j(s_1) - \Theta_j(s_2)) = \frac{1}{\varepsilon} \mathbb{1}_{\{|\Theta_j(s_1) - \Theta_j(s_2)| < \varepsilon\}}$.

For $k \in \{1, 2\}$, we infer from $\Theta_j(s_k) \in L^2(Q)$ that $\Theta_j(s_k) < \infty$ a.e. in $\tilde{Q}$. Consequently, we deduce the pointwise convergence $\text{sign}_\varepsilon(\Theta_j(s_1) - \Theta_j(s_2)) \to \text{sign}(\Theta_j(s_1) - \Theta_j(s_2))$ a.e. in $\tilde{Q}$ as $\varepsilon \to 0$. Using that Θ_j and Φ_j are strictly increasing, thus $\text{sign}(\Theta_j(s_1) - \Theta_j(s_2)) = \text{sign}(\Phi_j(s_1) - \Phi_j(s_2))$, and Lebesgue's dominated convergence theorem, we infer

$$I_1^\varepsilon \xrightarrow[\varepsilon \to 0]{} \int_{\tilde{Q}_j} \text{sign}[\Phi_j(s_1) - \Phi_j(s_2)] \left[\nabla \Phi_j(s_1) - \nabla \Phi_j(s_2) \right] \cdot \nabla \tilde{\gamma}.$$

Concerning I_2^ε, we introduce $E_j^\varepsilon := \tilde{Q}_j \cap \{|\Theta_j(s_1) - \Theta_j(s_2)| < \varepsilon\}$, use Assumption A2.4 and compute

$$\begin{aligned}
I_2^\varepsilon &= \frac{1}{\varepsilon} \int_{E_j^\varepsilon} \nabla[\Phi_j(s_1) - \Phi_j(s_2)] \cdot \nabla[\Theta_j(s_1) - \Theta_j(s_2)]\tilde{\gamma} \\
&= \frac{1}{\varepsilon} \int_{E_j^\varepsilon} (\Lambda_j(\Theta_j(s_1))\nabla[\Theta_j(s_1)] - \Lambda_j(\Theta_j(s_2))\nabla[\Theta_j(s_2)]) \cdot \nabla[\Theta_j(s_1) - \Theta_j(s_2)]\tilde{\gamma} \\
&= \frac{1}{\varepsilon} \int_{E_j^\varepsilon} \Lambda_j(\Theta_j(s_1)) \left| \nabla[\Theta_j(s_1) - \Theta_j(s_2)] \right|^2 \tilde{\gamma} \\
&\quad + \frac{1}{\varepsilon} \int_{E_j^\varepsilon} [\Lambda_j(\Theta_j(s_1)) - \Lambda_j(\Theta_j(s_2))]\nabla[\Theta_j(s_2)] \cdot \nabla[\Theta_j(s_1) - \Theta_j(s_2)]\tilde{\gamma}.
\end{aligned} \qquad (5.8)$$

Particularly, the boundedness of Λ_j implies that the integrals are well-defined. Since Λ_j is nonnegative, the first integral on the right-hand side of (5.8) is estimated by zero

from below. Let L_j denote the Lipschitz constant of Λ_j. On E_j^ε we find the pointwise estimate

$$\frac{1}{\varepsilon} \left|\Lambda_j(\Theta_j(s_1)) - \Lambda_j(\Theta_j(s_2))\right| \mathbb{1}_{\{|\Theta_j(s_1) - \Theta_j(s_2)| < \varepsilon\}} \leq L_j. \tag{5.9}$$

Together with (5.9), Stampacchia's lemma and the notation

$$E_j^{\varepsilon,0} := E_j^\varepsilon \cap \{|\Theta_j(s_1) - \Theta_j(s_2)| \neq 0\}$$

we obtain

$$\left| \frac{1}{\varepsilon} \int_{E_j^\varepsilon} [\Lambda_j(\Theta_j(s_1)) - \Lambda_j(\Theta_j(s_2))] \nabla[\Theta_j(s_2)] \cdot \nabla[\Theta_j(s_1) - \Theta_j(s_2)] \tilde{\gamma} \right|$$

$$\leq \int_{E_j^{\varepsilon,0}} L_j \left|\nabla[\Theta_j(s_2)]\right| \left|\nabla[\Theta_j(s_1) - \Theta_j(s_2)]\right| \tilde{\gamma} = L_j \mu_j(E_j^{\varepsilon,0}) \xrightarrow[\varepsilon \to 0]{} 0,$$

where μ_j is the nonnegative measure generated by $\left|\nabla[\Theta_j(s_2)]\right| \left|\nabla[\Theta_j(s_1) - \Theta_j(s_2)]\right| \tilde{\gamma}$. The convergence is inferred from $E_j^{\varepsilon_1,0} \subset E_j^{\varepsilon_2,0}$ for $\varepsilon_1 < \varepsilon_2$ and $\bigcap_{\varepsilon > 0} E_j^{\varepsilon,0} = \emptyset$ [Rud99, 1.19 and 1.29]. Hence, the previous considerations provide that the second integral on the right-hand side of (5.8) is nonnegative in the limit $\varepsilon \to 0$. Consequently, we obtain

$$\liminf_{\varepsilon \to 0} I_2^\varepsilon \geq 0. \tag{5.10}$$

The third integral one the left-hand side of (5.5) is treated similar to second integral. We compute

$$\int_{\tilde{Q}_j} [\lambda_j(s_1) - \lambda_j(s_2)] g_j \cdot \nabla[\mathrm{sign}_\varepsilon[\Theta_j(s_1) - \Theta_j(s_2)] \tilde{\gamma}]$$

$$= \int_{\tilde{Q}_j} \mathrm{sign}_\varepsilon[\Theta_j(s_1) - \Theta_j(s_2)][\lambda_j(s_1) - \lambda_j(s_2)] g_j \cdot \nabla \tilde{\gamma} \tag{5.11}$$

$$+ \int_{\tilde{Q}_j} \mathrm{sign}_\varepsilon'[\Theta_j(s_1) - \Theta_j(s_2)][\lambda_j(s_1) - \lambda_j(s_2)] g_j \cdot \nabla[\Theta_j(s_1) - \Theta_j(s_2)] \tilde{\gamma}$$

$$= I_3^\varepsilon + I_4^\varepsilon.$$

Concerning I_3^ε we proceed similarly to I_1^ε; we exploit the pointwise a.e. convergence $\mathrm{sign}_\varepsilon(\Theta_j(s_1) - \Theta_j(s_2)) \to \mathrm{sign}(\Theta_j(s_1) - \Theta_j(s_2)) = \mathrm{sign}(s_1 - s_2)$, as $\varepsilon \to 0$, and Lebesgue's dominated convergence theorem to infer

$$I_3^\varepsilon \xrightarrow[\varepsilon \to 0]{} \int_{\tilde{Q}_j} \mathrm{sign}(s_1 - s_2)[\lambda_j(s_1) - \lambda_j(s_2)] g_j \cdot \nabla \tilde{\gamma}.$$

As to I_4^ε, under consideration of Assumption A2.4, we denote the Lipschitz constant of $\lambda_j \circ \Theta_j^{-1}$ by l_j. We obtain (5.9) with Λ_j replaced by $\lambda_j \circ \Theta_j^{-1}$ and L_j by l_j, respectively. Using the notation of the set $E_j^{\varepsilon,0}$ introduced above and Stampacchia's lemma, we deduce

$$|I_4^\varepsilon| \leq \frac{1}{\varepsilon} \int_{E_j^{\varepsilon,0}} l_j |g_j| \left|\nabla[\Theta_j(s_1) - \Theta_j(s_2)]\right| |\tilde{\gamma}| \leq l_j \tilde{\mu}_j(E_j^{\varepsilon,0}) \xrightarrow[\varepsilon \to 0]{} 0,$$

where $\tilde{\mu}_j$ is the nonnegative measure generated by $|\nabla[\Theta_j(s_1) - \Theta_j(s_2)]| \, |\tilde{\gamma}| \, |g_j|$. The convergence is justified as that in I_2^ε.

Concerning the right-hand side of (5.5), we use $\operatorname{sign}_\varepsilon(\Theta_j(s_1) - \Theta_j(s_2)) \to \operatorname{sign}(\Theta_j(s_1) - \Theta_j(s_2)) = \operatorname{sign}(s_1 - s_2)$ almost everywhere on $\tilde{Q}_j$ as $\varepsilon \to 0$ and Lebesgue's dominated convergence theorem to obtain

$$\int_{\tilde{Q}} (f(s_1) - f(s_2)) \operatorname{sign}_\varepsilon(\Theta(s_1) - \Theta(s_2)) \tilde{\gamma} \xrightarrow[\varepsilon \to 0]{} \int_{\tilde{Q}} \operatorname{sign}(s_1 - s_2)(f(s_1) - f(s_2)) \tilde{\gamma}.$$

Collecting the previous estimates, we find

$$\begin{aligned}
&- \int_{\tilde{Q}} |s_1 - s_2| \, (\partial_{t_1} + \partial_{t_2}) \, \tilde{\gamma} \\
&\quad + \sum_{j \in \{l,r\}} \int_{\tilde{Q}_j} \operatorname{sign}(\Phi_j(s_1) - \Phi_j(s_2)) \nabla[\Phi_j(s_1) - \Phi_j(s_2)] \cdot \nabla \tilde{\gamma} \\
&\quad + \sum_{j \in \{l,r\}} \int_{\tilde{Q}_j} \operatorname{sign}(s_1 - s_2)(\lambda_j(s_1) - \lambda_j(s_2)) g_j \cdot \nabla \tilde{\gamma} \\
&\leq \int_{\tilde{Q}} \operatorname{sign}(s_1 - s_2)(f(s_1) - f(s_2)) \tilde{\gamma}.
\end{aligned} \tag{5.12}$$

Step 3: Reducing the doubling of the time variable

Let $\gamma \in C_c^\infty((0,T) \times \mathbb{R}^d)$ be nonnegative and for $\delta > 0$, let $\varphi_\delta \in C_c^\infty(\mathbb{R})$ be a Dirac sequence as introduced in section 1.4. For δ sufficiently small, the function

$$\gamma_\delta(t_1, t_2, x) := \varphi_\delta(t_1 - t_2) \, \gamma\left(\frac{t_1 + t_2}{2}, x\right).$$

is an admissible choice for $\tilde{\gamma}$ in (5.12). Thanks to

$$(\partial_{t_1} + \partial_{t_2}) \gamma_\delta(t_1, t_2, x) = \varphi_\delta(t_1 - t_2) \, \partial_t \gamma\left(\frac{t_1 + t_2}{2}, x\right),$$

$$\nabla \gamma_\delta(t_1, t_2, x) = \varphi_\delta(t_1 - t_2) \, \nabla \gamma\left(\frac{t_1 + t_2}{2}, x\right),$$

the derivatives of φ_δ, which are singular in the limit $\delta \to 0$, cancel. We change the variables in virtue of $t = t_1$ and $\tau = t_1 - t_2$. We find $\tau \in (-T, T)$ and the identity $t - \tau/2 = (t_1 + t_2)/2$. With the notation $w^\tau(t) := w(t - \tau)$, exploiting the compactness of the supports of φ and γ, and Fubini's theorem, we infer

$$\begin{aligned}
\int_{\mathbb{R}} \varphi_\delta(\tau) \Bigg[&- \int_Q |s_1 - s_2^\tau| \, \partial_t \gamma^{\tau/2} - \int_Q \operatorname{sign}(s_1 - s_2^\tau)(f(s_1) - f(s_2^\tau)) \gamma^{\tau/2} \, d\tau \\
&+ \sum_{j \in \{l,r\}} \int_{Q_j} \operatorname{sign}(\Phi_j(s_1) - \Phi_j(s_2^\tau)) \nabla[\Phi_j(s_1) - \Phi_j(s_2^\tau)] \cdot \nabla \gamma^{\tau/2} \\
&+ \sum_{j \in \{l,r\}} \int_{Q_j} \operatorname{sign}(s_1 - s_2^\tau)(\lambda_j(s_1) - \lambda_j(s_2^\tau)) g_j \cdot \nabla \gamma^{\tau/2} \Bigg] \, d\tau \leq 0.
\end{aligned} \tag{5.13}$$

Concerning the limit $\delta \to 0$ in (5.13) we show that the function $G : (-2T, 2T) \to \mathbb{R}$, extended trivially onto $\mathbb{R}$, defined by

$$
\begin{aligned}
G(\tau) := & -\int_Q |s_1 - s_2^\tau| \, \partial_t \gamma^{\tau/2} - \int_Q \text{sign}(s_1 - s_2^\tau)(f(s_1) - f(s_2^\tau))\gamma^{\tau/2} \\
& + \sum_{j \in \{l,r\}} \int_{Q_j} \text{sign}(\Phi_j(s_1) - \Phi_j(s_2^\tau))\nabla[\Phi_j(s_1) - \Phi_j(s_2^\tau)] \cdot \nabla\gamma^{\tau/2} \\
& + \sum_{j \in \{l,r\}} \int_{Q_j} \text{sign}(s_1 - s_2^\tau)(\lambda_j(s_1) - \lambda_j(s_2^\tau))g_j \cdot \nabla\gamma^{\tau/2} =: \sum_{k=1}^4 G_k(\tau)
\end{aligned}
\tag{5.14}
$$

is continuous in $\tau = 0$, where

$$
\begin{aligned}
G(0) = & -\int_Q |s_1 - s_2| \, \partial_t \gamma - \int_Q \text{sign}(s_1 - s_2)(f(s_1) - f(s_2))\gamma \\
& + \sum_{j \in \{l,r\}} \int_{Q_j} \text{sign}(\Phi_j(s_1) - \Phi_j(s_2))\nabla[\Phi_j(s_1) - \Phi_j(s_2)] \cdot \nabla\gamma \\
& + \sum_{j \in \{l,r\}} \int_{Q_j} \text{sign}(s_1 - s_2)(\lambda_j(s_1) - \lambda_j(s_2))g_j \cdot \nabla\gamma.
\end{aligned}
\tag{5.15}
$$

From the continuity of translation operator in $L^p(Q)$ for $1 \leq p < \infty$, cf. e.g. [Alt06, 2.14], and the Lipschitz continuity of the modulus function, we infer

$$
G_1(\tau) = -\int_Q |s_1 - s_2^\tau| \, \partial_t \gamma^{\tau/2} \xrightarrow[\tau \to 0]{} -\int_Q |s_1 - s_2| \, \partial_t \gamma.
$$

For $h \in C([0,1])$ define the function $F_h : [0,1]^2 \to \mathbb{R}$ in virtue of

$$
F_h(u,v) := \text{sign}(u-v)(h(u) - h(v)).
$$

Since F_h is continuous, we deduce from Assumption A2.2 the continuity of G_2 and G_4 in zero. In particular the integrands in $G_2(\tau)$ and $G_4(\tau)$ converge pointwise a.e. to the integrands of $G_2(0)$ and $G_4(0)$, respectively. Since the occuring functions are bounded, we apply Lebesgue's dominated convergence theorem and obtain $G_k(\tau) \to G_k(0)$ as $\tau \to 0$ for $k \in \{2, 4\}$.

The limiting relation for G_3 is more intricate to handle. To simplify notation and concentrate on the essential parts we exploit again the monotonicity of Φ_j and use $\text{sign}(\Phi_j(s_1) - \Phi_j(s_2)) = \text{sign}(s_1 - s_2)$. Realizing that $h(s^\tau) = h(s)^\tau$ for any time independent function h, we write

$$
\begin{aligned}
& \text{sign}(s_1 - s_2^\tau)\nabla(\Phi_j(s_1) - \Phi_j(s_2)^\tau) \cdot \nabla\gamma^{\tau/2} - \text{sign}(s_1 - s_2)\nabla(\Phi_j(s_1) - \Phi_j(s_2)) \cdot \nabla\gamma \\
& = [\text{sign}(s_1 - s_2^\tau) - \text{sign}(s_1 - s_2)]\nabla(\Phi_j(s_1) - \Phi_j(s_2)) \cdot \nabla\gamma \\
& \quad + \text{sign}(s_1 - s_2^\tau)[\nabla(\Phi_j(s_1) - \Phi_j(s_2)^\tau) \cdot \nabla\gamma^{\tau/2} - \nabla(\Phi_j(s_1) - \Phi_j(s_2)) \cdot \nabla\gamma] \\
& =: T_1^\tau + T_2^\tau.
\end{aligned}
$$

For T_2^τ, we exploit the Cauchy-Schwarz inequality and a standard argument to deduce

$$\int_{Q_j} |T_2^\tau| \le \left\| \nabla (\Phi_j(s_1) - \Phi_j(s_2^\tau)) \right\|_2 \left\| \nabla (\gamma - \gamma^{\tau/2}) \right\|_2 \tag{5.16}$$
$$+ \left\| \nabla \gamma \right\|_2 \left\| \nabla (\Phi_j(s_2) - \Phi_j(s_2^\tau)) \right\|_2 \to 0,$$

due to the uniform continuity of $\nabla \gamma$ and the continuity of the translation operator.

Concerning T_1^τ, we define

$$S_j := \{s_1 = s_2\} \cap Q_j,$$

realize that $\text{sign}(s_1 - s_2) = 0$ on S_j and use Stampacchia's lemma to compute

$$\int_{S_j} \left| [\text{sign}(s_1 - s_2^\tau) - \text{sign}(s_1 - s_2)] \right| \left| \nabla (\Phi_j(s_1) - \Phi_j(s_2)) \right| |\nabla \gamma|$$
$$\le \int_{S_j} \left| \nabla (\Phi_j(s) - \Phi_j(s_2)) \right| |\nabla \gamma| = 0. \tag{5.17}$$

On the complement $S_j^{\complement}$ of S_j, we use that $s_2^\tau \to s_2$ in $L^2(Q_j)$. Thus, for any sequence $\tau \to 0$, we obtain the convergence $s_2^\tau \to s_2$ a.e. on $S_j^{\complement}$. This implies a.e. on $S_j^{\complement}$ the convergence $|\text{sign}(s_1 - s_2^\tau) - \text{sign}(s_1 - s_2)| \to 0$ as $\tau \to 0$, up to a subsequence. Lebesgue's dominated convergence theorem yields the strong convergence

$$\int_{S_j^{\complement}} [\text{sign}(s_1 - s_2^\tau) - \text{sign}(s_1 - s_2)] \nabla (\Phi_j(s_1) - \Phi_j(s_2)) \cdot \nabla \gamma \xrightarrow[\tau \to 0]{} 0 \tag{5.18}$$

for this subsequence. With a standard argument [DiB02, I.1.1], we obtain the convergence in (5.18) for any sequence $\tau \to 0$. Combining (5.17) and (5.18), we infer

$$\limsup \int_{Q_j} |T_1^\tau| = 0.$$

Summarizing, we have shown the continuity of $G_k(\tau)$ in $\tau = 0$ for $k \in \{1, \dots, 4\}$ with $G(0)$ given by (5.15).

With the continuity of G in zero, $G \in L^1(\mathbb{R})$ and the properties of φ_δ, we find

$$\lim_{\delta \to 0} \int_{\mathbb{R}} \frac{1}{\delta} \varphi \left(\frac{\tau}{\delta} \right) G(\tau) \, d\tau = \lim_{\delta \to 0} \int_{\mathbb{R}} \frac{1}{\delta} \varphi \left(\frac{\tau}{\delta} \right) G(0) \, d\tau = G(0). \tag{5.19}$$

From (5.13) we derive $G(0) \le 0$, conclude the Kato inequality (5.2) and the proof is complete.

5.3 Extension of the Kato Inequality

In this section, we provide tools to close the gap between the Kato inequality (5.2) and (2.14) from Theorem 2.4. In Proposition 5.8 we show that $t = 0$ is a right Lebesgue point of s with $s(0) = s_0$. To prove Theorem 2.4 in the next section, this is used to dispose the restriction of γ vanishing in $t = 0$ in the Kato inequality (5.2) and yields the extended Kato inequality (2.14). To obtain the generalized L^1-contraction from the extended Kato inequality, we require the auxiliary Lemma 5.5 and a Gronwall inequality stated in Lemma 5.6.

Lemma 5.5 (Weak differential inequality). *Let functions $u, \theta \in L^1((0, T))$ and constants $u_0 \in \mathbb{R}$ and $L, K \geq 0$ be given. If*

$$- \int_0^T \alpha'(t)(u(t) - u_0) \, dt \leq \int_0^T \alpha(t)(Lu(t) + K\theta(t)) \, dt \tag{5.20}$$

for every nonnegative $\alpha \in C_c^\infty((-\infty, T))$, then there holds

$$u(t^*) \leq u_0 + \int_0^{t^*} Lu(s) + K\theta(s) \, ds \tag{5.21}$$

for every Lebesgue point t^* of $u(t)$ and thus for a.e. $t^* \in (0, T)$.

Proof. Let $t^* \in (0, T)$ be a Lebesgue point of u and let $\eta \in C^\infty(\mathbb{R})$ be non-decreasing and such that $\eta^+(\sigma) = 0$ for $\sigma \leq 0$ and $\eta(\sigma) = 1$ for $\sigma \geq 1$. For $\delta > 0$, we define the sequence $\tilde\eta_\delta(\sigma) = \eta((-\sigma + t^* + \delta)/\delta)$. With $C_\eta := \max |\eta'|$, we infer $|(\tilde\eta_\delta)'| \leq C_\eta/\delta$. Let $\beta \in C_c^\infty(\mathbb{R})$ be nonnegative and such that $\beta \equiv 1$ on $[0, T]$. Then $\alpha = \tilde\eta_\delta\beta$ is admissible in (5.20) for δ sufficiently small. We compute

$$- \int_t^{t^*+\delta} (\tilde\eta_\delta)'(s)u(s) \, ds \leq u_0\tilde\eta_\delta(0) + \int_0^{t^*+\delta} \tilde\eta_\delta(s)(Lu(s) + K\theta(s)) \, ds$$

$$= u_0 + \int_0^{t^*} (Lu(s) + K\theta(s)) \, ds + \int_{t^*}^{t^*+\delta} \tilde\eta_\delta(s)(Lu(s) + K\theta(s)) \, ds \tag{5.22}$$

$$\xrightarrow[\delta \to 0]{} u_0 + \int_0^{t^*} Lu(s) + K\theta(s) \, ds$$

To pass with $\delta \to 0$ on the left-hand side of (5.22), we claim that $u(t^*)$ is the desired limit. Indeed, we find

$$\left| \int_{t^*}^{t^*+\delta} -(\tilde\eta_\delta)'(s)u(s) \, ds - u(t^*) \right| \leq \frac{C_\eta}{\delta} \int_{t^*}^{t^*+\delta} |u(s) - u(t^*)| \, ds$$

$$\leq 2C_\eta \fint_{B_\delta(t^*)} |u(s) - u(t^*)| \, ds \xrightarrow[\delta \to 0]{} 0$$

since t^* is a Lebesgue point of u and (5.21) is proven. □

To obtain the L^1-contraction property, we need a simple version of Gronwall's inequality. In [Eva98, Appendix B.2] we find some variants of Gronwall's inequality and adapt the proof to our needs.

Lemma 5.6 (A Gronwall Inequality). *Let $u(t) \in L^1(0,T)$ with $u(t) \geq 0$ for almost every $t \in (0,T)$ and $u_0, L \geq 0$ be given. Furthermore, assume that*

$$u(t) \leq u_0 + L \int_0^t u(s) \, ds \quad \text{for a.e. } t \in (0,T). \tag{5.23}$$

Then there holds

$$u(t) \leq e^{Lt} u_0 \tag{5.24}$$

for almost every $t \in (0,T)$.

Proof. We define the absolutely continuous function $\eta(t) = \int_0^t u(s) \, ds$. Hence, for a.e. $\tau \in [0,T]$, we infer

$$\partial_s \left(\eta(s) e^{-Ls} \right) = e^{-Ls} \left(\eta'(s) - L\eta(s) \right) \leq e^{-Ls} u_0$$

using (5.23) in the last inequality. Integration over $s \in [0,t]$, multiplication by e^{Lt} and using that $\eta(0) = 0$ yields

$$\eta(t) \leq e^{Lt} \left(u_0 \int_0^t e^{-Ls} \, ds \right) = u_0 e^{Lt} \frac{1}{L} \left(1 - e^{-Lt} \right)$$

for a.e. $t \in [0,T]$. Inserting this in (5.23) yields

$$u(t) \leq u_0 + L \int_0^t u(s) \, ds \leq u_0 e^{Lt} \left(e^{-Lt} + 1 - e^{-Lt} \right) = u_0 e^{-Lt}$$

for a.e. $t \in [0,T]$ $\qquad\square$

In the following proposition, we use the concept of essential limit, which is the usual limit except for a set of zero measure.

Definition 5.7 (Essential limit). *Let $g \in L^1(0,T)$. We say that $c \in \mathbb{R}$ is the essential limit of g in zero, i.e.*

$$c := \operatorname{ess} \lim_{t \downarrow 0} g(t)$$

if there exists a subset $E \subset (0,T)$ with $|E| = 0$ such that

$$c = \lim_{\substack{t \downarrow 0, \\ t \notin E}} g(t).$$

Proposition 5.8 (Essential continuity in $t = 0$)**.** *Let s be a weak solution of* (TDR) *in the sense of Definition 2.2 with initial data $0 \leq s_0(x) \leq 1$ for a.e. $x \in \Omega$. Then there holds*

$$\operatorname*{ess\,lim}_{t \downarrow 0} \int_\Omega |s(t) - s_0| = 0. \tag{5.25}$$

Proof. We aim to use Lemma 5.3 with $v = s_0$. Since v has to fulfill $\Theta(v) \in H^1(\Omega)$ and $\Theta(s) - \Theta(v) \in L^2(0, T; V)$, we need to approximate $v = s_0$. Due to the density of $C_c^\infty(\Omega_j)$ in $L^1(\Omega_j)$, we find two sequences $C_c^\infty(\Omega_j) \ni v_{n,j} \to s_0 \in L^1(\Omega_j)$ for $j \in \{l, r\}$. The sequence $v_n := v_{n,l} \mathbb{1}_{\Omega_l} + v_{n,r} \mathbb{1}_{\Omega_r}$ is such that $v_n \to s_0$ in $L^1(\Omega)$, $v_n|\Gamma = 0$, $\Theta(v_n) \in H_0^1(\Omega)$. Moreover, extending $\Theta(v_n)$ constantly in time, we obtain $\Theta(s) - \Theta(v_n) \in L^2(0, T; V)$.

With the choice $v = v_n$ in (5.1), we fix sequences $\varepsilon = \varepsilon_k \to 0$ and $v_n \to s_0$ as $k, n \to \infty$, and define the functions

$$h_{k,n}(t) := \int_\Omega q_{\varepsilon_k}(s(t), v_n), \quad h_n(t) := \int_\Omega |s(t) - v_n| \text{ and the sets}$$

$$E_{k,n} := \{t \in [0, T] \mid t \text{ is not a Lebesgue point of } h_{k,n}(t)\}$$

for $k, n \in \mathbb{N}$. Since $|E_{k,n}| = 0$ for $k, n \in \mathbb{N}$, the countable union $E := \bigcup_{k=1}^\infty \bigcup_{n=1}^\infty E_{k,n}$ also fulfills $|E| = 0$. Consequently, any $t \notin E$ is a Lebesgue point of $h_{k,n}$ for every $k, n \in \mathbb{N}$. Choosing $\gamma(x, t) = \alpha(t)\beta(x)$ with $\beta \equiv 1$ on Ω and $\alpha \in C_c^\infty(\mathbb{R})$, α nonnegative, leads in equation (5.1) to the weak differential inequality

$$-\int_0^T \int_\Omega [q_{\varepsilon_k}(s, v_n) - q_{\varepsilon_k}(s_0, v_n)]\alpha'(t) \leq \int_0^T \alpha(t)\theta_{k,n}(t). \tag{5.26}$$

for some $\theta_{k,n}(t) \in L^1(0, T)$. Lemma 5.5 yields for every Lebesgue point t^* and consequently for every $t^* \notin E$ with $L = 0$ and $K = 1$ the relation

$$\int_\Omega q_{\varepsilon_k}(s(t^*), v_n) \leq \int_\Omega q_{\varepsilon_k}(s_0, v_n) + \int_0^{t^*} \theta_{k,n}(t). \tag{5.27}$$

Since $\theta_{k,n} \in L^1(0, T)$, we conclude

$$\limsup_{t \downarrow 0, t \notin E} \int_\Omega q_{\varepsilon_k}(s(t), v_n) \leq \int_\Omega q_{\varepsilon_k}(s_0, v_n). \tag{5.28}$$

Concerning the right-hand side of (5.28), we exploit item 2 of Lemma 5.2 and the strong convergence $v_n \to s_0$ to find

$$\limsup_{t \downarrow 0, t \notin E} \int_\Omega q_{\varepsilon_k}(s(t), v_n) \leq \int_\Omega q_{\varepsilon_k}(s_0, v_n) \leq \int_\Omega |s_0 - v_n| \,. \to 0 \tag{5.29}$$

We emphasize that the set E is independent of n. Concerning the left hand side of (5.28), we want to pass with the limit $k \to \infty$ first. With the uniform convergence $q_{\varepsilon_k,j} \to q_j$ from item 4 of Lemma 5.2, we find

$$
\left| \int_{\Omega} |s(t) - v_n| - q_{\varepsilon_k}(s(t), v_n) \right| = \sum_{j \in \{l,r\}} \left| \int_{\Omega_j} |s(t) - v_{n,j}| - q_{\varepsilon_k,j}(s(t), v_{n,j}) \right|
$$
$$
\leq \sum_{j \in \{l,r\}} |\Omega_j| \, \| |s(t) - v_{n,j}| - q_{\varepsilon_k,j}(s(t), v_{n,j}) \|_{\infty} \to 0
\tag{5.30}
$$

as $k \to \infty$ uniformly in $t \in (0,T) \setminus E$ and $n \in \mathbb{N}$. Due to the uniform convergence, we find for $\tilde{\varepsilon} > 0$ a $k_0 > 0$ such that for every $K \geq k_0$ and every $t \in (0,T) \setminus E$ there holds $|h_{K,n}(t) - h_n(t)| < \tilde{\varepsilon}$ and consequently $h_n(t) \leq h_{K,n}(t) + \tilde{\varepsilon}$. Since (5.29) holds for every k and since E was chosen independent of k and n, we find

$$
\lim_{n \to \infty} \limsup_{t \downarrow 0, t \notin E} h_n(t) \leq \lim_{n \to \infty} \limsup_{t \downarrow 0, t \notin E} h_{K,n}(t) + \tilde{\varepsilon} \leq \tilde{\varepsilon}.
$$

This implies

$$
\lim_{n \to \infty} \limsup_{t \downarrow 0, t \notin E} \int_{\Omega} |s(t) - v_n| \leq 0.
\tag{5.31}
$$

Combining (5.31) and the convergence $v_n \to s_0$ in $L^1(\Omega)$, using that the latter convergence is independent of $t \in (0,T) \setminus E$, that E is independent of n and exploiting the triangle inequality, we find

$$
\limsup_{t \downarrow 0, t \notin E} \int_{\Omega} |s(t) - s_0| \leq \lim_{n \to \infty} \limsup_{t \downarrow 0, t \notin E} \int_{\Omega} |s(t) - v_n| + \lim_{n \to \infty} \int_{\Omega} |v_n - s_0| \leq 0.
\tag{5.32}
$$

Hence, observing that

$$
0 = \limsup_{t \downarrow 0, t \notin E} \int_{\Omega} |s(t) - s_0| = \operatorname{ess\,lim}_{t \downarrow 0} \int_{\Omega} |s(t) - s_0|
$$

concludes the proof. $\qquad\square$

5.4 Proof of Theorem 2.4

To prove Theorem 2.4, we perform two steps. Firstly, we derive the extended Kato inequality (2.14) from Propositions 5.4 and 5.8. Secondly, we use Gronwall argument to infer (2.15), i.e. the generalized L^1-contraction property.

Proof of Theorem 2.4. We start from the Kato-inequality (5.2). An approximation yields that (5.2) holds for $\gamma \in H^1(\Omega_T)$ with $\gamma(0) = \gamma(T) = 0$. Specifically, the left hand side of (5.2) defines a bounded linear functional on a dense subset of $\{v \in H^1(\Omega_T) : v(T,x) = $

$v(0, x) = 0\}$. We define the function $\eta : \mathbb{R} \to \mathbb{R}$ with $\eta(s) = 0$ for $s \leq 0$, $\eta(s) = 1$ for $s \geq 1$ and linear in between. With $\eta_\delta(s) = \eta(s/\delta)$ and $\theta \in C_c^\infty((-\infty, T) \times \mathbb{R}^d)$, we find the admissibility of $\gamma = \theta\eta_\delta$ in (5.2). By Lebesgue's dominated convergence theorem we infer

$$\int_{Q_j} \eta_\delta \, \text{sign}(\Phi_j(s_1) - \Phi_j(s_2)) \nabla[\Phi_j(s_1) - \Phi_j(s_2)] \cdot \nabla\theta$$

$$\xrightarrow[\delta \to 0]{} \int_{Q_j} \text{sign}(\Phi_j(s_1) - \Phi_j(s_2)) \nabla[\Phi_j(s_1) - \Phi_j(s_2)] \cdot \nabla\theta,$$

$$\int_{Q_j} \eta_\delta \, \text{sign}(s_1 - s_2)[\lambda_j(s_1) - \lambda_j(s_2)]g_j \cdot \nabla\theta$$

$$\xrightarrow[\delta \to 0]{} \int_{Q_j} \text{sign}(s_1 - s_2)[\lambda_j(s_1) - \lambda_j(s_2)]g_j \cdot \nabla\theta$$

for $j \in \{l, r\}$, as well as

$$\int_Q \eta_\delta \, \text{sign}(s_1 - s_2)[f(s_1) - f(s_2)]\theta \xrightarrow[\delta \to 0]{} \int_Q \text{sign}(s_1 - s_2)[f(s_1) - f(s_2)]\theta.$$

Concerning the integral containing the time derivative, we compute

$$\int_Q |s_1 - s_2| \, \partial_t(\eta_\delta\theta) = \int_Q |s_1 - s_2| \eta_\delta \partial_t\theta + \int_Q |s_1 - s_2| \theta\partial_t\eta_\delta$$

$$\leq \int_Q |s_1 - s_2| \eta_\delta \partial_t\theta + \int_0^\delta \int_\Omega \frac{1}{\delta}(|s_1 - s_{0,1}| + |s_{0,1} - s_{0,2}| + |s_2 - s_{0,2}|)\theta$$

$$\leq \int_Q |u| \, \eta_\delta\partial_t\theta + \frac{1}{\delta}\int_0^\delta \int_\Omega |s_{0,1} - s_{0,2}| \, \theta$$

$$+ \frac{C}{\delta}\int_0^\delta \int_\Omega (|s_1 - s_{0,1}| + |s_2 - s_{0,2}|)\theta$$

$$\xrightarrow[\delta \to 0]{} \int_\Omega |s_{0,1} - s_{0,2}| \, \theta(0) + \int_Q |s_1 - s_2| \, \partial_t\theta = \int_Q (|s_1 - s_2| - |s_{0,1} - s_{0,2}|)\partial_t\theta.$$

The convergence in the last step follows from the continuity of θ in zero and from Proposition 5.8. More precisely, we argue similar to (4.39) and obtain

$$\operatorname*{ess\,lim}_{t\downarrow 0} \int_\Omega |s_k(t) - s_{0,k}| = \lim_{\delta\to 0} \fint_0^h \int_\Omega |s_k(t) - s_{0,k}|$$

for $k = 1, 2$. Collecting the previous identities we infer (2.14).

To prove the L^1-contraction property (2.15), we choose $\gamma(t, x) = \alpha(t)\psi(x)$ with nonnegative $\alpha \in C_c^\infty((-\infty, T))$ and nonnegative $\psi \in C_c^\infty(\mathbb{R}^d)$ as a test function in the extended Kato-inequality (2.14). We even choose $\psi \equiv 1$ on Ω, use the properties of f to infer that $\text{sign}(s_1 - s_2)(f(s_1) - f(s_2)) \leq L |s_1 - s_2|$ and derive

$$\int_0^T \alpha'(t) \int_\Omega (|s_{0,1} - s_{0,2}| - |s_1(t) - s_2(t)|) \, dt \leq \int_0^T L\alpha(t) \int_\Omega |s_1(t) - s_2(t)| \, dt. \quad (5.33)$$

Application of Lemma 5.5 and the Gronwall inequality 5.6 concludes the proof. $\qquad \square$

5.5 Discussion of the Proof and Outlook

The presented proof also works in the case without an interface. In particular, the proof of the Kato inequality is more direct and we do not need the particular treatment of the integral I_2^ε in (5.8). In that situation, we obtain directly the expression

$$I_2^\varepsilon = \frac{1}{\varepsilon} \int_{E_j^\varepsilon} |\nabla[\Phi(s_1) - \Phi(s_2)]|^2 \, \tilde{\gamma} \geq 0.$$

Particularly, the result in the case without interfaces is a special case of [Ott95]. There more general elliptic are terms are considered.

In Cancès article [Can08] a specialized variant of Lemma 4.36 is stated implicitly in the proof of Theorem 3.1. In addition, the L^1-contraction is shown exploiting time continuity of solutions. This is a consequence of the Lipschitz continuity of the Kirchhoff transform $\Phi_j(s)$ and was shown by Cancès and Gallouët in [CG11].

In our setting Φ_j is not necessarily Lipschitz continuous. Hence, we prove the L^1-contraction by a different argument that is strongly inspired by [Ott95]. Of outmost importance is item 4 of Lemma 5.2, which is the uniform convergence of $q_\varepsilon \to q$. For bounded Θ and Φ as in Assumption A2.3, this convergence is immediate as long as there is only a finite amount of interfaces. We emphasize, that Dini's theorem is exploited to obtain the uniform convergence and no further assumptions on the growth of e.g. $\Theta_j(s)$ for $s \to 1$ are necessary.

The strict monotony of Θ_j is necessary to exploit the identities $\text{sign}_\varepsilon(\Theta_j(s_1) - \Theta_j(s_2)) \to \text{sign}(s_1 - s_2)$ and to invert Θ_j, which is used to exploit Assumption A2.4. We emphasize that the Lipschitz continuity of Λ_j is essential to obtain the Kato inequality. For example, the conclusion starting from (5.9) seems to be impossible if Λ_j merely Hölder continuous.

Similarly to [Can08], the L^1-contraction result we provide here can also be obtained for Ω divided by several interfaces into n subdomains. Another point of interest is to provide an existence result for (TDR) for degenerate capillary transformations as in Assumption A2.3. As far as we know, such a result is not available in the literature. However, due to the integration by parts formulae from chapter 4.4 there are several possibilities to tackle this problem. One could for example attempt to proceed with the idea introduced in [AL83] or start from [Can08] and regularize the pressures. However, the latter approach still enforces that $\lambda_j(1) = 0$. Hence, showing existence using a regularization argument, in the situation that $\lambda_j(1) \neq 0$ and $p_{c,j}$ is bounded, should be viable. In addition, the definition of Θ_j is not fixed. Different choices of Θ_j could make a modification of the proof in [Ott95] possible and lead to L^1-contraction results in the case of multivalued capillary pressures.

6 Local Hölder Continuity for the Two-Phase Flow Problem

In this chapter, (s, p) is always a weak solution of the two-phase problem (TP1) in the sense of Definition 3.5. We show that the saturation s is locally Hölder continuous. For convenience, we usually suppress the x-dependencies of the occurring functions.

A standard approach to obtain (local) Hölder continuity of a function $u : \mathbb{R}^d \to \mathbb{R}$ is to show that the oscillation of u on balls of radius $R > 0$ is proportional to a positive power of R. The oscillation of u is defined as the difference of its supremum and infimum on a given set. For example, the estimate

$$|u(x) - u(y)| \leq \operatorname*{osc}_{B_R(x)} u \leq CR^\alpha$$

implies local Hölder continuity of u immediately. Here, $x, y \in \mathbb{R}^d$, $C > 0$ and $\alpha \in (0, 1)$, and the estimate holds for every $B_R(x) \subset \Omega$ in some bounded open set $\Omega \subset \mathbb{R}^d$.

If it is possible to quantitatively measure the decrease of oscillation of u on nested and shrinking balls with the same center, then also Hölder continuity can be obtained. For example, if an estimate of the type

$$\operatorname*{osc}_{B_R(x)} u \leq (1 - \delta) \operatorname*{osc}_{B_{4R}(x)} u$$

holds for $0 < \delta < 1$ independent of x, then (local) Hölder continuity can be obtained by iteration argument.

The method of intrinsic scaling implements the iteration idea on certain cylinders that reflect the structure of the differential equation. Particularly, we consider cylinders that are related to the standard parabolic cylinders defined below and reflect the degeneracy of the equation. These cylinders depend on the oscillation of the solution itself, which makes them *intrinsic*. However, during the upcoming proofs we may transform the cylinders to the standard parabolic ones. To describe the method precisely, it is necessary to investigate its technical implementation. Summarizing the idea of the method in one sentence, we quote from [Urb08, p. 6]:

> The punch line of the theory is that *the equation behaves, in its own geometry, like the heat equation.*

This chapter is structured as follows. In section 6.1, we provide notation and some technical results needed later. Particularly, we infer from elliptic regularity theory that the pressure p is locally Hölder continuous. In section 6.2, we state the main proposition to infer the Hölder continuity. The proof of the main proposition unfolds along two alternatives, which are executed in sections 6.3 and 6.4. The proof is concluded in section 6.5. From there on, we prove Theorem 3.12 in the same section. In section 6.6, we discuss the literature and provide an outlook towards extensions of the local result.

6.1 Preliminaries

We define the parabolic cylinders and refer to Figure 6.1, where the cylinder $(x_0, t_0) + Q(\rho, \tau)$ is depicted.

Definition 6.1 (Parabolic Cylinders)**.** *Let $\rho, \tau, \theta > 0$ and $(x_0, t_0) \in \mathbb{R}^{d+1}$ be given. We define the cylinders*

$$Q(\rho, \tau) = B_\rho \times (-\tau, 0),\ (x_0, t_0) + Q(\rho, \tau) = B_\rho(x_0) \times (t_0 - \tau, t_0),$$
$$Q_\rho = Q(\rho, \rho^2)\ and\ Q_\rho(\theta) = Q(\rho, \theta\rho^2).$$

A cylinder of the type Q_ρ is often called standard parabolic cylinder and reflects the natural homogeneity between the space and time variables for the heat equation: For any solution $u(x, t)$ to the heat equation, the rescaled function $u(\varepsilon x, \varepsilon^2 t)$, $\varepsilon \in \mathbb{R}$, is again a solution to the heat equation, i.e. the equation remains invariant under similarity transformations of the space and time variables that leave the ratio $|x|^2 / t$ constant. In general, such an invariance is not recovered for degenerate parabolic equations. In particular, such a scaling may result in a factor that depends on the solution itself as is pointed out in [Urb08, section 3.1] for the parabolic p-Laplace equation.

Lemma 6.2 (Fast geometric convergence [DiB93, I.4.Lemma 4.1])**.** *Let X_n be a sequence of nonnegative real numbers, let $b > 1$ and $C, \alpha > 0$ be such that the recurrence relation*

$$X_{n+1} \leq Cb^n X_n^{1+\alpha} \tag{6.1}$$

holds. If

$$X_0 \leq C^{-1/\alpha} b^{-1/\alpha^2} \tag{6.2}$$

then $X_n \to 0$ as $n \to \infty$.

Since there seems to be no complete proof in the literature and the constant C is usually stated to be larger than 1, we provide a proof.

Proof. To prove the Lemma, it suffices to show that

$$X_n \leq \frac{X_0}{b^{\frac{n}{\alpha}}} \tag{6.3}$$

holds for $1 \leq n \in \mathbb{N}$ We show inequality (6.3) by induction. For the base case $n = 1$, we infer

$$X_1 \leq Cb^0 X_0^{1+\alpha} \leq X_0 \left(Cb^0 \left(C^{-\frac{1}{\alpha}} b^{-\frac{1}{\alpha^2}} \right)^\alpha \right) = \frac{X_0}{b^{\frac{1}{\alpha}}}.$$

For the inductive step, we assume that (6.3) holds for n. To show that the relation also holds for $n + 1$, we compute

$$X_{n+1} \leq Cb^n X_n^{1+\alpha} \leq Cb^n \left(\frac{X_0}{b^{\frac{n}{\alpha}}} \right)^{1+\alpha} \leq X_0 Cb^{-\frac{n}{\alpha}} X_0^\alpha \leq \frac{X_0}{b^{\frac{n+1}{\alpha}}}$$

and conclude the proof.

$\square$

From elliptic theory, in particular from [GT98, Theorem 8.24], we infer the local Hölder regularity of the pressure

Theorem 6.3 (Hölder Regularity of the pressure p)**.** *Let Assumptions A3.1, A3.2, A3.3 and A3.5 hold and let (s, p) be a weak solution of* (TP1) *in the sense of Definition 3.5. Then $p(t)$ is locally Hölder continuous in Ω uniformly in $t \in (0, T)$. More precisely, for C_* and C^* from Assumption A3.7 and some set $K \subset\subset \Omega$, there exist $\gamma_p > 0$ and $\beta \in (0, 1)$ independent of t and depending only the data and the distance of K to $\partial\Omega$, such that*

$$\|p(t)\|_{C^\beta(\overline{K})} \leq \gamma_p \left(1 + \|p\|_{L^2(\Omega)} \right) \tag{6.4}$$

for almost every $t \in (0, T)$.

Remark 6.4. We emphasize that only the bounds on κ, λ and E from Assumptions A3.2, A3.3 and A3.5 enter in the statement of the last result. Those bounds are contained in the definitions of C_* and C^*. Furthermore, due to Assumption A3.7, the Dirichlet data p_D are also estimated by C^*. Thus, we can estimate the $L^2(\Omega)$ of $p(t)$ only in terms of the data. To see this, use $p - p_D$ as a test function in (3.22). Absorption yields that $\|\nabla p(t)\|_2 \leq C$ for some C depending only on the data. Then, an application of Poincaré's inequality provides the estimate $\|p(t)\|_2 \leq C$ for some different C depending only on the data.

After a redefinition of γ_p, we find that equation (6.4) reads

$$|p(x, t) - p(y, t)| \leq \gamma_p |x - y|^\beta \text{ for every } x, y \in \overline{K} \text{ and } \|p(t)\|_{L^\infty(\overline{K})} \leq \gamma_p \tag{6.5}$$

for almost every $t \in (0, T)$. Consequently, the Hölder norm of $p(t)$ on a given set $K \subset\subset \Omega$ can be determined only in terms of the data and the distance from K to $\partial\Omega$ uniformly in t. $\triangleleft$

Using (6.5), the redefined γ_p and the properties on E from equation (3.29) we obtain the following Caccioppoli estimate.

Proposition 6.5 (Caccioppoli estimate of p)**.** *Let Assumptions A3.1–A3.7 hold and let (s, p) be a weak solution of* (TP1) *in the sense of Definition 3.5. For any open set $K \subset\subset \Omega$ with $\mathrm{diam}(K) \leq 2r$, there is a constant $C_p \geq 1$ that depends only the data and the distance of K to $\partial\Omega$, such that*

$$\int_K |\nabla p(t)|^2 \, f^2 \leq C_p \left(r^{2\beta} \int_K |\nabla f|^2 + \int_K f^2 \right), \tag{6.6}$$

for any $f \in H_0^1(K)$ and almost every $t \in (0, T)$. Particularly, C_p is independent of t.

Proof. Fix $x_0 \in K$ and use $w = [p(x, t) - p(x_0, t)]f^2(x)$ as a test function in equation (3.22). Using the definition of C_* and C^*, particularly using $|E(x, s)| \leq C^*$ for every $x \in \overline{\Omega}$ and $s \in [0, 1]$, we obtain

$$C_*^2 \int_K |\nabla p(x, t)|^2 \, f^2(x) \, \mathrm{d}x \leq (C^*)^2 \int_K |\nabla p(x, t)| \, |p(x, t) - p(x_0, t)| \left| \nabla f^2(x) \right| \, \mathrm{d}x \\ + (C^*)^2 \int_K \left| \nabla \left([p(x, t) - p(x_0, t)]f^2(x) \right) \right| \, \mathrm{d}x$$

for a.e. $t \in (0, T)$. For $\varepsilon > 0$ and using Cauchy's inequality, we compute

$$C_*^2 \int_K |\nabla p(x, t)|^2 \, f^2(x) \, \mathrm{d}x \\ \leq 2\varepsilon \int_K |\nabla p(x, t)|^2 \, |f(x)|^2 \, \mathrm{d}x + \left(\frac{(C^*)^4}{4\varepsilon} + (C^*)^2 \right) \int_K f^2(x) \, \mathrm{d}x \\ + \left(\frac{(C^*)^4}{4\varepsilon} + (C^*)^2 \right) \int_K |p(x, t) - p(x_0, t)|^2 \, |\nabla f(x)|^2 \, \mathrm{d}x \tag{6.7}$$

for a.e. $t \in (0, T)$. We absorb the first integral on the right-hand side of (6.7) choosing $4\varepsilon = C_*^2$. To fill the gap between (6.7) and (6.6), we exploit Theorem 6.3. More precisely, we use (6.5) and and $|p(x, t) - p(x_0, t)| \leq \gamma_p |x - x_0|^{2\beta} \leq \gamma_p(2r)^{2\beta}$. Thus, we find a constant C_p depending on the same quantities as γ_p and β such that (6.6) holds. $\qquad\square$

The following Lemma is usually referred to as DeGiorgi's lemma.

Lemma 6.6 (DeGiorgi's lemma [DiB93, I.2.Lemma 2.2])**.** *Let $u \in W^{1,1}(B_r(y))$ and let $k < l \in \mathbb{R}$ be given. There exists a constant C depending only on d, p, not depending on r, y, k, l, such that*

$$(l - k) \, |\{u > l\}| \leq C \frac{r^{d+1}}{|\{u < k\}|} \int_{\{k < u < l\}} |\nabla u| \tag{6.8}$$

Proof. From Stampacchia's lemma, more precisely from Corollary 4.5, we infer that $v := (\min\{u, l\} - k)_+ \in W^{1,1}(B_r(y))$. We assume that $|\{v < k\}| > 0$, otherwise the right-hand side of (6.8) is interpreted to be equal to ∞ and the conclusion holds for any $C > 0$.

We apply Proposition 1.2 to v with $\varphi \equiv 1$ and $p = 1$. We remark that $\nabla v = \nabla u \, \mathbb{1}_{\{k < u < l\}}$, due to Stampacchia's lemma. Since $|\{u < k\}| \leq |B_1(0)| \, r^d$ on $B_r(y)$, we compute

$$(l - k) \, |\{u > l\}| \leq \int_{B_r(y)} |v| \leq C \frac{2^d r^d}{|\{u < k\}|^{1 - \frac{1}{d}}} \int_{B_r(y)} |\nabla v|$$

$$\leq C 2^d \frac{r^{d+1}}{|\{u < k\}|} \int_{\{k < u < l\}} |\nabla u|$$

and remark that the integrals are only extended on $B_r(y)$. $\qquad \square$

In the next sections we also make use of so-called logarithmic estimates. To this end, we introduce a certain logarithmic function. We write $\log_+(v) = (\log(v))_+$ for $v > 0$ and extend the function trivially to $(-\infty, 0]$.

Definition 6.7. *Let $0 < a < b$ be given, we define for $v < a + b$ the function*

$$\Psi_{a,b}(v) = \ln_+ \left(\frac{b}{b - v + a} \right). \tag{6.9}$$

Lemma 6.8 (Properties of $\Psi_{a,b}$). *Let a, b, v and $\Psi_{a,b}$ be as in Definition 6.7. For $a < v < a + b$, there holds*

$$\Psi'_{a,b}(v) = \frac{1}{b - v + a}, \quad \Psi''_{a,b}(v) = \left(\Psi'_{a,b}(v) \right)^2 \tag{6.10}$$

and

$$\left(\Psi^2_{a,b}(v) \right)'' = 2(1 + \Psi_{a,b}(v))(\Psi'_{a,b}(v))^2, \tag{6.11}$$

as well as the estimate

$$2\Psi_{a,b}(v)\Psi'_{a,b}(v) \leq \Psi_{a,b}(v) + \Psi_{a,b}(v)\Psi'^2_{a,b}(v) \leq \Psi_{a,b}(v) + (1 + \Psi_{a,b}(v))\Psi'^2_{a,b}(v). \tag{6.12}$$

Furthermore, we find

$$\Psi^2_{a,b} \in C^{1,1}([0,b]), \quad \max_{0 \leq v \leq b}\{\Psi_{a,b}(v)\} \leq \ln\left(\frac{b}{a}\right) \quad \text{and} \quad \max_{a < v \leq b}\{\Psi'_{a,b}(v)\} \leq \frac{1}{a}. \tag{6.13}$$

Proof. Since $\Psi_{a,b} \in C^\infty(a, a+b)$, we obtain (6.10) and (6.11). Using Cauchy's inequality we deduce (6.12). The inequalities in (6.13) are derived by monotonicity.

Since $\Psi_{a,b}(a) = 0$ we deduce that $\Psi_{a,b} \in C([0, a + b))$. For $v < a$, we find $(\Psi^2_{a,b})'(v) = 0$ and for $v > a$ we find $(\Psi^2_{a,b})'(v) = 2\Psi'_{a,b}(v)\Psi_{a,b}(v) \to 0$ as $v \searrow a$. Consequently, since $\Psi^2_{a,b} \in C^1([0,b] \setminus \{a\})$ we infer $\Psi^2_{a,b} \in C^1([0,b])$. Since $2\Psi_{a,b}\Psi'_{a,b}(a) = 0$ we infer from (6.10), the bounds in (6.13) and the mean value theorem that $2\Psi_{a,b}\Psi'_{a,b} \in C^{0,1}([0,b])$ and conclude. $\qquad \square$

To justify the use of test functions in the following section, we exploit the integration by parts formula from Lemma 4.32. The following lemma is used to calculate the primitives of certain functions. We emphasize, that a and b are not related to the logarithmic function.

Lemma 6.9 (Primitives). *Let $0 < a < b$ and $\alpha, \beta \in \mathbb{R}$ be given. Let $g : [0, 1] \to \mathbb{R}$ be Lipschitz continuous and such that $g|_{[0,a]} = \alpha$ and $g|_{[b,1]} = \beta$. If $\alpha = 0$, a primitive G of g with $G(a) = G(0) = 0$ is given by*

$$G(u) = \int_0^u g(v)\,\mathrm{d}v = \int_a^{T_a^b(u)} g(v)\,\mathrm{d}v + g(b)(u - b)_+ \tag{6.14}$$

for any $u \in [0, 1]$. If $\beta = 0$, a primitive G of g with $G(b) = G(1) = 0$ is given by

$$G(u) = \int_1^u g(v)\,\mathrm{d}v = -\int_{T_a^b(u)}^b g(v)\,\mathrm{d}v - g(a)(u - a)_- \tag{6.15}$$

for any $u \in [0, 1]$.

Proof. We only prove (6.14). The proof of (6.15) is similar and uses that $(u - a)_- = a - T^a(u)$. With $T_b(u) - b = (u - b)_+$, we compute

$$G(u) = \int_0^u g(v)\,\mathrm{d}v = \int_a^{T_a^b(u)} g(v)\,\mathrm{d}v + \int_b^{T_b(u)} g(b)\,\mathrm{d}v = \int_a^{T_a^b(u)} g(v)\,\mathrm{d}v + g(b)(u - b)_+.$$

$\square$

6.2 Main Proposition and Rescaled Cylinders

Let $(x_0, t_0) \in Q$ be given. We fix sets $K \subset\subset \Omega$ and $[\tau_1, \tau_2] \subset (0, T)$ such that $(x_0, t_0) \in K \times [\tau_1, \tau_2] =: \tilde{\mathcal{K}}$. We assume that $d_2(\tilde{\mathcal{K}}, \partial_p Q) = \mathrm{dist}(K, \partial\Omega)$, where d_2 was introduced in Definition 3.9. Then the constants β, γ_p and C_p from Theorem 6.3, equation (6.5) and Proposition 6.5 depend only on the data and on $\tilde{\mathcal{K}}$ in the sense of Definition 3.10. Otherwise, $\tilde{\mathcal{K}}$ needs to be suitably redefined.

Due to a translation argument, we may assume that $(x_0, t_0) = (0, 0)$. We recall the cylinders from Definition 6.1. For some $\epsilon \in (0, 1)$ to be chosen, let $R > 0$ be small enough to ensure that $Q(2R, (2R)^{2-\epsilon}) \subset \tilde{\mathcal{K}} \subset Q$. In addition, we assume for sake of simplicity that $R \leq 1$. This is only used to show that for $0 < r \leq R$ and $p < q$ also $r^p \leq r^p$ holds. If $R > 1$, we infer with $\overline{R} := \mathrm{diam}\, K$ and any $0 < r \leq R \leq \overline{R}$ the inequality $r^p \leq \overline{R}^{p-q} r^q$. $\overline{R}$ is bounded from above by the size of Ω and depends on $d_2(\tilde{\mathcal{K}}, \partial_p Q)$.

On $Q(2R, (2R)^{2-\epsilon})$, we define the essential supremum, infimum and oscillation of s to be

$$\mu_+ = \operatorname*{ess\,sup}_{Q(2R,(2R)^{2-\epsilon})} s, \ \mu_- = \operatorname*{ess\,inf}_{Q(2R,(2R)^{2-\epsilon})} s \ \text{ and } \ \omega = \mu_+ - \mu_- := \operatorname*{ess\,osc}_{Q(2R,(2R)^{2-\epsilon})} s. \quad (6.16)$$

To take the degeneracies of Φ into account, we introduce subcylinders of $Q(2R, (2R)^{2-\epsilon})$ of the type $Q_R(\theta)$ for a suitable θ. Concerning the degeneracy near $s = 0$, we introduce the cylinder

$$Q_R(\theta_m), \quad \text{where} \quad \frac{1}{\theta_m} := \frac{1}{\theta_m(\omega)} = \Phi_{0,l}\left(\frac{\omega}{2^m}\right) \le \min_{x\in\overline{\Omega}} \Phi'\left(x, \frac{\omega}{2^m}\right). \quad (6.17)$$

The number $0 < m \in \mathbb{N}$ is chosen later and is determined depending only on the data and $\tilde{\mathcal{K}}$. In particular, m is going to be independent of ω. For such fixed m, we may assume $Q_R(\theta_m) \subset Q(2R, (2R)^{2-\epsilon})$, that is $-(2R)^{2-\epsilon} \le -\theta_m R^2$. This inclusion implies

$$\operatorname*{ess\,osc}_{Q_R(\theta_m)} s \le \omega. \quad (6.18)$$

If the inclusion does not hold, we infer

$$-\theta_m R^2 < -(2R)^{2-\epsilon} \iff \Phi_{0,l}\left(\frac{\omega}{2^m}\right) = \theta_m^{-1} < 2^{\epsilon-2} R^\epsilon. \quad (6.19)$$

If m and ϵ can be determined depending only on the data and on $\tilde{\mathcal{K}}$, then (6.19) yields

$$\operatorname*{ess\,osc}_{Q(2R,(2R)^{2-\epsilon})} s = \omega < \gamma R^{\frac{\epsilon}{\alpha_0}} \quad (6.20)$$

for some $\gamma = \gamma(data, \tilde{\mathcal{K}})$ and α_0 from Assumption A3.7. Equation (6.20) coincides with the first alternatives in Proposition 6.10 and there is nothing to show. Later, we also assume that $Q(2R, \theta_m R^2) \subset Q(2R, (2R)^{2-\epsilon})$. If this inclusion fails to hold, we also obtain (6.20).

To accommodate the degeneracy at $s = 1$, we choose $m_0 > 3$ to be the smallest integer such that

$$\frac{\omega}{2^{m_0}} \le \frac{1}{2^{m_0}} < \delta_0, \quad (6.21)$$

where δ_0 was introduced in Assumption A3.7. Having chosen m_0, consider m to be large enough to ensure

$$\frac{1}{\theta_m} \le \Phi_{0,u}\left(\frac{\omega}{2^m}\right) \le \frac{1}{2}\Phi_{1,l}\left(\frac{\omega}{2^{m_0+1}}\right) =: \frac{1}{2}\frac{1}{\theta_{m_0}}. \quad (6.22)$$

For any $t^* < 0$, we define the intrinsic subcylinders $Q_R^{t^*}$ of $Q_R(\theta_m)$ in virtue of

$$Q_R^{t^*} := (0, t^*) + Q_R(\theta_{m_0}) = B_R \times (t^* - \theta_{m_0} R^2, t^*). \quad (6.23)$$

Indeed, $Q_R^{t^*} \subset Q_R(\theta_m)$ provided

$$0 > t^* - R^2\theta_{m_0} > -R^2\theta_m \iff R^2(\theta_{m_0} - \theta_m) < t^* < 0. \tag{6.24}$$

Before we state the main proposition, we fix the setting we intend to work in.

Assumption A6.1. *Let Assumptions A3.1–A3.7 hold and let (s,p) be a weak solution of (TP1) in the sense of Definition 3.5. Let $(x_0,t_0) \in \tilde{\mathcal{K}} \subset\subset Q$ be given, with the structure $\tilde{\mathcal{K}} = [\tau_1,\tau_2] \times K \subset\subset Q$. Without loss of generality, we assume that $(x_0,t_0) = (0,0)$. On the set $\tilde{\mathcal{K}}$, we determine, as described above, β be from Theorem 6.3, γ_p from (6.5) and C_p from Proposition 6.5. Furthermore, we recall α_0 from Assumption A3.7.*

Furthermore, let $\epsilon \in (0,1)$ be given and let $0 < R \le 1$ be such that $Q(2R,(2R)^{2-\epsilon}) \subset \tilde{\mathcal{K}}$. Let μ_+, μ_- and ω be as in (6.16) and let $m_0 > 3$ be the smallest integer such that (6.21) holds. Lastly, let $\theta_m(\omega) = \theta_m$ be as in (6.17) and assume that m is large enough to ensure (6.22).

Proposition 6.10. *Assume the setting from Assumption A6.1 is fulfilled. For arbitrary*

$$0 < \epsilon \le \frac{\beta\alpha_0}{4\max\{\alpha_0,1\}}, \tag{6.25}$$

consider the cylinder $Q(2R,(2R)^{2-\epsilon})$. The constant m can be determined depending only on the data and on $\tilde{\mathcal{K}}$, independent of ϵ and ω, such that at least one of the following alternatives hold. Either there exists a constant $\gamma = \gamma(data, \tilde{\mathcal{K}})$ with

$$\omega \le \gamma R^{\frac{\epsilon}{\alpha_0}} \tag{6.26}$$

or there are positive constants C, γ and $\sigma \in (0,1)$ that depend only on the data and on $\tilde{\mathcal{K}}$ with the following properties. Defining for $n = 0,1,2,\ldots$ the sequence

$$R_n := C^{-n}R,$$

we find sequences $\omega_n \searrow 0$ and $\theta_{m,n} \nearrow \infty$ with the properties

$$\omega_{n+1} = \max\left\{\sigma\omega_n, \gamma R_n^{\frac{\epsilon}{\alpha_0}}\right\} \quad and \quad \theta_{m,n} = \theta_m(\omega_n), \tag{6.27}$$

and cylinders $Q_n := Q_{R_n}(\theta_{m,n})$ such that $Q_{n+1} \subset Q_n$ and

$$\operatorname*{ess\,osc}_{Q_n} s \le \omega_n.$$

Remark 6.11. We emphasize that the constants from Proposition 6.10 that depend only on the data and $\tilde{\mathcal{K}}$ are independent of the vertex (x_0,t_0). ◁

The proof of Proposition 6.10 unfolds along two alternatives. So far, ϵ and m still need to be determined and we increase complexity by adding a constant $\nu_0 \in (0,1)$. For such fixed ν_0, we need to examine the following two alternatives.

First Alternative: There exists a cylinder of type $Q_R^{t^*} \subset Q_R(\theta_m)$ such that

$$\left| \left\{ s > \mu_+ - \frac{\omega}{2^{m_0}} \right\} \cap Q_R^{t^*} \right| \leq \nu_0 \left| Q_R^{t^*} \right|. \tag{6.28}$$

Second Alternative: For every cylinder of type $Q_R^{t^*} \subset Q_R(\theta_m)$, there holds

$$\left| \left\{ s > \mu_+ - \frac{\omega}{2^{m_0}} \right\} \cap Q_R^{t^*} \right| > \nu_0 \left| Q_R^{t^*} \right|. \tag{6.29}$$

We emphasize that these are all the cases. Since we require several constants and points in time in the next two sections, we provide a brief overview on these objects. Furthermore, we introduce a convention for the constant γ.

Remark 6.12. In the subsequent calculations, γ denotes a generic constant, that can be determined only in terms of the data and $\tilde{\mathcal{K}}$. Particularly, γ is independent of the solution itself.

- m_0 is defined in (6.21), appears in Lemma 6.14 and Lemma 6.16 and depends only on δ_0.

- m has been introduced in (6.19) with the restriction that (6.22) holds. In Proposition 6.31 the additional restriction $m > m_1$ is made. The definition of m is performed in Definition 6.25 under consideration of Remark 6.28 by a bootstrap argument. The definition of m yields the constants μ and n_0. The purpose of the alternative argument is to show that m can be determined depending only on the data and on $\tilde{\mathcal{K}}$.

- m_1 is introduced in Proposition 6.31 depending on the data, $\tilde{\mathcal{K}}$ and a temporary parameter λ_0. For the determination of λ_0 and thus m_1, we refer to Remark 6.28. Furthermore, there holds $m_1 > q_2$.

- ν_0 has been introduced to state the two alternatives (6.28) and (6.29). It is determined in Lemma 6.14 and depends only on the data and on $\tilde{\mathcal{K}}$.

- l_1 is introduced in Lemma 6.16 and l_2 in Proposition 6.18. l_1 is only used in an intermediate step to determine l_2. To this end, the constant ν_1 in Lemma 6.16 is introduced. l_2 depends on the data, $\tilde{\mathcal{K}}$ and m.

- q_1 and q_2 are introduced in Lemma 6.22 and in Proposition 6.24, respectively. Both constants depend only on the data and on $\tilde{\mathcal{K}}$.

- The numbers q_3 and q_4 are introduced in Lemma 6.29 and in Proposition 6.31. q_3 is only used in an intermediate step to determine q_4. To this end, the constant ν_1 in Lemma is 6.29 introduced. m is determined before q_4. Thus, q_4 can be determined only in terms of the data and $\tilde{\mathcal{K}}$.

- t^* is chosen such that (6.24) holds, $\hat{t}$ is defined in (6.51) and σ_1 is defined in Corollary 6.32 and depends in particular on m. t° is defined in Corollary 6.21, t_σ is defined in (6.115) and depends on j_0. The number j_0 depends on n_0 and λ_0 and is introduced in Lemma 6.26. $\tilde{t}$ is defined in (6.123) and σ_0 is defined in Corollary 6.21. σ and $t^\diamond$ are determined in Proposition 6.33.

$\triangleleft$

Remark 6.13 (Remark on upcoming proofs)**.** We will restrict ourselves to the cases where $\mu_+ \in [1 - \delta_0]$ and $\mu_- \in [0, \delta_0]$. The other cases are more favorable since at most one degeneracy occurs. We comment on these cases at the beginning of section 6.5. In addition, the result is trivial if $\omega = 0$. Hence, we assume that $\omega > 0$. Additionally, the result $\omega = 0$ is also contained in the case (6.26).

The lemmas and propositions stated in the treatment of the two alternatives are usually based on their predecessors and are linked through the properties of the constants that are enlisted in Remark 6.12.

In the upcoming proofs, we perform similar estimates and calculations as those found in the proof of Lemma 6.14. We will not point out these arguments as explicitly as in the proof of that lemma. In particular, the chain rule from Proposition 4.21, Lemma 4.19 and Corollary 4.23 are used. In addition, we will not keep track of the constants as explicit as in the proof of Lemma 6.14, but rather write γ under consideration of Remark 6.12, reuse the constant C_ξ and the notations I_1 to I_4, etc., for integrals containing the spatial derivatives. $\triangleleft$

Assumption A6.2. *We assume $\mu_+ \in [1 - \delta_0]$, $\mu_- \in [0, \delta_0]$ and $\omega > 0$. Additionally, for given $0 < R \leq 1$ we assume that m is large enough to ensure $Q_R(\theta_m) \subset Q(2R, \theta_m R^2) \subset Q(2R, (2R)^{2-\epsilon})$.*

6.3 The First Alternative — Degeneracy at $s = 1$

In this section, we examine the first alternative and assume that, for some $\nu_0 \in (0, 1)$ to be determined, there exists a cylinder of type $Q_R^{t^*}$ such that (6.28) holds. We proceed as follows. In Lemma 6.14 we determine ν_0 and infer that s is bounded away from μ_+ on $Q_{R/2}^{t^*}$, see Figure 6.1. This information is extended by means of logarithmic estimates from Lemma 6.16 and with Proposition 6.18 up to the subcylinder $Q(\frac{R}{8}, \hat{t}) \subset Q_R(\theta_m)$. We emphasize, that neither m nor ϵ are determined during the first alternative.
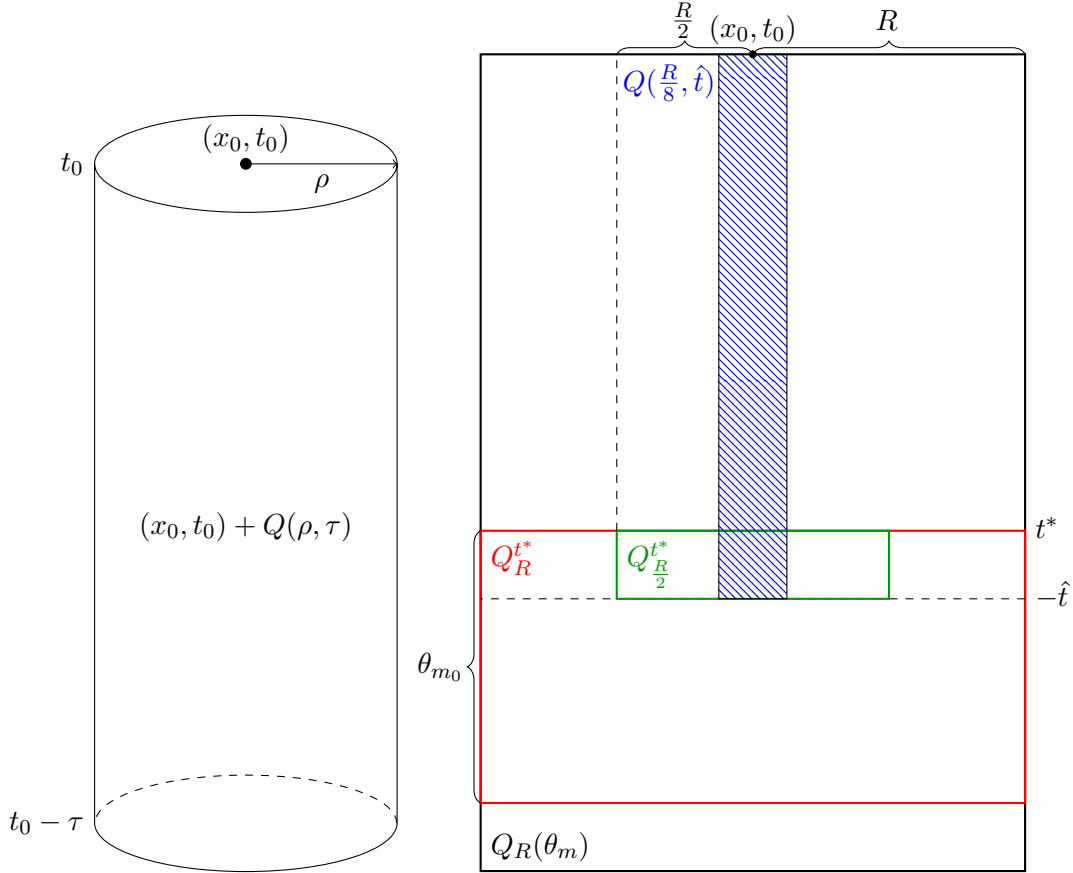
Figure 6.1: *The left picture shows a parabolic cylinder introduced in Definition 6.1. In the right picture, the cylinders occurring in the first alternative are presented. These cylinders are distorted in the x direction. The outer, black cylinder $Q_R(\theta_m) \subset Q(2R, (2R)^{2-\epsilon})$ contains the red cylinder $Q_R^{t^*}$. On the green cylinder $Q_{R/2}^{t^*}$, constructed in Lemma 6.14, we find that s is bounded away from μ_+. This result is extended in Proposition 6.18 up to $t = t_0$ at the cost of shrinking the cylinder in the x-direction and leads to the blue, striped cylinder $Q(\frac{R}{8}, \hat{t})$.*

6.3.1 Determination of ν_0.

Lemma 6.14 (Truncated Energy Estimate)**.** *Let Assumptions A6.1 and hold A6.2. There exists a constant $\nu_0 \in (0, 1)$, with $\nu_0 = \nu_0(data, \tilde{\mathcal{K}})$, such that if (6.28) is in force for some cylinder $Q_R^{t^*}$, then either*

$$\min\left\{\Phi_{1,l}\left(\frac{\omega}{2^{m_0+1}}\right), \frac{\omega}{2^{m_0}}\right\} \leq \gamma R^{\beta/4} \tag{6.30}$$

for a specific constant $\gamma = \gamma(data, \tilde{\mathcal{K}})$, or

$$s < \mu_+ - \frac{\omega}{2^{m_0+1}} \ a.e. \ on \ Q_{\frac{R}{2}}^{t^*}. \tag{6.31}$$

Proof. **Step 1. Preliminaries:** Let m_0 be as in (6.21) and $a, b \in \mathbb{R}$, such that

$$\mu_+ - \frac{\omega}{2^{m_0}} \leq a < \mu_+ - \frac{\omega}{2^{m_0+1}} = b \leq 1 - \frac{\omega}{2^{m_0+1}}. \tag{6.32}$$

With $s_b := \min\{s, b\}$, we infer from Proposition 4.21 that $(s_b - a)_+ = T_a^b(s) - a \in L^2(0, T; H^1(\Omega))$. In particular, we find $\nabla(s_b - a)_+ = \nabla[T_a^b(s)]$ pointwise almost everywhere in Q. The definition of δ_0, the minimality of m_0 and the assumption on μ_+ imply

$$b > a > \mu_+ - \frac{\omega}{2^{m_0}} \geq 1 - \delta_0 - \frac{\omega}{2^{m_0}} > 1 - 2\delta_0 > 1 - \frac{\omega}{2^{m_0-2}} > 1 - 4\delta_0, \tag{6.33}$$

which leads to the estimate

$$\max\{(s - b)_+, (b - a)\} \leq (\mu_+ - b) \leq \frac{\omega}{2^{m_0+1}} < 1. \tag{6.34}$$

Due the monotony of $\Phi_{1,l}$ and $\Phi_{1,u}$ on $[1 - 4\delta_0, 1]$ and equations (3.31), (6.32) and (6.33), we infer

$$\Phi_{1,l}\left(\frac{\omega}{2^{m_0+1}}\right) \leq \Phi'(x, s) \leq \Phi_{1,u}\left(\frac{\omega}{2^{m_0-2}}\right) \ a.e. \ on \ \{s > a\}. \tag{6.35}$$

For later purposes, we write $r = R$ and introduce, for $0 < \sigma < \eta \leq 1$, a nonnegative smooth cutoff function ξ on $Q_{\eta r}^{t^*}$ that equals one on $Q_{\sigma r}^{t^*}$, vanishes on $\partial_p Q_{\eta r}^{t^*}$ and is such that $0 \leq \xi \leq 1$. Moreover, we demand the form $\xi(x, t) = \xi_1(x)\xi_2(t)$, where

$$\begin{cases} \xi_1 = 1 & \text{on } B_{\sigma r}, \\ \xi_1 = 0 & \text{on } \partial B_{\eta r}, \\ |\nabla \xi_1| \leq \frac{C_\xi}{(\eta-\sigma)r}, \quad |D^2 \xi_1| \leq \frac{C_\xi}{(\eta-\sigma)^2 r^2} \end{cases} \tag{6.36}$$

and

$$\begin{cases} \xi_2 = 0 & \text{for } t \leq t^* - \theta_{m_0}(\eta r)^2 \\ \xi_2 = 1 & \text{for } t \geq t^* - \theta_{m_0}(\sigma r)^2 \\ 0 \leq \xi_{2,t} \leq \frac{C_\xi}{(\eta^2-\sigma^2)r^2\theta_{m_0}} & \text{for } t \in (t^* - \theta_{m_0}(\eta r)^2, t^* - \theta_{m_0}(\sigma r)^2). \end{cases} \tag{6.37}$$

The constant $C_\xi > 1$ depends only on the dimension d and on the order of derivatives, see [Alt06, 2.18]. On the range of η and σ we find

$$(\eta - \sigma)^2 \leq (\eta - \sigma)(\eta + \sigma) = \eta^2 - \sigma^2. \tag{6.38}$$

We introduce the abbreviations $\tau_\eta := t^* - \theta_{m_0}(\eta r)^2$ and $U_t := B_r \times (\tau_\eta, t)$, define the set

$$M_{a,\eta} := \{s > a\} \cap Q_{\eta r}^{t^*} = \{s_b > a\} \cap Q_{\eta r}^{t^*} = \{s > a\} \cap U_{t^*} \tag{6.39}$$

and mention that $\{s > a\} \cap U_t \subset M_{a,\eta}$.

Step 2: Truncated energy estimates. We infer from the previous considerations that $v := (s_b - a)_+ \xi^2 \mathbb{1}_{[\tau_\eta, t]} \in L^2(0, T; H_0^1(\Omega)) \subset L^2(0, T; V)$ for $t \in (\tau_\eta, t^*)$. Hence, v is admissible in equation (3.23).

We remark that τ_η is a Lebesgue point of $v(t)$ due to Lemma 4.30. To estimate the terms containing the time derivative, we apply Lemma 4.32 under consideration of Remark 4.38. Particularly, we use Lemma 6.9 with $g(u) = T_a^b(u) - a$ which gives $G(s) = \frac{1}{2}(s_b - a)_+^2 + (b - a)(s - b)_+$. We obtain for almost every $t \in (\tau_\eta, t^*)$ the identity

$$
\begin{aligned}
\int_{\tau_\eta}^t \langle \phi \partial_t s, (s_b - a)_+ \xi^2 \rangle &= \frac{1}{2} \int_{B_{\eta r} \times \{t\}} \phi (s_b - a)_+^2 \xi^2 - \frac{1}{2} \int_{B_{\eta r} \times \{\tau_\eta\}} \phi (s_b - a)_+^2 \xi^2 \\
&\quad + (b - a) \int_{B_{\eta r} \times \{t\}} \phi (s - b)_+ \xi^2 - (b - a) \int_{B_{\eta r} \times \{\tau_\eta\}} \phi (s - b)_+ \xi^2 \\
&\quad - \int_{U_t} \phi (s_b - a)_+^2 \xi \xi_t - 2(b - a) \int_{U_t} \phi (s - b)_+ \xi \xi_t.
\end{aligned}
\tag{6.40}
$$

The third term on the right-hand side is nonnegative and, exploiting the properties of ξ, we see that the second and fourth term on the right-hand side vanish. With the properties of $M_{a,\eta}$ and from equation (6.38), we infer the lower estimate

$$
\begin{aligned}
&\int_{\tau_\eta}^t \langle \phi \partial_t s, (s_b - a)_+ \xi^2 \rangle \\
&\geq \frac{C_*}{2} \int_{B_{\eta r} \times \{t\}} (s_b - a)_+^2 \xi^2 - \frac{C_\xi C^*}{(\eta - \sigma)^2 r^2} \left(\frac{\omega}{2^{m_0}} \right)^2 \Phi_{1,l} \left(\frac{\omega}{2^{m_0+1}} \right) |M_{a,\eta}|.
\end{aligned}
\tag{6.41}
$$

Before we consider the terms containing the spatial derivatives, we refer to the main tools required to perform the following estimates. Without further mentioning, we exploit the results of chapter 4.2 applied to $(s_b - a)_+$; particularly, the chain rule from Proposition 4.21, Lemma 4.19 and Corollary 4.23 to characterize the support of the occurring functions. We emphasize that integrals containing $(s_b - a)_+$ and $\nabla(s_b - a)_+$ are only extended onto $\{s > a\}$ and $\{b > s > a\}$, respectively. Thus, (6.35) may be used to estimate expressions containing $\Phi'(s)$.

Considering the integrals containing the spatial derivatives, we split the expression according to

$$
\int_{U_t} [\kappa (\nabla[\Phi(s)] - \nabla_x \Phi(s) + B(s)) + D(s)u] \cdot \nabla \left[(s_b - a)_+ \xi^2 \right] = I_1 + I_2 + I_3 + I_4
$$

and consider the integrals I_1 to I_4 separately. Concerning I_1, we write

$$
I_1 = \int_{U_t} (\kappa \nabla[\Phi(s)]\xi^2) \cdot \nabla(s_b - a)_+ + 2(\kappa \nabla[\Phi(s)](s_b - a)_+ \xi) \cdot \nabla \xi = I_{11} + I_{12}
$$

and estimate I_{11} in virtue of

$$I_{11} = \int_{U_t} (\kappa \xi^2 [\Phi'(s) \nabla (s_b - a)_+ + \nabla_x \Phi(s)]) \cdot \nabla (s_b - a)_+$$

$$\geq C_* \int_{U_t} \Phi'(s) |\nabla (s_b - a)_+|^2 \xi^2 + \int_{U_t} (\kappa \nabla_x \Phi(s) \xi^2) \cdot \nabla (s_b - a)_+.$$

For I_{12}, we compute

$$I_{12} = 2 \int_{U_t} \left(\kappa \nabla \left[(\Phi(s) - \Phi(b))_+ + \Phi(T^b(s)) \right] (s_b - a)_+ \xi \right) \cdot \nabla \xi$$

$$= 2 \int_{U_t} (\kappa (b - a) \xi \nabla (\Phi(s) - \Phi(b))_+) \cdot \nabla \xi + \left(\kappa (s_b - a)_+ \xi \nabla [\Phi(T_a^b(s))] \right) \cdot \nabla \xi$$

$$= 2 \int_{U_t} (\kappa (b - a) \xi \nabla (\Phi(s) - \Phi(b))_+) \cdot \nabla \xi + (\kappa (s_b - a)_+ \xi \Phi'(s) \nabla (s_b - a)_+) \cdot \nabla \xi$$

$$+ 2 \int_{U_t} \left(\kappa (s_b - a)_+ \nabla_x \Phi(T_a^b(s)) \xi \right) \cdot \nabla \xi$$

$$= J_{11} + J_{12} + J_{13}.$$

Since ξ_1 is compactly supported and since $\xi, \kappa, \nabla \xi$ are sufficiently smooth we may integrate by parts in J_{11} with respect to x. To this end, we use equation (1.19) with the notation of Definition 1.4, the smoothness of ξ_1 and deduce the upper estimate

$$|J_{11}| = 2(b - a) \left| \int_{U_t} \nabla (\Phi(s) - \Phi(b))_+ \cdot (\xi \kappa \nabla \xi) \right|$$

$$\leq 2(b - a) \int_{U_t} (\Phi(s) - \Phi(b))_+ \left| (\kappa \nabla \xi) \cdot \nabla \xi + \xi \kappa : D^2 \xi + \xi (\nabla \cdot \kappa) \cdot \nabla \xi \right|$$

$$\leq 2 C^* (b - a) \int_{U_t} \left(\int_b^s \Phi'(v) \, dv \right)_+ \left(|\nabla \xi|^2 + \xi \left| D^2 \xi \right| + \xi |\nabla \xi| \right)$$

$$\leq C^* C_\xi^2 \left(\frac{\omega}{2^{m_0}} \right)^2 \Phi_{1,u} \left(\frac{\omega}{2^{m_0 - 2}} \right) \frac{1 + r}{(\eta - \sigma)^2 r^2} |M_{a,\eta}|.$$

Application of Cauchy's inequality to J_{12} yields

$$|J_{12}| \leq 2 \int_{U_t} C^* \Phi'(s) (s_b - a)_+ |\nabla (s_b - a)_+| \xi |\nabla \xi|$$

$$\leq \frac{C_*}{2} \int_{U_t} \Phi'(s) |\nabla (s_b - a)_+|^2 \xi^2 + \frac{C^*}{C_*} \Phi'(s) (s_b - a)_+^2 |\nabla \xi|^2$$

$$\leq \frac{C^*}{2} \int_{U_t} \Phi'(s) |\nabla (s_b - a)_+|^2 \xi^2 + \frac{C^* C_\xi^2}{C_*} \left(\frac{\omega}{2^{m_0}} \right)^2 \Phi_{1,u} \left(\frac{\omega}{2^{m_0 - 2}} \right) \frac{1}{(\eta - \sigma)^2 r^2} |M_{a,\eta}|.$$

Concerning I_2, we find

$$I_2 = - \int_{U_t} \left(\kappa \nabla_x \Phi(s) \xi^2 \right) \cdot \nabla (s_b - a)_+ + (2 \xi (s_b - a)_+ \kappa \nabla_x \Phi(s)) \cdot \nabla \xi$$

$$= - \int_{U_t} \left(\kappa \nabla_x \Phi(s) \xi^2 \right) \cdot \nabla (s_b - a)_+ - 2 \int_{U_t} \left(\xi (s_b - a)_+ \kappa \nabla_x \Phi(T_a^b(s)) \right) \cdot \nabla \xi$$

$$- 2 \int_{U_t} (\xi (b - a) \kappa [\nabla_x \Phi(s) - \nabla_x \Phi(b)]) \cdot \nabla \xi \, \mathbb{1}_{\{s > b\}} = I_{21} + I_{22} + I_{23}$$

The integrals I_{21} and I_{22} cancel with the second integral in I_{11} and with J_{13}, respectively. We use $\nabla_x \Phi(s) = \int_0^s \nabla_x \Phi'$ and $|\nabla_x \Phi'| \leq C^*$ from (3.30), and equation (6.34) to obtain

$$|I_{23}| \leq 2C^*(b-a) \int_{U_t} \left(\int_b^s |\nabla_x \Phi'(v)| \, dv \right)_+ \xi \, |\nabla \xi| \leq C_\xi (C^*)^2 \left(\frac{\omega}{2^{m_0}} \right)^2 \frac{r}{(\eta - \sigma)^2 r^2} |M_{a,\eta}|.$$

To estimate I_3, we use the bound $|B(x,s)| \leq C^* \Phi'(x,s)$ for $(x,s) \in \overline{\Omega} \times [0,1]$, as derived in Remark 3.8, and Cauchy's inequality to obtain

$$
\begin{aligned}
|I_3| &\leq \frac{(C^*)^2}{2} \int_{U_t} \Phi'(s) \, |\nabla(s_b - a)_+| \xi^2 + \Phi'(s)(s_b - a)_+ \xi \, |\nabla \xi| \\
&\leq \frac{C_*}{4} \int_{U_t} \Phi'(s) \, |\nabla(s_b - a)_+|^2 \, \xi^2 + \frac{(C^*)^4}{4C_*} \frac{r^2}{(\eta - \sigma)^2 r^2} \Phi_{1,u} \left(\frac{\omega}{2^{m_0 - 2}} \right) |M_{a,\eta}| \\
&\quad + \frac{C_\xi (C^*)^2}{2} \frac{r}{(\eta - \sigma)^2 r^2} \left(\frac{\omega}{2^{m_0}} \right) \Phi_{1,u} \left(\frac{\omega}{2^{m_0 - 2}} \right) |M_{a,\eta}|.
\end{aligned}
$$

Concerning I_4, we split the integrals according to

$$I_4 = \int_{U_t} D(s) u \cdot \nabla(s_b - a)_+ \xi^2 + 2 \int_Q \kappa(s_b - a)_+ \xi D(s) u \cdot \nabla \xi = I_{41} + I_{42}.$$

We treat I_{41}, exploiting the identity

$$\nabla \left(\int_a^{T_a^b(s)} D(v) \, dv \right) = D(s) \nabla(s_b - a)_+ + \int_a^{T_a^b(s)} \nabla_x D(v) \, dv,$$

integrate by parts and use that $\nabla \cdot u = 0$ to obtain

$$
\begin{aligned}
I_{41} &= \int_{U_t} \xi^2 u \cdot \nabla \left(\int_a^{T_a^b(s)} D(v) \, dv \right) - \xi^2 u \cdot \left(\int_a^{T_a^b(s)} \nabla_x D(v) \, dv \right) \\
&= -\int_{U_t} 2 \left(\int_a^{T_a^b(s)} D(v) \, dv \right) u \cdot \xi \nabla \xi - \xi^2 u \cdot \left(\int_a^{T_a^b(s)} \nabla_x D(v) \, dv \right).
\end{aligned}
$$

Consequently, we estimate I_4 as

$$
\begin{aligned}
|I_4| &\leq 2 \int_{U_t} \left[\left(\int_a^{T_a^b(s)} |D(v)| \, dv \right) + |D(s)| \, (s_b - a)_+ \right] |u| \, \xi \, |\nabla \xi| \\
&\quad + 2 \int_{U_t} \xi^2 \, |u| \left(\int_a^{T_a^b(s)} |\nabla_x D(v)| \, dv \right) \\
&\leq 2C^* \int_{U_t} (s_b - a)_+ \, |u| \, [\xi \, |\nabla \xi| + \xi^2] \leq 2(C^*)^3 \int_{U_t} (s_b - a)_+ (|\nabla p| + 1)[\xi \, |\nabla \xi| + \xi^2] \\
&\leq \frac{C_*}{16} r^{-\beta} \int_{U_t} |\nabla p|^2 \, (s_b - a)_+^2 \xi^2 + 132 \frac{C_\xi^2 (C^*)^6}{C_*} r^\beta \frac{1 + r^2}{(\eta - \sigma)^2 r^2} |M_{a,\eta}| \\
&\quad + C_\xi (C^*)^3 \left(\frac{\omega}{2^{m_0}} \right) \frac{r + r^2}{(\eta - \sigma)^2 r^2} |M_{a,\eta}|,
\end{aligned}
$$

where we exploit Cauchy's inequality and the estimates on $D(s)$ presented in Remark 3.8. The Caccioppoli estimate from Proposition 6.5 leads to

$$
\begin{aligned}
|I_4| \leq \; & \frac{C_*}{8} C_p r^\beta \int_{U_t} |\nabla (s_b - a)_+|^2 \, \xi^2 + \frac{C_\xi^2 C_* C_p}{8} \left(\frac{\omega}{2^{m_0}} \right)^2 \frac{r^\beta}{(\eta - \sigma)^2 r^2} |M_{\alpha,\eta}| \\
& + \frac{C_* C_p}{16} \left(\frac{\omega}{2^{m_0}} \right)^2 \frac{r^{2-\beta}}{(\eta - \sigma)^2 r^2} |M_{a,\eta}| + 32 \frac{C_\xi^2 (C^*)^6}{C_*} r^\beta \frac{1 + r^2}{(\eta - \sigma)^2 r^2} |M_{a,\eta}| \\
& + C_\xi (C^*)^3 \left(\frac{\omega}{2^{m_0}} \right) \frac{r + r^2}{(\eta - \sigma)^2 r^2} |M_{a,\eta}| \, .
\end{aligned}
$$

We collect the previous estimates, absorb the terms containing $|\nabla (s_b - a)_+|^2$, divide by C_*, estimate all constants from above by

$$
\bar{C} := 32 \frac{C_\xi^2 (C^*)^6 C_p}{C_*^2},
$$

to obtain

$$
\begin{aligned}
& \frac{1}{2} \int_{B_{\eta r} \times \{t\}} (s_b - a)_+^2 \xi^2 + \frac{1}{4} \Phi_{1,l} \left(\frac{\omega}{2^{m_0+1}} \right) \int_{U_t} |\nabla (s_b - a)_+|^2 \, \xi^2 \\
& \leq \frac{\bar{C}}{(\eta - \sigma)^2 r^2} \Bigg[\left(\frac{\omega}{2^{m_0}} \right)^2 \Phi_{1,u} \left(\frac{\omega}{2^{m_0-2}} \right) (2 + r) + \left(\frac{\omega}{2^{m_0}} \right)^2 \Phi_{1,l} \left(\frac{\omega}{2^{m_0+1}} \right) \\
& \qquad + \left(\frac{\omega}{2^{m_0}} \right)^2 (r + r^\beta + r^{2-\beta}) + \Phi_{1,u} \left(\frac{\omega}{2^{m_0-2}} \right) r^2 \\
& \qquad + \frac{\omega}{2^{m_0}} \Phi_{1,u} \left(\frac{\omega}{2^{m_0-2}} \right) r + \frac{\omega}{2^{m_0}} \left(r + r^2 \right) + r^\beta + r^{2+\beta} \Bigg] |M_{a,\eta}| \\
& + \frac{1}{8} C_p r^\beta \int_{U_t} |\nabla (s_b - a)_+|^2 \, \xi^2 \, .
\end{aligned}
\tag{6.42}
$$

From now on, we assume that (6.30) is violated with the choice $\gamma = C_p$, i.e.

$$
C_p r^{\beta/4} \leq \min \left\{ \Phi_{1,l} \left(\frac{\omega}{2^{m_0+1}} \right), \frac{\omega}{2^{m_0}} \right\} \leq \min\{C^*, 1\} = 1,
\tag{6.43}
$$

which implies $r^\alpha \leq 1$ for all $\alpha \in [0,1]$. Hence, we may absorb the last term on the right-hand side of (6.42). Estimating the terms on the right hand side of (6.42), exploiting (6.35) and (6.43), and taking afterwards the essential supremum over all $t \in (\tau_\eta, t^*)$ yields

$$
\begin{aligned}
& \operatorname*{ess\,sup}_{\tau_\eta \leq t \leq t^*} \int_{B_{\eta r} \times \{t\}} (s_b - a)_+^2 \xi^2 + \Phi_{1,l} \left(\frac{\omega}{2^{m_0+1}} \right) \int_{Q_{\eta r}^{t^*}} |\nabla (s_b - a)_+|^2 \, \xi^2 \\
& \quad \leq \frac{104 \bar{C}}{(\eta - \sigma)^2 r^2} \left(\frac{\omega}{2^{m_0}} \right)^2 \Phi_{1,u} \left(\frac{\omega}{2^{m_0-2}} \right) |M_{a,\eta}| \, .
\end{aligned}
\tag{6.44}
$$

The change of variables $\bar{t} = (t - t^*)\theta_{m_0}^{-1}$ transforms the interval (τ_η, t^*) into $(-(\eta r)^2, 0)$. Define $\bar{s}(\cdot, \bar{t}) := s(\cdot, t)$, $\bar{\xi}(\cdot, \bar{t}) = x(\cdot, t)$ and $\bar{M}_{a,\eta} = \{\bar{s} > a\} \cap Q_{\eta r}$. Applying this transformation to (6.44), adding on both sides the term

$$\int_{Q_{\eta r}} (\bar{s}_b - a)_+^2 \left|\nabla\bar{\xi}\right|^2$$

and using $C_\xi^2 \leq \bar{C}$, we deduce the estimate

$$\left\|(\bar{s}_b - a)_+\bar{\xi}\right\|_{V^2(Q_{\eta r})}^2 \leq \frac{104\bar{C}}{(\eta - \sigma)^2 r^2} \left(\frac{\omega}{2^{m_0}}\right)^2 \left[\frac{\Phi_{1,u}\left(\omega/2^{m_0-2}\right)}{\Phi_{1,l}\left(\omega/2^{m_0+1}\right)} + 1\right] \left|\bar{M}_{a,\eta}\right|. \tag{6.45}$$

Step 3: Iteration. Inequality (6.44) is the starting point of an iteration process. We define for $k = 0, 1, 2, \ldots$ the numbers

$$a_k := b - \frac{\omega}{2^{m_0+1}}2^{-k}, \quad \eta_k := \frac{1}{2} + 2^{-(k+1)}, \quad r_k := \eta_k r \quad \text{and} \quad A_k := \left|\bar{M}_{a_k,r_k}\right|,$$

and choose $a = a_k$, $\eta = \eta_k$ and $\sigma = \eta_{k+1}$ in (6.44) to deduce

$$\begin{aligned}
2^{-2(k+1)} \left(\frac{\omega}{2^{m_0+1}}\right)^2 A_{k+1} &= |a_{k+1} - a_k|^2 A_{k+1} \\
&\leq \int_{Q_{r_{k+1}}} (\bar{s}_b - a_k)_+^2 \mathbb{1}_{\{\bar{s}_b > a_{k+1}\}} \leq \|(\bar{s}_b - a_k)_+\|_{2,Q_{r_{k+1}}}^2 \\
&\leq \left\|(\bar{s}_b - a_k)_+\bar{\xi}\right\|_{2,Q_{r_k}}^2 \leq C \left\|(\bar{s}_b - a_k)_+\bar{\xi}\right\|_{V^2(Q_{r_k})}^2 A_k^{\frac{2}{d+2}} \\
&\leq \tilde{\gamma}\frac{2^{2(k+2)}}{r_k^2} \left(\frac{\omega}{2^{m_0}}\right)^2 \left[\frac{\Phi_{1,u}\left(\omega/2^{m_0-2}\right)}{\Phi_{1,l}\left(\omega/2^{m_0+1}\right)} + 1\right] A_k^{1+\frac{2}{d+2}}.
\end{aligned} \tag{6.46}$$

We exploit Theorem 1.3 in the fourth inequality and emphasize that the constant C only depends on the dimension d and the exponent $p = 2$. In the last inequality we use the abbreviation $\tilde{\gamma} = 104C\bar{C}$ and $\eta_k - \sigma_k = 2^{-(k+2)}$ as well as $r_k \leq r$. Isolating A_{k+1} on the left-hand side of (6.46) yields

$$A_{k+1} \leq \tilde{\gamma}\frac{2^{4(k+2)}}{r_k^2} \left[\frac{\Phi_{1,u}\left(\omega/2^{m_0-2}\right)}{\Phi_{1,l}\left(\omega/2^{m_0+1}\right)} + 1\right] A_k^{1+\frac{2}{d+2}}. \tag{6.47}$$

We define the numbers

$$X_k := \frac{A_k}{|Q_{r_k}|}$$

and divide (6.47) by $|Q_{r_{k+1}}|$. To obtain the term X_k on the right-hand side, we observe that

$$\begin{aligned}
|Q_{r_{k+1}}|\, r_k^2 &= |B_1|\, (r_{k+1})^{d+2}r_k^2 \geq |B_1| \frac{1}{2^{(d+2)}}(r_k)^{d+4} \\
&= 2^{-(d+2)} |B_1|^{-\frac{2}{d+2}} (|B_1| r_k^{d+2})^{\frac{d+4}{d+2}} = 2^{-(d+2)} |B_1|^{-\frac{2}{d+2}} |Q_{r_k}|^{1+\frac{2}{d+2}}
\end{aligned} \tag{6.48}$$

where the estimate $2r_{k+1} \geq r_k$ is used. Combining (6.47) and (6.48) we infer

$$X_{k+1} \leq \tilde{\gamma} 2^{(d+10)} |B_1|^{\frac{2}{d+2}} \left[\frac{\Phi_{1,u} \left(\omega / 2^{m_0 - 2} \right)}{\Phi_{1,l} \left(\omega / 2^{m_0 + 1} \right)} + 1 \right] 4^{2k} X_k^{1 + \frac{d}{d+2}}.$$

From Lemma 6.2 we obtain $X_k \to 0$ as $k \to \infty$, provided that

$$X_0 \leq \left(\tilde{\gamma} 2^{(d+10)} |B_1|^{\frac{2}{d+2}} \left[\frac{\Phi_{1,u} \left(\omega / 2^{m_0 - 2} \right)}{\Phi_{1,l} \left(\omega / 2^{m_0 + 1} \right)} + 1 \right] \right)^{-\frac{d+2}{2}} 4^{-2 \left(\frac{d+2}{2} \right)^2} =: \nu_0 \in (0,1). \quad (6.49)$$

But the first alternative (6.28) is identical to (6.49). Hence, we infer $A_k \to 0$ and observing that $r_k \to \frac{R}{2}$ and $a_k \to b$ we find

$$\left| \{ \bar{s} \geq b \} \cap Q_{\frac{R}{2}} \right| = \left| \{ s \geq b \} \cap Q_{\frac{R}{2}}^{t^*} \right| = 0.$$

Thus, Lemma 6.14 is proven if the choice of ν_0 is independent of ω. In fact, this follows since the quotient

$$\frac{\Phi_{1,u} \left(\omega / 2^{m_0 - 2} \right)}{\Phi_{1,l} \left(\omega / 2^{m_0 + 1} \right)} = \frac{c_{1,u}}{c_{1,l}} 2^{3\alpha_1}. \quad (6.50)$$

is independent of ω due to (3.31) from Assumption A3.7. $\qquad \square$

Remark 6.15. The independence of ν_0 on ω depends crucially on the structure of $\Phi_{1,l}$ and $\Phi_{1,u}$ and particularly on both functions being power functions near one. $\qquad \triangleleft$

6.3.2 A Logarithmic Estimate

To continue the treatment of the first alternative, we intend to extend the statement of Lemma 6.14 from $Q_{R/2}^{t^*}$ up to $t = 0$ eventually shrinking the ball in that procedure. We define the time level

$$-\hat{t} := t^* - \theta_{m_0} \left(\frac{R}{2} \right)^2 \quad (6.51)$$

and refer to Figure 6.1. In a first step, the information of Lemma 6.14 is used to show that the measure of the set where $s(t)$ is near its supremum on $B_{R/4}$ is relatively small independent of $t \in (-\hat{t}, 0)$. To this end we utilize the logarithmic function from Definition 6.7 and derive so-called logarithmic estimates in the next lemma.

Lemma 6.16 (Logarithmic estimates)**.** *Let Assumptions A6.1 and A6.2 hold. Given* $\nu_1 \in (0,1)$, *there exists* $\mathbb{N} \ni l_1 > m_0 + 2$, $l_1 = l_1(data, \tilde{\mathcal{K}}, \nu_1, m)$ *such that either*

$$\min \left\{ \Phi_{1,u} \left(\frac{\omega}{2^{m_0 - 2}} \right), \frac{\omega}{2^{l_1}} \right\} \leq \gamma R^{\beta/4} \quad (6.52)$$

for some $\gamma = \gamma(data, \tilde{\mathcal{K}})$ *or*

$$\left| \left\{ s(t) > \mu_+ - \frac{\omega}{2^{l_1}} \right\} \cap B_{\frac{R}{4}} \right| \leq \nu_1 \left| B_{\frac{R}{4}} \right|, \quad \text{for a.e. } t \in (-\hat{t}, 0). \quad (6.53)$$

Proof. **Step 1: Preliminaries.** With $\bar{a} = \frac{\omega}{2^n}$ and $\bar{b} = \frac{\omega}{2^{m_0+1}}$ we abbreviate $\Psi = \Psi_{\bar{a},\bar{b}}$, where Ψ is the function of Definition 6.7 and $m_0 + 2 < n \in \mathbb{N}$ is to be chosen later. Let $r = {R}/{2}$, $\sigma \in (0,1)$ and ξ be a smooth, time-independent cut-off function on B_r that equals one on $B_{\sigma r}$ and is such that $0 \leq \xi \leq 1$, $|\nabla \xi| \leq C_\xi \frac{1}{(1-\sigma)r}$ and $|D^2 \xi| \leq \frac{C_\xi}{(1-\sigma)^2 r^2}$.

Let $a = \mu_+ - \frac{\omega}{2^{m_0+1}}$ and $b = \mu_+ - \frac{\omega}{2^n}$. Then $(s_b - a)_+ \leq \frac{\omega}{2^{m_0+1}} - \frac{\omega}{2^n} \leq \frac{\omega}{2^{m_0+1}}$ and analogously to (6.33) we find $b > a \geq 1 - \frac{\omega}{2^{m_0-2}}$. Using the notation and properties of $(s_b - a)_+$ indicated in the proof of Lemma 6.14 we deduce that $2\Psi((s_b - a)_+)\Psi'((s_b - a)_+)\xi^2 \in L^2(0,T; H_0^1(\Omega))$ since $2\Psi\Psi' \in C^{0,1}([0,\bar{b}])$ by Lemma 6.8. To keep the notation compact, we introduce $\psi = \psi(s) = \Psi((s_b - a)_+)$ and, with abuse of notation, also $\psi' := \psi'(s) := \Psi'((s_b - a)_+)$. We emphasize that the formal derivative of ψ vanishes for $s > b$ but our choice of ψ' does not. We find the identity

$$2\psi(s)\psi'(s) = \begin{cases} (\psi^2)'(s) = (\Psi^2)'((s_b - a)_+) & s < b, \\ 2\Psi(b - a)\Psi'(b - a) & s \geq b, \end{cases} \tag{6.54}$$

as well as $2\psi\psi'\xi^2 \in L^2(0,T; H_0^1(\Omega))$. With the properties of Ψ from Lemma 6.8, we infer $\psi \leq \ln(2^{n-m_0-1}) \leq (n - m_0 - 1)\ln(2) \leq n$ and $\psi' \leq \frac{2^n}{\omega}$. For $t \in (-\hat{t}, 0)$, we define the cylinders $U_t := B_r \times (-\hat{t}, t)$. Due to the definition of $\hat{t}$ and choice of t^*, we find

$$\hat{t} < (2r)^2 \theta_m \quad \text{and} \quad |U_t| \leq \hat{t}|B_r| \leq 4\theta_m r^2 |B_r|. \tag{6.55}$$

Step 2: Logarithmic Estimates. For $t_1 \in (-\hat{t}, t^*)$ and $t \in (t_1, 0)$, the function $v = 2\psi\psi'\xi^2 \mathbb{1}_{[t_1,t]}$ is admissible in equation (3.23). Using Lemma 4.32 under consideration of Remark 4.38 and Lemma 6.9, we integrate by parts to obtain

$$\int_{t_1}^t \langle \phi \partial_t s, 2\psi\psi'\xi^2 \rangle = \int_{B_r} \phi\psi^2\xi^2 + 2\phi\Psi(b - a)\Psi'(b - a)(s - b)_+\xi^2 \Big|_{t_1}^t$$
$$= \int_{B_r \times \{t\}} \phi\psi^2\xi^2 + 2\phi\Psi(b - a)\Psi'(b - a)(s - b)_+\xi^2 \geq C_* \int_{B_r \times \{t\}} \psi^2\xi^2 \tag{6.56}$$

for a.e. $t_1 \in (-\hat{t}, t^*)$ and $t \in (t_1, 0)$. For such t_1, the integral evaluated at t_1 vanishes due to Lemma 6.14, i.e. $s < \mu_+ - \frac{\omega}{2^{m_0+1}}$ a.e. in $Q_r^{t^*}$. In particular, the function $v = 0$ a.e. on $B_r \times (-\hat{t}, t^*)$

As in the second step in the proof of Lemma 6.14 we exploit the properties of $(s_b - a)_+$ and its gradient. Furthermore, since $v = 0$ a.e. on $B_r \times (-\hat{t}, t^*)$ we may extend the integrals from $B_r \times (t_1, t)$ onto U_t.

Concerning the integrals containing the spatial derivatives, we write

$$\int_{U_t} [\kappa(\nabla[\Phi(s)] - \nabla_x \Phi(s) + B(s)) + D(s)u] \cdot \nabla\left[2\psi\psi'\xi^2\right] = I_1 + I_2 + I_3 + I_4$$

and consider the integrals I_1 to I_4 separately. From the properties of Ψ we infer the pointwise identity

$$\nabla\left[2\psi\psi'\xi^2\right] = 2(\psi')^2(1+\psi)\nabla(s_b-a)_+\xi^2 + 4\psi\psi'\xi\nabla\xi \tag{6.57}$$

almost everywhere on U_t and emphasize that the first term on the right-hand side vanishes a.e. on the sets $\{(s_b-a)_+ = c\}$, for $c \in \{a, b, \bar{a}\}$, due to Corollary 4.23. Concerning I_1, we write

$$I_1 = \int_{U_t}(\kappa\nabla[\Phi(s)]\xi^2)\cdot\left[2(\psi')^2[1+\psi]\nabla(s_b-a)_+\xi^2 + 4\psi\psi'\xi\nabla\xi\right] = I_{11} + I_{12}$$

and estimate I_{11} in virtue of

$$I_{11} \geq C_*\int_{U_t}(\psi')^2(1+\psi)\Phi'(s)\left|\nabla(s_b-a)_+\right|^2\xi^2$$
$$+ 2\int_{U_t}(\kappa\nabla_x\Phi(s)\xi^2)(\psi')^2(1+\psi)\cdot\nabla(s_b-a)_+$$

For I_{12}, we compute

$$I_{12} = 4\int_{U_t}\left(\kappa\nabla\left[(\Phi(s)-\Phi(b))_+ + \Phi(T^b(s))\right]\psi\psi'\xi\right)\cdot\nabla\xi$$
$$= 4\int_{U_t}\left(\kappa\Psi(b-a)\Psi'(b-a)\xi\nabla(\Phi(s)-\Phi(b))_+\right)\cdot\nabla\xi$$
$$+ 4\int_{U_t}\left(\kappa\psi\psi'\Phi'(s)\nabla(s_b-a)_+\xi\right)\cdot\nabla\xi + 4\int_{U_t}\left(\kappa\psi\psi'\nabla_x\Phi(T_a^b(s))\xi\right)\cdot\nabla\xi$$
$$= J_{11} + J_{12} + J_{13}$$

analogously to I_{12} in the previous proof. In the spirit of the treatment of J_{11} in the proof of Lemma 6.14, we use integration by parts to obtain

$$|J_{11}| \leq \gamma\Psi(b-a)\Psi'(b-a)\Phi_{1,u}(b)(\mu_+ - b)\left|\int_{U_t}\nabla\cdot(\xi\kappa\nabla\xi)\right|$$
$$\leq \gamma\Psi(b-a)\Psi'(b-a)\Phi_{1,u}(b)(\mu_+ - b)\frac{1+r}{(1-\sigma)^2r^2}|U_t|.$$

Application of Cauchy's inequality to J_{12} under consideration of (6.12) yields

$$|J_{12}| \leq \varepsilon\int_{U_t}\Phi'(s)\left|\nabla(s_b-a)_+\right|^2\xi^2(\psi')^2(1+\psi) + \frac{\gamma}{(1-\sigma)^2r^2}\int_{U_t}\Phi'(s)\psi$$

for some $\varepsilon > 0$. Concerning I_2, we write

$$I_2 = -2\int_{U_t}\left(\kappa(\psi')^2(1+\psi)\nabla_x\Phi(s)\xi^2\right)\cdot\nabla(s_b-a)_+ - 4\int_{U_t}\left(\kappa\psi\psi'\nabla_x\Phi\left(T_a^b(s)\right)\xi\right)\cdot\nabla\xi$$
$$- 4\int_{U_t}\left(\kappa\psi\psi'\left[\nabla_x\Phi(s)-\nabla_x\Phi(b)\right]\xi\right)\cdot\nabla\xi\,\mathbb{1}_{\{s>b\}} = I_{21} + I_{22} + I_{23}$$

The integrals I_{21} and I_{22} cancel with the second integral in I_{11} and with J_{13}, respectively. Using equation (3.30) from Assumption A3.7, we estimate

$$|I_{23}| \leq \gamma \Psi(b-a) \Psi'(b-a) \int_{U_t} \left(\int_b^s |\nabla_x \Phi'(v)| \, dv \right)_+ \xi \, |\nabla \xi|$$

$$\leq \gamma \Psi(b-a) \Psi'(b-a)(\mu_+ - b) \frac{r}{(1-\sigma)^2 r^2} |U_t|$$

Concerning I_3, we use the bound $|B(s)| \leq C^* \Phi'(s)$ from Assumption A3.7 and Cauchy's inequality under consideration of (6.12), to derive

$$|I_3| \leq \gamma \int_{U_t} \Phi'(s) \, |\nabla(s_b - a)_+| \, (\psi')^2 (1+\psi) \xi^2 + \Phi'(s) \psi \psi' \xi \, |\nabla \xi|$$

$$\leq \varepsilon \int_{U_t} \Phi'(s) \, |\nabla(s_b - a)_+|^2 \, (\psi')^2 (1+\psi) \xi^2 + \gamma \int_{U_t} \Phi'(s)(\psi')^2 (1+\psi) \xi^2$$

$$+ \frac{\gamma}{(1-\sigma)^2 r^2} \int_{U_t} \Phi'(s) \psi.$$

The reamining integral I_4 is treated differently in comparison to the proof of Lemma 6.14. From Remark 3.8 and due to the support of $(s_b - a)_+$, we estimate $|D(s)| \leq C^* \Phi'(s)$. We split I_4 according to

$$I_4 = \int_{U_t} D(s) u \cdot \left[2(\psi')^2(1+\psi)\nabla(s_b - a)_+ \xi^2 + 4\psi \psi' \xi \nabla \xi \right] = I_{4,1} + I_{4,2}$$

With Cauchy's inequality, we infer

$$|I_{41}| \leq \varepsilon \int_{U_t} \Phi'(s) \, |\nabla(s_b - a)_+|^2 \, (\psi')^2 (1+\psi) \xi^2 + \gamma \int_{U_t} \Phi'(s) \xi^2 \, |u|^2 \, (\psi')^2 (1+\psi)$$

and

$$|I_{42}| \leq \frac{\gamma}{(1-\sigma)^2 r^2} \int_{U_t} \Phi'(s) \psi + \gamma \int_{U_t} \Phi'(s)(\psi')^2(1+\psi) \, |u|^2 \, \xi^2.$$

We choose $\varepsilon = \frac{C_*}{3}$ and collect the previous estimates to obtain

$$\int_{B_r \times \{t\}} \psi^2 \xi^2 \leq \frac{\gamma}{(1-\sigma)^2 r^2} \Psi(b-a)\Psi'(b-a)(\mu_+ - b)_+ |U_t| \, (r + \Phi_{1,u}(b)(1+r))$$

$$+ \frac{\gamma}{(1-\sigma)^2 r^2} \int_{U_t} \Phi'(s)\psi + \gamma \int_{U_t} \Phi'(s)(\psi')^2(1+\psi)\xi^2 \qquad (6.58)$$

$$+ \gamma \int_{U_t} \Phi'(s)(\psi')^2(1+\psi) \, |u|^2 \, \xi^2$$

for a.e. $t \in (-\hat{t}, 0)$. From Proposition 6.5 with $f = \xi$ and $\xi \leq 1$, we derive

$$\int_{U_t} |\nabla p|^2 \, f^2 \leq C_p \left(\frac{r^{2\beta}}{(1-\sigma)^2 r^2} |U_t| + |U_t| \right). \qquad (6.59)$$

Exploiting this, the pointwise identity $|u|^2 \leq 4(|\nabla p|^2 + 1)$, using the preliminaries of the proof, particularly that $(\mu_+ - b)_+ \leq \omega 2^{-n}$, $b \geq 1 - \frac{\omega}{2^{m_0-1}}$ as well as the properties of ψ, and that $\xi = 1$ on $B_{\sigma r}$ we deduce from (6.58) the estimate

$$\int_{B_{\sigma r} \times \{t\}} \psi^2 \leq \frac{\gamma n}{(1-\sigma)^2 r^2} \left[r + \Phi_{1,u}\left(\frac{\omega}{2^{m_0-2}}\right)(1+r) \right] |U_t|$$
$$+ \frac{\gamma n}{(1-\sigma)^2 r^2} \Phi_{1,u}\left(\frac{\omega}{2^{m_0-2}}\right)\left(\frac{2^n}{\omega}\right)^2 \left(r^{2\beta} + r^2\right)|U_t| \tag{6.60}$$

for a.e. $t \in (-\hat{t}, 0)$.

Assume that

$$\gamma r^{\beta/4} < \min\left\{ \Phi_{1,u}\left(\frac{\omega}{2^{m_0-2}}\right), \frac{\omega}{2^n} \right\} \tag{6.61}$$

holds. Otherwise, equation (6.52) is fulfilled and there is nothing left to show. With $r^\alpha \leq 1$ for any $\alpha \in [0,1]$ and $r^{2\beta}(2^n/\omega)^2 \leq 1$, equation (6.60) transforms into

$$\int_{B_{\sigma r} \times \{t\}} \psi^2 \leq \frac{\gamma n}{(1-\sigma)^2 r^2} \Phi_{1,u}\left(\frac{\omega}{2^{m_0-2}}\right)|U_t| \overset{(6.55)}{\leq} \frac{\gamma n \theta_m}{(1-\sigma)^2}\Phi_{1,u}\left(\frac{\omega}{2^{m_0-2}}\right)|B_r|. \tag{6.62}$$

Step 3: Conclusion. We choose $\sigma = \frac{1}{2}$ and define the sets

$$S(t) := \left\{ s(t) > \mu_+ - \frac{\omega}{2^n} \right\} \cap B_{\frac{R}{4}} \subset B_{\frac{R}{2}}$$

for a.e. every $t \in (-\hat{t}, 0)$. On $S(t)$ we find $\xi = 1$, $(s_b - a)_+ = \frac{\omega}{2^{m_0+1}} - \frac{\omega}{2^n}$ and, together with the properties of ψ, also

$$\psi^2 \geq \ln_+^2(2^{n-m_0-2}) = \ln^2(2)(n - m_0 - 2)^2.$$

Application to (6.62) yields

$$\ln^2(2)(n - m_0 - 2)^2 |S(t)| \leq 4\gamma n \theta_m \Phi_{1,u}\left(\frac{\omega}{2^{m_0-2}}\right)\left|B_{\frac{R}{2}}\right|.$$

Since $\left|B_{\frac{R}{2}}\right| = 2^d \left|B_{\frac{R}{4}}\right|$ we find

$$\left| \left\{ s(t) > \mu_+ - \frac{\omega}{2^n} \right\} \cap B_{\frac{R}{4}} \times \{t\} \right| \leq \gamma \frac{n}{(n - m_0 - 2)^2} \frac{\Phi_{1,u}\left(\omega/2^{m_0-2}\right)}{\Phi_{0,l}\left(\omega/2^m\right)}\left|B_{\frac{R}{4}}\right|$$

for a.e. $t \in (-\hat{t}, 0)$.

Note that $n \geq 2p + \frac{1}{\alpha}$ for $p, \alpha > 0$ implies

$$2p + \frac{1}{\alpha} \leq n \Rightarrow 2p + \frac{1}{\alpha} \leq n + \frac{p^2}{n} \Leftrightarrow \frac{n}{(n-p)^2} \leq \alpha. \tag{6.63}$$

Hence, choosing

$$n \geq 2(m_0 - 1) + \frac{\gamma}{\nu_1} \frac{\Phi_{1,u} \left(\omega/2^{m_0 - 2} \right)}{\Phi_{0,l} \left(\omega/2^m \right)} =: H_n \tag{6.64}$$

provides (6.53) with $l_1 = n$ modulo the independence of ω. Observe also that with such a choice of l_1, the complementary case to (6.61) is nothing but (6.52).

Concerning the independence of ω, we observe that $\omega \leq 1$ and $\alpha_1 \geq \alpha_0$ from Assumption A3.7. Consequently, the quotient

$$\frac{\Phi_{1,u} \left(\frac{\omega}{2^{m_0 - 2}} \right)}{\Phi_{0,l} \left(\frac{\omega}{2^m} \right)} = \frac{c_{1,u} \left(\frac{\omega}{2^{m_0 - 2}} \right)^{\alpha_1}}{c_{0,l} \left(\frac{\omega}{2^m} \right)^{\alpha_0}} = \frac{c_{1,u}}{c_{0,l}} 2^{m\alpha_0 - (m_0 - 2)\alpha_1} \omega^{\alpha_1 - \alpha_0} \leq \frac{c_{1,u}}{c_{0,l}} 2^{m\alpha_0}$$

is estimated independent of ω. The choice

$$H_n \leq 2(m_0 - 2) + \frac{\gamma}{\nu_1} \frac{c_{1,u}}{c_{0,l}} 2^{m\alpha_0} \leq l_1 \tag{6.65}$$

for l_1 is independent of ω, depends only on ν_1, m, C_p, and thus only the data and $\tilde{\mathcal{K}}$. This implies (6.64). $\qquad \square$

Remark 6.17 (On the choice of b). In the previous proof it suffices to choose $a < b < 1$ such that $(\mu_+ - b)_+ \leq \omega 2^{-n}$. In particular, if $\mu_+ < 1$ we may choose $b > \mu_+$ and the integrals containing $(\mu_+ - b)$ vanish. However, in these cases the definition of ψ and ψ' is more complicated. This remark also applies to the logarithmic estimates we perform later. $\qquad \triangleleft$

6.3.3 The First Alternative Concluded

Proposition 6.18. *Assume that the setting described Assumption A6.1 is given and that Assumption A6.2 holds. There exists a constant $\mathbb{N} \ni l_2 > l_1$, $l_2 = l_2(data, \tilde{\mathcal{K}}, m)$, such that if (6.28) is in force for some cylinder $Q_R^{t^*}$, then there holds either*

$$\min \left\{ \frac{\omega}{2^{l_2}}, \Phi_{1,l} \left(\frac{\omega}{2^{l_2}} \right) \right\} \leq \gamma R^{\beta/4}$$

for some $\gamma = \gamma(data, \tilde{\mathcal{K}})$, or there holds

$$s < \mu_+ - \frac{\omega}{2^{l_2}} \ a.e \ on \ Q \left(\frac{R}{8}, \hat{t} \right).$$

Proof. **Step 1: Preliminaries.** For $n \geq l_1$ to be chosen, consider $b = \mu_+ - \frac{\omega}{2^{n+1}}$ and $\mu_+ - \frac{\omega}{2^n} \leq a < b$. As in (6.33) we obtain $a > 1 - \frac{\omega}{2^{m_0 - 2}}$. For $0 < \sigma < \eta \leq 1$ and with $r = R/4$, consider a time-independent cut-off function ξ on $B_{\eta r}$ that equals one on $B_{\sigma r}$, satisfying

$$0 \leq \xi \leq 1, \quad |\nabla \xi| \leq \frac{C_\xi}{(\eta - \sigma)r} \ \text{and} \ \left| D^2 \xi \right| \leq \frac{C_\xi}{(\eta - \sigma)^2 r^2}.$$

We define

$$M_{a,\eta} := \{s > a\} \cap Q(\eta r, \hat{t}) \tag{6.66}$$

and derive an energy estimate on $Q(\eta r, \hat{t})$ and proceed similar to the proof of Lemma 6.14.

Step 2: Energy estimates. We emphasize that, in contrast to the proof of Lemma 6.14, we consider time-independent functions ξ and that we seek to integrate with respect to a different cylinder. Observe that Lemma 6.14 yields

$$s(\cdot, t) < \mu_+ - \frac{\omega}{2^{m_0}} \le a \text{ a.e. on } B_r \text{ for a.e. } t \in (-\hat{t} - \delta, -\hat{t} + \delta)$$

for some $\delta > 0$, which implies that

$$(s_b - a)_+(\cdot, t) = 0 \text{ a.e. on } x \in B_r \text{ for a.e. } t \in (-\hat{t} - \delta, -\hat{t} + \delta).$$

With these considerations, similarly to the decution of (6.40), we obtain

$$\int_{t_1}^t \langle \phi \partial_t s, (s_b - a)_+ \xi^2 \rangle = \frac{1}{2} \int_{B_{\eta r} \times \{t\}} \phi(s_b - a)_+^2 \xi^2 + (b - a) \int_{B_{\eta r} \times \{t\}} \phi(s - b)_+ \xi^2 \tag{6.67}$$

for a.e. $-\hat{t} < t_1 < t < 0$. The integrals concerning the spatial derivatives are treated as in the second step of the proof of Lemma 6.14. Hence, in comparison to (6.43) and (6.44), we find that either

$$\min\left\{\frac{\omega}{2^n}, \Phi_{1,l}\left(\frac{\omega}{2^{n+1}}\right)\right\} \le \gamma r^{\beta/4} \tag{6.68}$$

or

$$\operatorname*{ess\,sup}_{-\hat{t} < t < 0} \int_{B_{\eta r} \times \{t\}} (s_b - a)_+^2 \xi^2 + \Phi_{1,l}\left(\frac{\omega}{2^{n+1}}\right) \int_{Q(\eta r, \hat{t})} |\nabla(s_b - a)_+|^2 \xi^2$$
$$\le \frac{\gamma}{(\eta - \sigma)^2 r^2} \left(\frac{\omega}{2^n}\right)^2 \Phi_{1,u}\left(\frac{\omega}{2^{m_0-2}}\right) |M_{a,\eta}|. \tag{6.69}$$

We change the time variable according to

$$\bar{t} = \left(\frac{R}{8}\right)^2 \frac{t}{\hat{t}}$$

and define the transformed function $\bar{s}(\cdot, \bar{t}) = s(\cdot, t)$ as well as the transformed set $\bar{M}_{a,\eta} = \{\bar{s} > a\} \cap Q(\eta r, (\frac{R}{8})^2)$. After multiplication of the whole expression (6.69) with

$$H(n) := \frac{(R/8)^2}{\Phi_{1,l}\left(\omega/2^{n+1}\right)\hat{t}} > 0,$$

this change of time variables leads to

$$H(n) \operatorname*{ess\,sup}_{-(R/8)^2 < t < 0} \int_{B_{\eta r} \times \{t\}} (\bar{s}_b - a)_+^2 \xi^2 + \int_{Q(\eta r, (\frac{R}{8})^2)} |\nabla (\bar{s}_b - a)_+|^2 \xi^2 \\ \leq \frac{\gamma}{(\eta - \sigma)^2 r^2} \left(\frac{\omega}{2^n} \right)^2 \frac{\Phi_{1,u} \left(\omega/2^{m_0 - 2} \right)}{\Phi_{1,l} \left(\omega/2^{n+1} \right)} \left| \bar{M}_{a,\eta} \right|. \tag{6.70}$$

For n large enough, we find $H(n) \geq 1$ and obtain, after adding the term

$$\int_{Q(\eta r, (\frac{R}{8})^2)} (\bar{s}_b - a)_+^2 |\nabla \xi|^2$$

on both sides, the simplified inequality

$$\| (\bar{s}_b - a)_+ \xi \|_{V^2(Q(\eta r, (R/8)^2))}^2 \leq \frac{\gamma}{(\eta - \sigma)^2 r^2} \left(\frac{\omega}{2^n} \right)^2 \left[\frac{\Phi_{1,u} \left(\omega/2^{m_0 - 2} \right)}{\Phi_{1,l} \left(\omega/2^{n+1} \right)} + 1 \right] \left| \bar{M}_{a,\eta} \right|. \tag{6.71}$$

Step 3: Iteration. We procceed similarly to the iteration presented in the third step of the proof of Lemma 6.14. For $k = 0, 1, 2, \ldots$, define

$$a_k := b - \frac{\omega}{2^{n+1}} 2^{-k}, \qquad \eta_k := \frac{1}{2} + 2^{-(k+1)}, \qquad r_k := \eta_k r \qquad \text{and} \qquad A_k := \left| \bar{M}_{a_k, r_k} \right|,$$

and choose $a = a_k$, $\eta = \eta_k$ and $\sigma = \eta_{k+1}$ in (6.71). With $r_k \geq \frac{R}{8}$ we obtain, analogously to (6.46), the estimate

$$2^{-2(k+1)} \left(\frac{\omega}{2^{n+1}} \right)^2 A_{k+1} \leq \gamma \frac{2^{2(k+2)}}{(R/8)^2} \left(\frac{\omega}{2^{m_0}} \right)^2 \left[\frac{\Phi_{1,u} \left(\omega/2^{m_0 - 2} \right)}{\Phi_{1,l} \left(\omega/2^{n+1} \right)} + 1 \right] A_k^{1 + \frac{2}{d+2}}. \tag{6.72}$$

Define the numbers

$$X_k := \frac{A_k}{|Q(r_k, (R/8)^2)|}$$

and divide (6.72) by $|Q(r_k, R^2)|$. Similarly to (6.48), we derive

$$\left| Q \left(r_{k+1}, \left(\frac{R}{8} \right)^2 \right) \right| \left(\frac{R}{8} \right)^2 \geq 2^{-d} |B_1|^{-\frac{2}{d+2}} \left| Q(r_k, (R/8)^2) \right|^{1 \frac{2}{d+2}}$$

and find

$$X_{k+1} \leq 2^{d+10} |B_1|^{\frac{2}{d+2}} \gamma \left[\frac{\Phi_{1,u} \left(\omega/2^{m_0 - 2} \right)}{\Phi_{1,l} \left(\omega/2^{n+1} \right)} + 1 \right] 4^{2k} X_k^{1 + \frac{d}{d+2}}.$$

From Lemma 6.2 we conclude that if

$$X_0 \leq \left(2^{d+10} |B_1|^{\frac{2}{d+2}} \gamma \left[\frac{\Phi_{1,u} \left(\omega/2^{m_0 - 2} \right)}{\Phi_{1,l} \left(\omega/2^{n+1} \right)} + 1 \right] \right)^{-\frac{d+2}{2}} 4^{-2 \left(\frac{d+2}{2} \right)^2} =: \nu_1 \in (0, 1) \tag{6.73}$$

holds, then $X_k \to 0$ as $k \to \infty$. Apply Lemma 6.16 with this ν_1 and infer the existence of $l_1 = l_1(data, \tilde{\mathcal{K}}, \nu_1, m)$, such that

$$M(t, l_1) := \left| \left\{ s(t) > \mu_+ - \frac{\omega}{2^{l_1}} \right\} \cap B_{\frac{R}{4}} \right| \leq \nu_1 \left| B_{\frac{R}{4}} \right|, \text{ for a.e. } t \in (-\hat{t}, 0). \tag{6.74}$$

Step 4: Conclusion. We need to verify, that indeed the choice $X_0 \leq \nu_1$ is possible. To this end, observe $M(t, n) \subset M(t, l_1)$ since $n \geq l_1$. Recall that M is defined in terms of s whereas A_0 is defined in terms of $\bar{s}$, and conclude

$$X_0 = \frac{A_0}{\left| Q\left(\frac{R}{4}, (\frac{R}{8})^2 \right) \right|} = \frac{(\frac{R}{8})^2}{\hat{t}} \frac{\int_{-\hat{t}}^0 |M(t, n)|}{\left| Q\left(\frac{R}{4}, (\frac{R}{8})^2 \right) \right|} \leq \frac{(\frac{R}{8})^2}{\hat{t}} \hat{t} \operatorname*{ess\,sup}_{-\hat{t} < t < 0} \frac{|M(t, l_1)|}{\left(\frac{R}{8} \right)^2 \left| B_{\frac{R}{4}} \right|} \leq \nu_1. \tag{6.75}$$

Since $r_k \searrow R$, $a_k \nearrow \mu_+ - \frac{\omega}{2^{n+1}}$ and $X_k \to 0$ implies $A_k \to 0$, we obtain

$$\left| \left\{ \bar{s} \geq \mu_+ - \frac{\omega}{2^{n+1}} \right\} \cap Q\left(\frac{R}{8}, \left(\frac{R}{8} \right)^2 \right) \right| = \left| \left\{ s \geq \mu_+ - \frac{\omega}{2^{n+1}} \right\} \cap Q\left(\frac{R}{8}, \hat{t} \right) \right| = 0$$

and the proposition is proved if we choose $l_2 = n + 1 > l_1$ modulo the independence of ν_1 and l_2 on ω.

Concerning the definition of ν_1, we observe that ν_1 is independent of ω since, due to Assumption A3.7, the quotient

$$\frac{\Phi_{1,u}\left(\frac{\omega}{2^{m_0 - 2}} \right)}{\Phi_{1,l}\left(\frac{\omega}{2^{n+1}} \right)} = \frac{c_{1,u}}{c_{1,l}} 2^{\alpha_1(n - m_0 - 1)}$$

is independent of ω. To take the independence of l_2 on ω into account we recall that $H(l_2) \geq 1$. With $\hat{t} \leq \theta_m R^2$, $\alpha_1 \geq \alpha_0$ and $0 < \omega \leq 1$, it suffices to choose

$$l_2 \geq \max\left\{ l_1, \frac{m\alpha_0 + 3 + \log_2(\frac{c_{1,l}}{c_{0,l}})}{\alpha_1} \right\}$$

to ensure

$$H(l_2) \geq 2^{-3} \frac{\Phi_{0,l}\left(\frac{\omega}{2^m} \right)}{\Phi_{1,l}\left(\frac{\omega}{2^{l_2}} \right)} = \frac{c_{0,l}}{c_{1,l}} \omega^{\alpha_0 - \alpha_1} 2^{l_2 \alpha_1 - m\alpha_0 - 3} \geq 1.$$

The choice of l_2 is independent of ω but still depends on m and we conclude the proof. $\quad\square$

Corollary 6.19. *Let Assumptions A6.1 and A6.2 hold. Then there exists a constant $\sigma_1 \in (0, 1)$ depending only on the data, $\tilde{\mathcal{K}}$ and l_2, thus on m, such that if (6.28) holds for some cylinder $Q_R^{t^*}$, then either*

$$\min\left\{ \frac{\omega}{2^{l_2}}, \Phi_{1,l}\left(\frac{\omega}{2^{l_2}} \right) \right\} \leq \gamma R^{\beta/4} \tag{6.76}$$

for some $\gamma = \gamma(data, \tilde{\mathcal{K}})$, or

$$\operatorname*{ess\,osc}_{Q(\frac{R}{8}, \hat{t})} s \leq \sigma_1 \omega. \tag{6.77}$$

Proof. Proposition 6.18 provides an l_2 such that

$$\operatorname*{ess\,sup}_{Q(\frac{R}{8}, \hat{t})} s \le \mu_+ - \frac{\omega}{2^{l_1+1}}.$$

From this we get

$$\operatorname*{ess\,osc}_{Q(\frac{R}{8}, \hat{t})} s = \operatorname*{ess\,sup}_{Q(\frac{R}{8}, \hat{t})} s - \operatorname*{ess\,inf}_{Q(\frac{R}{8}, \hat{t})} s \le \mu_+ - \frac{\omega}{2^{l_2+1}} - \mu_- = \left(1 - \frac{\omega}{2^{l_2+1}}\right)\omega.$$

Collection of the alternatives in Lemma 6.14, Lemma 6.16 and Proposition 6.18 and choosing γ as the maximum of the corresponding constants provides (6.76). $\qquad\square$

6.4 The Second Alternative — Degeneracy at $s = 0$

In this section, we examine the situation where the first alternative (6.28) fails to hold. In this case, the second alternative is in force, i.e. for every cylinder of type $Q_R^{t^*} \subset Q_R(\theta_m)$ there holds (6.29), where $\nu_0 \in (0, 1)$ is given by Lemma 6.14.

Let us fix such a cylinder $Q_R^{t^*}$ for the moment. Since $m_0 > 1$ we find

$$\mu_+ - \frac{\omega}{2^{m_0}} \ge \omega + \mu_- + \frac{2^{m_0} - 1}{2^{m_0}} \ge \mu_- + \frac{2^{m_0} - 1}{2^{m_0}}\omega > \mu_- + \frac{\omega}{2^{m_0}}$$

and under consideration of

$$\left\{s < \mu_- + \frac{\omega}{2^{m_0}}\right\} \subset \left\{s \le \mu_+ - \frac{\omega}{2^{m_0}}\right\} = \left\{s > \mu_+ - \frac{\omega}{2^{m_0}}\right\}^{\complement},$$

we infer

$$\left|\left\{s < \mu_- + \frac{\omega}{2^{m_0}}\right\} \cap Q_R^{t*}\right| \le (1 - \nu_0)\left|Q_R^{t^*}\right|. \tag{6.78}$$

For convenience, we define the intervals $I(t^*) := (t^* - \theta_{m_0}R^2, t^* - \frac{\nu_0}{2}\theta_{m_0}R^2)$.

Lemma 6.20. *Let Assumptions A6.1 and A6.2 hold. For any cylinder $Q_R^{t^*} \subset Q_r(\theta_m)$ as in (6.23) and such that (6.78) is in force, there is a subset $E \subset I(t^*)$ with $|E| \ne 0$ such that for all $t \in E$ there holds*

$$\left|\left\{s(t) < \mu_- + \frac{\omega}{2^{m_0}}\right\} \cap B_R\right| \le \left(\frac{1 - \nu_0}{1 - \frac{\nu_0}{2}}\right)|B_R|.$$

Proof. Assume $|E| = 0$. Hence for a.e. $t \in I(t^*)$, we infer

$$\left|\left\{s(t) < \mu_- + \frac{\omega}{2^{m_0}}\right\} \cap B_R\right| > \left(\frac{1 - \nu_0}{1 - \frac{\nu_0}{2}}\right)|B_R|.$$

Together with $|I(t^*)| = (1 - \frac{\nu_0}{2})R^2\theta_{m_0}$, we derive

$$
\left| \left\{ s < \mu_- + \frac{\omega}{2^{m_0}} \right\} \cap Q_R^{t^*} \right|
$$
$$
\geq \int_I \left| \left\{ s(t) < \mu_- + \frac{\omega}{2^{m_0}} \right\} \cap B_R \right| \, dt
$$
$$
> (1 - \nu_0) \, |B_R| \, \theta_{m_0} R^2 = (1 - \nu_0) \left| Q_R^{t^*} \right|
$$

which contradicts (6.78). $\qquad\square$

Corollary 6.21. *Let the setting from Assumption A6.1 be given and let Assumption A6.2 hold. For any cylinder $Q_R^{t^*} \subset Q_R(\theta_m)$ such that (6.78) holds, there is a Lebesgue point $t^\circ \in I(t^*)$ of $s(t)$ such that*

$$
\left| \left\{ s(t^\circ) < \mu_- + \frac{\omega}{2^{m_0}} \right\} \cap B_R \right| \leq \left(\frac{1 - \nu_0}{1 - \frac{\nu_0}{2}} \right) |B_R| .
$$

6.4.1 A Second Logarithmic Estimate

Lemma 6.22 (A second logarithmic estimate)**.** *Under Assumptions A6.1 and A6.2 consider a cylinder $Q_R^{t^*} \subset Q_R(\theta_m)$ as in (6.23) and let (6.78) hold. There exists a constant $m_0 + 2 < q_1 \in \mathbb{N}$ with $q_1 = q_1(data, \tilde{\mathcal{K}})$, such that either*

$$
\min \left\{ \frac{\omega}{2^{q_1}}, \Phi_{0,u} \left(\frac{\omega}{2^{m_0-2}} \right) \right\} \leq \gamma R^{\beta/4} \tag{6.79}
$$

for some specific constant $\gamma = \gamma(data, \tilde{\mathcal{K}})$, or

$$
\left| \left\{ s(t) < \mu_- + \frac{\omega}{2^{q_1}} \right\} \cap B_R \right| \leq \left[1 - \left(\frac{\nu_0}{2} \right)^2 \right] |B_R| \tag{6.80}
$$

for a.e. $t \in [t^* - \frac{\nu_0}{2}\theta_{m_0}R^2, t^*]$.

Proof. **Step 1: Preliminaries** Let $\mathbb{N} \ni n > m_0 + 2$ to be chosen and abbreviate $\Psi = \Psi_{\bar{a},\bar{b}}$ with $\bar{a} = \frac{\omega}{2^n}$ and $\bar{b} = \frac{\omega}{2^{m_0}}$, where Ψ is the function of Definition 6.7. Choose $r = R$ and for $\sigma \in (0,1)$ let ξ be a smooth, time-independent cut-off function on B_r that equals one on $B_{(1-\sigma)r}$ where and such that $0 \leq \xi \leq 1$, $|\nabla \xi| \leq C_\xi \frac{1}{\sigma r}$ and $|D^2\xi| \leq \frac{C_\xi}{\sigma^2 r^2}$.

Let $a = \mu_- + \frac{\omega}{2^{m_0}}$ and $b = \bar{a}$. In contrast to the first alternative, we use the different notation $s_b := \max\{s, b\}$. Then $(s_b - a)_- \leq \frac{\omega}{2^{m_0}} - \frac{\omega}{2^n}$ and, analogous to (6.33), we find $b < a \leq \frac{\omega}{2^{m_0-2}}$. Due to Proposition 4.21 we find $(s_b - a)_- = a - T_b^a(s) \in L^2(0,T; H^1(\Omega))$ and hence $\nabla(s_b - a)_- = -\nabla T_b^s(s)$. As in the proof of Lemma 6.16, we abbreviate $\psi :=$

$\psi(s) := \Psi((s_b - a)_-)$ and, again with abuse of notation, $\psi' := \psi'(s) := -\Psi'((s_b - a)_-)$. We emphasize that ψ' is only the formal derivative of ψ for $b < s$. Particularly, we find

$$\psi'(s) := \begin{cases} -\Psi'((s_b - a)_-) & b < s \\ -\Psi'(a - b) & s \leq b \end{cases}$$

as well as

$$2\psi(s)\psi'(s) = \begin{cases} (\psi^2)'(s) = -\Psi((s_b - a)_-)\Psi'((s_b - a)_-) & s > b \\ -2\Psi(a - b)\Psi'(a - b) & s \leq b \end{cases}$$

and $2\psi\psi'\xi^2 \in L^2(0, T; H_0^1(\Omega))$. Analogously to (6.57), we infer the pointwise identity

$$\begin{aligned} \nabla[\psi\psi'] &= \nabla[-\Psi((s_b - a)_-)\Psi'((s_b - a)_-)] \\ &= -\Psi'^2((s_b - a)_-)(1 + \Psi)((s_b - a)_-)\nabla(s_b - a)_- = -\psi'^2(1 + \psi)\nabla(s_b - a)_-. \end{aligned} \tag{6.81}$$

With the properties of Ψ from Lemma 6.8, we infer $\psi \leq \ln(2^{n-m_0}) \leq (n - m_0)\ln(2) \leq n$ and $|\psi'| \leq \frac{2^n}{\omega}$. For $t \in (t^\circ, t^*)$, with t° from Corollary 6.21, we define the cylinder $U_t := B_r \times (t^\circ, t)$ and remark that

$$0 \leq t^* - t^\circ \leq t^* - t^* + \theta_{m_0}r^2 \leq \theta_{m_0}r^2. \tag{6.82}$$

Step 2: Logarithmic Estimates. From Lemma 4.31 we infer that t° is also a Lebesgue point of $2\psi\psi'\xi^2$ since $\psi\psi'$ is Lipschitz continuous in s. For t° and $t \in (t^\circ, t^*)$, the function $v = 2\psi\psi'\xi^2 \mathbb{1}_{[t^\circ, t]}$ is admissible in equation (3.23). In comparison to the proof of the first logarithmic estimate in Lemma 6.16, we need to take certain integrals evaluated in t° into account. Similar to (6.56) and with (6.15), we obtain

$$\begin{aligned} &\int_{t^\circ}^t \langle \phi \partial_t s, 2\psi\psi'\xi^2 \rangle \\ &\geq C_* \int_{B_r \times \{t\}} \psi^2\xi^2 - \int_{B_r \times \{t^\circ\}} \psi^2\xi^2 - 2C^*\Psi(a - b)\Psi'(a - b)(\mu_- - b)_- |B_r|. \end{aligned} \tag{6.83}$$

In difference to the proof of Lemma 6.16, definition of ψ' and (6.81) contain a sign. However, this is not an issue since terms containing this sign cancel or are estimated by its absolute value. We do not comment on this issue in the following and, furthermore, we only point out the differences to the proof of Lemma 6.16.

Concerning the integrals containing the spatial derivatives, the first difference is spotted in the first integral of I_{12} and we replace the corresponding term by $[-(\Phi(s) - \Phi(b))_- + \Phi(T_b(s))]$ which leads to the modified estimate

$$|J_{11}| \leq \gamma\Psi(a - b)\Psi'(a - b)\Phi_{0,u}(b)(\mu_- - b)_- \frac{1 + r}{\sigma^2 r^2} |U_t|.$$

We emphasize that $(\mu_- - b)_- \leq \omega 2^{-n}$ and $\Phi_{0,u}(b) \leq \Phi_{0,u}(\omega 2^{-(m_0-2)})$. The second difference is found in the integral I_2, where we replace the corresponding term by $\nabla_x \Phi(T_b^a(s)) + [\nabla_x \Phi(s) - \nabla_x \Phi(b)] \mathbb{1}_{\{s<b\}}$ and estimate

$$|\nabla_x \Phi(s) - \nabla_x \Phi(b)| \, \mathbb{1}_{\{s<b\}} = \int_s^b |\nabla_x \Phi'(v) \, \mathrm{d}v| \, \mathbb{1}_{\{s<b\}} \leq C^*(s-b)_-$$

to obtain

$$|I_{23}| \leq \gamma \Psi(a-b) \Psi'(a-b) (\mu_- - b)_- \frac{r}{\sigma^2 r^2} |U_t| \, .$$

We infer from Remark 3.8 that I_4 may be estimated as before. With these considerations, we obtain, assuming that

$$\gamma r^{\beta/4} < \min \left\{ \Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right), \frac{\omega}{2^n} \right\}, \tag{6.84}$$

the estimate

$$\fint_{B_{(1-\sigma)r} \times \{t\}} \psi^2 \leq \fint_{B_r \times \{t^\circ\}} \psi^2 \xi^2 + \gamma \ln(2^{n-m_0}) \, |B_r|$$
$$+ \frac{\gamma n}{\sigma^2 r^2} \Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right) (t^* - t^\circ) \, |B_r| \, . \tag{6.85}$$

The first and second term on the right-hand side originate from (6.83) under consideration of $\psi'(a-b)(\mu_- - b)_- \leq 1$.

Step 3: Conclusion. With the properties of ψ from (6.82) and from Corollary 6.21, we infer

$$\fint_{B_{(1-\sigma)r} \times \{t\}} \psi^2 \leq \ln^2(2)(n-m_0)^2 \left(\frac{1-\nu_0}{1-\frac{\nu_0}{2}}\right) |B_r| + \gamma n \ln(2) \, |B_r|$$
$$+ \frac{\gamma n \theta_{m_0}}{\sigma^2} \Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right) |B_r| \tag{6.86}$$

and

$$S(t) := \left\{ s(t) < \mu_- + \frac{\omega}{2^n} \right\} \cap B_{(1-\sigma)r} \subset B_r$$

for a.e. every $t \in (t^\circ, t^*)$. On $S(t)$ we find $\xi = 1$ and $(s_b - a)_- = \frac{\omega}{2^{m_0}} - \frac{\omega}{2^n}$, which implies

$$\psi^2 \geq \ln_+^2(2^{n-m_0-1}) = \ln^2(2)(n-m_0-1)^2 \, .$$

Application to (6.86) yields

$$|S(t)| \leq \left(\frac{n-m_0}{n-m_0-1}\right)^2 \frac{1-\nu_0}{1-\frac{\nu_0}{2}} |B_r| + \gamma \frac{n-m_0}{(n-m_0-1)^2} |B_r|$$
$$+ \frac{\gamma n \theta_{m_0}}{\ln^2(2)(n-m_0-1)^2 \sigma^2} \Phi_{0,u}\left(\frac{\omega}{2^{m_0-1}}\right) |B_r| \, .$$

We use the identity $\left| B_r \setminus B_{(1-\sigma)r} \right| = |B_1| \, r^d \sigma^d = |B_r| \, \sigma^d$ and observe that

$$
\left| \left\{ s(t) < \mu_- + \frac{\omega}{2^n} \right\} \cap B_r \right|
$$

$$
\leq \left| \left\{ s(t) < \mu_- + \frac{\omega}{2^n} \right\} \cap B_{(1-\sigma)r} \right| + \left| B_r \setminus B_{(1-\sigma)r} \right|
$$

$$
\leq |S| + \sigma^d |B_r| .
$$

With $\sigma^d \leq d\sigma$ for $\sigma \in (0, 1)$, we conclude that

$$
\left| \left\{ s(t) < \mu_- + \frac{\omega}{2^n} \right\} \cap B_r \right|
$$
$$
\leq \left[\left(\frac{n - m_0}{n - m_0 - 1} \right)^2 \frac{1 - \nu_0}{1 - \frac{\nu_0}{2}} + d\sigma \right] |B_r| \tag{6.87}
$$
$$
+ \left[\frac{\gamma n}{(n - m_0 - 1)^2} + \frac{\gamma n \theta_{m_0}}{(n - m_0 - 2)^2 \sigma^2} \Phi_{0,u} \left(\frac{\omega}{2^{m_0 - 1}} \right) \right] |B_r| .
$$

Choose σ such that $d\sigma \leq \frac{1}{4} \nu_0^2$ and then $q_1 = n$ so large that

$$
\left(\frac{n - m_0}{n - m_0 - 1} \right)^2 \leq (1 + \nu_0) \left(1 - \frac{\nu_0}{2} \right), \quad \frac{\gamma n}{(n - m_0 - 1)^2} \leq \frac{1}{4} \nu_0^2
$$
$$
\text{and} \quad \frac{\gamma n \theta_{m_0}}{(n - m_0 - 1)^2 \sigma^2} \Phi_{0,u} \left(\frac{\omega}{2^{m_0 - 1}} \right) \leq \frac{1}{4} \nu_0^2. \tag{6.88}
$$

Since ν_0 is a constant that only depends on the data and on $\tilde{\mathcal{K}}$, so is σ. Furthermore, the first condition for the choice of q_1 is possible independent of ω since $\nu_0 \in (0, 1)$ and $(1 - \nu_0/2)(1 + \nu_0) = 1 + \nu_0/2 - (\nu_0^2/2) > 1$. Trivially, the second condition for q_1 is possible and independent of ω. To consider the third condition for q_1 we consider the quotient

$$
\frac{\Phi_{0,u} \left(\omega/2^{m_0 - 2} \right)}{\Phi_{1,l} \left(\omega/2^{m_0 + 1} \right)} = \frac{c_{0,u}}{c_{1,l}} 2^{3\alpha_0} \tag{6.89}
$$

which is independent of ω. Hence, choosing $q_1 = n$ such that even

$$
\frac{\gamma q_1}{(q_1 - m_0 - 1)^2 \sigma^2} \frac{c_{0,u}}{c_{1,l}} 2^{3\alpha_0} \leq \frac{3}{8} \nu_0^2,
$$

we obtain from (6.87) the conclusion

$$
\left| \left\{ s(t) < \mu_- + \frac{\omega}{2^{q_1}} \right\} \cap B_r \right| \leq \left(1 - \left(\frac{\nu_0}{2} \right)^2 \right) |B_r|
$$

for a.e. $t \in (t^\circ, t^*)$. We emphasize that the choice of q_1 is independent of ω and the lemma is shown. $\qquad \square$

Corollary 6.23. *Let Assumptions A6.1 and A6.2 hold. Then, there exists a constant $q_1 = q_1(data, \tilde{\mathcal{K}})$ such that either*

$$\min\left\{\frac{\omega}{2^{q_1}}, \Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right)\right\} \leq \gamma R^{\beta/4}$$

for some specific constant $\gamma = \gamma(data, \tilde{\mathcal{K}})$ or

$$\left|\left\{s(t) < \mu_- - \frac{\omega}{2^{q_1}}\right\} \cap B_R\right| \leq \left[1 - \left(\frac{\nu_0}{2}\right)^2\right]|B_R|$$

for a.e. $t \in \left(-\frac{\theta_m}{2}R^2, 0\right)$.

Proof. Since the first alternative fails to hold, the conclusion of Lemma 6.22 holds for every cylinder of type $Q_R^{t^*}$. Recalling from (6.24) that $0 \geq t^* \geq R^2(\theta_{m_0} - \theta_m)$, we conclude that the assertion of Lemma 6.22 holds for a.e.

$$0 \geq t \geq t^* - \frac{\nu_0}{2}\theta_{m_0}R^2 \geq R^2\left[\left(1 - \frac{\nu_0}{2}\right)\theta_{m_0} - \theta_m\right].$$

Above all, using (6.22), i.e. $2\theta_{m_0} \leq \theta_m$, the conclusion of Lemma 6.22 holds for a.e.

$$0 \geq t \geq -\frac{\theta_m}{2}R^2 \geq -\left(1 + \frac{\nu_0}{2}\right)\frac{\theta_m}{2}R^2 = R^2\left[\frac{1}{2}\left(1 - \frac{\nu_0}{2}\right) - 1\right]\theta_m$$

$$\geq R^2\left[\left(1 - \frac{\nu_0}{2}\right)\theta_{m_0} - \theta_m\right]$$

and we conclude the corollary. $\qquad\square$

6.4.2 Energy Estimates in Terms of Φ

The information from Corollary 6.23 is used to derive a statement that resembles (6.28) from the first alternative written in terms of μ_-.

We emphasize that, contrary to the usual approach found in the literature, we will not be able to use DeGiorgi's lemma in the following proof. Essentially, we cannot apply the lemma since we need to cut-off at contour lines of $\Phi(x, s)$. To overcome this problem, we use the more general Proposition 1.2.

Proposition 6.24. *Assume that the setting described in Assumption A6.1 is given and that Assumption A6.2 holds. For every $\lambda_0 \in (0, 1)$, there exist constants $q_2 > q_1 \in \mathbb{N}$, $q_2 = q_2(data, \tilde{\mathcal{K}})$ and $q_2 < m_1 \in \mathbb{N}$, $m_1 = m_1(data, \tilde{\mathcal{K}}, \lambda_0)$, such that, if $m > m_1$, then either*

$$\min\left\{\Phi_{0,u}\left(\frac{\omega}{2^{m_1}}\right), \Phi_{0,l}\left(\frac{\omega}{2^{q_2}}\right), \frac{\omega}{2^{q_2}}\right\} \leq \gamma R^{\beta/4} \tag{6.90}$$

for some specific constant $\gamma = \gamma(data, \tilde{\mathcal{K}})$, or

$$\left| \left\{ s < \mu_- - \frac{\omega}{2^{q_2}} \right\} \cap Q\left(R, \frac{\theta_m}{2} R^2\right) \right| \leq \lambda_0 \left| Q\left(R, \frac{\theta_m}{2} R^2\right) \right|. \tag{6.91}$$

Proof. **Step 1. Preliminaries:** We abbreviate $k = k(n) = \mu_- + \frac{\omega}{2^n}$ for $n \geq q_1$ and find, similar to (6.33), that $k(n) \leq \frac{\omega}{2^{m_0-2}}$. Since $\Phi \in C^1(\overline{\Omega} \times [0, 1])$ and $\Phi(s) \in L^2(0, T; H^1(\Omega))$, we infer from Lemma 4.4 that $-(\Phi(s) - \Phi(k))_- \in L^2(0, T; H^1(\Omega))$.

Consider a smooth cut-off function ξ on $Q(2R, \theta_m R^2)$ that equals one on $Q(R, \frac{\theta_m}{2} R^2)$ and zero on the parabolic boundary $\partial_p Q(2R, \theta_m R^2)$ and is such that

$$0 \leq \xi \leq 1, \quad |\nabla \xi| \leq \frac{C_\xi}{R} \quad \text{and } 0 \leq \partial_t \xi \leq \frac{C_\xi}{\theta_m R^2}.$$

We define the set $U_t = B_{2R} \times (-\theta_m R^2, t)$.

Step 2. Energy Estimates. We use $-(\Phi(s) - \Phi(k))_- \xi^2 \in L^2(0, T; H_0^1(\Omega))$ in (3.23). Considering the integral concerning the time derivative, we intend to exploit Lemma 4.36 under consideration of Remark 4.38. To use the lemma, we need to introduce g and h as in Definition 4.34.

We choose the function $g : (x, s) \mapsto -(\Phi(x, s) - \Phi(x, k))_-$ which is nondecreasing for every $x \in \overline{\Omega}$. With $h(x) = k$, we find $g(x, h(x)) = 0$. Consequently, with (4.34) and the non-negativity of this expression, we infer

$$\int_{-\theta_m R^2}^t \langle \phi \partial_t s, -(\Phi(s) - \Phi(k))_- \xi^2 \rangle \geq - \int_{U_t} \phi \int_s^k (\Phi(\sigma) - \Phi(k))_- \, d\sigma \, \xi \partial_t \xi \tag{6.92}$$

for a.e. $t \in (-\theta_m R^2, 0)$. Concerning the spatial integrals, we write

$$\int_{U_t} [\kappa(\nabla[\Phi(s)] - \nabla_x \Phi(s) + B(s)) + D(s)u] \cdot \nabla\left[-(\Phi(s) - \Phi(k))_- \xi^2\right] = I_1 + I_2 + I_3 + I_4$$

and consider the integrals I_1 to I_4 separately.

We recall the identity $\nabla(u)_- = -\nabla u \, \mathbb{1}_{\{u<0\}}$ from Lemma 4.4 and obtain under consideration of the support of $(\Phi(s) - \Phi(k))_-$, Cauchy's inequality, from $|\nabla[\Phi(x, k)]| \leq C^*$ and with an analogous notation in comparison to the previous proof, the estimates

$$I_{11} \int_{U_t} \kappa \nabla[\Phi(s)] \cdot (-\nabla(\Phi(s) - \Phi(k))_- \xi^2)$$

$$= \int_{U_t} \kappa(-\nabla[\Phi(s) - \Phi(k)]) \cdot \nabla(\Phi(s) - \Phi(k))_- \xi^2 - \kappa \nabla[\Phi(k)] \cdot \nabla(\Phi(s) - \Phi(k))_- \xi^2$$

$$\geq \frac{C_*}{2} \int_{U_t} |\nabla(\Phi(s) - \Phi(k))_- \xi|^2 - \gamma \left| Q(2R, \theta_m R^2) \right|$$

and

$$|I_{12}| \leq \left| \int_{U_t} \kappa \nabla[\Phi(s)] \cdot (-(\Phi(s) - \Phi(k))_- 2\xi \nabla \xi) \right|$$

$$= 2 \left| \int_{U_t} \kappa [\nabla(\Phi(s) - \Phi(k))_- - \nabla \Phi(k)] \cdot [(\Phi(s) - \Phi(k))_- \xi \nabla \xi] \right|$$

$$\leq \varepsilon \int_{U_0} |\nabla(\Phi(s) - \Phi(k))_- \xi|^2 + \frac{\gamma}{R^2} \int_{U_0} (\Phi(s) - \Phi(k))_-^2 + \gamma \left| Q(2R, \theta_m R^2) \right|$$

For I_2 and I_3, we exploit the regularity of Φ, $|B(s)| \leq C^*$ and Cauchy's inequality to obtain

$$|I_2| + |I_3| \leq \varepsilon \int_{U_t} |\nabla(\Phi(s) - \Phi(k))_-|^2 \xi^2 + \gamma \left| Q(2R, \theta_m R^2) \right| + \frac{\gamma}{R^2} \int_{U_0} (\Phi(s) - \Phi(k))_-^2.$$

Concerning I_4, we use $|D(x,s)| \leq C^*$, Cauchy's inequality and (6.59) to infer

$$|I_4| \leq \varepsilon \int_{U_0} |\nabla(\Phi(s) - \Phi(k))_- \xi|^2 + \gamma \int_{U_0} u^2 \xi^2 + \frac{\gamma}{R^2} \int_{U_0} (\Phi(s) - \Phi(k))_-^2$$

$$\leq \varepsilon \int_{U_0} |\nabla(\Phi(s) - \Phi(k))_- \xi|^2 + \left(\frac{R^{2\beta}}{R^2} + 1 \right) \left| Q(2R, \theta_m R^2) \right|$$

$$+ \frac{\gamma}{R^2} \int_{U_0} (\Phi(s) - \Phi(k))_-^2.$$

Collecting the previous estimates and choosing ε appropriately yields

$$\int_{U_0} |\nabla(\Phi(s) - \Phi(k))_-|^2 \leq \left(\frac{R^{2\beta}}{R^2} + 1 \right) \left| Q(2R, \theta_m R^2) \right| + \frac{\gamma}{R^2} \int_{U_0} (\Phi(s) - \Phi(k))_-^2$$

$$+ \frac{\gamma}{\theta_m R^2} \int_{U_0} \int_s^k (\Phi(\sigma) - \Phi(k))_- \, d\sigma. \tag{6.93}$$

Since Φ' is increasing on $[0, 4\delta_0]$ for every $x \in \Omega$ and since $k < \frac{\omega}{2^{m_0 - 2}}$, we obtain

$$(\Phi(\sigma) - \Phi(k))_- \leq \Phi'(k)(\sigma - k)_- \leq \Phi_{0,u}(k)(\sigma - k)_-.$$

Application to (6.93) yields

$$\int_{U_0} |\nabla(\Phi(s) - \Phi(k))_- \xi|^2 \leq \frac{\gamma}{R^2} \Phi_{0,u}(k) \left(\frac{w}{2^n} \right)^2 \left(\Phi_{0,u}(k) + \frac{1}{\theta_m} \right) \left| Q(2R, \theta_m R^2) \right|$$

$$+ \left(\frac{R^{2\beta}}{R^2} + 1 \right) \left| Q(2R, \theta_m R^2) \right|. \tag{6.94}$$

We assume that

$$\gamma R^{\beta/4} \leq \min \left\{ \Phi_{0,u} \left(\frac{\omega}{2^{m_1}} \right), \Phi_{0,l} \left(\frac{\omega}{2^{n+1}} \right), \frac{\omega}{2^n} \right\} < 1 \tag{6.95}$$

for $m_1 > m_0 - 2$ sufficiently large and chosen in the next step. We use

$$\left| Q(2R, \theta_m R^2) \right| = 2^{d-1} \left| Q(R, \frac{\theta_m}{2} R^2) \right|,$$

integrate on the left-hand side over the smaller set where $\xi = 1$ and exploit

$$\Phi_{0,l}\left(\frac{\omega}{2^{n+1}} \right) \leq \Phi_{0,u}\left(\frac{\omega}{2^{n+1}} \right) \leq \Phi_{0,u}(k)$$

to obtain

$$\begin{aligned}
&\| \nabla (\Phi(s) - \Phi(k))_- \|^2_{Q(R, \frac{\theta_m}{2} R^2)} \\
&\leq \frac{\gamma}{R^2} \Phi_{0,u}(k) \left(\frac{w}{2^n} \right)^2 \left(\Phi_{0,u}(k) + \frac{1}{\theta_m} \right) \left| Q\left(R, \frac{\theta_m}{2} R^2 \right) \right|.
\end{aligned} \tag{6.96}$$

Step 3: Iteration and Conclusion. Since $\Phi(x, \cdot)$ is increasing for every $x \in \overline{\Omega}$, there holds

$$\{ \Phi(s) < \Phi(k(n)) \} = \{ s < k(n) \} \tag{6.97}$$

and analogous results for different relations. The notation $k(n)$ was introduced, but not yet used, in the preliminaries of this proof.

From Corollary 6.23 and since $n \geq q_1$, we deduce

$$\begin{aligned}
\left| \left\{ s(t) \geq \mu_- + \frac{\omega}{2^n} \right\} \cap B_R \right| &\geq |B_R| - \left| \left\{ s(t) < \mu_- + \frac{\omega}{2^{q_1}} \right\} \cap B_R \right| \\
&\geq |B_r| - \left(1 - \left(\frac{\nu_0}{2} \right)^2 \right) |B_R| = \left(\frac{\nu_0}{2} \right)^2 |B_R|
\end{aligned} \tag{6.98}$$

for a.e. $t \in \left(-\frac{\theta_m}{2} R^2, 0 \right)$. We define

$$A_n(t) := |\{ s(t) < k(n) \} \cap B_R| \text{ and } A_n = \int_{-\frac{\theta_m}{2} R^2}^{0} |A_n(t)| \, dt.$$

As in (6.97), these sets can be interpreted in terms of Φ; particularly, we find

$$\begin{aligned}
A_n(t) - A_{n+1}(t) &= \{ k(n+1) \leq s < k(n) \} \cap B_R \\
&= \{ \Phi(k(n+1)) \leq \Phi(s) < \Phi(k(n)) \} \cap B_R.
\end{aligned} \tag{6.99}$$

Pointwise a.e. on $A_{n+1}(t)$ we find

$$\begin{aligned}
[\max\{ \Phi(s(t)), \Phi(k(n+1)) \} - \Phi(k(n))]_- &= \left[\Phi\left(T_{k(n)}^{k(n+1)}(s(t)) \right) - \Phi(k(n)) \right]_- \\
&= [\Phi(k(n+1)) - \Phi(k(n))]_- \geq \Phi_{0,l}(k(n+1))[k(n) - k(n+1)].
\end{aligned}$$

Hence, for a.e. $t \in \left(-\frac{\theta_m}{2}R^2, 0\right)$ and with the definition of $k(n)$ we deduce

$$
\frac{\omega}{2^{n+1}}\Phi_{0,l}(k(n+1))\,|A_{n+1}(t)| \leq \int_{A_{n+1}(t)} \left[\Phi\left(T_{k(n)}^{k(n+1)}(s(t))\right) - \Phi(k(n))\right]_-
$$
$$
\leq \int_{B_R} \left[\Phi\left(T_{k(n)}^{k(n+1)}(s(t))\right) - \Phi(k(n))\right]_- .
$$

Next, we apply Proposition 1.2 with $\varphi \equiv 1$, $p = 1$ and define for the moment

$$
u(t) := \left(\Phi\left(T_{k(n)}^{k(n+1)}(s(t))\right) - \Phi(k(n))\right)_- .
$$

We find $\mathcal{E} = \{u(t) = 0\} = \{s(t) \geq k(n)\} \cap B_R$ and infer from (6.98) that $|B_R| \geq |\mathcal{E}| \geq \left(\frac{\nu_0}{2}\right)^2 |B_R|$. Thus, we obtain

$$
\frac{\omega}{2^{n+1}}\Phi_{0,l}(k(n+1))\,|A_{n+1}(t)| \leq \gamma\frac{R^d\,|B_R|^{\frac{1}{d}}}{\left(\frac{\nu_0}{2}\right)^2 |B_R|}\int_{B_R}|\nabla u(t)| \leq \gamma\frac{R}{\nu_0^2}\int_{B_R}|\nabla u(t)| \quad (6.100)
$$

for a.e. $t \in \left(-\frac{\theta_m}{2}R^2, 0\right)$. With Lemma 4.19, Stampacchia's lemma and (6.97), we infer for a.e. $t \in \left(-\frac{\theta_m}{2}R^2, 0\right)$ the identity

$$
\begin{aligned}
\nabla u(t) &= -\nabla\left(\Phi\left(T_{k(n)}^{k(n+1)}(s(t))\right) - \Phi(k(n))\right)\mathbb{1}_{A_n(t)}\\
&= \nabla\left[\Phi(k(n)) - \Phi(k(n+1))\right]\mathbb{1}_{A_{n+1}(t)} + \nabla[\Phi(k(n)) - \Phi(s(t))]\mathbb{1}_{A_n(t)\setminus A_{n+1}(t)} \quad (6.101)\\
&= \nabla\left[\Phi(k(n)) - \Phi(k(n+1))\right]\mathbb{1}_{A_{n+1}(t)} + \nabla(\Phi(s(t)) - \Phi(k(n)))_-\mathbb{1}_{A_n(t)\setminus A_{n+1}(t)}
\end{aligned}
$$

a.e. on B_R. We integrate (6.100) in time over $t \in \left(-\frac{\theta_m}{2}R^2, 0\right)$, insert (6.101) and use the regularity of Φ to infer

$$
\begin{aligned}
\frac{\omega}{2^n}\Phi_{0,l}(k(n+1))A_{n+1} &\leq \gamma\frac{R}{\nu_0^2}\int_{-\frac{\theta_m}{2}R^2}^0\int_{A_{n+1}(t)}\nabla\left[\Phi(k(n)) - \Phi(k(n+1))\right]\\
&\quad + \gamma\frac{R}{\nu_0^2}\int_{-\frac{\theta_m}{2}R^2}^0\int_{A_n(t)\setminus A_{n+1}(t)}\nabla(\Phi(s) - \Phi(k(n)))_- \quad (6.102)\\
&\leq \gamma\frac{R}{\nu_0^2}A_{n+1} + \gamma\frac{R}{\nu_0^2}\int_{-\frac{\theta_m}{2}R^2}^0\int_{A_n(t)\setminus A_{n+1}(t)}\nabla(\Phi(s) - \Phi(k(n)))_-
\end{aligned}
$$

We absorb the first term on the right-hand side of (6.102) using (6.95). Particularly, we choose m_1 large enough to ensure that

$$
\begin{aligned}
\frac{\gamma R}{\nu_0^2} &\leq \frac{\omega}{2^n}\Phi_{0,l}\left(\frac{\omega}{2^{n+1}}\right)\frac{\Phi_{0,u}\left(\frac{\omega}{2^{m_1}}\right)^{1-\beta}}{\nu_0^2}\\
&\leq \frac{\omega}{2^n}\Phi_{0,l}\left(k(n+1)\right)\frac{\Phi_{0,u}\left(\frac{1}{2^{m_1}}\right)^{1-\beta}}{\nu_0^2} \leq \frac{1}{2}\frac{\omega}{2^n}\Phi_{0,l}\left(k(n+1)\right).
\end{aligned}
\quad (6.103)
$$

Up to now, the choice of m_1 depends only on ν_0, on β and on the structure conditions of Φ from Assumption A3.7, thus $m_1 = m_1(data, \tilde{\mathcal{K}})$. We insert this into (6.102), square both sides and use Hölder's inequality. This yields

$$\left(\frac{\omega}{2^n}\right)^2 \Phi_{0,l}(k(n+1))^2 A_{n+1}^2 \leq \gamma \frac{R^2}{\nu_0^4}(A_n - A_{n+1}) \left\|\nabla(\Phi(s) - \Phi(k))_-\right\|_{Q\left(R, \frac{\theta_m}{2}R^2\right)}^2$$

$$\leq \frac{\gamma}{\nu_0^4}(A_n - A_{n+1})\Phi_{0,u}(k(n)) \left(\frac{w}{2^n}\right)^2 \left(\Phi_{0,u}(k(n)) + \frac{1}{\theta_m}\right) \left|Q\left(R, \frac{\theta_m}{2}R^2\right)\right|,$$

where we exploited the energy estimate (6.96) in the second inequality. Division by $\omega 2^{-n}\Phi(k(n+1))^2$ leads to

$$\begin{aligned}
A_{n+1}^2 &\leq \frac{\gamma}{\nu_0^4}(A_n - A_{n+1}) \left(\frac{\Phi_{0,u}(k(n))}{\Phi_{0,l}(k(n+1))}\right)^2 \left|Q\left(R, \frac{\theta_m}{2}R^2\right)\right| \\
&\quad + \frac{\gamma}{\nu_0^4}(A_n - A_{n+1}) \left(\frac{\Phi_{0,u}(k(n))}{\Phi_{0,l}(k(n+1))} \frac{\Phi_{0,l}\left(\frac{\omega}{2^m}\right)}{\Phi_{0,l}(k(n+1))}\right) \left|Q\left(R, \frac{\theta_m}{2}R^2\right)\right|
\end{aligned} \tag{6.104}$$

Recall that $k(n) = \mu_- + \omega 2^{-n}$ and, with the elementary inequality $\frac{a+b}{a+c} \leq \frac{b}{c}$ for $0 \leq a$ and $0 < c \leq b$, we find

$$\frac{\Phi_{0,u}(k(n))}{\Phi_{0,l}(k(n+1))} \leq \frac{\Phi_{0,u}\left(\frac{\omega}{2^n}\right)}{\Phi_{0,l}\left(\frac{\omega}{2^{n+1}}\right)} \leq \frac{c_{0,u}}{c_{0,l}} 2^{\alpha_0}.$$

Using that $\Phi_{0,l}$ is increasing on $[0, 4\delta_0]$, we infer also

$$\frac{\Phi_{0,l}\left(\frac{\omega}{2^m}\right)}{\Phi_{0,l}(k(n+1))} \leq \frac{\Phi_{0,l}\left(\frac{\omega}{2^m}\right)}{\Phi_{0,l}\left(\frac{\omega}{2^{n+1}}\right)} \leq 2^{n+1-m} \leq 1 \tag{6.105}$$

for $m \geq m_1 \geq n+1$. This choice of m is independent of ω modulo the potential dependence of n on ω. With these considerations, (6.104) simplifies to

$$A_{n+1}^2 \leq \frac{\gamma}{\nu_0^4}(A_n - A_{n+1}) \left|Q\left(R, \frac{\theta_m}{2}R^2\right)\right|. \tag{6.106}$$

Adding these inequalities for $n = q_1, q_1 + 1, \ldots, q_2 - 1$, where q_2 needs to be chosen, yields

$$\sum_{n=q_1}^{q_2-1} A_{n+1}^2 \leq \frac{\gamma}{\nu_0^4}(A_{q_1} - A_{q_2}) \left|Q\left(R, \frac{\theta_m}{2}R^2\right)\right|. \tag{6.107}$$

Since $A_{q_1} - A_{q_2} \leq \left|Q\left(R, \frac{\theta_m}{2}R^2\right)\right|$ and

$$\sum_{n=q_1}^{q_2-1} A_{n+1}^2 \geq (q_2 - q_1)A_{q_2}^2$$

we conclude that

$$A_{q_2} \leq \frac{\gamma}{\nu_0^2}(q_2 - q_1)^{-\frac{1}{2}} \left| Q\left(R, \frac{\theta_m}{2}R^2\right) \right|$$

To prove the result, we choose q_2 so large that

$$\frac{\gamma}{\nu_0^2}(q_2 - q_1)^{-\frac{1}{2}} \leq \lambda_0 \tag{6.108}$$

This choice depends only on the data, on $\tilde{\mathcal{K}}$, on λ_0, and on q_1. Thus, q_2 independent of ω. Consequently, the choice of $m_1 > q_2$ is independent of ω but still depends on λ_0. $\quad\square$

6.4.3 Defining the Geometry

In this section, we determine the value for m and thus fix the geometry of the initial cylinder $Q_r(\theta_m)$. Once m is determined, we are also in the position to conclude the first alternative, i.e. we can determine l_2 and σ_1 from Corollary 6.19.

We determine m with a kind of bootstrap argument. Assume for the moment that λ_0 is already determined. Then, by Proposition 6.24, we are able to choose m_1.

Definition 6.25 (Choice of m). *Assume that the setting described Assumption A6.1 is given and that Assumption A6.2 holds. Let*

$$\phi_1 := \Phi_{0,u}\left(\frac{\omega}{2^{q_2}}\right), \quad \phi_2 := \Phi_{0,l}\left(\frac{\omega}{2^{q_2+1}}\right), \quad \Gamma := \frac{\phi_1}{\phi_2}, \quad and \quad \mu := \phi_2\Gamma^{\frac{d+2}{2}}$$

and choose $m > m_1$ as the smallest real number such that

$$\frac{\mu}{\Phi_{0,l}\left(\frac{\omega}{2^m}\right)} = n_0 \tag{6.109}$$

for some integer $n_0 \in \mathbb{N}$.

Since $\Phi_{0,l}$ is a power function, m is independent of ω; in particular, we find that

$$\Gamma = \frac{c_{0,u}}{c_{0,l}}2^{\alpha_0}. \tag{6.110}$$

is independent of ω. Thus, also m is independent of ω and we find

$$\frac{\mu}{\Phi_{0,l}\left(\frac{\omega}{2^m}\right)} = \Gamma^{\frac{d+2}{2}}2^{m-(q_2+1)} = n_0. \tag{6.111}$$

For fixed m, we break $Q\left(R, \frac{\theta_m}{2}R^2\right)$ into n_0 subcylinders of the form

$$Q_R^j := B_R \times \left(-j\frac{R^2}{\mu}, -(j-1)\frac{R^2}{\mu}\right) \text{ for } j = 1, 2, \ldots, n_0. \tag{6.112}$$

Since these cylinders are disjoint and exhaust $Q\left(R, \frac{\theta_m}{2}R^2\right)$, we infer from Proposition 6.24, assuming that is in force, the following statement.

Lemma 6.26. *Let Assumptions A6.1 and A6.2 hold and assume that (6.91) is in force. For any $\lambda_0 \in (0,1)$, there exists $j_0 \in \{1, \dots, n_0\}$ such that*

$$\left| \left\{ s < \frac{\omega}{2^{q_2}} \right\} \cap Q_R^{j_0} \right| \leq \lambda_0 \left| Q_R^{j_0} \right|. \tag{6.113}$$

Proof. Let $\lambda_0 \in (0,1)$ be given. Assume that for every cylinder Q_R^j, with $j = 1, 2, \dots, n_0$, the condition (6.113) fails to hold, i.e. there exists $\varepsilon > 0$ such that for every such j there holds

$$\left| \left\{ s < \mu_- + \frac{\omega}{2^{q_2}} \right\} \cap Q_R^j \right| > \lambda_0 \left| Q_R^j \right| \geq (\lambda_0 + \varepsilon) \left| Q_R^j \right| \tag{6.114}$$

The choice for ε may be independent of j since n_0 is finite. The cylinders Q_R^j are disjoint and exhaust $Q\left(R, \frac{\theta_m}{2} R^2\right)$. We add (6.114) over $j = 1, 2, \dots, n_0$ and infer

$$\left| \left\{ s < \frac{\omega}{2^{q_2}} \right\} \cap Q\left(R, \frac{\theta_m}{2} R^2\right) \right| \geq (\lambda_0 + \varepsilon) Q\left(R, \frac{\theta_m}{2} R^2\right)$$

which contradicts Proposition 6.24. $\qquad \square$

We fix j_0 for the moment and introduce the notation

$$t_\sigma := \frac{-(\sigma R)^2 - (j_0 - 1)R^2}{\mu} \text{ for } \sigma \in [0,1] \tag{6.115}$$

and introduce the cylinder

$$Q_{R,\eta}^{j_0} = (0, t_0) + Q\left(\eta R, \frac{(\eta R)^2}{\mu}\right) \subset \mathbb{R}^{d+1}$$

for $0 < \eta \leq 1$ and emphasize that $Q_{R,1}^{j_0} = Q_R^{j_0}$.

Lemma 6.27. *Under Assumptions A6.1 and A6.2, there exists $\lambda_0 \in (0,1)$, $\lambda_0 = \lambda_0(data, \tilde{\mathcal{K}})$, such that if (6.113) is in force for $Q_R^{j_0}$, then either*

$$\min\left\{ \Phi_{0,l}\left(\frac{\omega}{2^{q_2+1}}\right), \frac{\omega}{2^{q_2}} \right\} \leq \gamma R^{\frac{\beta}{4}}$$

for some $\gamma = \gamma(data, \tilde{\mathcal{K}})$, or

$$s(x,t) > \mu_- + \frac{\omega}{2^{q_2+1}} \ a.e. \ on \ Q_{R,\frac{1}{2}}^{j_0}.$$

Proof. **Step 1: Energy estimates.** The proof is similar to the proof of Lemma 6.14 and, basically, we only point out the differences. Let $\mu_- + \frac{\omega}{2^{q_2}} \leq a < b := \mu_- + \frac{\omega}{2^{q_2+1}}$ be

given. For $0 < \sigma < \eta \le 1$, let ξ be a smooth, nonnegative cut-off function with $0 \le \xi \le 1$ and we demand $\xi(x,t) = \xi_1(x)\xi_2(t)$ with

$$
\begin{cases}
\xi_1 = 1 & \text{on } B_{\sigma R}, \\
\xi_1 = 0 & \text{on } \partial B_{\eta R}, \\
|\nabla \xi_1| \le \frac{C_\xi}{(\eta - \sigma)R}, \quad |D^2\xi_1| \le \frac{C_\xi}{(\eta - \sigma)^2 R^2}
\end{cases}
$$

and, with the notation (6.115), also

$$
\begin{cases}
\xi_2 = 0 & \text{for } t \le t_\eta \\
\xi_2 = 1 & \text{for } t \ge t_\sigma \\
0 \le \xi_{2,t} \le \frac{C_\xi \mu}{(\eta^2 - \sigma^2)R^2} & \text{for } t_\eta < t < t_\sigma.
\end{cases}
$$

We choose $v = -(s_b - a)_- \xi^2$ with $s_b = \max\{s, b\}$ in (3.23), define $M_{a,\eta} := \{s > a\} \cap Q_{R,\eta}^{j_0}$ and proceed as in Lemma 6.14 with the obvious modifications due to the sign. We assume that

$$
\gamma R^{\beta/4} < \min\left\{ \Phi_{0,l}\left(\frac{\omega}{2^{q_2+1}} \right), \frac{\omega}{2^{q_2}} \right\} \tag{6.116}
$$

and obtain

$$
\operatorname*{ess\,sup}_{t_\eta < t < t_0} \int_{B_{\eta R} \times \{t\}} (s_b - a)_-^2 \xi^2 + \Phi_{0,l}\left(\frac{\omega}{2^{q_2+1}} \right) \int_{Q_{R,\eta}^{j_0}} |\nabla(s_b - a)_-|^2 \xi^2
$$
$$
\le \frac{\gamma}{(\eta - \sigma)^2 R^2} \left(\frac{\omega}{2^{q_2}} \right)^2 \left[\Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}} \right) + \mu \right] |M_{a,\eta}|. \tag{6.117}
$$

We emphasize that, in contrast to the proof of Lemma 6.14, that the term containing $\Phi_{1,l}(\omega 2^{-(m_0+1)})$ coming from the $\partial_t \xi$-term is not estimated by $\Phi_{1,u}(\omega 2^{-(m_0-2)})$ from above. This explains the appearance of μ on the right hand side of (6.117).

We perform a change in the time variable; putting $\bar{t} = \left(t + (j_0 - 1)\frac{R^2}{\mu} \right)$ transforms (t_η, t_0) to $(-(\eta R)^2, 0)$ and $Q_{R,\eta}^{j_0}$ into $Q_{\eta R}$. Define $\bar{s}(\cdot, \bar{t}) = s(\cdot, t)$, $\bar{\xi}(\cdot, \bar{t}) = \xi(\cdot, t)$ as well as $\bar{M}_{a,\eta} = \{\bar{s} > a\} \cap Q_{\eta R}$ and obtain the inequality

$$
\operatorname*{ess\,sup}_{-(\eta R)^2 < t < t_0} \int_{B_{\eta R} \times \{t\}} (s_b - a)_-^2 \xi^2 + \frac{\Phi_{0,l}\left(\frac{\omega}{2^{q_2+1}} \right)}{\mu} \int_{Q_{R,\eta}^{j_0}} |\nabla(s_b - a)_-|^2 \xi^2
$$
$$
\le \frac{\gamma}{(\eta - \sigma)^2 R^2} \left(\frac{\omega}{2^{q_2}} \right)^2 \left[\frac{\Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}} \right)}{\mu} + 1 \right] |\bar{M}_{a,\eta}|. \tag{6.118}
$$

With Γ from Definition 6.25, we multiply by $\Gamma^{\frac{d+2}{2}}$ and obtain with the properties of ϕ_1 and ϕ_2, and with (6.110) that

$$
\Gamma^{\frac{d+2}{2}} \operatorname*{ess\,sup}_{-(\eta R)^2 < t < t_0} \int_{B_{\eta R} \times \{t\}} (s_b - a)_-^2 \xi^2 + \int_{Q_{R,\eta}^{j_0}} |\nabla(s_b - a)_-|^2 \xi^2
$$
$$
\le \frac{\gamma}{(\eta - \sigma)^2 r^2} \left(\frac{\omega}{2^{q_2}} \right)^2 \left[2^{q_2 - m_0} + 1 \right] |\bar{M}_{a,\eta}|. \tag{6.119}
$$

We add on both sides the term

$$\int_{Q_{\eta r}} (\bar{s}_b - a)_+^2 \left| \nabla \bar{\xi} \right|^2$$

to infer

$$\left\| (\bar{s}_b - a)_- \bar{\xi} \right\|_{V^2(Q_{\eta r})}^2 \leq \frac{\gamma}{(\eta - \sigma)^2 r^2} \left[2^{q_2 - m_0} + 1 \right] \left| \bar{M}_{a,\eta} \right|. \tag{6.120}$$

Step 2: Iteration. The iteration is performed as in the proof of Lemma 6.14. For $k = 0, 1, 2, \ldots$, define

$$a_k := b - \frac{\omega}{q_2 + 1} 2^{-k}, \quad \eta_k := \frac{1}{2} + 2^{-(k+1)}, \quad R_k = \eta_k R \quad \text{and} \quad A_k := \left| \bar{M}_{a_k, R_k} \right|,$$

and choose $a = a_k$, $\eta = \eta_k$ and $\sigma = \eta_{k+1}$ in (6.120). We infer, as in (6.46) and (6.47), the estimate

$$A_{k+1} \leq \gamma \frac{2^{4(k+2)}}{R_k^2} \left[2^{q_2 - m_0} + 1 \right] A_k^{1 + \frac{2}{d+2}}. \tag{6.121}$$

Define

$$X_k := \frac{A_k}{|Q_{R_k}|}$$

and divide (6.121) by $|Q_{R_k}|$. With the calculation in (6.48), we obtain

$$X_{k+1} \leq \gamma \left[2^{q_2 - m_0} + 1 \right] 4^{2k} X_k^{1 + \frac{2}{d+2}}$$

and Lemma 6.2 implies that $X_k \to 0$ as $k \to \infty$, provided that

$$X_0 \leq \left(\gamma \left[2^{q_2 - m_0} + 1 \right] \right)^{-\frac{d+2}{2}} 4^{-2\left(\frac{d+2}{2} \right)^2} =: \lambda_0 \in (0, 1). \tag{6.122}$$

This choice of λ_0 is independent of ω and depends only on the data, $\tilde{\mathcal{K}}$, q_2 and m_0. Hence, for this λ_0, we choose m_1 as in Proposition 6.24 and then m as above. Thus

$$X_0 \leq \lambda_0 \iff \left| \left\{ s < \frac{\omega}{2^{q_2}} \right\} \cap Q_R^{j_0} \right| \leq \lambda_0 \left| Q_R^{j_0} \right|$$

is true. Consequently, we infer $A_k \to 0$ and, observing that $R_k \to \frac{R}{2}$ and $a_k \to b$, we find

$$\left| \{ \bar{s} \leq b \} \cap Q_{\frac{R}{2}} \right| = \left| \{ s \leq b \} \cap Q_{R, \frac{1}{2}}^{j_0} \right| = 0.$$

$\square$

Remark 6.28 (Bootstrapping)**.** Let us examine the reasoning in this section. The number $q_2 = q_2(data, \tilde{\mathcal{K}})$ was determined independent of m in Proposition 6.24. With q_2, we choose λ_0 as in (6.122) independent of m. Once λ_0 is given, m_1 in Proposition 6.24 can be chosen depending only on the data and $\tilde{\mathcal{K}}$, and thus, we define m in terms

of Definition 6.25. Consequently, n_0 is given by (6.109) and we may split the cylinder $Q(R, \frac{\theta_m}{2}R^2)$ as in (6.112).

Then Lemma 6.26 provides the existence of a cylinder $Q_R^{j_0}$ such that (6.113) holds and we may apply the reasoning of Lemma 6.27 to this cylinder. We emphasize that (6.113) plays the same role as (6.28) in the first alternative. In addition Lemma 6.27 is the corresponding analogue of Lemma 6.14. ◁

6.4.4 The Second Alternative Concluded

In view of the previous section and, particularly, of Remark 6.28, we conclude the second alternative starting from Lemma 6.27 with analogues of Lemma 6.16 and Proposition 6.18.

For convenience, we define

$$\tilde{t} = \left(j_0 - \frac{3}{4} \right) \frac{R^2}{\mu} = -t_{\frac{1}{4}} \tag{6.123}$$

and emphasize that q_2, m, m_1 and λ_0 are completely determined by means of the data and $\tilde{\mathcal{K}}$. Additionally, we remark that for $t \in (\tilde{t}, t_0)$ there holds $s(t) > \mu_- + \frac{\omega}{2^{q_2+1}}$ a.e. on $B_{\frac{R}{2}} \times \{t\}$ due to Lemma 6.27. This is used in the proof of the next lemma.

Lemma 6.29. *Let Assumptions A6.1 and A6.2 hold and let $\nu_1 \in (0, 1)$ be given. There exists $\max\{l_2, m\} < q_3 \in \mathbb{N}$, $q_3 = q_3(data, \tilde{\mathcal{K}}, \nu_1)$, such that either*

$$\min \left\{ \Phi_{0,u} \left(\frac{\omega}{2^{m_0-2}} \right), \frac{\omega}{2^{q_3}} \right\} \leq \gamma R^{\frac{\beta}{4}},$$

for some $\gamma = \gamma(data, \tilde{\mathcal{K}})$, or

$$\left| \left\{ s(t) \leq \mu_- + \frac{\omega}{2^{q_3}} \right\} \cap B_{\frac{R}{4}} \right| \leq \nu_1 \left| B_{\frac{R}{4}} \right| \text{ for a.e. } t \in (\tilde{t}, 0). \tag{6.124}$$

Remark 6.30. The choice $q_3 > \max\{l_2, m\}$ is made for convenience and is used in Proposition 6.33. ◁

Proof. We choose $\bar{a} = \frac{\omega}{2^n}$, $\bar{b} = \frac{\omega}{2^{q_2+1}}$ in the function $\Psi_{\bar{a}, \bar{b}}$ from Definition 6.7, where $n > q_2 + 2$ is to be chosen later. Let $a = \mu_- + \frac{\omega}{2^{q_2+1}}$ and $b = \mu_- + \frac{\omega}{2^n}$. Let $\frac{R}{2} = r$ and $\sigma \in (0, 1)$ be given. Let ξ be a time-independent cut-off function that equals one on $B_{\sigma r}$, vanishes on ∂B_r and is such that

$$0 \leq \xi \leq 1, \quad |\nabla \xi| \leq C_\xi \frac{1}{(1-\sigma)r} \quad \text{and} \quad \left| D^2 \xi \right| \leq \frac{C_\xi}{(1-\sigma)^2 r^2}.$$

Define the cylinders $U_t := B_r \times (-\tilde{t}, t)$ for $t \in (-\tilde{t}, 0)$. With ψ as in the proof of Lemma 6.22, we use $-\psi(s)\psi'(s)\xi^2$ in (3.23). We proceed as in the proof of Lemma 6.16 and treat the differences as in the proof of Lemma 6.22. We assume

$$\gamma r^{\frac{\beta}{4}} < \min\left\{\Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right), \frac{\omega}{2^n}\right\}$$

and infer from (6.109)

$$\tilde{t} \le \frac{n_0}{\mu} R^2 \le \frac{R^2}{\Phi_{0,l}(\frac{\omega}{2^m})}.$$

This leads to the estimate

$$
\begin{aligned}
\fint_{B_{\sigma r} \times \{t\}} \psi^2 &\le \frac{\gamma n}{(1-\sigma)^2 r^2} \Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right) |U_t| \le \frac{\gamma \tilde{t} n}{(1-\sigma)^2 r^2} \Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right) |B_r| \\
&\le \frac{\gamma n}{(1-\sigma)^2} \frac{\Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right)}{\Phi_{0,l}\left(\frac{\omega}{2^m}\right)} |B_r|.
\end{aligned}
\tag{6.125}
$$

for a.e. $t \in (-\tilde{t}, 0)$. We used the definition of $\tilde{t}$ and (6.109) in the last inequality. As in the third step of the proof of Lemma 6.16, we choose $\sigma = \frac{1}{2}$ and define the sets

$$S_n(t) := \left\{s(t) < \mu_- + \frac{\omega}{2^n}\right\} \cap B_{\frac{R}{4}} \subset B_{\frac{R}{2}} \text{ for a.e. } t \in (\tilde{t}, 0).$$

On $S_n(t)$ we infer the estimate $\psi^2 \ge \ln^2(2)(n - q_2 - 2)$ and obtain also

$$|S_n(t)| \le \frac{\gamma n}{(n - q_2 - 2)^2} \frac{\Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right)}{\Phi_{0,l}\left(\frac{\omega}{2^m}\right)} \left|B_{\frac{R}{4}}\right|$$

for a.e. $t \in (-\tilde{t}, 0)$ after rescaling the ball. To prove the lemma we choose, under consideration of (6.63) and

$$\frac{\Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right)}{\Phi_{0,l}\left(\frac{\omega}{2^m}\right)} \le \frac{c_{0,u}}{c_{0,l}} 2^{m-m_0-2},$$

q_3 such that

$$q_3 = n \ge 2(q_2 - 2) + \frac{\gamma}{\nu_1} \frac{c_{0,u}}{c_{0,l}} 2^{m-m_0-2} \quad \text{and} \quad q_3 > \max\{l_2, m\}.$$

This choice is indepenedent of ω and clearly depends only on the data, on $\tilde{\mathcal{K}}$ and on ν_1. $\qquad\square$

Proposition 6.31. *Under Assumptions A6.1 and A6.2, there exists a constant $q_3 \in \mathbb{N}$ with $\max\{l_2, m\} < q_3 < q_4 \in \mathbb{N}$ and $q_4 = q_4(data, \tilde{\mathcal{K}})$ such that either*

$$\min\left\{ \frac{\omega}{2^{q_4}}, \Phi_{0,l}\left(\frac{\omega}{2^{q_4}} \right) \right\} \leq \gamma R^{\frac{\beta}{4}},$$

for some $\gamma = \gamma(data, \tilde{\mathcal{K}})$, or

$$s > \mu_- + \frac{\omega}{2^{q_4}}, \quad a.e. \ in \ Q\left(\frac{R}{8}, \tilde{t} \right).$$

Proof. We proceed analogous to the proof of Proposition 6.18.

Step 1: Preliminaries. For $n \geq q_3$ to be chosen, consider $b = \mu_+ - \frac{\omega}{2^{n+1}}$ and $\mu_+ - \frac{\omega}{2^n} \leq a < b$. As in (6.33) we obtain $a > 1 - \frac{\omega}{2^{m_0-2}}$. For $0 < \sigma < \eta \leq 1$ and with $r = \frac{R}{4}$, we consider a time-independent cut-off function ξ on $B_{\eta r}$ that equals one on $B_{\sigma r}$, satisfying

$$0 \leq \xi \leq 1, \quad |\nabla \xi| \leq \frac{C_\xi}{(\eta - \sigma)r} \ \text{and} \ \left| D^2 \xi \right| \leq \frac{C_\xi}{(\eta - \sigma)^2 r^2}.$$

We define

$$M_{a,\eta} := \{s > a\} \cap Q(\eta r, \tilde{t}) \tag{6.126}$$

and derive an energy estimate on $Q(\eta r, \hat{t})$.

Step 2: Energy estimates. With the same reasoning as in the proof of Proposition 6.18, we infer, using $-(s_b - a)_- \xi^2$ as a test function, that either

$$\min\left\{ \frac{\omega}{2^n}, \Phi_{1,l}\left(\frac{\omega}{2^{n+1}} \right) \right\} \leq \gamma r^{\beta/4} \tag{6.127}$$

or

$$\operatorname*{ess\,sup}_{-\tilde{t} < t < 0} \int_{B_{\eta r} \times \{t\}} (s_b - a)^2_- \xi^2 + \Phi_{0,l}\left(\frac{\omega}{2^{n+1}} \right) \int_{Q(\eta r, \tilde{t})} |\nabla (s_b - a)_-|^2 \xi^2$$
$$\leq \frac{\gamma}{(\eta - \sigma)^2 r^2} \left(\frac{\omega}{2^n} \right)^2 \Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}} \right) |M_{a,\eta}|. \tag{6.128}$$

We change the time variable according to

$$\bar{t} = \left(\frac{R}{8} \right)^2 \frac{t}{\tilde{t}} = \left(\frac{r}{2} \right)^2 \frac{t}{\tilde{t}} \tag{6.129}$$

and define the transformed function $\bar{s}(\cdot, \bar{t}) = s(\cdot, t)$ as well as the transformed set $\bar{M}_{a,\eta} = \{\bar{s} > a\} \cap Q(\eta r, (\frac{R}{8})^2)$. With n such that

$$H(n) := \frac{(R/8)^2}{\Phi_{0,l}(\omega/2^{n+1}) \tilde{t}} \geq 1,$$

we infer, as in the transition from (6.70) to (6.71), the estimate

$$\|(\bar{s}_b - a)_- \xi\|^2_{V^2(Q(\eta r, (R/8)^2))} \le \frac{\gamma}{(\eta - \sigma)^2 r^2} \left(\frac{\omega}{2^n}\right)^2 \left[\frac{\Phi_{0,u}\left(\omega/2^{m_0-2}\right)}{\Phi_{0,l}\left(\omega/2^{n+1}\right)} + 1\right] \left|\bar{M}_{a,\eta}\right|. \quad (6.130)$$

Step 3: Iteration. We proceed similarly to the iteration presented in the third step of the proof of Proposition 6.18. With the same notation for a_k, η_k, r_k and A_k, with

$$X_k := \frac{A_k}{|Q(r_k, (R/8)^2)|}$$

and under consideration of

$$\left|Q\left(r_{k+1}, \left(\frac{R}{8}\right)^2\right)\right| \left(\frac{R}{8}\right)^2 \ge 2^{-d} |B_1|^{-\frac{2}{d+2}} \left|Q(r_k, (R/8)^2)\right|^{1\frac{2}{d+2}},$$

we find

$$X_{k+1} \le 2^{d+10} |B_1|^{\frac{2}{d+2}} \gamma \left[\frac{\Phi_{0,u}\left(\omega/2^{m_0-2}\right)}{\Phi_{0,l}\left(\omega/2^{n+1}\right)} + 1\right] 4^{2k} X_k^{1+\frac{d}{d+2}}.$$

From Lemma 6.2 we conclude that if

$$X_0 \le \left(2^{d+10} |B_1|^{\frac{2}{d+2}} \gamma \left[\frac{\Phi_{0,u}\left(\omega/2^{m_0-2}\right)}{\Phi_{0,l}\left(\omega/2^{n+1}\right)} + 1\right]\right)^{-\frac{d+2}{2}} 4^{-2\left(\frac{d+2}{2}\right)^2} =: \nu_1 \in (0,1) \quad (6.131)$$

then $X_k \to 0$ as $k \to \infty$. Apply Lemma 6.29 with this ν_1 and conclude the existence of q_3, depending only on the data, $\tilde{\mathcal{K}}$ and ν_1, such that

$$M(t, q_3) := \left|\left\{s(t) < \mu_- + \frac{\omega}{2^{q_3}}\right\} \cap B_{\frac{R}{4}}\right| \le \nu_1 \left|B_{\frac{R}{4}}\right|, \text{ for a.e. } t \in (-\tilde{t}, 0). \quad (6.132)$$

Step 4: Conclusion. We proceed as in (6.75) with l_1 and $\hat{t}$ replaced by q_3 and $\tilde{t}$, respectively, and we obtain $X_0 \le \nu_1$. Since $r_k \searrow R$, $a_k \nearrow \mu_- + \frac{\omega}{2^{n+1}}$ and $X_k \to 0$ implies $A_k \to 0$, we obtain

$$\left|\left\{\bar{s} \ge \mu_- + \frac{\omega}{2^{n+1}}\right\} \cap Q\left(\frac{R}{8}, \left(\frac{R}{8}\right)^2\right)\right| = \left|\left\{s \ge \mu_- + \frac{\omega}{2^{n+1}}\right\} \cap Q\left(\frac{R}{8}, \hat{t}\right)\right| = 0$$

and the proposition is proved if we choose $q_4 = n + 1 \ge q_3$ modulo the independence of ν_1 and q_4 on ω.

Concerning the definition of ν_1, we observe that ν_1 only depends on the data and $\tilde{\mathcal{K}}$. In addition, ν_1 is independent of ω since the quotient

$$\frac{\Phi_{0,u}\left(\frac{\omega}{2^{m_0-2}}\right)}{\Phi_{0,l}\left(\frac{\omega}{2^{n+1}}\right)} = \frac{c_{0,u}}{c_{0,l}} 2^{\alpha_0(n-m_0-1)}.$$

is independent of ω, due to Assumption A3.7. To take the independence of q_4 on ω into account, we recall that $H(q_4) \geq 1$ was exploited to show (6.130). With $\tilde{t} \leq \frac{n_0 R^2}{\mu}$ and (6.110), we find

$$H(q_4) \geq 2^{q_4-q_2} \left(\frac{c_{0,u}}{c_{0,l}} \right)^{\frac{d+2}{2}}.$$

Since q_2 already has been determined in terms of the same quantities, this expression can be made larger than one independent of ω and depending only on the data and $\tilde{\mathcal{K}}$, $\qquad\qquad\qquad\square$

We gather the results of the second alternative in the following corollary.

Corollary 6.32. *Let Assumptions A6.1 and A6.2 hold. The constant m, and thus also l_2 and σ_1, depend only on the data and on $\tilde{\mathcal{K}}$. Furthermore, there exists a constant $\sigma_0 \in (0,1)$, $\sigma_0 = \sigma_0(data, \tilde{\mathcal{K}})$, such that if (6.29) holds for every cylinder of type $Q_R^{t^*} \subset Q_R(\theta_m)$, then either*

$$\min \left\{ \frac{\omega}{2^{q_4}}, \Phi_{0,l} \left(\frac{\omega}{2^{q_4}} \right) \right\} \leq \gamma R^{\beta/4} \qquad (6.133)$$

for some $\gamma = \gamma(data, \tilde{\mathcal{K}})$, or

$$\operatorname*{ess\,osc}_{Q(\frac{R}{8}, \tilde{t})} s \leq \sigma_0 \omega. \qquad (6.134)$$

Proof. From Proposition 6.31 we find q_4 such that even

$$\operatorname*{ess\,inf}_{Q(\frac{R}{8}, \tilde{t})} s \geq \mu_- + \frac{\omega}{2^{q_4}}.$$

From this we get

$$\operatorname*{ess\,osc}_{Q(\frac{R}{8}, \tilde{t})} s = \operatorname*{ess\,sup}_{Q(\frac{R}{8}, \tilde{t})} s - \operatorname*{ess\,inf}_{Q(\frac{R}{8}, \tilde{t})} s \leq 1 - \frac{\omega}{2^{q_4}} = \left(1 - \frac{\omega}{2^{q_4}} \right) \omega.$$

As in the proof of Corollary 6.19 we collect the alternatives appearing in the second alternative and obtain (6.133) under consideration of Remark 6.30. $\qquad\square$

6.5 Proof of Proposition 6.10 and Theorem 3.12

Before we link the results from Corollary 6.19 and Corollary 6.32, we discuss briefly the cases $\mu_- > \delta_0$ or $\mu_+ < 1 - \delta_0$, i.e. the cases where at most one degeneracy occurs. In [PV93] and [Iva91] local Hölder regularity to certain parabolic equations with a degeneracy only at zero saturation is shown. Adapting these results to our situation, provides an extension to the case where $\mu_- = 0$. Due to symmetry also the case where $\mu_+ = 1$

can be treated. In the case where no degeneracy occurs, we may even use the classical result provided in [LSU88, V.Theorem 1.1].

However, at the moment there seems to be no complete result present in the literature for our problem. Thus, we provide an elementary idea on how to extend our result to the case $\mu_- = 0$. We assume that the growth conditions from (3.31) hold on $[0, 1-\delta_0]$ for $\Phi_{0,l}$ and $\Phi_{0,u}$ and on $[\delta_0, 1]$ for $\Phi_{1,l}$ and $\Phi_{1,u}$. Applying this in the arguments in sections 6.3 and 6.4, provides always an upper estimate of Φ' in terms of $\Phi_{j,u}(\omega)$ for at least one $j \in \{0,1\}$ and a lower estimate in term of $\Phi_{j,l}(\omega 2^{-n})$ for at least one $j \in \{0,1\}$ and a suitable n. Hence, the previous proofs can be performed completely analogous and we only need to modify the constant $c_{j,k}$ for $j \in \{0,1\}$ and $k \in \{l, u\}$.

In the following, we take the parameter ϵ introduced in the definition of $Q(2R, (2R)^{2-\epsilon})$ into account and require smallness of this parameter. With the previous considerations in this section, we may dispose the assumptions on μ_+, μ_- and ω. Since m is independent of R, we may even dispose Assumption A6.2 completely.

Proposition 6.33. *Assume the setting from Assumption A6.1 is fulfilled. For arbitrary*

$$0 < \epsilon \le \frac{\beta \alpha_0}{4 \max\{\alpha_0, 1\}},$$

consider the cylinder $Q(2R, (2R)^{2-\epsilon})$. The constant m can be determined depending only on the data and on $\tilde{\mathcal{K}}$, independent of ϵ, such that at least one of the following alternatives hold. Either there exists a constant $\gamma = \gamma(data, \tilde{\mathcal{K}})$

$$\omega \le \gamma R^{\frac{\epsilon}{\alpha_0}} \tag{6.135}$$

or there is a constant $\sigma \in (0,1)$ that depends only on the data and on $\tilde{\mathcal{K}}$ and a cylinder $Q(\frac{R}{8}, t^\diamond) \subset Q_R(\theta_m)$ such that

$$\operatorname*{ess\,osc}_{Q(\frac{R}{8}, t^\diamond)} s \le \sigma \omega. \tag{6.136}$$

Proof. The previous two sections, showed that $m = m(data, \tilde{\mathcal{K}})$. Particularly, m does not depend on R and ϵ. Remark 6.30 implies that $q_4 \ge q_3 \ge \max\{l_2, m\}$. As we mentioned at the beginning of this paragraph, the assumptions from A6.2 on μ_+, μ_- and ω can be disposed.

Concerning the assumption on the inclusions $Q_R(\theta_m) \subset Q(2R, \theta_m R^2) \subset Q(2R, (2R)^{2-\epsilon})$, we recall equations (6.19) and (6.20) for the case that any of these inclusions fails to hold.

Otherwise, if both inclusions hold, we may perform the alternative argument performed in the last sections and use either the results of Corollaries 6.19 or 6.32.

Assume that either (6.77) from Corollary 6.19 or that (6.134) from Corollary 6.32 holds. With $t^\diamond = \min\{\hat{t}, \tilde{t}\}$ and $\sigma = \max\{\sigma_0, \sigma_1\}$, we conclude (6.136) in both cases.

On the other hand, if (6.77) and (6.134) both fail to hold, we find the estimate

$$\min\left\{\frac{\omega}{2^{q_4}}, \Phi_{1,l}\left(\frac{\omega}{2^{q_4}}\right), \Phi_{0,l}\left(\frac{\omega}{2^{q_4}}\right)\right\} \leq \gamma R^{\frac{\beta}{4}}. \tag{6.137}$$

We combine (6.137) with (6.19) and (6.20), use the structure condition of Φ from Assumption A3.7 and estimate the constants roughly from above. Using $0 < R \leq 1$, this implies

$$\omega \leq \gamma \max\left\{R^{\frac{\beta}{4\max\{\alpha_0,1\}}}, R^{\frac{\epsilon}{\alpha_0}}\right\} \leq \gamma R^{\frac{\epsilon}{\alpha_0}}$$

and we conclude. $\qquad\square$

6.5.1 Proof of the Main Proposition

Having linked the results of the alternatives, we are in the position to prove the main proposition.

Proof of Proposition 6.10. We start from Proposition 6.33 and assume that (6.135) is violated. Thus, also (6.26) is violated.

Consequently, the "starting" condition (6.18) is fulfilled, i.e.

$$\operatorname*{ess\,osc}_{Q_R(\theta_m)} s \leq \omega.$$

We estimate $t^\diamond = \min\{\hat{t}, \tilde{t}\} \geq 0$ from below. We recall the definition of $\tilde{t}$ from (6.123), the definition from μ and Γ from Definition 6.25 and that $j_0 \geq 1$. Since $m > m_1 > q_2$, we infer from (6.111) that $n_0 \geq \Gamma^{\frac{d+2}{d}}$. Summarizing, we obtain

$$\tilde{t} \geq \frac{1}{4}\frac{\theta_m}{\Gamma^{\frac{d+2}{2}}}R^2 \overset{(6.22)}{\geq} \frac{\theta_{m_0}}{2\Gamma^{\frac{d+2}{2}}}R^2. \tag{6.138}$$

Concerning $\hat{t}$, defined in (6.51), we realize that

$$\hat{t} = \theta_{m_0}\left(\frac{R}{2}\right)^2 - t^* \geq \frac{\theta_{m_0}}{4}R^2. \tag{6.139}$$

Exploiting that $\Gamma > 1$ and combining (6.138) and (6.139), we obtain

$$t^\diamond > \frac{\theta_{m_0}}{4\Gamma^{\frac{d+2}{2}}}R^2.$$

With the notation from (6.17), we infer

$$t^\diamond > \frac{\theta_{m_0}}{4\Gamma^{\frac{d+2}{2}}}R^2 = \theta_m(\sigma\omega)\frac{\theta_{m_0}}{4\Gamma^{\frac{d+2}{2}}\theta_m(\sigma\omega)}R^2 = \theta_m(\sigma\omega)\left(\sigma^{\alpha_0}\frac{c_{0,l}}{c_{1,l}\Gamma^{\frac{d+2}{2}}}2^{m_0-m-1}\right)R^2.$$

Choosing $C > 0$ such that

$$\frac{1}{C^2} < \min\left\{\sigma^{\alpha_0}\frac{c_{0,l}}{c_{1,l}\Gamma^{\frac{d+2}{2}}}2^{m_0-m-1}, 64\right\},$$

we infer $Q_{\frac{R}{C}}(\theta_m(\sigma\omega)) \subset Q\left(\frac{R}{8}, t^\diamond\right)$ and thus

$$\operatorname*{ess\,osc}_{Q_{\frac{R}{C}}(\theta_m(\sigma\omega))} s \leq \operatorname*{ess\,osc}_{Q\left(\frac{R}{8}, t^\diamond\right)} s \leq \sigma\omega. \tag{6.140}$$

The previous consideration shows how to decrease the oscillation once (6.135) is violated.

Thus, with $R_0 = R$, $R_1 = \frac{R}{C}$, $\omega_0 = \omega$, $\omega_1 = \sigma\omega$, $\theta_{m,0} = \theta_m(\omega)$ and $\theta_{m,1} = \theta_m(\omega_1)$ we find the sequences stated in Proposition 6.10 for $n = 0$ and $n = 1$.

To pass from n to $n+1$ for $n \geq 1$, we define $R_n = C^{-n}R$. Assuming that ω_n and $\theta_{m,n}$ are determined, we make the following choices. If we find that (6.135) is violated on $Q_{R_n}(\theta_{m,n})$, i.e.

$$\omega_n \leq \gamma R_n^{\epsilon/\alpha_0} \text{ on } Q_{R_n}(\theta_{m,n}),$$

we conclude from the previous step and from

$$\operatorname*{ess\,osc}_{Q_{R_n}(\theta_{m,n})} s \leq \omega_n \text{ that also } \operatorname*{ess\,osc}_{Q_{R_{n+1}}(\theta_{m,n})} s \leq \omega_n.$$

We emphasize that, the cylinders are included in each other and the oscillation may only decrease in that case. Hence, we choose $\omega_{n+1} = \omega_n$ and $\theta_{m,n+1} = \theta_{m,n}$. In the other case, we argue as above and choose $\omega_{n+1} = \sigma\omega_n$ and $\theta_{m,n+1} = \theta_m(\omega_{n+1})$. Particularly, we obtain $\omega_n \searrow 0$, $Q_{n+1} \subset Q_n$ and also $\theta_{m,n} \nearrow \infty$. □

Remark 6.34. We remark that the introduction of the cylinder $Q(2R, (2R)^{2-\epsilon})$ was necessary to obtain a subcylinder of type $Q_R(\theta_m)$ such that (6.18) holds; thus the iteration process could be started. In general, this relation is not verifiable for a cylinder of type $Q_R(\theta_m)$ since its definition depends on the essential oscillation within it. ◁

6.5.2 Proof of the Main Theorem

To derive the Hölder continuity we need an adaption of [DiB93, III.Lemma 3.1]. To this end, we fix ϵ, determine α from Theorem 3.12 and obtain in the situation of Proposition 6.10 the following quantitative estimate.

Lemma 6.35. *Let the setting of Proposition 6.10 be given. Then there exists $\varepsilon_0 = \varepsilon_0(data, \tilde{\mathcal{K}})$, such that for the cylinder $Q(2R, (2R)^{2-\varepsilon_0})$ we find either*

$$\omega \leq \gamma R^{\varepsilon_0} \tag{6.141}$$

for some $\gamma = \gamma(data, \tilde{\mathcal{K}})$*, or there exist constants* $\gamma > 1$ *and* $\alpha \in (0, \frac{\varepsilon_0}{2}]$ *that can be determined depending only on the data and on* $\tilde{\mathcal{K}}$*, such that for the cylinders*

$$Q_\rho(\theta_m(\omega)) \text{ with } 0 < \rho \le R$$

there holds

$$\operatorname*{ess\,osc}_{Q_\rho(\theta_m(\omega))} s \le \gamma(\omega + R^{2\alpha}) \left(\frac{\rho}{R}\right)^\alpha. \tag{6.142}$$

Proof. We start from Proposition 6.10, choose some ϵ such that (6.25) holds and define

$$\varepsilon_0 := \frac{1}{2} \min\left\{\frac{-\ln(\sigma)}{\ln(C)}, \frac{\epsilon}{\alpha_0}\right\}.$$

This implies $\sigma \le C^{-\varepsilon_0}$ which is used in the following.

From the estimate on ω_n from Proposition 6.10, we infer $\omega_{n+1} \le \sigma\omega_n + \gamma R_n^{\varepsilon_0}$. Iterating this estimate and using the definition of R_n leads to

$$\omega_n \le \sigma^n \omega + \gamma R^{\varepsilon_0} \sum_{k=1}^n \sigma^{k-1} C^{-(n-k)\varepsilon_0}. \tag{6.143}$$

Thus, (6.143) together with the definition of ε_0 implies

$$\omega_n \le \sigma^n \omega + \gamma n \left(\frac{R}{C^n}\right)^{\varepsilon_0}. \tag{6.144}$$

For fixed $0 < \rho \le R$, there exists a nonnegative integer n such that

$$C^{-(n+1)} R \le \rho \le C^{-n} R. \tag{6.145}$$

Since $C > 1$, this is equivalent to

$$n \le -\frac{\ln \frac{\rho}{R}}{\ln C} \le n + 1. \tag{6.146}$$

With

$$\tilde{\alpha} = -\frac{\ln C}{\ln \sigma} \text{ there holds } C = \sigma^{-\tilde{\alpha}}$$

and from the first inequality in (6.145), we infer

$$\sigma^{\frac{n+1}{\tilde{\alpha}}} \le \frac{\rho}{R} \text{ which implies } \sigma^n \le \sigma^{-1} \left(\frac{\rho}{R}\right)^{\tilde{\alpha}}.$$

In addition, we find

$$n \left(\frac{R}{C^n}\right)^{\varepsilon_0} = n \left(\frac{R}{\rho C^n}\right)^{\varepsilon_0} \rho^{\varepsilon_0} \le C^{\varepsilon_0} \rho^{\varepsilon_0} \left(-\frac{\ln \frac{\rho}{R}}{\ln C}\right) \le C^{\varepsilon_0} \rho^{\frac{\varepsilon_0}{2}} \left(\frac{R}{C^n}\right)^{\frac{\varepsilon_0}{2}} n + 1$$

$$\le \tilde{\gamma}(\varepsilon_0) R^{\frac{\varepsilon_0}{2}} \rho^{\frac{\varepsilon_0}{2}} \le \tilde{\gamma}(\varepsilon_0) R^{\varepsilon_0} \left(\frac{\rho}{R}\right)^{\frac{\varepsilon_0}{2}},$$

since $C^{-n\frac{\varepsilon_0}{2}}(n+1)$ is bounded from above by a constant depending only on C and ε_0. Therefore, for a constant $\gamma = \gamma(data, \tilde{\mathcal{K}})$ we infer from (6.144) the estimate

$$\omega_n \leq \gamma(\omega + R^{2\alpha})\left(\frac{\rho}{R}\right)^\alpha \text{ for } \alpha := \min\left\{\tilde{\alpha}, \frac{\varepsilon_0}{2}\right\}.$$

Since $\omega_n \leq \omega$ and $\rho \leq R_n$, we find $Q_\rho(\theta_m) \subset Q_n$ and

$$\underset{Q_\rho(\theta_m)}{\text{ess osc}}\, s \leq \underset{Q_n}{\text{ess osc}}\, s \leq \omega_n.$$

This concludes the proof. □

As was already stated in Remark 6.34, the inclusion of $Q_R(\theta_m)$ in $Q(2R, (2R)^{2-\varepsilon})$ is necessary to start the iteration process stated in Proposition 6.10. The estimate (6.142) has been derived in Lemma 6.35 using this inclusion. We also require an oscillation estimate if the inclusion $Q_R(\theta_m) \subset Q(2R, (2R)^{2-\varepsilon})$ fails to hold. To this end, we investigate (6.141) more extensively.

Lemma 6.36. *Let the setting of Proposition 6.10 be given, consider $Q(2R, (2R)^{2-\varepsilon_0})$ from Proposition 6.10 and assume that (6.141) holds. For C from Proposition 6.10, we define the sequence $R_n = RC^{-n}$. Then, either there exists n_0 such that*

$$\underset{Q_\rho(\theta_m)}{\text{ess osc}}\, s \leq \gamma(R^{\varepsilon_0} + R^{2\alpha})\left(\frac{\rho}{R}\right)^\alpha \tag{6.147}$$

for every $\rho \leq R_{n_0}$ and for every $n \leq n_0 - 1$ there holds

$$\underset{Q(2R_n,(2R_n)^{2-\varepsilon_0})}{\text{ess osc}}\, s \leq \gamma C^{\varepsilon_0} R_{n+1}^\alpha, \tag{6.148}$$

or (6.148) holds for every $n \in \mathbb{N}$.

Proof. In this proof we consider the cylinders $Q_n := (2R_n, (2R_n)^{2-\varepsilon_0})$ and define the numbers

$$\omega_n := \underset{Q(2R_n,(2R_n)^{2-\varepsilon_0})}{\text{ess osc}}\, s.$$

With these numbers, we find either that $\omega_n \leq \gamma R_n^{\varepsilon_0}$ for every $n \in \mathbb{N}$ and, since $\alpha \leq \varepsilon_0$, we infer

$$\underset{Q(2R_n,(2R_n)^{2-\varepsilon_0})}{\text{ess osc}}\, s \leq \gamma R_n^{\varepsilon_0} \leq \gamma C^{\varepsilon_0} R_{n+1}^\alpha,$$

which is (6.148) for every n. Elsewise, there exists $n_0 \in \mathbb{N}$ such that $\omega_n \geq \gamma R_n$. Particularly, $n_0 > 0$ since (6.141) holds. Consequently, we may apply Lemma 6.35 for the cylinder Q_{n_0} with ω_{n_0}. Hence, we infer

$$\underset{Q_\rho(\theta_m(\omega_{n_0}))}{\text{ess osc}}\, s \leq \gamma(\omega_{n_0} + R_{n_0}^{2\alpha})\left(\frac{\rho}{R_{n_0}}\right)^\alpha \tag{6.149}$$

for every $0 < \rho \leq R_{n_0}$. Since $Q_{n+1} \subset Q_n$, we infer also $\omega_{n+1} \leq \omega_n$ for any $n \in \mathbb{N}$. Particularly, we find $\omega_{n_0} \leq \omega$ and thus, since θ_m is decreasing, even

$$\operatorname*{ess\,osc}_{Q_\rho(\theta_m(\omega))} s \leq \operatorname*{ess\,osc}_{Q_\rho(\theta_m(\omega_{n_0}))} s.$$

To conclude (6.147), we estimate the right hand side of (6.149). Since

$$\omega_{n_0} \leq \omega_{n_0-1} \leq \gamma R_{n_0-1}^{\varepsilon_0} \leq \gamma C^{\varepsilon_0} R_{n_0}^{\varepsilon_0},$$

$C > 1$ and $\alpha \leq \frac{\varepsilon_0}{2}$, we obtain

$$\gamma(\omega_{n_0} + R_{n_0}^{2\alpha})\left(\frac{\rho}{R_{n_0}}\right)^\alpha \leq \gamma(\gamma c^{\varepsilon_0} R_{n_0}^{\varepsilon_0} + R_{n_0}^{2\alpha})\left(\frac{\rho}{R_{n_0}}\right)^\alpha$$

$$\leq \gamma\left(\gamma c^{\varepsilon_0} \frac{R^{\varepsilon_0}}{C^{n\varepsilon_0}} + \frac{R^{2\alpha}}{C^{2n\alpha}}\right)\left(\frac{\rho}{R}\right)^\alpha C^{n\alpha} \leq \gamma(R^{\varepsilon_0} + R^{2\alpha})\left(\frac{\rho}{R}\right)^\alpha.$$

Summarizing, we obtain (6.147) and (6.148) is concluded as before. $\qquad\square$

Exploiting both previous lemmas, we conclude the local Hölder continuity of the saturation s with a covering argument. This proves the main theorem.

Proof of Theorem 3.12 Let $\mathcal{K} \subset Q$ be given and determine $4\tilde{R} = \operatorname{dist}_2(\mathcal{K}, \partial_p Q)$. We cover $\mathcal{K}$ by cylinders of type $Q_{2\tilde{R}}$, take the closure of this cover and denote it by $\tilde{\mathcal{K}}$. On the set $\tilde{\mathcal{K}}$, we determine the constants γ, α and m stated in Lemma 6.35. We define $R := \min\{\tilde{R}, 1\}$.

For convenience, we assume that we have already shown that a continuous representative of s exists. This follows with a standard argument from e.g Proposition 6.10 or even from the estimate we present in the following.

Furthermore, we will only show the Hölder continuity in time. The reasoning to obtain the Hölder regularity in space follows along the same lines. Due to the triangle inequality, we can combine both results to conclude.

Let (x, t_1) and $(x, t_2) \in \mathcal{K}$ be arbitrary. We assume that $t_1 < t_2$ and define $\rho := \sqrt{t_2 - t_1}$. We introduce the cylinder $(x, t_2) + Q(2R, (2R)^{2-\varepsilon_0})$ and we need to take several cases for t_1, t_2 related to this cylinder into account. If $\rho > R$ we obtain, since $s \in L^\infty(Q, [0, 1])$, directly

$$|s(x, t_1) - s(x, t_2)| \leq \left(\frac{\rho}{R}\right)^\alpha = \left(\frac{\sqrt{t_2 - t_1}}{R}\right)^\alpha.$$

For the other cases, we exploit Lemma 6.35 and Lemma 6.36. Thus, ρ is such that one of the estimates (6.142), (6.148) or (6.149) holds. In all of these cases we obtain an estimate of the form

$$|s(x, t_1) - s(x, t_2)| \leq \gamma\left(\frac{\rho}{R}\right)^\alpha = \gamma\left(\frac{\sqrt{t_2 - t_1}}{R}\right)^\alpha,$$

where γ is a constant that can be determined a priori only in terms of the data and $\tilde{\mathcal{K}}$. From here on, we conclude as stated above.

6.6 Comparison to the Literature

In [Urb08, chapter 5], we find a local Hölder continuity result for the model Problem (MP) without x-dependence. The proof of the local Hölder continuity for the saturation of the two-phase flow problem (TP1) presented here follows closely the arguments of Urbano. However, we require several modifications, due to the different structure of the equation. In particular, the additional issues we need to face were the following: The saturation s is not necessarily continuous in time, the occurrence of lower order and divergence free terms, and the x-dependence of the coefficient functions, Φ and κ. We provide a brief overview of the literature on continuity for the two-phase flow problem and compare also to these additional issues.

In [Che01] the local Hölder continuity of the saturation for the two-phase flow problem with x-dependence in the form of (TP2) under the assumption that at most one degeneracy occurs and that λ is independent of s is stated. The growth of the Φ' near the degeneracy must be bounded above and below by different power functions. The proof is not complete, and, essentially, only the first alternative is investigated. From the Hölder continuity and the regularity of the pressure a uniqueness result is obtained in virtue of a method introduced in [Fri10, Theorem 10.1].

The next articles we comment on, inspired our treatment of the lower order and divergence free terms. Essentially, differences only occur due to the x-dependence of the coefficient functions.

In [Che02], the local Hölder continuity of the saturation for the two-phase flow problem in the form of (TP1) is stated. It is assumed that the growth of Φ' is bounded above and below by some power function near the degeneracies of Φ. An x-dependence only occurs in κ and, since κ is differentiated during the derivation of energy estimates, one should restrict the assumptions from $\kappa \in L^\infty(\Omega, \mathbb{R}^{d \times d})$ to $\kappa \in W^{1,\infty}(\Omega, \mathbb{R}^{d \times d})$ as we do. The treatment of the second alternative and thus the definition of the geometry is skipped in this article; essentially the reference [DiB93] is given where only equations of p-Laplace type are investigated. Such equations are of a different type.

In [AD85b; DiB87] and in the more recent article [DGV10] continuity of the saturation for the two-phase flow problem without x-dependence in the form of (TP1) is shown. Only strict monotony of Φ' near zero and one is required. In [AD85b] a Hölder continuity result is stated without providing a proof.

In [AD85a] the continuity of the saturation for a problem similar to (TP1) with x-dependence and one or two degeneracies is shown. Requirements on Φ' are strict monotony near one degeneracy and a log like growth near the other. In particular,

the equation considered is obtained as a limit problem and we also refer to Remarks 3.1, 3.2 and 4.42.

6.6.1 Discussion of the Proof of the Main Proposition

In this section, we compare the proof of the previous results to the analogues found in the literature. A first difference, when compared to the literature presented above, is that we do not assume that the saturation s is continuous in time. The existence result in [AD85a], [Arb92] and [Che01] do not state such a continuity. Also the result of [AL83, Theorem 2.3] is not applicable since Φ^{-1} is in general not Lipschitz continuous.

Justification of the test functions   To justify the use of the test functions in, e.g. Lemma 6.14, Lemma 6.16 and Proposition 6.24 we used the integration by parts formulae from Lemma 4.32 or Lemma 4.36. To show these formulae, we exploited the Steklov average. Except for [AD85a, Remark 3.4] and [Urb08], this justification is not performed in the references given above. In [Urb08, p. 56] the justification seems to have a flaw. The averaging operation $(\cdot)_h$ is interchanged with the positive part operation $(\cdot)_+$. Since the positive part is not a linear function, this is not possible.

In addition we needed to justify that truncations of s are regular, which is studied in section 4.2. In all the references given above this justification is missing and formally $\nabla[\Phi(s)] = \Phi'(s)\nabla s$ is used. In particular, though the necessary condition $s \in H^1(\Omega)$ is not established, Stampacchia's Lemma is applied to s. For examples, we refer to [Urb08, p. 57f], [Che02, p. 354ff] and [DGV10, p. 2051] and compare this to the corresponding terms in the proof of Lemma 6.14. Nevertheless, thanks to the results of section 4.2 these formal calculations can be justified and we refer to section 4.5.1.

To show the logarithmic estimates, for example see [Urb08, Lemma 5.4], [DGV10, Proposition 3.3] or [Che02, Lemma 2.4], we find as a part of the test function the expression $\Psi((s-a)_\pm)\Psi'((s-a)_\pm)$. Since s is only truncated from below, if the positive part is chosen, or from above, if the negative part is chosen, it is not apparent if these test functions are weakly differentiable. For example, in the proof of Lemma 6.16 we truncate s a second time to end up with $\Psi((s_b-a)_+)\Psi'((s_b-a)_+)$ and we may use the regularity of these truncations. A hint on this already has been given in [AD85a, p. 389]. These truncations lead to additional terms, e.g. to the term J_{11}. Those terms are estimated using that b is sufficiently close near μ_+. In addition, if $\mu_+ < 1$, we could truncate at the level $1 > b \geq \mu_+$ and this makes J_{11} disappear directly.

Differences to the literature   The main difference, when comparing the presented proof to the literature occurs in Proposition 6.24. The differences are highly influenced by the x-dependence of Φ. Comparing the proof of the proposition to the proofs of [Urb08, Propositions 5.10 and 5.12], [DGV10, Proposition 4.6] or [AD85b, Lemma 4.4] we see that, in contrast to our proof, the parabolic equation is transformed into an equation of type

$$\partial_t b(v) = \nabla \cdot (\nabla v + h(v, u)), \qquad (6.150)$$

with the inverse $b = \Phi^{-1}$ and h contains the lower order terms. We emphasize that the function b is independent of x. In the following steps of the proofs found in the references, certain truncations of u are used as test functions and Lemma 6.6 plays a major role. To use the lemma it is necessary to truncate functions at constant values.

Transferring these proofs to the case with an x-dependent Φ and not using a transformation like (6.150), leads to x-dependent truncations at e.g. $\Phi(x, k)$ for an appropriate k. Hence, it is not possible to apply Lemma 6.6. To overcome this issue, we exploit the more general Proposition 1.2.

Another minor difference occurs in (6.119) in the proof of Lemma 6.27, due to the special choice $\alpha_0 = \alpha_1$ from Assumption A3.7. This special choice yields in a simpler proof compared to the literature. In particular, we do not need to use [AD85b, Lemma 6.1] as it is done in [AD85b, Lemma 4.7] or referred to in [Urb08, Lemma 5.13].

Further comments   It seems that Urbano's proof of local Hölder regularity, though stated for $\alpha_0 < \alpha_1$ only works for the case $\alpha_0 = \alpha_1$, and, essentially, that is why we impose this strong assumption.

First, we demonstrate Urbano's approach in view of our proof and notation. In (6.89) we proceed differently and exploit (6.22) to infer

$$\frac{\Phi_{0,u}\left(\omega/2^{m_0 - 2}\right)}{\Phi_{1,l}\left(\omega/2^{m_0 + 1}\right)} \leq 2^{\alpha_0(m - m_0 + 2) + 1}. \qquad (6.151)$$

In that case, one does not use that $\alpha_1 = \alpha_0$ at the cost of a dependence of q_1 on m, as can be seen from (6.88). Roughly speaking we find the relation $2^m \lesssim q_1$, where $\lesssim$ denotes $<$ up to a constant. In Proposition 6.24, we need to choose $m > m_1$ and $q_2 > q_1$ such that (6.105) and (6.108) are fulfilled. The last relation necessitates that $q_2 > q_1$, whereas the first relation requires that $q_2 \leq m_1 \leq m$. In addition, also (6.103) requires m_1 to be large. Combining these results we obtain

$$q_1 < q_2 \leq m \leq 2^m \lesssim q_1$$

which yields a contradiction since the intention is to choose m large.

In Urbano's argumentation and notation, we find in the proof of [Urb08, Lemma 5.8 and p. 71], i.e. the corresponding analogue of Lemma 6.22, the relation $2^m \lesssim s_2$. Due to the transformation of the equation described in the previous paragraph a new constant $s_2 \leq s_4$ is introduced, see [Urb08, p. 72]. Thus, also $2^m \lesssim s_4$ and s_2 and s_4 correspond to q_1 in our notation. In [Urb08, Proposition 5.12], i.e. the corresponding analogue to Proposition 6.24, the existence of $s_5 > s_4$ and $m_0 \in \mathbb{N}$ with $m > m_0$ is claimed, such that particularly

$$\frac{C}{\nu_0^2 \sqrt{s_5 - s_4}} \leq \frac{\lambda_0}{\sqrt{2}}, \tag{6.152}$$

see [Urb08, p. 75] and

$$(p_2 + 1)2^{s_5 - \frac{s_4}{p_2+1}} \leq 2^{m_0 p_2}, \tag{6.153}$$

see [Urb08, p. 76], are fulfilled. Here m_0, s_5 and p_2 corresponds to m_1, q_2 and α_0 in our notation. Summarizing these results, we obtain

$$s_4 < s_5 \lesssim m_0 \leq m \leq 2^m \lesssim s_4$$

which is again a contradiction since m needs to be chosen large.

In [Che02, p. 349, (A3)], the local Hölder regularity is stated for the cases that the function $\Phi_{j,k}$ from Assumption A3.7 have a different power like growth, i.e. $\Phi_{j,k}(s) = c_{j,k} s^{\alpha_{j,k}}$ for $j \in \{0, 1\}$ and $k \in \{l, u\}$, which is clearly a more desirable configuration in comparison to our assumptions. Furthermore, the proof is only presented in the case $\alpha_{j,k} = \alpha_{\tilde{j},\tilde{k}}$ for $j, \tilde{j} \in \{0, 1\}$ and $k, \tilde{k} \in \{l, u\}$, i.e. for the case $\alpha_0 = \alpha_1$ in our notation. It is claimed that at least the proof for the case $\alpha_{0,l} = \alpha_{0,u} = \alpha_0 \neq \alpha_1 = \alpha_{1,l} = \alpha_{1,u}$ can be traced back to the case $\alpha_0 = \alpha_1$, see [Che02, Remark 2.1, Remark 2.2]. However, as was stated above, the second alternative, i.e. the part where the flaw in Urbano's proof occurs, is not executed. Thus, the statement seems questionable and the implementation of this idea yields to ω dependent constants as we are going to see.

Let us assume $c_{j,k} = 1$ for $j \in \{0, 1\}$ and $k \in \{l, u\}$ for simplicity, as well as $\Phi_{0,l}(s) = \Phi_{0,u}(s) = s^{\alpha_0} =: \Phi_0(s)$ and $\Phi_{1,l}(s) = \Phi_{1,u}(s) = s^{\alpha_1} =: \Phi_1(s)$ with $\alpha_0 \neq \alpha_1$. Following [Che02, Remark 2.1], we define the cylinder θ_m for the cylinder $Q_R(\theta_m)$ as

$$\frac{1}{\theta_m} := \Phi_0^{\alpha_1}\left(\frac{\omega}{2^m}\right),$$

and we adapted to the situation that we treat the degeneracy at one instead of zero first. The degeneracies at zero and one are connected through

$$\Phi_0^{\alpha_1}\left(\frac{\omega}{2^m}\right) \leq \frac{1}{2}\Phi_1^{\alpha_0}\left(\frac{\omega}{2^{m_0+1}}\right) =: \frac{1}{\theta_{m_0}}$$

instead of (6.22).

Let us examine the proof of Lemma 6.14 with this choice of θ_{m_0}. Since terms containing $\Phi'(s)$ can only be estimated in terms of $\Phi_1(s)$ or $\Phi_0(s)$ the only occurrence of θ_m originates from (6.41), the term containing the time derivatives. Continuing the proof, we find in (6.45) an additional term of the form

$$\frac{\Phi_0^{\alpha_1}(\omega/2^{m_0+1})}{\Phi_1(\omega/2^{m_0+1})} = \frac{\omega^{\alpha_1\alpha_0}2^{(m_0+1)\alpha_1}}{\omega^{\alpha_1}2^{\alpha_0\alpha_1(m_0+1)}}$$

which clearly depends on ω. Thus, ν_0 depends on ω and the argument breaks down.

We remark that the result we obtained also holds if only one degeneracy of Φ, say at zero, is present. Essentially, the idea on how to obtain this result is explained at the beginning of section 6.5.

6.6.2 Outlook

In the literature, the method of intrinsic scaling is also suitable to provide continuity and Hölder continuity results up to the boundary or up to $t = 0$, see [AD85a; DGV10] or [PV93]. To obtain these results, analogous estimates of those in e.g. Lemma 6.14, Lemma 6.16 and Proposition 6.24 for cylinders that contain a part of the parabolic boundary of Q need to be shown. To obtain such results, a certain regularity of the boundary or initial data is needed.

In the remaining explanations of [Che02], the interior Hölder regularity was used to provide gradient regularity for the pressure p, that is regularity for the flux u, from elliptic regularity theory. These local results were assumed to hold on the full domain Q. From there on, under some assumptions on the coefficients [Che02, p. 366 (A11) and Proposition 3.5], a uniqueness result for the system (TP1) exploiting a dual problem was shown. The extension of the uniqueness result to the x-dependent case as well as possible relaxations of the assumptions on the coefficients still need to be shown.

As we saw in the previous section, a local Hölder continuity result for the saturation in two-phase flow problem only seems to be available under the assumption that the two degeneracies of Φ' are power functions with same exponent. Consequently, extension of the regularity to the cases stated in [Che02, p.349 (A3)] or [Urb08, p. 53 (A3)] or even to degeneracies with an e.g. exponential growth are still open questions.

Comparing our result to [DGV10], we see that the continuity of the saturation without any assumptions on the growth of Φ' near zero and one in the case of x-dependent coefficient functions is also a desirable result. Regarding the extension of the continuity for two-phase flow problems up to the boundary, one could also ask if such an extension holds with interfaces inside the domain under the assumption of continuity of the pressure. This leads, as seen before, to a discontinuity of saturation across the interface,

but the height of the discontinuity of the saturation is in some way controlled by the two capillary pressure functions $p_{c,1}$ and $p_{c,2}$. Hence, a decrease of the oscillation of the saturation on one side of the interface should effect in a decrease of the oscillation of the saturation on the other side of the interface.

7 Summary

The work at hand provides regularity and uniqueness results for certain equations occurring in porous medium flow. The main focus lies on x-dependent coefficient functions. Starting from the generalized porous medium equation

$$\partial_t s = \Delta[\Phi(x, s)] \text{ on } \Omega \times (0, T)$$

with $\Phi'(x, 0) = 0$ in chapter 4, we have shown that truncations of s are H^1-regular in space and proved integration by parts formulae of the form

$$\int_Q \partial_t s g(\cdot, s)\xi = \int_Q G(\cdot, s)\partial_t \xi \tag{7.1}$$

for certain functions g with primitive G. Both results are essential for the consideration of the main problems of this thesis. In chapter 2, we stated the problem for the discontinuous Richards equation, that is we consider, on a domain Ω divided by an interface Γ into two subdomains Ω_l and Ω_r, the Richards equation

$$\partial_t s = \nabla \cdot (\nabla[\Phi_j(s)] + \lambda_j(s)g_j) + f_j \text{ on } \Omega_j \times (0, T)$$

assuming the continuity of flux and pressure across the interface. For this equation, we have proven an L^1-contraction and uniqueness result in chapter 5. To this end, we exploited (7.1) and used the method of doubling the variables to obtain a Kato inequality away from $t = 0$. With a Gronwall argument, the Kato inequality was extended up to $t = 0$ and an L^1-contraction could be concluded. In this regime, the new contribution of this thesis is the L^1-contraction result also if $\lim_{s \to 1} \Phi_j'(s) = \infty$. Particularly, Φ_j needs not to be Lipschitz continuous.

In the regime of two-phase flows, we used a standard transformation to obtain a system of the form

$$\partial_t s = \nabla \cdot (\nabla[\Phi(x, s)] - \nabla_x \Phi(x, s) + B(x, s) + D(x, s)u)$$
$$0 = \nabla \cdot u = \nabla \cdot (\lambda(s)\nabla p + E(s))$$

on $\Omega \times (0, T)$ (see chapter 3). From there on, we recalled and compared solution concepts found in the literature and stated the local Hölder continuity result under the assumption that $\Phi'(x, s) \sim s^{\alpha_0}$ and $\Phi'(x, s) \sim (1 - s)^{\alpha_0}$ for s near zero and one and $\alpha_0 > 0$, respectively. The local Hölder continuity was proven in chapter 6 using the method of intrinsic

scaling. The regularity of truncations of s as well as (7.1) is essential to justify the calculations presented there. The new contribution of this thesis is the local Hölder continuity for an x-dependent Φ. Particularly, the method of intrinsic scaling needs to be modified since DeGiorgi's lemma, Lemma 6.6, can not be applied and one has to apply the more general Poincaré type inequality from Proposition 1.2.

In the first chapter we provided a brief introduction and derived the equations by means of physical principles.

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