Analysis and optimal control of a damage model with penalty

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Dissertation

Analysis and optimal control of a damage model with penalty

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Contents

Introduction

This thesis is devoted to the analysis of a viscous *two-field* gradient damage model. The particular feature thereof is that it contains two damage variables, which are connected through a penalty term in the free energy functional. While one damage variable provides a local character and carries the time evolution, the other one accounts for nonlocal effects, cf. [12].

The problem under consideration describes the evolution of damage in an elastic body under the influence of an external loading. It is inspired by the model presented in [12], which is widely used in computational mechanics. From a mathematical perspective, the damage model in [12] provides two main drawbacks. Firstly, the coupling between the damage evolution and balance of momentum equation is realized via the less regular one of the two damage variables. To guarantee satisfying solvability results, we slightly modify the original problem, by coupling these equations through the more regular damage variable. This enables the use of compact embeddings, which are essential for establishing solvability results. It turns out that this modification has only little influence in practice, as we later obtain that both damage variables coincide when the penalty term becomes ∞. Secondly, [12] deals with a rate-independent model where the corresponding dissipation functional is unbounded. In this situation, the existence of classical solutions (also known as differential solutions, see [52]) has been proven under smoothness and convexity assumptions on the reduced energy functional, see [59] (uniform convexity) and [44] (quadratic case), whereas uniqueness results can be found in [59], as well as in [8, 53] (generalization of the result in [59] under restrictive assumptions). On the other hand, in the nonconvex case, solutions may often be discontinuous is time, and thus, weak solvability concepts are required, such as global energetic solutions, see [52, Section 3.3] for the abstract main existence result. Another popular approach in this context consists of performing a viscous regularization. Weak solutions for the rate-independent model are then found via a vanishing viscosity analysis. For references and more details, see the introduction of Chapter 1. Since after modifying the original model as mentioned above, our reduced energy functional is not necessarily convex, we follow this standard procedure, namely, we add a viscosity term to the damage evolution, which turns the rate-independent model into a rate-dependent one.

For the viscous penalized damage model under consideration, see (P) below, we establish in this thesis the following main results:

- The well-posedness from a mathematical point of view. We prove the unique solvability of (P), make statements about the regularity of the unique solution and derive a system of differential equations which characterizes it (Chapter 1). These results can be also found in [49];

- The viability of the penalty approach. We show that the two damage variables become equal when the penalty term tends to infinity. It turns out that the limit damage variable satisfies a version of a classical viscous damage model analyzed by [41]. We derive conditions for the data under which both penalized damage variables and their limit are almost everywhere bounded by a desired positive value (Chapter 2). These results can be also found in [51];

- An optimality system for a class of optimal control problems governed by (P), where the external load acts as control. We prove the existence and directional differentiability of the associated control-to-state operator. We derive necessary optimality conditions in primal form and show that these are equivalent to an optimality system, if *strict complementarity* is fulfilled (Chapter 3).

To the best of our knowledge, a damage model containing two damage variables has never been investigated so far with regard to a rigorous mathematical analysis, although these models are frequently used for numerical simulations cf. e.g. [67, 76, 46, 64, 83]. This concerns the existence and regularity of solutions, let alone the behaviour as the penalty term tends to ∞ and the optimal control thereof.

Motivation

The problem analyzed throughout this thesis was inspired by a damage model proposed in [12]. Therein two damage variables are introduced, which the authors call 'local' and 'nonlocal' damage. Time-dependent volume and boundary forces, denoted by ℓ, are applied during the process upon an elastic body which has a part of its boundary clamped. The body is described by the domain $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$, on which we impose mild smoothness assumptions, see Assumption 0.5 below. The load induces a certain displacement $\boldsymbol{u} : [0, T] \times \Omega \to \mathbb{R}^N$, as well as local and nonlocal damage. The latter one is denoted by $\varphi : [0, T] \times \Omega \to \mathbb{R}$, while the local damage is called $d : [0, T] \times \Omega \to \mathbb{R}$. Its values measure the degree of the material rigidity loss. Therefore, $d(t, x) = 0$ means that the body is completely sound, while $d(t, x) = \infty$ means that the body is fully damaged.

In [12] the free energy function is enhanced by including a gradient term and a term which penalizes the difference between local and nonlocal damage. To be precise, the energy functional $\hat{\mathcal{E}} : [0, T] \times V \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ according to [12, (4) and (5)] is given by

$$\hat{\mathcal{E}}(t, \boldsymbol{u}, \varphi, d) := \frac{1}{2} \int_\Omega g(d)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\, dx - \langle \ell(t), \boldsymbol{u} \rangle_V + \frac{\alpha}{2} \|\nabla \varphi\|_2^2 + \frac{\beta}{2} \|\varphi - d\|_2^2,$$

where V is an appropriate Sobolev space and $\varepsilon(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)$ is the linearized strain tensor. Note that the prefactor γ_1 from [12] is entirely numerically motivated and thus, not considered here. The function g cf. [12, (2)] is supposed to be smooth and it measures the influence of the damage on the elastic behaviour of the body, see Assumption 0.6 below. $\mathbb{C}$ is the elasticity tensor, which is assumed to be coercive and bounded, see Assumption 0.7 below. The parameters $\alpha, \beta > 0$ are weighting parameters for the gradient regularization and for the penalization, respectively, see [12, Section 2] for more details.

The model in [12] can be formulated as follows. At (almost) all time points, the displacement and the nonlocal damage are supposed to minimize the stored energy, i.e.,

$$(\boldsymbol{u}(t), \varphi(t)) \in \underset{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)}{\arg\min} \; \hat{\mathcal{E}}(t, \boldsymbol{u}, \varphi, d(t)) \quad \text{f.a.a. } t \in (0, T). \tag{1}$$

Further, the evolution of the local damage is modeled by the differential inclusion

$$-\partial_d \hat{\mathcal{E}}(t, \boldsymbol{u}(t), \varphi(t), d(t)) \in \partial \mathcal{R}_1(\dot{d}(t)) \quad \text{f.a.a. } t \in (0, T), \tag{2}$$

where the function $\mathcal{R}_1$ denotes the dissipated energy. This is given by

Definition 0.1 (Dissipation functional)**.** *The dissipation $\mathcal{R}_1 : L^2(\Omega) \to [0, \infty]$ is defined as*

$$\mathcal{R}_1(\eta) := \begin{cases} r \int_\Omega \eta \, dx & \text{if } \eta \geq 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise,} \end{cases}$$

where $r > 0$ stands for the fracture toughness of the material.

Due to the positive homogeneity of $\mathcal{R}_1$, the considered model is *rate-independent*, which means that the values of the damage do not depend on the rate with which ℓ changes in time. As a consequence, one ignores inertial effects and assumes a slow external process. Definition 0.1 also tells us that the damage can only increase in time, in view of (2). Thus, we deal with an irreversible damage process, i.e., healing is not allowed.

We observe that in (1) one solves (at almost all time points) a convex optimization problem. Thus, the global minimum is (at almost all time points) characterized via the necessary and sufficient conditions. An easy computation shows that these are in fact given by [12, (6) and (7)]. On the other hand, (2) is equivalent to [12, (18) combined with (19) and (20)]. To see this, we refer to Subsection 1.1.2, where a similar result is proven by employing the positive homogeneity of $\mathcal{R}_1$. Altogether, we remark that the system (1)-(2) describes indeed the damage model introduced in [12].

The mathematical model

Because of theoretical reasons, we modify the energy functional $\hat{\mathcal{E}}$ such that the function g depends on the nonlocal damage φ instead of the local damage d. This modification is motivated by the fact that the local damage possesses less regularity, so that working further with the functional $\hat{\mathcal{E}}$ would lead to undesirable solvability results. Instead, we consider the energy functional

Definition 0.2 (Energy functional)**.** *The stored energy $\mathcal{E} : [0,T] \times V \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is given by*

$$\mathcal{E}(t, \boldsymbol{u}, \varphi, d) := \frac{1}{2} \int_\Omega g(\varphi) \mathbb{C} \varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) \, dx - \langle \ell(t), \boldsymbol{u} \rangle_V + \frac{\alpha}{2} \|\nabla \varphi\|_2^2 + \frac{\beta}{2} \|\varphi - d\|_2^2.$$

Remark 0.3. *As the penalty approach aims to minimize the deviation between φ and d, we expect the two models to yield similar results, at least for large values of β. This is confirmed by the limit analysis for $\beta \to \infty$ in Chapter 2, which shows that both damage variables equal in the limit.*

We will also work with a different dissipation functional, namely a viscous regularization of the functional from Definition 0.1. Although weak solvability results for rate-independent damage processes with nonconvex energy functional as in our case may be proven, one can neither expect the solutions to be unique nor smooth in time, see [52] for more details. To overcome this issue, we add an L^2-viscosity term in the dissipation functional, which leads to a rate-dependent process, since the dissipation loses its positive homogeneity.

Definition 0.4 (Viscous dissipation functional)**.** *We define $\mathcal{R}_\delta : L^2(\Omega) \to [0, \infty]$ as*

$$\mathcal{R}_\delta(\eta) := \begin{cases} r \int_\Omega \eta \, dx + \frac{\delta}{2} \|\eta\|_2^2 & \text{if } \eta \geq 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise,} \end{cases}$$

where $\delta > 0$ denotes the viscosity parameter.

Note that replacing $\mathcal{R}_1$ by $\mathcal{R}_\delta$ corresponds to smoothing the indicator functional in the dual formulation of the damage evolution by means of the Moreau-Yosida regularization, see Remark 1.31 below for more details.

To summarize, the viscous damage model with penalty arising from the above considerations reads:

$$\left.\begin{aligned} (\boldsymbol{u}(t), \varphi(t)) &\in \operatorname*{arg\,min}_{(\boldsymbol{u},\varphi) \in V \times H^1(\Omega)} \mathcal{E}(t, \boldsymbol{u}, \varphi, d(t)), \\ 0 &\in \partial \mathcal{R}_\delta(\dot{d}(t)) + \partial_d \mathcal{E}(t, \boldsymbol{u}(t), \varphi(t), d(t)) \quad \text{f.a.a. } t \in (0,T), \end{aligned}\right\} \tag{P}$$

with the initial condition $d(0) = d_0$ a.e. in Ω.

Notation

In all what follows, $T > 0$ is fixed and $\Omega \subset \mathbb{R}^N$ is a bounded domain, where $N \in \{2, 3\}$ denotes the spatial dimension. By bold-face case letters we denote vector-valued variables and vector-valued spaces. The Frobenius norm on $\mathbb{R}^{N \times N}$, as well as the euclidean norm on $\mathbb{R}^N$ are denoted by $|\cdot|$, whereas the inducing scalar product in $\mathbb{R}^{N \times N}$ is represented by $(\cdot : \cdot)$. Let X and Y be Banach spaces. The open ball in X around $x \in X$ with radius $R > 0$ is denoted by $B_X(x, R)$. The space of linear and bounded operators from X to Y is called $\mathcal{L}(X, Y)$ and if $X = Y$, it is called $\mathcal{L}(X)$. The dual of the space

X will be denoted by X^* and for the dual pairing between X and X^* we write $\langle .,.\rangle_X$. If X is compactly embedded in Y, we write $X \hookrightarrow\hookrightarrow Y$ and $X \overset{d}{\hookrightarrow} Y$ means that X is dense in Y. Let $s \in [1,\infty]$ and s' be the conjugate exponent of s, i.e., $1/s + 1/s' = 1$. By $\|\cdot\|_s$ we abbreviate the notation for the $L^s(\Omega)$-norm and by $(\cdot,\cdot)_2$, the $L^2(\Omega)$-scalar product. For frequently used function spaces we introduce the following abbreviations:

$$
\begin{aligned}
\boldsymbol{W}^{1,s}_{\widetilde{\Gamma}}(\Omega) &:= \{\boldsymbol{v} \in \boldsymbol{W}^{1,s}(\Omega) : \boldsymbol{v}|_{\widetilde{\Gamma}} = \boldsymbol{0}\},\\
\boldsymbol{W}^{1,s}_{D}(\Omega) &:= \boldsymbol{W}^{1,s}_{\Gamma_D}(\Omega),\\
\boldsymbol{W}^{1,s}_{0}(\Omega) &:= \boldsymbol{W}^{1,s}_{\Gamma}(\Omega),\\
V &:= \boldsymbol{W}^{1,2}_{D}(\Omega),\\
\boldsymbol{W}^{-1,s}(\Omega) &:= \boldsymbol{W}^{1,s'}_{D}(\Omega)^*,\\
W^{1,s}_0(0,T;X) &:= \{z \in W^{1,s}(0,T;X) : z(0) = 0\},\\
W^{1,s}_T(0,T;X) &:= \{z \in W^{1,s}(0,T;X) : z(T) = 0\},\\
H^1_0(0,T;X) &:= W^{1,2}_0(0,T;X),
\end{aligned}
$$

where $\widetilde{\Gamma}$, Γ_D and Γ stand for a part of the boundary, the Dirichlet boundary and the entire boundary of Ω, respectively. For the precise assumption on Γ_D see Assumption 0.5 below. By $\operatorname{div} : L^s(\Omega;\mathbb{R}^{N\times N}_{sym}) \to \boldsymbol{W}^{1,s'}(\Omega)^*$ we denote the distributional vector-valued divergence, i.e.,

$$\langle \operatorname{div} \boldsymbol{\sigma}, \boldsymbol{v}\rangle := -\int_\Omega \boldsymbol{\sigma} : \varepsilon(\boldsymbol{v})\, dx \quad \forall\, \boldsymbol{v} \in \boldsymbol{W}^{1,s'}(\Omega)$$

and $\Delta : W^{1,s}(\Omega) \to W^{1,s'}(\Omega)^*$ is the distributional Laplace operator, i.e.,

$$\langle \Delta\phi, \psi\rangle := -\int_\Omega \nabla\phi\nabla\psi\, dx \quad \forall\, \psi \in W^{1,s'}(\Omega).$$

Throughout the thesis, c and C denote generic positive constants, which depend only on the given data, unless otherwise specified. Time derivatives are frequently denoted by a dot. $\mathbb{R}^+$ and $\mathbb{N}^+$ represent the sets of positive real and natural numbers, respectively. By χ_M we denote the characteristic function associated to the set M. For $a, b \in \mathbb{R}$ we write $a \perp b$ if $ab = 0$. For $f : [0,T] \times \Omega \to \mathbb{R} \cup \{\infty\}$ and $a \in \mathbb{R}$ the abbreviation $f \geq a$ means that $f(t,x) \geq a$ f.a.a. $(t,x) \in (0,T) \times \Omega$. Given $A : X \to Y$, the symbol $A^{-1}y$ stands for the inverse at a fixed point $y \in Y$ and does not necessarily mean that $A : X \to Y$ is invertible. For a mapping $A : X \to L^1(0,T;Y)$ and $U \subset Y$ we say that $A(M)(t) \in U$ if and only if $A(x)(t) \in U$ for all $x \in M \subset X$, where $t \in [0,T]$. Finally, we point out that the notation is not strictly consistent, but it is always clear from the context. To improve the readability, we sometimes neglect the subscripts when working with dual pairings, norms, scalar products and variables as long as it is clear from the context which one is meant. Nevertheless, to emphasize the dependency on the penalty term β and on the load ℓ, respectively, the problem (P) is renamed (P_β) in Chapter 2 and (P_ℓ) in Chapter 3.

General assumptions

Assumption 0.5. *The domain* $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, *is a bounded Lipschitz domain, see [22, Chap. 1.2]. Its boundary is denoted by* Γ *and consists of two disjoint measurable parts* Γ_N *and* Γ_D *such that* $\Gamma = \Gamma_N \cup \Gamma_D$. *While* Γ_N *is an open subset,* Γ_D *is a closed subset of* Γ. *Moreover,* Γ_D *is assumed to have positive measure.*

In addition, the set $\Omega \cup \Gamma_N$ *is regular in the sense of Gröger, cf. [23, Definition 2]. That is, for every point* $\boldsymbol{x} \in \Gamma$, *there exists an open neighborhood* $\mathcal{U}_{\boldsymbol{x}} \subset \mathbb{R}^N$ *of* $\boldsymbol{x}$ *and a bi-Lipschitz map (a Lipschitz continuous and bijective map with Lipschitz continuous inverse)* $\Psi_{\boldsymbol{x}} : \mathcal{U}_{\boldsymbol{x}} \to \mathbb{R}^N$ *such that* $\Psi_{\boldsymbol{x}}(\boldsymbol{x}) = \mathbf{0} \in \mathbb{R}^N$ *and* $\Psi_{\boldsymbol{x}}\big(\mathcal{U}_{\boldsymbol{x}} \cap (\Omega \cup \Gamma_N)\big)$ *equals one of the following sets:*

$$
\begin{aligned}
E_1 &:= \left\{\boldsymbol{y} \in \mathbb{R}^N : |y| < 1,\ y_N < 0\right\}, \\
E_2 &:= \left\{\boldsymbol{y} \in \mathbb{R}^N : |y| < 1,\ y_N \leq 0\right\}, \\
E_3 &:= \left\{\boldsymbol{y} \in E_2 : y_N < 0 \text{ or } y_1 > 0\right\}.
\end{aligned}
$$

A detailed characterization of Gröger regular sets in two and three spatial dimensions is given in [24, Section 5].

Assumption 0.6. *The function* $g : \mathbb{R} \to [\epsilon, 1]$, *where* $\epsilon \in (0,1]$, *satisfies*

$$g \in C^{1,1}(\mathbb{R}). \tag{3}$$

With a little abuse of notation, the Nemytskii operators associated to g *and* g', *considered with different domains and ranges, will be denoted by the same symbol.*

The coefficient function g measures how the elastic properties of the body are preserved depending on the value of the damage. Since with increasing damage, the material becomes weaker, it would make sense to impose that g is monotonically decreasing. This property of g is needed e.g. if one aims to show that the nonlocal damage variable admits just non-negative values, as the local damage variable does. For this, it suffices in fact that g decreases only on its negative domain, cf. Remark 2.40 on page 92 below. However, since we do not need this result in our analysis, we do not require here that g has this property.

Letting the smoothness assumption aside, we shortly compare the assumptions we made on g with those in [12, (2)]. Firstly, we observe that in [12] it holds $g(0) = 1$, i.e., the elastic properties of the body are not affected at all (pure elastic behaviour) when no damage is present. This precise condition is however not needed for the upcoming analysis and from a mathematical point of view, it would have actually sufficed to impose that $g(\cdot) \leq \epsilon_1$ (instead of $g(\cdot) \leq 1$) with some arbitrary (but fixed) $\epsilon_1 \geq \epsilon$. Secondly, $\lim_{\varphi\to\infty} g(\varphi) = 0$ is assumed in [12, (2)], which means that complete material rigidity loss occures in the case of complete damage. By contrast, we impose the condition $g(\cdot) \geq \epsilon > 0$. This is essential for the solvability of (P) or to be more precise, for the coercivity of the bilinear form associated with the balance of momentum equation (1.11), see also proof of Lemma 1.3 below. Thus, as in the most mathematical literature, we investigate a *partial damage model.* For methods for dealing with complete desintegration we refer here to [27, Chapter 6], see also [68] for rate-dependent complete damage systems in thermoviscoelastic materials.

Assumption 0.7. *The fourth-order tensor* $\mathbb{C} \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{N\times N}_{\text{sym}}))$ *is symmetric and uniformly coercive, i.e., there is a constant* $\gamma_\mathbb{C} > 0$ *such that*

$$\mathbb{C}(x)\boldsymbol{\sigma} : \boldsymbol{\sigma} \geq \gamma_\mathbb{C} |\boldsymbol{\sigma}|^2 \quad \forall \boldsymbol{\sigma} \in \mathbb{R}^{N\times N}_{\text{sym}} \text{ and f.a.a. } x \in \Omega. \tag{4}$$

Assumption 0.8. *The load* ℓ *is a mapping from* $[0,T]$ *to* V^*. *The initial damage is supposed to satisfy* $d_0 \in L^2(\Omega)$.

For convenience of the reader, the often used notations are collected in the following table:

Variables, functionals, (solution) operators, integrability exponents and sets

Symbol	Meaning	Definition
$\mathcal{R}_1$	Dissipation functional	Definition 0.1, p. 3
$\mathcal{E}$	Stored energy functional	Definition 0.2, p. 4
$\mathcal{R}_\delta$	Viscous dissipation functional	Definition 0.4, p. 4
A_φ	Linear elliptic operator in (1.25a)	Definition 1.2, p. 12
$\boldsymbol{W}^{1,p}_D(\Omega)$	Space regularity of the optimal displacement	Lemma 1.3, p. 12
$\mathcal{U}$	Solution operator of (1.25a)	Definition 1.8, p. 15
B	Linear part in (1.25b)	Definition 1.15, p. 18
F	Nonlinear part in (1.25b)	Definition 1.15, p. 18
Φ	Solution operator of (1.25b)	Definition 1.24, p. 24
$W^{1,q}(\Omega)$	Space regularity of the optimal nonlocal damage	Theorem 1.37, p. 31
$\mathcal{I}$	Reduced energy functional	Definition 2.1, p. 55
$\widetilde{\mathcal{E}}$	Energy functional without penalty	(2.47), p. 70
$\widetilde{\mathcal{I}}$	Reduced energy functional without penalty	Definition 2.19, p. 70
$\widetilde{\mathcal{R}}_1$	Dissipation functional after passing to the limit $\beta \to \infty$	(2.76), p. 78
$\widetilde{\mathcal{R}}_\delta$	Viscous dissipation functional after passing to the limit $\beta \to \infty$	Definition 2.21, p. 71
$\mathcal{B}_M$	Set of admissible loads (time-independent case)	Definition 3.3, p. 103
$\mathfrak{B}_M$	Set of admissible loads (time-dependent case)	Definition 3.21, p. 118
$\mathcal{S}$	Control-to-state operator	Definition 3.22, p. 119
$\mathfrak{L}$	Control set	Assumption 3.29, p. 128
$\mathcal{J}$	Objective functional	Assumption 3.33, p. 129
$\boldsymbol{u}$	Displacement	
φ	Nonlocal damage	
d	Local damage	

Chapter 1

Analysis of the damage model with penalty

In this chapter we show that the problem (P) admits a unique solution, and thus, that the modified damage model with penalty is well posed from a mathematical point of view. Moreover, we investigate the regularity of the unique solution on different levels. We also derive an equivalent formulation of (P), which consists of an elliptic system for the displacement field and the nonlocal damage and an operator differential equation for the local damage. The problem under consideration is analyzed throughout this chapter in the context of a smooth given load and a fixed penalization parameter. The herein achieved results constitute the basis for the rest of the thesis. In Chapter 2 we employ in particular a final existence result for (P), see Theorem 1.62 below, as well as various intermediate results. Concerning Chapter 3, we mention Sections 3.1 and 3.2, which rely on arguments similar to the proofs given in the present chapter.

Let us put our work into perspective. Numerous damage models have been addressed by many authors under different aspects. In [4, 5, 6, 16, 26] various viscous damage models are analyzed regarding existence of solutions. The concept of viscosity plays an important role in the mathematical treatment of rate-independent damage models, as the vanishing viscosity approach is a prominent method to find solutions for rate-independent problems, such as balanced viscosity (BV) solutions, see [55, 56], and parametrized solutions, see [13, 41, 42, 61]. These are obtained by considering a viscous dissipation in the initial model, which becomes rate-dependent, and by then letting the viscosity parameter tend to zero. We further refer here to [1, 39, 43, 45, 47, 48, 54, 78] and the references therein. There are also other solution concepts for evolutionary equations in the context of rate-independent damage modelling. An overview thereof can be found in [52], in the framework of generalized gradient systems. A particular attention is given in the literature to discontinuous solutions, such as energetic solutions and BV solutions, which were introduced in [58, 60] and [54, 55], respectively.

Unlike in the above contributions, we deal with a viscous damage model containing two damage variables. To the best of our knowledge, such a model has not been investigated so far with regard to a rigorous mathematical analysis, although these models are frequently used for numerical simulations, cf. e.g. [46, 64, 67, 76, 83]. We approach the problem (P) in this chapter mostly by classical arguments, such as elements of calculus of variations, a contraction argument, boot strapping and the implicit function theorem. An essential tool in the context of proving the unique solvability of (P) is the $W^{1,p}$-theory with $p > 2$ from [29].

Outline of the chapter

The chapter is organized as follows. In Section 1.1 we first address the existence and uniqueness of solutions for the minimization problem in (P) and show that it is equivalent to an elliptic system. Based on these results, we then deal with the complete model including the evolutionary equation for the local damage. This turns out to be equivalent to an operator differential equation and the unique solvability of (P) follows by standard arguments. Sections 1.2 and 1.3 are devoted to improve the regularity of the solution. In Section 1.2 we show a higher spatial regularity result for the nonlocal damage, as well as a corresponding Lipschitz continuity condition. In Section 1.3 we prove that the solution operators of the elliptic system are continuously Fréchet-differentiable. This finally allows us to establish that the overall solution of (P) is continuously differentiable in appropriate spaces, as a mapping in time.

1.1 Existence and uniqueness of solutions

In this section we mainly focus on finding unique solutions $\boldsymbol{u}, \varphi, d$ to the problem (P) for a given load ℓ. For this purpose, we first show that the optimization problem in (P) admits solutions for fixed t and d. However, the existence cannot be demonstrated by the direct method of the calculus of variations, since in the first place the displacement $\boldsymbol{u}$ and the nonlocal damage φ do not provide sufficient regularity, see also Definition 0.2 and Lemma 5.2. Therefore we proceed as follows. Starting from

$$\min_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{E}(t,\boldsymbol{u},\varphi,d) = \min_{\varphi\in H^1(\Omega)} \min_{\boldsymbol{u}\in V} \mathcal{E}(t,\boldsymbol{u},\varphi,d), \tag{1.1}$$

we first show that, for every $\varphi \in H^1(\Omega)$, the problem $\min_{\boldsymbol{u}\in V} \mathcal{E}(t,\boldsymbol{u},\varphi,d)$ admits a unique solution, denoted by $\mathcal{U}(t,\varphi)$, which possesses improved regularity. In the second part of Subsection 1.1.1 this allows us to show existence of solutions for the outer optimization problem on the right-hand side in (1.1), see also Remark 1.13 below. Such solutions will turn out to satisfy (together with the corresponding optimal displacement) the elliptic system (1.25) below, as necessary optimality condition. As this system is uniquely solvable under suitable assumptions, we obtain the unique solvability of the optimization problem in (P), with solutions characterized by (1.25). After concluding the uniqueness, the Lipschitz continuity of the resulting solution maps is proven. Finally, by means of the latter ones, the problem (P) can be reduced to an evolutionary equation, which is in fact an ordinary differential equation in Banach space. This aspect is addressed in Subsection 1.1.2, where one establishes the unique solvability of the reduced problem (P). The section ends with a first existence and uniqueness result for (P), which will be improved in the upcoming sections.

1.1.1 The elliptic system

Throughout this section we work with a fixed $(t, d) \in [0, T] \times L^2(\Omega)$ and deal with the optimization problem

$$\min_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{J}(\boldsymbol{u}, \varphi), \tag{1.2}$$

where $\mathcal{J} : V \times H^1(\Omega) \to \mathbb{R}$ is defined as

$$\mathcal{J}(\boldsymbol{u}, \varphi) := \mathcal{E}(t, \boldsymbol{u}, \varphi, d), \tag{1.3}$$

This means in view of Definition 0.2 that for all $(\boldsymbol{u}, \varphi) \in V \times H^1(\Omega)$ it holds

$$\mathcal{J}(\boldsymbol{u}, \varphi) = \frac{1}{2}\int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\; dx - \langle \ell(t), \boldsymbol{u}\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2 + \frac{\beta}{2}\|\varphi - d\|_2^2.$$

Displacement

As indicated above, we first fix $\varphi \in H^1(\Omega)$ and investigate the problem

$$\min_{\boldsymbol{u}\in V} \mathcal{J}(\boldsymbol{u}, \varphi). \tag{1.4}$$

The unique solvability of (1.4) can be followed by very standard arguments, while proving the improved regularity of the optimal displacement requires some preparatory work. The key tool therefor is the next result, which is based on Assumption 0.5:

Lemma 1.1. *[29, Theorem 1.1, Proposition 1.2] Let $\{b_\iota\}$ be a family of nonlinearities, so that each $b_\iota\colon \Omega \times \mathbb{R}^{N\times N}_{\mathrm{sym}} \to \mathbb{R}^{N\times N}_{\mathrm{sym}}$ satisfies*

$$b_\iota(\cdot, \mathbf{0}) \in L^\infty(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}}), \tag{1.5a}$$

$$b_\iota(\cdot, \boldsymbol{\sigma}) \text{ is measurable}, \tag{1.5b}$$

$$(b_\iota(x, \boldsymbol{\sigma}) - b_\iota(x, \boldsymbol{\varsigma})) : (\boldsymbol{\sigma} - \boldsymbol{\varsigma}) \geq \underline{m}|\boldsymbol{\sigma} - \boldsymbol{\varsigma}|^2, \tag{1.5c}$$

$$|b_\iota(x, \boldsymbol{\sigma}) - b_\iota(x, \boldsymbol{\varsigma})| \leq \overline{m}|\boldsymbol{\sigma} - \boldsymbol{\varsigma}| \tag{1.5d}$$

f.a.a. $x \in \Omega$ and all $\boldsymbol{\sigma}, \boldsymbol{\varsigma} \in \mathbb{R}^{N\times N}_{\mathrm{sym}}$ with constants $\underline{m}, \overline{m} > 0$ independent of ι. Furthermore, let $B_{p,\iota}\colon \boldsymbol{W}^{1,p}_D(\Omega) \to \boldsymbol{W}^{-1,p}(\Omega)$ be defined as

$$\langle B_{p,\iota}(\boldsymbol{u}), \boldsymbol{v}\rangle := \int_\Omega b_\iota(\cdot, \varepsilon(\boldsymbol{u})) : \varepsilon(\boldsymbol{v})\; dx \quad \forall\, \boldsymbol{v} \in \boldsymbol{W}^{1,p'}_D(\Omega).$$

Then there exists $p > 2$ such that for all $\bar{p} \in [2, p]$, all operators $B_{\bar{p},\iota}$ are continuously invertible. Moreover, their inverses are globally Lipschitz continuous and share a uniform Lipschitz constant independent of $\bar{p} \in [2, p)$ and ι.

The above result will be applied in the upcoming lemma for the following operator(s):

Definition 1.2. *Given $\varphi \in L^1(\Omega)$, we define the linearity $A_\varphi : V \to V^*$ as*

$$\langle A_\varphi \boldsymbol{u}, \boldsymbol{v} \rangle_V := \int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}) \, dx \quad \forall \, \boldsymbol{v} \in V.$$

The operator A_φ considered with different domains and ranges will be denoted by the same symbol for the sake of convenience. Note that A_φ is well defined in view of Hölder's inequality combined with Assumption 0.6 and $\mathbb{C} \in L^\infty(\Omega; \mathcal{L}(\mathbb{R}^{N\times N}_{\text{sym}}))$, cf. Assumption 0.7.

Lemma 1.3. *There exists $p > 2$ such that, for all $\bar{p} \in [2, p]$ and all $\varphi \in L^1(\Omega)$, the operator $A_\varphi : \boldsymbol{W}^{1,\bar{p}}_D(\Omega) \to \boldsymbol{W}^{-1,\bar{p}}(\Omega)$ is continuously invertible. Moreover, there exists a constant $c > 0$, independent of φ and $\bar{p} \in [2, p)$, such that*

$$\|A_\varphi^{-1} h\|_{\boldsymbol{W}^{1,\bar{p}}_D(\Omega)} \leq c \, \|h\|_{\boldsymbol{W}^{-1,\bar{p}}(\Omega)} \quad \forall \, h \in \boldsymbol{W}^{-1,\bar{p}}(\Omega) \tag{1.6}$$

holds for all $\varphi \in L^1(\Omega)$ and $\bar{p} \in [2, p]$.

Proof. The result follows by applying Lemma 1.1. To this end, we define the family of functions $\{b_\varphi\}_{\varphi \in L^1(\Omega)}$, $b_\varphi : \Omega \times \mathbb{R}^{N\times N}_{\text{sym}} \to \mathbb{R}^{N\times N}_{\text{sym}}$, by

$$b_\varphi(x, \boldsymbol{\sigma}) := g(\varphi(x))\mathbb{C}(x)\boldsymbol{\sigma} \tag{1.7}$$

and verify (1.5) therefor. The condition (1.5a) is obviously fulfilled as $\mathbb{C}(x) \in \mathcal{L}(\mathbb{R}^{N\times N}_{\text{sym}})$ f.a.a. $x \in \Omega$, and thus, $\mathbb{C}(\cdot)\boldsymbol{0} = \boldsymbol{0}$, while (1.5b) follows by Assumptions 0.6 and 0.7. From the latter ones we also deduce the uniform coercivity, i.e., (1.5c) and boundedness, i.e., (1.5d), with $\underline{m} = \epsilon \, \gamma_{\mathbb{C}} > 0$ and $\overline{m} = \|\mathbb{C}\|_\infty$, respectively. Since these constants are independent of φ, we can now apply Lemma 1.1, which yields the existence of a $p > 2$ such that, for all $\bar{p} \in [2, p]$ and all $\varphi \in L^1(\Omega)$, the operator $A_\varphi : \boldsymbol{W}^{1,\bar{p}}_D(\Omega) \to \boldsymbol{W}^{-1,\bar{p}}(\Omega)$ is continuously invertible, as a result of (1.7). Note that since $A_\varphi \in \mathcal{L}(\boldsymbol{W}^{1,\bar{p}}_D(\Omega), \boldsymbol{W}^{-1,\bar{p}}(\Omega))$, the global Lipschitz continuity of its inverse holds true anyway. However, Lemma 1.1 gives us the additional information that the norm of $A_\varphi^{-1} \in \mathcal{L}(\boldsymbol{W}^{-1,\bar{p}}(\Omega), \boldsymbol{W}^{1,\bar{p}}_D(\Omega))$ can be estimated independently of φ and $\bar{p} \in [2, p)$. This finalizes the proof. □

Remark 1.4. *In view of Definition 1.2, the operator $A_\varphi \in \mathcal{L}(\boldsymbol{W}^{1,\zeta}_D(\Omega), \boldsymbol{W}^{-1,\zeta}(\Omega))$ is the adjoint of $A_\varphi \in \mathcal{L}(\boldsymbol{W}^{1,\zeta'}_D(\Omega), \boldsymbol{W}^{-1,\zeta'}(\Omega))$ for any $\zeta \in (1, \infty)$ and any $\varphi \in L^1(\Omega)$. This follows by the definition of the adjoint, where one employs the reflexivity of $\boldsymbol{W}^{1,\zeta}_D(\Omega)$. Therefore, in view of Lemma 1.3, $A_\varphi : \boldsymbol{W}^{1,\zeta}_D(\Omega) \to \boldsymbol{W}^{-1,\zeta}(\Omega)$ is continuously invertible for $p' \leq \zeta \leq 2$ as well, and*

$$\|A_\varphi^{-1}\|_{\mathcal{L}(\boldsymbol{W}^{-1,\zeta}(\Omega), \boldsymbol{W}^{1,\zeta}_D(\Omega))} = \|A_\varphi^{-1}\|_{\mathcal{L}(\boldsymbol{W}^{-1,\zeta'}(\Omega), \boldsymbol{W}^{1,\zeta'}_D(\Omega))} \leq c,$$

where $c > 0$ is independent of φ and $\zeta' \in [2, p)$, and thus of $\zeta \in (p', 2]$.

In the rest of the thesis, $p > 2$ stands for the exponent given by Lemma 1.3.

We can now state the assumption for the load, which is supposed to hold throughout this entire chapter and the next one. This reads as follows:

Assumption 1.5. *For the applied volume and boundary load we require*

$$\ell \in C^{0,1}([0,T]; \boldsymbol{W}^{-1,p}(\Omega)),$$

see also Assumptions 1.14 and 1.17.1 below.

We point out that the choice of the space for the load is not random at all, as explained in Remark 1.26 below.

The next lemma covers some differentiability properties of the energy functional $\mathcal{E}$. These will be in particular needed when deriving optimality conditions for the optimal displacement and optimal nonlocal damage.

Lemma 1.6. *Then the functional $\mathcal{E}$ is partially Fréchet-differentiable w.r.t. $\boldsymbol{u}$ and d on $[0,T] \times V \times H^1(\Omega) \times L^2(\Omega)$, and the partial derivatives are given by*

$$\partial_{\boldsymbol{u}}\mathcal{E}(t,\boldsymbol{u},\varphi,d) = A_\varphi \boldsymbol{u} - \ell(t) \quad \text{in } V^*, \tag{1.8}$$

$$\partial_d \mathcal{E}(t,\boldsymbol{u},\varphi,d) = \beta(d-\varphi) \quad \text{in } L^2(\Omega). \tag{1.9}$$

Furthermore, if considered as a mapping on $[0,T] \times \boldsymbol{W}^{1,r}_D(\Omega) \times H^1(\Omega) \times L^2(\Omega)$ with $r > 2$ for $N = 2$ and $r > 12/5$ for $N = 3$, then $\mathcal{E}$ is also partially Fréchet-differentiable w.r.t. φ. The partial derivative reads

$$\begin{aligned}\partial_\varphi \mathcal{E}(t,\boldsymbol{u},\varphi,d)(\delta\varphi) = &\frac{1}{2}\int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\delta\varphi \, dx \\ &+ \int_\Omega \alpha\nabla\varphi\cdot\nabla\delta\varphi + \beta(\varphi - d)\delta\varphi \, dx \quad \forall\, \delta\varphi \in H^1(\Omega).\end{aligned} \tag{1.10}$$

Proof. Let $(t,\varphi,d) \in [0,T] \times H^1(\Omega) \times L^2(\Omega)$ be arbitrary, but fixed. To prove the partial Fréchet-differentiability w.r.t. $\boldsymbol{u}$ we first observe that the mapping

$$\boldsymbol{u} \mapsto \frac{1}{2}\int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) \, dx$$

is Fréchet-differentiable on V by means of product rule. Keep in mind that g maps $H^1(\Omega)$ to $L^\infty(\Omega)$, cf. Lemma 5.1, and $\varepsilon \in \mathcal{L}(V; L^2(\Omega;\mathbb{R}^{N\times N}))$. In view of Definition 1.2 and since $\ell(t) \in V^*$, cf. Assumption 0.8, we get (1.8).

The partial Fréchet-differentiability w.r.t. d follows immediately from the Fréchet-differentiability of $\|\cdot\|_2^2$ on $L^2(\Omega)$.

Let now $(t,\boldsymbol{u},d) \in [0,T] \times \boldsymbol{W}^{1,r}_D(\Omega) \times L^2(\Omega)$ be arbitrary, but fixed. On account of Hölder's inequality with $(r-2)/r + 2/r = 1$ we deduce that the linear functional

$$L^{\frac{r}{r-2}}(\Omega) \ni w \mapsto \frac{1}{2}\int_\Omega w\, \mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) \, dx \in \mathbb{R}$$

is bounded and thus an element of $L^{r/(r-2)}(\Omega)^*$. Moreover, the conditions on r and Sobolev embeddings imply $H^1(\Omega) \hookrightarrow L^s(\Omega)$ with some $s > r/(r-2)$ so that, in view of Lemma 5.3, g is Fréchet-differentiable from $H^1(\Omega)$ to $L^{r/(r-2)}(\Omega)$. The identity (1.10) then follows from chain rule, in combination with the Fréchet-differentiability of $\|\nabla\cdot\|_2^2$ and $\|\cdot\|_2^2$ on $H^1(\Omega)$. □

We are now finally in the position to state the unique solvability of (1.4), as well as the (improved) regularity of the optimal solution:

Proposition 1.7 (Existence and uniqueness of the optimal displacement)**.** *Let Assumption 1.5 hold. Then, for every $\varphi \in H^1(\Omega)$, the optimization problem* (1.4) *is convex and admits a unique solution $\bar{\boldsymbol{u}} \in \boldsymbol{W}_D^{1,p}(\Omega)$, which is characterized by*

$$\langle A_\varphi \bar{\boldsymbol{u}}, \boldsymbol{v} \rangle_{\boldsymbol{W}_D^{1,p'}(\Omega)} = \langle \ell(t), \boldsymbol{v} \rangle_{\boldsymbol{W}_D^{1,p'}(\Omega)} \quad \forall\, \boldsymbol{v} \in \boldsymbol{W}_D^{1,p'}(\Omega). \tag{1.11}$$

Proof. Let $\varphi \in H^1(\Omega)$ be arbitrary, but fixed and define the functional

$$f_\varphi : V \ni \boldsymbol{u} \mapsto \mathcal{J}(\boldsymbol{u}, \varphi) \in \mathbb{R},$$

which is just the objective in (1.4). First we establish that f_φ is strictly convex, by checking if

$$f_\varphi(\boldsymbol{u}_1) - f_\varphi(\boldsymbol{u}_2) > f'_\varphi(\boldsymbol{u}_2)(\boldsymbol{u}_1 - \boldsymbol{u}_2) \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in V,\, \boldsymbol{u}_1 \neq \boldsymbol{u}_2, \tag{1.12}$$

holds true. In view of (1.3) and (1.8), (1.12) is equivalent to

$$\int_\Omega g(\varphi)\mathbb{C}[\varepsilon(\boldsymbol{u}_1) - \varepsilon(\boldsymbol{u}_2)] : [\varepsilon(\boldsymbol{u}_1) - \varepsilon(\boldsymbol{u}_2)]\, dx > 0, \quad \forall\, \boldsymbol{u}_1, \boldsymbol{u}_2 \in V,\, \boldsymbol{u}_1 \neq \boldsymbol{u}_2, \tag{1.13}$$

where we also employed Definitions 0.2 and 1.2. Since (1.13) can be deduced from Assumptions 0.6 and 0.7, we obtain that f_φ is indeed strictly convex. Thus, if existent, the solution $\bar{\boldsymbol{u}}$ of (1.4) is unique and, since we optimize on V, $\bar{\boldsymbol{u}}$ is characterized by

$$A_\varphi \bar{\boldsymbol{u}} = \ell(t) \quad \text{ in } V^*, \tag{1.14}$$

in view of (1.8). Lemma 1.3 now gives the (unique) solvability as well as the improved regularity of $\bar{\boldsymbol{u}}$, which is guaranteed by Assumption 1.5. In light of $V \overset{d}{\hookrightarrow} \boldsymbol{W}_D^{1,p'}(\Omega)$ and (1.14) combined with Definition 1.2, the unique solution $\bar{\boldsymbol{u}}$ is (also) characterized by (1.11). This completes the proof. ☐

The unique solvability of (1.11) leads to the following

Definition 1.8 (Solution operator of (1.11))**.** *Under Assumption 1.5, we define the operator $\mathcal{U} : [0,T] \times H^1(\Omega) \to \boldsymbol{W}_D^{1,p}(\Omega)$ by*

$$\mathcal{U}(t, \varphi) := A_\varphi^{-1} \ell(t).$$

As an immediate consequence of Lemma 1.3 one obtains the following

Corollary 1.9. *If Assumption 1.5 holds true, then there exists a constant $c > 0$, independent of t and φ such that*

$$\|\mathcal{U}(t, \varphi)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \leq c \quad \forall\, (t, \varphi) \in [0,T] \times H^1(\Omega).$$

We also observe that c is independent of β, which will be crucial in Chapter 2.

The constant c depends on the given data, and in particular on $\|\ell\|_{C([0,T];\boldsymbol{W}^{-1,p}(\Omega))}$. In Chapter 3 this will play an essential role, as therein the load is no longer fixed, but a variable. A special attention to this situation is given in Section 3.1, where counterpart results (of the ones in this chapter) are derived. For example, the estimate corresponding to the one in Corollary 1.9 is given by (3.5), see page 102 below. We refer here also to Section 3.2. As in this chapter and in the next one, ℓ remains the whole time fixed, any dependence of the upcoming constants on the load is not of concern.

We now address the continuity properties of the solution operator $\mathcal{U}$. We begin with the Lipschitz continuity.

Proposition 1.10 (Lipschitz continuity of $\mathcal{U}$)**.** *Suppose that Assumption 1.5 is fulfilled and let $r \in \big[2p/(p-2), \infty\big]$ be given. Then there exists $L > 0$ such that for all $\varphi_1, \varphi_2 \in H^1(\Omega) \cap L^r(\Omega)$ and all $t_1, t_2 \in [0,T]$ it holds*

$$\|\mathcal{U}(t_1,\varphi_1) - \mathcal{U}(t_2,\varphi_2)\|_{\boldsymbol{W}^{1,\pi}_D(\Omega)} \leq L\,(\|\varphi_1 - \varphi_2\|_r + |t_1 - t_2|), \tag{1.15}$$

where $1/\pi = 1/p + 1/r \in [1/p, 1/2]$.

Proof. The proof follows the lines of the proof of [41, Lemma 2.5]. For simplicity, we abbreviate in what follows $\boldsymbol{u}_i := \mathcal{U}(t_i, \varphi_i)$, $i = 1,2$. The idea is to write down an equation of the type (1.11) which characterizes $\boldsymbol{u}_1 - \boldsymbol{u}_2$ and apply (1.6) therefor. This is done by subtracting the equations associated with $\boldsymbol{u}_i$, $i = 1,2$, which yields

$$A_{\varphi_1}(\boldsymbol{u}_1 - \boldsymbol{u}_2) = (A_{\varphi_2} - A_{\varphi_1})\boldsymbol{u}_2 + \ell(t_1) - \ell(t_2) \quad \text{in } \boldsymbol{W}^{-1,p}(\Omega). \tag{1.16}$$

Let us first notice that for given $\mu, \rho, \tau \geq 1$ such that $1/\mu = 1/\rho + 1/\tau$, Hölder's inequality and Assumption 0.7 imply

$$\|\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{w})\|_\mu \leq C\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,\rho}_D(\Omega)}\|\boldsymbol{w}\|_{\boldsymbol{W}^{1,\tau}_D(\Omega)} \quad \forall\, \boldsymbol{u} \in \boldsymbol{W}^{1,\rho}_D(\Omega), \forall\, \boldsymbol{w} \in \boldsymbol{W}^{1,\tau}_D(\Omega). \tag{1.17}$$

We now apply Hölder's inequality with $1/r + 1/p + 1/\pi' = 1$ to the first term on the right-hand side in (1.16). This gives together with Lemma 5.1, (1.17), and Corollary 1.9 the following estimate

$$\begin{aligned} \|(A_{\varphi_2} - A_{\varphi_1})\boldsymbol{u}_2\|_{\boldsymbol{W}^{-1,\pi}(\Omega)} &\leq C\,\|g(\varphi_1) - g(\varphi_2)\|_r\,\|\boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}_D(\Omega)} \\ &\leq C\,\|\varphi_1 - \varphi_2\|_r. \end{aligned} \tag{1.18}$$

Since $0 \leq 1/r \leq (p-2)/(2p)$, by assumption, it holds $\pi \in [2,p]$. Thus, we are allowed to apply estimate (1.6) to A_{φ_1} when considered as an operator from $\boldsymbol{W}^{1,\pi}_D(\Omega)$ to $\boldsymbol{W}^{-1,\pi}(\Omega)$. Therewith we deduce by means of (1.16) and (1.18) the estimate

$$\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}^{1,\pi}_D(\Omega)} \leq L\,(\|\varphi_1 - \varphi_2\|_r + |t_1 - t_2|),$$

where we used $\ell \in C^{0,1}([0,T]; \boldsymbol{W}^{-1,\pi}(\Omega))$, as a result of Assumption 1.5. Note that the constant $L > 0$ is independent of (t_i, φ_i), $i = 1,2$. This completes the proof. $\square$

We finish the discussion concerning the optimal displacement with a result which is essential for proving the existence of minimizers for (1.2).

Lemma 1.11 (Continuity of $\mathcal{U}$)**.** *Suppose that Assumption 1.5 holds. Let* $\{(t_n, \varphi_n)\} \subset [0,T]\times H^1(\Omega)$ *and* $(t,\varphi) \in [0,T]\times H^1(\Omega)$ *be given such that* $(t_n,\varphi_n) \to (t,\varphi)$ *in* $\mathbb{R}\times L^1(\Omega)$ *as* $n \to \infty$*. Then* $\mathcal{U}(t_n,\varphi_n) \to \mathcal{U}(t,\varphi)$ *in* $\boldsymbol{W}^{1,s}_D(\Omega)$ *as* $n\to\infty$ *for every* $s \in [2,p)$*.*

Proof. We again abbreviate $\boldsymbol{u} := \mathcal{U}(t,\varphi)$ and $\boldsymbol{u}_n := \mathcal{U}(t_n,\varphi_n)$, where $n \in \mathbb{N}$. By subtracting the equations associated with $\boldsymbol{u}$ and $\boldsymbol{u}_n$ we obtain for all $n \in \mathbb{N}$

$$A_\varphi(\boldsymbol{u} - \boldsymbol{u}_n) = (A_{\varphi_n} - A_\varphi)\boldsymbol{u}_n + \ell(t) - \ell(t_n) \quad \text{in } \boldsymbol{W}^{-1,p}(\Omega). \tag{1.19}$$

Completely analogously to (1.18), one derives for all $n \in \mathbb{N}$ the estimate

$$\|(A_{\varphi_n} - A_\varphi)\boldsymbol{u}_n\|_{\boldsymbol{W}^{-1,s}(\Omega)} \le C\,\|g(\varphi_n) - g(\varphi)\|_\varrho \|\boldsymbol{u}_n\|_{\boldsymbol{W}^{1,p}_D(\Omega)}, \tag{1.20}$$

with $\varrho \in [1,\infty)$ such that $1/\varrho + 1/p + 1/s' = 1$. Notice that the existence of ϱ is due to $1/s' \in [1/2, 1/p')$. Lemma 5.2, Corollary 1.9, (1.20) and Assumption 1.5 now lead to

$$\|(A_{\varphi_n} - A_\varphi)\boldsymbol{u}_n + \ell(t) - \ell(t_n)\|_{\boldsymbol{W}^{-1,s}(\Omega)} \to 0 \text{ as } n \to \infty.$$

In view of (1.19), applying (1.6) to $A_\varphi : \boldsymbol{W}^{1,s}_D(\Omega) \to \boldsymbol{W}^{-1,s}(\Omega)$ then gives the assertion. □

Nonlocal damage

We now turn our attention to the outer optimization problem on the right-hand side of (1.1), i.e., we investigate

$$\min_{\varphi\in H^1(\Omega)} \mathcal{J}(\mathcal{U}(t,\varphi),\varphi). \tag{1.21}$$

For convenience of the reader we recall that $\mathcal{J} : V \times H^1(\Omega) \to \mathbb{R}$ is given by

$$\mathcal{J}(\boldsymbol{u},\varphi) = \frac{1}{2}\int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\,dx - \langle \ell(t), \boldsymbol{u}\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2 + \frac{\beta}{2}\|\varphi - d\|_2^2.$$

Note that $\varphi \mapsto \mathcal{J}(\mathcal{U}(t,\varphi),\varphi)$ is not necessarily convex, and thus, we cannot directly derive a characterization of a (possible existent) solution of (1.21). As we will see, even deriving necessary optimality conditions is without further ado not possible. The next result covers only the solvability of (1.21). As already mentioned at the beginning of this section, the uniqueness can be first deduced after writing down the necessary optimality conditions.

Proposition 1.12 (Existence of the optimal nonlocal damage)**.** *Suppose that Assumption 1.5 holds. Then, the optimization problem* (1.21) *admits at least one solution, and therefore* (1.2) *possesses a solution as well.*

Proof. By means of Definitions 1.2 and 1.8, the objective in (1.21) can be rewritten as

$$f : H^1(\Omega) \to \mathbb{R}, \quad f(\varphi) := \mathcal{J}(\mathcal{U}(t,\varphi),\varphi) = -\frac{1}{2}\langle \ell(t), \mathcal{U}(t,\varphi)\rangle + \frac{\alpha}{2}\|\nabla\varphi\|_2^2 + \frac{\beta}{2}\|\varphi - d\|_2^2.$$

The existence of solutions for (1.21) follows by classical arguments of the direct method of variational calculus. To this end, notice that f is radially unbounded because of Corollary 1.9. Moreover, it is weakly lower semicontinuous. To see this, consider a sequence $\{\varphi_n\} \subset H^1(\Omega)$ with $\varphi_n \rightharpoonup \varphi$ in $H^1(\Omega)$ as $n \to \infty$. The compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$ and Lemma 1.11 then imply

$$\mathcal{U}(t,\varphi_n) \to \mathcal{U}(t,\varphi) \text{ in } V \quad \text{as } n \to \infty.$$

This together with the weak lower semicontinuity of $\|\nabla \cdot\|_2^2$ and $\|\cdot\|_2^2$ on $H^1(\Omega)$ gives that f is indeed weakly lower semicontinuous. Since $H^1(\Omega)$ is a reflexive Banach space, a standard argument yields now that (1.21) admits solutions. In consequence, so does (1.2), as the set of solutions for (1.2) is given by $\{(\mathcal{U}(t,\varphi),\varphi) : \ \varphi \text{ solves } (1.21)\}$, which is straight forward to see. □

Remark 1.13. *We point out that the continuity of the operator* $\mathcal{U}(t,\cdot)$ *in Lemma 1.11 is crucial for proving the existence of solutions for* (1.21)*. One should however notice that, if we assume that the optimal displacement has only* V*-regularity, then the result in Lemma 1.11 holds true for any* $s \in [p', 2)$*, see the proof thereof and Remark 1.4. Since by assumption* $\ell(t) \in \boldsymbol{W}^{-1,p}(\Omega) \hookrightarrow (\boldsymbol{W}_D^{1,s}(\Omega))^*$*, Proposition 1.12 can still be concluded in the exact same way. Note that this alternative proof does not (directly) use the improved regularity of the optimal displacement. However, it uses Lemma 1.3 (in form of Remark 1.4) and Assumption 1.5, which were the tools needed for establishing the improved regularity in Proposition 1.7. Hence, these are the essential ingredients for proving the existence of the optimal nonlocal damage.*

As the upcoming analysis shows, the $\boldsymbol{W}_D^{1,p}(\Omega)$*-regularity of the optimal displacement is also crucial for deriving necessary optimality conditions, and thus, for proving the uniqueness of solutions for* (1.21)*.*

Next we concentrate on deriving necessary optimality conditions for the optimal nonlocal damage. For this purpose one has to differentiate the function $\mathcal{J}$ w.r.t. φ at $(\mathcal{U}(t,\bar{\varphi}),\bar{\varphi})$, where $\bar{\varphi}$ is a solution of (1.21). This can be done in view of Lemma 1.6 and Definition 1.8 only under the following additional

Assumption 1.14. *From now on we assume that, in case of* $N = 3$*, the assertion in Lemma 1.3 holds with* $p > 12/5$*.*

The existence of p in Assumption 1.14 is ensured by further conditions on the data, which are stated in Remark 1.27 below. We emphasize that one can go without this additional assumption, if one replaces the H^1-seminorm in the energy functional in Definition 0.2 by the $H^{3/2}$-seminorm, see Remark 1.28 below for more details.

The following definition will be useful in the sequel:

Definition 1.15 (The linear and nonlinear part of (1.25b)). *Suppose that Assumptions 1.5 and 1.14 are fulfilled. Then we define the mappings* $B : H^1(\Omega) \to H^1(\Omega)^*$ *and* $F : [0,T] \times H^1(\Omega) \to H^1(\Omega)^*$ *by*

$$\langle B\varphi, \psi\rangle_{H^1(\Omega)} := \int_\Omega \alpha\nabla\varphi \cdot \nabla\psi + \beta\varphi\psi \, dx \quad \forall\, \psi \in H^1(\Omega), \tag{1.22}$$

$$\langle F(t,\varphi), \psi\rangle_{H^1(\Omega)} := \frac{1}{2}\int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi))\psi \, dx \quad \forall\, \psi \in H^1(\Omega). \tag{1.23}$$

We emphasize that F *is well defined. To see this, first note that the Sobolev embedding* $H^1(\Omega) \hookrightarrow L^{p/(p-2)}(\Omega)$ *holds true, since* $p > 2$ *for* $N = 2$ *and since, thanks to Assumption 1.14,* $p > 12/5$ *in case of* $N = 3$*. Then, by Assumptions 0.6 and 0.7, and in view of Definition 1.8, applying Hölder's inequality with* $1/(p/2) + 1/(p/(p-2)) = 1$ *yields that* F *has range in* $L^{p/(p-2)}(\Omega)^*$ *and thus, in* $H^1(\Omega)^*$*. On account of* $1 \leq p/(p-2) < \infty$*, we can identify* $F(t,\varphi)$ *with the term* $1/2\, g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi)) \in L^{p/2}(\Omega)$ *for all* $(t,\varphi) \in [0,T] \times H^1(\Omega)$*.*

With a little abuse of notation, the operators B *and* F *considered with different domains and ranges will be denoted by the same symbol.*

Now we can state the necessary optimality conditions as follows:

Proposition 1.16. *Under Assumptions 1.5 and 1.14, every local minimizer* $(\bar{\boldsymbol{u}}, \bar{\varphi})$ *of* (1.2) *fulfills* $\bar{\boldsymbol{u}} = \mathcal{U}(t,\bar{\varphi}) \in \boldsymbol{W}^{1,p}_D(\Omega)$ *and*

$$B\bar{\varphi} + F(t,\bar{\varphi}) = \beta d \quad \text{in } H^1(\Omega)^*, \tag{1.24}$$

which is equivalent to the following optimality system:

$$-\operatorname{div} g(\bar{\varphi})\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}) = \ell(t) \quad \text{in } \boldsymbol{W}^{-1,p}(\Omega), \tag{1.25a}$$

$$-\alpha\Delta\bar{\varphi} + \beta\,\bar{\varphi} + \frac{1}{2}\,g'(\bar{\varphi})\mathbb{C}\,\varepsilon(\bar{\boldsymbol{u}}) : \varepsilon(\bar{\boldsymbol{u}}) = \beta d \quad \text{in } H^1(\Omega)^*. \tag{1.25b}$$

Proof. The local optimality of $(\bar{\boldsymbol{u}}, \bar{\varphi})$ in particular implies that $\bar{\boldsymbol{u}}$ is a local minimizer of

$$\min_{\boldsymbol{u}\in V} \mathcal{J}(\boldsymbol{u}, \bar{\varphi}),$$

which is a convex problem according to Proposition 1.7. Therefore, $\bar{\boldsymbol{u}}$ is a global minimizer of this problem, and Proposition 1.7 yields $\bar{\boldsymbol{u}} = \mathcal{U}(t,\bar{\varphi})$.

Similarly, the local optimality of $(\bar{\boldsymbol{u}}, \bar{\varphi})$ also implies that $\bar{\varphi}$ is a local minimizer of

$$\min_{\varphi\in H^1(\Omega)} \mathcal{J}(\bar{\boldsymbol{u}}, \varphi). \tag{1.26}$$

Thanks to the improved regularity of $\bar{\boldsymbol{u}}$ by Proposition 1.7 and thanks to Assumption 1.14, one can differentiate $\mathcal{J}$ w.r.t. φ at $(\bar{\boldsymbol{u}}, \bar{\varphi})$ by means of Lemma 1.6. Thus, we can write $\partial_\varphi\mathcal{J}(\bar{\boldsymbol{u}}, \bar{\varphi}) = 0$ as necessary optimality condition for a local minimizer of (1.26). In view of (1.10), Definition 1.15 and $\bar{\boldsymbol{u}} = \mathcal{U}(t,\bar{\varphi})$, this is equivalent to (1.24). The equivalence to (1.25) follows directly from the definitions of $A_{\bar{\varphi}}$, B, and F. □

From Propositions 1.12 and 1.16 we know that (1.24) has at least one solution. In the following we aim for showing that this solution is unique, which will give in turn the unique solvability of (1.2). Unfortunately, Assumption 1.14 does not suffice to prove the uniqueness of solutions to (1.25). In order to show strong monotonicity of the operator on the left-hand side of (1.24), we additionally need that $H^1(\Omega) \hookrightarrow L^r(\Omega)$ with $r > 2p/(p-2)$, see proof of Lemma 1.20 below for more details. This motivates the first part of the following

Assumption 1.17. *From now on we require:*

1. *The assertion of Lemma 1.3 holds for some $p > N$.*
2. *The penalization parameter β is sufficiently large, depending only on the given data, see* (1.36)*,* (1.102)*,* (2.38) *and* (3.17) *on page 106 below, as well as Remark 1.22 and Remark 3.9, p. 107, for Chapters 1, 2 and for Chapter 3, respectively.*

Note that Assumption 1.17.1 is automatically fulfilled if $N = 2$, see Lemma 1.3. For $N = 3$ this is guaranteed by imposing additional conditions on the data, see Remark 1.27 below for more details. Moreover, as in case of Assumption 1.14, one does not need Assumption 1.17.1, if one replaces the H^1-seminorm in the energy functional by the $H^{3/2}$-seminorm, see Remark 1.28 below.

Assumption 1.17.2 is not restrictive at all, since β is a penalization parameter, which is supposed to be large anyway and will be send to ∞ in the next chapter. Of course, the dependency of β on the given data, and in particular on the load, cf. Remark 1.22, does not affect the analysis in this chapter or in Chapter 2, while in Chapter 3, the dependency on the load asks for a new threshold, namely (3.17) on page 106 below, see Remark 1.22 for more details.

We can now start the discussion of the uniqueness of the optimal nonlocal damage. The main idea is to show the strong monotonicity of the operator on the left-hand side in (1.24), by making use of the fact that β is large, which allows us to absorb (possible bounded) terms. We begin with the Lipschitz continuity of the mapping F. For later purposes, we prove a slightly more general result.

Lemma 1.18. *Let $r \geq 2p/(p-2)$ and define s via $1/s + 2/p + 1/r = 1$. Under Assumptions 1.5 and 1.14 we have for all $t_1, t_2 \in [0, T]$, $\varphi_1, \varphi_2 \in H^1(\Omega) \cap L^r(\Omega)$ and $\psi \in L^s(\Omega)$ the following estimate*

$$|\langle F(t_1, \varphi_1) - F(t_2, \varphi_2), \psi\rangle| \leq C\,(\|\varphi_1 - \varphi_2\|_r + |t_1 - t_2|)\|\psi\|_s,$$

with a constant $C > 0$ depending only on the given data.

Proof. We again abbreviate $\boldsymbol{u}_i := \mathcal{U}(t_i, \varphi_i)$ for $i = 1, 2$. First thing to notice is that $s \in [p/(p-2), 2p/(p-2)]$, by definition. Hence $\psi \in L^s(\Omega) \hookrightarrow L^{p/(p-2)}(\Omega)$. Now, the definition of F in (1.23) implies

$$\begin{aligned}|\langle F(t_1,\varphi_1)-F(t_2,\varphi_2),\psi\rangle| \\ \leq \int_\Omega \big|(g'(\varphi_1)-g'(\varphi_2))\mathbb{C}\varepsilon(\boldsymbol{u}_1):\varepsilon(\boldsymbol{u}_1)\psi\big|\,dx \\ + \int_\Omega \big|g'(\varphi_2)[\mathbb{C}\varepsilon(\boldsymbol{u}_1):\varepsilon(\boldsymbol{u}_1)-\mathbb{C}\varepsilon(\boldsymbol{u}_2):\varepsilon(\boldsymbol{u}_2)]\psi\big|\,dx.\end{aligned} \tag{1.27}$$

We discuss the two terms on the right-hand side of (1.27) separately:

(i) In view of (1.17) and Corollary 1.9 we have

$$\|\mathbb{C}\varepsilon(\boldsymbol{u}_1):\varepsilon(\boldsymbol{u}_1)\|_{\frac{p}{2}} \leq c, \tag{1.28}$$

where $c > 0$ is a constant independent of (t_1, φ_1). In addition, the function $g' : L^r(\Omega) \to L^r(\Omega)$ is Lipschitz continuous according to Lemma 5.1. Thus, applying Hölder's inequality with $1/r + 2/p + 1/s = 1$ for the first term on the right-hand side in (1.27) gives

$$\int_\Omega \big|(g'(\varphi_1)-g'(\varphi_2))\mathbb{C}\varepsilon(\boldsymbol{u}_1):\varepsilon(\boldsymbol{u}_1)\psi\big|\,dx \leq C_1\|\varphi_1-\varphi_2\|_r\,\|\psi\|_s, \tag{1.29}$$

where $C_1 > 0$ depends only on the given data.

(ii) Define π and ω through $1/\pi = 1/p + 1/r$ and $1/\omega = 1/p + 1/\pi$, respectively. Then (1.17), Corollary 1.9, and Proposition 1.10 result in

$$\begin{aligned}\|\mathbb{C}\varepsilon(\boldsymbol{u}_1):\varepsilon(\boldsymbol{u}_1)-\mathbb{C}\varepsilon(\boldsymbol{u}_2):\varepsilon(\boldsymbol{u}_2)\|_\omega &= \|\mathbb{C}[\varepsilon(\boldsymbol{u}_1)+\varepsilon(\boldsymbol{u}_2)]:[\varepsilon(\boldsymbol{u}_1)-\varepsilon(\boldsymbol{u}_2)]\|_\omega \\ &\leq C_2\|\boldsymbol{u}_1+\boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}_D(\Omega)}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{\boldsymbol{W}^{1,\pi}_D(\Omega)} \\ &\leq C_2(\,\|\varphi_1-\varphi_2\|_r+|t_1-t_2|).\end{aligned} \tag{1.30}$$

Now, in light of Assumption 0.6, Hölder's inequality with $1/\omega + 1/s = 1$ yields

$$\begin{aligned}\int_\Omega \big|g'(\varphi_2)[\mathbb{C}\varepsilon(\boldsymbol{u}_1):\varepsilon(\boldsymbol{u}_1)-\mathbb{C}\varepsilon(\boldsymbol{u}_2):\varepsilon(\boldsymbol{u}_2)]\psi\big|\,dx \\ \leq C_2(\|\varphi_1-\varphi_2\|_r+|t_1-t_2|)\|\psi\|_s.\end{aligned} \tag{1.31}$$

Notice that $C_2 > 0$ depends only on the given data. Inserting (1.29) and (1.31) in (1.27) finally gives the assertion. □

We observe that, if $p > N$, then

$$r := \frac{2p}{p-2} \in \Big(2, \frac{2N}{N-2}\Big), \tag{1.32}$$

so that Sobolev embeddings give in turn $H^1(\Omega) \hookrightarrow L^r(\Omega)$. Since by construction, r satisfies $2/r + 2/p = 1$, Lemma 1.18 is applicable with $s = r$, which results in the following

Corollary 1.19. *Under Assumptions 1.5 and 1.17.1 it holds*

$$|\langle F(t_1,\varphi_1)-F(t_2,\varphi_2),\psi\rangle| \le C\big(\|\varphi_1-\varphi_2\|_{\frac{2p}{p-2}}+|t_1-t_2|\big)\|\psi\|_{\frac{2p}{p-2}} \qquad \forall\, t_1,t_2\in[0,T],\ \forall\,\varphi_1,\varphi_2,\psi\in H^1(\Omega),$$

where $C>0$ depends only on the given data.

The following lemma is an essential tool for proving the unique solvability of (1.2).

Lemma 1.20. *Let $m>0$ be given. Then, under Assumption 1.17.1, it holds*

$$m\|\varphi\|^2_{\frac{2p}{p-2}} \le k\,\|\varphi\|_2^2+\widetilde{c}(k)\,\|\varphi\|^2_{H^1(\Omega)} \quad \forall\,\varphi\in H^1(\Omega) \text{ and } \forall\,k>0,$$

where $\widetilde{c}:\mathbb{R}^+\to\mathbb{R}^+$ is a function depending on m, p and N, which satisfies $\widetilde{c}(k)\searrow 0$ as $k\nearrow\infty$.

Proof. For convenience we again set $r:=2p/(p-2)$. First note that, because of Assumption 1.17.1, there is an index ϱ such that $r\in(2,\varrho)$ and $H^1(\Omega)\hookrightarrow L^\varrho(\Omega)$. For instance, take $\varrho=(2p+1)/(p-2)$ for $N=2$ and $\varrho=6$ if $N=3$, see also (1.32). Therefore, there exists $\theta\in(0,1)$ such that $1/r=\theta/2+(1-\theta)/\varrho$, whence by Lyapunov's inequality we have

$$m\|\varphi\|_r^2 \le m\|\varphi\|_2^{2\theta}\|\varphi\|_\varrho^{2-2\theta} \le mC\|\varphi\|_2^{2\theta}\|\varphi\|_{H^1(\Omega)}^{2-2\theta} \quad \forall\,\varphi\in H^1(\Omega). \tag{1.33}$$

Note that C is the embedding constant, and thus, it depends on ϱ and N, or to be more precise, on p and N. Thanks to the generalized Young inequality, (1.33) can be continued as

$$m\|\varphi\|_r^2 \le k\,\|\varphi\|_2^2+\frac{(vk)^{1-w}m^wC^w}{w}\,\|\varphi\|^2_{H^1(\Omega)} \quad \forall\,k>0,\ \forall\,\varphi\in H^1(\Omega),$$

where $v:=1/\theta$ and $w:=1/(1-\theta)$. We remark that v and w depend on p and N, as well. Note that the mapping

$$\widetilde{c}:\mathbb{R}^+\ni k\mapsto \frac{(vk)^{1-w}m^wC^w}{w}\in\mathbb{R}^+$$

satisfies $\widetilde{c}(k)\searrow 0$ as $k\nearrow\infty$, since $k\mapsto k^{1-w}$ does so, in view of $w>1$. The proof is now complete. ☐

We are now finally in the position to state the 'strong monotonicity' of $B+F$, from which the uniqueness of the optimal nonlocal damage will follow easily.

Lemma 1.21. *Let Assumptions 1.5 and 1.17 be satisfied. Then, for all $t_1,t_2\in[0,T]$ and all $\varphi_1,\varphi_2\in H^1(\Omega)$, $\varphi_1\ne\varphi_2$, it holds*

$$\frac{\langle B(\varphi_1-\varphi_2)+F(t_1,\varphi_1)-F(t_2,\varphi_2),\varphi_1-\varphi_2\rangle_{H^1(\Omega)}}{\|\varphi_1-\varphi_2\|_{H^1(\Omega)}} \ge \alpha/2\|\varphi_1-\varphi_2\|_{H^1(\Omega)}-C|t_1-t_2|,$$

where $C>0$ depends only on the given data.

Proof. Let $(t_i, \varphi_i)_{i=1,2} \in [0,T] \times H^1(\Omega)$ be arbitrary, but fixed with $\varphi_1 \neq \varphi_2$. Then, Corollary 1.19 and Lemma 1.20 yield

$$
\begin{aligned}
|\langle F(t_1,\varphi_1) - F(t_2,\varphi_2), \varphi_1 - \varphi_2\rangle_{H^1(\Omega)}| \leq k\|\varphi_1 - \varphi_2\|_2^2 + \widetilde{c}(k)\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}^2 \\
+ C\,|t_1 - t_2|\,\|\varphi_1 - \varphi_2\|_{H^1(\Omega)} \quad \forall\, k > 0,
\end{aligned}
\tag{1.34}
$$

with $\widetilde{c}$ as in Lemma 1.20. Note that we also employed that $H^1(\Omega) \hookrightarrow L^{2p/(p-2)}(\Omega)$, in view of Assumption 1.17.1. Using the definition of B in (1.22) we further infer from (1.34) that for all $k > 0$ it holds

$$
\begin{aligned}
\frac{\langle B(\varphi_1 - \varphi_2) + F(t_1,\varphi_1) - F(t_2,\varphi_2), \varphi_1 - \varphi_2\rangle}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}} &\geq (\alpha - \widetilde{c}(k))\|\varphi_1 - \varphi_2\|_{H^1(\Omega)} - C|t_1 - t_2| \\
&\quad + (\beta - \alpha - k)\frac{\|\varphi_1 - \varphi_2\|_2^2}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}}.
\end{aligned}
\tag{1.35}
$$

Keeping in mind the properties of $\widetilde{c}$, we can choose $k > 0$ (large enough) such that $\alpha - \widetilde{c}(k) \geq \alpha/2$. Furthermore, if

$$\beta \geq \alpha + k, \tag{1.36}$$

cf. Assumption 1.17.2, then (1.35) gives the assertion. □

Remark 1.22. *Note that in the above proof, Lemma 1.20 is applied with $m = C$, where $C > 0$ is the constant from Corollary 1.19. An inspection of the proof of Lemma 1.18 shows that this depends on the (supremum) norm of the load, see e.g.* (1.28) *and Corollary 1.9. Hence, the function $\widetilde{c}$ depends, cf. Lemma 1.20 and cf. Remark 1.27 below, only on the given data, and in particular on the load, which implies in view of* (1.36) *that β does so as well.*

This becomes a central concern in Chapter 3, where the load is no longer fixed, but a variable. Roughly speaking, this is the reason why in Section 3.1 below, the (time-independent) variable loads have to be uniformly bounded to guarantee unique solvability of the time-independent version of the minimization problem in (P)*. The threshold* (1.36) *is then replaced by* (3.17) *(see page 106 below), where M is the (given) constant which uniformly bounds the variable loads. We refer here also to Section 3.2 below, where one deals with the same problem in the time-dependent case.*

The unique solvability of the minimization problem in (P) is covered by the following

Theorem 1.23 (Unique solvability of (1.2))**.** *Under Assumptions 1.5 and 1.17, the optimization problem* (1.2) *admits a unique solution, which is characterized by* (1.25)*. Consequently, the unique optimal nonlocal damage is characterized by* (1.24)*.*

Proof. We focus on showing a Lipschitz estimate for solutions of (1.24), as this will be also needed for later purposes. Then, the desired assertion follows immediately. To this end, let φ_i denote solutions of (1.24) associated with given $(t_i, d_i) \in [0,T] \times L^2(\Omega)$, $i = 1, 2$. Note that the existence thereof is ensured by Propositions 1.12 and 1.16. By assuming $\varphi_1 \neq \varphi_2$, we obtain from Lemma 1.21 and Cauchy-Schwarz inequality the estimate

$$\begin{aligned}
&\|\varphi_1 - \varphi_2\|_{H^1(\Omega)} \\
&\quad \leq \frac{2}{\alpha}\Big(\frac{\langle B(\varphi_1 - \varphi_2) + F(t_1, \varphi_1) - F(t_2, \varphi_2), \varphi_1 - \varphi_2\rangle_{H^1(\Omega)}}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}} + C|t_1 - t_2|\Big) \\
&\quad = \frac{2}{\alpha}\Big(\beta\frac{(d_1 - d_2, \varphi_1 - \varphi_2)_2}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}} + C|t_1 - t_2|\Big) \\
&\quad \leq C\big(\|d_1 - d_2\|_2 + |t_1 - t_2|\big).
\end{aligned} \tag{1.37}$$

Note that the estimate (1.37) holds trivially also for $\varphi_1 = \varphi_2$. If we set $t_1 = t_2$ and $d_1 = d_2$, then (1.37) implies uniqueness for (1.24), and thus the unique solvability of (1.25), cf. Proposition 1.16. As this system constitutes the necessary optimality condition for (1.2), which admits global solutions by Proposition 1.12, we deduce that (1.2) is uniquely solvable too, and that every local minimizer must be the global minimizer. This completes the proof. □

The unique solvability of (1.24) leads to

Definition 1.24 (Solution operator of (1.24))**.** *Let Assumptions 1.5 and 1.17 be fulfilled. We define the operator* $\Phi : [0, T] \times L^2(\Omega) \to H^1(\Omega)$ *as*

$$\Phi(t, d) := \big(B + F(t, \cdot)\big)^{-1}(\beta d).$$

As a result of (1.37) we have the following

Corollary 1.25 (Lipschitz continuity of Φ)**.** *Under Assumptions 1.5 and 1.17, there exists a constant* $L > 0$ *such that*

$$\|\Phi(t_1, d_1) - \Phi(t_2, d_2)\|_{H^1(\Omega)} \leq L(\|d_1 - d_2\|_2 + |t_1 - t_2|) \tag{1.38}$$

holds true for all $t_1, t_2 \in [0, T]$ *and all* $d_1, d_2 \in L^2(\Omega)$.

To summarize our results so far, we have proven that, under Assumptions 1.5 and 1.17, the optimization problem

$$\min_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{E}(t, \boldsymbol{u}, \varphi, d)$$

is uniquely solvable for any $(t, d) \in [0, T] \times L^2(\Omega)$. The optimal displacement is given by $\mathcal{U}(t, \bar{\varphi}) \in \boldsymbol{W}^{1,p}_D(\Omega)$, where $\bar{\varphi} = \Phi(t, d) \in H^1(\Omega)$ is the optimal nonlocal damage.

We conclude this subsection with some remarks concerning the assumptions we made on the load and on p, respectively.

Remark 1.26. *We point out that it was necessary in Assumption 1.5 to impose that ℓ has range in $\boldsymbol{W}^{-1,p}(\Omega)$, in order to obtain the improved regularity of the optimal displacement, or better said, maximal regularity therefor, see Definition 1.8 and Lemma 1.3. Since the threshold for β depends on $\|\ell\|_{C([0,T];\boldsymbol{W}^{-1,p}(\Omega))}$, cf. Remark 1.22, it makes sense to work with a space for the load so that the 'maximal value' of ℓ is independent of the variable t (at least almost everywhere). For example, in Section 3.2 below one finds $L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$ as a proper (more general) choice for the space for the load. As we are interested in solution operators which are smooth w.r.t. time, and later even continuously differentiable, we require in Assumption 1.5 that the load is (for the beginning) Lipschitz continuous.*

Remark 1.27. *We emphasize that by the results of [29], the maximal value of p depends only on the domain, the partition of its boundary, and the boundedness and monotonicity constants of the stress-strain relation, which in our case is depicted via* (1.7). *Roughly speaking, a sufficiently smooth domain, no mixed boundary conditions and a smaller difference between the above mentioned constants account for larger values for p. We refer here to the proof of Lemma 1.1 in [29]. Thus, from the proofs of Lemmata 1.1 and 1.3 we infer that p depends on $\epsilon\gamma_{\mathbb{C}}$, $\|\mathbb{C}\|_\infty$, and the domain. We moreover infer that the existence of a p fulfilling Assumption 1.17.1 is ensured provided that the values $\underline{m} = \epsilon\,\gamma_{\mathbb{C}}$ and $\overline{m} = \|\mathbb{C}\|_\infty$ are close enough to each other and if the domain is smooth enough (C^1-boundary) with Γ_D as entire boundary.*

Recall that, in the two-dimensional case, Assumption 1.17.1 is automatically fulfilled.

Remark 1.28. *Alternatively to Assumption 1.17.1 one can proceed as in [41] and use in case of $N = 3$ the Sobolev-Slobodeckij space $H^{3/2}(\Omega)$ for the nonlocal damage. To this end, one replaces the gradient term in the energy functional by the seminorm generated by [41, (2.4b)] so that the radial unboundedness in the proof of Proposition 1.12 is still ensured, and thus, the existence of the optimal nonlocal damage. Note that the rest of the proof remains unaffected, as $H^{3/2}(\Omega)$ (endowed with the same norm as in [41]) is a reflexive Banach space.*

The advantage thereof is that the embedding $H^{3/2}(\Omega) \hookrightarrow L^r(\Omega)$ holds for every $r \in [1,\infty)$ in the three-dimensional case, which means that there is no longer need for making extra assumptions on p if $N = 3$. This is shown by a closer inspection of the preceding analysis, which reveals that the embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ for all $r \in [1,\infty)$ in case of $N = 2$ is the key ingredient to prove unique solvability for (1.2) *without any additional assumptions on the integrability exponent p. Thus, working with $H^{3/2}(\Omega)$ instead of $H^1(\Omega)$ in three dimensions allows to do the same in case of $N = 3$, so that no extra assumptions on p are required. However, we chose not to work with $H^{3/2}(\Omega)$, as the bilinear form associated with the $H^{3/2}(\Omega)$-seminorm is difficult to realize in numerical computations.*

1.1.2 The evolution equation

This subsection is devoted to prove existence and uniqueness for the complete damage model (P). We suppose that Assumptions 1.5 and 1.17 hold true in what follows. Then, with the results from Subsection 1.1.1 at hand, the problem (P) can be reformulated as

$$-\partial_d \mathcal{E}(t, \boldsymbol{u}(t), \varphi(t), d(t)) \in \partial \mathcal{R}_\delta(\dot{d}(t)) \quad \text{f.a.a. } t \in (0,T), \quad d(0) = d_0, \tag{1.39}$$

where $\boldsymbol{u} = \mathcal{U}(\cdot, \varphi(\cdot))$ and $\varphi = \Phi(\cdot, d(\cdot))$, while $d : [0,T] \to L^2(\Omega)$ is the local damage. Moreover, due to (1.9), the evolutionary equation (1.39) reads

$$-\beta(d(t) - \varphi(t)) \in \partial \mathcal{R}_\delta(\dot{d}(t)) \quad \text{f.a.a. } t \in (0,T), \quad d(0) = d_0. \tag{1.40}$$

We approach the reduced problem (P) by showing that (1.40) is equivalent to the following operator differential equation, which can then be solved by standard arguments.

Theorem 1.29 (Operator differential equation)**.** *Let Assumptions 1.5 and 1.17 hold true. Then, the evolution equation* (1.40) *is equivalent to*

$$\dot{d}(t) = \frac{1}{\delta} \max\left(-\beta(d(t) - \varphi(t)) - r\right) \quad \textit{f.a.a. } t \in (0,T), \quad d(0) = d_0, \tag{1.41}$$

where $\varphi = \Phi(\cdot, d(\cdot))$.

Proof. First we note that $\mathcal{R}_\delta$ is the sum of two convex functionals, namely $\mathcal{R}_1$ and $\frac{\delta}{2}\|\cdot\|_2^2$. As the latter one is Fréchet-differentiable, we can apply the sum rule for convex subdifferentials, so that

$$\partial \mathcal{R}_\delta(\eta) = \partial \mathcal{R}_1(\eta) + \delta \eta \quad \text{for all } \eta \in L^2(\Omega).$$

Further, since $\mathcal{R}_1$ is positively homogeneous, we have for all $\xi, \eta \in L^2(\Omega)$ with $\eta \geq 0$ the equivalence

$$\xi \in \partial \mathcal{R}_1(\eta) \quad \Longleftrightarrow \quad \begin{cases} (\xi, \eta)_2 = \mathcal{R}_1(\eta), \\ (\xi, v)_2 \leq \mathcal{R}_1(v) \quad \forall\, v \in L^2(\Omega). \end{cases} \tag{1.42}$$

This can be easily seen by writing down the definition of the subdifferential and testing therein with 0 and 2η, respectively.

In view of the above, we can now rewrite the evolution in (1.40) as

$$(-\beta(d(t) - \varphi(t)) - \delta \dot{d}(t), \dot{d}(t))_2 = \mathcal{R}_1(\dot{d}(t)), \tag{1.43a}$$

$$(-\beta(d(t) - \varphi(t)) - \delta \dot{d}(t), v)_2 \leq \mathcal{R}_1(v) \quad \forall\, v \in L^2(\Omega) \tag{1.43b}$$

for almost all $t \in (0,T)$. Note that from (1.40) we know that $\dot{d}(t) \in \operatorname{dom}(\mathcal{R}_1)$, i.e., $\dot{d}(t) \geq 0$ f.a.a. $t \in (0,T)$. As a result of Definition 0.1 and fundamental lemma of the calculus of variations we further obtain

$$(1.43\text{b}) \quad \Longleftrightarrow \quad -\beta(d(t) - \varphi(t)) - \delta \dot{d}(t) - r \leq 0 \quad \text{a.e. in } \Omega, \ \text{f.a.a. } t \in (0,T).$$

By employing again Definition 0.1, we also get

$$(1.43\text{a}) \quad \Longleftrightarrow \quad \big(\underbrace{-\beta(d(t) - \varphi(t)) - \delta \dot{d}(t) - r}_{\leq 0}, \underbrace{\dot{d}(t)}_{\geq 0}\big)_2 = 0 \quad \text{f.a.a. } t \in (0,T).$$

Therefore, the system (1.43) and thus, the evolution in (1.40), is equivalent to the following complementarity system

$$0 \leq \delta \dot{d}(t) \perp -\beta(d(t) - \varphi(t)) - r - \delta \dot{d}(t) \leq 0 \quad \text{a.e. in } \Omega, \text{ f.a.a. } t \in (0, T), \tag{1.44}$$

where we used $\delta > 0$ for the left inequality. Since the max-function is a well known complementarity function, (1.44) gives the assertion. □

Remark 1.30. *There are multiple equivalent formulations of the penalized damage model in reduced form. An overview thereof is given in Proposition 2.7 on page 59 below. The operator differential equation* (1.41) *is certainly one of the most significant, as it has various advantages. Firstly, it may provide a useful starting point for a numerical solution of* (P)*, as one could solve (the discretized version of)* (1.41) *by means of a semi-smooth Newton algorithm, see also* ($P_k^{\beta,\tau}$) *on page 82 below. Moreover, it facilitates deriving necessary optimality conditions in Chapter 3, due to the directional differentiability of the operator* max*, cf. Lemma 5.6. Besides, by describing the evolution of the local damage via* (1.41)*, the unique solvability of* (1.39)*, and thus of* (P)*, follows easily by Picard-Lindelöf's theorem, as we will next see. Furthermore,* (1.41) *will be the starting point for performing a time-discretization for* (P) *in Subsection 2.4 below, which will lead to finding an L^∞-bound for the local damage. However,* (1.41) *does not help us when it comes to passing to the limit $\beta \to \infty$ in Chapter 2, as explained at the beginning of Section 2.1 below. For this reason, one derives therein a new equivalent formulation for* (1.39)*, i.e., the energy identity, by means of which the passage to the limit is possible.*

Remark 1.31. *The result in Theorem 1.29 can be as well proven by employing the dual formulation of* (1.40)*, see* (2.12) *(a.e. in* $(0,T)$*), p. 59 below, and by showing that the conjugate functional $\mathcal{R}_\delta^*$ equals the δ-Moreau-Yosida regularization of the indicator functional associated to the set $\{v \in L^2(\Omega) : v \leq r\}$. According to [28, Lemma 4.1(a)], the latter one is Fréchet-differentiable. Moreover, its derivative can be expressed by means of the* max*-function, so that* (2.12) *(a.e. in* $(0,T)$*), p. 59, becomes* (1.41)*.*

One could also prove Theorem 1.29 by rewriting (1.40) *as a minimization problem, see* (2.14)*, p. 59 below. As this is convex, $\dot{d}(t)$ can be described a.e. in $(0,T)$ by the necessary and sufficient optimality condition, which is nothing else as the complementarity system* (1.44)*, i.e.,* (1.41)*.*

The unique solvability of the problem (P) in reduced form is covered by the following

Theorem 1.32 (Existence and uniqueness for the evolution equation)**.** *Under Assumptions 1.5 and 1.17 there exists a unique function $d \in C^{1,1}([0,T]; L^2(\Omega))$ satisfying* (1.39)*, and thus,* (1.40) *and* (1.41)*, for all $t \in [0,T]$.*

Proof. Theorem 1.29 tells us that (1.39) and (1.40) are equivalent to the operator differential equation given by (1.41). We intend to solve the latter one by means of Picard-Lindelöf's theorem. For this purpose, we define the mapping $f : [0,T] \times L^2(\Omega) \to L^2(\Omega)$ as

$$f(t,d) := \frac{1}{\delta} \max(-\beta(d - \Phi(t,d)) - r). \tag{1.45}$$

Of course, f is well defined in view of Lemma 5.6.(i). Due to the Lipschitz continuity of $\max : L^2(\Omega) \to L^2(\Omega)$, cf. Lemma 5.6.($i$), and (1.38), it holds for all $(t_1, d_1), (t_2, d_2) \in [0, T] \times L^2(\Omega)$

$$\begin{aligned} \|f(t_1, d_1) - f(t_2, d_2)\|_2 &\leq \frac{\beta}{\delta}\big(\|\Phi(t_1, d_1) - \Phi(t_2, d_2)\|_{H^1(\Omega)} + \|d_1 - d_2\|_2\big) \\ &\leq \frac{\beta}{\delta}\,(L+1)\,\|d_1 - d_2\|_2 + \frac{\beta}{\delta}\,L\,|t_1 - t_2|, \end{aligned} \tag{1.46}$$

where L is the Lipschitz constant of Φ. Therefore, f is globally Lipschitz continuous, and we can conclude with [14, Theorem 7.2.6] that there exists a unique $d \in C^1([0, T]; L^2(\Omega))$ satisfying

$$\dot{d}(t) = f(t, d(t)) \quad \forall\, t \in [0, T], \quad d(0) = d_0. \tag{1.47}$$

In particular, it holds $d \in C^{0,1}([0, T]; L^2(\Omega))$, which in view of (1.46) means that $t \mapsto f(t, d(t)) \in C^{0,1}([0, T]; L^2(\Omega))$ as well, and thus, $d \in C^{1,1}([0, T]; L^2(\Omega))$, on account of (1.47). This together with (1.45) finally gives the assertion. □

Remark 1.33 (Improved space regularity for the local damage)**.** *An inspection of the above proof shows the following: If we assume that the initial datum is more regular, in the sense that* $d_0 \in L^\rho(\Omega)$, *where* $\rho \in (2, \infty]$, *and if* $\Phi : [0, T] \times L^2(\Omega) \to L^\rho(\Omega)$ *is Lipschitz continuous, then* $d \in C^{1,1}([0, T]; L^\rho(\Omega))$. *Note that this is due to the fact that* $\max : L^\rho(\Omega) \to L^\rho(\Omega)$ *is well defined and Lipschitz continuous, as a consequence of the Lipschitz continuity of* $\max : \mathbb{R} \to \mathbb{R}$.

Hence, if $d_0 \in L^r(\Omega)$, *then the local damage belongs to* $C^{1,1}([0, T]; L^r(\Omega))$, *where* $r < \infty$ *and* $r = 6$ *for* $N = 2$ *and* $N = 3$, *respectively, as a result of the embedding* $H^1(\Omega) \hookrightarrow L^r(\Omega)$. *As in the next section the Lipschitz continuity of the nonlocal damage improves, the same holds for the space regularity of* d, *see Remark 1.46 below.*

Let us point out that the Lipschitz continuity of the local damage d readily transfers to $\varphi = \Phi(\cdot, d(\cdot))$ and $\boldsymbol{u} = \mathcal{U}(\cdot, \varphi(\cdot))$, as explained in the sequel. First of all, (1.38) and the Lipschitz continuity of d imply the Lipschitz continuity of φ. Further, in view of $H^1(\Omega) \hookrightarrow L^r(\Omega)$ with $r \in [2p/(p-2), \infty)$ and $r \in [2p/(p-2), 6]$ for $N = 2$ and $N = 3$, respectively, we obtain from Proposition 1.10 that $\boldsymbol{u} \in C^{0,1}([0, T]; \boldsymbol{W}^{1,\pi}_D(\Omega))$, with $\pi \in [2, p)$ for $N = 2$ and $\pi \in [2, 6p/(p+6)]$ in case of $N = 3$. Note that the intervals $[2p/(p-2), \infty)$ and $[2p/(p-2), 6]$ are not empty thanks to $p > 2$ for $N = 2$ and $p \geq 3$ for $N = 3$, respectively.

We observe that as long as $\ell \in C^{0,1}([0, T]; \boldsymbol{W}^{-1,p}(\Omega))$, one cannot expect more time regularity for $\boldsymbol{u}$ and φ, see also Remark 3.16 on page 115 below. The time regularity thereof can be however further improved, provided that the load is more regular in time, as we will see in Section 1.3 below, in the case of a continuously differentiable load.

To summarize our results so far, we have proven that, under Assumptions 1.5 and 1.17, there exists a unique solution $(\boldsymbol{u}, \varphi, d)$ for the penalized damage model (P) satisfying $\boldsymbol{u} \in C^{0,1}([0, T]; \boldsymbol{W}^{1,\pi}_D(\Omega))$ with π as above, $\boldsymbol{u}(t) \in \boldsymbol{W}^{1,p}_D(\Omega)$ for all $t \in [0, T]$,

$\varphi \in C^{0,1}([0,T]; H^1(\Omega))$ and $d \in C^{1,1}([0,T]; L^2(\Omega))$, and the following system of differential equations:

$$-\operatorname{div} g(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) = \ell(t) \quad \text{in } \boldsymbol{W}^{-1,p}(\Omega), \tag{1.48a}$$

$$-\alpha\Delta\varphi(t) + \beta\,\varphi(t) + \frac{1}{2}\,g'(\varphi(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) = \beta d(t) \quad \text{in } H^1(\Omega)^*, \tag{1.48b}$$

$$\dot{d}(t) - \frac{1}{\delta}\max(-\beta(d(t) - \varphi(t)) - r) = 0, \qquad d(0) = d_0 \tag{1.48c}$$

for all $t \in [0,T]$.

1.2 Improved regularity and Lipschitz continuity of the nonlocal damage

In this section we show that the optimal nonlocal damage possesses higher regularity and satisfies a corresponding Lipschitz condition. The key tool therefor is a known result of $W^{1,q}$-theory, with $q > 2$, of which we can make use mainly thanks to the space regularity of the nonlinearity in (1.48b). These new findings can be used to improve many results throughout this work, see Remarks 1.46, 1.61, 1.63, 3.12 and 3.23 below. Of course, Assumptions 1.5 and 1.17 are supposed to hold throughout this section.

1.2.1 Improved regularity

The starting point for proving the improved regularity is the equation (1.24), which characterizes the optimal nonlocal damage, cf. Theorem 1.23. Applying a classical boot strapping argument therefor will then easily give the result. However, we first need to do a little preparatory work, of which we also make use in the upcoming subsection. For convenience, we introduce

Definition 1.34. *We define the operator $-\Delta + I : H^1(\Omega) \to H^1(\Omega)^*$ by*

$$\langle(-\Delta + I)w, \psi\rangle_{H^1(\Omega)} := \int_\Omega \nabla w \cdot \nabla\psi + w\,\psi\,dx \quad \forall\,\psi \in H^1(\Omega).$$

The operator $-\Delta + I$ considered with different domains and ranges will be denoted by the same symbol for the sake of simplicity.

The key result mentioned at the beginning of the section is covered by the following

Lemma 1.35. *[23, Theorem 3, Lemma 1, Definition 4, Remark 6] Let $\widetilde{\Gamma}_N \subset \Gamma$ be given so that $\Omega\cup\widetilde{\Gamma}_N$ is regular in the sense of Gröger cf. [23, Definition 2], see also Assumption 0.5. Then there exists $q > 2$ such that for all $\nu \in [2, q]$ the operator $-\Delta+I : W^{1,\nu}_{\Gamma\setminus\widetilde{\Gamma}_N}(\Omega) \to W^{1,\nu'}_{\Gamma\setminus\widetilde{\Gamma}_N}(\Omega)^*$ is continuously invertible. The norms of their inverses are bounded by a constant depending only on q, the domain Ω and $\widetilde{\Gamma}_N$ at most.*

In the remaining of the subsection we work with an arbitrary, but fixed $(t,d) \in [0,T] \times L^2(\Omega)$ and use for simplicity the notations $\bar\varphi := \Phi(t,d)$ and

$$f := \beta(d - \bar\varphi) + \alpha\,\bar\varphi - F(t,\bar\varphi). \tag{1.49}$$

We consider the equation

$$(-\Delta + I)w = \frac{1}{\alpha} f \quad \text{in } H^1(\Omega)^*, \tag{1.50}$$

which is solved by $\bar\varphi$, in view of (1.49) and Definition 1.24. Then, taking advantage of the fact that f possesses higher regularity than $H^1(\Omega)^*$, we show by means of Lemma 1.35 that $\bar\varphi \in W^{1,q}(\Omega)$, where $q > 2$.

The regularity of the linear form on the right-hand side in (1.50) is given by the following

Lemma 1.36. *Under Assumptions 1.5 and 1.17, it holds $f \in W^{1,\varrho'}(\Omega)^*$, where*

$$\frac{1}{\varrho} = \max\Big\{\frac{2}{p} - \frac{1}{N}, \frac{1}{2} - \frac{1}{N}\Big\} < \frac{1}{N}. \tag{1.51}$$

Proof. From (1.49) and Sobolev embeddings we infer that $f \in L^2(\Omega) \cap L^{\frac{p}{2}}(\Omega)$ (see Definition 1.15), which implies $f \in L^r(\Omega)$, where $r := \min\{2, p/2\}$. Thus, the idea of the proof is to find $\varrho \in (2,\infty)$ (as large as possible) so that

$$L^r(\Omega) \hookrightarrow W^{1,\varrho'}(\Omega)^*. \tag{1.52}$$

To this end, we employ Sobolev embeddings, which tell us in view of $\varrho' \in (1,2)$ that $W^{1,\varrho'}(\Omega) \hookrightarrow L^{\frac{N\varrho'}{N-\varrho'}}(\Omega)$, whence $L^{\frac{N\varrho}{N+\varrho}}(\Omega) \hookrightarrow W^{1,\varrho'}(\Omega)^*$, as $\frac{N\varrho'}{N-\varrho'} \in [1,\infty)$. The embedding (1.52) is then guaranteed if

$$r \geq \frac{N\varrho}{N+\varrho},$$

which is equivalent to

$$\frac{1}{\varrho} \geq \frac{1}{r} - \frac{1}{N}.$$

In light of the definition of r, this gives the identity in (1.51), as we searched for the largest value for ϱ. From Assumption 1.17.1 and $N < 4$ we finally deduce $\varrho > N$. $\square$

We can now conclude the main result of this subsection:

Theorem 1.37 (Improved regularity of the nonlocal damage)**.** *Suppose that Assumptions 1.5 and 1.17 hold true. Then, there exists a $q > 2$ such that $\Phi(t,d) \in W^{1,q}(\Omega)$ for every $(t,d) \in [0,T] \times L^2(\Omega)$.*

Proof. In view of Lemma 1.36, we want to apply Lemma 1.35 for $\widetilde{\Gamma}_N = \Gamma$, which means that we first have to verify that $\bar{\Omega}$ is regular in the sense of Gröger. Thanks to [24, Theorem 5.2, 5.4], this is indeed the case, since Ω is a bounded Lipschitz domain, cf. Assumption 0.5. Thus, by virtue of Lemma 1.35, there exists $q_\Omega > 2$ such that for all $\nu \in [2, q_\Omega]$ the operator $-\Delta + I : W^{1,\nu}(\Omega) \to W^{1,\nu'}(\Omega)^*$ is continuously invertible. Hence, in view of Lemma 1.36, (1.50) admits a unique solution $\varphi \in W^{1,q}(\Omega)$, where $q := \min\{q_\Omega, \varrho\} > 2$, with ϱ given by (1.51). Since $\bar{\varphi}$ solves (1.50) as well, we obtain $\varphi = \bar{\varphi}$, which completes the proof. □

1.2.2 Improved Lipschitz continuity

As a consequence of the higher regularity of the solution of (1.24), one expects that Φ satisfies a corresponding Lipschitz condition. For this reason, we investigate in what follows the $W^{1,q}(\Omega)$-Lipschitz continuity of the solution map of (1.24). Throughout this subsection, q, q_Ω and ϱ stand for the numbers given by Theorem 1.37, Lemma 1.35, where $\widetilde{\Gamma}_N := \Gamma$, and Lemma 1.36, respectively. We work with $(t_i, d_i) \in [0, T] \times L^2(\Omega)$ arbitrary, but fixed and define $\varphi_i := \Phi(t_i, d_i) \in W^{1,q}(\Omega)$, where $i = 1, 2$. In addition, we abbreviate

$$\iota := \frac{1}{\alpha}\big(\beta(d_1 - d_2) - (\beta - \alpha)(\varphi_1 - \varphi_2) - (F(t_1, \varphi_1) - F(t_2, \varphi_2))\big). \tag{1.53}$$

Note that $\iota \in W^{1,\varrho'}(\Omega)^*$, on account of Lemma 1.36. Moreover, notice that, by construction, $\varphi_1 - \varphi_2$ solves

$$(-\Delta + I)w = \iota.$$

Thus, according to Lemma 1.35 applied for $\widetilde{\Gamma}_N = \Gamma$, the following estimate holds true

$$\|\varphi_1 - \varphi_2\|_{W^{1,\mu}(\Omega)} \le c\|\iota\|_{W^{1,\mu'}(\Omega)^*} \quad \forall\, 2 \le \mu \le q = \min\{q_\Omega, \varrho\}. \tag{1.54}$$

Unfortunately, the desired Lipschitz continuity condition cannot be directly proven by setting $\mu = q$ in (1.54), as one cannot directly derive an estimate of the form $\|\iota\|_{W^{1,q'}(\Omega)^*} \le L(\|d_1 - d_2\|_2 + |t_1 - t_2|)$. However, by applying a finite number of boot strapping steps, it is possible to prove the result, as we will next see. The starting point therefor is the Lipschitz estimate (1.38).

The main idea in each step is to search for $\nu \in [2, \varrho]$ as large as possible such that $\|\iota\|_{W^{1,\nu'}(\Omega)^*} \le C(\|d_1 - d_2\|_2 + |t_1 - t_2|)$ holds. Such an estimate will be established by using the Lipschitz continuity result for the solution map Φ, which was proven in the previous boot strapping step. Then, applying (1.54) with $\mu = \min\{q_\Omega, \nu\}$ leads to an improved Lipschitz continuity result, as we will later see.

In the proof of Theorem 1.45 below we establish that the number of boot strapping steps, which we hereafter call m, is finite, see (1.67). Therein we also show that, starting from the $H^1(\Omega)$-Lipschitz continuity of Φ, the arising sequence of Lipschitz exponents reads

$$\{2 = \nu_0(p), ..., \nu_{m-1}(p), \min\{q_\Omega, \nu_m(p)\} = q\},$$

with $\nu_j(p)_{j=0,\dots,m}$ as in Definition 1.39 below. As confirmed by Lemma 1.41 below, the result improves in each step. Now, assuming that the $W^{1,\nu_n(p)}(\Omega)$-Lipschitz continuity of Φ has already been proven, the $n+1$-st boot strapping step consists of

(i) showing the estimate

$$\|\iota\|_{W^{1,\nu_{n+1}(p)'}(\Omega)^*} \le C(\|\varphi_1 - \varphi_2\|_{W^{1,\nu_n(p)}(\Omega)} + \|d_1 - d_2\|_2 + |t_1 - t_2|), \tag{1.55}$$

see Lemma 1.44 below,

(ii) using the $W^{1,\nu_n(p)}(\Omega)$-Lipschitz continuity of Φ in (1.55),

(iii) applying (1.54) for $\mu = \nu_{n+1}(p) \le q$, which yields the $W^{1,\nu_{n+1}(p)}(\Omega)$-Lipschitz continuity.

We point out that the number of steps m, depends on p, N and q_Ω, in view of (1.67) and (1.59). The formula (1.59) is in particular essential in three dimensions and gives, in light of (1.68), the maximal number of required steps. Thus, in the two-dimensional case, the $W^{1,q}(\Omega)-$Lipschitz continuity is established in maximal two steps, cf. (1.59). From Lemma 1.40 below we further deduce that small values of p account for small values of $\nu_n(p)$. Thus, the closer p is to N, the more difficult it is to come closer to q, which is also confirmed by (1.59), since n^* is larger for smaller values of p.

The subsection is structured as follows. First we lay the foundations for proving (1.55) by introducing the sequence $\{\nu_n(p)\}_n$ (Definitions 1.38-1.39) and by enumerating some useful properties thereof (Lemmata 1.40-1.42). Afterwards, we proceed towards showing the main result by first deriving an estimate of the type (1.55) in a more general form (Lemma 1.43), of which we make use to establish Lemma 1.44. The subsection ends with the statement of the desired result (Thereom 1.45) and some comments regarding its consequences (Remark 1.46).

Definition 1.38. *We define the sequence $\{a_n\}_{n\in\mathbb{N}^+}$ as*

$$a_n := \frac{4Nn}{4n - N + 2}.$$

Note that $\{a_n\}_{n\in\mathbb{N}^+}$ is strictly decreasing in three dimensions and that it satisfies

$$\begin{cases} a_n = 2 & \text{for } N = 2, \\ 4 = a_1 \ge a_n > 3 & \text{for } N = 3. \end{cases} \tag{1.56}$$

for all $n \in \mathbb{N}^+$. This can be easily shown by straight forward computation.

In the next definition we give a general formula for $\nu_n(p)$, by means of which the Lipschitz exponent achieved in the n-th step can be exactly specified, see the proof of Theorem 1.45 below.

Definition 1.39. *Let Assumption 1.17.1 hold true. Given $n \in \mathbb{N}, n \geq 2$, we define*

$$\nu_n(p) := \begin{cases} \dfrac{2Np}{4Nn - (4n-N)p} & \text{if } p \in (N, a_{n-1}), \\ \in (N, \varrho) & \text{if } p = a_{n-1}, \\ \varrho & \text{if } p \in (a_{n-1}, \infty). \end{cases}$$

Further, define $\nu_0(p) := 2$ and

$$\nu_1(p) := \begin{cases} \in (N, \varrho) & \text{if } p \in (N, 4] \\ \varrho & \text{if } p \in (4, \infty) \end{cases}, \qquad \text{for } N = 2,$$

$$\nu_1(p) := \begin{cases} \dfrac{6p}{12-p} & \text{if } p \in (N, 6) \\ \varrho & \text{if } p \in [6, \infty) \end{cases}, \qquad \text{for } N = 3.$$

The notation '$:=\in (N, \varrho)$' stands for '$:= \kappa$, where $\kappa \in (N, \varrho)$ is given'.

Observe that for $n \geq 2$ and $N = 2$ it holds $\nu_n(p) = \varrho$, due to (1.56) and $p > 2$. We have chosen however to define $\nu_n(p)$, $n \geq 2$, by a formula which can be used for both space dimensions, in order to distinguish as least as possible between the cases $N = 2$ and $N = 3$ in the upcoming analysis.

The following lemma has a key role when making the connection between the Lipschitz exponents in Lemma 1.44 below.

Lemma 1.40. *Under Assumption 1.17.1, it holds for all $n \in \mathbb{N}^+$*

$$\left.\begin{aligned} 2 < \nu_n(p) < N \quad & \text{if } p \in (N, a_n), \\ \nu_n(p) = N \quad & \text{if } p = a_n, \\ N < \nu_n(p) \leq \varrho \quad & \text{if } p \in (a_n, \infty). \end{aligned}\right\} \tag{1.57}$$

Consequently, the embedding $W^{1,\nu_n(p)}(\Omega) \hookrightarrow L^{S_n(p)}(\Omega)$ is true, where

$$S_n(p) = \begin{cases} \dfrac{N\nu_n(p)}{N - \nu_n(p)} & \text{if } p \in (N, a_n), \\ \kappa & \text{if } p = a_n, \\ \infty & \text{if } p \in (a_n, \infty), \end{cases} \tag{1.58}$$

with any $\kappa \in [1, \infty)$.

Proof. We only focus on proving (1.57), as the last statement is just a result of Sobolev embeddings. For $n = 1$ and $N = 2$, (1.57) is immediately given by (1.56) and Definition 1.39, since $p > 2$. In case of $n = 1$ and $N = 3$, we observe that straight forward computation leads to the fact that $(3, 6) \ni p \mapsto \frac{6p}{12-p} \in (2, 6)$ is strictly increasing, which ensures (1.57) in view of $a_1 = 4$ and Definition 1.39. Note that $\nu_1(p) > 2$ is ensured by $p > 3$, cf. Assumption 1.17.1. Let now $n \in \mathbb{N}$, $n \geq 2$ be arbitrary, but fixed. For $N = 2$ the system (1.57) is then automatically fulfilled, due to (1.56). In fact, $\nu_n(p) = \varrho$ by Definition 1.39. Let now $N = 3$. Again, we see that $\nu_n(p) > 2$, as a result of Assumption 1.17.1. From Definition 1.39 we further deduce $\nu_n'(p) = \frac{8N^2 n}{(4Nn-(4n-N)p)^2} > 0$ for $p \in (N, a_{n-1})$, and thus, ν_n is strictly increasing on (N, a_{n-1}). Recall that for $N = 3$ we have $a_{n-1} > a_n > N$. With $\nu_n(a_n) = \frac{8N^2 n}{8Nn} = N$ and by means of Definition 1.39 we then conclude (1.57). This completes the proof. □

The next result confirms that the Lipschitz continuity of Φ improves in each boot strapping step.

Lemma 1.41. *Under Assumption 1.17.1 it holds $\nu_n(p) \geq \nu_{n-1}(p)$ for all $n \in \mathbb{N}^+$. The equality is satisfied only if $\nu_{n-1}(p) = \varrho$.*

Proof. One can easily see that $\nu_1(p) > \nu_0(p)$ in view of Definition 1.39. Note that in the two-dimensional case it holds $\nu_n(p) = \varrho \geq \nu_{n-1}(p)$ for all $n \in \mathbb{N}, n \geq 2$, since $p > 2$. In the three-dimensional case we further obtain

$$\begin{aligned} \frac{6p}{24-5p} = \nu_2(p) > \nu_1(p) = \frac{6p}{12-p} \quad & \text{if } p \in (3,4), \\ (3,\varrho) \ni \nu_2(p) > \nu_1(p) = 3 \quad & \text{if } p = a_1 = 4, \\ \varrho = \nu_2(p) \geq \nu_1(p) \quad & \text{if } p \in (4,\infty). \end{aligned}$$

Let now $n \in \mathbb{N}, n \geq 3$, be arbitrary, but fixed and $N = 3$. With $a_{n-1} < a_{n-2}$ and (1.57) we conclude

$$\begin{aligned} \frac{6p}{12n-(4n-3)p} = \nu_n(p) > \nu_{n-1}(p) = \frac{6p}{12(n-1)-(4n-7)p} \quad & \text{if } p \in (3, a_{n-1}), \\ (3,\varrho) \ni \nu_n(p) > \nu_{n-1}(p) = 3 \quad & \text{if } p = a_{n-1}, \\ \varrho = \nu_n(p) \geq \nu_{n-1}(p) \quad & \text{if } p \in (a_{n-1},\infty). \end{aligned}$$

This gives the desired assertion. □

An inspection of the proof of Theorem 1.45 below shows that, if q_Ω is small enough, the number of steps m can be considerably reduced in the three dimensional case, while in the two-dimensional case we have $m \leq 2$ anyway. We refer here to (1.67) and (1.68), respectively. The situation is far more challenging in case of $N = 3$ and $q_\Omega > \varrho$. Here a_n plays an important role, as it tells us for given p and n, if one should expect $W^{1,q}(\Omega)$-Lipschitz continuity in the near future, see Lemma 1.40. Moreover, it facilitates treating different situations separately. One sees in Lemma 1.40 that the smaller p, the smaller the improvement of the Lipschitz-exponent in the n-th step. However, since $\{a_n\}$ is strictly decreasing, the interval $(3, a_n)$ becomes smaller with each step and regardless of how small the difference between p and 3 is, one can achieve ϱ after a finite number of boot strapping steps, in light of Definition 1.39. This result is stated for both dimensions in the following

Lemma 1.42. *Under Assumption 1.17.1, it holds $\nu_{n^*}(p) = \varrho$, where*

$$n^* = \left[\frac{p(N+2) - 4N}{4(p-N)}\right] + 1. \tag{1.59}$$

Proof. First note that $\frac{p(N+2)-4N}{4(p-N)} \geq 1 \Leftrightarrow p(N-2) \geq 0$. Thus, $n^* \geq 2$. On account of Definition 1.38 we have

$$a_{n^*-1} = \frac{4N(n^*-1)}{4n^* - N - 2}.$$

By using (1.59) in the above identity we then obtain

$$a_{n^*-1} < p.$$

This leads in view of Definition 1.39 to $\nu_{n^*}(p) = \varrho$, whence the desired assertion. □

We can now proceed towards proving (1.55). Prior to this, it is useful to derive a more general estimate, of which we make use in the proof of Lemma 1.44 below.

Lemma 1.43. *Let Assumptions 1.5, 1.17 hold and let $r \in [\frac{2p}{p-2}, \infty]$ be given such that $\varphi_1, \varphi_2 \in L^r(\Omega)$. Then the following estimate holds true*

$$\|\iota\|_{W^{1,\nu'}(\Omega)^*} \leq L(\|\varphi_1 - \varphi_2\|_r + \|d_1 - d_2\|_2 + |t_1 - t_2|), \tag{1.60}$$

where $L > 0$ and

$$\frac{1}{\nu} = \max\left\{\frac{1}{r} + \frac{2}{p} - \frac{1}{N}, \frac{1}{2} - \frac{1}{N}\right\}. \tag{1.61}$$

Proof. The idea of the proof is to search for the largest $\nu \in (2, \varrho]$ such that (1.60) is fulfilled. Since $\nu' < 2 \leq N$, Sobolev embeddings imply $W^{1,\nu'}(\Omega) \hookrightarrow L^{\frac{N\nu'}{N-\nu'}}(\Omega)$. Then, by applying Lemma 1.18 and Hölder's inequality, we have in view of (1.53) for all $\psi \in W^{1,\nu'}(\Omega)$

$$\begin{aligned} |\langle \iota, \psi \rangle_{W^{1,\nu'}(\Omega)}| &\leq C(\|\varphi_1 - \varphi_2\|_r + |t_1 - t_2| + \|d_1 - d_2\|_2 + \|\varphi_1 - \varphi_2\|_2)\|\psi\|_{\frac{N\nu'}{N-\nu'}} \\ &\leq L(\|\varphi_1 - \varphi_2\|_r + \|d_1 - d_2\|_2 + |t_1 - t_2|)\|\psi\|_{W^{1,\nu'}(\Omega)}, \end{aligned} \tag{1.62}$$

provided that

$$\left.\begin{aligned} &\frac{N-\nu'}{N\nu'} + \frac{2}{p} + \frac{1}{r} \leq 1, \\ &\frac{1}{2} + \frac{N-\nu'}{N\nu'} \leq 1. \end{aligned}\right\} \tag{1.63}$$

Note that the second inequality in (1.62) follows from $r \geq 2p/(p-2) \geq 2$, or alternatively, from (1.38). The system (1.63) is further equivalent to

$$\left.\begin{aligned} &\frac{1}{\nu'} - \frac{1}{N} + \frac{2}{p} + \frac{1}{r} \leq 1, \\ &\frac{1}{2} + \frac{1}{\nu'} - \frac{1}{N} \leq 1 \end{aligned}\right\} \Longleftrightarrow \left\{\begin{aligned} &\frac{1}{r} + \frac{2}{p} - \frac{1}{N} \leq \frac{1}{\nu}, \\ &\frac{1}{2} - \frac{1}{N} \leq \frac{1}{\nu}. \end{aligned}\right.$$

By taking the definition of the $\|\cdot\|_{W^{1,\nu'}(\Omega)^*}$-norm into account and by keeping in mind that we search for ν as large as possible, we now deduce that (1.60) holds true for ν as in (1.61). $\square$

The next lemma is the essential tool for proving the $W^{1,q}(\Omega)$-Lipschitz continuity, as it connects the Lipschitz exponents of Φ achieved in consecutive boot strapping steps by (1.55).

Lemma 1.44. *Let Assumptions 1.5, 1.17 hold and let $n \in \mathbb{N}$ be given with $\nu_n(p) \leq q$. Then there exists $L > 0$ such that*

$$\|\iota\|_{W^{1,\nu_{n+1}(p)'}(\Omega)^*} \leq L(\|\varphi_1 - \varphi_2\|_{W^{1,\nu_n(p)}(\Omega)} + \|d_1 - d_2\|_2 + |t_1 - t_2|).$$

Proof. The idea of the whole proof is to find $r \geq 2p/(p-2)$ as large as possible such that $W^{1,\nu_n(p)}(\Omega) \hookrightarrow L^r(\Omega)$ and then use (1.61) in order to obtain ν for which the estimate

$$\begin{aligned} \|\iota\|_{W^{1,\nu'}(\Omega)^*} \leq & L(\|\varphi_1 - \varphi_2\|_r + \|d_1 - d_2\|_2 + |t_1 - t_2|) \\ \leq & L(\|\varphi_1 - \varphi_2\|_{W^{1,\nu_n(p)}(\Omega)} + \|d_1 - d_2\|_2 + |t_1 - t_2|) \end{aligned} \tag{1.64}$$

is true. Note that once r is found, it holds $\varphi_1, \varphi_2 \in L^r(\Omega)$, since $\varphi_1, \varphi_2 \in W^{1,q}(\Omega) \hookrightarrow W^{1,\nu_n(p)}(\Omega)$, by assumption. Thus, when applying Lemma 1.43 in what follows, we only check that $r \geq 2p/(p-2)$. In view of Lemma 1.40, we treat the cases $n = 0$ and $n \in \mathbb{N}^+$ separately.
(i) $n = 0$.
If $N = 2$, the embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ is satisfied for any $r \in [2p/(p-2), \infty)$ and therefore (1.64) holds in view of Lemma 1.43 when

$$\frac{1}{\nu} = \max\left\{\frac{1}{r} + \frac{2}{p} - \frac{1}{2}, 0\right\} \quad \forall\, r \in [2p/(p-2), \infty).$$

We now search for $r \in [2p/(p-2), \infty)$ so that $\nu = \nu_1(p)$. If $p \leq 4$, we have $\frac{1}{r} + \frac{2}{p} - \frac{1}{2} > \frac{2}{p} - \frac{1}{2} \geq 0$, in which case $\frac{1}{r} + \frac{2}{p} - \frac{1}{2} = \frac{1}{\nu} > \frac{1}{\varrho} = \frac{2}{p} - \frac{1}{2}$, by (1.51). In view of $\frac{1}{N} > \frac{1}{\varrho}$, we then choose r as large as possible such that $\frac{1}{N} > \frac{1}{\nu} > \frac{1}{\varrho}$. If $p > 4$, we have $\frac{2}{p} - \frac{1}{2} < 0$, so that $\varrho = \infty$ by (1.51). In view of the definition of $\nu_1(p)$ we choose r large enough such that $\frac{1}{r} + \frac{2}{p} - \frac{1}{2} \leq 0 = \frac{1}{\varrho}$, which leads to $\nu = \infty = \varrho$. Altogether, it follows from the above that $\nu = \nu_1(p)$, in light of Definition 1.39.
In three dimensions we obtain $r = 6 \geq 2p/(p-2)$ and (1.61) reads

$$\begin{aligned} \frac{1}{\nu} &= \max\left\{\frac{1}{6} + \frac{2}{p} - \frac{1}{3}, \frac{1}{2} - \frac{1}{3}\right\} \\ &= \max\left\{\frac{2}{p} - \frac{1}{6}, \frac{1}{6}\right\}. \end{aligned}$$

If $p < 6$, we arrive at $\nu = \nu_1(p)$, since $\frac{2}{p} - \frac{1}{6} > \frac{1}{6}$. In case of $p \geq 6$ we obtain from (1.51) that $\frac{1}{\nu} = \frac{1}{6} = \frac{1}{\varrho}$, which gives the assertion in view of Definition 1.39.

(ii) $n \in \mathbb{N}^+$ arbitrary, but fixed.
On account of Lemma 1.40 we set $r := S_n(p)$ and first check that $r \geq 2p/(p-2)$. To this end, note that, if $N = 3$, $x \mapsto Nx/(N-x)$ is strictly increasing on $[2, N)$, which in view of $\nu_n(p) \in (2, N)$ leads to $S_n(p) > 2N/(N-2) > 2p/(p-2)$ if $p \in (N, a_n)$, see also (1.32). Note further that, due to $p > N$, the interval $[2p/(p-2), \infty)$ is not empty, so that, in light of (1.58) we have $r \geq 2p/(p-2)$ in both dimensions. Now Lemma 1.43 ensures that (1.64) holds for

$$\begin{aligned}\frac{1}{\nu} &= \max\left\{\frac{1}{S_n(p)} + \frac{2}{p} - \frac{1}{N}, \frac{1}{2} - \frac{1}{N}\right\}\\ &= \begin{cases} \max\left\{\dfrac{N-\nu_n(p)}{N\nu_n(p)} + \dfrac{2}{p} - \dfrac{1}{N}, \dfrac{1}{2} - \dfrac{1}{N}\right\} & \text{if } p \in (N, a_n), \\ \max\left\{\dfrac{1}{\kappa} + \dfrac{2}{p} - \dfrac{1}{N}, \dfrac{1}{2} - \dfrac{1}{N}\right\} & \text{if } p = a_n, \\ \max\left\{\dfrac{1}{\infty} + \dfrac{2}{p} - \dfrac{1}{N}, \dfrac{1}{2} - \dfrac{1}{N}\right\} & \text{if } p \in (a_n, \infty), \end{cases}\end{aligned} \tag{1.65}$$

where $\kappa < \infty$ can be chosen large enough. We distinguish between

- $p \in (N, a_n)$: This situation can occur only in three dimensions, see (1.56). In view of (1.57) and (1.56) we have $\frac{1}{\nu_n(p)} + \frac{2}{p} - \frac{2}{N} > \frac{1}{N} + \frac{2}{p} - \frac{2}{N} = \frac{2}{p} - \frac{1}{N} > \frac{2}{a_n} - \frac{1}{N} \geq \frac{2}{a_1} - \frac{1}{N} = \frac{1}{2} - \frac{1}{N}$. This implies

  $$\frac{1}{\nu} = \frac{1}{\nu_n(p)} + \frac{2}{p} - \frac{2}{N} \Longleftrightarrow \nu = \frac{2Np}{4N(n+1) - (4n+4-N)p},$$

  in view of the definition of $\nu_n(p)$ and by keeping in mind that $a_1 = 4 < 6$ if $n = 1$ and $a_n < a_{n-1}$ in case of $n \geq 2$. Thus, by Definition 1.39, we now have $\nu = \nu_{n+1}(p)$.

- $p = a_n$: This can take place only in the three-dimensional case, in view of (1.56), which also yields $\frac{2}{a_n} \geq \frac{1}{2}$. This means that

  $$\frac{1}{\nu} = \frac{1}{\kappa} + \frac{2}{a_n} - \frac{1}{N} \quad \text{with } \kappa < \infty, \tag{1.66}$$

  and $\frac{1}{\varrho} = \frac{2}{p} - \frac{1}{N}$, on account of (1.65) and (1.51), respectively. Since $\kappa < \infty$, we have $\frac{1}{\nu} > \frac{1}{\varrho}$. Moreover, due to (1.56), it holds $\frac{2}{N} - \frac{2}{a_n} > 0$ and thus, it is possible to choose κ (large enough) such that $\frac{1}{\kappa} \in \left(0, \frac{2}{N} - \frac{2}{a_n}\right)$. Then, $\frac{1}{\kappa} + \frac{2}{a_n} - \frac{1}{N} < \frac{1}{N}$ and hence, in view of (1.66), we get $\nu > N$. Summarizing, we have $\nu \in (N, \varrho)$ and Definition 1.39 combined with (1.64) finally give the desired estimate.

- $p \in (a_n, \infty)$: In this case, the assertion is immediately given by (1.65), (1.51) and Definition 1.39.

□

We are now in the position to establish the main result of this subsection:

Theorem 1.45 (Improved Lipschitz continuity of Φ)**.** *Let Assumptions 1.5 and 1.17 hold. Then, there exists $L > 0$ such that the estimate*

$$\|\Phi(t_1, d_1) - \Phi(t_2, d_2)\|_{W^{1,q}(\Omega)} \leq L(\|d_1 - d_2\|_2 + |t_1 - t_2|)$$

holds true for all $t_1, t_2 \in [0, T]$ and all $d_1, d_2 \in L^2(\Omega)$, where $q > 2$ is given by Theorem 1.37.

Proof. Throughout this proof, $(t_i, d_i) \in [0, T] \times L^2(\Omega)$ is arbitrary, but fixed and φ_i denotes $\Phi(t_i, d_i)$ for $i = 1, 2$. We show that after executing a finite number of boot strapping steps starting from the $W^{1,\nu_0(p)}(\Omega)-$Lipschitz continuity of Φ, one obtains the $W^{1,q}(\Omega)-$Lipschitz continuity thereof. As it will turn out, the number of steps is given by

$$m := \begin{cases} \bar{n} & \text{if } \min\{q_\Omega, \varrho\} = q_\Omega (= q), \\ n^* & \text{if } \min\{q_\Omega, \varrho\} = \varrho (= q), \end{cases} \tag{1.67}$$

where

- $\bar{n} := \min\{n \in \mathbb{N} : \nu_n(p) \geq q_\Omega\}$,
- n^* as in (1.59), i.e., $\nu_{n^*}(p) = \varrho$.

We remark that

$$m \leq n^*, \tag{1.68}$$

since $n^* \geq \bar{n}$ if $q_\Omega \leq \varrho$. Moreover, note that $\{n \in \mathbb{N} : \nu_n(p) \geq q_\Omega\} \neq \emptyset$ if $q_\Omega \leq \varrho$. Otherwise, we have $\nu_n(p) < q_\Omega \leq \varrho$ for all $n \in \mathbb{N}$, which is in contradiction with Lemma 1.42. Additionally, the latter one yields on account of (1.67)

$$\begin{cases} \nu_m(p) = \nu_{\bar{n}}(p) \geq q_\Omega & \text{if } \min\{q_\Omega, \varrho\} = q_\Omega (= q), \\ \nu_m(p) = \nu_{n^*}(p) = \varrho & \text{if } \min\{q_\Omega, \varrho\} = \varrho (= q), \end{cases} \tag{1.69}$$

whence

$$\min\{\nu_m(p), q_\Omega\} = q. \tag{1.70}$$

With Lemma 1.41, (1.69) and from the definition of $\bar{n}$ we further imply

$$\nu_j(p) \leq q \quad \text{for } j \in \{0, ..., m-1\},$$

which allows us to make use of Lemma 1.44 in what follows. Let us assume that $m \geq 2$. Then, after applying $j \in [1, m-1]$ times Lemma 1.44 and (1.54), we have the improved Lipschitz condition

$$\begin{aligned} \|\varphi_1 - \varphi_2\|_{W^{1,\nu_j(p)}(\Omega)} &\leq c\|\iota\|_{W^{1,\nu_j(p)'}(\Omega)^*} \\ &\leq L(\|\varphi_1 - \varphi_2\|_{W^{1,\nu_{j-1}(p)}(\Omega)} + \|d_1 - d_2\|_2 + |t_1 - t_2|) \\ &\leq L(\|\varphi_1 - \varphi_2\|_{W^{1,\nu_0(p)}(\Omega)} + \|d_1 - d_2\|_2 + |t_1 - t_2|) \\ &\leq L(\|d_1 - d_2\|_2 + |t_1 - t_2|), \end{aligned} \tag{1.71}$$

where in the last inequality we used the already known $H^1(\Omega)$-Lipschitz continuity of Φ, cf. (1.38). We set $j := m-1$ in (1.71) and apply one last time Lemma 1.44 and (1.54). This gives in turn

$$\begin{aligned}\|\varphi_1 - \varphi_2\|_{W^{1,q}(\Omega)} &\le c\|\iota\|_{W^{1,q'}(\Omega)^*} \\ &\le c\|\iota\|_{W^{1,\nu_m(p)'}(\Omega)^*} \\ &\le L(\|\varphi_1 - \varphi_2\|_{W^{1,\nu_{m-1}(p)}(\Omega)} + \|d_1 - d_2\|_2 + |t_1 - t_2|) \\ &\le L(\|d_1 - d_2\|_2 + |t_1 - t_2|),\end{aligned}$$

where we also employed (1.70). Note that the above estimate holds true in case of $m = 1$ as well. This completes the proof. □

Remark 1.46. *The results in Theorems 1.37 and 1.45 lead to an improvement of the existence result at the end of Section 1.1 in both space dimensions, and especially if $N = 2$, as in this case we have $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$, thanks to $q > 2$. To be more precise, one can now state that the solution $(\boldsymbol{u}, \varphi, d)$ of the damage model* (P) *satisfies $\boldsymbol{u} \in C^{0,1}([0,T]; \boldsymbol{W}_D^{1,p}(\Omega))$, $\varphi \in C^{0,1}([0,T]; W^{1,q}(\Omega))$ and $d \in C^{1,1}([0,T]; L^2(\Omega))$ in two dimensions. The improved result for the displacement is due to the fact that $\mathcal{U} : [0,T] \times W^{1,q}(\Omega) \to \boldsymbol{W}_D^{1,p}(\Omega)$ is Lipschitz continuous, in view of Proposition 1.10 applied for $r = \infty$.*

Moreover, as a result of Remark 1.33, it holds $d \in C^{1,1}([0,T]; L^\infty(\Omega))$, if, in addition, $d_0 \in L^\infty(\Omega)$. This means that if $N = 2$ and $d_0 \in L^\infty(\Omega)$, then there exists $c > 0$ so that the local and nonlocal damage fulfill $d(t,x), \varphi(t,x) \le c$ a.e. in $(0,T) \times \Omega$, as a consequence of $C^{1,1}([0,T]; L^\infty(\Omega)), C^{0,1}([0,T]; W^{1,q}(\Omega)) \hookrightarrow L^\infty((0,T) \times \Omega)$. However, the constant c depends on β. This is no longer the case in Subsection 2.4 below, where the same estimate holds true in both dimensions and with c independent of β (under additional nonrestrictive assumptions). Furthermore, as explained in Remark 1.61 below, the results in this section yield in two dimensions the best differentiability result one can expect for $\mathcal{U}$, as well as for Φ, in the sense that both operators can be differentiated without norm gap.

1.3 Fréchet-differentiability of the elliptic system

This section is dedicated to studying the differentiability properties of the solution operators $\mathcal{U}$ and Φ, which were already introduced in Definitions 1.8 and 1.24. To be more specific, we aim to show that $\mathcal{U}$ and Φ are continuously Fréchet-differentiable. This is essential for deriving the results in the next chapter, in particular Lemmata 2.2 and 2.20, which will be needed when applying chain rule, see Sections 2.1 and 2.3 below. As we will see at the end of this section, these new findings also ensure the continuously differentiability in time of the optimal displacement and optimal nonlocal damage, which leads to an improved existence result for (P).

We begin by observing that in light of (1.25) the time dependency of $\mathcal{U}$ and Φ is due only to the time dependency of ℓ. Moreover, an inspection thereof shows that, within our scope of proving the continuously differentiability (with respect to time) of $\mathcal{U}$ and Φ, it is necessary to impose the following additional assumption

Assumption 1.47. *From now on we assume that the applied volume and boundary load satisfies*

$$\ell \in C^1([0,T]; \boldsymbol{W}^{-1,p}(\Omega)),$$

see also Assumptions 1.17.1 and 2.29 below.

Assumptions 1.5 and 1.17 are supposed to hold throughout this section as well.

1.3.1 Differentiability of $\mathcal{U}$

By relying on standard arguments, we establish in the sequel that $\mathcal{U}$ possesses continuous partial derivatives, which then gives in turn the main result of this subsection.

Lemma 1.48 (Partial differentiability of $\mathcal{U}$ w.r.t. time)**.** *Under Assumption 1.47, the operator $\mathcal{U}$ is partially differentiable w.r.t. time. Its partial derivative $\partial_t \mathcal{U}$ belongs to $C([0,T] \times H^1(\Omega); V)$ and satisfies the equation*

$$A_\varphi\big(\partial_t \mathcal{U}(t,\varphi)\big) = \dot{\ell}(t) \quad \textit{for all } (t,\varphi) \in [0,T] \times H^1(\Omega). \tag{1.72}$$

Proof. Let $\varphi \in H^1(\Omega)$ be arbitrary, but fixed. From Definition 1.2 and Lemma 1.3 we know that $A_\varphi^{-1} \in \mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega), \boldsymbol{W}_D^{1,p}(\Omega))$ and therefore, it is continuously Fréchet-differentiable. By employing Definition 1.8, Assumption 1.47, and chain rule, we thus obtain that $\mathcal{U}(\cdot,\varphi) : [0,T] \to \boldsymbol{W}_D^{1,p}(\Omega)$ is differentiable and the derivative fulfills (1.72). Completely analogously to the proof of Lemma 1.11 one deduces in view of Assumption 1.47 that

$$\partial_t \mathcal{U}(t_n,\varphi_n) \to \partial_t \mathcal{U}(t,\varphi) \quad \text{in } V, \tag{1.73}$$

for any sequence $\{(t_n,\varphi_n)\} \subset [0,T] \times H^1(\Omega)$ which converges towards $(t,\varphi) \in [0,T] \times H^1(\Omega)$ as $n \to \infty$. This completes the proof. $\square$

Note that as a consequence of (1.6) and (1.72), one obtains on account of Assumption 1.47 the following estimate

$$\|\partial_t \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \leq c \quad \forall\, (t,\varphi) \in [0,T] \times H^1(\Omega), \tag{1.74}$$

where $c > 0$ is independent of t and φ.

For a better overview, the partial differentiability w.r.t. φ and the continuity of the partial derivative are proven separately.

Lemma 1.49 (Partial differentiability of $\mathcal{U}$ w.r.t. φ)**.** *Let Assumptions 1.5 and 1.17.1 be fulfilled. Then there exists an index $\nu \in (2,p)$ such that, for every $t \in [0,T]$, the map $\mathcal{U}(t,\cdot) : H^1(\Omega) \to \boldsymbol{W}_D^{1,\nu}(\Omega)$ is Fréchet-differentiable. Moreover, for all $t \in [0,T]$ and all $\varphi, \delta\varphi \in H^1(\Omega)$ it holds*

$$A_\varphi\big(\partial_\varphi \mathcal{U}(t,\varphi)(\delta\varphi)\big) = \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi))\big) \quad \textit{in } \boldsymbol{W}^{-1,\nu}(\Omega). \tag{1.75}$$

Proof. We first investigate (1.75) by showing the existence of the solution operator $\mathcal{W}$ in (1.79) below, which will be the candidate for the partial derivative of $\mathcal{U}$ w.r.t. φ. Prior to this, it is useful to define some integrability exponents, of which we make use throughout this proof. We set $r := 2p/(p-2)$ and recall that Assumption 1.17.1 guarantees the existence of an index ϱ such that $r \in (2, \varrho)$ and $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$, as shown at the beginning of the proof of Lemma 1.20. Since $\varrho > r$, there exists another index κ with $r < \kappa < \varrho$, say $\kappa = (r + \varrho)/2$. Then one defines ν through

$$\frac{1}{\nu} = \frac{1}{\kappa} + \frac{1}{p}. \tag{1.76}$$

Notice that in view of $\kappa \in (r, \infty)$ we have

$$\nu \in (2, p). \tag{1.77}$$

Let now $t \in [0, T]$ and $\varphi, \delta\varphi \in H^1(\Omega)$ be arbitrary, but fixed. For the right-hand side in (1.75), Hölder's inequality with $1/\nu' + 1/\kappa + 1/p = 1$, in combination with Assumption 0.6, $H^1(\Omega) \hookrightarrow L^\kappa(\Omega)$ and Corollary 1.9 imply

$$\begin{aligned} \| \operatorname{div} \big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi))\big) \|_{\boldsymbol{W}^{-1,\nu}(\Omega)} &\leq \|g'(\varphi)\|_\infty \|\delta\varphi\|_\kappa \|\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi))\|_p \\ &\leq C\|\delta\varphi\|_\kappa. \end{aligned} \tag{1.78}$$

Due to (1.77), Lemma 1.3 is applicable with the exponent ν, which gives in turn that the operator

$$\mathcal{W} : H^1(\Omega) \ni \delta\varphi \mapsto A_\varphi^{-1} \operatorname{div} \big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi))\big) \in \boldsymbol{W}_D^{1,\nu}(\Omega) \tag{1.79}$$

is well defined. Moreover, it is linear and bounded, on account of (1.6) and (1.78), i.e.,

$$\mathcal{W} \in \mathcal{L}(H^1(\Omega), \boldsymbol{W}_D^{1,\nu}(\Omega)). \tag{1.80}$$

This operator is the candidate for the partial derivative of $\mathcal{U}$ w.r.t. φ at the point (t, φ). For this reason, we now consider the remainder term

$$R_\varphi(\delta\varphi) := \mathcal{U}(t, \varphi + \delta\varphi) - \mathcal{U}(t,\varphi) - \mathcal{W}(\delta\varphi) \tag{1.81}$$

and show an appropriate estimate therefor, which will ultimately give the claim. By employing Definitions 1.2, 1.8 and (1.79) we obtain

$$\begin{aligned} A_\varphi(R_\varphi(\delta\varphi)) &= A_\varphi\big(\mathcal{U}(t,\varphi+\delta\varphi)\big) - \ell(t) - A_\varphi\big(\mathcal{W}(\delta\varphi)\big) \\ &= A_\varphi\big(\mathcal{U}(t,\varphi+\delta\varphi)\big) - A_{\varphi+\delta\varphi}\big(\mathcal{U}(t,\varphi+\delta\varphi)\big) \\ &\quad - \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi+\delta\varphi))\big) + \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi+\delta\varphi))\big) \\ &\quad - \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi))\big) \\ &= \operatorname{div}\Big(\big(\underbrace{g(\varphi+\delta\varphi) - g(\varphi) - g'(\varphi)(\delta\varphi)}_{=:\, r_\varphi(\delta\varphi)}\big)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi+\delta\varphi))\Big) \\ &\quad + \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon\big(\mathcal{U}(t,\varphi+\delta\varphi) - \mathcal{U}(t,\varphi)\big)\big). \end{aligned}$$

Let us now define s via $1/s = 1/\nu - 1/\varrho$. Relying on (1.76), (1.77) and $\varrho > \kappa$, we infer that $1/2 > 1/s > 1/\nu - 1/\kappa = 1/p$, whence

$$2 < s < p.$$

Applying Hölder's inequality with $1/\nu = 1/\kappa + 1/p = 1/\varrho + 1/s$, in combination with Corollary 1.9, Assumption 0.6 and $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$ then gives

$$\begin{aligned}\|A_\varphi(R_\varphi(\delta\varphi))\|_{\boldsymbol{W}^{-1,\nu}(\Omega)} &\leq C\|r_\varphi(\delta\varphi)\|_\kappa \|\mathcal{U}(t,\varphi+\delta\varphi)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \\ &\quad + C\|g'(\varphi)\|_\infty \|\delta\varphi\|_\varrho \|\mathcal{U}(t,\varphi+\delta\varphi) - \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}_D^{1,s}(\Omega)} \\ &\leq C\big(\|r_\varphi(\delta\varphi)\|_\kappa + \|\delta\varphi\|_{H^1(\Omega)} \|\mathcal{U}(t,\varphi+\delta\varphi) - \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}_D^{1,s}(\Omega)}\big).\end{aligned}$$

Together with (1.6), this further implies

$$\|R_\varphi(\delta\varphi)\|_{\boldsymbol{W}_D^{1,\nu}(\Omega)} \leq C\big(\|r_\varphi(\delta\varphi)\|_\kappa + \|\delta\varphi\|_{H^1(\Omega)} \|\mathcal{U}(t,\varphi+\delta\varphi) - \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}_D^{1,s}(\Omega)}\big). \quad (1.82)$$

Moreover, we recall that $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$ with $\varrho > \kappa$, which allows us to deduce from Lemma 5.3 that $g : H^1(\Omega) \to L^\kappa(\Omega)$ is Fréchet-differentiable. Together with Lemma 1.11 and (1.82), one concludes

$$\frac{\|R_\varphi(\delta\varphi)\|_{\boldsymbol{W}_D^{1,\nu}(\Omega)}}{\|\delta\varphi\|_{H^1(\Omega)}} \leq C\Big(\frac{\|r_\varphi(\delta\varphi)\|_\kappa}{\|\delta\varphi\|_{H^1(\Omega)}} + \|\mathcal{U}(t,\varphi+\delta\varphi) - \mathcal{U}(t,\varphi)\|_{\boldsymbol{W}_D^{1,s}(\Omega)}\Big) \to 0 \quad \text{as } \|\delta\varphi\|_{H^1(\Omega)} \to 0.$$

As a result of (1.80) and (1.81), this yields the Fréchet-differentiability of $\mathcal{U}(t,\cdot) : H^1(\Omega) \to \boldsymbol{W}_D^{1,\nu}(\Omega)$ and $\partial_\varphi \mathcal{U}(t,\varphi) = \mathcal{W}$. In view of (1.79), the proof is now complete. □

Let us now derive an estimate which will be needed in the next subsection, in the proof of Lemma 1.59 below. We observe that analogously to (1.78), Hölder's inequality with $1/2 + 1/p + 1/r = 1$, where again $r = 2p/(p-2)$, leads to

$$\|\operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi))\big)\|_{V^*} \leq C\|\delta\varphi\|_r \quad \forall\, t \in [0,T],\ \forall\, \varphi, \delta\varphi \in H^1(\Omega).$$

Here we used that $H^1(\Omega) \hookrightarrow L^r(\Omega)$, by Assumption 1.17.1. Therefore, we deduce from (1.75) and (1.6) the following

Lemma 1.50. *Under Assumptions 1.5 and 1.17.1, there exists* $C > 0$ *so that*

$$\|\partial_\varphi \mathcal{U}(t,\varphi)(\delta\varphi)\|_V \leq C\,\|\delta\varphi\|_r \quad \forall\, t \in [0,T],\ \forall\, \varphi, \delta\varphi \in H^1(\Omega),$$

where $r = 2p/(p-2)$.

To prove the continuously Fréchet-differentiability of $\mathcal{U}$, we also have to establish

Lemma 1.51 (Continuity of $\partial_\varphi\mathcal{U}$)**.** *Under Assumptions 1.5 and 1.17.1 the operator* $\partial_\varphi \mathcal{U} : [0,T] \times H^1(\Omega) \to \mathcal{L}(H^1(\Omega), V)$ *is continuous.*

Proof. Let $(t_i, \varphi_i)_{i=1,2} \in [0,T] \times H^1(\Omega)$ and $\delta\varphi \in H^1(\Omega)$ be arbitrary, but fixed with $\delta\varphi \neq 0$. Further, let us abbreviate $\boldsymbol{u}_i' := \partial_\varphi \mathcal{U}(t_i, \varphi_i)\delta\varphi$ and $\boldsymbol{u}_i := \mathcal{U}(t_i, \varphi_i)$ for $i = 1, 2$. Moreover, define $f_1 := A_{\varphi_2}\boldsymbol{u}_2' - A_{\varphi_1}\boldsymbol{u}_2' \in V^*$ and $f_2 := A_{\varphi_1}\boldsymbol{u}_1' - A_{\varphi_2}\boldsymbol{u}_2' \in V^*$ such that

$$A_{\varphi_1}(\boldsymbol{u}_1' - \boldsymbol{u}_2') = f_1 + f_2. \tag{1.83}$$

Our purpose is to find an appropriate estimate for f_1 and f_2, respectively, which will then give the assertion in view of (1.6).

Thanks to Lemma 1.49, there is an index $\nu \in (2,p)$ such that $\mathcal{U}(t_2, \cdot) : H^1(\Omega) \to \boldsymbol{W}_D^{1,\nu}(\Omega)$ is Fréchet-differentiable. We set $\rho = 2\nu/(\nu - 2) \in [1, \infty)$. From Definition 1.2 and Hölder's inequality with $1/\rho + 1/\nu + 1/2 = 1$ we then have

$$\|f_1\|_{V^*} \leq C_1 \|g(\varphi_2) - g(\varphi_1)\|_\rho \|\boldsymbol{u}_2'\|_{\boldsymbol{W}_D^{1,\nu}(\Omega)} \leq C_1 \|g(\varphi_2) - g(\varphi_1)\|_\rho \, \|\delta\varphi\|_{H^1(\Omega)},$$

where for the last inequality we used (1.75), (1.78) and (1.6) applied for ν. Due to Lemma 5.2, this gives

$$\sup_{\substack{\delta\varphi \in H^1(\Omega) \\ \delta\varphi \neq 0}} \frac{\|f_1\|_{V^*}}{\|\delta\varphi\|_{H^1(\Omega)}} \to 0 \quad \text{as } \varphi_1 \to \varphi_2 \text{ in } H^1(\Omega). \tag{1.84}$$

It remains to find an estimate for f_2, which will lead to a similar result. From the definitions of $\boldsymbol{u}_i$ and $\boldsymbol{u}_i'$ it follows that

$$A_{\varphi_i}\boldsymbol{u}_i' = \operatorname{div}\big(g'(\varphi_i)(\delta\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}_i)\big) \quad \text{for } i = 1, 2,$$

in view of equation (1.75). This allows us to rewrite f_2 as

$$\begin{aligned} f_2 = &\operatorname{div}\big(g'(\varphi_1)(\delta\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}_1)\big) - \operatorname{div}\big(g'(\varphi_1)(\delta\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}_2)\big) \\ &+ \operatorname{div}\big(g'(\varphi_1)(\delta\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}_2)\big) - \operatorname{div}\big(g'(\varphi_2)(\delta\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}_2)\big). \end{aligned}$$

We again abbreviate $r := 2p/(p-2)$ and as at the beginning of the proof of Lemma 1.49, we infer that there is an index ϱ such that $r \in (2, \varrho)$ and $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$. Moreover, we define s via $1/s = 1/2 - 1/\varrho$ and observe that $s \in (2,p)$, as a result of $r > \varrho$ and $1/2 - 1/r = 1/p$. By applying Hölder's inequality with $1/\varrho + 1/s + 1/2 = 1$ and $1/r + 1/p + 1/2 = 1$, one then arrives at

$$\begin{aligned} \|f_2\|_{V^*} \leq &\, C_2 \|g'(\varphi_1)(\delta\varphi)\|_\varrho \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,s}(\Omega)} \\ &+ \|(g'(\varphi_2) - g'(\varphi_1))(\delta\varphi)\|_r \|\boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \\ \leq &\, C_2 \|\delta\varphi\|_{H^1(\Omega)} \big(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,s}(\Omega)} + \|g'(\varphi_2) - g'(\varphi_1)\|_{\mathcal{L}(H^1(\Omega), L^r(\Omega))}\big). \end{aligned}$$

Note that for the second inequality we used Assumption 0.6, the embedding $H^1(\Omega) \hookrightarrow L^\varrho(\Omega) \hookrightarrow L^r(\Omega)$, the fact that $g : H^1(\Omega) \to L^r(\Omega)$ is Fréchet-differentiable, cf. Lemma 5.3, and Corollary 1.9. Lemma 1.11 and the continuity of $g' : H^1(\Omega) \to \mathcal{L}(H^1(\Omega), L^r(\Omega))$, according to Lemma 5.3, now ensure that

$$\sup_{\substack{\delta\varphi \in H^1(\Omega) \\ \delta\varphi \neq 0}} \frac{\|f_2\|_{V^*}}{\|\delta\varphi\|_{H^1(\Omega)}} \to 0 \quad \text{as } (t_1, \varphi_1) \to (t_2, \varphi_2) \text{ in } \mathbb{R} \times H^1(\Omega). \tag{1.85}$$

Altogether, it follows from (1.83), (1.84), (1.85) and (1.6) that

$$\sup_{\substack{\delta\varphi\in H^1(\Omega)\\ \delta\varphi\neq 0}} \frac{\|\boldsymbol{u}_1' - \boldsymbol{u}_2'\|_V}{\|\delta\varphi\|_{H^1(\Omega)}} \leq C \sup_{\substack{\delta\varphi\in H^1(\Omega)\\ \delta\varphi\neq 0}} \frac{\|f_1 + f_2\|_{V^*}}{\|\delta\varphi\|_{H^1(\Omega)}} \to 0$$

for $(t_1, \varphi_1) \to (t_2, \varphi_2)$ in $\mathbb{R} \times H^1(\Omega)$. In light of the definitions of $\boldsymbol{u}_1'$ and $\boldsymbol{u}_2'$, this completes the proof. □

Remark 1.52. *We point out that it was crucial that the partial derivative of $\mathcal{U}$ w.r.t. φ (at any point, in any direction) has higher regularity than V, in order to obtain the continuity of $\partial_\varphi\mathcal{U} : [0,T] \times H^1(\Omega) \to \mathcal{L}(H^1(\Omega), V)$. This norm gap arises in the estimate for $\|f_1\|_{V^*}$ in the proof of Lemma 1.51 and is a result of the fact that g is not necessarily continuous from $H^1(\Omega)$ to $L^\infty(\Omega)$, but to $L^\rho(\Omega)$, where $\rho \in [1,\infty)$. However, by choosing ρ large enough, one can show that there is an index $\xi \in (2,\nu)$, i.e., ξ defined via $1/\xi = 1/\rho + 1/\nu \in (1/\nu, 1/2)$, so that $\|f_1\|_{V^*}$ can be replaced by $\|f_1\|_{\boldsymbol{W}^{-1,\xi}(\Omega)}$ in* (1.84). *Note that $\nu \in (2,p)$ denotes here the index from Lemma 1.49. With the exact same arguments as in the proof of Lemma 1.51, one then estimates $\|f_2\|_{\boldsymbol{W}^{-1,\xi}(\Omega)}$ correspondingly, which gives in turn that $\partial_\varphi\mathcal{U}$ is continuous as a mapping from $[0,T] \times H^1(\Omega)$ to $\mathcal{L}(H^1(\Omega), \boldsymbol{W}^{1,\xi}_D(\Omega))$, with $\xi \in (2,\nu)$, where ν is the index from Lemma 1.49.*

We are now in the position to state the main result of this subsection.

Proposition 1.53 (Fréchet-differentiability of the operator $\mathcal{U}$)**.** *Under Assumptions 1.17.1 and 1.47 it holds $\mathcal{U} \in C^1([0,T] \times H^1(\Omega); V)$.*

Proof. From Lemma 1.11 we know that $\mathcal{U} \in C([0,T] \times H^1(\Omega); V)$, while Lemmata 1.48, 1.49 and 1.51 tell us that $\mathcal{U}$ possesses partial derivatives with $\partial_t\mathcal{U} \in C([0,T] \times H^1(\Omega); V)$ and $\partial_\varphi\mathcal{U} \in C([0,T] \times H^1(\Omega); \mathcal{L}(H^1(\Omega), V))$, respectively. Hence, we can apply [9, Theorem 3.7.1], which gives $\mathcal{U} \in C^1((0,T) \times H^1(\Omega); V)$. Note that, for every $\varphi \in H^1(\Omega)$, the derivative of $\mathcal{U}$ can be continuously extended to $(0,\varphi)$ and (T,φ), due to the continuity of $\partial_t\mathcal{U}$ and $\partial_\varphi\mathcal{U}$ at $(0,\varphi)$ and (T,φ), respectively. □

Remark 1.54. *The result in Proposition 1.53 is not optimal, since actually it holds $\mathcal{U} \in C^1([0,T] \times H^1(\Omega); \boldsymbol{W}^{1,\xi}_D(\Omega))$ with an index $\xi \in (2,\nu)$, where ν is given by Lemma 1.49. To see this, first note that $\mathcal{U} : [0,T] \times H^1(\Omega) \to \boldsymbol{W}^{1,p}_D(\Omega)$ is differentiable w.r.t. time, see proof of Lemma 1.48. Moreover, the convergence* (1.73) *holds true in $\boldsymbol{W}^{1,\nu}_D(\Omega)$ as well, by Lemma 1.11, since $\nu \in (2,p)$. Hence, $\partial_t\mathcal{U} \in C([0,T] \times H^1(\Omega); \boldsymbol{W}^{1,\nu}_D(\Omega))$. In addition, cf. Remark 1.52, $\partial_\varphi\mathcal{U} : [0,T] \times H^1(\Omega) \to \mathcal{L}(H^1(\Omega), \boldsymbol{W}^{1,\xi}_D(\Omega))$ is continuous, and thus, by the same arguments as in the proof of Proposition 1.53, we deduce that $\mathcal{U}$ belongs indeed to $C^1([0,T] \times H^1(\Omega); \boldsymbol{W}^{1,\xi}_D(\Omega))$.*

Working with these integrability exponents would lead to a lack of overview in the upcoming analysis. As Proposition 1.53 proves to be sufficient for differentiating Φ (without norm gap), we do not employ in what follows the above mentioned improved differentiability results.

1.3.2 Differentiability of Φ

Although it is possible to differentiate the operator Φ by means of the same standard arguments used for differentiating $\mathcal{U}$, we choose not to do so, in view of the more complicated structure of the equation for the nonlocal damage, see (1.24) and (1.25b). Instead, we employ the implicit function theorem, see e.g. [84, Theorem 4.B(d)].

For this purpose, we introduce the following

Definition 1.55. *Let Assumptions 1.5 and 1.14 be fulfilled. We define the mapping* $\Psi : [0,T] \times L^2(\Omega) \times H^1(\Omega) \to H^1(\Omega)^*$ *by* $\Psi(t,d,\varphi) := B\varphi + F(t,\varphi) - \beta d$. *Note that* B *and* F *are given by Definition 1.15.*

By construction, it holds $\Psi(t,d,\Phi(t,d)) = 0$ for all $(t,d) \in [0,T] \times L^2(\Omega)$, in light of Definition 1.24. Our goal is to prove that Ψ satisfies all the premises of the implicit function theorem. First we focus on showing its continuously Fréchet-differentiability. To this end, we have to differentiate the Nemytskii operator g', in view of Definition 1.55. Therefor we need the following

Assumption 1.56. *In addition to Assumption 0.6, we require that* $g \in C^2(\mathbb{R})$.

The differentiability of the nonlinear part of Ψ is covered by

Lemma 1.57 (Fréchet-differentiability of F)**.** *Let Assumptions 1.17.1, 1.47 and 1.56 hold. Then the function* $F : [0,T] \times H^1(\Omega) \to H^1(\Omega)^*$ *is continuously Fréchet-differentiable. Its derivative at* $(t,\varphi) \in [0,T] \times H^1(\Omega)$ *in direction* $(\delta t, \delta\varphi) \in \mathbb{R} \times H^1(\Omega)$ *is given by*

$$\begin{aligned} \langle F'(t,\varphi)(\delta t,\delta\varphi), z\rangle_{H^1(\Omega)} &= \frac{1}{2}\int_\Omega g''(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi))z \, dx \\ &+ \int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon\big(\mathcal{U}'(t,\varphi)(\delta t,\delta\varphi)\big) z \, dx \quad \forall\, z \in H^1(\Omega). \end{aligned} \tag{1.86}$$

Proof. We begin by writing F as a product, where one of the factors is a product as well. We then prove the result in two steps, by applying Lemma 5.4 for these. For this purpose, one defines

$$\mathcal{H} : (0,T) \times H^1(\Omega) \to L^{p/2}(\Omega), \quad \mathcal{H}(t,\varphi) := \mathbb{C}\varepsilon(\mathcal{U}(t,\varphi)) : \varepsilon(\mathcal{U}(t,\varphi)) \tag{1.87}$$

and

$$\begin{aligned} &P_1 : L^\infty(\Omega) \times L^{p/2}(\Omega) \to H^1(\Omega)^*, \\ &\langle P_1(y_1,y_2), z\rangle_{H^1(\Omega)} := \frac{1}{2}\int_\Omega y_1 \cdot y_2 \cdot z \, dx \quad \forall\, z \in H^1(\Omega), \end{aligned} \tag{1.88}$$

such that

$$F : (t,\varphi) \mapsto P_1\big(g'(\varphi), \mathcal{H}(t,\varphi)\big). \tag{1.89}$$

With a little abuse of notation, the mapping P_1 considered with different domains will be denoted by the same symbol. Notice that Hölder's inequality ensures that $\mathcal{H}$ and P_1 are indeed well defined, on account of Definition 1.8 and since $H^1(\Omega) \hookrightarrow L^{2p/(p-2)}(\Omega)$ by Assumption 1.17.1, respectively.

One now proves the assertion by first applying the product rule from Lemma 5.4 to $\mathcal{H}$ in form (1.92), see below (step (i)), and then to F in form (1.89) (step (ii)).

(i) We first show that $\mathcal{H}$ is continuous as an operator with range in $L^\omega(\Omega)$ and continuously Fréchet-differentiable, if considered as an operator with range in $L^\varrho(\Omega)$. Here ω and ϱ are defined through

$$\frac{1}{\omega} = \frac{1}{p} + \frac{1}{s} \quad \text{and} \quad \frac{1}{\varrho} = \frac{1}{2} + \frac{1}{s}, \tag{1.90}$$

where $s \in (2, p)$ is arbitrary, but fixed. Notice that due to the latter one, there holds $1 < \varrho < \omega < p/2$. Concerning the continuity, we estimate similarly to (1.30) by using (1.90), which gives for all $(t_i, \varphi_i)_{i=1,2} \in (0, T) \times H^1(\Omega)$

$$\|\mathcal{H}(t_1, \varphi_1) - \mathcal{H}(t_2, \varphi_2)\|_\omega \leq C \, \|\mathcal{U}(t_1, \varphi_1) + \mathcal{U}(t_2, \varphi_2)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \|\mathcal{U}(t_1, \varphi_1) - \mathcal{U}(t_2, \varphi_2)\|_{\boldsymbol{W}_D^{1,s}(\Omega)}.$$

Lemma 1.11 in combination with Corollary 1.9 then imply the desired continuity of $\mathcal{H}$. To prove the differentiability, consider the mapping

$$P_2 : \boldsymbol{W}_D^{1,s}(\Omega) \times V \to L^\varrho(\Omega), \quad P_2(\boldsymbol{u}, \boldsymbol{v}) := \mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{v}), \tag{1.91}$$

such that

$$\mathcal{H} : (t, \varphi) \mapsto P_2\big(\mathcal{U}(t, \varphi), \mathcal{U}(t, \varphi)\big). \tag{1.92}$$

With (1.90) and (1.17) we infer that P_2 is a well defined product. In light of Lemma 5.4, we set

$$\begin{aligned} &U := (0, T) \times H^1(\Omega), \quad X := \mathbb{R} \times H^1(\Omega), \quad W := L^\varrho(\Omega), \\ &P := P_2, \quad f_i := \mathcal{U}, \quad Y_i := \boldsymbol{W}_D^{1,s}(\Omega), \quad Z_i := V, \quad i = 1, 2. \end{aligned}$$

From Lemma 1.11 and Proposition 1.53 we know that $\mathcal{U} : (0, T) \times H^1(\Omega) \to \boldsymbol{W}_D^{1,s}(\Omega)$ is continuous and $\mathcal{U} : (0, T) \times H^1(\Omega) \to V$ is continuously Fréchet-differentiable, respectively. Hence, we can apply Lemma 5.4 to (1.92) giving in turn that $\mathcal{H} : (0, T) \times H^1(\Omega) \to L^\varrho(\Omega)$ is continuously Fréchet-differentiable with

$$\mathcal{H}'(t, \varphi)(\delta t, \delta\varphi) = 2\, \mathbb{C}\varepsilon(\mathcal{U}(t, \varphi)) : \varepsilon\big(\mathcal{U}'(t, \varphi)(\delta t, \delta\varphi)\big) \tag{1.93}$$

for all $(t, \varphi) \in (0, T) \times H^1(\Omega)$ and all $(\delta t, \delta\varphi) \in \mathbb{R} \times H^1(\Omega)$.

(ii) The result from the previous step allows us now to prove the continuously Fréchet-differentiability of F. We again apply the product rule from Lemma 5.4, this time to (1.89). To fix the setting, let $\kappa > 2p/(p-2)$ satisfy

$$H^1(\Omega) \hookrightarrow L^\kappa(\Omega), \tag{1.94}$$

e.g. let $\kappa := (2p+1)/(p-2)$ for $N = 2$ and $\kappa := 6$ if $N = 3$, see also the beginning of the proof of Lemma 1.20. Since the interval $(1/p, 1/2 - 1/\kappa)$ is not empty, the number $s \in (2, p)$ from step (i) can be chosen such that

$$\frac{1}{p} < \frac{1}{s} < \frac{1}{2} - \frac{1}{\kappa} < \frac{1}{2}, \tag{1.95}$$

which is assumed in the following. Given s and κ, we then define ρ and τ via

$$\frac{1}{\rho} = \frac{1}{2} - \frac{1}{s} - \frac{1}{\kappa} \quad \text{and} \quad \frac{1}{\tau} = \frac{1}{2} - \frac{1}{p}, \quad \text{respectively.} \tag{1.96}$$

Because of (1.95), these indices satisfy

$$1 \leq \rho < \infty \quad \text{and} \quad 1 \leq \tau < \kappa. \tag{1.97}$$

In view of Lemma 5.4, we then set

$$\begin{aligned} U &:= (0,T) \times H^1(\Omega), \quad X := \mathbb{R} \times H^1(\Omega), \quad W := H^1(\Omega)^*, \\ P = P_1, \quad f_1 &:= g', \quad Y_1 := L^\rho(\Omega), \quad Z_1 := L^\tau(\Omega), \\ f_2 &:= \mathcal{H}, \quad Y_2 := L^\omega(\Omega), \quad Z_2 := L^\varrho(\Omega), \end{aligned}$$

where we considered g' with domain U, with a little abuse of notation. From step (i) we already know that $f_2 = \mathcal{H}$ fulfills the required continuity and differentiability conditions. Moreover, due to (1.97) and (1.94), Assumption 1.56 together with Lemmata 5.2 and 5.3 yields that $f_1 = g'$ is continuous from $H^1(\Omega)$ to $L^\rho(\Omega)$ and continuously Fréchet-differentiable from $H^1(\Omega)$ to $L^\tau(\Omega)$. Further, the mapping P_1 from (1.88) considered with domains $L^\tau(\Omega) \times L^\omega(\Omega)$ and $L^\rho(\Omega) \times L^\varrho(\Omega)$ is well defined, in view of Hölder's inequality applied with $1/\tau + 1/\omega + 1/\kappa < 1$ and $1/\rho + 1/\varrho + 1/\kappa = 1$, respectively. Note that therefor we used (1.94), (1.95) and (1.96) combined with (1.90), as well as (1.97). As a byproduct, the bilinear form P_1 satisfies

$$\begin{aligned} \|P_1(y_1, y_2)\|_{H^1(\Omega)^*} &\leq C\|y_1\|_\tau \|y_2\|_\omega \quad \forall\, (y_1, y_2) \in L^\tau(\Omega) \times L^\omega(\Omega), \\ \|P_1(y_1, y_2)\|_{H^1(\Omega)^*} &\leq C\|y_1\|_\rho \|y_2\|_\varrho \quad \forall\, (y_1, y_2) \in L^\rho(\Omega) \times L^\varrho(\Omega), \end{aligned}$$

and is therefore continuous on the required spaces. Thus, Lemma 5.4 gives the continuous Fréchet-differentiability of $F : (0,T) \times H^1(\Omega) \to H^1(\Omega)^*$ and (1.86), as a result of (1.88), (1.87) and (1.93). Note that the derivative of F can be extended continuously to $(0, \varphi)$ and (T, φ) for every $\varphi \in H^1(\Omega)$ due to Lemmata 1.11, 5.2, 5.3 and Proposition 1.53. The proof is now complete. $\square$

With Lemma 1.57 at hand, the continuously Fréchet-differentiability of Ψ follows immediately from Definition 1.55, by keeping in mind that $B \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$.

Corollary 1.58 (Fréchet-differentiability of Ψ)**.** *Under Assumptions 1.17.1, 1.47 and 1.56 it holds* $\Psi \in C^1([0,T] \times L^2(\Omega) \times H^1(\Omega); H^1(\Omega)^*)$.

The last result required for the application of the implicit function theorem is the following

Lemma 1.59. *Under Assumptions 1.5, 1.17 and 1.56 the operator* $\partial_\varphi \Psi(t, d, \varphi) : H^1(\Omega) \to H^1(\Omega)^*$ *is bijective for all* $(t, d, \varphi) \in [0,T] \times L^2(\Omega) \times H^1(\Omega)$.

Proof. Throughout this proof let $(t, d, \varphi) \in [0, T] \times L^2(\Omega) \times H^1(\Omega)$ be arbitrary, but fixed. On account of Definition 1.55 we have to show that for every $h \in H^1(\Omega)^*$ the equation

$$B\delta\varphi + \partial_\varphi F(t, \varphi)\delta\varphi = h \tag{1.98}$$

admits a unique solution $\delta\varphi \in H^1(\Omega)$. Clearly, when deriving Ψ w.r.t. φ, one does not need Assumption 1.47, as this was above necessary only when employing the differentiability of $\mathcal{U}$ with respect to time. We prove the unique solvability of (1.98) by means of Lax-Milgram's lemma. Thanks to $B, \partial_\varphi F(t, \varphi) \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ one obtains

$$|\langle B\delta\varphi + \partial_\varphi F(t, \varphi)(\delta\varphi), z\rangle_{H^1(\Omega)}| \le C\|\delta\varphi\|_{H^1(\Omega)}\|z\|_{H^1(\Omega)} \quad \forall\, \delta\varphi, z \in H^1(\Omega), \tag{1.99}$$

whence the boundedness of the bilinear form $B + \partial_\varphi F(t, \varphi) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$. Note that (at first glance) $C > 0$ depends on (t, φ), which in this context is not problematic. Anyway, an estimate similar to (1.100) below shows that C actually depends only on the given data. In order to prove the coercivity of $B + \partial_\varphi F(t, \varphi)$, we follow the ideas of the proof of Lemma 1.21. From (1.86) we read

$$\begin{aligned}\langle \partial_\varphi F(t, \varphi)z, z\rangle_{H^1(\Omega)} &= \frac{1}{2}\int_\Omega g''(\varphi)z\mathbb{C}\varepsilon(\mathcal{U}(t, \varphi)) : \varepsilon(\mathcal{U}(t, \varphi))z\, dx \\ &\quad + \int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t, \varphi)) : \varepsilon\big(\partial_\varphi\mathcal{U}(t, \varphi)(z)\big)z\, dx \quad \forall\, z \in H^1(\Omega).\end{aligned}$$

As frequently mentioned before, Assumption 1.17.1 guarantees that $H^1(\Omega) \hookrightarrow L^r(\Omega)$ with $r := 2p/(p-2)$, see also (1.32). Thus, Hölder's inequality with $1/r + 2/p + 1/r = 1$ and $1/p + 1/2 + 1/r = 1$, respectively, leads to

$$\begin{aligned}|\langle \partial_\varphi F(t, \varphi)z, z\rangle_{H^1(\Omega)}| &\le C\big(\|g''(\varphi)\|_\infty\|z\|_r\|\mathbb{C}\varepsilon(\mathcal{U}(t, \varphi)) : \varepsilon(\mathcal{U}(t, \varphi))\|_{\frac{p}{2}}\|z\|_r \\ &\quad + \|g'(\varphi)\|_\infty\|\varepsilon(\mathcal{U}(t, \varphi))\|_p\|\varepsilon(\partial_\varphi\mathcal{U}(t, \varphi)(z))\|_2\|z\|_r\big) \\ &\le C\, \|z\|_r^2 \quad \text{for all } z \in H^1(\Omega),\end{aligned} \tag{1.100}$$

where for the first estimate we also used Lemma 5.1. Notice that the second estimate follows from $g', g'' \in L^\infty(\mathbb{R})$, cf. Assumptions 0.6 and 1.56, respectively, Corollary 1.9, and Lemma 1.50. By Lemma 1.20, (1.100) can be continued as follows

$$|\langle \partial_\varphi F(t, \varphi)z, z\rangle_{H^1(\Omega)}| \le k\|z\|_2^2 + \widetilde{c}(k)\|z\|_{H^1(\Omega)}^2 \quad \forall\, z \in H^1(\Omega) \text{ and } \forall\, k > 0, \tag{1.101}$$

where $\widetilde{c} : \mathbb{R}^+ \to \mathbb{R}^+$ depends on C from (1.100), p, and N, and satisfies $\widetilde{c}(k) \searrow 0$ as $k \nearrow \infty$. Note that due to Remark 1.27, $\widetilde{c}$ depends only on the given data. Further, (1.101) and the definition of B in (1.22) imply for all $k > 0$ that

$$\langle Bz + \partial_\varphi F(t, \varphi)z, z\rangle_{H^1(\Omega)} \ge (\alpha - \widetilde{c}(k))\|z\|_{H^1(\Omega)}^2 + (\beta - \alpha - k)\|z\|_2^2 \quad \forall\, z \in H^1(\Omega).$$

Taking into account the characteristics of $\widetilde{c}$, we can choose $k > 0$ sufficiently large such that $\alpha - \widetilde{c}(k) \ge \alpha/2$. Moreover, if we require

$$\beta \ge \alpha + k, \tag{1.102}$$

cf. Assumption 1.17.2, we finally arrive at

$$\langle Bz + \partial_\varphi F(t,\varphi)z, z\rangle_{H^1(\Omega)} \geq \alpha/2\|z\|^2_{H^1(\Omega)} \quad \forall\, z \in H^1(\Omega), \tag{1.103}$$

i.e., the coercivity of $B+\partial_\varphi F(t,\varphi)$. In view of (1.99) and (1.103), Lax-Milgram's lemma now gives the unique solvability of (1.98), whence the desired assertion. □

We now have all the necessary tools to state the main result of this subsection.

Proposition 1.60 (Fréchet-differentiability of the operator Φ)**.** *Let Assumptions 1.17, 1.47 and 1.56 hold true. Then* $\Phi \in C^1([0,T] \times L^2(\Omega); H^1(\Omega))$ *and its derivative at* $(t,d) \in [0,T] \times L^2(\Omega)$ *in direction* $(\delta t, \delta d) \in \mathbb{R} \times L^2(\Omega)$ *satisfies*

$$B\Phi'(t,d)(\delta t,\delta d) + F'(t,\varphi)\big(\delta t, \Phi'(t,d)(\delta t,\delta d)\big) = \beta\delta d, \tag{1.104}$$

where we abbreviate $\varphi := \Phi(t,d)$.

Proof. Let $(t,d) \in (0,T) \times L^2(\Omega)$ be arbitrary, but fixed and $\varphi := \Phi(t,d)$. We apply the implicit function theorem to Ψ given by Definition 1.55, cf. e.g. [84, Theorem 4.B(d)]. Due to Corollary 1.58 and Lemma 1.59, $\Psi : (0,T) \times L^2(\Omega) \times H^1(\Omega) \to H^1(\Omega)^*$ is continuously Fréchet-differentiable and $\partial_\varphi\Psi(t,d,\varphi) : H^1(\Omega) \to H^1(\Omega)^*$ is bijective. Thus, the implicit function theorem is applicable. This implies that Φ is continuously Fréchet-differentiable from $(0,T) \times L^2(\Omega)$ to $H^1(\Omega)$. Moreover, its derivative is given by

$$\Phi'(t,d)(\delta t,\delta d) = -[\partial_\varphi\Psi(t,d,\varphi)]^{-1}\partial_{(t,d)}\Psi(t,d,\varphi)(\delta t,\delta d) \quad \forall\,(\delta t,\delta d) \in \mathbb{R} \times L^2(\Omega),$$

which is equivalent to (1.104) in view of Definition 1.55.

It remains to prove that the derivative can be continuously extended to $(0,d)$ and (T,d) for any $d \in L^2(\Omega)$. From Corollary 1.58 we know that $\partial_{(t,d)}\Psi$ and $\partial_\varphi\Psi$ can be continuously extended to $(0,d,\varphi)$, where φ denotes $\Phi(0,d)$ in the rest of the proof. Furthermore, in light of Lemma 1.59, we are allowed to define

$$\Phi'(0,d)(\delta t,\delta d) := -[\partial_\varphi\Psi(0,d,\varphi)]^{-1}\partial_{(t,d)}\Psi(0,d,\varphi)(\delta t,\delta d) \quad \forall\,(\delta t,\delta d) \in \mathbb{R} \times L^2(\Omega),$$

which is equivalent to (1.104) in case of $t = 0$. Note that the inversion $A \mapsto A^{-1}$ is continuous from the set of isomorphisms in $\mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ to $\mathcal{L}(H^1(\Omega)^*, H^1(\Omega))$, see e.g. [74, Ch. III.8], which now yields the continuity of Φ' at $(0,d)$, in view of the continuity of $\partial_{(t,d)}\Psi$, $\partial_\varphi\Psi$ at $(0,d,\varphi)$ and of Φ at $(0,d)$, cf. (1.38). The continuity of Φ' at (T,d) can then be shown in the exact same way. □

Remark 1.61. *We point out that in two dimensions one can show, by proceeding as in Subsection 1.3.1, that* $\mathcal{U} \in C^1([0,T] \times W^{1,q}(\Omega); \boldsymbol{W}^{1,p}_D(\Omega))$*, with* $q > 2$ *given by Theorem 1.37. This is mainly due to the Sobolev embedding* $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$*, the continuously Fréchet-differentiability of* $g : W^{1,q}(\Omega) \to L^\infty(\Omega)$*, as a result of Lemma 5.1, Assumption 0.6 and [21, Theorem 7], combined with the fact that* $\mathcal{U} : [0,T] \times W^{1,q}(\Omega) \to \boldsymbol{W}^{1,p}_D(\Omega)$ *is Lipschitz continuous, see Remark 1.46.*

If in addition to Assumption 1.56 it holds $g'' \in C^{0,1}(\mathbb{R})$, then $\Phi \in C^1([0,T] \times L^2(\Omega); W^{1,q}(\Omega))$, in case of $N = 2$. To see this, one follows the lines of the above proofs, with the goal to apply the implicit function theorem to $\Psi : [0,T] \times L^2(\Omega) \times W^{1,q}(\Omega) \to W^{1,q'}(\Omega)^*$. By arguing as in the proof of Lemma 1.36, one can show that actually Ψ maps $[0,T] \times L^2(\Omega) \times W^{1,q}(\Omega)$ to $W^{1,\varrho'}(\Omega)^*$, and thus it is well defined, as $\varrho \geq q$, see proof of Theorem 1.37. A closer inspection of the proof of Lemma 1.57 then shows that $F \in C^1([0,T] \times W^{1,q}(\Omega); W^{1,q'}(\Omega)^*)$, due to $\mathcal{H} \in C^1([0,T] \times W^{1,q}(\Omega); L^{p/2}(\Omega))$ and since $g' : W^{1,q}(\Omega) \to L^\infty(\Omega)$ is continuously Fréchet-differentiable, by assumption and [21, Theorem 7]. Thus, $\Psi \in C^1([0,T] \times L^2(\Omega) \times W^{1,q}(\Omega); W^{1,q'}(\Omega)^*)$, as $B \in \mathcal{L}(W^{1,q}(\Omega), W^{1,q'}(\Omega)^*)$ and $L^2(\Omega) \hookrightarrow W^{1,\varrho'}(\Omega)^* \hookrightarrow W^{1,q'}(\Omega)^*$, see (1.52)*. Further, we know from Lemma 1.59 that $\partial_\varphi \Psi(t,d,\varphi) : H^1(\Omega) \to W^{1,q'}(\Omega)^*$ is bijective for all $(t,d,\varphi) \in [0,T] \times L^2(\Omega) \times W^{1,q}(\Omega)$. Now, to ensure that $\partial_\varphi \Psi(t,d,\varphi)^{-1}$ maps $W^{1,q'}(\Omega)^*$ to $W^{1,q}(\Omega)$, we rely again on a boot strapping argument similar to Subsection 1.2.2. In each boot strapping step, the regularity of the solution of* (1.98) *(with right-hand side in $W^{1,q'}(\Omega)^*$) improves, and as in Subsection 1.2.2, the desired regularity is achieved in maximal two steps. With all these results at hand, one can prove Proposition 1.60 in this new setting completely analogously.*

We end this chapter with a final theorem on the regularity of the solution of the viscous penalized damage model:

Theorem 1.62. *Let Assumptions 1.17, 1.47 and 1.56 be fulfilled. Then there exists a unique solution $(\boldsymbol{u}, \varphi, d)$ of the problem* (P)*, satisfying $d \in C^{1,1}([0,T]; L^2(\Omega))$, $\varphi \in C^{0,1}([0,T]; W^{1,q}(\Omega)) \cap C^1([0,T]; H^1(\Omega))$, $\boldsymbol{u} \in C([0,T]; \boldsymbol{W}^{1,s}_D(\Omega)) \cap C^1([0,T]; V)$, with $q > 2$ given by Theorem 1.37 and $s \in (2,p)$. Moreover, this satisfies the system of differential equations* (1.48)*.*

Proof. At the end of Section 1.1 we already established that the unique solution of (P) satisfies (1.48), as well as a first regularity result, and in particular the regularity of the local damage d, see also Theorem 1.32. We recall that the optimal nonlocal damage and the optimal displacement are given by $\varphi = \Phi(\cdot, d(\cdot))$ and $\boldsymbol{u} = \mathcal{U}(\cdot, \varphi(\cdot))$, respectively. The additional regularity for φ follows from Theorem 1.45, the regularity of d and Proposition 1.60 combined with chain rule, while for $\boldsymbol{u}$ we employ Lemma 1.11, the fact that $\varphi \in C([0,T]; H^1(\Omega)) \cap C^1((0,T); H^1(\Omega))$, and Proposition 1.53, as well as chain rule. The resulting derivatives can be continuously extended to $t = 0$ and $t = T$, as $\dot{d}$ does so, and in view of the continuity of d, φ, and Propositions 1.60, 1.53. Therewith all the desired assertions are proven. □

Remark 1.63. *From Remark 1.61 and with arguments similar to the proof of Theorem 1.62 we infer that, in case of $N = 2$, the unique solution $(\boldsymbol{u}, \varphi, d)$ of the problem* (P) *satisfies $d \in C^{1,1}([0,T]; L^2(\Omega))$, $\varphi \in C^1([0,T]; W^{1,q}(\Omega))$, with $q > 2$ given by Theorem 1.37, and $\boldsymbol{u} \in C^1([0,T]; \boldsymbol{W}^{1,p}_D(\Omega))$, provided that $g'' \in C^{0,1}(\mathbb{R})$ holds true.*

Chapter 2

Penalization limit

In this chapter we establish the viability of the penalty approach. To be more precise, we study the behaviour of the penalized damage model for β approaching ∞ and show that in the limit both damage variables coincide. Moreover, we prove that the resulting limit model is in accordance with a class of classical damage models introduced in [17]. For convenience of the reader, we recall here the damage model with penalty

$$\left.\begin{aligned} (\boldsymbol{u}(t), \varphi(t)) \in \underset{(\boldsymbol{u},\varphi) \in V \times H^1(\Omega)}{\arg\min} \ \mathcal{E}(t, \boldsymbol{u}, \varphi, d(t)), \\ 0 \in \partial \mathcal{R}_\delta(\dot{d}(t)) + \partial_d \mathcal{E}(t, \boldsymbol{u}(t), \varphi(t), d(t)), \quad d(0) = d_0 \end{aligned}\right\} \tag{P_β}$$

for all $t \in [0, T]$. This was investigated regarding its unique solvability in the previous chapter, in the context where the penalization parameter was fixed. To emphasize the dependency on the penalty term of its unique solution, we denote this throughout this chapter by $(\boldsymbol{u}_\beta, \varphi_\beta, d_\beta)$, unless otherwise specified.

The passage to the limit $\beta \to \infty$ is performed by means of an equivalent reformulation of (P_β) in terms of an *energy identity*. The latter one plays an essential role in the present chapter, as it not only ensures the convergence of the penalized solutions, but it is also crucial for deriving the *energy inequality* which characterizes the limit damage variable. In combination with well known convex analysis results, this allows for deriving a single-field gradient damage model, that can be ultimately transformed into a classical partial damage model. To make sure that these two viscous damage models are compatible with each other, we also show, by employing a *time-discretization* technique for (P_β), that the limit damage variable is bounded a.e. in $(0, T) \times \Omega$.

The *energy inequality*, often met in the form of an identity, plays a crucial role in the context of rate-independent damage models, as various notions of solutions are based on it, such as for example energetic and local solutions, see [57] for an overview. It is also an essential tool for the vanishing viscosity limit analysis as performed in [41]. The *energy inequality* turns out to be very useful also in the context of rate-dependent damage models when it comes to proving existence of viscous solutions. We mention here the contributions of [41] (classical viscous solutions for a gradient damage model), [42] (weak viscous solutions) and [26] (weak viscous solutions for Cahn-Larché systems coupled with damage). In all these contributions, the existence of viscous solutions is shown by passing to the limit in a time-discretization scheme coupled with (a discrete version of) the energy inequality. An alternative approach consists of regularizing the viscous damage model, see [40]. The resulting (weak) solutions are then described in terms of an *energy inequality.* Our final result shows that the existence of viscous solutions may also be obtained via penalization. In other words, we show the existence of solutions (for a version of the viscous model in [41]) by following a similar approach, but unlike in the above mentioned contributions, we do not regularize nor discretize in time, but penalize.

Outline of the chapter

Within our scope of passing to the limit $\beta \to \infty$ in (P_β), we derive in Section 2.1 an equivalent formulation of the evolution in (P_β), namely the *energy identity.* This will be the starting point for the limit analysis, which is performed in Section 2.2 in two steps. We first prove that the solution of (P_β) is bounded by a constant independent of β in suitable spaces, which ensures the existence of the limit variables. We then pass to the limit in the elliptic system which characterizes the minimization problem in (P_β), whence we deduce that the local damage equals the nonlocal damage for $\beta = \infty$. We also pass to the limit in the *energy identity*, which results in an *energy inequality* that describes the evolution of the *single* limit damage variable. Based on these results, Section 2.3 deals with deriving a single-field gradient damage model in terms of an evolutionary equation and addresses the unique solvability thereof. In Section 2.4 we prove, under rather nonrestrictive assumptions, that the limit damage variable belongs to $L^\infty((0,T) \times \Omega)$. By means thereof, we establish in Section 2.5 that the single-field gradient damage model is a version of a viscous damage model analyzed in [41]. Note that throughout this chapter, Assumptions 1.17, 1.47 and 1.56 are supposed to hold true, in order to guarantee the unique solvability of (P_β) and the C^1-regularity in time of the solution, cf. Theorem 1.62.

2.1 Energy identity

The purpose of this section is to derive a characterization of the optimal local damage, which allows us to pass to the limit in (P_β) as $\beta \to \infty$. In Subsection 1.1.2 we have already seen that the evolutionary equation (1.39) can be equivalently expressed as an operator differential equation. There are also other various equivalent formulations thereof. These are collected in Proposition 2.7 below, where one sees that, excepting (2.13), they all feature the term $\beta(d_\beta - \varphi_\beta)$. This provides a significant disadvantage, as $\beta(d_\beta - \varphi_\beta)$ is not necessarily uniformly bounded w.r.t. β in suitable spaces. Hence, in the presence thereof, a passage to the limit $\beta \to \infty$ in the penalized damage model is not to be expected.

By contrast, the formulation (2.13), also known as *energy identity*, has the advantage of not containing the above mentioned problematic term. This is eliminated in the process of deriving (2.13), via a convex analysis result and chain rule, see (2.2b), (2.7) and

(2.10) below. Thereby one obtains an alternative description of the evolution (1.40), cf. Proposition 2.5 and Remark 2.6 below. As it will turn out, all the addends therein, and in particular those involving d_β, are uniformly bounded w.r.t. the penalization parameter, which is essential for the upcoming analysis. We refer here to the proof of Lemma 2.11 in Section 2.2 below. For this reason, the *energy identity* constitutes the starting point for proving the uniform boundedness w.r.t. β of the overall solution of (P_β) in the next section. Moreover, it is the fundamental tool for studying the limit behaviour $\beta \to \infty$ of the local damage d_β, see Subsection 2.2.3 below.

We show the *energy identity* by following the lines of the proof of [41, Proposition 3.2]. This consists of two main steps: rewriting the differential inclusion, in our case (1.40), as a Fenchel identity and using chain rule to substitute the left-hand side in (1.40). For the latter one, we first need to introduce the reduced energy functional and investigate its differentiability properties. We recall that, for a fixed pair $(t, d) \in [0, T] \times L^2(\Omega)$, the functional $\mathcal{E}$ admits a unique global minimizer on $V \times H^1(\Omega)$, namely $\big(\mathcal{U}(t, \Phi(t, d)), \Phi(t, d)\big)$, cf. Theorem 1.23. This gives rise to the following

Definition 2.1 (Reduced energy functional)**.** *Let Assumptions 1.5 and 1.17 hold. Then the reduced energy functional* $\mathcal{I} : [0, T] \times L^2(\Omega) \to \mathbb{R}$ *is given by*

$$\mathcal{I}(t, d) := \mathcal{E}(t, \mathcal{U}(t, \Phi(t, d)), \Phi(t, d), d).$$

On account of Definitions 0.2, 1.2 and 1.8 we can rewrite the reduced energy functional at all $(t, d) \in [0, T] \times L^2(\Omega)$ as

$$\begin{aligned}\mathcal{I}(t, d) &= \frac{1}{2}\langle A_{\Phi(t,d)}\big(\mathcal{U}(t, \Phi(t, d))\big), \mathcal{U}(t, \Phi(t, d))\rangle_V - \langle \ell(t), \mathcal{U}(t, \Phi(t, d))\rangle_V \\ &\quad + \frac{\alpha}{2}\|\nabla\Phi(t, d)\|_2^2 + \frac{\beta}{2}\|\Phi(t, d) - d\|_2^2 \\ &= -\frac{1}{2}\langle \ell(t), \mathcal{U}(t, \Phi(t, d))\rangle_V + \frac{\alpha}{2}\|\nabla\Phi(t, d)\|_2^2 + \frac{\beta}{2}\|\Phi(t, d) - d\|_2^2,\end{aligned} \tag{2.1}$$

which will come in handy in the sequel. In particular, this reformulation allows us to establish

Lemma 2.2 (Fréchet-differentiability of $\mathcal{I}$)**.** *Under Assumptions 1.17, 1.47 and 1.56, it holds* $\mathcal{I} \in C^1([0, T] \times L^2(\Omega))$ *with*

$$\partial_t \mathcal{I}(t, d) = -\langle \dot{\ell}(t), \mathcal{U}(t, \Phi(t, d))\rangle_V, \tag{2.2a}$$

$$\partial_d \mathcal{I}(t, d) = \beta(d - \Phi(t, d)) \tag{2.2b}$$

at all $(t, d) \in [0, T] \times L^2(\Omega)$.

Proof. First note that the mapping

$$f : [0, T] \times L^2(\Omega) \to \mathbb{R}, \quad f(t, d) := \langle \ell(t), \mathcal{U}(t, \Phi(t, d))\rangle_V$$

can be seen as product of the functions $[0, T] \ni t \mapsto \ell(t) \in V^*$ and $[0, T] \times L^2(\Omega) \ni (t, d) \mapsto \mathcal{U}(t, \Phi(t, d)) \in V$. The latter one is continuously Fréchet-differentiable, thanks to Propositions 1.53 and 1.60. Together with Assumption 1.47 we obtain $f \in C^1([0,T] \times L^2(\Omega))$, as a result of product rule. From the continuous Fréchet-differentiability of $\|\cdot\|_2^2 : L^2(\Omega) \to \mathbb{R}$ and $\nabla : H^1(\Omega) \to L^2(\Omega;\mathbb{R}^N)$, combined with Proposition 1.60, we can now deduce in view of (2.1) that $\mathcal{I} \in C^1([0,T] \times L^2(\Omega))$. Notice that the derivative of $\mathcal{I}$ can be continuously extended to $(0,d)$ and (T,d) for any $d \in L^2(\Omega)$, as Φ' does so and since $\mathcal{U}'$ can be continuously extended to $(0,\varphi)$ and (T,φ) for any $\varphi \in H^1(\Omega)$, cf. Proposition 1.53.

To prove (2.2), let $(t,d) \in [0,T] \times L^2(\Omega)$ be arbitrary, but fixed. By product and chain rule we obtain from (2.1) for all $(\delta t, \delta d) \in \mathbb{R} \times L^2(\Omega)$

$$\begin{aligned}\mathcal{I}'(t,d)(\delta t,\delta d) = &-\frac{1}{2}\langle \dot{\ell}(t)\delta t, \mathcal{U}(t,\Phi(t,d))\rangle_V - \frac{1}{2}\langle \ell(t), \mathcal{U}'(t,\Phi(t,d))(\delta t,\delta\varphi)\rangle_V \\ &+ \alpha(\nabla\Phi(t,d),\nabla\delta\varphi)_2 + \beta(\Phi(t,d)-d,\delta\varphi-\delta d)_2,\end{aligned} \tag{2.3}$$

where we abbreviate $\delta\varphi = \Phi'(t,d)(\delta t,\delta d)$. In view of (1.72) tested with $\mathcal{U}(t,\Phi(t,d))$ and Definitions 1.2 and 1.8 we have

$$\langle \dot{\ell}(t), \mathcal{U}(t,\Phi(t,d))\rangle_V = \langle A_{\Phi(t,d)}\partial_t\mathcal{U}(t,\Phi(t,d)), \mathcal{U}(t,\Phi(t,d))\rangle_V = \langle \ell(t), \partial_t\mathcal{U}(t,\Phi(t,d))\rangle_V,$$

whence

$$-\frac{1}{2}\langle \dot{\ell}(t), \mathcal{U}(t,\Phi(t,d))\rangle_V - \frac{1}{2}\langle \ell(t), \partial_t\mathcal{U}(t,\Phi(t,d))\rangle_V = -\langle \dot{\ell}(t), \mathcal{U}(t,\Phi(t,d))\rangle_V. \tag{2.4}$$

Relying on Definition 1.8 and (1.75) tested with $\mathcal{U}(t,\Phi(t,d)) \in V$, we further obtain

$$\begin{aligned}&-\frac{1}{2}\langle \ell(t), \partial_\varphi\mathcal{U}(t,\Phi(t,d))\delta\varphi\rangle_V + \alpha(\nabla\Phi(t,d),\nabla\delta\varphi)_2 + \beta(\Phi(t,d)-d,\delta\varphi-\delta d)_2 \\ &= -\frac{1}{2}\langle \operatorname{div}\big(g'(\Phi(t,d))(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(t,\Phi(t,d))\big), \mathcal{U}(t,\Phi(t,d))\rangle_V + \alpha(\nabla\Phi(t,d),\nabla\delta\varphi)_2 \\ &\quad + \beta(\Phi(t,d)-d,\delta\varphi-\delta d)_2 \\ &= \langle F\big(t,\Phi(t,d)\big),\delta\varphi\rangle_{H^1(\Omega)} + \langle B\Phi(t,d),\delta\varphi\rangle_{H^1(\Omega)} - \beta(d,\delta\varphi)_2 + \beta(d-\Phi(t,d),\delta d)_2 \\ &= \beta(d-\Phi(t,d),\delta d)_2 \quad \forall\, \delta d \in L^2(\Omega),\end{aligned} \tag{2.5}$$

where for the last two equalities we used Definitions 1.15 and 1.24, respectively. Inserting (2.4) and (2.5) in (2.3) finally leads to

$$\mathcal{I}'(t,d)(\delta t,\delta d) = -\langle \dot{\ell}(t)\delta t, \mathcal{U}(t,\Phi(t,d))\rangle_V + \beta(d-\Phi(t,d),\delta d)_2 \quad \forall (\delta t,\delta d) \in \mathbb{R} \times L^2(\Omega),$$

which gives the desired assertion. □

As an immediate consequence of Lemma 2.2, we have

Corollary 2.3 (Differentiability of $\mathcal{I}(\cdot,d(\cdot))$)**.** *Let $d \in C^1([0,T];L^2(\Omega))$ be given. Under Assumptions 1.17, 1.47 and 1.56, the map $[0,T] \ni t \mapsto \mathcal{I}(t,d(t))$ is continuously differentiable and it holds*

$$\frac{d}{dt}\mathcal{I}(t,d(t)) = \partial_t\mathcal{I}(t,d(t)) + (\partial_d\mathcal{I}(t,d(t)),\dot{d}(t))_2 \quad \forall\, t \in [0,T].$$

Proof. The result follows easily by chain rule, Lemma 2.2 and since the map $h : [0,T] \ni t \mapsto (t, d(t)) \in [0,T] \times L^2(\Omega)$ is continuously differentiable, in light of $d \in C^1([0,T]; L^2(\Omega))$. From $\dot h(t) = (1, \dot d(t))$ for all $t \in [0,T]$ we deduce the desired identity, which completes the proof. □

Note that in view of (2.2b) and Theorem 1.32, the optimal local damage d can also be characterized by

$$-\partial_d \mathcal{I}(t, d(t)) \in \partial \mathcal{R}_\delta(\dot d(t)) \quad \forall\, t \in [0,T], \quad d(0) = d_0. \tag{2.6}$$

The evolutionary equation (2.6) and Corollary 2.3 will be essential for proving the *energy identity* in Proposition 2.5 below. Prior to this, we state a last auxiliary result.

Lemma 2.4. *Let Assumptions 1.5, 1.17 and 1.56 hold true. Then, for any $t \in [0,T]$, we have*

$$\mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(t, d(t))\big) = \frac{\delta}{2}\|\dot d(t)\|_2,$$

where d is the unique solution of (2.6)*.*

Proof. Note that here we can go without Assumption 1.47, since this is not needed for the existence of the derivative of $\mathcal{I}$ w.r.t. d, see also proof of Lemma 2.2. We also observe that, due to the characterization of d via (1.41), it holds $\dot d(t) \geq 0$ for all $t \in [0,T]$. Since $\mathcal{R}_\delta$ is convex and proper, see Definition 0.4, we obtain from (2.6) by means of e.g. [69, Theorem 23.5] the following Fenchel identity

$$\mathcal{R}_\delta(\dot d(t)) + \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(t, d(t))\big) = (-\partial_d \mathcal{I}(t, d(t)), \dot d(t))_2 \quad \forall\, t \in [0,T]. \tag{2.7}$$

Using Definitions 0.1, 0.4 and (2.2b) in (2.7) leads to

$$\mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(t, d(t))\big) = (-\beta(d(t) - \varphi(t)), \dot d(t))_2 - \mathcal{R}_1(\dot d(t)) - \frac{\delta}{2}\|\dot d(t)\|_2^2 \quad \forall\, t \in [0,T], \tag{2.8}$$

where we abbreviate $\varphi = \Phi(\cdot, d(\cdot))$. Finally, we deduce from (1.43a) that (2.8) can be continued as

$$\mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(t, d(t))\big) = \delta\|\dot d(t)\|_2^2 - \frac{\delta}{2}\|\dot d(t)\|_2 \quad \forall\, t \in [0,T],$$

which gives the assertion. □

The main result of this section reads as follows:

Proposition 2.5 (The energy identity)**.** *Under Assumptions 1.17, 1.47 and 1.56, the unique solution d of* (2.6)*, and thus of* (1.39)*, fulfills for all $0 \leq s \leq t \leq T$ the energy identity*

$$\int_s^t \mathcal{R}_\delta(\dot d(\tau))d\tau + \int_s^t \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d(\tau))\big)d\tau + \mathcal{I}(t, d(t)) = \mathcal{I}(s, d(s)) + \int_s^t \partial_t \mathcal{I}(\tau, d(\tau))d\tau \tag{2.9}$$

and the initial datum condition $d(0) = d_0$.

Proof. First we recall that, cf. Theorem 1.32, d belongs to $C^{1,1}([0,T];L^2(\Omega))$, so that Corollary 2.3 yields the continuity of $\frac{d}{dt}\mathcal{I}(\cdot,d(\cdot))$. In combination with (2.7), this also gives the identity

$$\mathcal{R}_\delta(\dot d(\tau)) + \mathcal{R}_\delta^*\big(-\partial_d\mathcal{I}(\tau,d(\tau))\big) = \partial_t\mathcal{I}(\tau,d(\tau)) - \frac{d}{dt}\mathcal{I}(\tau,d(\tau)) \quad \forall\,\tau\in[0,T]. \tag{2.10}$$

Since $\dot d \geq 0$, as a result of (1.41), we obtain by Definition 0.4 and $\dot d \in C([0,T];L^2(\Omega))$, that the map $[0,T]\ni\tau\mapsto\mathcal{R}_\delta(\dot d(\tau))\in\mathbb{R}$ is continuous. From Lemma 2.4 and Corollary 2.3 we deduce the continuity in time of the other terms in (2.10), and therefore, the integrability thereof. From (2.10) we now arrive at

$$\int_s^t \mathcal{R}_\delta(\dot d(\tau))d\tau + \int_s^t \mathcal{R}_\delta^*\big(-\partial_d\mathcal{I}(\tau,d(\tau))\big)d\tau + \mathcal{I}(t,d(t)) = \mathcal{I}(s,d(s)) + \int_s^t \partial_t\mathcal{I}(\tau,d(\tau))d\tau$$

for all $0\leq s\leq t\leq T$, which completes the proof, as the initial datum condition is automatically fulfilled. □

Remark 2.6. *We point out that the energy identity is just another equivalent formulation of the evolution in* (2.6)*. To be more precise, one can show as in the proof of [41, Proposition 3.2], that if $d\in C^1([0,T];L^2(\Omega))$ satisfies* (2.9) *and $d(0)=d_0$, then d is the unique solution of* (2.6)*. This follows by Corollary 2.3 combined with the definition of the Fenchel conjugate, i.e. $\mathcal{R}_\delta^*(\xi)=\sup_{v\in L^2(\Omega)}(\xi,v)_2-\mathcal{R}_\delta(v)\ \forall\,\xi\in L^2(\Omega)$, which give in turn*

$$\begin{aligned}\mathcal{R}_\delta^*\big(-\partial_d\mathcal{I}(t,d(t))\big) &\geq (-\partial_d\mathcal{I}(t,d(t)),\dot d(t))_2 - \mathcal{R}_\delta(\dot d(t))\\ &= \partial_t\mathcal{I}(t,d(t)) - \frac{d}{dt}\mathcal{I}(t,d(t)) - \mathcal{R}_\delta(\dot d(t)) \quad \forall\,t\in[0,T].\end{aligned} \tag{2.11}$$

Using (2.9) *for $s=0$ and $t=T$ then leads to equality in* (2.11) *at almost all $t\in(0,T)$, which by convex analysis (see e.g. [69, Theorem 23.5]) is equivalent to the evolutionary equation in* (2.6) *f.a.a. $t\in(0,T)$. The latter one holds however for any $t\in[0,T]$, in view of the regularity of d and the equivalency of* (2.6) *(f.a.a. $t\in(0,T)$) with* (1.41)*, cf. Theorem 1.29. Since throughout this chapter we need only the implication stated in Proposition 2.5, we do not go here into more details.*

We end this section with a result which collects the various equivalent formulations of the reduced problem (P_β), namely (1.39).

Proposition 2.7. *Suppose that Assumptions 1.17, 1.47 and 1.56 hold true. Let $d\in C^1([0,T];L^2(\Omega))$ and denote $\Phi(\cdot,d(\cdot))$ by φ. Then , the following are equivalent:*

(i) d solves the reduced problem (P_β) in primal form, i.e.,

$$-\beta(d(t)-\varphi(t))\in\partial\mathcal{R}_\delta(\dot d(t)) \Leftrightarrow -\partial_d\mathcal{I}(t,d(t))\in\partial\mathcal{R}_\delta(\dot d(t)) \quad \forall\,t\in[0,T];$$

(ii) d solves the reduced problem (P_β) in dual form, i.e.,

$$\dot d(t)\in\partial\mathcal{R}_\delta^*\big(-\beta(d(t)-\varphi(t))\big) \quad \forall\,t\in[0,T]; \tag{2.12}$$

(iii) d solves the operator differential equation

$$\dot{d}(t) = \frac{1}{\delta} \max\left(-\beta(d(t) - \varphi(t)) - r\right) \quad \forall\, t \in [0, T];$$

(iv) d satisfies the following complementarity system at all $t \in [0, T]$

$$0 \leq \dot{d}(t) \perp -\beta(d(t) - \varphi(t)) - r - \delta\dot{d}(t) \leq 0 \quad a.e. \ in \ \Omega;$$

(v) d fulfills for all $0 \leq s \leq t \leq T$ *the energy identity*

$$\int_s^t \mathcal{R}_\delta(\dot{d}(\tau))d\tau + \int_s^t \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d(\tau))\big)d\tau + \mathcal{I}(t, d(t)) = \mathcal{I}(s, d(s)) + \int_s^t \partial_t \mathcal{I}(\tau, d(\tau))d\tau; \tag{2.13}$$

(vi) for all $t \in [0, T]$*,* $\dot{d}(t)$ *is the unique solution of the minimization problem*

$$\min_{\substack{v \in L^2(\Omega) \\ v \geq 0}} \mathcal{R}_\delta(v) + (\beta(d(t) - \varphi(t)), v)_2 \ ; \tag{2.14}$$

(vii) d satisfies for all $t \in [0, T]$ *the variational inequality of first and second kind*

$$\begin{aligned} \delta \int_\Omega \dot{d}(t)(v - \dot{d}(t)) \, dx + \int_\Omega \beta(d(t) - \varphi(t))(v - \dot{d}(t)) \, dx \\ + \mathcal{R}_1(v) - \mathcal{R}_1(\dot{d}(t)) \geq 0 \quad \forall\, v \in L^2(\Omega) : v \geq 0 \ a.e. \ in \ \Omega. \end{aligned} \tag{2.15}$$

Proof. The equivalence in (*i*) is due to (2.2b), as already mentioned above. The equivalence (*i*) ⇔ (*ii*) follows by convex analysis, see e.g. [69, Corollary 23.5.1], as $\mathcal{R}_\delta$ is convex, lower semicontinuous and proper. Note that the lower semicontinuity is implied by the continuity of the norms, see Definition 0.4. Moreover, we have (*i*) ⇔ (*iii*) ⇔ (*iv*) ⇔ (*v*), as a result of Theorem 1.29, the proof thereof, and Proposition 2.5 combined with Remark 2.6.

Further, the implication (*i*) ⇒ (*vii*) is a consequence of the equivalence of (1.40) with (1.43) at any $t \in [0, T]$, see the proof of Theorem 1.29. On the other hand, from (*vii*) one deduces by Definition 0.1 that $\dot{d} \geq 0$. Testing (2.15) at all $t \in [0, T]$ with 0 and $2\dot{d}(t)$, respectively, yields (1.43a), whence we have (1.43b), and thus, (1.40) at all $t \in [0, T]$. Note that therefor we used again the definition of $\mathcal{R}_1$, as well as its positive homogenity. Thus, it holds (*i*) ⇔ (*vii*). We observe that (2.15) is a VI of first and second kind, as $\mathcal{R}_1$ restricted on its domain has linear behaviour.

It remains to prove (*i*) ⇔ (*vi*). The fact that $\dot{d}(t)$ solves the minimization problem (2.14) for all $t \in [0, T]$ is equivalent with (*i*), in light of the definition of the subdifferential. Regarding the unique solvability of (2.14) for all $t \in [0, T]$, we rely on Definition 0.4, which tells us that the objective functional is strictly convex. The proof is now complete. □

2.2 Limit analysis

This section proves the viability of the penalty approach, that is, it shows that one can pass to the limit $\beta \to \infty$ in (P_β) and obtain a damage model that features only one damage variable. The latter one is nothing else as the limit of the local and nonlocal damage, respectively, as $\beta \to \infty$. In the next section we will see that the damage model without penalty is equivalent to a version of a classical viscous partial damage model which was analyzed by [41].

In the first part of the present section we entirely focus on finding bounds independent of β, in suitable spaces, for the local and nonlocal damage. Note that for the displacement such a bound is already given by Corollary 1.9. Altogether, Subsection 2.2.1 allows us to find weakly convergent subsequences. The limiting behaviour thereof, as β approaches ∞, is studied in Subsections 2.2.2 and 2.2.3. As it will turn out, the limit problem reduces to an *energy inequality*, which is in fact an *energy identity*, cf. Remark 2.27 below.

2.2.1 Uniform boundedness

Throughout this subsection $(\boldsymbol{u}, \varphi, d)$ denotes the unique solution of the problem (P_β), that is,

$$\boldsymbol{u}(t) = \mathcal{U}(t, \varphi(t)) = A_{\varphi(t)}^{-1}\ell(t), \tag{2.16a}$$

$$\varphi(t) = \Phi(t, d(t)) = \big(B + F(t, \cdot)\big)^{-1}(\beta d(t)), \tag{2.16b}$$

$$\dot{d}(t) = \frac{1}{\delta}\max\left(-\beta(d(t) - \varphi(t)) - r\right), \quad d(0) = d_0 \tag{2.16c}$$

for all $t \in [0, T]$, cf. Theorem 1.62, see also the definitions of the operators $\mathcal{U}$ and Φ, as well as Definitions 1.2 and 1.15. For later purposes, it is sufficient to recall here that $(\boldsymbol{u}, \varphi, d) \in C^1([0, T]; V) \times C^1([0, T]; H^1(\Omega)) \times C^{1,1}([0, T]; L^2(\Omega))$. The starting point for deriving the upcoming results is the *energy identity* in Proposition 2.5. To make use thereof, we need the following additional assumption, which is rather self-evident in many practical applications.

Assumption 2.8. *In the rest of this chapter we assume that at the beginning of the process the body is completely sound, i.e., $d_0 = 0$ and that there is no load acting upon the body at the initial time, i.e., $\ell(0) = 0$.*

Remark 2.9 (Active damage process)**.** *To ensure the dynamic of the system, i.e., $\dot{d} \not\equiv 0$ on $[0, T]$, one can impose various assumptions on the coefficient function g and on the load. For example, one can show by way of contradiction that $\dot{d} \not\equiv 0$, if there is a time point t for which $\ell(t) \neq 0$ and if g is monotonically decreasing on $(-\infty, 0)$ and in addition, satisfies $g'(x) \leq -C \;\forall\, x \in [0, r/\beta]$, where $C > 0$ is a constant which can be computed depending on the given data. Let us mention that the larger the value of $\|\ell\|_{C([0,T];V^*)}$, the smaller the constant C, and thus, a mild assumption on g is guaranteed when one requires that the maximal value of the load is large. We also observe that with increasing β the assumption on g becomes less restrictive. Recall that the monotonicity assumption is in any case reasonable from a practical point of view.*

Remark 2.10 (Inactive damage process)**.** *We remark that if $g'(0) = 0$ is allowed, then the solution of (P_β) is given by $(\mathcal{U}(t,0),0,0)$ at any time point $t \in [0,T]$, which in particular means that even if we assume that $\|\ell\|_{C([0,T];V^*)}$ is very large, the body remains sound the whole time.*

As a first consequence of Assumption 2.8, we obtain in view of (2.16), by employing the linearity of $A^{-1}_{\varphi(0)}$, as well as (1.22) and (1.23), that

$$\boldsymbol{u}(0) = \varphi(0) = 0, \tag{2.17a}$$

$$\dot{d}(0) = 0. \tag{2.17b}$$

This allows us to establish

Lemma 2.11 (Boundedness of the local damage)**.** *Let Assumptions 1.17, 1.47, 1.56 and 2.8 hold. Then there exists a constant $C > 0$ independent of β, such that*

$$\|d\|_{H^1(0,T;L^2(\Omega))} \leq C,$$

where d is the optimal local damage associated to (P_β).

Proof. The result follows mainly from Proposition 2.5. In order to see this, we first set $s := 0$ and $t := T$ in (2.9) and employ the fact that $\dot{d} \geq 0$, Definition 0.4, Lemma 2.4, (2.1) and (2.2a), as well as Assumption 2.8 and (2.17a). These yield the following identity

$$\underbrace{r\int_0^T \|\dot{d}(\tau)\|_1 \, d\tau}_{\geq 0} + \underbrace{\delta\int_0^T \|\dot{d}(\tau)\|_2^2 \, d\tau}_{\geq 0} - \frac{1}{2}\langle \ell(T), \boldsymbol{u}(T)\rangle_V + \underbrace{\frac{\alpha}{2}\|\nabla\varphi(T)\|_2^2}_{\geq 0} + \underbrace{\frac{\beta}{2}\|\varphi(T) - d(T)\|_2^2}_{\geq 0}$$
$$= \int_0^T \langle -\dot{\ell}(\tau), \boldsymbol{u}(\tau)\rangle_V \, d\tau. \tag{2.18}$$

Besides, thanks to Corollary 1.9 and Assumption 1.47, one has

$$\int_0^T \langle -\dot{\ell}(\tau), \boldsymbol{u}(\tau)\rangle_V \, d\tau + \frac{1}{2}\langle \ell(T), \boldsymbol{u}(T)\rangle_V \leq \int_0^T \|\dot{\ell}(\tau)\|_{V^*}\|\boldsymbol{u}(\tau)\|_V \, d\tau + \frac{1}{2}\|\ell(T)\|_{V^*}\|\boldsymbol{u}(T)\|_V$$
$$\leq C, \tag{2.19}$$

with $C > 0$ independent of β. Inserting (2.19) in (2.18) leads now to

$$\int_0^T \|\dot{d}(\tau)\|_2^2 \, d\tau \leq C, \tag{2.20}$$

where $C > 0$ is a constant independent of β. From (2.20) and $d(0) = 0$, by assumption, we infer the desired assertion from Poincaré-Friedrich's inequality, see (5.39). This completes the proof. □

In the rest of this subsection we focus on finding a constant $C > 0$ independent of β such that the estimate

$$\|\varphi\|^2_{H^1(0,T;H^1(\Omega))} \leq C \tag{2.21}$$

holds true. In view of (2.17a) and Poincaré-Friedrich's inequality, see (5.39), we only need to show that there is $C > 0$ independent of β such that

$$\|\dot{\varphi}\|^2_{L^2(0,T;H^1(\Omega))} = \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + \int_0^T \|\nabla\dot{\varphi}(\tau)\|_2^2 \, d\tau \leq C. \tag{2.22}$$

The starting point herefor is the equation which characterizes the time derivative of the nonlocal damage, see (2.23) and (2.24) below. By means thereof, we first bound the second addend in (2.22) by $\|\dot{\varphi}\|_{L^2(0,T;L^2(\Omega))}$, see Lemma 2.14 below. Relying again on (2.24), one then finds $C > 0$ independent of β such that $\|\dot{\varphi}\|^2_{L^2(0,T;L^2(\Omega))} \leq C$ holds, see Lemma 2.15 below. This will then have as immediate consequence (2.22), and thus (2.21).

Since $\varphi = \Phi(\cdot, d(\cdot))$, we get by Proposition 1.60 and chain rule that the (time) derivative of φ satisfies at all $t \in [0,T]$

$$B\dot{\varphi}(t) + \frac{d}{dt}F(t, \varphi(t)) = \beta \dot{d}(t) \quad \text{in } H^1(\Omega)^*. \tag{2.23}$$

Testing (2.23) with $\dot{\varphi}(t)$, integrating over $[0,T]$, and using (1.22) lead to

$$\int_0^T \alpha\|\nabla\dot{\varphi}(\tau)\|_2^2 \, d\tau = \beta \int_0^T (\dot{d}(\tau) - \dot{\varphi}(\tau), \dot{\varphi}(\tau))_2 \, d\tau - \int_0^T \langle \frac{d}{dt}F(\tau, \varphi(\tau)), \dot{\varphi}(\tau)\rangle \, d\tau. \tag{2.24}$$

Note that it is indeed possible to integrate over time, since all the above integrands are continuous, in light of $\dot{\varphi} \in C([0,T]; H^1(\Omega))$, $\dot{d} \in C([0,T]; L^2(\Omega))$ and Lemma 1.57.

The next two results yield suitable estimates for the first and second term on the right-hand side in (2.24), respectively, which will then allow us to deduce Lemma 2.14.

Lemma 2.12. *Under Assumptions 1.17, 1.47, 1.56 and 2.8, it holds*

$$\int_0^T (\dot{d}(\tau) - \dot{\varphi}(\tau), \dot{\varphi}(\tau))_2 \, d\tau \leq 0.$$

Proof. The proof follows mainly by the arguments employed for the proof of [40, (4.14)]. From Theorem 1.62 we recall that $\dot{d}$, d and φ are Lipschitz continuous from $[0,T]$ to $L^2(\Omega)$. Hence, the mapping $f : [0,T] \to L^2(\Omega)$ defined as

$$f(t) := \delta\dot{d}(t) + \beta(d(t) - \varphi(t)) + r \tag{2.25}$$

belongs to $W^{1,\infty}(0,T; L^2(\Omega))$, in view of [77, Proposition 2.28b)], and thus, it is almost everywhere differentiable, cf. [81, Theorem 3.1.40]. Let now $t \in (0,T)$ be arbitrary, but fixed and h small enough such that $t+h \in (0,T)$. From Proposition 2.7, $(i) \Rightarrow (iv)$, we deduce

$$(f(t+h), \dot{d}(t))_2 \geq 0, \tag{2.26a}$$

$$(f(t), \dot{d}(t))_2 = 0, \tag{2.26b}$$

where for the above inequality we used $\dot{d}(t) \geq 0$. Relying on (2.26) one arrives at

$$\begin{cases} \frac{1}{h}(f(t+h) - f(t), \dot{d}(t))_2 \geq 0 & \text{for } h > 0, \\ \frac{1}{h}(f(t+h) - f(t), \dot{d}(t))_2 \leq 0 & \text{for } h < 0. \end{cases}$$

Passing to the limits $h \searrow 0$ and $h \nearrow 0$, respectively, and keeping in mind the fact that f is almost everywhere differentiable further implies

$$(\dot{f}(t), \dot{d}(t))_2 = 0 \quad \text{f.a.a. } t \in (0,T). \tag{2.27}$$

With (2.25) we rewrite (2.27) as

$$\delta(\ddot{d}(t), \dot{d}(t))_2 + (\beta(\dot{d}(t) - \dot{\varphi}(t)), \dot{d}(t))_2 = 0,$$

which can be continued as

$$\frac{\delta}{2}\frac{d}{dt}\|\dot{d}(t)\|_2^2 + \beta\|\dot{d}(t) - \dot{\varphi}(t))\|_2^2 + \beta(\dot{d}(t) - \dot{\varphi}(t), \dot{\varphi}(t))_2 = 0 \tag{2.28}$$

for almost all $t \in (0,T)$, see also [81, Lemma 3.1.43]. Due to $\dot{f} \in L^\infty(0,T;L^2(\Omega))$ and $\dot{d} \in C([0,T];L^2(\Omega))$ we can integrate (2.27), i.e., (2.28), over $[0,T]$. This finally yields

$$\frac{\delta}{2}\|\dot{d}(T)\|_2^2 - \frac{\delta}{2}\|\dot{d}(0)\|_2^2 + \beta\int_0^T \|\dot{d}(\tau) - \dot{\varphi}(\tau))\|_2^2 \, d\tau + \beta\int_0^T (\dot{d}(\tau) - \dot{\varphi}(\tau), \dot{\varphi}(\tau))_2 \, d\tau = 0,$$

which on account of (2.17b) gives the assertion. □

Lemma 2.13. *Let Assumptions 1.17, 1.47 and 1.56 hold true. Then, for all $t \in [0,T]$, it holds*

$$-\langle \frac{d}{dt}F(t,\varphi(t)), \dot{\varphi}(t)\rangle_{H^1(\Omega)} \leq \widetilde{c}(k)\|\dot{\varphi}(t)\|_{H^1(\Omega)}^2 + k\|\dot{\varphi}(t)\|_2^2 + Ck \quad \forall\, k > 0,$$

where $\widetilde{c} : \mathbb{R}^+ \to \mathbb{R}^+$ is a function independent of β, which satisfies $\widetilde{c}(k) \searrow 0$ as $k \nearrow \infty$, and $C > 0$ is a constant independent of β.

Proof. Let $t \in [0,T]$ be arbitrary, but fixed. From Lemma 1.57 we know that $F \in C^1([0,T] \times H^1(\Omega); H^1(\Omega)^*)$, so that chain rule gives

$$\langle \frac{d}{dt}F(t,\varphi(t)), \dot{\varphi}(t)\rangle_{H^1(\Omega)} = \langle \partial_t F(t,\varphi(t)), \dot{\varphi}(t)\rangle_{H^1(\Omega)} + \langle \partial_\varphi F(t,\varphi(t))\dot{\varphi}(t), \dot{\varphi}(t)\rangle_{H^1(\Omega)}, \tag{2.29}$$

in view of $\varphi \in C^1([0,T];H^1(\Omega))$. From (1.86) we read

$$\langle \partial_t F(t,\varphi(t)), \dot\varphi(t)\rangle = -\langle \operatorname{div}\big(g'(\varphi(t))\dot\varphi(t)\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big), \partial_t \mathcal{U}(t,\varphi(t))\rangle_V.$$

Due to $p > N$, by Assumption 1.17.1, we have $H^1(\Omega) \hookrightarrow L^{\frac{2p}{p-2}}(\Omega)$, see (1.32). Now Hölder's inequality with $(p-2)/2p + 1/p + 1/2 = 1$ and (3) yield

$$\begin{aligned} |\langle \partial_t F(t,\varphi(t)), \dot\varphi(t)\rangle_{H^1(\Omega)}| &\le \|g'(\varphi(t))\|_\infty \|\dot\varphi(t)\|_{\frac{2p}{p-2}} \|\boldsymbol{u}(t)\|_{\boldsymbol{W}^{1,p}_D(\Omega)} \|\partial_t \mathcal{U}(t,\varphi(t))\|_V \\ &\le C\|\dot\varphi(t)\|_{H^1(\Omega)}, \end{aligned} \tag{2.30}$$

with $C > 0$ independent of β. Note that for the last estimate we used Corollary 1.9 and (1.74), where the constant depends only on the norm of A_φ^{-1} (see (1.6)) and $\dot\ell$, and it is thus independent of β. On account of the generalized Young inequality, (2.30) can be continued as follows

$$|\langle \partial_t F(t,\varphi(t)), \dot\varphi(t)\rangle_{H^1(\Omega)}| \le \frac{1}{4k}\|\dot\varphi(t)\|^2_{H^1(\Omega)} + Ck \quad \forall\, k > 0. \tag{2.31}$$

For the second term in (2.29), one obtains from (1.101) the estimate

$$|\langle \partial_\varphi F(t,\varphi(t))\dot\varphi(t), \dot\varphi(t)\rangle_{H^1(\Omega)}| \le k\|\dot\varphi(t)\|_2^2 + \widetilde{c_1}(k)\|\dot\varphi(t)\|^2_{H^1(\Omega)} \quad \forall\, k > 0, \tag{2.32}$$

where $\widetilde{c_1} : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies $\widetilde{c_1}(k) \searrow 0$ as $k \nearrow \infty$ and depends on the constant C from (1.100), p and N. A close inspection of the proof of (1.100) and in particular of the proof of Lemma 1.50 shows that $\widetilde{c_1}$ is a function independent of β. By inserting (2.31) and (2.32) in (2.29) we get

$$-\langle \frac{d}{dt}F(t,\varphi(t)), \dot\varphi(t)\rangle_{H^1(\Omega)} \le \underbrace{\Big(\frac{1}{4k} + \widetilde{c_1}(k)\Big)}_{=:\,\widetilde{c}(k)}\|\dot\varphi(t)\|^2_{H^1(\Omega)} + k\|\dot\varphi(t)\|_2^2 + Ck \quad \forall\, k > 0.$$

This gives the assertion, in view of the properties of $\widetilde{c_1}$ and since C was independent of β. □

As a result of Lemmata 2.12 and 2.13 we obtain

Lemma 2.14 (Boundedness of the gradient)**.** *Let Assumptions 1.17, 1.47, 1.56 and 2.8 hold. Then there exist constants $C_1, C_2 > 0$ independent of β such that*

$$\int_0^T \|\nabla\dot\varphi(\tau)\|_2^2\, d\tau \le C_1 \int_0^T \|\dot\varphi(\tau)\|_2^2\, d\tau + C_2.$$

Proof. Applying Lemmata 2.12 and 2.13 for the right-hand side in (2.24) leads to

$$\int_0^T \alpha\|\nabla\dot\varphi(\tau)\|_2^2\, d\tau \le \widetilde{c}(k)\int_0^T \|\dot\varphi(\tau)\|^2_{H^1(\Omega)}\, d\tau + k\int_0^T \|\dot\varphi(\tau)\|_2^2\, d\tau + Ck \quad \forall\, k > 0, \tag{2.33}$$

with $\widetilde{c}$ as in Lemma 2.13 and $C > 0$ independent of β. We employ the properties of $\widetilde{c}$ and set k (large enough) such that $\widetilde{c}(k) < \alpha$ holds. Subtracting the term $\widetilde{c}(k) \int_0^T \|\nabla\dot{\varphi}(\tau)\|_2^2 \, d\tau$ on both sides in (2.33) then gives

$$\underbrace{(\alpha - \widetilde{c}(k))}_{>0} \int_0^T \|\nabla\dot{\varphi}(\tau)\|_2^2 \, d\tau \le (\widetilde{c}(k) + k) \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + Ck.$$

Since the (above fixed) value of k depends only on $\widetilde{c}$ and α, and since $\widetilde{c}$ is independent of β cf. Lemma 2.13, and C as well, the proof is now complete. □

Lemma 2.15 (Boundedness of the L^2-component)**.** *Let Assumptions 1.17, 1.47, 1.56 and 2.8 hold. Then there exists a constant $C > 0$ independent of β such that*

$$\int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau \le C.$$

Proof. We begin with dividing (2.24) by β and bringing the term $\int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau$ on the left-hand side. This results in

$$\begin{aligned}\underbrace{\frac{\alpha}{\beta} \int_0^T \|\nabla\dot{\varphi}(\tau)\|_2^2 \, d\tau}_{\ge 0} + \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau = &\int_0^T (\dot{d}(\tau), \dot{\varphi}(\tau))_2 \, d\tau \\ &- \frac{1}{\beta} \int_0^T \langle \frac{d}{dt} F(\tau, \varphi(\tau)), \dot{\varphi}(\tau) \rangle \, d\tau,\end{aligned}$$

whence the estimate

$$\int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau \le \int_0^T (\dot{d}(\tau), \dot{\varphi}(\tau))_2 \, d\tau - \frac{1}{\beta} \int_0^T \langle \frac{d}{dt} F(\tau, \varphi(\tau)), \dot{\varphi}(\tau) \rangle \, d\tau. \tag{2.34}$$

Using generalized Young inequality for the first term on the right-hand side in (2.34) further yields

$$\int_0^T (\dot{d}(\tau), \dot{\varphi}(\tau))_2 \, d\tau \le j \int_0^T \|\dot{d}(\tau)\|_2^2 \, d\tau + \frac{1}{4j} \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau \quad \forall j > 0. \tag{2.35}$$

In order to estimate the second term on the right-hand side in (2.34), we apply Lemma 2.13 for some fixed $k > 0$. Thanks to Lemma 2.14, we then arrive at

$$\begin{aligned}-\frac{1}{\beta} \int_0^T \langle \frac{d}{dt} F(\tau, \varphi(\tau)), \dot{\varphi}(\tau) \rangle \, d\tau &\le \frac{C_1}{\beta} \int_0^T \|\nabla\dot{\varphi}(\tau)\|_2^2 \, d\tau + \frac{C_2}{\beta} \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + \frac{C_3}{\beta} \\ &\le \frac{C_1}{\beta} \Big(C_4 \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + C_5 \Big) + \frac{C_2}{\beta} \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + \frac{C_3}{\beta} \\ &\le \frac{\widetilde{C}}{\beta} \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + \frac{C}{\beta},\end{aligned} \tag{2.36}$$

where we abbreviate $\widetilde{C} := C_1C_4 + C_2$ and $C := C_1C_5 + C_3$. Note that C and $\widetilde{C}$ are independent of β on account of Lemmata 2.13 and 2.14. Further, inserting (2.35) and (2.36) in (2.34) implies

$$\int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau \leq j \int_0^T \|\dot{d}(\tau)\|_2^2 \, d\tau + \Big(\frac{1}{4j} + \frac{\widetilde{C}}{\beta}\Big) \int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau + \frac{C}{\beta} \quad \forall j > 0. \tag{2.37}$$

Now, we set $j > 1/2$ and impose that β satisfies

$$\beta > 2\widetilde{C}, \tag{2.38}$$

cf. Assumption 1.17.2, such that $1/4j + \widetilde{C}/\beta < 1$, which in view of (2.37) gives in turn

$$\int_0^T \|\dot{\varphi}(\tau)\|_2^2 \, d\tau \leq K_1 \int_0^T \|\dot{d}(\tau)\|_2^2 \, d\tau + K_2,$$

where $K_1, K_2 > 0$ are constants independent of β. Due to Lemma 2.11, the desired estimate is now proven. □

As a result of Lemmata 2.14 and 2.15, and by relying on Poincaré-Friedrich's inequality and (2.17a), see also (5.39), we can now conclude

Corollary 2.16 (Boundedness of the nonlocal damage)**.** *Under Assumptions 1.17, 1.47, 1.56 and 2.8, there exists a constant $C > 0$ independent of β such that*

$$\|\varphi\|_{H^1(0,T;H^1(\Omega))} \leq C,$$

where φ is the optimal nonlocal damage associated to (P_β).

2.2.2 Passage to the limit in the elliptic system

We begin studying the behaviour of the penalized model (P_β) as β approaches ∞ with the elliptic system given by (2.16a)-(2.16b). In order to emphasize the presence of the penalty term, we denote from now on by $(\boldsymbol{u}_\beta, \varphi_\beta, d_\beta)$ the unique solution of the problem (P_β).

Proposition 2.17 (Passage to the limit in (2.16a))**.** *Let Assumptions 1.17.1, 1.47, 1.56 and 2.8 hold. Then, for every sequence $\beta_n \to \infty$ as $n \to \infty$, there exists a (not relabeled) subsequence $\{\varphi_{\beta_n}\}_{n\in\mathbb{N}}$ and $\varphi \in H^1(0,T;H^1(\Omega))$ such that the following converges hold*

$$\varphi_{\beta_n} \rightharpoonup \varphi \quad \text{in } H^1(0,T;H^1(\Omega)), \tag{2.39}$$

$$\boldsymbol{u}_{\beta_n} \to \mathcal{U}(\cdot, \varphi(\cdot)) =: \boldsymbol{u} \quad \text{in } C([0,T];V) \tag{2.40}$$

as $n \to \infty$, where $\boldsymbol{u}_{\beta_n}$ and φ_{β_n} stand for the optimal displacement and optimal nonlocal damage, respectively, associated to the problem (P_{β_n}).

Proof. Since $H^1(0,T;H^1(\Omega))$ is a reflexive Banach space, see e.g. [81, Theorem 3.1.36], there exists in view of Corollary 2.16 a (not relabeled) weakly convergent subsequence of $\{\varphi_{\beta_n}\}_{n\geq m}$, i.e., there exists $\varphi \in H^1(0,T;H^1(\Omega))$ such that (2.39) holds. Here $m \in \mathbb{N}$ denotes an index so that β_n fulfills Assumption 1.17.2 for all $n \geq m$. Let us now abbreviate $r := 2p/(p-2)$ such that $1/2 = 1/p + 1/r$ holds. Note that due to Assumption 1.17.1, see (1.32), we have the compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$, which by [81, Corollary 3.1.42] implies the compact embedding $H^1(0,T;H^1(\Omega)) \hookrightarrow\hookrightarrow C([0,T];L^r(\Omega))$. Therewith we deduce from (2.39) the convergence

$$\varphi_{\beta_n} \to \varphi \text{ in } C([0,T];L^r(\Omega)) \quad \text{as } n \to \infty. \tag{2.41}$$

Additionally, as a result of Proposition 1.10, we have at all $t \in [0,T]$ the estimate

$$\|\mathcal{U}(t,\varphi_{\beta_n}(t)) - \mathcal{U}(t,\varphi(t))\|_V \leq L\|\varphi_{\beta_n}(t) - \varphi(t)\|_r \quad \forall\, n \in \mathbb{N}, \tag{2.42}$$

where $L > 0$ is a constant independent of n, since β does not appear in the equation associated to the solution operator $\mathcal{U}$, i.e., equation (1.48a), see also the proof of Proposition 1.10. Since $\boldsymbol{u}_{\beta_n} = \mathcal{U}(\cdot,\varphi_{\beta_n}(\cdot))$, (2.40) follows now from (2.41) and (2.42). □

Proposition 2.18 (Passage to the limit in (2.16b))**.** *Under Assumptions 1.17.1, 1.47, 1.56 and 2.8 it holds*

$$d_{\beta_n} \rightharpoonup \varphi \textit{ in } H^1(0,T;L^2(\Omega)) \quad \textit{as } n \to \infty,$$

where $\{\beta_n\}_{n\in\mathbb{N}}$ *denotes the subsequence from Proposition 2.17 and* φ *the corresponding limit, while* d_{β_n} *stands for the optimal local damage associated to the problem* (P_{β_n})*. This implies in particular that the local and nonlocal damage coincide in the limit.*

Proof. Let $n \in \mathbb{N}$ and $t \in [0,T]$ be arbitrary, but fixed. We start by showing a convergence result for the sequence given by $\xi_{\beta_n} := d_{\beta_n} - \varphi_{\beta_n} \in C^1([0,T];L^2(\Omega))$. From (2.16b) we know that

$$\beta_n \int_\Omega \xi_{\beta_n}(t)\psi \, dx = \int_\Omega \alpha\nabla\varphi_{\beta_n}(t)\nabla\psi \, dx + \langle F(t,\varphi_{\beta_n}(t)),\psi\rangle \quad \forall\, \psi \in H^1(\Omega). \tag{2.43}$$

Moreover, in view of Assumption 1.17.1 we have $H^1(\Omega) \hookrightarrow L^{\frac{p}{p-2}}(\Omega)$, which combined with Hölder's inequality and (1.23) gives in turn

$$\begin{aligned}
&\frac{1}{\beta_n}\Big|\int_\Omega \alpha\nabla\varphi_{\beta_n}(t)\nabla\psi \, dx\Big| + \frac{1}{\beta_n}\big|\langle F(t,\varphi_{\beta_n}(t)),\psi\rangle_{H^1(\Omega)}\big| \\
&\qquad \leq \frac{1}{\beta_n}\Big(\alpha\|\nabla\varphi_{\beta_n}(t)\|_2 + \frac{1}{2}\|g'(\varphi_{\beta_n}(t))\|_\infty\|\mathbb{C}\varepsilon(\boldsymbol{u}_{\beta_n}(t)) : \varepsilon(\boldsymbol{u}_{\beta_n}(t))\|_{\frac{p}{2}}\Big)\|\psi\|_{H^1(\Omega)} \\
&\qquad \leq \frac{C}{\beta_n}\|\psi\|_{H^1(\Omega)} \quad \forall\, \psi \in H^1(\Omega),
\end{aligned} \tag{2.44}$$

where $C > 0$ is independent of n. Note that for the last inequality we used the embedding $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$ combined with Corollary 2.16, as well as (3) and Corollary 1.9. Now (2.43) and (2.44) lead to

$$|\langle \xi_{\beta_n}(t), \psi \rangle_{H^1(\Omega)}| \leq \frac{C}{\beta_n} \|\psi\|_{H^1(\Omega)} \quad \forall\, \psi \in H^1(\Omega),$$

whence

$$\xi_{\beta_n} \to 0 \quad \text{in } C([0,T];H^1(\Omega)^*) \quad \text{as } n \to \infty. \tag{2.45}$$

By employing (2.41) and $L^{2p/(p-2)}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^1(\Omega)^*$ we further obtain

$$\varphi_{\beta_n} \to \varphi \quad \text{in } C([0,T];H^1(\Omega)^*) \quad \text{as } n \to \infty,$$

which together with (2.45) now gives

$$d_{\beta_n} \to \varphi \quad \text{in } C([0,T];H^1(\Omega)^*) \quad \text{as } n \to \infty. \tag{2.46}$$

On the other hand, as a result of Lemma 2.11, there exists a subsequence of $\{d_{\beta_n}\}_{n\in\mathbb{N}}$ and $d \in H^1(0,T;L^2(\Omega))$ such that

$$d_{\beta_{n_k}} \rightharpoonup d \text{ in } H^1(0,T;L^2(\Omega)) \quad \text{as } k \to \infty.$$

Thus, by the embedding $H^1(0,T;L^2(\Omega)) \hookrightarrow C([0,T];H^1(\Omega)^*)$ and (2.46) we arrive at $d = \varphi$. Arguing as above, one can show that any weakly convergent subsequence of $\{d_{\beta_n}\}_{n\in\mathbb{N}}$ possesses the same limit φ. This completes the proof. □

Let us conclude this subsection by pointing out the herein achieved results. It was established that after passing to the limit $\beta \to \infty$ in (2.16a), i.e., (1.48a), one obtains in light of (2.40) the same equation for the coupling between displacement and nonlocal damage. Further, it turns out that the equation (2.16b) for the nonlocal damage, i.e., (1.48b), reduces to the identity $d = \varphi$, where d and φ stand for the local and nonlocal damage after passing to the limit, respectively. Thus, both damage variables become equal when $\beta \to \infty$, which shows that the penalty approach makes sense from a mathematical point of view. Keep in mind that (at this point), the limit model depends on the chosen subsequence in Proposition 2.17. In the next section we will see that under special conditions, no matter what subsequence one chooses, the limit is the same and consequently the whole sequences $\{\varphi_{\beta_n}\}$ (and thus, $\{d_{\beta_n}\}$) and $\{\boldsymbol{u}_{\beta_n}\}$ converge for any $\beta_n \to \infty$ as $n \to \infty$.

2.2.3 Passage to the limit in the energy identity

After we exploited the behaviour of the elliptic system when $\beta \to \infty$, we now turn our attention to the dynamical component of the penalized damage model, i.e., (2.16c). However, as already indicated at the beginning of Section 2.1, we do not possess any useful information about the boundedness of the term $\beta(d_\beta - \varphi_\beta)$. That is why we again make use of the characterization of the local damage by means of the *energy identity*, see

(2.9), which has the advantage of not including this term. Passing to the limit therein will result in an *energy inequality*, where the original functionals slightly change, see Definitions 2.19 and 2.21 below. Moreover, as we will see in Section 2.3 below, this is equivalent to an evolutionary equation. Besides, the resulting *energy inequality* is in fact an *energy identity* as well, where, as in (2.9), one integrates over an arbitrary interval $[s,t] \subset [0,T]$, see Remark 2.27 below.

We begin by introducing the functionals that will arise after passing to the limit. We first define the energy functional without penalty $\widetilde{\mathcal{E}} : [0,T] \times V \times H^1(\Omega) \to \mathbb{R}$ by

$$\widetilde{\mathcal{E}}(t,\boldsymbol{u},\varphi) := \frac{1}{2}\int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\,dx - \langle \ell(t), \boldsymbol{u}\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2, \tag{2.47}$$

which has rather the purpose of motivating Definition 2.19 below. By comparing Definition 0.2 and (2.47) it is easy to see that at all $(t,\varphi,d) \in [0,T] \times H^1(\Omega) \times L^2(\Omega)$ it holds

$$\arg\min_{\boldsymbol{u}\in V} \widetilde{\mathcal{E}}(t,\boldsymbol{u},\varphi) = \arg\min_{\boldsymbol{u}\in V} \mathcal{E}(t,\boldsymbol{u},\varphi,d).$$

Hence, for a given pair $(t,\varphi) \in [0,T] \times H^1(\Omega)$, the functional $\widetilde{\mathcal{E}}$ reaches its minimum with respect to the variable $\boldsymbol{u}$ at $\mathcal{U}(t,\varphi)$, see also Proposition 1.7 and Definition 1.8. Therefore, after minimizing one obtains

Definition 2.19 (Reduced energy functional without penalty)**.** *We define the functional* $\widetilde{\mathcal{I}} : [0,T] \times H^1(\Omega) \to \mathbb{R}$ *as*

$$\widetilde{\mathcal{I}}(t,\varphi) := \min_{\boldsymbol{u}\in V} \widetilde{\mathcal{E}}(t,\boldsymbol{u},\varphi) = \widetilde{\mathcal{E}}(t,\mathcal{U}(t,\varphi),\varphi).$$

Note that Assumption 1.5 is not needed in order to define $\widetilde{\mathcal{I}}$, as $\mathcal{E}$, and thus $\widetilde{\mathcal{E}}$, has an unique minimum w.r.t. $\boldsymbol{u}$ in V characterized by (1.14) also when ℓ maps only to V^*. This is shown by a short inspection of the proof of Proposition 1.7. The only difference is that the solution operator $\mathcal{U}$ does no longer have range in $\boldsymbol{W}^{1,p}_D(\Omega)$, but only in V.

On account of (2.47) and Definitions 1.2 and 1.8 we can rewrite the reduced energy functional without penalty at all $(t,\varphi) \in [0,T] \times H^1(\Omega)$ as

$$\begin{aligned}\widetilde{\mathcal{I}}(t,\varphi) &= \frac{1}{2}\langle A_\varphi\mathcal{U}(t,\varphi),\mathcal{U}(t,\varphi)\rangle_V - \langle \ell(t),\mathcal{U}(t,\varphi)\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2 \\ &= -\frac{1}{2}\langle \ell(t),\mathcal{U}(t,\varphi)\rangle_V + \frac{\alpha}{2}\|\nabla\varphi\|_2^2.\end{aligned} \tag{2.48}$$

Lemma 2.20 (Fréchet-differentiability of $\widetilde{\mathcal{I}}$)**.** *Under Assumptions 1.17.1 and 1.47, it holds* $\widetilde{\mathcal{I}} \in C^1([0,T] \times H^1(\Omega))$ *and at all* $(t,\varphi) \in [0,T] \times H^1(\Omega)$ *we have*

$$\partial_t\widetilde{\mathcal{I}}(t,\varphi) = -\langle \dot{\ell}(t),\mathcal{U}(t,\varphi)\rangle_V, \tag{2.49a}$$

$$\partial_\varphi\widetilde{\mathcal{I}}(t,\varphi) = -\alpha\Delta\varphi + F(t,\varphi). \tag{2.49b}$$

Proof. The proof is very similar to the proof of Lemma 2.2. We apply product rule for the mapping $f : [0,T] \times H^1(\Omega) \ni (t,\varphi) \mapsto -1/2\langle \ell(t), \mathcal{U}(t,\varphi)\rangle_V \in \mathbb{R}$, which yields $f \in C^1([0,T]\times H^1(\Omega))$, in view of Assumption 1.47 and Proposition 1.53. The derivative thereof at all $(t,\varphi) \in [0,T] \times H^1(\Omega)$ reads

$$\begin{aligned} f'(t,\varphi)(\delta t,\delta\varphi) &= -\frac{1}{2}\langle \dot{\ell}(t)\delta t, \mathcal{U}(t,\varphi)\rangle_V - \frac{1}{2}\langle \ell(t), \mathcal{U}'(t,\varphi)(\delta t,\delta\varphi)\rangle_V \\ &= -\langle \dot{\ell}(t)\delta t, \mathcal{U}(t,\varphi)\rangle_V + \langle F(t,\varphi), \delta\varphi\rangle_{H^1(\Omega)} \quad \forall\, (\delta t,\delta\varphi) \in \mathbb{R}\times H^1(\Omega). \end{aligned} \tag{2.50}$$

Note that the last equality from above follows by the arguments used for deriving (2.4) and (2.5), i.e., (1.72), (1.75) and (1.23), as well as Definitions 1.2 and 1.8. From (2.50) combined with (2.48) and $\|\nabla\cdot\|_2^2 \in C^1(H^1(\Omega))$ we can now deduce that $\widetilde{\mathcal{I}} \in C^1([0,T] \times H^1(\Omega))$ and that its derivative at all $(t,\varphi) \in [0,T] \times H^1(\Omega)$ is given by

$$\begin{aligned} \mathcal{I}'(t,\varphi)(\delta t,\delta\varphi) = &-\langle \dot{\ell}(t)\delta t, \mathcal{U}(t,\varphi)\rangle_V + \langle F(t,\varphi), \delta\varphi\rangle_{H^1(\Omega)} \\ &+ \alpha(\nabla\varphi, \nabla\delta\varphi)_2 \quad \forall\, (\delta t,\delta\varphi) \in \mathbb{R}\times H^1(\Omega). \end{aligned}$$

This completes the proof. □

Next we introduce the viscous dissipation functional corresponding to the situation without penalty.

Definition 2.21 (Viscous dissipation functional)**.** *We define the functional $\widetilde{\mathcal{R}}_\delta : H^1(\Omega) \to [0,\infty]$ by*

$$\widetilde{\mathcal{R}}_\delta(\eta) := \begin{cases} r\int_\Omega \eta\, dx + \frac{\delta}{2}\|\eta\|_2^2 & \text{if } \eta \geq 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise,} \end{cases}$$

that is, $\widetilde{\mathcal{R}}_\delta = \mathcal{R}_\delta|_{H^1(\Omega)}$.

We now begin to analyze the behaviour of the terms in (2.9) as $\beta \to \infty$. In the sequel we again denote by $(\boldsymbol{u}_\beta, \varphi_\beta, d_\beta)$ the unique solution of the problem (P_β). Recall that, cf. Propositions 2.17 and 2.18 there exists a sequence $\beta_n \to \infty$ as $n \to \infty$ and $\varphi \in H^1(0,T;H^1(\Omega))$ such that the following converges hold

$$\varphi_{\beta_n} \rightharpoonup \varphi \quad \text{in } H^1(0,T;H^1(\Omega)), \tag{2.51a}$$

$$d_{\beta_n} \rightharpoonup \varphi \quad \text{in } H^1(0,T;L^2(\Omega)), \tag{2.51b}$$

$$\boldsymbol{u}_{\beta_n} \to \mathcal{U}(\cdot,\varphi(\cdot)) \text{ in } C([0,T];V) \tag{2.51c}$$

as $n \to \infty$, provided that Assumptions 1.17.1, 1.47, 1.56 and 2.8 are satisfied. In the rest of the section we denote by $\{\beta_n\}_{n\in\mathbb{N}}$ such a fixed sequence and by φ the corresponding limit.

Lemma 2.22. *Under Assumptions 1.17.1, 1.47, 1.56 and 2.8 it holds*

$$\int_0^t \widetilde{\mathcal{R}}_\delta(\dot{\varphi}(\tau))\, d\tau \leq \liminf_{n\to\infty} \int_0^t \mathcal{R}_\delta(\dot{d}_{\beta_n}(\tau))\, d\tau$$

for all $t \in [0,T]$.

Proof. Let $t \in [0,T]$ be arbitrary, but fixed. First note that (2.51b) implies

$$\dot{d}_{\beta_n} \rightharpoonup \dot{\varphi} \text{ in } L^2(0,t;L^2(\Omega)), \tag{2.52}$$

which further leads to

$$\dot{d}_{\beta_n} \rightharpoonup \dot{\varphi} \text{ in } L^1(0,t;L^1(\Omega)) \tag{2.53}$$

as $n \to \infty$. Note that here we also employed the fact that the operator $H^1(0,T;L^2(\Omega)) \ni \zeta \mapsto \zeta|_{[0,t]} \in H^1(0,t;L^2(\Omega))$ is linear and continuous. Now, since $\|\cdot\|_{L^1(0,t;L^1(\Omega))}$ and $\|\cdot\|^2_{L^2(0,t;L^2(\Omega))}$ are weakly lower semicontinuous, we infer from (2.52) and (2.53)

$$r\int_0^t \|\dot{\varphi}(\tau)\|_1 \, d\tau + \frac{\delta}{2}\int_0^t \|\dot{\varphi}(\tau)\|_2^2 \, d\tau \le \liminf_{n\to\infty} r\int_0^t \|\dot{d}_{\beta_n}(\tau)\|_1 \, d\tau + \frac{\delta}{2}\int_0^t \|\dot{d}_{\beta_n}(\tau)\|_2^2 \, d\tau. \tag{2.54}$$

Further, recall that from (2.16c) one reads $\dot{d}_{\beta_n} \ge 0$ for all $n \in \mathbb{N}$. Since the set $\{f \in L^2(0,t;L^2(\Omega)) : f \ge 0\}$ is weakly closed, one deduces from (2.52) that $\dot{\varphi}|_{[0,t]} \ge 0$. Now (2.54) gives in view of Definitions 0.4 and 2.21 the assertion. □

Lemma 2.23. *Let Assumptions 1.17.1, 1.47, 1.56 and 2.8 hold. Then for all $t \in [0,T]$ we have the convergence*

$$\partial_d \mathcal{I}(t, d_{\beta_n}(t)) \rightharpoonup \partial_\varphi \widetilde{\mathcal{I}}(t, \varphi(t)) \text{ in } H^1(\Omega)^* \quad \text{as } n \to \infty.$$

Proof. Let $n \in \mathbb{N}$ and $t \in [0,T]$ be arbitrary, but fixed. From (2.43) we know in view of (2.2b) and (2.49b) that

$$\partial_d \mathcal{I}(t, d_{\beta_n}(t)) = \partial_\varphi \widetilde{\mathcal{I}}(t, \varphi_{\beta_n}(t)) \;\text{ in } H^1(\Omega)^*. \tag{2.55}$$

As a result of Assumption 1.17.1 we have $H^1(\Omega) \hookrightarrow\hookrightarrow L^{\frac{2p}{p-2}}(\Omega)$, see (1.32), so that by the convergence (2.51a) and Lemma 5.9 we get

$$\varphi_{\beta_n}(t) \to \varphi(t) \;\text{ in } L^{\frac{2p}{p-2}}(\Omega), \tag{2.56a}$$

$$\nabla\varphi_{\beta_n}(t) \rightharpoonup \nabla\varphi(t) \;\text{ in } L^2(\Omega;\mathbb{R}^N) \tag{2.56b}$$

as $n \to \infty$. Employing again (2.49b) we obtain by means of Lemma 1.18 for all $v \in H^1(\Omega)$ the estimate

$$\begin{aligned} |\langle \partial_\varphi \widetilde{\mathcal{I}}(t, \varphi_{\beta_n}(t)) - \partial_\varphi \widetilde{\mathcal{I}}(t, \varphi(t)), v\rangle_{H^1(\Omega)}| \le C\|\varphi_{\beta_n}(t) - \varphi(t)\|_{\frac{2p}{p-2}} \|v\|_{\frac{2p}{p-2}} \\ + \alpha|(\nabla\varphi_{\beta_n}(t) - \nabla\varphi(t), \nabla v)_2|, \end{aligned} \tag{2.57}$$

where the constant C is independent of n, since β does not appear in the definition of F, see (1.23). We also refer to the proof of Lemma 1.18 for more details. From (2.57) we can now conclude with (2.56a), (2.56b) and (2.55) the desired assertion. □

Lemma 2.24. *Under Assumptions 1.17.1, 1.47, 1.56 and 2.8 it holds*

$$\int_0^t \widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi\widetilde{\mathcal{I}}(\tau,\varphi(\tau))\big)\,d\tau \le \liminf_{n\to\infty}\int_0^t \mathcal{R}_\delta^*\big(-\partial_d\mathcal{I}(\tau,d_{\beta_n}(\tau))\big)\,d\tau.$$

for all $t \in [0,T]$.

Proof. Recall that $\widetilde{\mathcal{R}}_\delta = \mathcal{R}_\delta|_{H^1(\Omega)}$, which by the definition of the Fenchel conjugate gives in turn

$$\widetilde{\mathcal{R}}_\delta^*(\xi) = \sup_{v\in H^1(\Omega)}(\xi(v) - \widetilde{\mathcal{R}}_\delta(v)) \le \sup_{v\in L^2(\Omega)}(\xi(v) - \mathcal{R}_\delta(v)) = \mathcal{R}_\delta^*(\xi). \tag{2.58}$$

for any $\xi \in L^2(\Omega)$. Let now again $t \in [0,T]$ be arbitrary, but fixed. Since $\widetilde{\mathcal{R}}_\delta^* : H^1(\Omega)^* \to (-\infty,\infty]$ is convex and lower semicontinuous, cf. e.g. [69, Theorem 12.2], and thus, weakly lower semicontinuous, Lemma 2.23 leads to

$$\widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi\widetilde{\mathcal{I}}(\tau,\varphi(\tau))\big) \le \liminf_{n\to\infty}\widetilde{\mathcal{R}}_\delta^*\big(-\partial_d\mathcal{I}(\tau,d_{\beta_n}(\tau))\big) \quad \forall\,\tau\in[0,t].$$

By setting $\xi := -\partial_d\mathcal{I}(\tau,d_{\beta_n}(\tau))$ in (2.58), where $\tau \in [0,t]$, the above estimate can be continued as

$$\widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi\widetilde{\mathcal{I}}(\tau,\varphi(\tau))\big) \le \liminf_{n\to\infty}\mathcal{R}_\delta^*\big(-\partial_d\mathcal{I}(\tau,d_{\beta_n}(\tau))\big) \quad \forall\,\tau\in[0,t]. \tag{2.59}$$

Further, one observes in view of Lemma 2.4, that for all $n \in \mathbb{N}$ the map $[0,t] \ni \tau \mapsto \mathcal{R}_\delta^*(-\partial_d\mathcal{I}(\tau,d_{\beta_n}(\tau))) = \delta/2\|\dot d_{\beta_n}(\tau)\|_2$ is non-negative and continuous, as $\dot d_{\beta_n} \in C([0,T];L^2(\Omega))$. Moreover, $\sup_{n\in\mathbb{N}}\|\dot d_{\beta_n}\|_{L^2(0,t;L^2(\Omega))} < \infty$, due to (2.51b). Hence, we are allowed to apply Fatou's lemma, cf. e.g. [7, Lemma 4.1], which tells us that

$$\int_0^t \liminf_{n\to\infty}\mathcal{R}_\delta^*\big(-\partial_d\mathcal{I}(\tau,d_{\beta_n}(\tau))\big)\,d\tau \le \liminf_{n\to\infty}\int_0^t \mathcal{R}_\delta^*\big(-\partial_d\mathcal{I}(\tau,d_{\beta_n}(\tau))\big)\,d\tau. \tag{2.60}$$

We also notice that $\partial_\varphi\widetilde{\mathcal{I}}(\cdot,\varphi(\cdot))$ is continuous, thanks to Proposition 2.20 and the embedding $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$. Since $\widetilde{\mathcal{R}}_\delta^*$ is lower semicontinuous, one concludes that the mapping $[0,t] \ni \tau \mapsto \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi\widetilde{\mathcal{I}}(\tau,\varphi(\tau))) \in \mathbb{R}$ is lower semicontinuous as well, and therefore, measurable. Note that $\widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi\widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)))$ is real-valued, since $\widetilde{\mathcal{R}}_\delta$ is proper and in view of (2.59), $\sup_{n\in\mathbb{N}}\|\dot d_{\beta_n}\|_{L^2(0,t;L^2(\Omega))} < \infty$, and Lemma 2.4. Now we can integrate (2.59) over $(0,t)$, which combined with (2.60) finally gives the assertion. □

We are now in the position to state the main result of this subsection.

Proposition 2.25 (The energy inequality without penalty)**.** *Let Assumptions 1.17.1, 1.47, 1.56 and 2.8 hold. Then the limit function* $\varphi \in H^1(0,T;H^1(\Omega))$ *from Proposition 2.17 fulfills for all* $t \in [0,T]$ *the estimate*

$$\int_0^t \widetilde{\mathcal{R}}_\delta(\dot\varphi(\tau))\,d\tau + \int_0^t \widetilde{\mathcal{R}}_\delta^*\big(-\partial_\varphi\widetilde{\mathcal{I}}(\tau,\varphi(\tau))\big)\,d\tau + \widetilde{\mathcal{I}}(t,\varphi(t)) \le \widetilde{\mathcal{I}}(0,\varphi(0)) + \int_0^t \partial_t\widetilde{\mathcal{I}}(\tau,\varphi(\tau))\,d\tau \tag{2.61}$$

and at the initial time point it holds $\varphi(0) = 0$.

Proof. Let $t \in [0,T]$ be arbitrary, but fixed. Setting $s := 0$ in (2.9) yields the following identity

$$\begin{aligned}\int_0^t \mathcal{R}_\delta(\dot{d}_{\beta_n}(\tau))\,d\tau + \int_0^t \mathcal{R}_\delta^*\big(-\partial_d \mathcal{I}(\tau, d_{\beta_n}(\tau))\big)\,d\tau + \mathcal{I}(t, d_{\beta_n}(t))\\ = \mathcal{I}(0, d_{\beta_n}(0)) + \int_0^t \partial_t \mathcal{I}\big(\tau, d_{\beta_n}(\tau)\big)\,d\tau \quad \forall\, n \in \mathbb{N},\end{aligned} \tag{2.62}$$

where we next pass to the limit $n \to \infty$. In view of Lemmata 2.22 and 2.24 we only need to discuss the last three terms in (2.62). To this end, we first notice that due to (2.51a), (2.51c) combined with Lemma 5.9, and the weak lower semicontinuity of $\|\cdot\|_2^2 : L^2(\Omega) \to \mathbb{R}$ we have

$$\begin{aligned}-\frac{1}{2}\langle \ell(t), \mathcal{U}(t,\varphi(t))\rangle_V + \frac{\alpha}{2}\|\nabla\varphi(t)\|_2^2 \le \liminf_{n\to\infty} -\frac{1}{2}\langle \ell(t), \boldsymbol{u}_{\beta_n}(t)\rangle_V + \frac{\alpha}{2}\|\nabla\varphi_{\beta_n}(t)\|_2^2\\ + \liminf_{n\to\infty} \underbrace{\frac{\beta_n}{2}\|\varphi_{\beta_n}(t) - d_{\beta_n}(t)\|_2^2}_{\ge 0},\end{aligned}$$

which in light of (2.1) and (2.48) reads

$$\widetilde{\mathcal{I}}(t, \varphi(t)) \le \liminf_{n\to\infty} \mathcal{I}(t, d_{\beta_n}(t)). \tag{2.63}$$

For the right-hand side in (2.62) we obtain by employing (2.1) and (2.2a), as well as Assumption 2.8 and (2.17a) the equality

$$\mathcal{I}(0, d_{\beta_n}(0)) + \int_0^t \partial_t \mathcal{I}(\tau, d_{\beta_n}(\tau))\,d\tau = \int_0^t \langle -\dot{\ell}(\tau), \boldsymbol{u}_{\beta_n}(\tau)\rangle\,d\tau \quad \forall\, n \in \mathbb{N}. \tag{2.64}$$

On the other side, from (2.17a) and (2.51a) together with Lemma 5.9 one deduces $\varphi(0) = 0$. By relying again on Assumption 2.8, we infer by (2.48) and (2.49a) that

$$\widetilde{\mathcal{I}}(0, \varphi(0)) + \int_0^t \partial_t \widetilde{\mathcal{I}}(\tau, \varphi(\tau))\,d\tau = \int_0^t \langle -\dot{\ell}(\tau), \mathcal{U}(\tau, \varphi(\tau))\rangle\,d\tau. \tag{2.65}$$

As a consequence of $\dot{\ell} \in C([0,T];V^*)$, by assumption, and (2.51c) one arrives at

$$\int_0^t \langle \dot{\ell}(\tau), \boldsymbol{u}_{\beta_n}(\tau) - \mathcal{U}(\tau, \varphi(\tau))\rangle\,d\tau \le t\|\dot{\ell}\|_{C([0,T];V^*)}\|\boldsymbol{u}_{\beta_n} - \mathcal{U}(\cdot, \varphi(\cdot))\|_{C([0,T];V)} \to 0$$

as $n \to \infty$, which on account of (2.64) and (2.65) gives in turn

$$\lim_{n\to\infty} \mathcal{I}(0, d_{\beta_n}(0)) + \int_0^t \partial_t \mathcal{I}(\tau, d_{\beta_n}(\tau))\,d\tau = \widetilde{\mathcal{I}}(0, \varphi(0)) + \int_0^t \partial_t \widetilde{\mathcal{I}}(\tau, \varphi(\tau))\,d\tau. \tag{2.66}$$

By using Lemmata 2.22, 2.24 and the convergences (2.63), (2.66), we ultimately obtain (2.61) from (2.62). The claim is now proven. □

In the next sections we employ the *energy inequality* (2.61) to show that the limit in (2.51a)-(2.51c) satisfies a system of equations, namely (2.75) below, which is equivalent to a classical viscous partial damage model containing only one damage variable, cf. Section 2.5 below. As a secondary result, we will also deduce that (2.61) is in fact equivalent to an *energy identity*, see Remark 2.27 below.

2.3 A single-field gradient damage model

The main purpose of this section is to show that the limit function φ from Proposition 2.17 satisfies an evolutionary equation, i.e., (2.67) below. This will then allow us to write our single-field limit damage model as a PDE system, which as it will turn out in Section 2.5 below, falls into the category of models analyzed in [41]. The section ends with the discussion of the unique solvability thereof.

By mainly following the lines of the proof of [41, Proposition 3.2], it can be shown that the *energy inequality* (2.61) is equivalent to the above mentioned evolutionary equation:

Proposition 2.26 (Evolution without penalty)**.** *Let Assumptions 1.17.1 and 1.47 hold true. Then, any $\varphi \in H^1(0,T;H^1(\Omega))$ which fulfills for all $t \in [0,T]$ the energy inequality* (2.61) *satisfies*

$$-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)) \in \partial \widetilde{\mathcal{R}}_\delta(\dot{\varphi}(t)) \quad \textit{f.a.a. } t \in (0,T). \tag{2.67}$$

The reverse assertion is true as well.

Proof. The proof is based on the same arguments as the proof of Proposition 2.5. We begin by proving two auxiliary results that will be needed for both implications. To this end, let $\varphi \in H^1(0,T;H^1(\Omega))$ be arbitrary, but fixed. Since $\widetilde{\mathcal{R}}_\delta$ is convex and proper, a well known convex analysis result, see e.g. [69, Theorem 23.5], leads to

$$\begin{gathered}\widetilde{\mathcal{R}}_\delta(\dot{\varphi}(t)) + \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t))) = -\langle \partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)), \dot{\varphi}(t)\rangle_{H^1(\Omega)} \\ \Longleftrightarrow \\ -\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)) \in \partial \widetilde{\mathcal{R}}_\delta(\dot{\varphi}(t))\end{gathered} \tag{2.68}$$

f.a.a. $t \in (0,T)$. Further, note that by (2.49), Assumption 1.47 and Corollary 1.9, in combination with (1.23) and (3), we have

$$|\partial_t \widetilde{\mathcal{I}}(t,\phi)|,\ \|\partial_\varphi \widetilde{\mathcal{I}}(t,\phi)\|_{H^1(\Omega)^*} \le C\|\phi\|_{H^1(\Omega)} + c \quad \forall\, (t,\phi) \in [0,T] \times H^1(\Omega),$$

where $C, c > 0$ are independent of (t,ϕ). We observe that this implies (5.4) for $\widetilde{\mathcal{I}}$. Thus, in view of Lemma 2.20, we can apply Lemma 5.5, which gives in turn that the function $[0,T] \ni t \mapsto \widetilde{\mathcal{I}}(t,\varphi(t)) \in \mathbb{R}$ belongs to $H^1(0,T)$. Moreover, the derivative thereof is given by

$$\frac{d}{dt}\widetilde{\mathcal{I}}(t,\varphi(t)) = \partial_t \widetilde{\mathcal{I}}(t,\varphi(t)) + \langle \partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)), \dot{\varphi}(t)\rangle_{H^1(\Omega)} \quad \text{f.a.a. } t \in (0,T). \tag{2.69}$$

Let us now assume that φ fulfills (2.61) for all $t \in [0,T]$. Due to $\widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)) \in H^1(0,T)$, the *energy inequality* reads for $t = T$ as follows

$$\int_0^T \widetilde{\mathcal{R}}_\delta(\dot{\varphi}(\tau))\, d\tau + \int_0^T \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)))\, d\tau \le -\int_0^T \frac{d}{dt}\widetilde{\mathcal{I}}(\tau,\varphi(\tau)) - \partial_t \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\, d\tau,$$

which in light of (2.69) has as consequence

$$\int_0^T \widetilde{\mathcal{R}}_\delta(\dot{\varphi}(\tau))\, d\tau + \int_0^T \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)))\, d\tau \le -\int_0^T \langle \partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)), \dot{\varphi}(\tau)\rangle_{H^1(\Omega)}\, d\tau. \tag{2.70}$$

On the other side, one deduces from the definition of the Fenchel conjugate the estimate

$$\widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) + \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t))) \geq -\langle \partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)), \dot\varphi(t)\rangle_{H^1(\Omega)} \quad \text{f.a.a. } t \in (0,T). \quad (2.71)$$

Combining (2.70) and (2.71) now yields

$$\widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) + \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t))) = -\langle \partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)), \dot\varphi(t)\rangle_{H^1(\Omega)} \quad \text{f.a.a. } t \in (0,T)$$

and with (2.68) we arrive at

$$-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t)) \in \partial \widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) \quad \text{f.a.a. } t \in (0,T).$$

The reverse assertion can be concluded by following the lines of the proof of Proposition 2.5. To this end, assume that φ satisfies (2.67). From (2.68) and (2.69) we then obtain

$$\widetilde{\mathcal{R}}_\delta(\dot\varphi(t)) + \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(t,\varphi(t))) = -\frac{d}{dt}\widetilde{\mathcal{I}}(t,\varphi(t)) + \partial_t \widetilde{\mathcal{I}}(t,\varphi(t)) \quad \text{f.a.a. } t \in (0,T). \quad (2.72)$$

Note that any φ which fulfills (2.67) satisfies automatically $\dot\varphi(t) \geq 0$ f.a.a. $t \in (0,T)$, in view of Definition 2.21. The latter one also accounts for the integrability of $\widetilde{\mathcal{R}}_\delta(\dot\varphi(\cdot))$, as $\dot\varphi \in L^2(0,T;L^2(\Omega))$. For the terms on the right-hand side in (2.72) we have due to Lemma 2.20, the embedding $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$ and since $\widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)) \in H^1(0,T)$ that $\partial_t \widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)) \in C[0,T]$ and $\frac{d}{dt}\widetilde{\mathcal{I}}(\cdot,\varphi(\cdot)) \in L^2(0,T)$, respectively. Thus, we are allowed to integrate (2.72) in time, which now implies for all $t \in [0,T]$ the following

$$\begin{aligned}\int_0^t \widetilde{\mathcal{R}}_\delta(\dot\varphi(\tau))\, d\tau &+ \int_0^t \widetilde{\mathcal{R}}_\delta^*(-\partial_\varphi \widetilde{\mathcal{I}}(\tau,\varphi(\tau)))\, d\tau \\ &= \widetilde{\mathcal{I}}(0,\varphi(0)) - \widetilde{\mathcal{I}}(t,\varphi(t)) + \int_0^t \partial_t \widetilde{\mathcal{I}}(\tau,\varphi(\tau))\, d\tau.\end{aligned}$$

This gives the desired assertion. □

Remark 2.27. *An inspection of the proof of Proposition 2.26 shows that, in order to prove* (2.67)*, it suffices that* (2.61) *holds only at* $t = T$*. In addition, at the end of the proof it can be seen that* (2.67) *leads to an energy equality where one can actually integrate over an arbitrary interval* $[s,t] \subset [0,T]$*. Altogether, this means that the energy inequality* (2.61) *for* $t = T$ *is equivalent to the evolution* (2.67)*, which is further equivalent to the corresponding energy identity on some arbitrary interval* $[s,t] \subset [0,T]$*. We refer here also to [41, Proposition 3.2], where a very similar result is proven.*

Thus, the passage to the limit $\beta \to \infty$ *in* (2.9) *preserves the structure of the energy identity with penalty, in light of Proposition 2.25 and the above comments.*

We summarize our results so far in the following

Theorem 2.28 (Single-field gradient damage model)**.** *Let Assumptions 1.17.1, 1.47, 1.56 and 2.8 hold and* $\{\beta_n\}_{n\in\mathbb{N}}$ *be a sequence with* $\beta_n \to \infty$ *as* $n \to \infty$*. Then there* *is a subsequence (denoted by the same symbol) and a pair* $(\boldsymbol{u}, \varphi) \in C([0,T];V) \times H^1(0,T;H^1(\Omega))$ *such that*

$$\begin{aligned} \boldsymbol{u}_{\beta_n} &\to \boldsymbol{u} \ \textit{in} \ C([0,T];V), \\ \varphi_{\beta_n} \rightharpoonup \varphi \ \textit{in} \ H^1(0,T;H^1(\Omega)), &\quad d_{\beta_n} \rightharpoonup \varphi \ \textit{in} \ H^1(0,T;L^2(\Omega)), \end{aligned} \tag{2.73}$$

as $n \to \infty$, *where* $(\boldsymbol{u}_{\beta_n}, \varphi_{\beta_n}, d_{\beta_n})$ *is the unique solution of the problem* (P_{β_n}). *Moreover, every limit of such a sequence satisfies* $\boldsymbol{u} = \mathcal{U}(\cdot, \varphi(\cdot))$ *and*

$$\alpha\Delta\varphi(t) - F(t,\varphi(t)) - \delta\dot{\varphi}(t) \in \partial\widetilde{\mathcal{R}}_1(\dot{\varphi}(t)) \quad \textit{f.a.a. } t \in (0,T), \quad \varphi(0) = 0, \tag{2.74}$$

which is equivalent to the following PDE system:

$$\left.\begin{aligned} -\operatorname{div} g(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) = \ell(t) \quad \textit{in } \boldsymbol{W}^{-1,p}(\Omega) \quad \forall\, t \in [0,T], \\ \alpha\Delta\varphi(t) - \frac{1}{2} g'(\varphi(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) - \delta\dot{\varphi}(t) \in \partial\widetilde{\mathcal{R}}_1(\dot{\varphi}(t)) \quad \textit{f.a.a. } t \in (0,T), \\ \varphi(0) = 0, \end{aligned}\right\} \tag{2.75}$$

with the (non-viscous) dissipation functional $\widetilde{\mathcal{R}}_1 : H^1(\Omega) \to [0,\infty]$ *defined by*

$$\widetilde{\mathcal{R}}_1(\eta) := \begin{cases} r\int_\Omega \eta\, dx & \textit{if } \eta \geq 0 \ \textit{a.e. in } \Omega, \\ \infty & \textit{otherwise.} \end{cases} \tag{2.76}$$

Proof. The first assertion, as well as $\boldsymbol{u} = \mathcal{U}(\cdot, \varphi(\cdot))$, were already proven in Propositions 2.17 and 2.18, while Propositions 2.25 and 2.26, combined with (2.49b) yield

$$\alpha\Delta\varphi(t) - F(t,\varphi(t)) \in \partial\widetilde{\mathcal{R}}_\delta(\dot{\varphi}(t)) \quad \text{f.a.a. } t \in (0,T), \quad \varphi(0) = 0. \tag{2.77}$$

The above mentioned results also show that for any given sequence with (2.73), the limit satisfies (2.77). In view of the sum rule for convex subdifferentials, Definition 2.21 and the Fréchet-differentiability of $\|\cdot\|_2^2 : L^2(\Omega) \to \mathbb{R}$, (2.77) is (2.74). The equivalence with (2.75) follows by the definitions of $\mathcal{U}$ and F, see Definition 1.8 and (1.23), respectively. □

The above theorem shows that (2.75) admits at least one solution. Of course, it would be desirable to prove its uniqueness too, since this guarantees in particular that, no matter which subsequence one chooses in Theorem 2.28, the limit function is the same, whence the (weak) convergence of the whole sequence. Unfortunately, for this purpose, we have to impose an additional assumption on the index p, which is rather restrictive. We underline that this is only needed to show the unique solvability of (2.75), while the rest of the analysis remains unaffected. For proving uniqueness for the evolution equation (2.74), it is necessary that the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{\frac{2p}{p-4}}(\Omega)$ holds true, as a closer inspection of the proof of Proposition 2.31 below shows. This is ensured if

Assumption 2.29. *In the rest of the section we require that the assertion in Lemma 1.3 on page 12 holds for some* $p > 4$ *in the two-dimensional case and* $p \geq 6$ *in the three-dimensional case.*

Remark 2.30. *Although restrictive, Assumption 2.29 is fulfilled cf. Remark 1.27 on page 25, provided that the domain possesses* C^1*-boundary, no mixed boundary conditions are present, and the difference between the boundedness and monotonicity constants of the stress-strain relation* (1.7) *is sufficiently small. Adapted to our situation, this means that the values* $\epsilon\,\gamma_{\mathbb{C}}$ *and* $\|\mathbb{C}\|_\infty$ *have to be close enough to each other, which is clearly rather restrictive (beside the assumptions on the domain). We refer here again to [29] for more details.*

These assumptions on the data can be weakened, if one proceeds as in [41, Section 2.4] and uses the Sobolev-Slobodeckij space $H^s(\Omega)$ *with* $s \in (N/2, 2)$*, instead of* $H^1(\Omega)$*, as space for the nonlocal damage in the penalized model* (P_β)*. To this end, one replaces the gradient term in the energy functional given in Definition 0.2 by the seminorm generated by [41, (2.4b)], see also Remark 1.28 on page 25. Note that this tells us that the additional assumptions on p in Chapter 1 are no longer needed. A close inspection of the preceding analysis then shows that analogous results to the throughout this chapter proven results can be reached for the modified damage model. In particular, the limit function* φ *will then belong to the space* $H^1(0,T;H^s(\Omega))$*. The advantage thereof is that* $H^s(\Omega) \hookrightarrow C^{0,\zeta}(\bar{\Omega})$ *for some* $\zeta \in (0,1]$ *in both space dimensions, see e.g. [79]. In this case, it suffices to impose that* $p = 4$ *in both dimensions instead of Assumption 2.29. This new assumption is fulfilled e.g. by imposing smoothness assumptions on the domain and on* $\mathbb{C}$ *and if no mixed boundary conditions are present, see [41, (2.40) and (2.43)]. One then obtains by [20, Theorem 10.17] and a classical density argument that for any* $\varrho \in [2,\infty)$*, the operator* $A_{\varphi(t)} : \boldsymbol{W}_0^{1,\varrho}(\Omega) \to \boldsymbol{W}_0^{1,\varrho'}(\Omega)^*$ *is continuously invertible for any* $t \in [0,T]$*. However, unlike in Lemma 1.3,* $\|A_{\varphi(t)}^{-1}\|$ *is now bounded by the term* $c\,\|\varphi\|_{H^1(0,T;H^s(\Omega))}$ *(for all* $t \in [0,T]$*), where* $c > 0$ *is a constant depending only on the given data. This is due to the fact that* $\|A_{\varphi(t)}^{-1}\|$ *is bounded by* $c\,\|g(\varphi(t))\|_{C^{0,\zeta}(\bar{\Omega})}$ *and thus by* $c\,\|\varphi(t)\|_{H^s(\Omega)}$*, in view of the Lipschitz continuity of* g *and* $H^s(\Omega) \hookrightarrow C^{0,\zeta}(\bar{\Omega})$*. This implies that the constant* C *in* (2.79) *below depends on* $\|\varphi_i\|_{H^1(0,T;H^s(\Omega))}$*,* $i = 1, 2$*. An inspection of the proof of Proposition 2.31 below shows however that this does not affect the uniqueness result for* (2.67)*. We also refer here to [41, Sections 2.4 and 3.2], where one deals with a very similar situation. Since the bilinear form associated with* $H^s(\Omega)$ *is harder to realize in numerical practice, we do not follow this approach.*

Proposition 2.31 (Unique solvability of (2.75))**.** *Suppose that Assumptions 1.47, 1.56, 2.8 and 2.29 hold true. Then, the evolutionary equation* (2.67) *admits a unique solution* $\varphi \in H_0^1(0,T;H^1(\Omega))$ *and thus,* (2.75) *is uniquely solvable.*

Proof. The proof is similar to the proof of [41, Proposition 3.6.]. We notice that it suffices to show that (2.74) is uniquely solvable. This is due to its equivalence with (2.67) supplemented with $\varphi(0) = 0$ and since (2.74) is just (2.75) written in compact from. To this end, let $\varphi_1, \varphi_2 \in H^1(0,T;H^1(\Omega))$ be two solutions of (2.74). Note that their existence is given by Theorem 2.28. Therefore, it holds $\partial\widetilde{\mathcal{R}}_1(\dot{\varphi}_i(t)) \neq \emptyset$ f.a.a. $t \in (0,T)$, whence $\dot{\varphi}_i(t) \geq 0$ f.a.a. $t \in (0,T)$, for any $i = 1,2$. By testing (2.74) for $i = 1$ with $\dot{\varphi}_2$ and vice versa, we arrive at

$$\langle \alpha\Delta\varphi_1(t) - F(t,\varphi_1(t)) - \delta\dot{\varphi}_1(t), \dot{\varphi}_2(t) - \dot{\varphi}_1(t)\rangle_{H^1(\Omega)} \leq r \int_\Omega \dot{\varphi}_2(t) - \dot{\varphi}_1(t)\, dx$$

$$\langle \alpha\Delta\varphi_2(t) - F(t,\varphi_2(t)) - \delta\dot{\varphi}_2(t), \dot{\varphi}_1(t) - \dot{\varphi}_2(t)\rangle_{H^1(\Omega)} \leq r \int_\Omega \dot{\varphi}_1(t) - \dot{\varphi}_2(t)\, dx$$

f.a.a. $t \in (0,T)$. Adding the above estimates and then inserting the term $\alpha(\varphi_1(t) - \varphi_2(t), \dot{\varphi}_1(t) - \dot{\varphi}_2(t))_2$ on both sides leads to

$$\begin{aligned} \delta\|\dot{\varphi}_1(t) - \dot{\varphi}_2(t)\|_2^2 &+ \alpha(\varphi_1(t) - \varphi_2(t), \dot{\varphi}_1(t) - \dot{\varphi}_2(t))_{H^1(\Omega)} \\ &\leq \langle F(t,\varphi_2(t)) - F(t,\varphi_1(t)), \dot{\varphi}_1(t) - \dot{\varphi}_2(t)\rangle_{H^1(\Omega)} \\ &\quad + \alpha(\varphi_1(t) - \varphi_2(t), \dot{\varphi}_1(t) - \dot{\varphi}_2(t))_2 \quad \text{f.a.a. } t \in (0,T). \end{aligned} \tag{2.78}$$

Thanks to Assumption 2.29, we can apply to the first term on the right-hand side in (2.78) Lemma 1.18 with $r := 2p/(p-4)$, so that $s = 2$. By Cauchy-Schwarz inequality and the embeddings $H^1(\Omega) \hookrightarrow L^r(\Omega)$ and $H^1(\Omega) \hookrightarrow L^2(\Omega)$ we then find f.a.a. $t \in (0,T)$ the following

$$\begin{aligned} \delta\|\dot{\varphi}_1(t) - \dot{\varphi}_2(t)\|_2^2 &+ \alpha(\varphi_1(t) - \varphi_2(t), \dot{\varphi}_1(t) - \dot{\varphi}_2(t))_{H^1(\Omega)} \\ &\leq C\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}\|\dot{\varphi}_1(t) - \dot{\varphi}_2(t)\|_2 \\ &\leq \frac{C}{4\varepsilon}\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 + C\varepsilon\|\dot{\varphi}_1(t) - \dot{\varphi}_2(t)\|_2^2 \quad \forall\, \varepsilon > 0, \end{aligned} \tag{2.79}$$

where the last estimate follows from the generalized Young inequality. Note that $C > 0$ denotes a constant independent of t, φ_1 and φ_2, in view of Lemma 1.18. Further, we set ε (small enough) in (2.79) such that $C\varepsilon < \delta$ holds, e.g. $\varepsilon := \delta/(2C)$, which now gives in turn

$$\alpha(\varphi_1(t) - \varphi_2(t), \dot{\varphi}_1(t) - \dot{\varphi}_2(t))_{H^1(\Omega)} \leq \frac{C^2}{2\delta}\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 \quad \text{f.a.a. } t \in (0,T). \tag{2.80}$$

On the other side, by e.g. [81, Lemma 3.1.43], we have for all $t \in [0,T]$

$$\int_0^t (\varphi_1(\tau) - \varphi_2(\tau), \dot{\varphi}_1(\tau) - \dot{\varphi}_2(\tau))_{H^1(\Omega)}\, d\tau = \frac{1}{2}\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 - \frac{1}{2}\|\varphi_1(0) - \varphi_2(0)\|_{H^1(\Omega)}^2$$

and due to $\varphi_1(0) = \varphi_2(0)$, we arrive after integrating (2.80) at

$$\frac{\alpha}{2}\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 \leq \frac{C^2}{2\delta}\int_0^t \|\varphi_1(\tau) - \varphi_2(\tau)\|_{H^1(\Omega)}^2\, d\tau \quad \forall\, t \in [0,T]. \tag{2.81}$$

In light of $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$, the map $t \mapsto \|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2$ is continuous and applying Gronwall's lemma for (2.81) leads to

$$\|\varphi_1(t) - \varphi_2(t)\|_{H^1(\Omega)}^2 \leq 0 \quad \forall\, t \in [0,T],$$

whence $\varphi_1 = \varphi_2$. This completes the proof. □

As an immediate consequence of Proposition 2.31 combined with Theorem 2.28 we infer

Corollary 2.32. *If in addition to Assumptions 1.47, 1.56 and 2.8, Assumption 2.29 is fulfilled, then the convergence in* (2.73) *is not only valid for a subsequence, but for the whole sequence* $\{(\boldsymbol{u}_{\beta_n}, \varphi_{\beta_n}, d_{\beta_n})\}$.

2.4 L^∞-bound for the limit damage variable

The last essential step before proving that the damage model without penalty is equivalent to a version of the model analyzed by [41] consists of showing that the limit function φ in (2.75) belongs to $L^\infty((0,T)\times\Omega)$. This is needed to perform the transformation of one model into the other in the upcoming section.

To this end, we turn back to the problem (P_β) and show that, under mild assumptions, the local and nonlocal damage are bounded a.e. in $(0,T)\times\Omega$ by a *constant independent of* β. The convergence (2.73) will then ensure that at least one of the solutions of (2.74) is essentially bounded. For the optimal damage variables associated to (P_β), the existence of L^∞-bounds was indeed already established in case of $N = 2$, see Remark 1.46, but we emphasize that these are dependent on β. To show the desired boundedness, we mostly follow the ideas of [41], where the authors derive a similar result in two dimensions under comparative assumptions, see [41, Proposition 4.5]. We prove the claim via a time-discretization procedure, by passing to the limit in a suitable time-discretization scheme for the problem (P_β) and by showing that the discrete local and nonlocal damage variables possess (a.e. in $(0,T)\times\Omega$) a uniform (w.r.t. β and time-step size) bound, see Lemma 2.38 below. As it will turn out, the value of the bound can be ultimately read from the properties of the function g.

In the sequel, β is fixed and large enough cf. Assumption 1.17.2. We omit for the sake of convenience the symbol for the dependence on the penalty term of the solution of (P_β) and of the associated discrete solutions. We only highlight the dependence of the latter ones on the different time-step sizes.

Recall that, by Proposition 2.7, the damage model with penalty reduces to the operator differential equation

$$\dot{d}(t) = \frac{1}{\delta}\max\big(-\beta(d(t)-\varphi(t)) - r\big) \quad \forall\, t\in[0,T], \quad d(0) = d_0, \tag{2.82}$$

where d and $\varphi = \Phi(\cdot, d(\cdot))$ stand for the local and nonlocal damage, respectively. This notation and the upcoming ones are valid throughout this section.

Starting from (2.82) we introduce the following time-discrete incremental problem: Given the number of time-steps $n\in\mathbb{N}^+$, we set $\tau := T/n$ and denote by $\{t_k^\tau = k\tau\}_{k=0,\ldots,n}$ the corresponding partition of the time interval $[0,T]$. Further we define, beginning with $d_0^\tau := d_0$, the approximation of the local damage at time point t_{k+1}^τ, $k\in\{0,\ldots,n-1\}$, as the unique solution of the fixed-point equation

$$d_{k+1}^\tau = d_k^\tau + \frac{\tau}{\delta}\max\big(-\beta(d_{k+1}^\tau - \Phi(t_{k+1}^\tau, d_{k+1}^\tau)) - r\big). \tag{$P_k^{\beta,\tau}$}$$

Note that the unique solvability of $(P_k^{\beta,\tau})$ can be easily concluded for any $k \in \{0, ..., n-1\}$ from Banach fixed-point theorem. To see this, we observe that

$$\begin{aligned}
&\frac{\tau}{\delta}\big\| \max\big(-\beta(z_1 - \Phi(t_{k+1}^\tau, z_1)) - r\big) - \max\big(-\beta(z_2 - \Phi(t_{k+1}^\tau, z_2)) - r\big)\big\|_2 \\
&\qquad \leq \frac{\tau}{\delta}\| -\beta(z_1 - \Phi(t_{k+1}^\tau, z_1)) + \beta(z_2 - \Phi(t_{k+1}^\tau, z_2))\|_2 \\
&\qquad \leq \frac{\tau}{\delta}(\beta + \beta L)\|z_1 - z_2\|_2 \quad \forall\, z_1, z_2 \in L^2(\Omega),
\end{aligned}$$

by the Lipschitz continuity of max and Φ (with constant L), cf. Lemma 5.6(i) and (1.38), respectively. Hence, if $\tau < \delta/(\beta + \beta L)$, then the function $L^2(\Omega) \ni z \mapsto d_k^\tau + \frac{\tau}{\delta} \max\big(-\beta(z - \Phi(t_{k+1}^\tau, z)) - r\big) \in L^2(\Omega)$ is a contraction. This is also well defined, and thus, $(P_k^{\beta,\tau})$ admits an unique solution $d_{k+1}^\tau \in L^2(\Omega)$, provided that τ is small enough. The latter one is supposed to hold throughout this section, as we however aim to pass to the limit $\tau \searrow 0$.

2.4.1 Passage to the limit in the time-discrete problem

Let the number of steps n be fixed and large enough. Before we proceed with analyzing the behaviour of the time-discrete solutions as $\tau \searrow 0$, we introduce, similarly to [41, Section 4], the notations

$$\bar{t}_\tau(t) := t_{k+1}^\tau \text{ for } t \in (t_k^\tau, t_{k+1}^\tau],\ \underline{t}_\tau(t) := t_k^\tau \quad \text{for } t \in [t_k^\tau, t_{k+1}^\tau), \quad k \in \{0, ..., n-1\},$$

$\bar{t}_\tau(0) := 0$ and $\underline{t}_\tau(T) := T$. We define the piecewise constant interpolation functions $\bar{d}_\tau, \underline{d}_\tau : [0, T] \to L^2(\Omega)$ by

$$\bar{d}_\tau(t) := d_{k+1}^\tau \text{ for } t \in (t_k^\tau, t_{k+1}^\tau],\ \underline{d}_\tau(t) := d_k^\tau \quad \text{for } t \in [t_k^\tau, t_{k+1}^\tau), \quad k \in \{0, ..., n-1\},$$

$\bar{d}_\tau(0) := d_0$ and $\underline{d}_\tau(T) := d_n$, as well as the piecewise linear interpolation function $d_\tau : [0, T] \to L^2(\Omega)$ as

$$d_\tau(t) := d_k^\tau + \frac{t - t_k^\tau}{\tau}(d_{k+1}^\tau - d_k^\tau) \quad \text{for } t \in [t_k^\tau, t_{k+1}^\tau], \quad k \in \{0, ..., n-1\}.$$

Notice that d_τ is differentiable on $[0, T] \setminus \{t_0, t_1, ..., t_n\}$ with

$$\dot{d}_\tau(t) = \frac{d_{k+1}^\tau - d_k^\tau}{\tau} \quad \text{for } t \in (t_k^\tau, t_{k+1}^\tau), \quad k \in \{0, ..., n-1\}, \tag{2.83}$$

which implies in view of $(P_k^{\beta,\tau})$ that for $t \in [0, T] \setminus \{t_0, t_1, ..., t_n\}$ it holds

$$\dot{d}_\tau(t) = \frac{1}{\delta} \max\big(-\beta\big(\bar{d}_\tau(t) - \Phi(\bar{t}_\tau(t), \bar{d}_\tau(t))\big) - r\big). \tag{2.84}$$

To prove convergence in the above time-discretization scheme, we need the following

Lemma 2.33. *Let Assumptions 1.5 and 1.17 hold true and suppose that $\tau < \delta/(\beta + \beta L)$ is small enough, depending only on the given data. Then, there exists a constant $C > 0$, independent of τ, such that for all $t \in [0, T] \setminus \{t_0, t_1, ..., t_n\}$ it holds*

$$\|\dot{d}_\tau(t)\|_2 \leq C.$$

Proof. The proof is inspired by the proof of [41, Proposition 4.2]. We aim to estimate $\|\dot{d}_\tau(t)\|_2^2$ in such a way that (the discrete version of) Gronwall's lemma can be applied. Let $k \in \{1, ..., n-1\}$ be arbitrary, but fixed, where n denotes the number of time steps. Then by testing (2.84) for fixed $t \in (t_k^\tau, t_{k+1}^\tau)$ and fixed $s \in (t_{k-1}^\tau, t_k^\tau)$ with $\dot{d}_\tau(t)$ we have

$$\begin{aligned}\|\dot{d}_\tau(t)\|_2^2 &= \frac{1}{\delta}\big(\max\big(-\beta\big(\bar{d}_\tau(t) - \Phi(\bar{t}_\tau(t), \bar{d}_\tau(t))\big) - r\big), \dot{d}_\tau(t)\big)_2,\\ (\dot{d}_\tau(s), \dot{d}_\tau(t))_2 &= \frac{1}{\delta}\big(\max\big(-\beta\big(\bar{d}_\tau(s) - \Phi(\bar{t}_\tau(s), \bar{d}_\tau(s))\big) - r\big), \dot{d}_\tau(t)\big)_2.\end{aligned}$$

By subtracting the second identity from the first one and by using

$$\|\dot{d}_\tau(t)\|_2^2 - \|\dot{d}_\tau(s)\|_2^2 \leq 2\|\dot{d}_\tau(t)\|_2^2 - 2(\dot{d}_\tau(s), \dot{d}_\tau(t))_2,$$

we arrive at

$$\begin{aligned}&\delta/2(\|\dot{d}_\tau(t)\|_2^2 - \|\dot{d}_\tau(s)\|_2^2)\\ &\leq \big(\max\big(-\beta\big(\bar{d}_\tau(t) - \Phi(\bar{t}_\tau(t), \bar{d}_\tau(t))\big) - r\big) - \max\big(-\beta\big(\bar{d}_\tau(s) - \Phi(\bar{t}_\tau(s), \bar{d}_\tau(s))\big) - r\big), \dot{d}_\tau(t)\big)_2\\ &\leq \big(\beta\|\bar{d}_\tau(t) - \bar{d}_\tau(s)\|_2 + \beta L(\tau + \|\bar{d}_\tau(t) - \bar{d}_\tau(s)\|_2)\big)\|\dot{d}_\tau(t)\|_2\\ &= \big(c\tau + C\|\bar{d}_\tau(t) - \bar{d}_\tau(s)\|_2\big)\|\dot{d}_\tau(t)\|_2,\end{aligned} \tag{2.85}$$

where $c = \beta L$ and $C = \beta(L+1)$ are constants independent of τ. Note that for the last inequality we used the Lipschitz continuity of max and Φ, cf. Lemma 5.6(i) and (1.38), respectively, as well as $\bar{t}_\tau(t) - \bar{t}_\tau(s) = \tau$. Relying on (2.83), $\bar{d}_\tau(t) = d_{k+1}^\tau$ and $\bar{d}_\tau(s) = d_k^\tau$, (2.85) can be continued as

$$\begin{aligned}\delta/2(\|\dot{d}_\tau(t)\|_2^2 - \|\dot{d}_\tau(s)\|_2^2) &\leq \big(c\tau + C\tau\|\dot{d}_\tau(t)\|_2\big)\|\dot{d}_\tau(t)\|_2\\ &\leq C\tau + C\tau\|\dot{d}_\tau(t)\|_2^2 \quad \forall\, t \in (t_k^\tau, t_{k+1}^\tau),\ \forall\, s \in (t_{k-1}^\tau, t_k^\tau),\end{aligned} \tag{2.86}$$

where for the last estimate we employed Young's inequality. Further, notice that as a consequence of (2.83), we can write

$$\dot{d}_\tau(\rho) = \dot{d}_\tau\Big(\frac{t_j^\tau + t_{j+1}^\tau}{2}\Big) \quad \forall\, \rho \in (t_j^\tau, t_{j+1}^\tau)_{j=0,\dots,n-1}. \tag{2.87}$$

Now let $t \in [t_1, T] \setminus \{t_1, ..., t_n\}$ be arbitrary, but fixed, which implies that there exists some $m \in \{1, ..., n-1\}$ such that $t \in (t_m, t_{m+1})$. By adding (2.86) for $k = 1, ..., m$ we arrive at

$$\|\dot{d}_\tau(t)\|_2^2 \leq \Big\|\dot{d}_\tau\Big(\frac{\tau}{2}\Big)\Big\|_2^2 + mC\tau + C\tau\sum_{k=1}^m \Big\|\dot{d}_\tau\Big(\frac{t_k^\tau + t_{k+1}^\tau}{2}\Big)\Big\|_2^2, \tag{2.88}$$

in view of (2.87). From the estimate (2.88) we want to conclude with the discrete version of Gronwall's lemma the assertion. Therefor, we need to find an independent of τ bound for $\big\|\dot{d}_\tau\big(\frac{\tau}{2}\big)\big\|_2$. To this end, we test (2.84) at time point $\tau/2$ with $\dot{d}_\tau(\tau/2)$ and deduce

$$
\begin{aligned}
&\|\dot{d}_\tau(\tau/2)\|_2^2 \\
&= \frac{1}{\delta}\big(\max\big(-\beta\big(\bar{d}_\tau(\tau/2) - \Phi\big(\bar{t}_\tau(\tau/2), \bar{d}_\tau(\tau/2)\big)\big) - r\big), \dot{d}_\tau(\tau/2)\big)_2 \\
&= \frac{1}{\delta}\big(\max\big(-\beta\big(\bar{d}_\tau(\tau/2) - \Phi\big(\bar{t}_\tau(\tau/2), \bar{d}_\tau(\tau/2)\big)\big) - r\big) - \max\big(-\beta\big(d_0 - \varphi_0\big) - r\big), \dot{d}_\tau(\tau/2)\big)_2 \\
&\quad + \frac{1}{\delta}\big(\max(-\beta(d_0 - \varphi_0) - r), \dot{d}_\tau(\tau/2)\big)_2,
\end{aligned}
$$

where $\varphi_0 := \Phi(0, d_0)$. By relying on the same arguments used for deriving (2.85) and (2.86), as well as Cauchy-Schwarz inequality, we then infer

$$
\begin{aligned}
\|\dot{d}_\tau(\tau/2)\|_2^2 &\leq c\tau + C\tau\|\dot{d}_\tau(\tau/2)\|_2^2 + C\|\beta\big(d_0 - \varphi_0\big) + r\|_2\|\dot{d}_\tau(\tau/2)\|_2 \\
&\leq c\tau + C\tau\|\dot{d}_\tau(\tau/2)\|_2^2,
\end{aligned}
\tag{2.89}
$$

with $c, C > 0$ independent of τ. Note that for the last estimate we employed again Young's inequality. Since $\tau \searrow 0$ later anyway, we may choose $\tau \leq \frac{1}{2C}$ in (2.89). This leads to $\|\dot{d}_\tau(\tau/2)\|_2^2 \leq 2c\tau \leq c/C$, which we insert in (2.88), giving in turn

$$
\|\dot{d}_\tau(t)\|_2^2 \leq c + C\tau \sum_{k=1}^{m} \Big\|\dot{d}_\tau\Big(\frac{t_k^\tau + t_{k+1}^\tau}{2}\Big)\Big\|_2^2 \quad \text{for } t \in (t_m, t_{m+1}),
$$

where we used $m \leq n-1$ and $\tau = T/n$. Since $\dot{d}_\tau(t) = \dot{d}_\tau\Big(\frac{t_m + t_{m+1}}{2}\Big)$, cf. (2.87), and $\tau \leq \frac{1}{2C}$, we further have

$$
\Big\|\dot{d}_\tau\Big(\frac{t_m + t_{m+1}}{2}\Big)\Big\|_2^2 \leq 2c + 2C\tau \sum_{k=1}^{m-1} \Big\|\dot{d}_\tau\Big(\frac{t_k^\tau + t_{k+1}^\tau}{2}\Big)\Big\|_2^2
\tag{2.90}
$$

with the convention $\sum_{k=1}^{0} = 0$. Note that (2.90) holds for any $m \in \{1, ..., n-1\}$, since $t \in [t_1, T] \setminus \{t_1, ..., t_n\}$ was arbitrary. Thus, the discrete version of Gronwall's lemma applied for the sequence $\{\omega_k\}$ with $\omega_k := \Big\|\dot{d}_\tau\Big(\frac{t_k^\tau + t_{k+1}^\tau}{2}\Big)\Big\|_2^2$ now tells us that

$$
\Big\|\dot{d}_\tau\Big(\frac{t_m + t_{m+1}}{2}\Big)\Big\|_2^2 \leq c\,\exp\Big(\sum_{k=1}^{m-1} C\tau\Big) \leq c\exp(CT) \quad \forall\, m \in \{1, ..., n-1\},
$$

in view of $\tau = T/n$. We observe that the constant on the right-hand side is independent of τ. Thanks to (2.87), this means that

$$
\|\dot{d}_\tau(t)\|_2^2 \leq C \qquad \text{for all } t \in [t_1, T] \setminus \{t_1, ..., t_n\},
$$

which together with $\|\dot{d}_\tau(\tau/2)\|_2^2 \leq c/C$ (see above), gives the assertion. $\square$

The main result in this subsection is given by

Proposition 2.34 (Convergence of the time-discretization)**.** *Suppose that Assumptions 1.5 and 1.17 are fulfilled and let d be the unique solution of* (2.82) *and* $\varphi := \Phi(\cdot, d(\cdot))$. *Then, the following convergence holds true*

$$d_\tau \to d \quad \text{in } W^{1,\infty}(0,T;L^2(\Omega)) \text{ as } \tau \searrow 0.$$

Moreover,

$$\bar{d}_\tau \to d \text{ in } L^\infty(0,T;L^2(\Omega)) \quad \text{and} \quad \bar{\varphi}_\tau \to \varphi \text{ in } L^\infty(0,T;H^1(\Omega)) \text{ as } \tau \searrow 0, \tag{2.91}$$

where $\bar{\varphi}_\tau(t) := \Phi\big(\bar{t}_\tau(t), \bar{d}_\tau(t)\big)$ *for all* $t \in [0,T]$.

Proof. Let $\tau < \delta/(\beta + \beta L)$ be arbitrary, but fixed. To make sure that the desired convergences make sense, let us first investigate the regularity of d_τ, $\bar{d}_\tau$ and $\bar{\varphi}_\tau$. Clearly, $\bar{d}_\tau : [0,T] \to L^2(\Omega)$ and $\bar{\varphi}_\tau : [0,T] \to H^1(\Omega)$ are Bochner simple functions, and thus, they belong to $L^\infty(0,T;L^2(\Omega))$ and $L^\infty(0,T;H^1(\Omega))$, respectively. Since $L^2(\Omega)$ is reflexive Banach space and d_τ is almost everywhere differentiable with $\dot{d}_\tau = 1/\tau(\bar{d}_\tau - \underline{d}_\tau)$ a.e. in $(0,T)$, we obtain that $d_\tau \in W^{1,\infty}(0,T;L^2(\Omega))$, see e.g. [81, p. 58]. Note that here we used that $\bar{d}_\tau - \underline{d}_\tau \in L^\infty(0,T;L^2(\Omega))$ as difference of Bochner simple functions. Thereby, we deduce that indeed

$$d_\tau \in W^{1,\infty}(0,T;L^2(\Omega)), \quad \bar{d}_\tau \in L^\infty(0,T;L^2(\Omega)) \quad \text{and} \quad \bar{\varphi}_\tau \in L^\infty(0,T;H^1(\Omega)). \tag{2.92}$$

We begin by deriving an estimate which will be very useful in what follows. First note that for all $t \in [0,T]$ we have

$$d_\tau(t) - \bar{d}_\tau(t) = \Big(1 - \frac{t - \underline{t}_\tau(t)}{\tau}\Big)(\underline{d}_\tau(t) - \bar{d}_\tau(t)).$$

Hence, due to $\frac{t - \underline{t}_\tau(t)}{\tau} \in [0,1]$ for all $t \in [0,T]$ and (2.83) it holds

$$\begin{aligned} \|d(t) - \bar{d}_\tau(t)\|_2 &\leq \|d(t) - d_\tau(t)\|_2 + \|d_\tau(t) - \bar{d}_\tau(t)\|_2 \\ &\leq \|d(t) - d_\tau(t)\|_2 + \|\bar{d}_\tau(t) - \underline{d}_\tau(t)\|_2, \\ &= \|d(t) - d_\tau(t)\|_2 + \tau\|\dot{d}_\tau(t)\|_2 \\ &\leq \|d(t) - d_\tau(t)\|_2 + \tau C \quad \text{f.a.a. } t \in (0,T), \end{aligned} \tag{2.93}$$

where $C > 0$ is the constant from Lemma 2.33.

Now, subtracting (2.84) from (2.82) yields f.a.a. $t \in (0,T)$

$$\dot{d}(t) - \dot{d}_\tau(t) = \frac{1}{\delta}\big(\max\big(-\beta(d(t) - \varphi(t)) - r\big) - \max\big(-\beta\big(\bar{d}_\tau(t) - \Phi(\bar{t}_\tau(t), \bar{d}_\tau(t))\big) - r\big)\big). \tag{2.94}$$

By making again use of the Lipschitz continuity of max and Φ we obtain from (2.94) and (2.93) the estimate

$$\begin{aligned} \|\dot{d}(t) - \dot{d}_\tau(t)\|_2 &\leq \frac{1}{\delta}\big(\beta\|d(t) - \bar{d}_\tau(t)\|_2 + \beta L(|t - \bar{t}_\tau(t)| + \|d(t) - \bar{d}_\tau(t)\|_2)\big) \\ &\leq \beta(L+1)/\delta\|d(t) - \bar{d}_\tau(t)\|_2 + \tau\beta L/\delta \\ &\leq c\|d(t) - d_\tau(t)\|_2 + \tau c \quad \text{f.a.a. } t \in (0,T), \end{aligned} \tag{2.95}$$

where $c > 0$ is independent of τ. Note that for the second inequality we employed the definition of $\bar{t}_\tau$. Since $d(0) = d_\tau(0)$ and $d - d_\tau \in W^{1,\infty}(0,T;L^2(\Omega))$, we can apply Lemma 5.10 for (2.95), which results in

$$\|d - d_\tau\|_{W^{1,\infty}(0,T;L^2(\Omega))} \leq K(c,T)c\tau,$$

whence the convergence

$$d_\tau \to d \quad \text{in } W^{1,\infty}(0,T;L^2(\Omega)) \text{ as } \tau \searrow 0.$$

From (2.93) we then conclude

$$\bar{d}_\tau \to d \text{ in } L^\infty(0,T;L^2(\Omega)) \text{ as } \tau \searrow 0.$$

The Lipschitz continuity of Φ, see (1.38), leads to

$$\|\bar{\varphi}_\tau(t) - \Phi\big(t, d(t)\big)\|_{H^1(\Omega)} \leq L\big(\tau + \|\bar{d}_\tau - d\|_{L^\infty(0,T;L^2(\Omega))}\big) \quad \text{f.a.a. } t \in (0,T),$$

and thus, $\bar{\varphi}_\tau \to \varphi$ in $L^\infty(0,T;H^1(\Omega))$ as $\tau \searrow 0$. This completes the proof. □

2.4.2 Uniform estimates for the discrete damage variables

In this subsection we focus on proving that the piecewise constant interpolation function $\bar{d}_\tau$ is bounded a.e. in $(0,T) \times \Omega$ by a *constant independent of* β *and of* τ. This constant can be a priori chosen by imposing corresponding assumptions on the data, from which one can read its precise value. As already mentioned above, this will also bound (a.e. in $(0,T) \times \Omega$) the limit function φ in (2.75), which is our final goal in this section.

To prove the result, we follow the ideas of the proof of [41, Proposition 4.5], that is, we show by induction on the index k that the solution of $(P_k^{\beta,\tau})$, i.e., d_{k+1}^τ, satisfies the desired boundedness condition for any $k \in \{0, ..., n-1\}$, see proof of Lemma 2.38 below. Therefor we first need to rewrite $(P_k^{\beta,\tau})$ as an equivalent (uniquely solvable) minimization problem:

Lemma 2.35. *Suppose that Assumptions 1.5, 1.17 and 1.56 are fulfilled. Let the number of time-steps n be large enough, fixed and let* $k \in \{0, ..., n-1\}$ *be given. Then, the solution of* $(P_k^{\beta,\tau})$ *is the unique minimizer of*

$$\min_{\substack{v \in L^2(\Omega), \\ v \geq d_k^\tau}} \mathcal{I}(t_{k+1}^\tau, v) + \tau \mathcal{R}_\delta\Big(\frac{v - d_k^\tau}{\tau}\Big). \tag{2.96}$$

Proof. (i) We begin by addressing the existence of solutions for (2.96). To this end, we define for simplicity $f : \mathcal{C} \to \mathbb{R}$ as

$$f(v) := \mathcal{I}(t_{k+1}^\tau, v) + r \int_\Omega v - d_k^\tau \, dx + \frac{\delta}{2} \frac{\|v - d_k^\tau\|_2^2}{\tau}, \tag{2.97}$$

where $\mathcal{C} := \{v \in L^2(\Omega) : v \geq d_k^\tau \text{ a.e. in } \Omega\}$. Note that f coincides with the objective in (2.96) on $\mathcal{C}$, as a result of Definition 0.4. We also observe that

$$f(v) = -\frac{1}{2}\langle \ell(t_{k+1}^\tau), \mathcal{U}(t_{k+1}^\tau, \Phi(t_{k+1}^\tau, v))\rangle_V + \frac{\alpha}{2}\|\nabla\Phi(t_{k+1}^\tau, v)\|_2^2 + \frac{\beta}{2}\|\Phi(t_{k+1}^\tau, v) - v\|_2^2$$
$$+ r\int_\Omega v - d_k^\tau \, dx + \frac{\delta}{2}\frac{\|v - d_k^\tau\|_2^2}{\tau} \quad \text{for all } v \in \mathcal{C}, \tag{2.98}$$

in view of (2.1). The existence of solutions for (2.96) follows by classical arguments of the direct method of variational calculus. We begin by noticing that $\mathcal{C}$ is convex and closed, and thus, weakly closed, while f is radially unbounded, in light of (2.98) combined with Corollary 1.9. Moreover, it is weakly lower semicontinuous, as we will next establish.

To see this, we first prove that $\Phi(t_{k+1}^\tau, \cdot)$ is weakly continuous. We consider a sequence $\{v_j\} \subset \mathcal{C}$ and $v \in \mathcal{C}$ such that $v_j \rightharpoonup v$ in $L^2(\Omega)$ as $j \to \infty$ and we abbreviate $\varphi_j := \Phi(t_{k+1}^\tau, v_j)$ for the sake of convenience. Since $\{v_j\}$ is bounded by a constant independent of j and since Φ is Lipschitz continuous, see (1.38), we get by the reflexivity of $H^1(\Omega)$ that there exists a subsequence of $\{\varphi_j\}$, denoted by the same symbol, and $\widetilde{\varphi} \in H^1(\Omega)$, with

$$\varphi_j \rightharpoonup \widetilde{\varphi} \quad \text{in } H^1(\Omega) \ \text{ as } j \to \infty. \tag{2.99}$$

Due to Assumption 1.17.1, the compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^{2p/(p-2)}(\Omega)$ holds true, cf. (1.32). Then, we obtain in view of Lemma 1.18 that

$$\|F(t_{k+1}^\tau, \varphi_j) - F(t_{k+1}^\tau, \widetilde{\varphi})\|_{H^1(\Omega)^*} \leq C\|\varphi_j - \widetilde{\varphi}\|_{2p/(p-2)} \to 0 \quad \text{as } j \to \infty,$$

whence

$$B\varphi_j + F(t_{k+1}^\tau, \varphi_j) \rightharpoonup B\widetilde{\varphi} + F(t_{k+1}^\tau, \widetilde{\varphi}) \quad \text{in } H^1(\Omega)^* \text{ as } j \to \infty,$$

where we also used (1.22) and (2.99). Further, the definition of φ_j and Definition 1.24 imply that $\beta v_j = B\varphi_j + F(t_{k+1}^\tau, \varphi_j)$, while the uniqueness of the above weak limit yields $\beta v = B\widetilde{\varphi} + F(t_{k+1}^\tau, \widetilde{\varphi})$, since $v_j \rightharpoonup v$ in $L^2(\Omega)$ by assumption. However, by Definition 1.24, this leads to $\widetilde{\varphi} = \Phi(t_{k+1}^\tau, v)$. From (2.99) we now have

$$\Phi(t_{k+1}^\tau, v_j) \rightharpoonup \Phi(t_{k+1}^\tau, v) \quad \text{in } H^1(\Omega) \ \text{ as } j \to \infty. \tag{2.100}$$

Note that (2.100) holds for the entire sequence, as the limit $\Phi(t_{k+1}^\tau, v)$ is independent of the chosen subsequence. Hence, $L^2(\Omega) \ni v \mapsto \Phi(t_{k+1}^\tau, v) \in H^1(\Omega)$ is weakly continuous, as claimed.

To ultimately conclude the weak lower semicontinuity of f we rely in view of (2.98) on (2.100) combined with the arguments used in the proof of Proposition 1.12, and the weak lower semicontinuity of the norm. As $L^2(\Omega)$ is reflexive Banach space, we finally have all the necessary tools for applying the direct method of variational calculus, from which we deduce that (2.96) admits solutions.

(ii) We now turn our attention to the unique solvability of (2.96). To this end, let z be an arbitrary, but fixed, solution thereof. By taking a look at (2.97), one sees in view of Lemma 2.2 that f is Fréchet-differentiable at $z \in \mathcal{C}$. Notice that here we need Assumption 1.56 to be fulfilled, as this guarantees the existence of the derivative of Φ, and thus of $\mathcal{I}$, w.r.t. the L^2-variable, as shown by a short inspection of Subsection 1.3.2 and the proof of Lemma 2.2, respectively. Thus, since $\mathcal{C}$ is convex, z satisfies the following necessary optimality condition

$$f'(z; v - z) \geq 0 \quad \forall\, v \in \mathcal{C},$$

which in view of (2.2b) and (2.97) reads

$$\big(\beta(z - \Phi(t^\tau_{k+1}, z)) + r + \frac{\delta}{\tau}(z - d^\tau_k), v - z\big)_2 \geq 0 \quad \text{for all } v \in \mathcal{C}. \tag{2.101}$$

Observe that $2z - d^\tau_k$, $d^\tau_k \in \mathcal{C}$ and by testing (2.101) therewith, one has

$$\big(\beta(z - \Phi(t^\tau_{k+1}, z)) + r + \frac{\delta}{\tau}(z - d^\tau_k), z - d^\tau_k\big)_2 = 0. \tag{2.102}$$

Now let $w \in L^2(\Omega)$ with $w \geq 0$ be arbitrary, but fixed and test (2.101) by $w + z \in \mathcal{C}$. Then, fundamental lemma of the calculus of variations yields

$$\beta(z - \Phi(t^\tau_{k+1}, z)) + r + \frac{\delta}{\tau}(z - d^\tau_k) \geq 0 \quad \text{a.e. in } \Omega,$$

which together with (2.102) and $z \geq d^\tau_k$ gives

$$0 \leq \frac{\delta}{\tau}(z - d^\tau_k) \perp \beta(z - \Phi(t^\tau_{k+1}, z)) + r + \frac{\delta}{\tau}(z - d^\tau_k) \geq 0 \quad \text{a.e. in } \Omega.$$

Since the max-function is a well known complementarity function, we infer that z solves $(P^{\beta,\tau}_k)$ and since the latter one is uniquely solvable, the claim is now proven. □

In order to obtain the desired boundedness result, we impose the following

Assumption 2.36. *There exists $M > 0$ such that $g(x) \geq g(M)$ for all $x \geq M$.*

Recall that the function $g : \mathbb{R} \to [\epsilon, 1]$ shows in what measure the elastic properties of the body are preserved under the influence of the damage. Thus, Assumption 2.36 says that whenever the damage is larger than M, the material rigidity of the body is at least $g(M)$. Note that from a mechanical point of view, it makes sense to impose (in addition to g monotonically decreasing, see page 6) that if the damage surpasses the value M, then the body preserves the degree of material rigidity $g(M)$, i.e., $g'(x) = 0$ for all $x \geq M$, which is just a special case of Assumption 2.36.

By taking a look at (2.115) below, which shows how the coefficient function g is connected to its counterpart in [41], we see that Assumption 2.36 corresponds to the assumption made on the analogous function in [41, Proposition 4.5], where one shows a similar boundedness result.

The following lemma plays a key role for proving the main result in this section.

Lemma 2.37. *Under Assumptions 1.5, 1.17 and 2.36, we have*

$$\mathcal{I}(t, \min(z, M)) \leq \mathcal{I}(t, z) \quad \forall\, (t, z) \in [0, T] \times L^2(\Omega).$$

Moreover, if there exists $z \in L^2(\Omega)$ *with* $\min(z, M) = z$*, then* $\min(\Phi(t, z), M) = \Phi(t, z)$ *for any* $t \in [0, T]$*. Here* $\min(\cdot, M)$ *stands for the Nemytskii operator associated to* $\min\{\cdot, M\} : \mathbb{R} \to \mathbb{R}$.

Proof. Let us first notice that the operator $\min(\cdot, M)$ maps $L^2(\Omega)$ to $L^2(\Omega)$ and is Lipschitz continuous with Lipschitz constant 1. This follows from the identity

$$\min\{a, M\} = -\max\{M - a, 0\} + M \quad \forall\, a \in \mathbb{R} \tag{2.103}$$

and the properties of the max-operator, see Lemma 5.6(i). Moreover, in view of [38, Theorem A.1] and (2.103), $\min(\cdot, M)$ maps $H^1(\Omega)$ to $H^1(\Omega)$ and it holds for all $\phi \in H^1(\Omega)$

$$\nabla \min(\phi, M) = -\nabla \max(M - \phi) = \begin{cases} \nabla\phi & \text{a.e. in } \{x \in \Omega : \phi(x) < M\}, \\ 0 & \text{a.e. in } \{x \in \Omega : \phi(x) \geq M\}, \end{cases}$$

whence

$$\|\nabla \min(\phi, M)\|_2^2 \leq \|\nabla\phi\|_2^2. \tag{2.104}$$

Let now $(t, z) \in [0, T] \times L^2(\Omega)$ be arbitrary, but fixed and let us for simplicity abbreviate $\bar{\varphi} := \Phi(t, z)$ in what follows. By Theorem 1.23 and Definition 2.1 one establishes that

$$\begin{aligned} \mathcal{I}(t, \min(z, M)) &\leq \mathcal{E}(t, \mathcal{U}(t, \bar{\varphi}), \min(\bar{\varphi}, M), \min(z, M)) \\ &= \frac{1}{2}\int_\Omega g(\min(\bar{\varphi}, M))\mathbb{C}\varepsilon(\mathcal{U}(t, \bar{\varphi})) : \varepsilon(\mathcal{U}(t, \bar{\varphi}))\, dx - \langle \ell(t), \mathcal{U}(t, \bar{\varphi})\rangle_V \\ &\quad + \frac{\alpha}{2}\|\nabla \min(\bar{\varphi}, M)\|_2^2 + \frac{\beta}{2}\|\min(\bar{\varphi}, M) - \min(z, M)\|_2^2, \end{aligned} \tag{2.105}$$

where for the equality we employed Definition 0.2. By Assumption 2.36, we further have $g(\min(\bar{\varphi}, M)) \leq g(\bar{\varphi})$ a.e. in Ω. Thanks to (4), (2.104), and since the function $\min(\cdot, M) : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous with Lipschitz constant 1, (2.105) can be continued as

$$\begin{aligned} \mathcal{I}(t, \min(z, M)) &\leq \frac{1}{2}\int_\Omega g(\bar{\varphi})\mathbb{C}\varepsilon(\mathcal{U}(t, \bar{\varphi})) : \varepsilon(\mathcal{U}(t, \bar{\varphi}))\, dx - \langle \ell(t), \mathcal{U}(t, \bar{\varphi})\rangle_V \\ &\quad + \frac{\alpha}{2}\|\nabla\bar{\varphi}\|_2^2 + \frac{\beta}{2}\|\bar{\varphi} - z\|_2^2 \\ &= \mathcal{I}(t, z), \end{aligned} \tag{2.106}$$

in view of Definitions 0.2 and 2.1. Therewith the first assertion is proven.

Now, if z satisfies $\min(z, M) = z$, then (2.106) holds as an equality and thus, the inequality in (2.105) is an equality. Since the minimization problem in (P_β) is uniquely solvable, cf. Theorem 1.23, it holds $\min(\bar{\varphi}, M) = \Phi(t, \min(z, M)) = \Phi(t, z)$, which completes the proof. □

The next result states that the damage variables in the time-discrete problem are bounded a.e. in $(0,T)\times\Omega$ by a constant independent of β and of τ, which is just the constant M from Assumption 2.36, provided that this is fulfilled.

Lemma 2.38 (Uniform estimates for the discrete local and nonlocal damage)**.** *Suppose that Assumptions 1.5, 1.17, 1.56 and 2.36 hold true. Moreover, let the number of time-steps in the discrete problem be large enough. If $d_0 \in L^\infty(\Omega)$ with $\|d_0\|_{L^\infty(\Omega)} \leq M$, then*

$$\bar{d}_\tau(t,x) \leq M, \quad \bar{\varphi}_\tau(t,x) \leq M \text{ a.e. in } \Omega,\ \forall\, t \in [0,T].$$

Here $\bar{\varphi}_\tau$ stands again for $\Phi\big(\bar{t}_\tau(\cdot), \bar{d}_\tau(\cdot)\big)$.

Proof. Let n denote again the number of time-steps. We first address the discrete local damage by following the lines of the proof of [41, Proposition 4.5], that is, we show that $d_k^\tau \leq M$ for all $k \in \{0, ..., n\}$ by induction on the index k. Note that for $k = 0$ the assertion is fulfilled, since $d_0(x) \leq M$ a.e. in Ω, by assumption. Now let $k \in \{0, ..., n-1\}$ be fixed and assume that

$$d_k^\tau(x) \leq M \quad \text{ a.e. in } \Omega. \tag{2.107}$$

The idea of the proof is to show that $(d_{k+1}^\tau)^- := \min(d_{k+1}^\tau, M)$ solves the problem (2.96), which will give in turn $d_{k+1}^\tau = \min(d_{k+1}^\tau, M)$, in view of Lemma 2.35. From $(P_k^{\beta,\tau})$ it is clear that $d_{k+1}^\tau \geq d_k^\tau$, and as a result of (2.107) we thus have $(d_{k+1}^\tau)^- \geq d_k^\tau$. With Definition 0.4 we then obtain

$$\begin{aligned}
\mathcal{R}_\delta\Big(\frac{(d_{k+1}^\tau)^- - d_k^\tau}{\tau}\Big) &= r\int_\Omega \frac{(d_{k+1}^\tau)^- - d_k^\tau}{\tau}\,dx + \frac{\delta}{2}\Big\|\frac{(d_{k+1}^\tau)^- - d_k^\tau}{\tau}\Big\|_2^2 \\
&\leq r\int_\Omega \frac{d_{k+1}^\tau - d_k^\tau}{\tau}\,dx + \frac{\delta}{2}\Big\|\frac{d_{k+1}^\tau - d_k^\tau}{\tau}\Big\|_2^2 \\
&= \mathcal{R}_\delta\Big(\frac{d_{k+1}^\tau - d_k^\tau}{\tau}\Big),
\end{aligned} \tag{2.108}$$

where for the inequality we used $0 \leq (d_{k+1}^\tau)^- - d_k^\tau \leq d_{k+1}^\tau - d_k^\tau$. From Lemma 2.37 we further deduce

$$\mathcal{I}(t_{k+1}^\tau, (d_{k+1}^\tau)^-) \leq \mathcal{I}(t_{k+1}^\tau, d_{k+1}^\tau),$$

which added to (2.108) multiplied by τ leads to

$$\mathcal{I}(t_{k+1}^\tau, (d_{k+1}^\tau)^-) + \tau\mathcal{R}_\delta\Big(\frac{(d_{k+1}^\tau)^- - d_k^\tau}{\tau}\Big) \leq \mathcal{I}(t_{k+1}^\tau, d_{k+1}^\tau) + \tau\mathcal{R}_\delta\Big(\frac{d_{k+1}^\tau - d_k^\tau}{\tau}\Big).$$

On the other side, Lemma 2.35 tells us that d_{k+1}^τ is the unique solution of (2.96) and as $(d_{k+1}^\tau)^-$ is in the admissible set of (2.96), we conclude from the above inequality that $(d_{k+1}^\tau)^- = d_{k+1}^\tau$ must hold. Hence, $d_{k+1}^\tau \leq M$, which ends the induction step. Therefore, we have

$$\bar{d}_\tau(t,x) \leq M \quad \text{ a.e. in } \Omega,\ \forall\, t \in [0,T].$$

Furthermore, according to Lemma 2.37, it also holds $\bar{\varphi}_\tau(t,x) = \Phi\big(\bar{t}_\tau(t), \bar{d}_\tau(t)\big)(x) \leq M$ a.e. in Ω, for all $t \in [0,T]$. The proof is now complete. □

With the above results at hand, one can easily show that the local and the nonlocal damage associated to (P_β), as well as their limit as $\beta \to \infty$, are a.e. in $(0,T)\times\Omega$ bounded by the constant M from Assumption 2.36. This is covered by the following

Theorem 2.39 (L^∞-bound for the limit damage variable)**.** *Suppose that Assumptions 1.17, 1.47, 1.56, 2.8 and 2.36 are satisfied. Then, the optimal local and the optimal nonlocal damage associated to* (P_β) *fulfill the estimates*

$$0 \le d_\beta(t,x) \le M, \quad \varphi_\beta(t,x) \le M \qquad \text{a.e. in } (0,T)\times\Omega. \tag{2.109}$$

Moreover, (2.74) *admits at least one solution* φ *in* $L^\infty((0,T)\times\Omega)$ *with*

$$0 \le \varphi(t,x) \le M \qquad \text{a.e. in } (0,T)\times\Omega. \tag{2.110}$$

Proof. By (1.40) and Proposition 2.7, (*i*) $\Leftrightarrow$ (*iii*), we know that

$$d_\beta(t) = d_0 + \frac{1}{\delta}\int_0^t \max\left(-\beta(d_\beta(s)-\varphi_\beta(s)) - r\right) ds \quad \forall\, t\in[0,T].$$

Thus, the first inequality for the local damage in (2.109) is a result of Assumption 2.8. In addition, we observe that the sets $\{f \in L^\infty(0,T;L^2(\Omega)) : f(t,x) \le M$ a.e. in $(0,T)\times\Omega\}$ and $\{f \in L^\infty(0,T;H^1(\Omega)) : f(t,x) \le M$ a.e. in $(0,T)\times\Omega\}$ are closed. The estimate (2.109) now follows from (2.91) and Lemma 2.38, in view of Assumption 2.8.

Furthermore, Theorem 2.28 tells us that (2.74) admits at least one solution $\varphi \in H^1(0,T;H^1(\Omega))$ and that there is a sequence $\{\beta_n\}$ with $\beta_n \to \infty$ as $n \to \infty$ so that

$$d_{\beta_n} \rightharpoonup \varphi \text{ in } H^1(0,T;L^2(\Omega)) \quad \text{as } n\to\infty, \tag{2.111}$$

where d_{β_n} is the optimal local damage associated to (P_{β_n}). The estimates in (2.110) are thus a consequence of (2.111), the estimate for the local damage in (2.109) and the fact that the set $\{f \in H^1(0,T;L^2(\Omega)) : 0 \le f(t,x) \le M$ a.e. in $(0,T)\times\Omega\}$ is convex and closed, and thus, weakly closed. Of course, φ belongs to $L^\infty((0,T)\times\Omega)$, as it is measurable and satisfies (2.110). This completes the proof. □

Remark 2.40 (Non-negative values for the nonlocal damage)**.** *To make sure that the nonlocal damage has only non-negative values, as the local damage does, one has to impose, in addition to Assumptions 1.5, 1.17 and* $d_0 = 0$*, that* g *decreases on the interval* $(-\infty, 0)$*, i.e.,* $g'(x) \le 0$ *for all* $x < 0$*. Recall that this assumption on* g *is in fact very reasonable from a practical point of view. Then, by testing* (1.48b) *with* $\min\{\varphi_\beta(t), 0\}$ *at all* $t \in [0,T]$*, one obtains in* (1.48b) *a non-negative and a non-positive term on the left-hand side and right-hand side, respectively. Therefrom one concludes that* $\varphi_\beta \ge 0$ *indeed holds true.*

2.5 Comparison to classical partial damage models

In this section we show that the single-field gradient damage model given by (2.74) falls into the category of classical partial damage models. To be more specific, we prove that in the two-dimensional case (2.74) is equivalent to a version of the viscous damage model studied in [41] for which the body is sound at the beginning of the process. In this situation, the viscous problem in [41] reads

$$-\partial_z \overline{\mathcal{I}}(t, z(t)) \in \partial \overline{\mathcal{R}}_{\bar{\delta}}(\dot{z}(t)) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = 1. \tag{2.112}$$

To see a first similarity with our damage model, recall that (2.74) is in fact equivalent to (2.67) suplemmented with the initial condition, as shown by the proof of Theorem 2.28.

In order to distinguish between the two models, we add in what follows the symbol $\bar{}$ to the notations used for the data, functionals and operators in [41], in case that the therein used notations coincide with ours or if we already used them for data or global operators in the present work. In the sequel, $N = 2$ and Assumptions 1.47, 1.56, 2.8 and 2.36 are supposed to be satisfied the whole time. Observe that Assumption 1.17.1 is automatically fulfilled, in view of Lemma 1.3. Moreover, $\varphi \in H^1(0,T;H^1(\Omega))$ denotes from now on a solution of (2.74) with

$$0 \le \varphi(t,x) \le M \quad \text{a.e. in } (0,T) \times \Omega,$$

where M is the constant from Assumption 2.36. Note that such a solution exists, thanks to Theorem 2.39, see also the proof thereof.

The main difference between (2.74) and (2.112) consists in the definition of the dissipation functional. To see this, compare Definition 2.21 to [41, (1.3) and (1.9)], which tells us that the viscous dissipation $\bar{\mathcal{R}}_{\bar{\delta}} : H^1(\Omega) \to [0,\infty]$ is given by $\bar{\mathcal{R}}_{\bar{\delta}} = \bar{\mathcal{R}}_1 + \frac{\bar{\delta}}{2}\|\cdot\|_2^2$, where

$$\bar{\mathcal{R}}_1(\eta) := \begin{cases} \kappa \int_\Omega |\eta| \, dx, & \text{if } \eta \le 0 \text{ a.e. in } \Omega, \\ \infty, & \text{otherwise.} \end{cases}$$

Here $\kappa > 0$ and $\bar{\delta} > 0$ denote the fracture toughness and the viscosity parameter, respectively. Therefore, unlike in our situation, the therein considered damage variable, which is denoted by z, can only decrease in time. This is due to the fact that in [41] the damage variable $z : [0,T] \times \Omega \to \mathbb{R}$ measures the soundness of the material, not the degree of the material rigidity loss, as in our case. This means that, the larger the values of z, the sounder the body. Moreover, in the designed model, this takes values only in the interval $[0,1]$, so that $z(t,x) = 0$ and $z(t,x) = 1$ when the system is fully damaged and completely sound, respectively. We refer here also to [17], where the gradient damage model which serves as basis for [41] was introduced. In [41, Proposition 4.5] it is shown under additional assumptions that one has $z(t,x) \in [0,1]$ throughout the whole process, for at least one of the solutions of (2.112), thus proving the viability of the mathematical model in this regard. As shown by the computations below, see (2.115), these assumptions, in particular [41, (4.26)], correspond to those required to show (2.110).

The above motivates the following transformation

$$z := 1 - \frac{\varphi}{M} \in H^1(0,T;H^1(\Omega)). \tag{2.113}$$

The case $z = 0$ corresponds to $\varphi = M$ (maximal damage), while $z = 1$ corresponds to $\varphi = 0$ (complete soundness), so that (2.113) is a very reasonable starting point for showing the equivalence between the two models.

The function spaces and the assumptions on the data for (2.112) are introduced in [41, Sections 2.1 and 2.2]. We observe that the setting coincides mostly with ours, excepting the following:

- the energy functional $\widetilde{\mathcal{E}}$ associated to (2.67), see (2.47), does not include a counterpart for the term featuring the nonlinearity f in [41, (1.1)]. For this reason, we consider in the following just the case $f = 0$. Although the condition [41, (2.8)] does not allow the function f to be the zero function, this is here not problematic, as explained in Remark 2.44 below;

- in the energy functional in [41], see [41, Section 2.2], the degree of gradient regularization is 1. As in (2.47), this can be however replaced by some parameter $\bar{\alpha} > 0$, which does not affect the analysis in [41] at all;

- since in our model the Dirichlet boundary remains fixed during the damage process, we consider the special case $\boldsymbol{u}_D = 0$ in [41], see [41, (2.16)].

Under the above considerations, the energy functional $\bar{\mathcal{E}} : [0,T] \times V \times H^1(\Omega) \to \mathbb{R}$ cf. [41, (1.1)], is given by

$$\bar{\mathcal{E}}(t, \boldsymbol{u}, z) := \frac{1}{2}\int_\Omega \bar{g}(z)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\, dx - \langle \bar{\ell}(t), \boldsymbol{u}\rangle_V + \frac{\bar{\alpha}}{2}\|\nabla z\|_2^2, \tag{2.114}$$

which now corresponds entirely to the definition of the energy functional $\widetilde{\mathcal{E}}$ in (2.47).

We can now proceed towards our goal, which is to show that the function z defined in (2.113) satisfies the considered version of the viscous model (2.112). To this end, we have to transform the function g and resize α, the viscosity parameter δ and the fracture toughness r, see (2.115) and (2.117) below.

In view of (2.113), we define $\bar{g} : \mathbb{R} \to \mathbb{R}$ by

$$\overline{g}(x) := g(M(1-x)), \tag{2.115}$$

such that the following holds true

$$\overline{g}(z(t)) = g(\varphi(t)) \quad \forall\, t \in [0,T]. \tag{2.116}$$

Note that $\overline{g}$ satisfies condition [41, (2.10)] due to Assumptions 0.6 and 1.56. Moreover, we observe that this transformation is reversible, in the sense that, given $\overline{g}$, one can reobtain g via $g(x) := \overline{g}(1 - x/M)\ \forall\, x \in \mathbb{R}$. Note that the properties of $\bar{g}$ transfer to g as well. Recall that the coefficient function g assesses the degree of the material elasticity loss (under the influence of damage). That is why, in practical applications, this is considered to be monotonically decreasing, unlike $\bar{g}$, which should monotonically increase cf. [41, Remark 4.6.]. Note that this aspect is also confirmed by (2.115).

In order to obtain (2.112) we still need to resize the following data:

$$\bar{\alpha} := \alpha\, M^2, \tag{2.117a}$$

$$\bar{\delta} := \delta\, M^2, \tag{2.117b}$$

$$\kappa := rM. \tag{2.117c}$$

Observe that the transformations in (2.117) are reversible and that the positivity is preserved.

Since it is interesting to see how both models behave with respect to each other under the influence of the same external load, we impose

$$\bar{\ell} := \ell. \tag{2.118}$$

We now have all the necessary tools for proving that the limit model (2.74) can be transformed into a version of (2.112) and vice versa.

Proposition 2.41. *Let $N = 2$ and suppose that Assumptions 1.47, 1.56 and 2.8 are satisfied. Moreover, let φ be a solution of* (2.74) *and $M > 0$ be given. Then, the function $z := 1 - \varphi/M$ solves the problem* (2.112) *with $f = 0, \boldsymbol{u}_D = 0$ and parameters $\bar{g}, \bar{\alpha}, \bar{\delta}, \kappa$ and $\bar{\ell}$, given by* (2.115), (2.117a), (2.117b), (2.117c) *and* (2.118), *respectively. If φ is obtained via* (2.73) *and Assumption 2.36 is fulfilled, then $z(t,x) \in [0,1]$ f.a.a. $(t,x) \in (0,T) \times \Omega$.*

Proof. We begin by enumerating some results which will be very useful in what follows. Firstly, as a consequence of (2.113), we have

$$\varphi(t) = M(1 - z(t)) \quad \forall\, t \in [0,T], \tag{2.119a}$$

$$\dot{\varphi}(t) = -M\, \dot{z}(t) \qquad \text{f.a.a. } t \in (0,T), \tag{2.119b}$$

$$\nabla\varphi(t) = -M\nabla z(t) \qquad \forall\, t \in [0,T]. \tag{2.119c}$$

Secondly, since in the two-dimensional case the operator $g : H^1(\Omega) \to L^\tau(\Omega)$ is continuously Fréchet-differentiable for $\tau \in [1,\infty)$, see Lemma 5.3, the same holds for the operator $\bar{g}$, as this preserves the properties of g. Thus, on account of (2.115) and (2.119a) it holds

$$\bar{g}'(z(t)) = -Mg'(\varphi(t)) \quad \forall\, t \in [0,T]. \tag{2.120}$$

Further, notice that [41, (2.28)] (adapted to the here considered situation) reads

$$\partial_z \overline{\mathcal{I}}(t, z(t)) = -\bar{\alpha}\Delta z(t) + \frac{1}{2}\bar{g}'(z(t))\mathbb{C}\varepsilon\big(\bar{\boldsymbol{u}}(t)\big) : \varepsilon\big(\bar{\boldsymbol{u}}(t)\big) \quad \forall\, t \in [0,T], \tag{2.121}$$

where $\bar{\boldsymbol{u}}(t)$ solves at $t \in [0,T]$ the balance of momentum equation

$$-\operatorname{div}\big(\bar{g}(z(t))\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t))\big) = \bar{\ell}(t) \quad \text{in } \boldsymbol{W}^{-1,p}(\Omega). \tag{2.122}$$

To see this, we first recall that g and $\bar{g}$ have the same properties, and thus, Proposition 1.7 tells us that for given $(t,z) \in [0,T] \times H^1(\Omega)$, the functional $\overline{\mathcal{E}}(t,\cdot,z)$ is minimized on V by the unique solution $\boldsymbol{u}$ of

$$-\operatorname{div}\big(\bar{g}(z)\mathbb{C}\varepsilon(\boldsymbol{u})\big) = \ell(t) \quad \text{in } \boldsymbol{W}^{-1,p}(\Omega), \tag{2.123}$$

in view of (2.118). Analogously to Definitions 1.8 and 2.19 we introduce the solution operator $\overline{\mathcal{U}} : [0,T] \times H^1(\Omega) \ni (t,z) \mapsto \boldsymbol{u} \in \boldsymbol{W}^{1,p}(\Omega)$ associated with (2.123) and the reduced energy functional $\overline{\mathcal{I}} : [0,T] \times H^1(\Omega) \ni (t,z) \mapsto \overline{\mathcal{E}}(t,\overline{\mathcal{U}}(t,z),z)$, respectively. We refer here also to [41, Lemma 2.4]. Following the lines of Lemma 2.20, one then shows that $\overline{\mathcal{I}}$ is Fréchet-differentiable w.r.t. z and the identity (2.121) can be deduced in the exact same way as (2.49b). Observe that by relying on (2.116) and (2.118), (2.122) gives in turn

$$\bar{\boldsymbol{u}}(t) = \mathcal{U}(t,\varphi(t)) \quad \forall\, t \in [0,T], \tag{2.124}$$

on account of Definition 1.8. With (2.117a), (2.119c), (2.120) and (2.124), the identity (2.121) then becomes

$$\partial_z\overline{\mathcal{I}}(t,z(t)) = M\big(\alpha\Delta\varphi(t) - \frac{1}{2}g'(\varphi(t))\mathbb{C}\varepsilon\big(\mathcal{U}(t,\varphi(t))\big) : \varepsilon\big(\mathcal{U}(t,\varphi(t))\big)\big),$$

i.e.,

$$\partial_z\overline{\mathcal{I}}(t,z(t)) = -M\,\partial_\varphi\widetilde{\mathcal{I}}(t,\varphi(t)) \quad \forall\, t \in [0,T]. \tag{2.125}$$

Note that (2.125) is a result of (2.49b) and (1.23). Further, by comparing (2.76) and [41, (1.3)], we find in view of (2.119b) f.a.a. $t \in (0,T)$ the following

$$\begin{aligned}
&\xi \in \partial\widetilde{\mathcal{R}}_1(\dot\varphi(t)) \\
\Longleftrightarrow\ &\langle \xi, v - \dot\varphi(t)\rangle \le \widetilde{\mathcal{R}}_1(v) - \widetilde{\mathcal{R}}_1(\dot\varphi(t)) = r/\kappa\big(\overline{\mathcal{R}}_1(-v) - \overline{\mathcal{R}}_1(-\dot\varphi(t))\big) \\
\Longleftrightarrow\ &\langle -\xi, v - M\dot z(t)\rangle \le r/\kappa\big(\overline{\mathcal{R}}_1(v) - \overline{\mathcal{R}}_1(M\dot z(t))\big) \quad \forall\, v \in H^1(\Omega) \\
\Longleftrightarrow\ &-\kappa\,\xi/r \in \partial\overline{\mathcal{R}}_1(M\dot z(t)) \Longleftrightarrow -M\xi \in \partial\overline{\mathcal{R}}_1(\dot z(t)),
\end{aligned}$$

where for the last equivalence we used (2.117c) and the positive homogeneity of $\overline{\mathcal{R}}_1$. Thus, we have

$$\partial\overline{\mathcal{R}}_1(\dot z(t)) = -M\,\partial\widetilde{\mathcal{R}}_1(\dot\varphi(t)) \quad \text{f.a.a. } t \in (0,T). \tag{2.126}$$

Now, by means of (2.125), (2.119b), (2.117b), (2.126) and (2.113), (2.74) can be rewritten as

$$1/M\big(\partial_z\overline{\mathcal{I}}(t,z(t)) + \bar\delta\dot z(t)\big) \in -1/M\partial\overline{\mathcal{R}}_1(\dot z(t)) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = 1,$$

where we kept (2.49b) in mind. Thereby we deduce that z solves

$$-\partial_z\overline{\mathcal{I}}(t,z(t)) \in \partial\overline{\mathcal{R}}_1(\dot z(t)) + \bar\delta\dot z(t) \quad \text{f.a.a. } t \in (0,T), \quad z(0) = 1$$

and by applying sum rule for convex subdifferentials we ultimately obtain (2.112), in view of the definition of the viscous dissipation functional in [41, (1.9)].

The proof of Theorem 2.39 shows that if φ is obtained via (2.73) and Assumption 2.36 is fulfilled, then $z(t,x) \in [0,1]$ f.a.a. $(t,x) \in (0,T) \times \Omega$, as a result of (2.113). The proof is now complete. □

A short inspection of the proof of Proposition 2.41 (in particular, (2.125) and (2.126)) shows that (2.74) can be deduced from (2.112) as well:

Corollary 2.42. *Let* $N = 2$ *and suppose that [41, (2.10), (2.16)] are satisfied. Moreover, let* $M > 0$ *be given. If* (2.112) *with* $f = 0$, $\bar{\alpha} > 0$ *and* $\boldsymbol{u}_D = 0$ *admits a solution* $z \in H^1(0,T;H^1(\Omega))$*, then* $\varphi := M(1-z)$ *solves* (2.74) *with parameters* $g(\cdot) = \bar{g}(1 - \cdot/M), \alpha = \bar{\alpha}/M^2, \delta = \bar{\delta}/M^2, r = \kappa/M$ *and* $\ell = \bar{\ell}$.

Remark 2.43. *Note that if one replaces the initial condition in* (2.112) *by* $z(0) = z_0$ *a.e. in* Ω*, where* z_0 *is a positive constant function, then* (2.74) *and* (2.112) *are still equivalent in the sense of Proposition 2.41 and Corollary 2.42 via the transformation* $z := z_0\big(1 - \frac{\varphi}{M}\big)$*. Of course, this calls for redefining* $\bar{g}$ *in* (2.115) *accordingly. However, if* $z_0 \not\equiv 1$*, then the models are no longer compatible from a practical point of view, as the soundness of the body (*$\varphi(t,x) = 0$*) would correspond to* $z(t,x) = z_0 \neq 1$*.*

Remark 2.44. *Proposition 2.41 tells us that, by means of the penalization approach, the existence of viscous solutions for the here considered version of the model in [41] can also be established when* $f \equiv 0$ *in the energy given by [41, (1.1)]. Note that this case is excluded in [41] because of the growth condition in [41, (2.8)].*

Anyway, the nonlinearity f *can be easily incorporated in the problem* (P_β) *as a function acting on the nonlocal damage. A closer inspection of the preceding analysis then shows that [41, (2.8)] is indeed needed to prove the essential results leading to Theorem 2.28. The result in Proposition 2.41 then holds true with* $\overline{f}(\cdot) := f(M(1-\cdot))$*. We however decided not to consider this additional term and to rely on the energy from [12] instead, as this serves as basis for our penalized damage model. Note that if we include* f *in our investigations, the transformed system* (2.74)*, i.e.,* (2.112)*, is just a special case of [41, (1.8)] in two dimensions. Thus, the crucial result in [41] concerning the vanishing viscosity analysis can be applied to our viscous limit model.*

Chapter 3

Optimal control of the damage model with penalty

In this chapter we turn our attention to the damage model with penalty, this time in the context of optimal control. Recall that this reads as follows

$$\left.\begin{aligned} (\boldsymbol{u}(t),\varphi(t)) \in \underset{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)}{\arg\min}\ \mathcal{E}(t,\boldsymbol{u},\varphi,d(t)), \\ 0 \in \partial\mathcal{R}_\delta(\dot{d}(t)) + \partial_d\mathcal{E}(t,\boldsymbol{u}(t),\varphi(t),d(t)), \quad d(0)=d_0 \end{aligned}\right\} \tag{P_ℓ}$$

for almost all $t \in (0,T)$. Moreover, recall that (P_ℓ) describes in terms of the displacement $\boldsymbol{u}$, the nonlocal damage φ and the local damage d the effect of a force ℓ on an elastic body. For more details, see the introduction of the thesis. In many practical applications it is of interest to gain information about those loads which (locally) minimize a given cost functional, e.g. one may be interested to minimize the damage or/and the distance to a desired displacement. This motivates our goal in this chapter: to derive necessary optimality conditions for an optimal control problem governed by the damage model with penalty, where the load is used as control. The throughout this chapter studied optimization problem is given by

$$\begin{aligned} \min_{\ell\in\mathfrak{L}} \quad & \mathcal{J}(\boldsymbol{u},\varphi,d,\ell) \\ \text{s.t.} \quad & (\boldsymbol{u},\varphi,d) \text{ solves } (P_\ell) \text{ with right-hand side } \ell, \end{aligned} \tag{P_{min}}$$

where the assumptions on the objective $\mathcal{J}$ and the control set $\mathfrak{L}$ are to be introduced and motivated in Section 3.4 below.

Recall that in Chapter 1 we already studied the damage model with penalty for a fixed, smooth in time ℓ. It turned out that (P_ℓ) is uniquely solvable and that the continuous differentiability in time of the load transfers to the unique solution $(\boldsymbol{u},\varphi,d)$, see Theorem 1.62, p. 52. This was ultimately essential for the limit analysis $\beta \to \infty$ in the previous chapter, e.g. as we employed the chain rule identity. By contrast, in this chapter we work with *variable, nonsmooth in time* loads, which calls for a careful reinvestigation of the constraint in (P_{min}). The reason is two-fold:

(I) as the natural way to approach (P_{min}) is to define the control-to-state operator, we have to ask the question if the main results in Chapter 1 remain unaffected by the fact that ℓ *variates*. The answer is negative, as we will see in Section 3.1 below;

(II) we will not restrict only on smooth loads, but we choose the admissible set as general as possible (so that unique solvability of (P_ℓ) is still guaranteed), see also Remark 1.26, p. 25. In this case, the unique solution possesses much less regularity, insufficient for performing the limit analysis in Chapter 2. We refer here to Section 3.2 below.

As we will see, the arguments employed to establish unique solvability for (P_ℓ) in this new context are very similar to those in Chapter 1 and that is why we will not entirely repeat them. However, because of the reasons enumerated above, an accurate inspection of (P_ℓ) from this new perspective cannot be omitted.

The problem (P_{min}) falls into the class of optimal control problems governed by time-dependent VIs. To be more precise, (P_ℓ) can be formulated (under suitable assumptions) as a time-dependent VI of first and second kind, namely (2.15), see Proposition 2.7 on page 59. While the optimal control of time-dependent VIs of second kind was very little investigated, there are many contributions in the field of time-dependent VIs of first kind. We mention [2, 3, 15, 18, 25, 34, 35], which focus on the optimal control of the parabolic obstacle problem. In [81] the optimal control problem of quasistatic plasticity is analyzed, while [15] and [33] deal with the optimal control of the Allen-Cahn and Cahn-Hilliard VIs, respectively. In all these contributions nonsmooth constraints are considered, and thus, the standard method of deriving necessary optimality conditions in the form of Karush-Kuhn-Tucker conditions is not applicable. As we will see, this is also the case when it comes to the optimal control of the damage model with penalty.

Deriving necessary optimality conditions is a challenging issue even in the finite dimensional case, where a special attention is given to MPECs (mathematical programs with equilibrium constraints). We refer here only to [37, 65, 66, 72] and the references therein. In [72] a detailed overview of various optimality conditions of different strength was introduced for a special class of MPECs, namely MPCCs (mathematical programs with complementarity constraints). In the spirit thereof, stationarity concepts for the infinite dimensional case are defined in [32]. The most rigorous stationarity concept is strong stationarity. Roughly speaking, the strong stationarity conditions involve an optimality system, which is equivalent to the purely primal conditions. Other stationarity concepts such as Clarke (C) stationarity are less rigorous compared to strong stationarity. The weakest concept is weak stationarity, which involves the existence of multipliers, but no sign conditions for the multipliers at all.

The most common way to overcome the lack of differentiability of the control-to-state operator consists of employing regularization and relaxation techniques, see [2, 3, 15, 18, 25, 33, 34, 35, 81]. The thereby derived optimality systems are in the best case of intermediate strength such as C stationarity, see e.g. [33] (time-dependent) and [30, 34] (time-independent). For the optimal control of the parabolic obstacle problem a strong stationarity system can be found in [62], but no rigorous proof is given there. Recently, an optimality system of strong stationary type was derived in [50] for an optimal control problem governed by a nonsmooth parabolic PDE. This was possible due to the presence of so-called ‘ample controls’, which are in most of the existing contributions necessary for deriving strong stationarity, see e.g. [10, 31, 63] (elliptic VIs). To the best of our knowledge, [82] is the only paper where a strong stationary optimality system is derived in the absence thereof. We refer here to Remark 3.47 below for more details. Hence, it is not surprising, that without regularizing, additional assumptions are needed in order to derive an optimality system for (P_{min}). Therefor we also make use of the special structure of the constraint in (P_{min}). To be more precise, we employ the fact that (P_ℓ) can be reduced to an ordinary differential equation in Banach space (see Theorem 1.29 on page 26 and (3.38) below).

Outline of the chapter

The chapter is organized as follows. Section 3.1 focuses on reanalyzing the minimization problem in (P_ℓ) in the context where ℓ is a (at first time-independent) variable. In Section 3.2 we introduce the control-to-state operator by carefully choosing its domain of definition, while in Section 3.3 we investigate its differentiability, as a preparatory step for deriving first order necessary optimality conditions for (P_{min}). These are established in Section 3.4, where our focus lies on deriving an optimality system in a rather straight forward way. As already mentioned above, this can be done only under additional assumptions. As always, Assumption 1.17.1 on page 19 is supposed to hold throughout this chapter.

3.1 The dependence of the elliptic system on the load

The purpose of this section is to gain information about the dependence of the displacement $\boldsymbol{u}$ and of the nonlocal damage φ on the load ℓ. Recall that, in Subsection 1.1.1 we approached the minimization problem in (P_ℓ) in the following way. We investigated

$$\min_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \mathcal{E}(t,\boldsymbol{u},\varphi,d) \tag{3.1}$$

for fixed $(t,d)\in[0,T]\times L^2(\Omega)$ and showed that it admits a unique solution characterized by (1.25) on page 19, provided that Assumptions 1.5, p. 13 and 1.17, p. 19 hold. We refer here to Theorem 1.23, p. 23. This allowed us to define the solution operators $\mathcal{U}$ and Φ, see Definitions 1.8, p. 15 and 1.24, p. 24, respectively. The optimal displacement and optimal nonlocal damage for (P_ℓ) were then ultimately given by $\big(\mathcal{U}(\cdot,\Phi(\cdot,d(\cdot))),\Phi(\cdot,d(\cdot))\big)$ where d denotes the unique solution of the evolutionary equation in (P_ℓ).

In this section we present an alternative approach, which has the advantage of highlighting how the minimizer of (3.1) is related to the load. We begin by considering the following optimization problem

$$\min_{(\boldsymbol{u},\varphi)\in V\times H^1(\Omega)} \bar{\mathcal{E}}(\ell,\boldsymbol{u},\varphi,d), \tag{3.2}$$

where $(\ell, d) \in \boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega)$ is fixed. The functional $\bar{\mathcal{E}} : \boldsymbol{W}^{-1,p}(\Omega) \times V \times H^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$ is just a slight modification of the energy functional $\mathcal{E}$ on page 4 and it is given by

$$\bar{\mathcal{E}}(\ell, \boldsymbol{u}, \varphi, d) := \frac{1}{2} \int_\Omega g(\varphi)\mathbb{C}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u}) \, dx - \langle \ell, \boldsymbol{u} \rangle_V + \frac{\alpha}{2} \|\nabla\varphi\|_2^2 + \frac{\beta}{2} \|\varphi - d\|_2^2.$$

In view of Definition 0.2, p. 4 one sees that at all $(t, \boldsymbol{u}, \varphi, d) \in [0,T] \times V \times H^1(\Omega) \times L^2(\Omega)$ it holds

$$\bar{\mathcal{E}}(\ell(t), \boldsymbol{u}, \varphi, d) = \mathcal{E}(t, \boldsymbol{u}, \varphi, d) \tag{3.3}$$

for some given $\ell : [0,T] \to \boldsymbol{W}^{-1,p}(\Omega)$. Since we know that for a fixed $(t, d) \in [0,T] \times L^2(\Omega)$, (3.1) admits a unique solution, it is clear in view of (3.3) that (3.2) is uniquely solvable for a fixed pair $(\ell, d) \in \boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega)$. Moreover, we can deduce from Theorem 1.23, p. 23, that the minimizer $(\bar{\boldsymbol{u}}, \bar{\varphi})$ of (3.2) is characterized by

$$-\operatorname{div} g(\bar{\varphi})\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}) = \ell \quad \text{in } \boldsymbol{W}^{-1,p}(\Omega) \tag{3.4a}$$

$$-\alpha\Delta\bar{\varphi} + \beta\,\bar{\varphi} + \frac{1}{2} g'(\bar{\varphi})\mathbb{C}\,\varepsilon(\bar{\boldsymbol{u}}) : \varepsilon(\bar{\boldsymbol{u}}) = \beta d \quad \text{in } H^1(\Omega)^*. \tag{3.4b}$$

This can be seen by imposing that the load in Assumption 1.5, p. 13 has at any time point the same value $\ell \in \boldsymbol{W}^{-1,p}(\Omega)$. Of course, the above statements are true as long as Assumption 1.17 on page 19 is satisfied (with the threshold for β depending on $\|\ell\|_{\boldsymbol{W}^{-1,p}(\Omega)}$, cf. Remark 1.22, p. 23).

The above discussion calls for redefining the solution operators $\mathcal{U}$ and Φ, by dropping the time dependence and replacing it with dependence on some (time-independent) load. Although the domain changes (from $[0,T] \times L^2(\Omega)$ to $\boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega)$), we choose for convenience to remain by the same notations. Most of the properties of the new solution operators, such as continuity, boundedness, can be easily transferred from Subsection 1.1.1 by readapting the corresponding estimates. By using arguments analogous to those employed in Section 1.3, Fréchet-differentiability of the new solution operators is proven in the sequel as well. This serves as important tool in the upcoming section. However, one has to keep in mind that in all the previous sections the load was a smooth in time given data, whereas in what follows the load ℓ is a time-independent variable. Therefore, not all results in Sections 1.1 and 1.3 can be readily transferred for the whole space $\boldsymbol{W}^{-1,p}(\Omega)$. As it will turn out, it is sometimes required that one works with loads from a bounded subset thereof.

We begin by defining the new solution operators and replacing the needed properties from Subsection 1.1.1 accordingly.

Definition 3.1 (Solution operator of (3.4a))**.** *We define the operator* $\mathcal{U} : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to \boldsymbol{W}_D^{1,p}(\Omega)$ *as*

$$\mathcal{U}(\ell, \varphi) := A_\varphi^{-1}\ell,$$

where A_φ *is given by Definition 1.2 on page 12. We emphasize that* $\mathcal{U}$ *is well defined in view of Lemma 1.3, p. 12.*

Lemma 1.3 also gives immediately

$$\|\mathcal{U}(\ell,\varphi)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \leq c\|\ell\|_{\boldsymbol{W}^{-1,p}(\Omega)} \quad \forall\,(\ell,\varphi) \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega), \tag{3.5}$$

where $c > 0$ is independent of ℓ and φ.

The next result follows by using the exact same arguments as in the proof of Lemma 1.11, p. 16.

Lemma 3.2 (Continuity of $(\ell,\varphi) \mapsto \mathcal{U}(\ell,\varphi)$)**.** *Let $\{(\ell_n,\varphi_n)\} \subset \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$ and $(\ell,\varphi) \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$ be given such that $(\ell_n,\varphi_n) \to (\ell,\varphi)$ in $\boldsymbol{W}^{-1,p}(\Omega) \times L^1(\Omega)$ as $n \to \infty$. Then it holds $\mathcal{U}(\ell_n,\varphi_n) \to \mathcal{U}(\ell,\varphi)$ in $\boldsymbol{W}_D^{1,s}(\Omega)$ as $n \to \infty$ for every $s \in [2,p)$.*

As for the Lipschitz continuity of $\mathcal{U}$, we are interested in an estimate of the type (1.15), p. 15, for $r := 2p/(p-2)$. This will ultimately lead to an estimate for all $\varphi_1,\varphi_2 \in H^1(\Omega)$, as a consequence of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$. Note that the latter one is guaranteed in the two-dimensional case thanks to $p > 2$, and since throughout this section Assumption 1.17.1 on page 19 is supposed to hold, the embedding holds true in the three-dimensional case as well. By taking a closer look at the proof of Proposition 1.10 on page 15, we see that all the arguments employed therein can be readily transferred. However, due to (3.5) and $H^1(\Omega) \hookrightarrow L^r(\Omega)$, the estimate (1.18), p. 16, is now replaced by

$$\begin{aligned} \|(A_{\varphi_2} - A_{\varphi_1})\mathcal{U}(\ell_2,\varphi_2)\|_{V^*} &\leq C\,\|g(\varphi_1) - g(\varphi_2)\|_r\,\|\mathcal{U}(\ell_2,\varphi_2)\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \\ &\leq C\,\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}\,\|\ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)} \\ &\qquad \forall\,(\ell_i,\varphi_i)_{i=1,2} \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega). \end{aligned} \tag{3.6}$$

Note that the constant $C > 0$ is independent of $(\ell_i,\varphi_i)_{i=1,2}$. We point out that the modified estimates hold true for the same integrability exponents as in the proof of Proposition 1.10, which in particular means that $\pi = 2$, if $r = 2p/(p-2)$ (see page 16). By further using the exact same arguments as in the proof of Proposition 1.10, p. 15, (3.6) together with $\boldsymbol{W}^{-1,p}(\Omega) \hookrightarrow V^*$ yields in the end for all $(\ell_i,\varphi_i) \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$

$$\begin{aligned} \|\mathcal{U}(\ell_1,\varphi_1) - \mathcal{U}(\ell_2,\varphi_2)\|_V &\leq L\big(\|\ell_1 - \ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)} + \|\ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)}\|\varphi_1 - \varphi_2\|_r\big) \\ &\leq L\big(\|\ell_1 - \ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)} + \|\ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)}\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}\big), \end{aligned} \tag{3.7}$$

where $r = 2p/(p-2)$ and $L > 0$ is a constant which depends only on the given data. Although $\mathcal{U}(\cdot,\varphi)$ and $\mathcal{U}(\ell,\cdot)$ are globally Lipschitz continuous for any fixed $(\ell,\varphi) \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$, the same holds no longer for $\mathcal{U} : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to V$. In view of (3.7), we introduce

Definition 3.3 (Time-independent admissible loads)**.** *For a given $M > 0$ we define the open ball*

$$\mathcal{B}_M := B_{\boldsymbol{W}^{-1,p}(\Omega)}(0,M).$$

Remark 3.4. *For the moment we deliberately choose to work with open (bounded) subsets of* $\boldsymbol{W}^{-1,p}(\Omega)$*, instead of closed ones, since in the upcoming section this turns out to be more convenient, see for example the proof of Lemma 3.25 on page 121.*

Based on the available control space, an appropriate value for the constant M *will be determined in Section 3.4 below. Until then,* M *is allowed to take any positive value.*

The estimate (3.7) now results in

Lemma 3.5 (Lipschitz continuity of $(\ell, \varphi) \mapsto \mathcal{U}(\ell, \varphi)$)**.** *Let Assumption 1.17.1 on page 19 hold. Then, for every* $M > 0$*, there exists a constant* $L(M) > 0$ *so that for all* $\ell_1, \ell_2 \in \mathcal{B}_M$ *and all* $\varphi_1, \varphi_2 \in H^1(\Omega)$ *it holds*

$$\|\mathcal{U}(\ell_1, \varphi_1) - \mathcal{U}(\ell_2, \varphi_2)\|_V \leq L(M)(\|\ell_1 - \ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)} + \|\varphi_1 - \varphi_2\|_{H^1(\Omega)}).$$

Remark 3.6. *Notice that the operators in Definitions 1.8 (page 15) and 3.1 both map to* $\boldsymbol{W}^{1,p}_D(\Omega)$*. Further, notice that the solution operator* $\mathcal{U}$ *in Definition 3.1 is bounded and satisfies continuity and Lipschitz continuity with the same integrability exponents as its corresponding solution operator in Subsection 1.1.1, see Corollary 1.9 on page 15, Proposition 1.10 on page 15 and Lemma 1.11 on page 16. This means that replacing estimates which feature* $\mathcal{U}$ *with domain* $[0, T] \times H^1(\Omega)$*, by estimates which feature* $\mathcal{U}$ *considered with domain* $\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$*, does not affect any of the integrability exponents involved.*

We now turn our attention to (3.4b). Similarly to (1.23), p. 18 we introduce the following useful

Definition 3.7 (The nonlinear part of (3.4b))**.** *Suppose that Assumption 1.14 on page 18 is fulfilled. Then we define the mapping* $F : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to H^1(\Omega)^*$ *by*

$$\langle F(\ell, \varphi), \psi \rangle_{H^1(\Omega)} := \frac{1}{2} \int_\Omega g'(\varphi) \mathbb{C} \varepsilon(\mathcal{U}(\ell, \varphi)) : \varepsilon(\mathcal{U}(\ell, \varphi)) \psi \, dx \quad \forall \psi \in H^1(\Omega).$$

Notice that F *is well defined for the same reasons why the nonlinearity in Definition 1.15, p. 18 is well defined. We recall here for convenience that this follows from Hölder's inequality combined with* (3)*, Assumption 0.7, Definition 3.1 and the Sobolev embedding* $H^1(\Omega) \hookrightarrow L^{p/(p-2)}(\Omega)$*. Observe that the latter one is guaranteed by Assumption 1.14.*

In view of Definitions 3.1 and 3.7, the elliptic system (3.4) can now be written, similarly to (1.24), p. 19 in the compact form

$$B\bar{\varphi} + F(\ell, \bar{\varphi}) = \beta d, \tag{3.8}$$

where $(\ell, d) \in \boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega)$ is fixed, B is given by (1.22) on page 18 and $\bar{\varphi}$ is the second component of the unique minimizer of (3.2). At the beginning of the section, we argued that (3.4) is uniquely solvable under Assumption 1.17, p. 19 (where this time the threshold for β depends on $\|\ell\|_{\boldsymbol{W}^{-1,p}(\Omega)}$). Hence, (3.8) is uniquely solvable as well. Thus, the operator $B + F(\ell, \cdot)$ is invertible, however for fixed $\ell \in \boldsymbol{W}^{-1,p}(\Omega)$. This can also be seen by taking a look at the proofs of Propositions 1.12 and Lemma 1.21 (see pages 17 and 22), which ultimately led to the unique solvability of (1.24), p. 19.

Further, note that the reduced functional in the proof of Proposition 1.12 is now replaced by

$$f : H^1(\Omega) \to \mathbb{R}, \quad f(\varphi) := -\frac{1}{2}\langle \ell, \mathcal{U}(\ell, \varphi)\rangle + \frac{\alpha}{2}\|\nabla\varphi\|_2^2 + \frac{\beta}{2}\|\varphi - d\|_2^2,$$

where $(\ell, d) \in \boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega)$. One sees that working with an arbitrary ℓ does not affect the radially unboundedness and the weakly lower semicontinuity of f, which are crucial for proving existence of solutions for $\min_{\varphi\in H^1(\Omega)} f(\varphi)$ and thus, for (3.8). Notice that the latter one is the necessary optimality condition for a local optimum of the above minimizing problem. We refer here to the proof of Proposition 1.16 on page 19. Thus, as the proof of Proposition 1.12 shows, the existence of solutions for (3.8) remains unaffected when ℓ is a variable. That is no longer the case when it comes to the uniqueness thereof. The latter one was established in Section 1.1.1 by means of Lemma 1.21, p. 22, where Assumption 1.17.2 on page 20 played an essential role, since it guaranteed that β exceeds a given threshold. However, as already stated at the beginning of the present section on page 101 (see also Remark 1.22, p. 23), the threshold for β depends on the load ℓ, which implies that the strong monotonicity of $B + F(\ell, \cdot)$, see Lemma 1.21, p. 22, is no longer ensured when ℓ is a variable in $\boldsymbol{W}^{-1,p}(\Omega)$. This is shown by the following

Lemma 3.8. *Suppose that Assumption 1.17.1 on page 19 holds true. Then, for all $\ell_1, \ell_2 \in \boldsymbol{W}^{-1,p}(\Omega)$ and all $\varphi_1, \varphi_2 \in H^1(\Omega)$, $\varphi_1 \neq \varphi_2$ we have*

$$\begin{aligned}\frac{\langle B(\varphi_1 - \varphi_2) + F(\ell_1, \varphi_1) - F(\ell_2, \varphi_2), \varphi_1 - \varphi_2\rangle_{H^1(\Omega)}}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}}&\\ \geq \big(\alpha - \widetilde{c}(k)L(\ell_1, \ell_2)^2\big)\|\varphi_1 - \varphi_2\|_{H^1(\Omega)} - C\, L(\ell_1, \ell_2)\, \|\ell_1 - \ell_2\|& \qquad (3.9)\\ + \big(\beta - \alpha - kL(\ell_1, \ell_2)^2\big)\frac{\|\varphi_1 - \varphi_2\|_2^2}{\|\varphi_1 - \varphi_2\|_{H^1(\Omega)}} \quad \forall\, k > 0,&\end{aligned}$$

where we use the notation $\|\cdot\|$ for the $\boldsymbol{W}^{-1,p}(\Omega)$-norm and abbreviate $L(\ell_1, \ell_2) := \|\ell_1\| + \|\ell_2\|$. In the above estimate, $\widetilde{c} : \mathbb{R}^+ \to \mathbb{R}^+$ is a function which depends only on the given data and which satisfies $\widetilde{c}(k) \searrow 0$ as $k \nearrow \infty$.

Proof. We address the properties of the function F considered with domain $\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$, by going through the properties of its corresponding function in Lemmata 1.18 and 1.21 (see pages 20 and 22). We do so by writing down only those estimates which are directly affected by the new definition of the solution operator of (3.4a). We point out that all the integrability exponents in the proofs of Lemmata 1.18 and 1.21 remain unchanged in view of Remark 3.6.

Let $\ell_1, \ell_2 \in \boldsymbol{W}^{-1,p}(\Omega)$ and $\varphi_1, \varphi_2 \in H^1(\Omega)$ be arbitrary, but fixed. We define $\boldsymbol{u}_i := \mathcal{U}(\ell_i, \varphi_i)$ for $i = 1, 2$ and set $r := 2p/(p-2)$ such that $H^1(\Omega) \hookrightarrow L^r(\Omega)$ holds true, in view of Assumption 1.17.1. We begin by noticing that the estimate (1.28), p. 20 is replaced in view of (3.5) by

$$\|\mathbb{C}\varepsilon(\boldsymbol{u}_1) : \varepsilon(\boldsymbol{u}_1)\|_{\frac{p}{2}} \leq C\|\ell_1\|^2. \tag{3.10}$$

Instead of the estimate (1.30), p. 21, we have due to triangle inequality, (3.5) and (3.7) the following

$$\begin{aligned} \|\mathbb{C}\varepsilon(\boldsymbol{u}_1) : \varepsilon(\boldsymbol{u}_1) - \mathbb{C}\varepsilon(\boldsymbol{u}_2) : \varepsilon(\boldsymbol{u}_2)\|_\omega &\leq C\|\boldsymbol{u}_1 + \boldsymbol{u}_2\|_{\boldsymbol{W}^{1,p}_D(\Omega)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \\ &\leq CL(\ell_1,\ell_2)(\|\ell_2\|\,\|\varphi_1 - \varphi_2\|_r + \|\ell_1 - \ell_2\|), \end{aligned} \tag{3.11}$$

where $\omega = 2p/(p+2)$, so that Hölder's inequality with $1/\omega = 1/p + 1/2$ can be applied. By further following the lines of the proof of Lemma 1.18, we deduce that (3.10) and (3.11) lead to modifying the estimates (1.29), p. 21 and (1.31), p. 21, such that in the end it holds for all $\psi \in H^1(\Omega)$

$$\begin{aligned} |\langle F(\ell_1,\varphi_1) - F(\ell_2,\varphi_2), \psi\rangle| &\leq CL(\ell_1,\ell_2)\big(\|\ell_2\|\,\|\varphi_1 - \varphi_2\|_r + \|\ell_1 - \ell_2\|\big)\|\psi\|_r \\ &\quad + C\|\ell_1\|^2\|\varphi_1 - \varphi_2\|_r\|\psi\|_r \\ &\leq C\big(L(\ell_1,\ell_2)^2\,\|\varphi_1 - \varphi_2\|_r + L(\ell_1,\ell_2)\,\|\ell_1 - \ell_2\|\big)\|\psi\|_r. \end{aligned} \tag{3.12}$$

Notice that for $r = 2p/(p-2)$ we have $s = 2p/(p-2)$ in Lemma 1.18, in view of $1/s + 2/p + 1/r = 1$ (see page 20). From (3.12) we infer by Lemma 1.20 on page 22

$$\begin{aligned} |\langle F(\ell_1,\varphi_1) - F(\ell_2,\varphi_2), \varphi_1 - \varphi_2\rangle| &\leq L(\ell_1,\ell_2)^2\big(k\|\varphi_1 - \varphi_2\|_2^2 + \widetilde{c}(k)\|\varphi_1 - \varphi_2\|^2_{H^1(\Omega)}\big) \\ &\quad + C\,L(\ell_1,\ell_2)\,\|\ell_1 - \ell_2\|\,\|\varphi_1 - \varphi_2\|_{H^1(\Omega)} \quad \forall\, k > 0. \end{aligned} \tag{3.13}$$

Note that, in view of Lemma 1.20, $\widetilde{c} : \mathbb{R}^+ \to \mathbb{R}^+$ depends on the constant C from (3.12), on p, and on N, and thus, only on the given data, see also Remark 1.27, p. 25. Now, the definition of B in (1.22), p. 18 and (3.13) finally give the assertion. □

Let us now take a look at $B + F(\ell,\cdot)$ regarding its strong monotonicity for some arbitrary $\ell \in \boldsymbol{W}^{-1,p}(\Omega)$. By setting $\ell_1 = \ell_2 = \ell$ in (3.9) we obtain for all $\varphi_1, \varphi_2 \in H^1(\Omega)$

$$\begin{aligned} \langle B(\varphi_1 - \varphi_2) + F(\ell,\varphi_1) - F(\ell,\varphi_2), \varphi_1 - \varphi_2\rangle &\geq \big(\alpha - 4\widetilde{c}(k)\|\ell\|^2\big)\|\varphi_1 - \varphi_2\|^2_{H^1(\Omega)} \\ &\quad + \big(\beta - \alpha - 4k\|\ell\|^2\big)\|\varphi_1 - \varphi_2\|_2^2 \quad \forall\, k > 0, \end{aligned} \tag{3.14}$$

where $\widetilde{c} : \mathbb{R}^+ \to \mathbb{R}^+$ has the same properties as in Lemma 3.8. We use for convenience again the notation $\|\cdot\|$ for the $\boldsymbol{W}^{-1,p}(\Omega)$-norm. From the estimate (3.14) we read that, in order to ensure the strong monotonicity of $B + F(\ell,\cdot)$, we must firstly choose k large enough such that $C_1 := \alpha - 4\widetilde{c}(k)\|\ell\|^2 > 0$. As in the proof of Lemma 1.21, p. 22, we again choose k such that $C_1 \geq \alpha/2$, i.e., such that $4\widetilde{c}(k)\|\ell\|^2 \leq \alpha/2$ holds. Note that such a k exists in view of the properties of $\widetilde{c}$ and it depends on the given data and $\|\ell\|$. Further, it is crucial that

$$\beta \geq \alpha + 4k\|\ell\|^2, \tag{3.15}$$

which in view of (3.14) would give in turn the strong monotonicity of $B + F(\ell, \cdot)$. Of course, (3.15) is fulfilled for a fixed load, if β is chosen sufficiently large, but, if the loads vary, as in case of optimal control problems, one cannot guarantee (3.15) with a fixed value of β when we allow the loads to be arbitrary functions in $\boldsymbol{W}^{-1,p}(\Omega)$. The situation changes however, if we turn to sets of admissible controls that are bounded in $\boldsymbol{W}^{-1,p}(\Omega)$, as we will next see. Observe that for $\ell \in \mathcal{B}_M$, k can be chosen independent of ℓ, such that in view of the properties of the function $\widetilde{c}$, it holds

$$\alpha/2 \geq 4\widetilde{c}(k)M^2 > 4\widetilde{c}(k)\|\ell\|^2. \tag{3.16}$$

Then, if additionally,

$$\beta \geq \alpha + 4kM^2 > \alpha + 4k\|\ell\|^2 \tag{3.17}$$

holds true by Assumption 1.17.2, p. 20, the estimate (3.14) can be continued in view of (3.16) as

$$\langle B(\varphi_1 - \varphi_2) + F(\ell, \varphi_1) - F(\ell, \varphi_2), \varphi_1 - \varphi_2 \rangle_{H^1(\Omega)} \geq \alpha/2\|\varphi_1 - \varphi_2\|^2_{H^1(\Omega)} \tag{3.18}$$

for all $\varphi_1, \varphi_2 \in H^1(\Omega)$ and all $\ell \in \mathcal{B}_M$.

Remark 3.9. *We point out that the threshold for β in* (3.17) *depends now only on the given data, and in particular on M, instead of the (variable) load, see also* (3.16) *and Lemma 3.8. The constant M, although fixed, may take different values depending on the context. That is why we will sometimes emphasize the dependency of β on M by writing $\beta = \beta(M)$, which basically means that β satisfies the first inequality in* (3.17)*, where k is given by the first inequality in* (3.16).

Since we already established above that (3.8) admits solutions for $(\ell, d) \in \boldsymbol{W}^{-1,p} \times L^2(\Omega)$, (3.18) together with Cauchy-Schwarz inequality yields now, similarly to Theorem 1.23 on page 23, the existence of the solution operator

Definition 3.10 (Solution operator of (3.8))**.** *Let Assumption 1.17 on page 19 hold and let $M > 0$ be given. Moreover, let $\beta = \beta(M)$. Then we define the operator $\Phi : \mathcal{B}_M \times L^2(\Omega) \to H^1(\Omega)$ as*

$$\Phi(\ell, d) := \big(B + F(\ell, \cdot)\big)^{-1}(\beta d),$$

where B is given by (1.22) *on page 18 .*

Lemma 3.11 (Lipschitz continuity of $(\ell, d) \mapsto \Phi(\ell, d)$)**.** *Let Assumption 1.17 on page 19 hold. Moreover, let $M > 0$ be given and $\beta = \beta(M)$. Then, there exists a constant $L(M) > 0$ so that for all $\ell_1, \ell_2 \in \mathcal{B}_M$ and all $d_1, d_2 \in L^2(\Omega)$ it holds*

$$\|\Phi(\ell_1, d_1) - \Phi(\ell_2, d_2)\|_{H^1(\Omega)} \leq L(M)(\|\ell_1 - \ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)} + \|d_1 - d_2\|_2).$$

Proof. Let $M > 0$ and $(\ell_i, d_i) \in \mathcal{B}_M \times L^2(\Omega)$, $i = 1, 2$, be arbitrary, but fixed. Further, let us abbreviate $\varphi_i := \Phi(\ell_i, d_i)$, $i = 1, 2$. Note that the existence of the solution operator

Φ is guaranteed by assumption. We suppose that $\varphi_1 \neq \varphi_2$, by keeping in mind that, otherwise, the Lipschitz continuity of Φ is trivial. We prove the result by using the exact same arguments as in the proof of Theorem 1.23, p. 23. To this end, we first have to derive an estimate similar to the one in Lemma 1.21, p. 22. Since Assumption 1.17.2, p. 20 is fulfilled with $\beta = \beta(M)$, we know in view of Remark 3.9 that $\beta \geq \alpha + 4kM^2$, where k is computed such that $\alpha/2 \geq 4\widetilde{c}(k)M^2$ holds. Since $\ell_1, \ell_2 \in \mathcal{B}_M$, we have

$$\alpha/2 > \widetilde{c}(k)L(\ell_1,\ell_2)^2, \quad \beta > \alpha + kL(\ell_1,\ell_2)^2.$$

Here, the symbols $\widetilde{c}$ and $L(\ell_1,\ell_2)$ have the same meaning as in Lemma 3.8, from which we now obtain

$$\begin{aligned}\frac{\langle B(\varphi_1-\varphi_2)+F(\ell_1,\varphi_1)-F(\ell_2,\varphi_2),\varphi_1-\varphi_2\rangle}{\|\varphi_1-\varphi_2\|_{H^1(\Omega)}} &\geq \alpha/2\|\varphi_1-\varphi_2\|_{H^1(\Omega)} \\ &\quad - C\,L(\ell_1,\ell_2)\,\|\ell_1-\ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)},\end{aligned} \tag{3.19}$$

where $C > 0$ depends only on the given data. The rest of the proof is entirely analogous to the proof of Theorem 1.23, p. 23, where instead of (1.37) we now have, in view of (3.19), Definition 3.10, and Cauchy-Schwarz inequality, the estimate

$$\begin{aligned}\alpha/2\|\varphi_1-\varphi_2\|^2_{H^1(\Omega)} &\leq \langle B(\varphi_1-\varphi_2)+F(\ell_1,\varphi_1)-F(\ell_2,\varphi_2),\varphi_1-\varphi_2\rangle_{H^1(\Omega)} \\ &\quad + C\,L(\ell_1,\ell_2)\,\|\ell_1-\ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)}\|\varphi_1-\varphi_2\|_{H^1(\Omega)} \\ &= \beta(d_1-d_2,\varphi_1-\varphi_2)_2 + C\,L(\ell_1,\ell_2)\,\|\ell_1-\ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)}\|\varphi_1-\varphi_2\|_{H^1(\Omega)} \\ &\leq \big(\beta\|d_1-d_2\|_2 + C\,L(\ell_1,\ell_2)\,\|\ell_1-\ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)}\big)\|\varphi_1-\varphi_2\|_{H^1(\Omega)}.\end{aligned}$$

Dividing by $\|\varphi_1-\varphi_2\|_{H^1(\Omega)}$ finally gives the assertion with $L(M) := 2/\alpha \max\{\beta, 2CM\}$. $\square$

Remark 3.12. *Actually, the solution operator Φ maps $\mathcal{B}_M \times L^2(\Omega)$ to $\boldsymbol{W}^{1,q}(\Omega)$, where $M > 0$ is fixed and $q > 2$ is given by Theorem 1.37 on page 31. Moreover, Φ satisfies the corresponding Lipschitz continuity condition. This is due to the fact that all the results in Section 1.2, p. 29, can be readily transferred for the operator in Definition 3.10.*

Remark 3.13. *So far, we have seen that, if $(\ell,d) \in \mathcal{B}_M \times L^2(\Omega)$, where $M > 0$ is fixed, then the dependence on (ℓ,d) of the optimal displacement and optimal nonlocal damage in* (3.2) *can be expressed under Assumption 1.17, p. 19 (where $\beta = \beta(M)$), via the mapping $\mathcal{B}_M \times L^2(\Omega) \ni (\ell,d) \mapsto (\mathcal{U}(\ell,\Phi(\ell,d)),\Phi(\ell,d)) \in \boldsymbol{W}^{1,p}_D(\Omega) \times H^1(\Omega)$. In other words, the latter one is the solution operator of the system* (3.4).

The rest of the section is devoted to proving that, as their corresponding operators in Subsection 1.1.1, p. 11, the new solution operators $\mathcal{U}$ and Φ are also Fréchet-differentiable. This is an essential tool in the upcoming section. We begin with the differentiability result for the operator $\mathcal{U}$, which is covered by the following

Lemma 3.14 (Fréchet-differentiability of $(\ell,\varphi) \mapsto \mathcal{U}(\ell,\varphi)$)**.** *Under Assumption 1.17.1 on page 19 it holds $\mathcal{U} \in C^1(\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega); V)$.*

Proof. We proceed as in Subsection 1.3.1, p. 41, that is, we show that $\mathcal{U}$ is partially continuously Fréchet-differentiable, which in view of [9, Theorem 3.7.1] will give the assertion. To this end, first notice that the operator $\mathcal{U} : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to \boldsymbol{W}^{1,p}_D(\Omega)$ is partially Fréchet-differentiable with respect to ℓ. This is a consequence of Definition 3.1 combined with Lemma 1.3, p. 12, which tells us that $A_\varphi^{-1} \in \mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega), \boldsymbol{W}^{1,p}_D(\Omega))$ for every $\varphi \in H^1(\Omega)$. Therefrom we also deduce

$$\partial_\ell \mathcal{U}(\ell, \varphi) = A_\varphi^{-1} \quad \forall\, (\ell, \varphi) \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega). \tag{3.20}$$

Now let $\{\varphi_n\} \subset H^1(\Omega)$ and $\varphi \in H^1(\Omega)$ be given such that $\varphi_n \to \varphi$ in $H^1(\Omega)$ as $n \to \infty$. Since Assumption 1.17.1 on page 19 is supposed to hold, we can apply (3.7), which together with Definition 3.1 leads to

$$\|A_{\varphi_n}^{-1}\ell - A_\varphi^{-1}\ell\|_V \leq L\|\ell\|_{\boldsymbol{W}^{-1,p}(\Omega)}\|\varphi_n - \varphi\|_{H^1(\Omega)} \quad \forall\, \ell \in \boldsymbol{W}^{-1,p}(\Omega),\ \forall\, n \in \mathbb{N},$$

whence

$$\|A_{\varphi_n}^{-1} - A_\varphi^{-1}\|_{\mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega),V)} \leq L\|\varphi_n - \varphi\|_{H^1(\Omega)} \to 0 \text{ as } n \to \infty.$$

Thus, in view of (3.20), $\mathcal{U} : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to V$ is partially continuously Fréchet-differentiable with respect to ℓ.

By comparing Definition 1.8 on page 15 with Definition 3.1, one sees that the partial Fréchet-differentiability with respect to φ of the operator $\mathcal{U}$ in Definition 3.1 is also given by Lemma 1.49 on page 42. To be more precise, the latter one guarantees that $\mathcal{U}(\ell, \cdot) : H^1(\Omega) \to \boldsymbol{W}^{1,\nu}_D(\Omega)$ is Fréchet-differentiable for every $\ell \in \boldsymbol{W}^{-1,p}(\Omega)$ for some $\nu \in (2, p)$ (see (1.76), p. 42). Moreover, the partial derivative at $(\ell, \varphi) \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$ in direction $\delta\varphi \in H^1(\Omega)$ fulfills

$$A_\varphi\big(\partial_\varphi \mathcal{U}(\ell, \varphi)(\delta\varphi)\big) = \operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(\ell, \varphi))\big) \text{ in } \boldsymbol{W}^{-1,\nu}(\Omega). \tag{3.21}$$

Let us now derive from the above identity an estimate which will be useful later in the ongoing proof. To this end, let ϱ and κ be the indices defined in the proof of Lemma 1.49, p. 42. Note that, due to $H^1(\Omega) \hookrightarrow L^\varrho(\Omega)$ and $\kappa < \varrho$, the embedding $H^1(\Omega) \hookrightarrow L^\kappa(\Omega)$ holds true. From Lemma 1.3, p. 12, an estimate similar to (1.78) on page 42, and (3.5) we then infer

$$\begin{aligned}\|\partial_\varphi \mathcal{U}(\ell, \varphi)(\delta\varphi)\|_{\boldsymbol{W}^{1,\nu}_D(\Omega)} &\leq C\,\|\operatorname{div}\big(g'(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(\ell, \varphi))\big)\|_{\boldsymbol{W}^{-1,\nu}(\Omega)}\\ &\leq C\,\|\ell\|_{\boldsymbol{W}^{-1,p}(\Omega)}\|\delta\varphi\|_\kappa\\ &\leq C\,\|\ell\|_{\boldsymbol{W}^{-1,p}(\Omega)}\|\delta\varphi\|_{H^1(\Omega)} \quad \forall\, \ell \in \boldsymbol{W}^{-1,p}(\Omega),\ \forall\, \varphi, \delta\varphi \in H^1(\Omega).\end{aligned} \tag{3.22}$$

In order to prove the continuity of $\partial_\varphi \mathcal{U} : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to \mathcal{L}(H^1(\Omega), V)$, we follow the lines of the proof of Lemma 1.51, p. 44. We emphasize that the arguments are exactly the same and, in view of Remark 3.6, there is no need to discuss the integrability exponents. These readily transfer and are denoted here by the same symbols as in the proof of Lemma 1.51, p. 44. Let $(\ell_i, \varphi_i)_{i=1,2} \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$ and $\delta\varphi \in H^1(\Omega)$ be arbitrary, but fixed with $\delta\varphi \neq 0$. Further, let us abbreviate $\boldsymbol{u}_i' := \partial_\varphi \mathcal{U}(\ell_i, \varphi_i)\delta\varphi$ and

$\boldsymbol{u}_i := \mathcal{U}(\ell_i, \varphi_i)$ for $i = 1, 2$. Moreover, we define the linear forms $f_1 := A_{\varphi_2}\boldsymbol{u}_2' - A_{\varphi_1}\boldsymbol{u}_2' \in V^*$ and $f_2 := A_{\varphi_1}\boldsymbol{u}_1' - A_{\varphi_2}\boldsymbol{u}_2' \in V^*$, which we now intend to estimate by relying on the arguments used in the proof of Lemma 1.51, p. 44. Hence, the estimate for f_1 is, in view of (3.22), given by

$$\begin{aligned}\|f_1\|_{V^*} &\le C_1 \|g(\varphi_2) - g(\varphi_1)\|_\rho \|\boldsymbol{u}_2'\|_{\boldsymbol{W}_D^{1,\nu}(\Omega)} \\ &\le C_1 \|g(\varphi_2) - g(\varphi_1)\|_\rho \ \|\ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)} \|\delta\varphi\|_{H^1(\Omega)},\end{aligned}$$

while, due to (3.5), the estimate for f_2 reads

$$\begin{aligned}\|f_2\|_{V^*} &\le C_2 \|g'(\varphi_1)(\delta\varphi)\|_\varrho \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,s}(\Omega)} + \|(g'(\varphi_2) - g'(\varphi_1))(\delta\varphi)\|_r \|\boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,p}(\Omega)} \\ &\le C_2 \|\delta\varphi\|_{H^1(\Omega)} \big(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{\boldsymbol{W}_D^{1,s}(\Omega)} + \|g'(\varphi_2) - g'(\varphi_1)\|_{\mathcal{L}(H^1(\Omega), L^r(\Omega))} \|\ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)} \big).\end{aligned}$$

By employing Lemmata 5.2, 3.2 and 5.3, the new estimates for f_1 and f_2 then give

$$\sup_{\delta\varphi \in H^1(\Omega), \delta\varphi \neq 0} \frac{\|f_1\|_{V^*} + \|f_2\|_{V^*}}{\|\delta\varphi\|_{H^1(\Omega)}} \to 0 \quad \text{as } (\ell_1, \varphi_1) \to (\ell_2, \varphi_2) \text{ in } \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega).$$

This allows us, in the exact same way as at the end of the proof of Lemma 1.51 on page 44, to conclude that $\partial_\varphi \mathcal{U} : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to \mathcal{L}(H^1(\Omega), V)$ is continuous. The above results now give the assertion in view of [9, Theorem 3.7.1]. □

Notice that $\mathcal{U}$ can be differentiated on the whole space $\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$, because the Lipschitz continuity thereof (see Lemma 3.5) is not needed in the above proof. This is of course no longer the case when it comes to Φ, as this is not even defined on $\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$.

Before addressing the differentiability thereof, we deduce by means of the arguments used for deriving Lemma 1.50 on page 44 and (3.5) that

$$\|\partial_\varphi \mathcal{U}(\ell, \varphi)(\delta\varphi)\|_V \le C \, \|\ell\|_{\boldsymbol{W}^{-1,p}(\Omega)} \|\delta\varphi\|_r \quad \forall\, \ell \in \boldsymbol{W}^{-1,p}(\Omega), \ \forall\, \varphi, \delta\varphi \in H^1(\Omega), \quad (3.23)$$

where $r = 2p/(p-2)$. This estimate turns out to be useful for proving the following

Lemma 3.15 (Fréchet-differentiability of $(\ell, d) \mapsto \Phi(\ell, d)$)**.** *Let $M > 0$ be given and suppose that Assumptions 1.17, p. 19 (with $\beta = \beta(M)$) and 1.56, p. 46 hold. Then $\Phi \in C^1(\mathcal{B}_M \times L^2(\Omega); H^1(\Omega))$. Moreover, its derivative satisfies at all $(\ell, d) \in \mathcal{B}_M \times L^2(\Omega)$ and all $(\delta\ell, \delta d) \in \boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega)$*

$$\left.\begin{aligned} -\alpha\Delta\delta\phi + \beta\delta\phi + \frac{1}{2} g''(\Phi(\ell, d))\delta\phi \mathbb{C}\varepsilon(\mathcal{U}(\ell, \Phi(\ell, d))) : \varepsilon(\mathcal{U}(\ell, \Phi(\ell, d))) \\ = \beta\delta d - g'(\Phi(\ell, d))\mathbb{C}\varepsilon(\mathcal{U}(\ell, \Phi(\ell, d))) : \varepsilon(\delta\boldsymbol{u}) \quad \text{in } H^1(\Omega)^*, \end{aligned}\right\} \quad (3.24)$$

where $\delta\phi$ and $\delta\boldsymbol{u}$ stand for $\Phi'(\ell, d)(\delta\ell, \delta d)$ and $\mathcal{U}'(\ell, \Phi(\ell, d))(\delta\ell, \delta\phi)$, respectively.

Proof. In order to differentiate the operator Φ with domain $\mathcal{B}_M \times L^2(\Omega)$ we carefully follow the lines of the proofs on pages 47-51, which, by the implicit function theorem, ultimately led to the continuously Fréchet-differentiability of the operator from Definition 1.24, p. 24. When transferring the results, we use for the sake of convenience the same notations for the involved functions, by keeping in mind that the time dependence is now replaced by the dependence on ℓ. In view of Remark 3.6 and since the new solution operator Φ has values in the same space as its corresponding operator in Chapter 1 (cf. Definitions 1.24 and 3.10), there is no need to discuss the integrability exponents. These will be in the following denoted by the same symbols as in the proofs from which they readily transfer. When readapting the results from Subsection 1.3.2 (pages 47-51) we go into detail only when it comes to results which are affected by the fact that $\mathcal{U}$ and Φ are now considered with other domains as in Subsection 1.1.1.

The ongoing proof is structured as follows. In the first part we establish the continuously Fréchet-differentiability of the mapping

$$\Psi : \boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega) \times H^1(\Omega) \to H^1(\Omega)^*, \quad \Psi(\ell, d, \varphi) := B\varphi + F(\ell, \varphi) - \beta d, \quad (3.25)$$

where B and F are given by (1.22), p. 18 and Definition 3.7, respectively. For purposes of comparison we refer here to Definition 1.55, Lemma 1.57 and Corollary 1.58 (pages 46-49). In the second part we show that $\partial_\varphi \Psi(\ell, d, \varphi) : H^1(\Omega) \to H^1(\Omega)^*$ is bijective for all $(\ell, d, \varphi) \in \mathcal{B}_M \times L^2(\Omega) \times H^1(\Omega)$, compare with Lemma 1.59 on page 49. As in Proposition 1.60, p. 51, these two results will finally give the assertion.

(I) Next we focus on proving that $\Psi \in C^1(\boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega) \times H^1(\Omega); H^1(\Omega)^*)$. As in Subsection 1.3.2, the challenging part is to show that $F : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to H^1(\Omega)^*$ is continuously Fréchet-differentiable. To differentiate F, we proceed as in the proof of Lemma 1.57, p. 47, i.e., we reformulate F by means of two products and apply Lemma 5.4 for these. To this end, we introduce

$$\mathcal{H} : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to L^{p/2}(\Omega), \quad \mathcal{H}(\ell, \varphi) := \mathbb{C}\varepsilon(\mathcal{U}(\ell, \varphi)) : \varepsilon(\mathcal{U}(\ell, \varphi))$$

such that

$$F : (\ell, \varphi) \mapsto P_1\big(g'(\varphi), \mathcal{H}(\ell, \varphi)\big), \quad (3.26)$$

where P_1 is the product defined in (1.88) on page 47. By doing so, we see that the domain of $\mathcal{H}$ changes in the proof of Lemma 1.57, p. 47. We now intend to show that the therein made statements about $\mathcal{H}$ still hold true, when this is considered with domain $\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$. In other words, we verify that $\mathcal{H} : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to L^{p/2}(\Omega)$ is well defined, continuous and continuously Fréchet-differentiable when considered as operator with ranges in the same spaces as in the proof of Lemma 1.57, p. 47.

(i) In view of Hölder's inequality and Definition 3.1, the operator $\mathcal{H} : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to L^{p/2}(\Omega)$ is well defined.

(ii) Regarding the continuity of $\mathcal{H} : \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega) \to L^{\omega}(\Omega)$, we estimate again similarly to (1.30), p. 21, by using (1.90), p. 47, which yields

$$\begin{aligned}\|\mathcal{H}(\ell_1,\varphi_1)-\mathcal{H}(\ell_2,\varphi_2)\|_\omega &\le C\,\|\mathcal{U}(\ell_1,\varphi_1)+\mathcal{U}(\ell_2,\varphi_2)\|_{\boldsymbol{W}^{1,p}_D(\Omega)}\|\mathcal{U}(\ell_1,\varphi_1)-\mathcal{U}(\ell_2,\varphi_2)\|_{\boldsymbol{W}^{1,s}_D(\Omega)}\\ &\le C\,\big(\|\ell_1\|_{\boldsymbol{W}^{-1,p}(\Omega)}+\|\ell_2\|_{\boldsymbol{W}^{-1,p}(\Omega)}\big)\|\mathcal{U}(\ell_1,\varphi_1)-\mathcal{U}(\ell_2,\varphi_2)\|_{\boldsymbol{W}^{1,s}_D(\Omega)}\\ &\qquad\forall\,(\ell_i,\varphi_i)_{i=1,2}\in \boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega).\end{aligned}$$

Note that for the last inequality we used (3.5). Lemma 3.2 then gives the desired continuity of $\mathcal{H}$, since $s\in(2,p)$ in the proof of Lemma 1.57, p. 47.

(iii) Let us now check if $\mathcal{H}:\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)\to L^{\varrho}(\Omega)$ is continuously Fréchet-differentiable. For this purpose, we write $\mathcal{H}$ as

$$\mathcal{H}:(\ell,\varphi)\mapsto P_2\big(\mathcal{U}(\ell,\varphi),\mathcal{U}(\ell,\varphi)\big), \tag{3.27}$$

where P_2 is the product defined in (1.91) on page 48. As in the proof of Lemma 1.57, p. 47, we then fix the setting in Lemma 5.4 accordingly. That is, we define

$$\begin{gathered}U:=X:=\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega),\quad W:=L^{\varrho}(\Omega),\\ P:=P_2,\quad f_i:=\mathcal{U},\quad Y_i:=\boldsymbol{W}^{1,s}_D(\Omega),\quad Z_i:=V,\quad i=1,2.\end{gathered}$$

From Lemmata 3.2 and 3.14 we know that $\mathcal{U}:\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)\to\boldsymbol{W}^{1,s}_D(\Omega)$ is continuous and $\mathcal{U}:\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)\to V$ is continuously Fréchet-differentiable, respectively. Hence, we can apply Lemma 5.4 to (3.27), giving in turn that $\mathcal{H}:\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)\to L^{\varrho}(\Omega)$ is continuously Fréchet-differentiable with

$$\mathcal{H}'(\ell,\varphi)(\delta\ell,\delta\varphi):=2\,\mathbb{C}\varepsilon(\mathcal{U}(\ell,\varphi)):\varepsilon\big(\mathcal{U}'(\ell,\varphi)(\delta\ell,\delta\varphi)\big) \tag{3.28}$$

for all $(\ell,\varphi)\in\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)$ and all $(\delta\ell,\delta\varphi)\in\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)$.

Altogether, we have shown that $\mathcal{H}$ with domain $\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)$ has the same properties as $\mathcal{H}$ with domain $(0,T)\times H^1(\Omega)$ in Lemma 1.57, p. 47. This brings us to the second step therein, where the product rule from Lemma 5.4 is this time applied to (3.26). But since g' does not depend on ℓ and since $\mathcal{H}$ with domain $\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)$ maintains the same properties with the exact same integrability exponents, all the results in the second step readily transfer. Of course, when fixing the setting in Lemma 5.4, one defines this time $U:=X:=\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)$, which does not affect the final result at all. All this being said, we can now conclude that $F:\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)\to H^1(\Omega)^*$ is continuously Fréchet-differentiable and thus, in view of (3.25), it holds $\Psi\in C^1(\boldsymbol{W}^{-1,p}(\Omega)\times L^2(\Omega)\times H^1(\Omega);H^1(\Omega)^*)$.

(II) By readapting the proof of Lemma 1.59 on page 49, we prove in what follows that $\partial_\varphi\Psi(\ell,d,\varphi):H^1(\Omega)\to H^1(\Omega)^*$ is bijective for every $(\ell,d,\varphi)\in\mathcal{B}_M\times L^2(\Omega)\times H^1(\Omega)$. We point out that this is not expected to hold for all $(\ell,d,\varphi)\in\boldsymbol{W}^{-1,p}(\Omega)\times L^2(\Omega)\times H^1(\Omega)$. This is due to the fact that, for the coercivity of $\partial_\varphi\Psi(\ell,d,\varphi)$, and thus for its invertibility, it is crucial that β exceeds a given threshold, which depends on $\|\ell\|_{\boldsymbol{W}^{-1,p}(\Omega)}$, see (3.31) below. On the other side, the value of β is fixed, and thus, it may only depend on the given data. As in a previous situation (see (3.17)), β fulfills these two conditions when

$\ell \in \mathcal{B}_M$, as we will next see in (3.32). This is not surprising at all, since the unique solvability of (3.8) can be alternatively followed from the implicit function theorem applied for the exact same setting as in the present proof.

On account of (3.25), we intend to show that for every $h \in H^1(\Omega)^*$, the equation

$$B\delta\varphi + \partial_\varphi F(\ell,\varphi)\delta\varphi = h \tag{3.29}$$

admits a unique solution $\delta\varphi \in H^1(\Omega)$, where $(\ell,\varphi) \in \mathcal{B}_M \times H^1(\Omega)$. First notice that, as a result of (3.26), (1.88) (see page 47) and (3.28), it can be deduced in view of (5.1), that the partial derivative of F at $(\ell,\varphi) \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$ in direction $\delta\varphi \in H^1(\Omega)$ reads

$$\begin{aligned}\langle \partial_\varphi F(\ell,\varphi)(\delta\varphi), z\rangle_{H^1(\Omega)} &= \frac{1}{2}\int_\Omega g''(\varphi)(\delta\varphi)\mathbb{C}\varepsilon(\mathcal{U}(\ell,\varphi)) : \varepsilon(\mathcal{U}(\ell,\varphi))z\,dx \\ &+ \int_\Omega g'(\varphi)\mathbb{C}\varepsilon(\mathcal{U}(\ell,\varphi)) : \varepsilon\big(\partial_\varphi\mathcal{U}(\ell,\varphi)(\delta\varphi)\big)z\,dx \quad \forall\, z \in H^1(\Omega).\end{aligned} \tag{3.30}$$

In order to solve (3.29) we employ again Lax-Milgram's lemma. The boundedness of $B + \partial_\varphi F(\ell,\varphi)$ is guaranteed for any $(\ell,\varphi) \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$, by the exact same arguments as in the proof of Lemma 1.59, p. 49. Further, the estimate (1.100) on page 50 is in view of (3.30) replaced by

$$\begin{aligned}|\langle \partial_\varphi F(\ell,\varphi)z, z\rangle_{H^1(\Omega)}| &\le C\big(\|g''(\varphi)\|_\infty \|z\|_r \|\mathbb{C}\varepsilon(\mathcal{U}(\ell,\varphi)) : \varepsilon(\mathcal{U}(\ell,\varphi))\|_{\frac{p}{2}}\|z\|_r \\ &\quad + \|g'(\varphi)\|_\infty \|\varepsilon(\mathcal{U}(\ell,\varphi))\|_p \|\varepsilon(\partial_\varphi\mathcal{U}(\ell,\varphi)(z))\|_2 \|z\|_r\big) \\ &\le C\,\|\ell\|^2_{\boldsymbol{W}^{-1,p}(\Omega)}\|z\|_r^2 \quad \forall\,(\ell,\varphi) \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega),\ \forall\, z \in H^1(\Omega),\end{aligned}$$

where for the last inequality we used (3.5) and (3.23). In the exact same way as in the proof of Lemma 1.59, p. 49, we then arrive for all $k > 0$ at

$$\begin{aligned}\langle Bz + \partial_\varphi F(\ell,\varphi)z, z\rangle \ge \big(\alpha - \widetilde{c}(k)\|\ell\|^2_{\boldsymbol{W}^{-1,p}(\Omega)}\big)\|z\|^2_{H^1(\Omega)} + \big(\beta - \alpha - k\|\ell\|^2_{\boldsymbol{W}^{-1,p}(\Omega)}\big)\|z\|_2^2 \\ \forall\,(\ell,\varphi) \in \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega),\ \forall\, z \in H^1(\Omega).\end{aligned} \tag{3.31}$$

where $\widetilde{c} : \mathbb{R}^+ \to \mathbb{R}^+$ depends only on the given data and satisfies $\widetilde{c}(k) \searrow 0$ as $k \nearrow \infty$. This brings us to a situation similar to (3.14), where the uniformly boundedness of the loads was essential for the coercivity. Hence, from now on, let $(\ell,\varphi) \in \mathcal{B}_M \times H^1(\Omega)$ be arbitrary, but fixed. Since $\ell \in \mathcal{B}_M$ and since Assumption 1.17.2, p. 20 holds true with $\beta = \beta(M)$, we have in view of Remark 3.9 that

$$\beta \ge \alpha + 4kM^2 > \alpha + kM^2 > \alpha + k\|\ell\|^2_{\boldsymbol{W}^{-1,p}(\Omega)}, \tag{3.32}$$

where k is large enough such that $\alpha/2 \ge 4\widetilde{c}(k)M^2 > \widetilde{c}(k)M^2 > \widetilde{c}(k)\|\ell\|^2_{\boldsymbol{W}^{-1,p}(\Omega)}$. In view of (3.31) and (3.32) we finally have

$$\langle Bz + \partial_\varphi F(\ell,\varphi)z, z\rangle_{H^1(\Omega)} \ge \alpha/2\|z\|^2_{H^1(\Omega)} \quad \forall\, z \in H^1(\Omega), \tag{3.33}$$

i.e., the coercivity of $B+\partial_\varphi F(\ell,\varphi)$ for any $(\ell,\varphi)\in\mathcal{B}_M\times H^1(\Omega)$. Due to the boundedness thereof, we obtain by Lax-Milgram's lemma that $\partial_\varphi\Psi(\ell,d,\varphi) : H^1(\Omega)\to H^1(\Omega)^*$ is bijective for all $(\ell,d,\varphi)\in\mathcal{B}_M\times L^2(\Omega)\times H^1(\Omega)$.

The continuously Fréchet-differentiability of $\Phi:\mathcal{B}_M\times L^2(\Omega)\to H^1(\Omega)$ is ultimately given, as in Proposition 1.60, p. 51, by applying the implicit function theorem, this time for Ψ in (3.25). Therefor we need the results shown in steps (I) and (II) above, as well as Definition 3.10. The identity (3.24) is proven in the same manner as (1.104), p. 51. From e.g. [84, Section 4.7] we know that

$$\Phi'(\ell,d) = -[\partial_\varphi\Psi(\ell,d,\Phi(\ell,d))]^{-1}\partial_{(\ell,d)}\Psi(\ell,d,\Phi(\ell,d)) \quad \forall\,(\ell,d)\in\mathcal{B}_M\times L^2(\Omega),$$

which, due to (3.25), can be continued as

$$\Phi'(\ell,d) = -[B+\partial_\varphi F(\ell,\Phi(\ell,d))]^{-1}\begin{pmatrix}\partial_\ell\Psi(\ell,d,\Phi(\ell,d))\\ \partial_d\Psi(\ell,d,\Phi(\ell,d))\end{pmatrix} \quad \forall\,(\ell,d)\in\mathcal{B}_M\times L^2(\Omega). \tag{3.34}$$

In order to write down the equation which characterizes $\delta\phi$ it remains us to determine the derivative of Ψ with respect to ℓ. To this end, notice that in view of (3.25) we have

$$\partial_\ell\Psi(\ell,d,\varphi) = \partial_\ell F(\ell,\varphi) \quad \forall\,(\ell,d,\varphi)\in\boldsymbol{W}^{-1,p}(\Omega)\times L^2(\Omega)\times H^1(\Omega).$$

Thanks to (3.26), the product rule (5.1) together with (3.28) then gives for all $(\ell,d,\varphi)\in\boldsymbol{W}^{-1,p}(\Omega)\times L^2(\Omega)\times H^1(\Omega)$

$$\partial_\ell\Psi(\ell,d,\varphi) = P_1(g'(\varphi),\partial_\ell\mathcal{H}(\ell,\varphi)(\cdot)) = P_1\big(g'(\varphi), 2\,\mathbb{C}\varepsilon(\mathcal{U}(\ell,\varphi)) : \varepsilon\big(\partial_\ell\mathcal{U}(\ell,\varphi)(\cdot)\big)\big), \tag{3.35}$$

where P_1 is the product defined in (1.88) on page 47. Using (3.35) in (3.34) and taking (3.30) into account then yields that $\delta\phi$ is the unique solution of (3.24). Note that since $\mathcal{B}_M\times L^2(\Omega)$ is an open set, we entirely skip the end of the proof of Proposition 1.60, p. 51. ☐

We finalize the discussion of the elliptic system (3.4) by pointing out that, given $M>0$, we have for all $(\ell,\varphi,d)\in\mathcal{B}_M\times H^1(\Omega)\times L^2(\Omega)$ the estimates

$$\|\mathcal{U}'(\ell,\varphi)\|_{\mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega);V)} \le L(M), \quad \|\Phi'(\ell,d)\|_{\mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega)\times L^2(\Omega);H^1(\Omega))} \le L(M), \tag{3.36}$$

where $L(M)$ is the maximum of the Lipschitz constants of $\mathcal{U}$ and Φ. These estimates are given by Lemma 5.8, where we employed Lemmata 3.14, 3.5 and 3.15, 3.11, respectively (provided that Assumptions 1.17, p. 19 (with $\beta=\beta(M)$) and 1.56, p. 46 hold).

3.2 The control-to-state operator $\mathcal{S}$

In this section we turn our attention to the constraint in (P_{min}), i.e., the problem (P_ℓ). Recall that the unique solvability thereof was already established in Chapter 1, under Assumption 1.17, p. 19, in the context where $\ell\in C^{0,1}([0,T];\boldsymbol{W}^{-1,p}(\Omega))$ was fixed. We refer here to the end of Section 1.1 and Theorem 1.62 on page 52, where

$\ell \in C^1([0,T]; \boldsymbol{W}^{-1,p}(\Omega))$. In the previous chapters it was reasonable to focus on the time dependence of the solution operators, not on the dependence on the load, which was just one of the given data. In particular, we were interested in the differentiability with respect to time of the solution operators, see Section 1.3, which in combination with the smoothness in time of the load played an essential role when deriving the *energy identity*. We refer here to Lemma 2.2. However, for the sole purpose of showing unique solvability of (P_ℓ) it suffices to consider some fixed load in $L^\infty(0,T; \boldsymbol{W}^{-1,p}(\Omega))$, as the next results show. The situation changes when we want to control the load in the context of analyzing the problem (P_{min}). As we will see, this calls for working with (variable) loads in a bounded subset of $L^\infty(0,T; \boldsymbol{W}^{-1,p}(\Omega))$, see Definition 3.21 below.

The aim of this section is to introduce the control-to-state operator, that is, the solution operator associated to the constraint in (P_{min}), where nonsmooth in time loads are allowed. To this end, we first discuss the transition from smooth to nonsmooth (fixed) loads. We end the section by addressing the transition from fixed to variable (nonsmooth) loads.

Unique solvability of (P_ℓ) for a nonsmooth load

From Remark 3.13 we deduce: given $\ell : [0,T] \to \boldsymbol{W}^{-1,p}(\Omega)$ and $d : [0,T] \to L^2(\Omega)$, the minimization problem in (P_ℓ) has a unique solution $(\bar{\boldsymbol{u}}, \bar{\varphi})$, which satisfies

$$\bar{\boldsymbol{u}}(t) = \mathcal{U}(\ell(t), \bar{\varphi}(t)), \quad \bar{\varphi}(t) = \Phi(\ell(t), d(t)) \quad \text{f.a.a. } t \in (0,T), \tag{3.37}$$

provided that there exists $M > 0$ so that $\ell(t) \in \mathcal{B}_M$ f.a.a. $t \in (0,T)$ and Assumption 1.17, p. 19 with $\beta = \beta(M)$ is satisfied. In view thereof, we work in what follows with a load in $L^\infty(0,T; \boldsymbol{W}^{-1,p}(\Omega))$ and impose that $\beta = \beta(\|\ell\| + 1)$ in Assumption 1.17.2, p. 20 (see also Remark 3.9) in order to ensure that $\bar{\varphi}(t)$, and consequently, $\bar{\boldsymbol{u}}(t)$, exist for almost all $t \in (0,T)$. Note that we have to set $M > \|\ell\|_{L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))} (\geq \|\ell(t)\|_{\boldsymbol{W}^{-1,p}(\Omega)}$ f.a.a. $t \in (0,T))$ in Definition 3.10 only because Φ is defined on an open ball. Due to the characterization of the optimal displacement and optimal nonlocal damage in (3.37), (P_ℓ) reduces to

$$\dot{d}(t) = \frac{1}{\delta} \max\big(-\beta(d(t) - \Phi(\ell(t), d(t)) - r\big) \quad \text{f.a.a. } t \in (0,T), \quad d(0) = d_0, \tag{3.38}$$

in view of Theorem 1.29 on page 26. Recall that the unique solvability of (3.38) was already addressed in Theorem 1.32 on page 28, however for $\ell \in C^{0,1}([0,T]; \boldsymbol{W}^{-1,p}(\Omega))$. As nonsmooth loads are now allowed, we have to readdress (3.38), this time by employing Picard-Lindelöf's theorem in abstract function spaces (Lemma 5.7). Prior to this, let us give some comments on the expected time regularity of $(\bar{\boldsymbol{u}}, \bar{\varphi})$, which will be useful when formulating Lemma 3.17 below.

Remark 3.16 (Expected time regularity)**.** *Since $(\bar{\boldsymbol{u}}(t), \bar{\varphi}(t))$ solves the elliptic system (3.4) with right-hand side $(\ell(t), d(t))$ for almost all $t \in (0,T)$, one does not expect $\bar{\boldsymbol{u}}$ and $\bar{\varphi}$ to possess more time regularity than ℓ does, regardless of how much time regularity d will turn out to have. One sees that $\bar{\boldsymbol{u}}$ and $\bar{\varphi}$ are differentiable in some sense with* *respect to time, if ℓ does so. This was the case in Section 1.3, where ℓ belonged to $C^1([0,T];\boldsymbol{W}^{-1,p}(\Omega))$. So, before having any information about the time regularity of the local damage, it is sufficient to investigate the time regularity of the solutions of the minimization problem in (P_ℓ) for $d \in L^\infty(0,T;L^2(\Omega))$.*

Lemma 3.17. *Suppose that Assumption 1.17 on page 19 is satisfied, with $\beta = \beta\big(\|\ell\|+1\big)$, where $\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$ is fixed. Then the solution $(\bar{\boldsymbol{u}},\bar{\varphi})$ of the optimization problem in (P_ℓ) belongs to $L^\infty(0,T;\boldsymbol{W}_D^{1,s}(\Omega)) \times L^\infty(0,T;H^1(\Omega))$, for any $s \in [2,p)$, provided that $d \in L^\infty(0,T;L^2(\Omega))$.*

Proof. As already mentioned at the beginning of the present section, the unique solution of the minimization problem in (P_ℓ) satisfies (3.37) for almost all $t \in (0,T)$ (if Assumption 1.17, p. 19, with $\beta = \beta\big(\|\ell\|+1\big)$ holds). Since ℓ and d are Bochner measurable by assumption, the mapping $t \mapsto (\ell(t),d(t))$ is Bochner measurable as well. Thanks to $\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$, it holds $\ell(t) \in B_{\boldsymbol{W}^{-1,p}(\Omega)}\big(0,\|\ell\|+1\big)$ f.a.a. $t \in (0,T)$, so that we can combine Lemma 3.11 with Lemma 5.11, where we set $f = \Phi$, $U = B_{\boldsymbol{W}^{-1,p}(\Omega)}\big(0,\|\ell\|+1\big) \times L^2(\Omega)$ and $Y = H^1(\Omega)$. This yields the Bochner measurability of $\bar{\varphi} : [0,T] \to H^1(\Omega)$. Hence, $t \mapsto (\ell(t),\bar{\varphi}(t))$ is Bochner measurable. By applying Lemma 5.11 again, this time for $f = \mathcal{U}$, $X = \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega)$ and $Y = \boldsymbol{W}_D^{1,s}(\Omega)$, where we take Lemma 3.2 into account, we obtain the Bochner measurability of $\bar{\boldsymbol{u}} : [0,T] \to \boldsymbol{W}_D^{1,s}(\Omega)$. Further, (3.5) gives immediately that $\bar{\boldsymbol{u}} \in L^\infty(0,T;\boldsymbol{W}_D^{1,s}(\Omega))$. From the Lipschitz continuity of the operator $\Phi : B_{\boldsymbol{W}^{-1,p}(\Omega)}\big(0,\|\ell\|+1\big) \times L^2(\Omega) \to H^1(\Omega)$, cf. Lemma 3.11, one deduces that f.a.a. $t \in (0,T)$ it holds

$$\|\Phi(\ell(t),d(t))\|_{H^1(\Omega)} \le L(\|\ell\|+1)(\|\ell(t)\|_{\boldsymbol{W}^{-1,p}(\Omega)} + \|d(t)\|_2) + \|\Phi(0,0)\|_{H^1(\Omega)}.$$

The fact that $(\ell,d) \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega))$ ensures that $\bar{\varphi} \in L^\infty(0,T;H^1(\Omega))$, which completes the proof. □

Remark 3.18. *Note that one cannot expect in Lemma 3.17 that $\bar{\boldsymbol{u}}$ belongs to the space $L^\infty(0,T;\boldsymbol{W}_D^{1,p}(\Omega))$. In view of Definition 3.1, one needs for the Bochner measurability of $\bar{\boldsymbol{u}} : [0,T] \to \boldsymbol{W}_D^{1,p}(\Omega)$ that $t \mapsto A_{\bar{\varphi}(t)}^{-1} \in \mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega),\boldsymbol{W}_D^{1,p}(\Omega))$ is Bochner measurable. Since $\bar{\varphi} \in L^\infty(0,T;H^1(\Omega))$, this asks for the continuity of $H^1(\Omega) \ni \varphi \mapsto A_\varphi^{-1}$, for which a norm gap is needed, on account of Lemma 5.2. In order to see this, observe that one can deduce as in Lemma 1.11 on page 16 that*

$$\|A_{\varphi_n}^{-1} - A_\varphi^{-1}\|_{\mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega),\boldsymbol{W}_D^{1,s}(\Omega))} \le C\,\|g(\varphi_n) - g(\varphi)\|_\varrho \quad \forall n \in \mathbb{N}, \tag{3.39}$$

with $\varrho \in [1,\infty)$ such that $1/\varrho + 1/p + 1/s' = 1$, where the sequence $\{\varphi_n\} \subset H^1(\Omega)$ and $\varphi \in H^1(\Omega)$ are arbitrary. Thus, by (3.39) and Lemma 5.2, the operator

$$H^1(\Omega) \ni \varphi \mapsto A_\varphi^{-1} \in \mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega),\boldsymbol{W}_D^{1,s}(\Omega)) \text{ is continuous}$$

only when $s < p$, which ultimately highlights why $\bar{\boldsymbol{u}} : [0,T] \to \boldsymbol{W}_D^{1,p}(\Omega)$ is not necessarily Bochner measurable.

The unique solvability of the evolutionary equation in (P_ℓ) (for nonsmooth in time loads) is given by the next

Theorem 3.19. *Let $\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$ be fixed. Under Assumption 1.17, p. 19, where $\beta = \beta(\|\ell\|+1)$, there exists a unique function $d \in W^{1,\infty}(0,T;L^2(\Omega))$ satisfying* (3.38).

Proof. As in Section 1.1 on page 28 we intend to solve (3.38) by means of Picard-Lindelöf's theorem, except that this time we need to apply the version for Bochner-Sobolev spaces, i.e., Lemma 5.7. To this end, we rewrite (3.38) as

$$\dot{d}(t) = f(t, d(t)) \quad \text{f.a.a. } t \in (0,T), \quad d(0) = d_0, \tag{3.40}$$

where $f : (0,T) \times L^2(\Omega) \to L^2(\Omega)$ is defined as

$$f(t,d) := \frac{1}{\delta} \max\big(-\beta\big(d - \Phi(\ell(t),d)\big) - r\big). \tag{3.41}$$

Note that for any $d \in L^2(\Omega)$, the value $\Phi(\ell(t),d) \in H^1(\Omega)$ exists f.a.a. $t \in (0,T)$, by assumption (see also Definition 3.10). Since max maps $L^2(\Omega)$ to $L^2(\Omega)$ in view of Lemma 5.6.(i), the value $f(t,d) \in L^2(\Omega)$ is well defined f.a.a. $t \in (0,T)$ and for any $d \in L^2(\Omega)$. Let us now check if the other two assumptions on f in Lemma 5.7 are satisfied. Lemma 3.17 tells us that for $d \in L^\infty(0,T;L^2(\Omega))$ the mapping $d - \Phi(\ell(\cdot),d(\cdot))$ belongs to $L^\infty(0,T;L^2(\Omega))$, so that from Lemma 5.6.(ii) we deduce

$$f(\cdot,d(\cdot)) \in L^\infty(0,T;L^2(\Omega)) \text{ for } d \in L^\infty(0,T;L^2(\Omega)),$$

on account of (3.41). Thereby the second assumption on f in Lemma 5.7 is fulfilled. From Lemma 5.6.(i) we know that the operator $\max : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous with Lipschitz constant 1, which in combination with Lemma 3.11 yields the estimate

$$\begin{aligned}
&\|f(t,d_1) - f(t,d_2)\|_2 \\
&\quad \le \frac{1}{\delta}\big\| \max\big(-\beta\big(d_1 - \Phi(\ell(t),d_1)\big) - r\big) - \max\big(-\beta\big(d_2 - \Phi(\ell(t),d_2)\big) - r\big)\big\|_2 \\
&\quad \le \frac{\beta}{\delta}\big(\|d_1 - d_2\|_2 + \|\Phi(\ell(t),d_1) - \Phi(\ell(t),d_2)\|_2\big) \\
&\quad \le \frac{\beta}{\delta}\,(L+1)\,\|d_1 - d_2\|_2 \quad \forall\, d_1,d_2 \in L^2(\Omega), \quad \text{f.a.a. } t \in (0,T),
\end{aligned}$$

where $L = L(\|\ell\|+1) > 0$ is independent of time. Therewith all the assumptions in Lemma 5.7 are verified. Now we can conclude the unique solvability of (3.40), and thus of (3.38), as well as the desired regularity of d. $\square$

Notice that Theorem 3.19 basically covers the unique solvability of the problem (P_ℓ) (for a nonsmooth load), since as already mentioned above, (3.38) is just the reduced form thereof. In view of (3.37), the unique solution of the problem (P_ℓ) is given by

$\big(\mathcal{U}\big(\ell(\cdot), \Phi(\ell(\cdot), d(\cdot))\big), \Phi\big(\ell(\cdot), d(\cdot)\big), d\big)$, where d solves (3.38). The regularity of the optimal displacement and optimal nonlocal damage can be read from Lemma 3.17 and Theorem 3.19, see also Remark 3.16. In light of Definitions 3.1 and 3.10, we can formulate the new solvability result as follows

Theorem 3.20. *Let $\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$ be fixed. Moreover, let Assumption 1.17 on page 19 hold true with $\beta = \beta(\|\ell\|+1)$. Then the problem (P_ℓ) admits a unique solution $(\boldsymbol{u}, \varphi, d)$, satisfying $\boldsymbol{u} \in L^\infty(0,T;\boldsymbol{W}_D^{1,s}(\Omega))$, for every $s \in [2,p)$, $\varphi \in L^\infty(0,T;H^1(\Omega))$, $d \in W^{1,\infty}(0,T;L^2(\Omega))$ and the system of differential equations f.a.a. $t \in (0,T)$*

$$\begin{aligned} -\operatorname{div} g(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) &= \ell(t) && \text{in } \boldsymbol{W}^{-1,p}(\Omega), \\ -\alpha\Delta\varphi(t) + \beta\,\varphi(t) + \frac{1}{2}\, g'(\varphi(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) &= \beta d(t) && \text{in } H^1(\Omega)^*, \\ \dot{d}(t) - \frac{1}{\delta}\max\big(-\beta(d(t)-\varphi(t)) - r\big) &= 0, && d(0) = d_0. \end{aligned}$$

The solution operator $\mathcal{S}$

Within our scope of analyzing (P_{min}), we are interested in defining the control-to-state operator associated to (P_ℓ), i.e., the solution operator that maps ℓ to $(\boldsymbol{u}, \varphi, d)$. This means that the load can no longer be seen as given data, but as variable. On the other side, as already pointed out above, (P_ℓ) is uniquely solvable provided that $\beta = \beta(\|\ell\|+1)$, see also Theorem 3.20 and Remark 3.9. Since β is fixed and may only depend on the given data, we introduce, as in Section 3.1 on page 103, the following

Definition 3.21 (Time-dependent admissible loads)**.** *For a given $M > 0$ we define*

$$\mathfrak{B}_M := B_{L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))}(0, M).$$

Note that this implies $\mathfrak{B}_M \subset \{\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega)) : \ell(t) \in \mathcal{B}_M \;\; f.a.a.\; t \in (0,T)\}$.

Thus, given $M > 0$ and some variable $\ell \in \mathfrak{B}_M$, we need to require that $\beta = \beta(M+1)$ in Assumption 1.17, p. 19. Then, β depends only on the given data and the unique solvability of (P_ℓ) is ensured, since for all $\ell \in \mathfrak{B}_M$ we have

$$\beta \geq \alpha + 4k(M+1)^2 > \alpha + 4k(\|\ell\|_{L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))} + 1)^2, \tag{3.43}$$

where k is chosen (large enough, depending only on M, α, p and N) such that

$$\alpha/2 \geq 4\widetilde{c}(k)(M+1)^2 > 4\widetilde{c}(k)(\|\ell\|_{L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))} + 1)^2.$$

Note that β satisfying (3.43) surpasses the threshold needed in Theorem 3.20, so that we can introduce

Definition 3.22 (Control-to-state operator associated to (P_ℓ))**.** *Let $M > 0$ be given and suppose that Assumption 1.17, p.19, with $\beta = \beta(M+1)$ holds true. Then we define $\mathcal{S} : \mathfrak{B}_M \to L^\infty(0,T;\boldsymbol{W}_D^{1,s}(\Omega)) \times L^\infty(0,T;H^1(\Omega)) \times W^{1,\infty}(0,T;L^2(\Omega))$, with $s \in [2,p)$, as*

$$\mathcal{S}(\ell) := (\boldsymbol{u}, \varphi, d),$$

where $(\boldsymbol{u}, \varphi, d)$ *is the unique solution of* (P_ℓ)*, i.e., it satisfies f.a.a.* $t \in (0, T)$ *the system*

$$\left.\begin{aligned} \boldsymbol{u}(t) = \mathcal{U}(\ell(t), \varphi(t)), \quad \varphi(t) = \Phi(\ell(t), d(t)), \\ \dot{d}(t) = \frac{1}{\delta}\max(-\beta(d(t) - \varphi(t)) - r), \quad d(0) = d_0, \end{aligned}\right\} \tag{3.44}$$

with right-hand side $\ell \in \mathfrak{B}_M$.

For $i \in \{1, 2, 3\}$ *we denote by* $\mathcal{S}_i$ *the operator which associates to any* $\ell \in \mathfrak{B}_M$ *the* i*-th component of* $\mathcal{S}(\ell)$.

Remark 3.23. *Note that in the two-dimensional case the control-to-state operator* $\mathcal{S}$ *has actually range in* $L^\infty(0, T; \boldsymbol{W}_D^{1,p}(\Omega)) \times L^\infty(0, T; W^{1,q}(\Omega)) \times W^{1,\infty}(0, T; L^2(\Omega))$*, with* $q > 2$ *given by Theorem 1.37, p. 31. In order to see this, observe that in view of Remark 3.12 one can show that* $\bar{\varphi} \in L^\infty(0, T; W^{1,q}(\Omega))$ *in Lemma 3.17 by using the exact same arguments. On the other side, in the two-dimensional case the embedding* $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ *holds true, and thanks to Lemma 5.1, one obtains as in Remark 3.18 that*

$$W^{1,q}(\Omega) \ni \varphi \mapsto A_\varphi^{-1} \in \mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega), \boldsymbol{W}_D^{1,p}(\Omega)) \text{ is continuous.}$$

Consequently, $t \mapsto A_{\bar{\varphi}(t)}^{-1} \in \mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega), \boldsymbol{W}_D^{1,p}(\Omega))$ *is Bochner measurable, which accounts for the Bochner measurability of* $\bar{\boldsymbol{u}} : [0, T] \to \boldsymbol{W}_D^{1,p}(\Omega)$ *in Lemma 3.17. For more details, we refer to Remark 3.18. Altogether, the improved regularity results in Lemma 3.17 yield in combination with Theorem 3.19 that, in two dimensions, the unique solution of* (P_ℓ) *belongs to* $L^\infty(0, T; \boldsymbol{W}_D^{1,p}(\Omega)) \times L^\infty(0, T; W^{1,q}(\Omega)) \times W^{1,\infty}(0, T; L^2(\Omega))$*, with* $q > 2$ *given by Theorem 1.37, p. 31.*

Remark 3.24 (Lipschitz continuity of $\mathcal{S}$)**.** *It is easy to see that the right-hand side in the operator differential equation in* (3.44) *is Lipschitz continuous w.r.t.* $(\ell(t), d(t)) \in \mathcal{B}_M \times L^2(\Omega)$ *f.a.a.* $t \in (0, T)$*, where* $M > 0$ *is given. This follows from the Lipschitz continuity of* $\max : L^2(\Omega) \to L^2(\Omega)$ *and* $\Phi : \mathcal{B}_M \times L^2(\Omega) \to H^1(\Omega)$*, see Lemmata 5.6.(i) and 3.11, respectively. In light of Lemma 5.10, it can then be shown that* $\mathcal{S}_3 : \mathfrak{B}_M \to W^{1,\infty}(0, T; L^2(\Omega))$ *is Lipschitz continuous, which as a result of the Lipschitz continuity of* $\mathcal{U}$ *and* Φ *(Lemmata 3.5 and 3.11) gives in turn the Lipschitz continuity of* $\mathcal{S} : \mathfrak{B}_M \to L^\infty(0, T; V) \times L^\infty(0, T; H^1(\Omega)) \times W^{1,\infty}(0, T; L^2(\Omega))$.

3.3 The directional differentiability of $\mathcal{S}$

Since in the next section we derive necessary optimality conditions in primal form for (P_{min}), we address in what follows the differentiability of the control-to-state operator $\mathcal{S}$. Notice that this is not expected to be Gâteaux-differentiable, since in (3.44) the evolution of the local damage d is described via the max-function, which is not Gâteaux-differentiable at 0 (see also (5.18) in Chapter 5).

For the sake of convenience we work in the rest of the section with a fixed pair $(\ell, \delta\ell) \in \mathfrak{B}_M \times L^\infty(0, T; \boldsymbol{W}^{-1,p}(\Omega))$, where $M > 0$ is given. Throughout this section Assumptions 1.17, p. 19 with $\beta = \beta(M + 1)$ and 1.56, p. 46 are supposed to hold. We prove the directional differentiability of $\mathcal{S}$ at ℓ in direction $\delta\ell$ in a standard manner, by deriving estimates for the components of $\left\|\frac{\mathcal{S}(\ell+\tau\delta\ell)-\mathcal{S}(\ell)}{\tau}-\mathcal{S}'(\ell;\delta\ell)\right\|$, which will ultimately converge towards 0 as $\tau \searrow 0$. Of course, in this context $\mathcal{S}'(\ell;\delta\ell)$ does not denote the derivative, but the candidate therefor. Note that this can be provided by 'linearizing' (3.44) at ℓ in direction $\delta\ell$. In order to have a better overview of the upcoming results, we use in what follows the abbreviations

$$\begin{aligned} \ell_\tau &:= \ell + \tau\delta\ell, \qquad (\boldsymbol{u},\varphi,d) := \mathcal{S}(\ell), \quad (\boldsymbol{u_\tau},\varphi_\tau,d_\tau) := \mathcal{S}(\ell_\tau), \\ \boldsymbol{\delta u} &:= \mathcal{U}'\big(\ell(\cdot),\varphi(\cdot)\big)\big(\delta\ell(\cdot),\delta\varphi(\cdot)\big), \quad \delta\varphi := \Phi'\big(\ell(\cdot),d(\cdot)\big)\big(\delta\ell(\cdot),\delta d(\cdot)\big), \end{aligned} \tag{3.45}$$

where $\delta d \in L^\infty(0,T;L^2(\Omega))$ is arbitrary, but fixed and $\tau > 0$ is small enough such that $\ell_\tau \in \mathfrak{B}_M$. Notice that such a τ exists, since $\mathfrak{B}_M$ is open, and all the operators in (3.45) are well defined. In particular, we remark that $\delta\varphi$ is a.e. in $(0,T)$ well defined, in view of Definition 3.21 and Lemma 3.15.

Before we proceed with the first estimates, we establish the time-integrability of $\boldsymbol{\delta u}$ and $\delta\varphi$. We know that $\mathcal{U}' \in C\big(\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega);\mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega);V)\big)$ and $\Phi' \in C\big(\mathcal{B}_M \times L^2(\Omega);\mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega);H^1(\Omega))\big)$, from Lemmata 3.14 and 3.15, respectively. Thus, with Lemma 5.11 we can deduce the measurability of $\mathcal{U}'\big(\ell(\cdot),\varphi(\cdot)\big)$ and $\Phi'\big(\ell(\cdot),d(\cdot)\big)$. Moreover, in view of (3.36), it is clear that $\mathcal{U}'\big(\ell(\cdot),\varphi(\cdot)\big)$ belongs to $L^\infty\big(0,T;\mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega);V)\big)$ and $\Phi'\big(\ell(\cdot),d(\cdot)\big) \in L^\infty\big(0,T;\mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega);H^1(\Omega))\big)$. Since $(\delta\ell,\delta d) \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega))$, we establish as in [81, Lemma 3.1.19] that

$$\delta\varphi \in L^\infty(0,T;H^1(\Omega)). \tag{3.46}$$

Thereby we have $(\delta\ell,\delta\varphi) \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega))$ and as in [81, Lemma 3.1.19] we also obtain

$$\boldsymbol{\delta u} \in L^\infty(0,T;V). \tag{3.47}$$

Lemma 3.25. *Let $(\ell,\delta\ell) \in \mathfrak{B}_M \times L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$, where $M > 0$ is given and let Assumptions 1.17, p. 19 with $\beta = \beta(M+1)$ and 1.56, p. 46 hold true. Then, there exist constants $C, c > 0$, so that for the quantities defined in* (3.45)*, we have for almost all $t \in (0,T)$ the estimates*

$$\Big\|\frac{\boldsymbol{u_\tau}(t)-\boldsymbol{u}(t)}{\tau} - \boldsymbol{\delta u}(t)\Big\|_V \le C\Big\|\frac{d_\tau(t)-d(t)}{\tau} - \delta d(t)\Big\|_2 + cR_\Phi(t,\tau) + R_{\mathcal{U}}(t,\tau), \tag{3.48a}$$

$$\Big\|\frac{\varphi_\tau(t)-\varphi(t)}{\tau} - \delta\varphi(t)\Big\|_{H^1(\Omega)} \le C\Big\|\frac{d_\tau(t)-d(t)}{\tau} - \delta d(t)\Big\|_2 + R_\Phi(t,\tau), \tag{3.48b}$$

where $\tau > 0$ is small enough, independent of t, and $R_{\mathcal{U}}, R_\Phi : (0,T) \times (0,1) \to [0,\infty)$ are mappings which satisfy

$$R_{\mathcal{U}}(\cdot,\tau), R_\Phi(\cdot,\tau) \to 0 \text{ in } L^\varrho(0,T) \quad \text{as } \tau \searrow 0, \tag{3.49}$$

for any $\varrho \in [1,\infty)$.

Proof. We intend to prove both estimates by making use of Lemma 5.12. In order to apply it for deriving (3.48a), we set

$$\begin{gathered} X := \boldsymbol{W}^{-1,p}(\Omega),\ Y := \boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega),\ Z := V, \\ \mathfrak{U} := \mathfrak{B}_M,\ U := \mathcal{B}_M \times H^1(\Omega), \\ \mathfrak{F} := (\cdot, \mathcal{S}_2(\cdot)),\ \mathcal{F} := \mathcal{U}, \\ x := \ell,\ \delta x := \delta\ell,\ \delta y := (\delta\ell, \delta\varphi). \end{gathered}$$

Note that $\mathfrak{U}$ is open in $L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$ in view of Definition 3.21. On account of Definition 3.3, U is a product of open sets, and thus, it is open as well. In view of Definition 3.22, $\mathfrak{F}$ maps $\mathfrak{B}_M$ to $\mathfrak{B}_M \times L^\infty(0,T;H^1(\Omega)) \subset L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega))$. As a result of Lemmata 3.5 and 3.14, $\mathcal{U} : \mathcal{B}_M \times H^1(\Omega) \to V$ is Lipschitz continuous and directionally differentiable. Due to $\ell \in \mathfrak{B}_M$ and Definition 3.21, $\varepsilon_1 > 0$ may be chosen small enough such that for any $z \in B_{L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))}\big(\ell, \varepsilon_1\|\delta\ell\|_{L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))}\big) \subset \mathfrak{B}_M$ it holds $z(t) \in \mathcal{B}_M$ f.a.a. $t \in (0,T)$. Thus, $\mathfrak{F}\big(B_{L^\infty(0,T;X)}(\ell, \varepsilon_1\|\delta\ell\|_{L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))})\big)(t) \subset \mathcal{B}_M \times H^1(\Omega)$ f.a.a. $t \in (0,T)$. Further, thanks to (3.46), it holds $(\delta\ell, \delta\varphi) \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega) \times H^1(\Omega))$. In view of Definition 3.21, there exists $\varepsilon > 0$, independent of t, such that $\|\ell(t)\|_{\boldsymbol{W}^{-1,p}(\Omega)} \le M - \varepsilon$ f.a.a. $t \in (0,T)$. Then, by choosing $\varepsilon_2 := \varepsilon/(2\|\delta\ell\|_{L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega)})$ (provided that $\delta\ell \neq 0$) we have f.a.a. $t \in (0,T)$

$$B_{\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)}\big((\ell(t), \mathcal{S}_2(\ell)(t)), \varepsilon_2\|(\delta\ell,\delta\varphi)\|_{L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega))}\big) \subset \mathcal{B}_M \times H^1(\Omega),$$

while for $\delta\ell = 0$ the above inclusion holds for any $\varepsilon_2 > 0$. Let now $\tau \in (0, \min\{\varepsilon_1, \varepsilon_2\})$ be arbitrary, but fixed. We point out that τ is independent of t, since ε_1 and ε_2 do so. Since above we covered all the assumptions in Lemma 5.12, we are now in the position to apply the latter one, which in view of (3.45) yields

$$\begin{aligned} &\Big\|\frac{\boldsymbol{u}_\tau(t) - \boldsymbol{u}(t)}{\tau} - \boldsymbol{\delta u}(t)\Big\|_V \\ &= \Big\|\frac{\mathcal{U}\big(\ell_\tau(t), \varphi_\tau(t)\big) - \mathcal{U}\big(\ell(t), \varphi(t)\big)}{\tau} - \mathcal{U}'\big(\ell(t), \varphi(t)\big)\big(\delta\ell(t), \delta\varphi(t)\big)\Big\|_V \\ &\le L_{\mathcal{U}}\Big\|\frac{\big(\ell_\tau(t), \varphi_\tau(t)\big) - \big(\ell(t), \varphi(t)\big)}{\tau} - (\delta\ell(t), \delta\varphi(t))\Big\|_{\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)} + R_{\mathcal{U}}(t,\tau) \\ &= L_{\mathcal{U}}\Big\|\frac{\varphi_\tau(t) - \varphi(t)}{\tau} - \delta\varphi(t)\Big\|_{H^1(\Omega)} + R_{\mathcal{U}}(t,\tau) \quad \text{f.a.a. } t \in (0,T), \end{aligned} \tag{3.50}$$

where $L_{\mathcal{U}} > 0$ and $R_{\mathcal{U}} : (0,T) \times (0,1) \to [0,\infty)$ satisfies

$$R_{\mathcal{U}}(\cdot,\tau) \to 0 \text{ in } L^\varrho(0,T) \quad \text{as } \tau \searrow 0,$$

for any $\varrho \in [1,\infty)$.

We now proceed in the exact same way in order to derive (3.48b), that is, we begin by fixing the setting in Lemma 5.12:

$$\begin{gathered} X := \boldsymbol{W}^{-1,p}(\Omega),\ Y := \boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega),\ Z := H^1(\Omega), \\ \mathfrak{U} := \mathfrak{B}_M,\ U := \mathcal{B}_M \times L^2(\Omega), \\ \mathfrak{F} := (\cdot, \mathcal{S}_3(\cdot)),\ \mathcal{F} := \Phi, \\ x := \ell,\ \delta x := \delta\ell,\ \delta y := (\delta\ell, \delta d). \end{gathered}$$

We verify if the above setting fulfills the assumptions in Lemma 5.12, by addressing only those which haven't been already checked in the first part of the proof. From Definition 3.22 we know that $\mathfrak{F}$ maps $\mathfrak{B}_M$ to $L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega))$. Due to Lemmata 3.11 and 3.15, $\Phi : \mathcal{B}_M \times L^2(\Omega) \to H^1(\Omega)$ is Lipschitz continuous and directionally differentiable. Note that we can choose the exact same value for $\varepsilon_1 > 0$ as in the first part of the proof. Further, the value of $\varepsilon_2 > 0$ can be again chosen independent of t, so that f.a.a. $t \in (0,T)$ we have

$$B_{\boldsymbol{W}^{-1,p}(\Omega)\times L^2(\Omega)}\big((\ell(t), \mathcal{S}_3(\ell)(t)), \varepsilon_2 \|(\delta\ell, \delta d)\|_{L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega)\times L^2(\Omega))}\big) \subset \mathcal{B}_M \times L^2(\Omega).$$

Hence, for $\tau \in (0, \min\{\varepsilon_1, \varepsilon_2\})$, independent of t, we can now apply Lemma 5.12, which in view of (3.45) allows us to finally deduce

$$\begin{aligned} &\Big\|\frac{\varphi_\tau(t) - \varphi(t)}{\tau} - \delta\varphi(t)\Big\|_{H^1(\Omega)} \\ &= \Big\|\frac{\Phi\big(\ell_\tau(t), d_\tau(t)\big) - \Phi\big(\ell(t), d(t)\big)}{\tau} - \Phi'\big(\ell(t), d(t)\big)\big(\delta\ell(t), \delta d(t)\big)\Big\|_{H^1(\Omega)} \\ &\le L_\Phi \Big\|\frac{\big(\ell_\tau(t), d_\tau(t)\big) - \big(\ell(t), d(t)\big)}{\tau} - (\delta\ell(t), \delta d(t))\Big\|_{\boldsymbol{W}^{-1,p}(\Omega)\times L^2(\Omega)} + R_\Phi(t,\tau) \\ &= L_\Phi \Big\|\frac{d_\tau(t) - d(t)}{\tau} - \delta d(t)\Big\|_2 + R_\Phi(t,\tau) \quad \text{f.a.a. } t \in (0,T), \end{aligned} \tag{3.51}$$

where $L_\Phi > 0$ and $R_\Phi : (0,T) \times (0,1) \to [0,\infty)$ fulfills

$$R_\Phi(\cdot,\tau) \to 0 \text{ in } L^\varrho(0,T) \quad \text{as } \tau \searrow 0,$$

for any $\varrho \in [1,\infty)$. Inserting estimate (3.51) in (3.50) now gives (3.48a). Since (3.51) is (3.48b), the proof is now complete. ☐

In view of the operator differential equation in (3.44), the next lemma provides the candidate for the derivative of $\mathcal{S}_3$ at ℓ in direction $\delta\ell$.

Lemma 3.26. *Let $(\ell, \delta\ell) \in \mathfrak{B}_M \times L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$, where $M > 0$ is given. Suppose that Assumptions 1.17, p. 19 with $\beta = \beta(M+1)$ and 1.56, p. 46 hold true. Then the equation*

$$\left.\begin{aligned} \dot\eta(t) = \frac{1}{\delta} \max{}'\big(-\beta(d(t) - \varphi(t)) - r\,; -\beta\big(\eta(t) - \Phi'\big(\ell(t), d(t)\big)\big(\delta\ell(t), \eta(t)\big)\big)\big) \\ \text{f.a.a. } t \in (0,T), \\ \eta(0) = 0 \end{aligned}\right\} \tag{3.52}$$

admits a unique solution $\eta \in W_0^{1,\infty}(0,T;L^2(\Omega))$, where $d = \mathcal{S}_3(\ell)$ and $\varphi = \mathcal{S}_2(\ell)$.

Proof. We solve the operator differential equation by means of Lemma 5.7. In view of (3.52), we define $f : (0,T) \times L^2(\Omega) \to L^2(\Omega)$ as

$$f(t,\eta) = \frac{1}{\delta} \max{}'\big(-\beta(d(t) - \varphi(t)) - r\,; -\beta\big(\eta - \Phi'\big(\ell(t), d(t)\big)\big(\delta\ell(t), \eta\big)\big)\big). \tag{3.53}$$

Note that for any $\eta \in L^2(\Omega)$, the value $f(t,\eta) \in L^2(\Omega)$ is well defined f.a.a. $t \in (0,T)$, as a consequence of Lemmata 5.6.(i) and 3.15. Let us now check if the other assumptions on f in Lemma 5.7 are satisfied. Definition 3.22 tells us that $d - \varphi \in L^\infty(0,T; L^2(\Omega))$ and for $\eta \in L^\infty(0,T;L^2(\Omega))$ we deduce from (3.45) and (3.46) that

$$\eta - \Phi'\big(\ell(\cdot), d(\cdot)\big)\big(\delta\ell(\cdot), \eta(\cdot)\big) \in L^\infty(0,T;L^2(\Omega)).$$

Now Lemma 5.6.(ii) gives $\max{}'\big(-\beta(d(\cdot)-\varphi(\cdot))-r; -\beta\big(\eta(\cdot)-\Phi'\big(\ell(\cdot),d(\cdot)\big)\big(\delta\ell(\cdot),\eta(\cdot)\big)\big)\big) \in L^\infty(0,T;L^2(\Omega))$, and hence

$$f(\cdot,\eta(\cdot)) \in L^\infty(0,T;L^2(\Omega)) \text{ for } \eta \in L^\infty(0,T;L^2(\Omega)).$$

Let now $\eta_i \in L^2(\Omega)$ be arbitrary, but fixed and let us abbreviate

$$\delta\varphi_i := \Phi'(\ell(\cdot), d(\cdot))(\delta\ell(\cdot), \eta_i) \text{ a.e. in } (0,T)$$

for $i = 1,2$. From Lemma 5.6.(i) we know that for any $y \in L^2(\Omega)$ the operator $\max{}'(y;\cdot) : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous with Lipschitz constant 1 and in view of (3.53) we have as a consequence of (3.36) the estimate

$$\begin{aligned}
\|f(t,\eta_1) - f(t,\eta_2)\|_2 &\le \frac{1}{\delta}\| -\beta\big(\eta_1 - \delta\varphi_1(t)\big) + \beta\big(\eta_2 - \delta\varphi_2(t)\big)\|_2 \\
&\le \frac{\beta}{\delta}\big(\|\eta_1 - \eta_2\|_2 + \|\Phi'(\ell(t),d(t))\|_{\mathcal{L}(\boldsymbol{W}^{-1,p}(\Omega)\times L^2(\Omega);H^1(\Omega))}\|\eta_1-\eta_2\|_2\big) \\
&\le \frac{\beta}{\delta}\,(L+1)\,\|\eta_1 - \eta_2\|_2 \quad \text{f.a.a. } t \in (0,T),
\end{aligned}$$

where $L = L(M) > 0$ depends only on the given data. Therewith all the assumptions in Lemma 5.7 are verified. Now we can conclude the unique solvability of (3.52), as well as the desired regularity of η. □

With the candidate for the derivative of $\mathcal{S}_3$ at hand, we can now prove its directional differentiability at ℓ in direction $\delta\ell$, which is covered by the following

Lemma 3.27 (Directional differentiability of $\mathcal{S}_3$)**.** *Let $\varrho \in [1,\infty)$ and $(\ell, \delta\ell) \in \mathfrak{B}_M \times L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$, where $M > 0$ is given. Under Assumptions 1.17, p. 19, where $\beta = \beta(M+1)$, and 1.56, p. 46, the following convergence holds true*

$$\frac{\mathcal{S}_3(\ell + \tau\delta\ell) - \mathcal{S}_3(\ell)}{\tau} \to \eta \quad \textit{in } W^{1,\varrho}(0,T;L^2(\Omega)) \quad \textit{as } \tau \searrow 0,$$

where $\eta \in W_0^{1,\infty}(0,T;L^2(\Omega))$ is the unique solution of (3.52).

Proof. For simplicity we use again the notations $d_\tau = \mathcal{S}_3(\ell + \tau\delta\ell)$ and $d = \mathcal{S}_3(\ell)$, where $\tau > 0$ is small enough such that $\ell + \tau\delta\ell \in \mathfrak{B}_M$. The result is proven in two steps. We first derive estimates for the term $\left\|\frac{\dot d_\tau(t)-\dot d(t)}{\tau} - \dot\eta(t)\right\|_2$ a.e. in $(0,T)$, by employing Lemma 5.12. This will lead to an estimate of the type (5.38) and Lemma 5.10 will ultimately give the assertion.

(i) In view of (3.45) and (3.52) we fix the following setting in Lemma 5.12:

$$\begin{gathered}
X := \boldsymbol{W}^{-1,p}(\Omega),\ Y := Z := L^2(\Omega),\\
\mathfrak{U} := \mathfrak{B}_M,\ U := L^2(\Omega),\\
\mathfrak{F} := -\beta\big(\mathcal{S}_3(\cdot) - \mathcal{S}_2(\cdot)\big) - r,\ \mathcal{F} := \max,\\
x := \ell,\ \delta x := \delta\ell,\ \delta y := -\beta\big(\eta - \Phi'\big(\ell(\cdot), d(\cdot)\big)\big(\delta\ell(\cdot), \eta(\cdot)\big)\big).
\end{gathered}$$

Clearly, $\mathfrak{U}$ and U are open. In view of Definition 3.22, $\mathfrak{F}$ maps $\mathfrak{B}_M$ to $L^\infty(0,T;L^2(\Omega))$, and according to Lemma 5.6.(i), $\max : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous and directionally differentiable. Moreover, in view of Lemma 3.26 and (3.46) it holds $\delta y \in L^\infty(0,T;L^2(\Omega))$. Since $U = L^2(\Omega)$, the assumptions on ε_1 and ε_2 in Lemma 5.12 are automatically fulfilled. Hence, for $\tau > 0$ small enough, depending on ℓ and $\delta\ell$, Lemma 5.12 together with the operator differential equation in (3.44) and (3.52) now gives

$$\begin{aligned}
\delta\left\|\frac{\dot d_\tau(t) - \dot d(t)}{\tau} - \dot\eta(t)\right\|_2 &= \left\|\frac{\max\big(\mathfrak{F}(\ell_\tau)(t)\big) - \max\big(\mathfrak{F}(\ell)(t)\big)}{\tau} - \max{}'\big(\mathfrak{F}(\ell)(t); \delta y(t)\big)\right\|_2\\
&\leq \left\|\frac{\mathfrak{F}(\ell_\tau)(t) - \mathfrak{F}(\ell)(t)}{\tau} - \delta y(t))\right\|_2 + R_{\max}(t,\tau) \quad \text{f.a.a. } t \in (0,T),
\end{aligned} \tag{3.54}$$

where $R_{\max} : (0,T) \times (0,1) \to [0,\infty)$ satisfies

$$R_{\max}(\cdot,\tau) \to 0 \text{ in } L^\varrho(0,T) \quad \text{as } \tau \searrow 0.$$

Further, in view of the definition of $\mathfrak{F}$ and δy, the estimate (3.54) can be continued, for $\tau > 0$ small enough, as follows

$$\begin{aligned}
&\delta\left\|\frac{\dot d_\tau(t) - \dot d(t)}{\tau} - \dot\eta(t)\right\|_2\\
&\leq \beta\left\|\frac{d_\tau(t) - d(t)}{\tau} - \eta(t)\right\|_2 + \beta\left\|\frac{\varphi_\tau(t) - \varphi(t)}{\tau} - \Phi'\big(\ell(t), d(t)\big)\big(\delta\ell(t), \eta(t)\big)\right\|_2 + R_{\max}(t,\tau)\\
&\leq 2\beta\left\|\frac{d_\tau(t) - d(t)}{\tau} - \eta(t)\right\|_2 + \beta R_\Phi(t,\tau) + R_{\max}(t,\tau) \quad \text{f.a.a. } t \in (0,T),
\end{aligned} \tag{3.55}$$

where the last inequality is ensured by (3.48b) and the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$. Note that $R(\cdot,\tau) := \beta R_\Phi(\cdot,\tau) + R_{\max}(\cdot,\tau) \geq 0$ f.a.a. $t \in (0,T)$ and $R(\cdot,\tau) \to 0$ in $L^\varrho(0,T)$ as $\tau \searrow 0$.

(ii) Furthermore, in view of Definition 3.22 and Lemma 3.26,

$$\tfrac{d_\tau - d}{\tau} - \eta \in W_0^{1,\infty}(0,T;L^2(\Omega)) \hookrightarrow W_0^{1,\varrho}(0,T;L^2(\Omega)).$$

Thanks to the properties of R and (3.55) we can now apply Lemma 5.10, which gives in turn

$$\Big\|\frac{d_\tau - d}{\tau} - \eta\Big\|_{W^{1,\varrho}(0,T;L^2(\Omega))} \leq C\|R(\cdot,\tau)\|_{L^\varrho(0,T)} \to 0 \text{ as } \tau \searrow 0,$$

whence the desired convergence. Note that the constant $C > 0$ depends only on β, δ, ϱ and T, in view of Lemma 5.10. □

We now have all the necessary tools in order to conclude the main result of this section.

Proposition 3.28 (Directional differentiability of the control-to-state operator)**.** *Let $M > 0$ and $\varrho \in [1,\infty)$ be given. Under Assumptions 1.17, p. 19, where $\beta = \beta(M+1)$, and 1.56, p. 46, the operator $\mathcal{S} : \mathfrak{B}_M \to L^\varrho(0,T;V) \times L^\varrho(0,T;H^1(\Omega)) \times W^{1,\varrho}(0,T;L^2(\Omega))$ is directionally differentiable. The derivative at $\ell \in \mathfrak{B}_M$ in direction $\delta\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$, which we denote by $\mathcal{S}'(\ell;\delta\ell) := (\boldsymbol{\delta u}, \delta\varphi, \delta d)$, belongs to $L^\infty(0,T;V) \times L^\infty(0,T;H^1(\Omega)) \times W_0^{1,\infty}(0,T;L^2(\Omega))$. Moreover, this satisfies f.a.a. $t \in (0,T)$ the following system*

$$\left\{\begin{aligned} \boldsymbol{\delta u}(t) &= \mathcal{U}'(\ell(t),\varphi(t))(\delta\ell(t),\delta\varphi(t)),\\ \delta\varphi(t) &= \Phi'(\ell(t),d(t))(\delta\ell(t),\delta d(t)),\\ \left.\begin{aligned}\dot{\delta d}(t) &= \frac{1}{\delta}\max{}'\big(-\beta(d(t)-\varphi(t)) - r; -\beta(\delta d(t) - \delta\varphi(t))\big),\\ \delta d(0) &= 0,\end{aligned}\right\}\end{aligned}\right. \tag{3.56}$$

where we abbreviate $\varphi := \mathcal{S}_2(\ell)$ and $d := \mathcal{S}_3(\ell)$.

Proof. Let $\ell \in \mathfrak{B}_M$ and $\delta\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$ be arbitrary, but fixed. From Lemma 3.27 we know that $\mathcal{S}_3 : \mathfrak{B}_M \to W^{1,\varrho}(0,T;L^2(\Omega))$ is directionally differentiable and $\mathcal{S}_3'(\ell;\delta\ell) \in W_0^{1,\infty}(0,T;L^2(\Omega))$ is the unique solution of the operator differential equation in (3.56). Hence, it remains to address the directional differentiability of $\mathcal{S}_1$ and $\mathcal{S}_2$, for which we employ Lemma 3.25. Recall that throughout this section, δd denoted some arbitrary element in $L^\infty(0,T;L^2(\Omega))$, which we now set $\delta d := \mathcal{S}_3'(\ell;\delta\ell)$ for the rest of this proof. By building $L^\varrho(0,T)$-norms in (3.48), we obtain the estimates

$$\Big\|\frac{\boldsymbol{u_\tau} - \boldsymbol{u}}{\tau} - \boldsymbol{\delta u}\Big\|_{L^\varrho(0,T;V)} \leq C\Big\|\frac{d_\tau - d}{\tau} - \delta d\Big\|_{L^\varrho(0,T;L^2(\Omega))} + \|cR_\Phi(\cdot,\tau) + R_{\mathcal{U}}(\cdot,\tau)\|_{L^\varrho(0,T)}, \tag{3.57a}$$

$$\Big\|\frac{\varphi_\tau - \varphi}{\tau} - \delta\varphi\Big\|_{L^\varrho(0,T;H^1(\Omega))} \leq C\Big\|\frac{d_\tau - d}{\tau} - \delta d\Big\|_{L^\varrho(0,T;L^2(\Omega))} + \|R_\Phi(\cdot,\tau)\|_{L^\varrho(0,T)}, \tag{3.57b}$$

where $\tau > 0$ is small enough, $R_{\mathcal{U}}, R_\Phi$ fulfill (3.49) and $C, c > 0$ are the constants from Lemma 3.25. Note that the abbreviations in (3.57) have the same meaning as in (3.45), where one sets $\delta d := \mathcal{S}_3'(\ell;\delta\ell)$. Moreover, note that the fact that τ is independent of t in Lemma 3.25 is crucial for deriving (3.57), and that, due to Definition 3.22, (3.47) and (3.46), the norms in (3.57) are well defined. In view of Lemma 3.27, combined with

$W^{1,\varrho}(0,T;L^2(\Omega)) \hookrightarrow L^\varrho(0,T;L^2(\Omega))$ and (3.49), we finally deduce from (3.57) that $\mathcal{S}_1 : \mathfrak{B}_M \to L^\varrho(0,T;V)$ and $\mathcal{S}_2 : \mathfrak{B}_M \to L^\varrho(0,T;H^1(\Omega))$ are directionally differentiable with

$$\mathcal{S}_1'(\ell;\delta\ell) = \boldsymbol{\delta u} = \mathcal{U}'\big(\ell(\cdot),\varphi(\cdot)\big)\big(\delta\ell(\cdot),\delta\varphi(\cdot)\big) \in L^\infty(0,T;V),$$
$$\mathcal{S}_2'(\ell;\delta\ell) = \delta\varphi = \Phi'\big(\ell(\cdot),d(\cdot)\big)\big(\delta\ell(\cdot),\delta d(\cdot)\big) \in L^\infty(0,T;H^1(\Omega)),$$

where we relied on (3.45). Note that the regularity thereof is given by (3.47) and (3.46). This completes the proof. □

We point out that $\mathcal{S} : \mathfrak{B}_M \to L^\infty(0,T;V) \times L^\infty(0,T;H^1(\Omega)) \times W^{1,\infty}(0,T;L^2(\Omega))$ is not expected to be directionally differentiable without norm gap. In order to see this, we first take a look at the last estimate in the proof of Lemma 3.27. Therein one observes that the convergence of $\frac{\mathcal{S}_3(\ell+\tau\delta\ell)-\mathcal{S}_3(\ell)}{\tau} - \mathcal{S}_3'(\ell;\delta\ell)$ in $W^{1,\varrho}(0,T;L^2(\Omega))$ is ultimately given by the convergence in $L^\varrho(0,T)$ of R_Φ and $R_{\max}$, where $\varrho < \infty$. We refer here also to the proof of Lemma 5.10. Nevertheless, one could make use of $W^{1,\varrho}(0,T;L^2(\Omega)) \hookrightarrow L^\infty(0,T;L^2(\Omega))$ in order to deduce from (3.48) convergence in L^∞-spaces, but this is not the case, since $R_{\mathcal{U}}$ and R_Φ do not necessarily converge in $L^\infty(0,T)$. Therefore, the norm gap is altogether caused by the fact that the convergence of $R_{\mathcal{U}}$, R_Φ and $R_{\max}$ was shown only in $L^\varrho(0,T)$ with $\varrho < \infty$. Recall that each of these convergences was deduced in Lemma 5.12 by means of Lebesgue's dominated convergence theorem, which does not hold true for L^∞-spaces, whence the norm gap.

3.4 Optimality system

The last part of this chapter is entirely concerned with deriving necessary optimality conditions for the optimal control problem governed by the damage model with penalty. For convenience let us recall that this reads as follows

$$\begin{aligned} \min_{\ell\in\mathfrak{L}} \quad & \mathcal{J}(\boldsymbol{u},\varphi,d,\ell) \\ \text{s.t.} \quad & (\boldsymbol{u},\varphi,d) \text{ solves } (P_\ell), \end{aligned} \tag{P_{min}}$$

where $\mathfrak{L}$ denotes the set of admissible loads and $\mathcal{J}$ is the objective functional. In this section we rely on the fact that (P_ℓ) can be reduced to an ordinary differential equation in Banach space. This was also earlier employed, in the context of defining and differentiating the control-to-state operator. With the results from Section 3.3 at hand, we shall first derive a purely primal optimality condition for (P_{min}), which is given by (VI) below. At the beginning of the last section we already mentioned that the control-to-state operator is not expected to be Gâteaux-differentiable, while Proposition 3.28 states only its directional differentiability. Indeed, as it will turn out, the Gâteaux-differentiability may be violated, if the *strict complementarity* condition (see Assumption 3.38 below) is not fulfilled. Therefore, the standard adjoint calculus for the derivation of optimality conditions from (VI) below is without further ado not applicable. By means of Assumption 3.38, one ultimately obtains an optimality system equivalent to the classical purely primal optimality condition. The section ends with a short discussion regarding the existence of solutions for (P_{min}) and with some remarks concerning *strict complementarity.*

We begin by fixing the assumptions on the optimal control problem. We choose to work with a very general setting, in the sense that, in view of practical applications, it would suffice for example that the objective is smoother as in Assumption 3.33 below.

Assumption 3.29 (The control set). *$\mathfrak{L}$ is a nonempty, convex and bounded subset of $L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$.*

Remark 3.30. *Note that the boundedness assumption on the control set implies that there exists some $M>0$ so that $\mathfrak{L}\subset\mathfrak{B}_M$. Hence, the results in Section 3.3 ensure that the control-to-state operator associated to (P_ℓ) is defined and directionally differentiable on $\mathfrak{L}$ (provided that $\beta=\beta(M+1)$). Notice that a reasonable choice for the set of admissible loads is*

$$\mathfrak{L} := \{\ell\in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega)) : \|\ell(t)\|_{\boldsymbol{W}^{-1,p}(\Omega)}\leq b \ \text{ f.a.a. } t\in(0,T)\},$$

where $b>0$ is a given bound.

Of course, throughout this section Assumptions 1.17, p. 19 and 1.56, p. 46 are supposed to further hold. Given the set $\mathfrak{L}$, the value of β in Assumption 1.17.2, p. 20, is supposed to satisfy from now on $\beta=\beta(M+1)$ (see Remark 3.9), where $M>0$ is chosen so that $\mathfrak{L}\subset\mathfrak{B}_M$.

The assumption on the objective $\mathcal{J}$ features the notion of Hadamard-differentiability, which is a concept of differentiability often found in the literature (see e.g. [75], [73]) and will be useful when applying chain rule in the proof of Lemma 3.35 below. For the convenience of the reader we give here the definition thereof, as well as an appropriate chain rule.

Definition 3.31 (Hadamard (directional) differentiability). *[73, Definition 3.1.1] Let X and Y be normed vector spaces. A function $f: X\to Y$ is said to be Hadamard directionally differentiable at $x\in X$, if the following limit exists*

$$f'_H(x;\delta x) := \lim_{\substack{\tau\searrow 0\\ z\to\delta x}} \frac{f(x+\tau z)-f(x)}{\tau} \quad \forall\,\delta x\in X.$$

In this situation, $f'_H(x;\cdot)$ is called Hadamard directional derivative of f at x. If $f'_H(x;\cdot)$ is linear, then f is called Hadamard-differentiable at x, in view of [73, Lemma 3.1.3 and Proposition 3.2.4 (ii)]. If existent, the Hadamard derivative of f at x is denoted by $f'_H(x)$ and belongs to $\mathcal{L}(X,Y)$, as a result of [73, Lemma 3.1.3]. Note that any Hadamard-differentiable function is Gâteaux-differentiable.

Lemma 3.32 (Chain rule for Hadamard directionally differentiable functions). *[75, Proposition 3.6 (i)] Let X,Y and Z be normed vector spaces and let $f_1: X\to Y$, $f_2: Y\to Z$ be some mappings. Assume that f_1 is directionally differentiable at $x\in X$ and*

f_2 is Hadamard directionally differentiable at $f_1(x) \in Y$. Then the composite mapping $f_2 \circ f_1 : X \to Z$ is directionally differentiable at $x \in X$ and the following chain rule holds

$$(f_2 \circ f_1)'(x; \delta x) = f_2'\big(f_1(x); (f_1'(x; \delta x))\big) \quad \forall\, \delta x \in X.$$

Now we can state the assumption on the functional $\mathcal{J}$:

Assumption 3.33 (The objective functional)**.** *$\mathcal{J} : L^2(0,T; \boldsymbol{W}_D^{1,\nu}(\Omega)) \times L^2(0,T; H^1(\Omega)) \times L^2(0,T; L^2(\Omega)) \times L^2(0,T; \boldsymbol{W}^{-1,p}(\Omega)) \to \mathbb{R}$ is Hadamard-differentiable, where $\nu \in (1,2)$ is given.*

The Hadamard-differentiability in Assumption 3.33 is self-evident in practice, since most of the objectives are even continuously Fréchet-differentiable, and hence Hadamard-differentiable, in view of [73, Proposition 3.4.2].

An example therefor is the functional $\mathcal{J}_{ex} : L^2(0,T; L^2(\Omega)) \times L^2(0,T; L^2(\Omega)) \times L^2(0,T; V^*)$ given by

$$\mathcal{J}_{ex}(\boldsymbol{u}, d, \ell) := \frac{1}{2}\,\|\boldsymbol{u} - \boldsymbol{u}_d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\alpha_1}{2}\|d\|^2_{L^2(0,T;L^2(\Omega))} + \frac{\alpha_2}{2}\|\ell\|^2_{L^2(0,T;V^*)},$$

where $\boldsymbol{u}_d \in L^2(0,T; L^2(\Omega))$ is a desired displacement and $\alpha_1, \alpha_2 > 0$. Note that $\mathcal{J}_{ex}$ fulfills Assumption 3.33 for $\nu \in (1,2)$ and $\nu \in [6/5, 2)$ in the two-dimensional and three-dimensional case, respectively. This is due to $\boldsymbol{W}_D^{1,\nu}(\Omega) \hookrightarrow L^2(\Omega)$ (see e.g. [80, Theorem 7.1]) and $\boldsymbol{W}^{-1,p}(\Omega) \hookrightarrow V^*$. Moreover note that $\mathcal{J}$ is continuously Fréchet-differentiable.

Remark 3.34. *Although it might seem more reasonable to consider in Assumption 3.33 the Hilbert space $L^2(0,T; V)$ for the displacement instead of $L^2(0,T; \boldsymbol{W}_D^{1,\nu}(\Omega))$, where $\nu \in (1,2)$, we choose to work with the latter one in order to obtain $L^2(0,T; V)$-regularity in both dimensions for the adjoint state $\boldsymbol{w}$ in Lemma 3.40 below. Otherwise, the adjoint state belongs to $L^2(0,T; \boldsymbol{W}_D^{1,\zeta}(\Omega))$, where $\zeta < 2$, in the three-dimensional case, while in the two-dimensional case it belongs to $L^2(0,T; V)$. For the sake of convenience we choose not to make this distinction. We point out that the condition for ν in Assumption 3.33 is essential only for the Bochner measurability of $\boldsymbol{w} : [0,T] \to V$. For the desired integrability of the latter one it suffices to work with $L^2(0,T; V)$ as space for the displacement. This is due to the estimate* (3.65) *in the proof of Lemma 3.40 below.*

We explain in what follows why the choice of ν in Assumption 3.33 is crucial for the Bochner measurability of $\boldsymbol{w} : [0,T] \to V$. Assuming that the space for the displacement is $L^2(0,T; \boldsymbol{W}_D^{1,\nu}(\Omega))$, where $\nu \in (1,p)$ is to be determined, we have in view of the Hadamard-differentiability of $\mathcal{J}$ and [14, Theorem 7.1.23 (vi)] that $\partial_{\boldsymbol{u}}\mathcal{J}(\cdot) \in L^2(0,T; \boldsymbol{W}^{-1,\nu'}(\Omega))$. Hence, $t \mapsto \partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t) \in \boldsymbol{W}^{-1,\nu'}(\Omega)$ is Bochner measurable. Note that here $(\cdot)$ denotes a point in the domain of $\mathcal{J}$. In view of (3.64) *below, one needs for the Bochner measurability of $\boldsymbol{w} : [0,T] \to V$ that $t \mapsto A^{-1}_{\varphi(t)} \in \mathcal{L}(\boldsymbol{W}^{-1,\nu'}(\Omega), V)$ is Bochner measurable, where $\varphi \in L^\infty(0,T; H^1(\Omega))$ and $\nu' \in [2,p]$. This asks for the continuity of $H^1(\Omega) \ni \varphi \mapsto A_\varphi^{-1} \in \mathcal{L}(\boldsymbol{W}^{-1,\nu'}(\Omega), V)$, for which a norm gap is needed, on account of Lemma 5.2. We point out that this is very similar to the situation described in Remark 3.18. Thus,*

$$H^1(\Omega) \ni \varphi \mapsto A_\varphi^{-1} \in \mathcal{L}(\boldsymbol{W}^{-1,\nu'}(\Omega), V) \text{ is continuous} \tag{3.58}$$

only when $\nu' > 2$, which ultimately motivates the assumption on $\mathcal{J}$. Similarly, it can be deduced that, working with $\nu = 2$ in Assumption 3.33 leads to $\boldsymbol{w} \in L^2(0,T;\boldsymbol{W}_D^{1,\zeta}(\Omega))$, where $\zeta < 2$. For this, one also needs Remark 1.4 on page 13.

As mentioned above, in the two-dimensional case it actually suffices to work with $L^2(0,T;V)$ as space for the displacement in order to obtain the Bochner measurability of $\boldsymbol{w} : [0,T] \to V$. This is due to the fact that, in Lemma 3.40 below, $\varphi = \mathcal{S}_2(\ell)$, where $\ell \in \mathfrak{L}$ and cf. Remark 3.23, there exists $q > 2$ so that $\varphi \in L^\infty(0,T;W^{1,q}(\Omega))$. The Bochner measurability of $\boldsymbol{w} : [0,T] \to V$ follows then by the arguments employed in Remark 3.23.

With the properties of $\mathfrak{L}$ and $\mathcal{J}$ at hand, we can now derive

Lemma 3.35 (Purely primal necessary optimality conditions)**.** *Let Assumptions 1.17, p. 19 and 1.56, p. 46 hold. Moreover, suppose that Assumptions 3.29 and 3.33 are fulfilled. Then, any local solution $\bar{\ell}$ of the problem (P_{min}) satisfies*

$$\partial_{(\boldsymbol{u},\varphi,d)}\mathcal{J}(\mathcal{S}(\bar{\ell}),\bar{\ell})\big(\mathcal{S}'(\bar{\ell};\delta\ell-\bar{\ell})\big) + \partial_\ell\mathcal{J}(\mathcal{S}(\bar{\ell}),\bar{\ell})(\delta\ell-\bar{\ell}) \geq 0 \quad \forall\, \delta\ell \in \mathfrak{L}. \tag{VI}$$

Proof. The proof is standard. In view of Definition 3.22 and Remark 3.30, the optimal control problem (P_{min}) can be rewritten as

$$\min_{\ell\in\mathfrak{L}} f(\ell),$$

where $f : \ell \mapsto \mathcal{J}(\mathcal{S}(\ell),\ell)$ is the reduced objective. Due to Proposition 3.28 and Assumption 3.33 we can apply Lemma 3.32 for f. This gives in turn that f is directionally differentiable at $\bar{\ell} \in \mathfrak{L}$ with

$$f'(\bar{\ell};\delta\ell-\bar{\ell}) = \mathcal{J}'(\mathcal{S}(\bar{\ell}),\bar{\ell})(\mathcal{S}'(\bar{\ell};\delta\ell-\bar{\ell}),\delta\ell-\bar{\ell}) \quad \forall\, \delta\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega)).$$

Note that the Hadamard derivative of $\mathcal{J}$ coincides with its Gâteaux derivative. From the convexity of $\mathfrak{L}$ we then deduce that any local minimizer of (P_{min}) satisfies (VI), which completes the proof. □

It is easy to see, in view of the Hadamard-differentiability of $\mathcal{J}$ and (3.56), that, if $\mathcal{S}_3$ is Gâteaux-differentiable at $\bar{\ell}$, then all the terms in (VI) are linear in $\delta\ell - \bar{\ell}$. Since, within our scope of deriving an optimality system, we want to write the left-hand side in (VI) as a linear form in $\delta\ell-\bar{\ell}$, we investigate in the following under which conditions the nonlinearity of $\mathcal{S}_3'(\bar{\ell};\cdot)$ can be overcome. Therefor, a closer inspection of the operator differential equation in (3.56) is required.

Prior to this, let us define for simplicity

Definition 3.36. *For a given $\ell \in \mathfrak{L}$ we define at $t \in (0,T)$ the following sets:*

- $\Omega_t^+ := \{x \in \Omega : -\beta(d(t,x) - \varphi(t,x)) - r > 0\}$,
- $\Omega_t^0 := \{x \in \Omega : -\beta(d(t,x) - \varphi(t,x)) - r = 0\}$,
- $\Omega_t^- := \{x \in \Omega : -\beta(d(t,x) - \varphi(t,x)) - r < 0\}$,

where we abbreviate $\varphi := \mathcal{S}_2(\ell)$ and $d := \mathcal{S}_3(\ell)$. Note that, for almost all $t \in (0,T)$, each of the above subsets of Ω is uniquely specified up to a set of measure zero (in Ω). We emphasize that the above defined sets ultimately depend on ℓ and the given data.

Let now $\ell \in \mathfrak{L}$ and $\delta\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$ be arbitrary, but fixed. In view of Proposition 3.28 and (5.17) combined with Definition 3.36, $\delta d := \mathcal{S}_3'(\ell;\delta\ell)$ is characterized as the unique solution of the operator differential equation

$$\dot{\delta d}(t) = \begin{cases} -\dfrac{\beta}{\delta}(\delta d(t) - \delta\varphi(t)) & \text{a.e. in } \Omega_t^+ \\ \max\big(-\dfrac{\beta}{\delta}(\delta d(t) - \delta\varphi(t))\big) & \text{a.e. in } \Omega_t^0 \\ 0 & \text{a.e. in } \Omega_t^- \end{cases} \quad \text{f.a.a. } t \in (0,T), \quad \delta d(0) = 0. \tag{3.59}$$

Here we use again the notations $\delta\varphi := \Phi'(\ell(\cdot), d(\cdot))(\delta\ell(\cdot), \delta d(\cdot))$ and $d := \mathcal{S}_3(\ell)$.

Remark 3.37. *From* (3.59) *we read, in view of the nonlinearity of* $\max\{\cdot, 0\}$*, that as long as there exist* $0 \le t_1 < t_2 \le T$ *such that* $\mu(\Omega_t^0) > 0$ *f.a.a.* $t \in (t_1, t_2)$*, the operator* $\mathcal{S}_3$ *is not necessarily Gâteaux-differentiable at* ℓ*. This is also shown by straight forward computation.*

Indeed, as we will next see, the linearity of $\mathcal{S}_3'(\ell;\cdot)$ is ensured if

Assumption 3.38 (Strict complementarity)**.** *The set* Ω_t^0 *associated to* $\ell \in \mathfrak{L}$ *has measure zero for almost all* $t \in (0,T)$*, i.e.,*

$$\mu(\Omega_t^0) = 0 \quad \textit{f.a.a. } t \in (0,T).$$

To prove that $\mathcal{S}_3'(\ell;\cdot)$ is linear under Assumption 3.38, note that (3.59) reads

$$\dot{\delta d}(t) = f_t(\delta\ell(t), \delta d(t)) \quad \text{f.a.a. } t \in (0,T), \quad \delta d(0) = 0, \tag{3.60}$$

where $f_t : \boldsymbol{W}^{-1,p}(\Omega) \times L^2(\Omega) \to L^2(\Omega)$ is given by

$$f_t(\delta\ell, \delta d) := -\frac{\beta}{\delta}\chi_{\Omega_t^+}\big(\delta d - \Phi'(\ell(t), d(t))(\delta\ell, \delta d)\big).$$

Observe that, since Φ is Gâteaux-differentiable at $(\ell(t), d(t))$, f_t is linear for almost all $t \in (0,T)$. Now let $\delta\ell_1, \delta\ell_2 \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$ and $a, b \in \mathbb{R}$. With the notation $\delta d_i := \mathcal{S}_3'(\ell;\delta\ell_i)$, where $i = 1,2$, (3.60) leads to

$$\begin{aligned} a\,\dot{\delta d}_1(t) + b\,\dot{\delta d}_2(t) &= a f_t(\delta\ell_1(t), \delta d_1(t)) + b f_t(\delta\ell_2(t), \delta d_2(t)) \\ &= f_t(a\,\delta\ell_1(t) + b\,\delta\ell_2(t), a\,\delta d_1(t) + b\,\delta d_2(t)) \quad \text{f.a.a. } t \in (0,T) \end{aligned}$$

and $a\delta d_1(0) + b\delta d_2(0) = 0$. Hence, $a\,\delta d_1 + b\,\delta d_2$ is the unique solution of (3.60) associated to $a\,\delta\ell_1 + b\,\delta\ell_2$, that is, $a\,\mathcal{S}_3'(\ell;\delta\ell_1) + b\,\mathcal{S}_3'(\ell;\delta\ell_2) = \mathcal{S}_3'(\ell; a\,\delta\ell_1 + b\,\delta\ell_2)$.

At the end of this chapter we make a few more comments on the *strict complementarity* assumption, including possible alternative approaches, such as regularization, see Remark 3.46 below.

Remark 3.39. *We emphasize that the above discussion concerning the linearity of the derivative has the sole purpose of giving an idea under which conditions one should expect an optimality system. The linearity result is rather indirectly used, e.g. by replacing the operator differential equation in* (3.56) *by the linear equation* (3.60)*, for constructing an 'artificial adjoint operator', see Lemma 3.40 and Remark 3.41 below. By [73, Lemma 3.1.2(b)] we infer from Remark 3.24 and Proposition 3.28 that $\mathcal{S}$ is Hadamard directionally differentiable, and hence, Hadamard differentiable, provided that Assumption 3.38 is satisfied. However, the standard approach of deducing that the solution operator of the adjoint equation is $\mathcal{S}'(\ell)^*$ cannot be employed here. That is so because, in view of Proposition 3.28, if existent, the adjoint operator*

$$\mathcal{S}'(\ell)^\star : L^\varrho(0,T;V)^* \times L^\varrho(0,T;H^1(\Omega))^* \times W^{1,\varrho}(0,T;L^2(\Omega))^* \to L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))^*$$

with $\varrho \in [1,\infty)$, gives in turn one adjoint state instead of three. This problem may be overcome by considering three controls instead of one, in which case the adjoint operator has range in $L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))^* \times X^* \times Y^*$, where X and Y are the spaces for the auxiliary controls. However, in this situation, a system of the type (3.61) *cannot be fulfilled by the adjoint states, as long as the first adjoint state does not possess higher regularity. This is due to the mere fact that $L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))^*$ cannot be identified with a Bochner-Lebesgue space. Altogether it does not make sense to adjoint $\mathcal{S}'(\ell)$, but rather to derive, for example via the formal Lagrange method, a candidate for the adjoint system. The latter one is a system with right-hand side $(\partial_{\boldsymbol{u}}\mathcal{J}(\mathcal{S}(\ell),\ell), \partial_\varphi\mathcal{J}(\mathcal{S}(\ell),\ell), \partial_d\mathcal{J}(\mathcal{S}(\ell),\ell))$, which is uniquely solved by what will turn out to be the adjoint states associated to ℓ in Theorem 3.42 below.*

The following result is an essential tool for deriving the main result of this chapter, as it provides the candidates for the adjoint states associated to $\ell \in \mathfrak{L}$.

Lemma 3.40 (Adjoint equation)**.** *Let Assumptions 1.17, p. 19, 1.56, p. 46, 3.29 and 3.33 hold. Moreover, let $\ell \in \mathfrak{L}$ be given and define $(\boldsymbol{u},\varphi,d) := \mathcal{S}(\ell)$. Then there exists a unique $(\boldsymbol{w},\upsilon,\xi) \in L^2(0,T;V) \times L^2(0,T;H^1(\Omega)) \times W^{1,2}_T(0,T;L^2(\Omega))$ which fulfills f.a.a. $t \in (0,T)$ the following system of equations*

$$-\operatorname{div}\big(g(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{w}(t)) + g'(\varphi(t))\upsilon(t)\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big) = \partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t) \quad \text{in } V^*, \tag{3.61a}$$

$$\left.\begin{aligned} -\alpha\Delta\upsilon(t) + \beta\big(\upsilon(t) - \tfrac{1}{\delta}\chi_{\Omega_t^+}\xi(t)\big) + g'(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{w}(t)) \\ + \tfrac{1}{2}g''(\varphi(t))\upsilon(t)\mathbb{C}\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) = \partial_\varphi\mathcal{J}(\cdot)(t) \quad \text{in } H^1(\Omega)^*, \end{aligned}\right\} \tag{3.61b}$$

$$\left.\begin{aligned} -\dot{\xi}(t) &= \beta\big(\upsilon(t) - \tfrac{1}{\delta}\chi_{\Omega_t^+}\xi(t)\big) + \partial_d\mathcal{J}(\cdot)(t) \quad \text{in } L^2(\Omega), \\ \xi(T) &= 0, \end{aligned}\right\} \tag{3.61c}$$

where $(\cdot)$ stands for $(\mathcal{S}(\ell),\ell)$.

Proof. We begin by noticing that (3.61) has a similar structure to (1.48) on page 29, as well as to its 'linearized' counterpart. Thus, well-known arguments are expected to be encountered throughout this proof. We approach the system (3.61) as in Section 1.1, in the sense that we first solve the elliptic system (3.61a)-(3.61b). Notice that this features the quantities $\boldsymbol{u}$ and φ, which are not well defined at any $t \in [0,T]$, but only a.e. in $(0,T)$ (see Definitions 3.22, 3.21 and 3.10). Hence, its solvability is addressed f.a.a. $t \in (0,T)$ and for any $\xi \in L^2(\Omega)$. Then, with the resulting 'solution operators' at hand, we rewrite the operator differential equation in (3.61) as depending only on ξ. The latter one is then solved by using Lemma 5.7, which practically yields the unique solvability of (3.61). The time regularity of ξ ultimately gives the time regularity of $\boldsymbol{w}$ and υ. For the sake of convenience we will denote the 'solution operators' involved in the elliptic system also by $\boldsymbol{w}$ and υ. Before we begin to discuss the solvability of (3.61), let us mention here that, since $\boldsymbol{W}^{1,\nu}_D(\Omega)$, $H^1(\Omega)$ and $L^2(\Omega)$ are reflexive Banach spaces, the partial derivatives $\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)$, $\partial_\varphi\mathcal{J}(\cdot)$ and $\partial_d\mathcal{J}(\cdot)$ belong to $L^2(0,T;\boldsymbol{W}^{-1,\nu'}(\Omega))$, $L^2(0,T;H^1(\Omega)^*)$ and $L^2(0,T;L^2(\Omega))$, respectively, see e.g. [14, Theorem 7.1.23 (vi)]. Note that, due to $\nu' > 2$, we have $\boldsymbol{W}^{-1,\nu'}(\Omega) \hookrightarrow V^*$, whence $\partial_{\boldsymbol{u}}\mathcal{J}(\cdot) \in L^2(0,T;V^*)$.

(i) Solvability of (3.61a). We search f.a.a. $t \in (0,T)$ and for any $\upsilon \in H^1(\Omega)$ for $\boldsymbol{w}$ such that

$$-\operatorname{div}\big(g(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{w})\big) = \partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t) + \operatorname{div}\big(g'(\varphi(t))\upsilon\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big) \quad \text{in } V^*. \tag{3.62}$$

The first thing to observe is that the right-hand side belongs to V^*. This follows from Assumption 3.33 and Hölder's inequality with $(p-2)/2p + 1/p = 1/2$, combined with Lemma 5.1, Definition 3.1 and the embedding $H^1(\Omega) \hookrightarrow L^{2p/(p-2)}(\Omega)$. Recall that the latter one is a consequence of Assumption 1.17.1, p. 19. Moreover, f.a.a. $t \in (0,T)$ and for any $\upsilon \in H^1(\Omega)$ it holds

$$\|\operatorname{div}\big(g'(\varphi(t))\upsilon\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big)\|_{V^*} \le C\|\upsilon\|_{2p/(p-2)}\|\mathcal{U}(\ell(t),\varphi(t))\|_{\boldsymbol{W}^{1,p}_D(\Omega)} \le C\|\upsilon\|_{H^1(\Omega)}. \tag{3.63}$$

Note that the last inequality was deduced from (3.5) and that $C > 0$ depends only on the given data, including the constant M from Remark 3.30. Further, in view of Definition 1.2, Lemma 1.3 on page 12 tells us that (3.62) is uniquely solvable at almost all $t \in (0,T)$ and for any $\upsilon \in H^1(\Omega)$ with

$$\boldsymbol{w}(t,\upsilon) = A^{-1}_{\varphi(t)}\big(\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t) + \operatorname{div}\big(g'(\varphi(t))\upsilon\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big)\big) \in V. \tag{3.64}$$

We now apply estimate (1.6) to $A_{\varphi(t)}$ and in view of (3.63) we get

$$\|\boldsymbol{w}(t,\upsilon)\|_V \le C\big(\|\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t)\|_{V^*} + \|\upsilon\|_{H^1(\Omega)}\big) \text{ f.a.a. } t \in (0,T),\ \forall\, \upsilon \in H^1(\Omega), \tag{3.65}$$

where again $C > 0$ depends only on the given data.

We now show that the Nemytskii operator associated to $\boldsymbol{w}$ maps $L^2(0,T;H^1(\Omega))$ to $L^2(0,T;V)$. This result will be useful at the end of the proof, after the time regularity of the solution of the operator differential equation (3.61c) is established. Let $\upsilon \in L^2(0,T;H^1(\Omega))$ be arbitrary, but fixed. In view of Remark 3.34, see also (3.58) and (3.64), it is necessary for the Bochner measurability of $t \mapsto \boldsymbol{w}(t,\upsilon(t)) \in V$ that

$$t \mapsto \partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t) + \operatorname{div}\big(g'(\varphi(t))\upsilon(t)\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big)$$

is Bochner measurable as a function mapping to $\boldsymbol{W}^{-1,\rho}(\Omega)$, with some $\rho > 2$. Hence, we prove next that there exists $\rho > 2$ such that at almost all $t \in (0,T)$ it holds

$$\operatorname{div}\big(g'(\varphi(t))\upsilon(t)\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big) \in \boldsymbol{W}^{-1,\rho}(\Omega).$$

To see this, recall that in view of Assumption 1.17.1, there exists $\kappa > 2p/(p-2)$ so that the embedding $H^1(\Omega) \hookrightarrow L^\kappa(\Omega)$ holds true. By choosing $\varrho < \infty$ and $s \in [2,p)$ large enough such that $1/\varrho + 1/\kappa + 1/s < (p-2)/2p + 1/p = 1/2$, we can then define the index $\rho > 2$ via

$$1/\rho := 1/\varrho + 1/\kappa + 1/s < 1/2.$$

Hölder's inequality with $1/\varrho + 1/\kappa + 1/s = 1/\rho$ now gives the desired regularity and from the boundedness of the operator $\operatorname{div} : L^\rho(\Omega;\mathbb{R}^{N\times N}) \to \boldsymbol{W}^{-1,\rho}(\Omega)$ we deduce

$$\|\operatorname{div}\big(g'(\varphi(t))\upsilon(t)\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big)\|_{\boldsymbol{W}^{-1,\rho}(\Omega)} \leq \|g'(\varphi(t))\|_\varrho \|\upsilon(t)\|_\kappa \|\mathbb{C}\varepsilon(\boldsymbol{u}(t))\|_s \tag{3.66}$$

for almost all $t \in (0,T)$. Further, we know from Lemmata 5.2 and 5.11 that the mapping $t \mapsto g'(\varphi(t)) \in L^\varrho(\Omega)$ is Bochner measurable, while $t \mapsto \mathbb{C}\varepsilon(\boldsymbol{u}(t)) \in L^s(\Omega;\mathbb{R}^{N\times N})$ is Bochner measurable as well, as a result of Definition 3.22. Thus, on account of the definition of the Bochner measurability and (3.66), we obtain the Bochner measurability of

$$t \mapsto \operatorname{div}\big(g'(\varphi(t))\upsilon(t)\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big) \in \boldsymbol{W}^{-1,\rho}(\Omega),$$

where we also employed that $\upsilon : [0,T] \to L^\kappa(\Omega)$ is Bochner measurable, by assumption. Besides, $t \mapsto \partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t) \in \boldsymbol{W}^{-1,\nu'}(\Omega)$ is Bochner measurable as well, in view of Assumption 3.33, whence we conclude that

$$t \mapsto \partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t) + \operatorname{div}\big(g'(\varphi(t))\upsilon(t)\mathbb{C}\varepsilon(\boldsymbol{u}(t))\big) \in \boldsymbol{W}^{-1,\omega}(\Omega)$$

is Bochner measurable, where $\omega := \min\{\rho, \nu'\} > 2$. From (3.58) and (3.64) we now deduce the Bochner measurability of $t \mapsto \boldsymbol{w}(t,\upsilon(t)) \in V$. Assumption 3.33 together with (3.65) ultimately yields

$$\boldsymbol{w}(\cdot,\upsilon(\cdot)) \in L^2(0,T;V) \text{ for } \upsilon \in L^2(0,T;H^1(\Omega)). \tag{3.67}$$

(ii) Solvability of (3.61b). For almost all $t \in (0,T)$ and for any $\xi \in L^2(\Omega)$, we now search for $\upsilon \in H^1(\Omega)$ that solves

$$\left.\begin{aligned} B\upsilon + g'(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{w}(t,\upsilon)) + \frac{1}{2}g''(\varphi(t))\upsilon\mathbb{C}\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) \\ = \partial_\varphi\mathcal{J}(\cdot)(t) + \frac{\beta}{\delta}\chi_{\Omega_t^+}\xi \quad \text{in } H^1(\Omega)^*. \end{aligned}\right\} \tag{3.68}$$

Recall that the linearity B is given by (1.22) on page 18. Let us begin by rewriting (3.68) in a compact form. To this end, notice that in view of (3.64) and (3.21) it holds

$$\boldsymbol{w}(t,\upsilon) = A_{\varphi(t)}^{-1}\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t) + \partial_\varphi\mathcal{U}(\ell(t),\varphi(t))(\upsilon) \quad \text{f.a.a. } t \in (0,T),\ \forall\, \upsilon \in H^1(\Omega),$$

where we relied on the linearity of $A^{-1}_{\varphi(t)}$. By taking a look at (3.30) one sees now that (3.68) is equivalent to

$$Bv + \partial_\varphi F(\ell(t), \varphi(t))v = \iota(t, \xi) \quad \text{in } H^1(\Omega)^*,$$

where the mapping $\iota : (0, T) \times L^2(\Omega) \to H^1(\Omega)^*$ is given by

$$\iota(t, \xi) := \partial_\varphi \mathcal{J}(\cdot)(t) + \frac{\beta}{\delta}\chi_{\Omega_t^+}\xi - g'(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) : \varepsilon\big(A^{-1}_{\varphi(t)}\big(\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t)\big)\big). \tag{3.69}$$

In order to see that $\iota(t, \xi)$ is well defined f.a.a. $t \in (0, T)$ and all $\xi \in L^2(\Omega)$ we argue as follows. From part (i) we recall that $A^{-1}_{\varphi(t)}\big(\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t)\big) \in V$, since $A^{-1}_{\varphi(t)} \in \mathcal{L}(V^*, V)$, f.a.a. $t \in (0, T)$. Then, on account of Lemma 5.1, Definition 3.1 and the embedding $H^1(\Omega) \hookrightarrow L^{2p/(p-2)}(\Omega)$, we can apply Hölder's inequality with $1/p+1/2+(p-2)/2p = 1$ for the last term on the right-hand side in (3.69), which gives in turn the $H^1(\Omega)^*$-regularity thereof. Additionally, by employing again (3.5) and (1.6), we deduce here

$$\big\|g'(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) : \varepsilon\big(A^{-1}_{\varphi(t)}\big(\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t)\big)\big)\big\|_{H^1(\Omega)^*} \leq C\|\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t)\|_{V^*} \quad \text{f.a.a. } t \in (0, T), \tag{3.70}$$

which is needed only later in the proof. Note that $C > 0$ depends only on the given data, including the constant M from Remark 3.30. As $\chi_{\Omega_t^+} \in L^\infty(\Omega)$ f.a.a. $t \in (0, T)$, we finally conclude with Assumption 3.33 and $L^2(\Omega) \hookrightarrow H^1(\Omega)^*$ that $\iota(t, \xi)$ is well defined f.a.a. $t \in (0, T)$ and all $\xi \in L^2(\Omega)$. On the other side, $B + \partial_\varphi F(\ell(t), \varphi(t)) \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$ is f.a.a. $t \in (0, T)$ continuously invertible with

$$\|\big(B + \partial_\varphi F(\ell(t), \varphi(t))\big)^{-1}\|_{\mathcal{L}(H^1(\Omega)^*, H^1(\Omega))} \leq 2/\alpha \quad \text{f.a.a. } t \in (0, T). \tag{3.71}$$

We refer here to page 113, where the unique solvability of (3.29) is shown. Note that the estimate (3.71) follows from (3.33). Hence, (3.68) is uniquely solvable f.a.a. $t \in (0, T)$ and for any $\xi \in L^2(\Omega)$ with

$$v(t, \xi) = \big(B + \partial_\varphi F(\ell(t), \varphi(t))\big)^{-1}\iota(t, \xi) \in H^1(\Omega). \tag{3.72}$$

Moreover, due to (3.72), (3.71) and (3.69) we have f.a.a. $t \in (0, T)$ and for all $\xi_1, \xi_2 \in L^2(\Omega)$ the estimate

$$\|v(t, \xi_1) - v(t, \xi_2)\|_{H^1(\Omega)} \leq \frac{2}{\alpha}\|\iota(t, \xi_1) - \iota(t, \xi_2)\|_{H^1(\Omega)^*} \leq \frac{2\beta}{\alpha\delta}\|\xi_1 - \xi_2\|_2. \tag{3.73}$$

In preparation for the next part of the proof, we prove in the following that v belongs to $L^2(0, T; H^1(\Omega))$ if $\xi \in L^2(0, T; L^2(\Omega))$. Note that the above time regularity of v does not improve, even when ξ turns out to have higher regularity than $L^2(0, T; L^2(\Omega))$. This can be seen by taking a look at (3.72), (3.71) and (3.69), while keeping in mind that $\partial_\varphi \mathcal{J}(\cdot) \in L^2(0, T; H^1(\Omega)^*)$. We do not go here into any more details and refer to Remark 3.16, where a similar observation was made in the context of establishing the time regularity of the solutions of the minimization problem in (P_ℓ). Firstly, we want to prove that

$$\big(B+\partial_\varphi F(\ell(t),\varphi(t))\big)^{-1}\in L^\infty\big(0,T;\mathcal{L}(H^1(\Omega)^*,H^1(\Omega))\big). \tag{3.74}$$

To this end, recall that $\partial_\varphi F:\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)\to\mathcal{L}(H^1(\Omega),H^1(\Omega)^*)$ is continuous. This is stated at the end of part (I) of the proof of Lemma 3.15. Since $t\mapsto(\ell(t),\varphi(t))\in\boldsymbol{W}^{-1,p}(\Omega)\times H^1(\Omega)$ is Bochner measurable, Lemma 5.11 yields the Bochner measurability of $t\mapsto B+\partial_\varphi F(\ell(t),\varphi(t))\in\mathcal{L}(H^1(\Omega),H^1(\Omega)^*)$. Cf. e.g. [74, Ch. III.8], the inversion $A\mapsto A^{-1}$ is continuous from the set of isomorphisms in $\mathcal{L}(H^1(\Omega),H^1(\Omega)^*)$ to $\mathcal{L}(H^1(\Omega)^*,H^1(\Omega))$. This together with the definition of Bochner measurability then gives the measurability of the mapping in (3.74). The desired Bochner integrability thereof follows now from estimate (3.71). Secondly, it holds

$$\iota(\cdot,\xi(\cdot))\in L^2(0,T;H^1(\Omega)^*)\quad\text{ for }\xi\in L^2(0,T;L^2(\Omega)). \tag{3.75}$$

To see this, we address the time regularity of each term in (3.69) separately. We already know that $\partial_\varphi\mathcal{J}(\cdot)\in L^2(0,T;H^1(\Omega)^*)$, in view of Assumption 3.33. Further, Definition 3.36 allows us to write

$$\chi_{\Omega_t^+}(x)\xi(t)(x)=(\chi_{Q_+}\xi)(t,x)\quad\text{for almost all }(t,x)\in(0,T)\times\Omega,$$

where $Q_+:=\{(t,x)\in(0,T)\times\Omega:-\beta(d(t,x)-\varphi(t,x))-r>0\}$ and $\xi\in L^2(0,T;L^2(\Omega))$. Note that Q_+ depends only on ℓ and it is well defined up to a set of measure zero in $(0,T)\times\Omega$. Since $\chi_{Q_+}\in L^\infty((0,T)\times\Omega)$ and $L^2((0,T)\times\Omega)=L^2(0,T;L^2(\Omega))$, see e.g. [14, Theorem 7.1.24], we have $\chi_{Q_+}\xi\in L^2((0,T)\times\Omega)$, whence

$$t\mapsto\chi_{\Omega_t^+}\xi(t)\in L^2(0,T;L^2(\Omega))\text{ for }\xi\in L^2(0,T;L^2(\Omega)). \tag{3.76}$$

Furthermore, recall that in view of Assumption 1.17.1, there exists $\kappa>2p/(p-2)$ so that $H^1(\Omega)\hookrightarrow L^\kappa(\Omega)$. In light of $1/2+1/\kappa<1-1/p$, we find $s\in[2,p)$ and $\varrho<\infty$ large enough, so that

$$1/\varrho+1/s+1/2+1/\kappa=1. \tag{3.77}$$

The mapping $t\mapsto g'(\varphi(t))\in L^\varrho(\Omega)$ is Bochner measurable thanks to Lemmata 5.2 and 5.11, while $t\mapsto\mathbb{C}\varepsilon(\boldsymbol{u}(t))\in L^s(\Omega;\mathbb{R}^{N\times N})$ is Bochner measurable as well, as a result of Definition 3.22. Arguing as at the end of part (i), it follows further that $t\mapsto\varepsilon\big(A^{-1}_{\varphi(t)}\big(\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t)\big)\big)\in L^2(\Omega;\mathbb{R}^{N\times N})$ is also Bochner measurable. Altogether we deduce from the above that

$$t\mapsto g'(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)):\varepsilon\big(A^{-1}_{\varphi(t)}\big(\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t)\big)\big)\in H^1(\Omega)^*$$

is Bochner measurable, on account of Hölder's inequality with (3.77) and the definition of Bochner measurability. From (3.70) we then obtain

$$t\mapsto g'(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)):\varepsilon\big(A^{-1}_{\varphi(t)}\big(\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t)\big)\big)\in L^2(0,T;H^1(\Omega)^*).$$

Therewith we addressed all the terms in (3.69), and thus, we can now infer (3.75). By means of (3.74) and (3.75) one can finally establish as in [81, Lemma 3.1.19] that

$$\upsilon(\cdot,\xi(\cdot)) \in L^2(0,T;H^1(\Omega)) \text{ for } \xi \in L^2(0,T;L^2(\Omega)), \tag{3.78}$$

where we relied on (3.72).

(iii) Solvability of (3.61c). By means of the 'solution operator' $(t,\xi) \mapsto \upsilon(t,\xi)$ of the elliptic system, (3.61) reduces to

$$\left.\begin{aligned} -\dot{\xi}(t) &= \beta\big(\upsilon(t,\xi(t)) - \frac{1}{\delta}\chi_{\Omega_t^+}\xi(t)\big) + \partial_d \mathcal{J}(\cdot)(t) \quad \text{f.a.a. } t \in (0,T), \\ \xi(T) &= 0. \end{aligned}\right\} \tag{3.79}$$

We intend to solve (3.79) by employing again Lemma 5.7. For this purpose, note first that via the (at this point informal) transformation

$$\widetilde{\xi}(\cdot) = \xi(T-\cdot),$$

(3.79) is equivalent to

$$\dot{\widetilde{\xi}}(t) = f(t,\widetilde{\xi}(t)) \text{ f.a.a. } t \in (0,T), \quad \widetilde{\xi}(0) \ = 0, \tag{3.80}$$

where $f : (0,T) \times L^2(\Omega) \to L^2(\Omega)$ is given by

$$f(t,\widetilde{\xi}) = \beta\Big(\upsilon(T-t,\widetilde{\xi}) - \frac{1}{\delta}\chi_{\Omega_{T-t}^+}\widetilde{\xi}\Big) + \partial_d \mathcal{J}(\cdot)(T-t). \tag{3.81}$$

Note that for any $\widetilde{\xi} \in L^2(\Omega)$, the value $f(t,\widetilde{\xi}) \in L^2(\Omega)$ is well defined f.a.a. $t \in (0,T)$, as a consequence of (3.72), Definition 3.36 and Assumption 3.33. Let now $\widetilde{\xi} \in L^2(0,T;L^2(\Omega))$ be arbitrary, but fixed. It is easy to see that this implies $t \mapsto \widetilde{\xi}(T-t) \in L^2(0,T;L^2(\Omega))$. According to (3.78), we then have $\upsilon(\cdot,\widetilde{\xi}(T-\cdot)) \in L^2(0,T;H^1(\Omega))$, which in particular means that

$$t \mapsto \upsilon(T-t,\widetilde{\xi}(t)) \in L^2(0,T;H^1(\Omega)). \tag{3.82}$$

With (3.76) we establish that $t \mapsto \chi_{\Omega_t^+}\widetilde{\xi}(T-t) \in L^2(0,T;L^2(\Omega))$, and consequently

$$t \mapsto \chi_{\Omega_{T-t}^+}\widetilde{\xi}(t) \in L^2(0,T;L^2(\Omega)). \tag{3.83}$$

From Assumption 3.33 we infer $t \mapsto \partial_d \mathcal{J}(\cdot)(T-t) \in L^2(0,T;L^2(\Omega))$, which combined with (3.82) and (3.83) now yields

$$f(\cdot,\widetilde{\xi}(\cdot)) \in L^2(0,T;L^2(\Omega)) \quad \text{ for } \widetilde{\xi} \in L^2(0,T;L^2(\Omega)),$$

in view of (3.81). Therewith the second assumption on f in Lemma 5.7 is satisfied. As (3.81) and (3.73) imply

$$\begin{aligned} \|f(t,\widetilde{\xi}_1) - f(t,\widetilde{\xi}_2)\|_2 &\leq \beta\|\upsilon(T-t,\widetilde{\xi}_1) - \upsilon(T-t,\widetilde{\xi}_2)\|_2 + \beta/\delta\|\widetilde{\xi}_1 - \widetilde{\xi}_2\|_2 \\ &\leq L\|\widetilde{\xi}_1 - \widetilde{\xi}_2\|_2 \quad \forall \widetilde{\xi}_1,\widetilde{\xi}_2 \in L^2(\Omega), \ \text{ f.a.a. } t \in (0,T), \end{aligned}$$

the third assumption on f in Lemma 5.7 is satisfied as well. Note that $L > 0$ depends only on α, β and δ. Lemma 5.7 now tells us that (3.80) is uniquely solvable with $\widetilde{\xi} \in W_0^{1,2}(0,T;L^2(\Omega))$, whence the unique solvability of (3.79) with

$$t \mapsto \xi(t) = \widetilde{\xi}(T-t) \in W^{1,2}_T(0,T; L^2(\Omega)).$$

Note that the time regularity of ξ is due to $t \mapsto \widetilde{\xi}(T-t) \in L^2(0,T;L^2(\Omega))$ and $t \mapsto \dot{\xi}(t) = -\dot{\widetilde{\xi}}(T-t) \in L^2(0,T;L^2(\Omega))$. This gives the unique solvability of (3.61) with the desired regularity of solutions, in view of (3.67) and (3.78). The proof is now complete. □

Remark 3.41. *Clearly, given $\ell \in \mathfrak{L}$, the system* (3.61) *admits a unique solution (with the exact same regularity) for an arbitrary right-hand side $h \in L^2(0,T;\boldsymbol{W}^{-1,\nu'}(\Omega)) \times L^2(0,T;H^1(\Omega)^*) \times L^2(0,T;L^2(\Omega))$ (instead of $\partial_{(\boldsymbol{u},\varphi,d)}\mathcal{J}(\mathcal{S}(\ell),\ell)$), where $\nu \in (1,2)$ is given. The thereby induced solution operator, which we here call Υ, can be interpreted as an 'artificial adjoint operator' of $\mathcal{S}'(\ell)$, provided that Assumption 3.38 holds true for ℓ. This can be seen by linearizing the state equation* (3.84) *below at ℓ in some arbitrary direction $\delta\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$ and by arguing as in the proof of Theorem 3.42 below, where we replace the partial derivatives of $\mathcal{J}$ in* (3.85) *by h as above. Then, one has instead of* (3.93) *below, the equality*

$$\int_0^T \sum_{i=1}^3 \langle h_i(t), \mathcal{S}_i'(\ell)(\delta\ell)(t)\rangle \, dt = \int_0^T \langle \delta\ell(t), \Upsilon_1(h)(t)\rangle_V \, dt,$$

i.e.,

$$\langle h, \mathcal{S}'(\ell)(\delta\ell)\rangle_{L^2(0,T;\boldsymbol{W}^{1,\nu}_D(\Omega)\times H^1(\Omega)\times L^2(\Omega))} = \langle \Upsilon_1 h, \delta\ell\rangle_{L^2(0,T;\boldsymbol{W}^{-1,p}(\Omega))}$$

for any $h \in L^2(0,T;\boldsymbol{W}^{-1,\nu'}(\Omega) \times H^1(\Omega)^* \times L^2(\Omega))$ and any $\delta\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$. Here Υ_1 stands for the first component of the operator Υ.

We point out that Υ_1 is the classical adjoint operator of $\mathcal{S}'(\ell) : L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega)) \to L^2(0,T;\boldsymbol{W}^{1,\nu}_D(\Omega) \times H^1(\Omega) \times L^2(\Omega))$, as a result of the above identity and as it maps as follows

$$L^2(0,T;\boldsymbol{W}^{-1,\nu'}(\Omega) \times H^1(\Omega)^* \times L^2(\Omega)) \overset{\Upsilon_1}{\longmapsto} L^2(0,T;V) \subset L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))^*,$$

in view of Lemma 3.40 and $L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega)) \hookrightarrow L^2(0,T;V^*)$.

The main result of this chapter is covered by the following

Theorem 3.42 (Optimality system)**.** *Let Assumptions 1.17, p. 19, 1.56, p. 46, 3.29 and 3.33 hold. Moreover, let $\bar{\ell}$ be a local solution of (P_{min}) with associated states*

$$(\bar{\boldsymbol{u}}, \bar{\varphi}, \bar{d}) = \mathcal{S}(\bar{\ell}) \in L^\infty(0,T;\boldsymbol{W}^{1,s}_D(\Omega)) \times L^\infty(0,T;H^1(\Omega)) \times W^{1,\infty}(0,T;L^2(\Omega)),$$

where $s \in [2,p)$, and suppose that Assumption 3.38 is fulfilled for $\bar{\ell}$. Then there exist unique adjoint states

$$(\boldsymbol{w}, \upsilon, \xi) \in L^2(0,T;V) \times L^2(0,T;H^1(\Omega)) \times W^{1,2}_T(0,T;L^2(\Omega))$$

so that the following optimality system is f.a.a. $t \in (0,T)$ satisfied:

$$-\operatorname{div} g(\bar\varphi(t))\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t)) = \bar\ell(t) \quad \text{in } \boldsymbol{W}^{-1,p}(\Omega), \tag{3.84a}$$

$$-\alpha\Delta\bar\varphi(t) + \beta\,\bar\varphi(t) + \frac{1}{2}\,g'(\bar\varphi(t))\mathbb{C}\,\varepsilon(\bar{\boldsymbol{u}}(t)) : \varepsilon(\bar{\boldsymbol{u}}(t)) = \beta\bar d(t) \quad \text{in } H^1(\Omega)^*, \tag{3.84b}$$

$$\dot{\bar d}(t) = \frac{1}{\delta}\max(-\beta(\bar d(t) - \bar\varphi(t)) - r), \quad \bar d(0) = d_0, \tag{3.84c}$$

$$-\operatorname{div}\big(g(\bar\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{w}(t)) + g'(\bar\varphi(t))\upsilon(t)\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t))\big) = \partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t) \quad \text{in } V^*, \tag{3.85a}$$

$$\left.\begin{aligned} &-\alpha\Delta\upsilon(t) + \beta\big(\upsilon(t) - \frac{1}{\delta}\chi_{\Omega_t^+}\xi(t)\big) + g'(\bar\varphi(t))\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t)) : \varepsilon(\boldsymbol{w}(t)) \\ &\qquad + \frac{1}{2}g''(\bar\varphi(t))\upsilon(t)\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t)) : \varepsilon(\bar{\boldsymbol{u}}(t)) = \partial_\varphi\mathcal{J}(\cdot)(t) \quad \text{in } H^1(\Omega)^*, \end{aligned}\right\} \tag{3.85b}$$

$$\left.\begin{aligned} &-\dot\xi(t) = \beta\big(\upsilon(t) - \frac{1}{\delta}\chi_{\Omega_t^+}\xi(t)\big) + \partial_d\mathcal{J}(\cdot)(t) \quad \text{in } L^2(\Omega), \\ &\xi(T) = 0, \end{aligned}\right\} \tag{3.85c}$$

$$\langle \boldsymbol{w} + \partial_\ell\mathcal{J}(\cdot), \delta\ell - \bar\ell\rangle_{L^2(0,T;\boldsymbol{W}^{-1,p}(\Omega))} \geq 0 \quad \forall\,\delta\ell \in \mathfrak{L}, \tag{3.86}$$

where $(\cdot)$ *denotes* $(\mathcal{S}(\bar\ell), \bar\ell)$.

Proof. The state equation (3.84) is automatically fulfilled in view of Definition 3.22, combined with Definitions 3.1 and 3.10, while from Lemma 3.40 we know that (3.85) admits a unique solution $(\boldsymbol{w}, \upsilon, \xi)$ with the desired regularity. Hence, it remains to prove the gradient inequality (3.86), for which we employ standard arguments. Roughly speaking, we test the linearized counterpart of (3.84) with the adjoint states and show that the (integrated over time) sum of the resulting equations is the (integrated over time) sum of the equations in (3.85) tested with the directional derivatives of $\mathcal{S}$. Using this in (VI) will ultimately give the claim. For a better overview, we split the rest of the proof in two parts: in the first one we derive a linearization for the state equation, while in the second one we test as depicted above and finalize the proof.

(i) Let $\delta\ell \in L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$ be arbitrary, but fixed. In view of Assumption 3.38, (3.84) can be linearized, such that the resulting system is uniquely solved by $\mathcal{S}'(\bar\ell;\delta\ell - \bar\ell)$. For the sake of simplicity we denote the latter one by $(\boldsymbol{\delta u}, \delta\varphi, \delta d)$ in what follows. To be more precise, Proposition 3.28 and Assumption 3.38 ensure that $(\boldsymbol{\delta u}, \delta\varphi, \delta d)$ satisfies f.a.a. $t \in (0,T)$ the system

$$\boldsymbol{\delta u}(t) = \mathcal{U}'(\bar\ell(t), \bar\varphi(t))(\delta\ell(t) - \bar\ell(t), \delta\varphi(t)), \tag{3.87a}$$

$$\delta\varphi(t) = \Phi'(\bar\ell(t), \bar d(t))(\delta\ell(t) - \bar\ell(t), \delta d(t)), \tag{3.87b}$$

$$\dot{\delta d}(t) = -\frac{\beta}{\delta}\chi_{\Omega_t^+}(\delta d(t) - \delta\varphi(t)), \quad \delta d(0) = 0. \tag{3.87c}$$

From the differentiability results in Section 3.1, we can deduce the equations which characterize $\boldsymbol{\delta u}(t)$ and $\delta\varphi(t)$ f.a.a. $t \in (0,T)$. Firstly, from (3.20) and (3.21) we infer that

$$\boldsymbol{\delta u}(t) = A^{-1}_{\bar\varphi(t)}\Big(\delta\ell(t) - \bar\ell(t) + \operatorname{div}\big(g'(\bar\varphi(t))\delta\varphi(t)\mathbb{C}\varepsilon\big(\mathcal{U}(\bar\ell(t), \bar\varphi(t))\big)\big)\Big) \text{ f.a.a. } t \in (0,T),$$

on account of (3.87a). By Definition 1.2 on page 12 we then obtain that $\boldsymbol{\delta u}(t)$ is the unique solution of

$$-\operatorname{div}\big(g(\bar\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{\delta u}(t))\big) = \delta\ell(t) - \bar\ell(t) + \operatorname{div}\big(g'(\bar\varphi(t))\delta\varphi(t)\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t))\big) \quad \text{in } V^* \tag{3.88}$$

f.a.a. $t \in (0,T)$. Secondly, using (3.87b) in (3.24) yields that $\delta\varphi(t)$ is the unique solution of

$$\left.\begin{aligned} -\alpha\Delta\delta\varphi(t) + \beta\delta\varphi(t) + \frac{1}{2}g''(\bar\varphi(t))\delta\varphi(t)\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t)) : \varepsilon(\bar{\boldsymbol{u}}(t))& \\ = \beta\delta d(t) - g'(\bar\varphi(t))\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t)) : \varepsilon(\boldsymbol{\delta u}(t)) \quad \text{in } H^1(\Omega)^*& \end{aligned}\right\} \tag{3.89}$$

f.a.a. $t \in (0,T)$.

To summarize, the linearized counterpart of (3.84) (at $\bar\ell$ in direction $\delta\ell-\bar\ell$) consists of (3.88), (3.89) and (3.87c). Let us recall here that $(\boldsymbol{\delta u}, \delta\varphi, \delta d)$ belongs to $L^\infty(0,T;V) \times L^\infty(0,T;H^1(\Omega)) \times W_0^{1,\infty}(0,T;L^2(\Omega))$, cf. Proposition 3.28.

(ii) We test (3.88), (3.89) and (3.87c) with $\boldsymbol{w}(t) \in V$, $\upsilon(t) \in H^1(\Omega)$ and $\xi(t) \in L^2(\Omega)$, respectively, at almost all $t \in (0,T)$. For the sake of convenience, we bring all the resulting terms containing $\boldsymbol{\delta u}$ and $\delta\varphi$ together, by abbreviating

$$\begin{aligned} \iota(t) :=& -\langle \operatorname{div}\big(g(\bar\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{\delta u}(t)) + g'(\bar\varphi(t))\delta\varphi(t)\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t))\big), \boldsymbol{w}(t)\rangle_V \\ &+ \langle -\alpha\Delta\delta\varphi(t) + \beta\delta\varphi(t) + \frac{1}{2}g''(\bar\varphi(t))\delta\varphi(t)\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t)) : \varepsilon(\bar{\boldsymbol{u}}(t)), \upsilon(t)\rangle_{H^1(\Omega)} \\ &+ \langle g'(\bar\varphi(t))\mathbb{C}\varepsilon(\bar{\boldsymbol{u}}(t)) : \varepsilon(\boldsymbol{\delta u}(t)), \upsilon(t)\rangle_{H^1(\Omega)} - \frac{\beta}{\delta}(\chi_{\Omega_t^+}\delta\varphi(t), \xi(t))_2 \quad \text{f.a.a. } t \in (0,T). \end{aligned}$$

Thereby, the sum of the above tested equations reads f.a.a. $t \in (0,T)$ as follows

$$\iota(t) + (\dot{\delta d}(t), \xi(t))_2 = \beta(\delta d(t), \upsilon(t))_2 - \frac{\beta}{\delta}\chi_{\Omega_t^+}\delta d(t), \xi(t))_2 + \langle \delta\ell(t) - \bar\ell(t), \boldsymbol{w}(t)\rangle_V. \tag{3.90}$$

Furthermore, as a result of (3.85a) and (3.85b), combined with Definition 3.36, we observe that

$$\iota(t) = \langle \partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t), \boldsymbol{\delta u}(t)\rangle_V + \langle \partial_\varphi\mathcal{J}(\cdot)(t), \delta\varphi(t)\rangle_{H^1(\Omega)} \quad \text{f.a.a. } t \in (0,T), \tag{3.91}$$

while testing (3.85c) with $\delta d(t)$ gives in turn

$$(\partial_d\mathcal{J}(\cdot)(t), \delta d(t))_2 = -(\dot\xi(t), \delta d(t))_2 - \beta(\upsilon(t), \delta d(t))_2 + \frac{\beta}{\delta}(\chi_{\Omega_t^+}\xi(t), \delta d(t))_2 \quad \text{f.a.a. } t \in (0,T).$$

Assumption 3.33 and the regularity of δd, see Proposition 3.28, imply further that $t \mapsto (\partial_d\mathcal{J}(\cdot)(t), \delta d(t))_2 \in L^2(0,T)$. Hence, we can integrate the above identity over time, which results in

$$\begin{aligned}\int_0^T(\partial_d\mathcal{J}(\cdot)(t),\delta d(t))_2\,dt &= -\int_0^T(\dot{\xi}(t),\delta d(t))_2\,dt - \int_0^T\beta(\upsilon(t),\delta d(t))_2 - \frac{\beta}{\delta}(\chi_{\Omega_t^+}\xi(t),\delta d(t))_2\,dt \\ &= -(\xi(T),\delta d(T))_2 + (\xi(0),\delta d(0))_2 + \int_0^T(\dot{\delta d}(t),\xi(t))_2 \\ &\qquad - \int_0^T\beta(\upsilon(t),\delta d(t))_2 - \frac{\beta}{\delta}(\chi_{\Omega_t^+}\xi(t),\delta d(t))_2\,dt \\ &= \int_0^T(\dot{\delta d}(t),\xi(t))_2 - \int_0^T\beta(\upsilon(t),\delta d(t))_2 - \frac{\beta}{\delta}(\chi_{\Omega_t^+}\xi(t),\delta d(t))_2\,dt,\end{aligned} \tag{3.92}$$

where we employed $\xi \in W_T^{1,2}(0,T;L^2(\Omega))$, $\delta d \in W_0^{1,\infty}(0,T;L^2(\Omega))$ and integrated by parts cf. [81, Lemma 3.1.43]. Moreover, note that the terms on the right-hand side in (3.91) are integrable over time, by Assumption 3.33 and Proposition 3.28. Inserting (3.91) and (3.92) in (3.90) leads to

$$\begin{aligned}\int_0^T\langle\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t),\boldsymbol{\delta u}(t)\rangle_V + \langle\partial_\varphi\mathcal{J}(\cdot)(t),\delta\varphi(t)\rangle_{H^1(\Omega)}+(\partial_d\mathcal{J}(\cdot)(t),\delta d(t))_2\,dt \\ = \int_0^T\langle\delta\ell(t)-\bar{\ell}(t),\boldsymbol{w}(t)\rangle_V\,dt.\end{aligned} \tag{3.93}$$

On the other side, by Assumption 3.33, we are allowed to write

$$\begin{aligned}&\partial_{(\boldsymbol{u},\varphi,d)}\mathcal{J}(\mathcal{S}(\bar{\ell}),\bar{\ell})\big(\mathcal{S}'(\bar{\ell};\delta\ell-\bar{\ell})\big) \\ &\quad= \langle\partial_{\boldsymbol{u}}\mathcal{J}(\cdot),\boldsymbol{\delta u}\rangle_{L^2(0,T;V)} + \langle\partial_\varphi\mathcal{J}(\cdot),\delta\varphi\rangle_{L^2(0,T;H^1(\Omega))} + \langle\partial_d\mathcal{J}(\cdot),\delta d\rangle_{L^2(0,T;L^2(\Omega))} \\ &\quad= \int_0^T\langle\partial_{\boldsymbol{u}}\mathcal{J}(\cdot)(t),\boldsymbol{\delta u}(t)\rangle_V + \langle\partial_\varphi\mathcal{J}(\cdot)(t),\delta\varphi(t)\rangle_{H^1(\Omega)} + (\partial_d\mathcal{J}(\cdot)(t),\delta d(t))_2\,dt,\end{aligned} \tag{3.94}$$

due to the reflexivity of V, $H^1(\Omega)$ and $L^2(\Omega)$, see e.g. [14, Theorem 7.1.23(vi)]. As $\bar{\ell}$ is a local minimizer of (P_{min}), it satisfies the variational inequality in Lemma 3.35, which in light of (3.93), (3.94) and Assumption 3.33 now reads

$$\begin{aligned}&\int_0^T\langle\delta\ell(t)-\bar{\ell}(t),\boldsymbol{w}(t)\rangle_V\,dt + \langle\partial_\ell\mathcal{J}(\mathcal{S}(\bar{\ell}),\bar{\ell}),\delta\ell-\bar{\ell}\rangle_{L^2(0,T;\boldsymbol{W}^{-1,p}(\Omega))} \\ &\quad= \langle\boldsymbol{w},\delta\ell-\bar{\ell}\rangle_{L^2(0,T;\boldsymbol{W}^{-1,p}(\Omega))} + \langle\partial_\ell\mathcal{J}(\mathcal{S}(\bar{\ell}),\bar{\ell}),\delta\ell-\bar{\ell}\rangle_{L^2(0,T;\boldsymbol{W}^{-1,p}(\Omega))} \geq 0 \quad \forall\delta\ell\in\mathfrak{L},\end{aligned} \tag{3.95}$$

where we employed the reflexivity of V, the embedding $V \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)^*$, as well as the regularity of $\boldsymbol{w}$. Since (3.95) is (3.86), the proof is now complete. □

With the next result we can conclude that the optimality system (3.84)-(3.86) is in fact equivalent to the first order necessary optimality condition in Lemma 3.35, provided that the *strict complementarity* assumption is fulfilled:

Proposition 3.43. *Suppose that Assumptions 1.17, p. 19, 1.56, p. 46, 3.29 and 3.33 hold true. Moreover, let Assumption 3.38 be fulfilled for some* $\bar{\ell} \in \mathfrak{L}$. *Then, if* $\bar{\ell}$ *together with its states*

$$(\bar{\boldsymbol{u}}, \bar{\varphi}, \bar{d}) \in L^\infty(0,T; \boldsymbol{W}_D^{1,s}(\Omega)) \times L^\infty(0,T; H^1(\Omega)) \times W^{1,\infty}(0,T; L^2(\Omega)),$$

where $s \in [2, p)$, *and adjoint states*

$$(\boldsymbol{w}, \upsilon, \xi) \in L^2(0,T; V) \times L^2(0,T; H^1(\Omega)) \times W_T^{1,2}(0,T; L^2(\Omega)),$$

satisfies the optimality system (3.84)-(3.86), *it also satisfies the variational inequality* (VI).

Proof. In view of (VI), it suffices to show

$$\langle \boldsymbol{w}, \delta\ell - \bar{\ell} \rangle_{L^2(0,T;\boldsymbol{W}^{-1,p}(\Omega))} = \partial_{(\boldsymbol{u},\varphi,d)} \mathcal{J}(\mathcal{S}(\bar{\ell}), \bar{\ell}) \mathcal{S}'(\bar{\ell}; \delta\ell - \bar{\ell}) \quad \forall \delta\ell \in \mathfrak{L}. \tag{3.96}$$

Since in the proof of Theorem 3.42 one derives (3.93) and (3.94) regardless of the (therein imposed) local optimality assumption on $\bar{\ell}$, these two equalities can be used here to infer (3.96). Note that therefor we employ again the reflexivity of V, the embedding $V \hookrightarrow \boldsymbol{W}^{-1,p}(\Omega)^*$, as well as the regularity of $\boldsymbol{w}$. Moreover, note that it is essential that the *strict complementarity* assumption on $\bar{\ell}$ is fulfilled in order to conclude (3.96), see the proof of Theorem 3.42 for more details. The proof is now complete. □

Remark 3.44. *Notice that* (3.86) *holds as an equality on* $\mathfrak{L}$ *if the local optimum* $\bar{\ell}$ *is an inner point of* $\mathfrak{L}$, *and in particular, if* $\mathfrak{L}$ *is an open set. To be more precise,* (3.86) *is then replaced by*

$$\langle \boldsymbol{w} + \partial_\ell \mathcal{J}(\cdot), \delta\ell \rangle_{L^2(0,T;\boldsymbol{W}^{-1,p}(\Omega))} = 0 \quad \forall\, \delta\ell \in \mathfrak{L}. \tag{3.97}$$

In order to see this, we argue as follows. Since $\bar{\ell} \in int(\mathfrak{L})$, *there exists* $\tau > 0$ *small enough such that* $\bar{\ell} \pm \tau\delta\ell \in \mathfrak{L}$ *for all* $\delta\ell \in \mathfrak{L}$. *Testing in* (3.86) *then yields*

$$\langle \boldsymbol{w} + \partial_\ell \mathcal{J}(\cdot), \pm\tau\delta\ell \rangle_{L^2(0,T;\boldsymbol{W}^{-1,p}(\Omega))} \geq 0 \quad \forall\, \delta\ell \in \mathfrak{L},$$

whence (3.97).

Let us now make some comments regarding the existence of solutions for (P_{min}). To be more precise, we derive a setting so that the direct method of variational calculus can be applied for (P_{min}). To this end, let us recall that the latter one can be written as

$$\min_{\ell \in \mathfrak{L}} f(\ell), \tag{3.98}$$

where $f : \ell \mapsto \mathcal{J}(\mathcal{S}(\ell), \ell)$ is the reduced objective functional. The first thing to observe is that the direct method of variational calculus cannot be applied to solve (3.98) without further ado. The control set $\mathfrak{L}$ is indeed a bounded subset of the reflexive Banach space $L^2(0,T;V^*)$, in light of $L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega)) \hookrightarrow L^2(0,T;V^*)$. Since cf. Assumption 3.29, $\mathfrak{L}$ is convex, it would suffice to impose that it is also closed in $L^2(0,T;V^*)$, in order to obtain its weak compactness (in $L^2(0,T;V^*)$). However, $f : \mathfrak{L} \subset L^2(0,T;V^*) \to \mathbb{R}$ is not necessarily weakly lower semicontinuous. This would be the case if $\mathcal{S}$ were (at least) weakly continuous, which is not to be expected due to the structure of (3.44), see also Theorem 3.20. As cf. Remark 3.24, $\mathcal{S}$ is Lipschitz continuous, it makes sense to require that the control set is first of all a (bounded) subset of a (reflexive Banach) space which compactly embeds in $L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$. An example for such a control set, which as we will see, satisfies all the conditions needed for showing existence of solutions, is

$$\mathfrak{L} := \{\ell \in H_0^1(0,T;L^p(\Omega;\mathbb{R}^N)) : \|\dot{\ell}\|_{L^2(0,T;L^p(\Omega;\mathbb{R}^N))} \leq b\}, \tag{3.99}$$

where $b > 0$ is a given bound. Clearly, $\mathfrak{L}$ satisfies Assumption 3.29 and although for the sole purpose of finding global minimizers for (3.98), weaker assumptions for the functional $\mathcal{J}$ would suffice, we let Assumption 3.33 further hold, such that all the results proven in this section still apply.

We now prove that for $\mathfrak{L}$ given by (3.99) (and $\mathcal{J}$ as in Assumption 3.33) the optimal control problem (P_{min}) admits solutions. We begin by noticing that $\mathfrak{L}$ is a nonempty, convex, closed and bounded subset of the reflexive Banach space $H^1(0,T;L^p(\Omega;\mathbb{R}^N))$, which in particular means that $\mathfrak{L}$ is weakly compact. Moreover, $f : \mathfrak{L} \to \mathbb{R}$ is weakly lower semicontinuous. To see this, consider a sequence $\{\ell_n\} \subset \mathfrak{L}$ with $\ell_n \rightharpoonup \ell$ in $H^1(0,T;L^p(\Omega;\mathbb{R}^N))$ as $n \to \infty$. Due to $\boldsymbol{W}_D^{1,p'}(\Omega) \hookrightarrow\hookrightarrow L^{p'}(\Omega;\mathbb{R}^N)$ and Schauder's theorem we have the compact embedding $H^1(0,T;L^p(\Omega;\mathbb{R}^N)) \hookrightarrow\hookrightarrow L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$, see e.g. [81, Corollary 3.1.42]. With Remark 3.24 we then infer

$$\mathcal{S}(\ell_n) \to \mathcal{S}(\ell) \quad \text{in } L^\infty(0,T;V) \times L^\infty(0,T;H^1(\Omega)) \times W^{1,\infty}(0,T;L^2(\Omega))$$

as $n \to \infty$. As a Hadamard-differentiable function, the objective $\mathcal{J}$ is continuous on $L^2(0,T;\boldsymbol{W}_D^{1,\nu}(\Omega)) \times L^2(0,T;H^1(\Omega)) \times L^2(0,T;L^2(\Omega)) \times L^2(0,T;\boldsymbol{W}^{-1,p}(\Omega))$, cf. [73, Proposition 3.2.5]. This together with the above convergence gives that f is indeed weakly lower semicontinuous. Finally, a standard argument yields that (3.98) admits solutions and as an immediate consequence, so does (P_{min}).

We end this chapter with some remarks related with Assumption 3.38. We first point out that one can reformulate (P_{min}) as an MPCC, provided that the set $\mathfrak{L}$ is chosen accordingly.

Remark 3.45 ((P_{min}) as MPCC)**.** *In view of Theorem 3.20 and since the* $\max\{\cdot,0\}$*-function is a complementarity function, the unique solution* $(\boldsymbol{u},\varphi,d)$ *of the problem* (P_ℓ) *with right-hand side* $\ell \in \mathfrak{L}$ *is characterized by*

$$\begin{aligned} -\operatorname{div} g(\varphi(t))\mathbb{C}\varepsilon(\boldsymbol{u}(t)) &= \ell(t) && \text{in } \boldsymbol{W}^{-1,p}(\Omega), \\ -\alpha\Delta\varphi(t) + \beta\,\varphi(t) + \frac{1}{2}\,g'(\varphi(t))\mathbb{C}\,\varepsilon(\boldsymbol{u}(t)) : \varepsilon(\boldsymbol{u}(t)) &= \beta d(t) && \text{in } H^1(\Omega)^*, \\ 0 \leq \dot{d}(t) \perp \beta(d(t)-\varphi(t)) + r + \delta\dot{d}(t) \geq 0 \quad \text{a.e. in } \Omega, & && d(0) = d_0 \end{aligned}$$

f.a.a. $t \in (0,T)$, *see also* *(1.44), p. 27.* *Thus, if the control set can be described e.g. only by inequalities, e.g.* $\mathfrak{L} = \overline{\mathfrak{B}_M}$, *then the problem* ($P_{min}$) *falls into the class of MPCCs, see [72] for the definition thereof in the finite-dimensional case.*

Remark 3.46 (Strict complementarity)**.** *In contrast to our approach, the most authors dealing with time-dependent MPCCs make use of regularization and penalization techniques, see e.g. [2, 35] (parabolic obstacle problem), [81] (quasistatic plasticity) and [15, 33] (Allen-Cahn and Cahn-Hilliard VIs). In [81] the complementarity condition is relaxed by regularizing, so that the biactive set, i.e., the set where both inequalities in the complementarity condition hold with equality, vanishes. As already mentioned in the introduction of this chapter, the optimality systems obtained by applying the above mentioned methods are in the best case of intermediate strength. This is not surprising at all, since one always loses information when passing to the limit (in the regularized/penalized problem). Hence, an optimality system of strong stationary type is not to be expected when applying this technique.*

Recall that, cf. Remark 3.39, the operator $\mathcal{S}$ is Hadamard-differentiable in those points for which Assumption 3.38 is fulfilled. Thus, conditions for Gâteaux-differentiability are in our situation simpler described as in the above contributions, since therein the complementarity conditions feature dual variables, see e.g. [81, (4.18)] and [15, (2.1)-(2.5)] (Allen-Cahn VIs).

Remark 3.47 ('Ample controls')**.** *When it comes to the optimal control of nonsmooth problems, strong stationary optimality systems have been mostly derived in the presence of 'ample controls', i.e., (distributed) controls that are not restricted by additional constraints. The literature here is rather scarce. We refer to [10, 31, 63] (elliptic VIs) and to [50] (nonsmooth parabolic equations). In [31] a modified version of the static plasticity problem is considered, in the context of distributed controls without control constraints. After deriving the strong stationarity conditions for the optimal control thereof, the authors state a strong stationary optimality system for the original problem (without proving its necessity).*

Until recently, it was an open question whether such a system can be derived in the absence of 'ample controls', see also [63, Section 4]. It turns out, that indeed, the necessity of strong stationarity can be proven for the obstacle problem with pointwise constraints on the control. This was shown in [82], however by requiring that the (unknown) optimizer satisfies certain assumptions (constraint qualifications). There one obtains a strong stationary optimality system, which is a generalization of the optimality system for the optimal control of the obstacle problem derived by [63] in the more restrictive case of 'ample controls'.

Chapter 4

Conclusions and outlook

In the present thesis we obtained satisfying results regarding the viability of the damage model with penalty (P). This is two-fold. Firstly, we proved the well-posedness of the problem from a mathematical point of view. Secondly, we showed that the penalty approach makes sense, as both damage variables become equal in the limiting case $\beta \to \infty$, whereas the limit model turns out to be a classical viscous damage model. Finally, we derived an optimality system for an optimal control problem governed by (P). Unfortunately, this was possible only under an additional assumption.

In Chapter 1 we saw that (P) admits an unique solution, which is characterized by a system consisting of two elliptic PDEs and an operator differential equation. This was shown under nonrestrictive assumptions in two dimensions, whereas for the unique solvability of the minimization problem in (P) in case of $N = 3$, additional assumptions on the data are required. Without imposing further assumptions, we established that the $H^1(\Omega)$-regularity of the nonlocal damage can be improved, which is useful especially in two dimensions, as it leads to significantly better differentiability results for the solution operators, and thus for the entire solution of (P). All in all, we stated different regularity results for the latter one, which take many forms, depending on the space regularity of d_0, the smoothness of the coefficient function g and the smoothness in time of the load. We recall here also the findings in Chapter 3, see Section 3.2, where the unique solvability of (P) was addressed in a more general setting.

A noteworthy result in Chapter 1 was the equivalence of the evolution equation in (P) with an ordinary differential equation in Banach space. This is particularly advantageous from a numerical point of view, as well as from a theoretical point of view e.g. for performing the time-discretization in Chapter 2 or for investigating the differentiability of the control-to-state operator in Chapter 3. We saw that, in fact, the evolution of the local damage can be expressed by means of various equivalent formulations. Here we recall the *energy identity* which was essential for proving the viability of the penalty approach. In this context, we established the following. The two damage variables are equal when $\beta \to \infty$ and the resulting single-field gradient damage model is just a version of a viscous damage model analyzed in [41]. Of course, this is the case only in two dimensions, as in three dimensions the designated space for the damage variable in [41] no longer coincides with $H^1(\Omega)$. In particular, we deduced that the vanishing viscosity analysis from [41] may be transferred to our limit model, so that there is no need to analyze the behaviour of the non-penalized problem (2.74) as $\delta \searrow 0$. An interesting question arises here: what happens to the penalized model (P) when the viscosity vanishes, i.e., $\delta \searrow 0$? This is of particular interest, since the non-viscous damage model with penalty forms the basis for mechanical applications, see [12].

In Chapter 3 we considered an optimal control problem governed by the penalized damage model, where we used the load as control. A reinvestigation of the constraint in (P_{min}) was necessary, which among others led us to a general suitable admissible set for the loads. The control-to-state operator associated to (P) turned out to be only (Hadamard) directionally differentiable. Therefore, standard adjoint calculus could not be applied for the derivation of necessary optimality conditions in form of an optimality system. This was however possible under the *strict complementarity* assumption. Unfortunately, the latter one cannot be a priori verified, as it depends on the searched local optimum. Anyway, this type of assumption is not surprising at all, if we take a look at the existing results in the literature. Roughly speaking, it is the price one has to pay for not regularizing, and thus for obtaining a stronger optimality system as in the case of regularizing. In what concerns the existence of global solutions for (P_{min}), we saw that this is guaranteed if the control set is a (bounded) subset of a space which compactly embeds in $L^\infty(0,T;\boldsymbol{W}^{-1,p}(\Omega))$. It remains an open question if (P_{min}) possesses solutions in the original control set.

Chapter 5

Auxiliary results

5.1 Nemytskii operators g and g'

In this section we establish some useful results on the Nemytskii operators associated to the function g and its derivatives, see Assumption 0.6. The latter one is supposed to hold throughout this section.

Lemma 5.1. *For all $\rho \in [1, \infty]$, the Nemytskii operators $g : L^\rho(\Omega) \to L^\infty(\Omega)$ and $g' : L^\rho(\Omega) \to L^\infty(\Omega)$ are well defined and Lipschitz continuous from $L^\rho(\Omega)$ to $L^\rho(\Omega)$. If in addition, $g \in C^2(\mathbb{R})$, then the operator $g'' : L^\rho(\Omega) \to L^\infty(\Omega)$ is well defined as well.*

Proof. We prove the result just for the function g'. The results for g and g'' follow completely analogously. The first thing to notice is that g' transforms measurable functions into measurable functions, since g' is continuous in view of (3). Moreover, $g' \in L^\infty(\mathbb{R})$ and hence, $g' : L^\rho(\Omega) \to L^\infty(\Omega)$ is well defined for all $\rho \in [1, \infty]$. The Lipschitz continuity from $L^\rho(\Omega)$ to $L^\rho(\Omega)$ is a direct consequence of the Lipschitz continuity of $g' : \mathbb{R} \to \mathbb{R}$. $\square$

Lemma 5.2. *The Nemytskii operators $g, g' : L^1(\Omega) \to L^\rho(\Omega)$ are continuous for all $\rho \in [1, \infty)$.*

Proof. The functions g and g' are continuous on account of (3) and the associated Nemytskii operators $g, g' : L^1(\Omega) \to L^\infty(\Omega)$ are well defined by means of Lemma 5.1. Thus, the assumptions in [21, Theorem 4] are fulfilled, which gives the assertion. $\square$

Lemma 5.3. *The operator $g : L^\rho(\Omega) \to L^\tau(\Omega)$ is continuously Fréchet-differentiable for $1 \le \tau < \rho < \infty$. If we assume that the map g satisfies $g \in C^2(\mathbb{R})$, then the operator $g' : L^\rho(\Omega) \to L^\tau(\Omega)$ is continuously Fréchet-differentiable as well.*

Proof. We prove the continuously Fréchet-differentiability by means of [21, Theorem 7]. We address just the second part of the statement, since the first one follows with the exactly same arguments. From Lemma 5.2 and by employing [21, Theorem 4], we obtain that g'' is continuous from $L^\rho(\Omega)$ to $L^{\frac{\rho\tau}{\rho-\tau}}(\Omega)$ for $1 \le \tau < \rho < \infty$. Since $g' \in C^1(\mathbb{R})$, all the assumptions in [21, Theorem 7] are fulfilled. This completes the proof. $\square$

5.2 Product and chain rule

This section consists of a generalization of the product rule in Banach spaces and a generalization of the chain rule, where the inner function is only weakly differentiable.

Lemma 5.4 (Product rule)**.** *Let X, W and Y_i, Z_i, $i = 1, 2$, be Banach spaces. Moreover, let $U \subset X$ be an open set and $f_i : U \to Y_i$, $i = 1, 2$, be continuous mappings, which are continuously Fréchet-differentiable, when considered as mappings from U to Z_i. Additionally, let $P : Z_1 \times Y_2 \to W$ be a product, i.e., a continuous bilinear mapping, and assume that P possesses the same properties, when considered as a mapping from $Y_1 \times Z_2$ to W. Then, the map $h : x \in U \to P(f_1(x), f_2(x)) \in W$ is continuously Fréchet-differentiable with*

$$h'(x)(\delta x) = P(f_1'(x)(\delta x), f_2(x)) + P(f_1(x), f_2'(x)(\delta x)) \quad \forall\, x \in U,\ \forall\, \delta x \in X. \tag{5.1}$$

Proof. Let $x \in U$ be arbitrary, but fixed and $\delta x \in X$ with $\|\delta x\|_X \neq 0$ small enough such that $x + \delta x \in U$. Straight forward computation yields

$$\begin{aligned}
&\|R(\delta x)\|_W := \\
&:= \|h(x+\delta x) - h(x) - P(f_1'(x)(\delta x), f_2(x)) - P(f_1(x), f_2'(x)(\delta x))\|_W \\
&\le \|P(f_1(x+\delta x), f_2(x)) - P(f_1(x), f_2(x)) - P(f_1'(x)(\delta x), f_2(x))\|_W \\
&\quad + \|P(f_1(x+\delta x), f_2(x+\delta x)) - P(f_1(x+\delta x), f_2(x)) - P(f_1(x+\delta x), f_2'(x)(\delta x))\|_W \\
&\quad + \|P(f_1(x+\delta x), f_2'(x)(\delta x)) - P(f_1(x), f_2'(x)(\delta x))\|_W.
\end{aligned}$$

Since $P : Z_1 \times Y_2 \to W$, $P : Y_1 \times Z_2 \to W$ are continuous bilinear mappings, we obtain in view of the Fréchet-differentiability of $f_i : U \to Z_i$ for every $i \in \{1, 2\}$, combined with the continuity of $f_1 : U \to Y_1$ that

$$\begin{aligned}
\frac{\|R(\delta x)\|_W}{\|\delta x\|_X} \le C\Big(&\frac{\|R_{f_1}(\delta x)\|_{Z_1}}{\|\delta x\|_X}\|f_2(x)\|_{Y_2} + \frac{\|R_{f_2}(\delta x)\|_{Z_2}}{\|\delta x\|_X}\|f_1(x+\delta x)\|_{Y_1} \\
&+ \|f_1(x+\delta x) - f_1(x)\|_{Y_1}\frac{\|f_2'(x)(\delta x)\|_{Z_2}}{\|\delta x\|_X}\Big) \to 0, \quad \text{as } \|\delta x\|_X \to 0,
\end{aligned}$$

where $R_{f_i}(\delta x) := f_i(x+\delta x) - f_i(x) - f_i'(x)(\delta x)$ for every $i \in \{1, 2\}$. Therefore, h is Fréchet-differentiable at $x \in U$, with derivative given by (5.1).

In order to show the continuity thereof, let $\{x_n\} \subset U$ with $x_n \xrightarrow{n\to\infty} x$ in X be given. By employing the properties of P we obtain for all $\delta x \in X$

$$\begin{aligned}
&\|P(f_1'(x_n)(\delta x), f_2(x_n)) - P(f_1'(x)(\delta x), f_2(x))\|_W \\
&\le \|P(f_1'(x_n)(\delta x) - f_1'(x)(\delta x), f_2(x_n))\|_W + \|P(f_1'(x)(\delta x), f_2(x_n) - f_2(x))\|_W \\
&\le C\big(\|f_1'(x_n)(\delta x) - f_1'(x)(\delta x)\|_{Z_1}\|f_2(x_n)\|_{Y_2} + \|f_1'(x)(\delta x)\|_{Z_1}\|f_2(x_n) - f_2(x)\|_{Y_2}\big) \\
&\le C\big(\|f_1'(x_n) - f_1'(x)\|_{\mathcal{L}(X,Z_1)}\|\delta x\|_X\|f_2(x_n)\|_{Y_2} \\
&\qquad + \|f_1'(x)\|_{\mathcal{L}(X,Z_1)}\|\delta x\|_X\|f_2(x_n) - f_2(x)\|_{Y_2}\big).
\end{aligned}$$

The continuity of $f_1' : U \to \mathcal{L}(X, Z_1)$ and $f_2 : U \to Y_2$ thus implies

$$\begin{aligned}\sup_{\|\delta x\|_X=1} &\|P(f_1'(x_n)(\delta x), f_2(x_n)) - P(f_1'(x)(\delta x), f_2(x))\|_W \\ &\leq C\big(\|f_1'(x_n) - f_1'(x)\|_{\mathcal{L}(X,Z_1)}\|f_2(x_n)\|_{Y_2} \\ &\qquad + \|f_1'(x)\|_{\mathcal{L}(X,Z_1)}\|f_2(x_n) - f_2(x)\|_{Y_2}\big) \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{5.2}$$

Completely analogously we obtain

$$\sup_{\|\delta x\|_X=1} \|P(f_1(x_n), f_2'(x_n)(\delta x)) - P(f_1(x), f_2'(x)(\delta x))\|_W \to 0 \quad \text{as } n \to \infty. \tag{5.3}$$

Finally, (5.1), (5.2) and (5.3) result in

$$\sup_{\|\delta x\|_X=1} \|h'(x_n)(\delta x) - h'(x)(\delta x)\|_W \to 0 \quad \text{as } n \to \infty,$$

which completes the proof. □

Lemma 5.5 (Chain rule)**.** *Let $\mathfrak{F} \in C^1([0,T] \times H^1(\Omega))$. Assume that for any $K > 0$ there is $C_K > 0$ such that*

$$\|\partial_t \mathfrak{F}(\cdot, \phi(\cdot))\|_{L^2(0,T)}, \|\partial_z \mathfrak{F}(\cdot, \phi(\cdot))\|_{L^\infty(0,T;H^1(\Omega)^*)} \leq C_K \tag{5.4}$$

for all $\phi \in H^1(0,T;H^1(\Omega))$ with $\|\phi\|_{H^1(0,T;H^1(\Omega))} \leq K$. Then, for $z \in H^1(0,T;H^1(\Omega))$, the map $t \mapsto \mathfrak{F}(t, z(t))$ belongs to $H^1(0,T)$ with weak derivative given by

$$\frac{d}{dt}\mathfrak{F}(t, z(t)) = \partial_t \mathfrak{F}(t, z(t)) + \langle \partial_z \mathfrak{F}(t, z(t)), \dot{z}(t)\rangle_{H^1(\Omega)} \quad \text{f.a.a. } t \in (0,T).$$

Proof. We intend to prove the assertion by employing a density argument, as well as the fact that chain rule holds true for C^1-functions. First, we observe that a short inspection of the proof of [70, Lemma 7.2] shows that

$$C^\infty([0,T]; H^1(\Omega)) \overset{d}{\hookrightarrow} H^1(0,T; H^1(\Omega)).$$

In view thereof, there exists a sequence $\{z_n\} \subset C^\infty([0,T]; H^1(\Omega))$ such that

$$z_n \to z \text{ in } H^1(0,T;H^1(\Omega)) \quad \text{as } n \to \infty. \tag{5.5}$$

By $H^1(0,T;H^1(\Omega)) \hookrightarrow C([0,T];H^1(\Omega))$, cf. [81, Theorem 3.1.41], this leads to

$$z_n(t) \to z(t) \text{ in } H^1(\Omega) \;\; \forall\, t \in [0,T] \quad \text{as } n \to \infty. \tag{5.6}$$

Moreover, as a result of chain rule, it holds for all $n \in \mathbb{N}$

$$\begin{aligned}\mathfrak{F}(t, z_n(t)) &= \mathfrak{F}(0, z_n(0)) + \int_0^t \frac{d}{dt}\mathfrak{F}(s, z_n(s))ds \\ &= \mathfrak{F}(0, z_n(0)) + \int_0^t \partial_t \mathfrak{F}(s, z_n(s)) + \langle \partial_z \mathfrak{F}(s, z_n(s)), \dot{z}_n(s)\rangle_{H^1(\Omega)} ds \quad \forall\, t \in [0,T].\end{aligned} \tag{5.7}$$

We now focus on passing to the limit in (5.7). The continuity of $\mathfrak{F}$ combined with (5.6) yields

$$\mathfrak{F}(t, z_n(t)) - \mathfrak{F}(0, z_n(0)) \overset{n\to\infty}{\longrightarrow} \mathfrak{F}(t, z(t)) - \mathfrak{F}(0, z(0)) \quad \text{for all } t \in [0, T], \tag{5.8}$$

so that it remains to show the convergence of the last term on the right-hand side in (5.7). Let us first abbreviate for simplicity

$$\begin{aligned} f_n &:= \partial_t \mathfrak{F}(\cdot, z_n(\cdot)) + \langle \partial_z \mathfrak{F}(\cdot, z_n(\cdot)), \dot{z}_n(\cdot)\rangle_{H^1(\Omega)} \quad \forall\, n \in \mathbb{N}, \\ f &:= \partial_t \mathfrak{F}(\cdot, z(\cdot)) + \langle \partial_z \mathfrak{F}(\cdot, z(\cdot)), \dot{z}(\cdot)\rangle_{H^1(\Omega)}. \end{aligned}$$

We observe that due to (5.5) it holds

$$\dot{z}_n \to \dot{z} \quad \text{in } L^2(0, T; H^1(\Omega)) \;\text{ as } n \to \infty. \tag{5.9}$$

Thus, there exists a subsequence $\{z_{n_k}\}$ so that

$$\dot{z}_{n_k}(t) \to \dot{z}(t) \quad \text{in } H^1(\Omega) \;\text{ f.a.a. } t \in (0, T) \;\text{ as } k \to \infty. \tag{5.10}$$

Since $\mathfrak{F} \in C^1([0, T] \times H^1(\Omega))$, one further deduces from (5.6) the convergences

$$\begin{aligned} &\partial_t \mathfrak{F}(t, z_n(t)) \overset{n\to\infty}{\longrightarrow} \partial_t \mathfrak{F}(t, z(t)) \quad \text{for all } t \in [0, T], \\ &\partial_z \mathfrak{F}(t, z_n(t)) \overset{n\to\infty}{\longrightarrow} \partial_z \mathfrak{F}(t, z(t)) \text{ in } H^1(\Omega)^* \quad \text{for all } t \in [0, T]. \end{aligned} \tag{5.11}$$

Now, (5.10) and (5.11) give in turn

$$f_{n_k}(t) \to f(t) \quad \text{f.a.a. } t \in (0, T) \;\text{ as } k \to \infty. \tag{5.12}$$

Further, for every $n \in \mathbb{N}$, Hölder's inequality leads to

$$\|f_n\|_{L^2(0,T)} \le \|\partial_t \mathfrak{F}(\cdot, z_n(\cdot))\|_{L^2(0,T)} + \|\partial_z \mathfrak{F}(\cdot, z_n(\cdot))\|_{L^\infty(0,T;H^1(\Omega)^*)} \|\dot{z}_n\|_{L^2(0,T;H^1(\Omega))}. \tag{5.13}$$

In view of (5.5) and (5.9) there exists $K > 0$, independent of n, so that

$$\|z_n\|_{H^1(0,T;H^1(\Omega))}, \|\dot{z}_n\|_{L^2(0,T;H^1(\Omega))} \le K \text{ for all } n \in \mathbb{N}.$$

Thanks to (5.4), the estimate (5.13) can be continued as

$$\|f_n\|_{L^2(0,T)} \le C \quad \text{for all } n \in \mathbb{N}, \tag{5.14}$$

where $C > 0$ is independent of n. Since $L^2(0, T)$ is a reflexive Banach space, we deduce from (5.14) that $\{f_{n_k}\}$ possesses a weakly convergent subsequence, denoted by the same symbol, which together with (5.12) gives in turn

$$f_{n_k} \rightharpoonup f \text{ in } L^2(0, T) \;\text{ as } k \to \infty. \tag{5.15}$$

In particular, we have for all $t \in [0, T]$ the convergence

$$\int_0^t f_{n_k}(s)\, ds \rightharpoonup \int_0^t f(s)\, ds \;\text{ as } k \to \infty, \tag{5.16}$$

since the integral operator is linear and continuous. Passing to the limit in (5.7), where we rely on (5.8) and (5.16), gives in turn

$$\mathfrak{F}(t, z(t)) = \mathfrak{F}(0, z(0)) + \int_0^t \partial_t \mathfrak{F}(s, z(s)) + \langle \partial_z \mathfrak{F}(s, z(s)), \dot{z}(s) \rangle_{H^1(\Omega)} ds \quad \forall\, t \in [0, T].$$

Since $f \in L^2(0, T)$, cf. (5.15), we now deduce that $\mathfrak{F}(\cdot, z(\cdot)) \in H^1(0, T)$ with $\frac{d}{dt}\mathfrak{F}(\cdot, z(\cdot)) = f$, see e.g. [81, Lemma 3.1.37]. The proof is now complete. □

5.3 Properties of the max-operator

Lemma 5.6.

(i) The Nemytskii operator associated to $\max : \mathbb{R} \to \mathbb{R}$, $\max(\cdot) := \max\{\cdot, 0\}$, *maps* $L^2(\Omega)$ *to* $L^2(\Omega)$. *Moreover,* $\max : L^2(\Omega) \to L^2(\Omega)$ *is Lipschitz continuous with Lipschitz constant* 1 *and directionally differentiable. For any* $y, h \in L^2(\Omega)$, *the derivative satisfies*

$$\max{}'(y; h)(x) = \max{}'(y(x); h(x)) = \begin{cases} h(x) & \text{if } y(x) > 0 \\ \max\{h(x), 0\} & \text{if } y(x) = 0 \\ 0 & \text{if } y(x) < 0 \end{cases} \quad \text{f.a.a. } x \in \Omega. \tag{5.17}$$

In addition, at any $y \in L^2(\Omega)$, *the operator* $\max{}'(y; \cdot) : L^2(\Omega) \to L^2(\Omega)$ *is Lipschitz continuous with Lipschitz constant* 1.

(ii) The Nemytskii operator associated to $\max : L^2(\Omega) \to L^2(\Omega)$ *maps* $L^\infty(0, T; L^2(\Omega))$ *to* $L^\infty(0, T; L^2(\Omega))$. *Moreover,* $\max : L^\infty(0, T; L^2(\Omega)) \to L^\infty(0, T; L^2(\Omega))$ *is Lipschitz continuous with constant* 1. *The operator* $\max : L^\infty(0, T; L^2(\Omega)) \to L^\varrho(0, T; L^2(\Omega))$ *is directionally differentiable for any* $\varrho \in [1, \infty)$, *with* $\max{}'(\cdot; \cdot) \in L^\infty(0, T; L^2(\Omega))$. *For any* $y, h \in L^\infty(0, T; L^2(\Omega))$ *the derivative is given by*

$$\begin{aligned} \max{}'(y; h)(t, x) &= \max{}'(y(t); h(t))(x) = \max{}'(y(t, x); h(t, x)) \\ &= \begin{cases} h(t, x) & \text{if } y(t, x) > 0 \\ \max\{h(t, x), 0\} & \text{if } y(t, x) = 0 \\ 0 & \text{if } y(t, x) < 0 \end{cases} \quad \text{f.a.a. } (t, x) \in (0, T) \times \Omega. \end{aligned} \tag{5.18}$$

Proof. (i) The first thing to notice is that $\mathbb{R} \ni v \mapsto \max\{v, 0\} \in \mathbb{R}$ is globally Lipschitz continuous with constant 1 and satisfies $|\max\{v, 0\}| \leq |v|$ for all $v \in \mathbb{R}$. As a continuous function, max maps measurable functions into measurable functions, and from the above estimate we deduce that the Nemytskii operator $\max : L^2(\Omega) \to L^2(\Omega)$ is well defined. In particular,

$$\|\max(y)\|_2 \leq \|y\|_2 \quad \forall\, y \in L^2(\Omega). \tag{5.19}$$

The desired Lipschitz continuity of $\max : L^2(\Omega) \to L^2(\Omega)$ is an immediate consequence of the Lipschitz continuity of $\max : \mathbb{R} \to \mathbb{R}$.

Furthermore, straight forward computation shows that $\max : \mathbb{R} \to \mathbb{R}$ is directionally differentiable with

$$\max{}'(v; \delta v) = \begin{cases} \delta v & \text{if } v > 0 \\ \max\{\delta v, 0\} & \text{if } v = 0 \\ 0 & \text{if } v < 0 \end{cases} \quad \forall\, v, \delta v \in \mathbb{R}. \tag{5.20}$$

As a consequence thereof (or by employing Lemma 5.8), we deduce that

$$|\max{}'(v; \delta v)| \leq |\delta v| \quad \forall\, v, \delta v \in \mathbb{R}. \tag{5.21}$$

With the above results at hand we next show by Lebesgue's dominated convergence theorem that the Nemytskii operator $\max : L^2(\Omega) \to L^2(\Omega)$ is directionally differentiable, too. To this end, let $y, h \in L^2(\Omega)$ be arbitrary, but fixed. The directional differentiability of $\max : \mathbb{R} \to \mathbb{R}$ yields

$$\Big|\frac{\max(y(x) + \tau h(x)) - \max(y(x))}{\tau} - \max{}'(y(x); h(x))\Big| \overset{\tau \searrow 0}{\longrightarrow} 0 \quad \text{f.a.a. } x \in \Omega.$$

Note that $\Omega \ni x \mapsto \max(y(x) + \tau h(x)) - \max(y(x)) \in \mathbb{R}$ is measurable for any $\tau \in \mathbb{R}$ and, as a.e. limit of measurable functions, $\Omega \ni x \mapsto \max{}'(y(x); h(x)) \in \mathbb{R}$ is measurable as well. On the other side, the global Lipschitz continuity of $\max : \mathbb{R} \to \mathbb{R}$ with constant 1 together with (5.21) implies for $\tau \neq 0$ that

$$\Big|\frac{\max(y(x) + \tau h(x)) - \max(y(x))}{\tau} - \max{}'(y(x); h(x))\Big| \leq 2\,|h(x)| \quad \text{f.a.a. } x \in \Omega.$$

Now, Lebesgue's dominated convergence theorem gives that $\max : L^2(\Omega) \to L^2(\Omega)$ is directionally differentiable with directional derivative given by (5.17), in view of (5.20).

It now remains to show that at any $y \in L^2(\Omega)$, the operator $\max{}'(y; \cdot) : L^2(\Omega) \to L^2(\Omega)$ is Lipschitz continuous with Lipschitz constant 1. This follows immediately from the definition of the directional derivative and the Lipschitz continuity of the operator max, which imply that

$$\begin{aligned} \|\max{}'(y; h_1) - \max{}'(y; h_2)\|_2 &= \lim_{\tau \searrow 0} \Big\|\frac{\max(y + \tau h_1) - \max(y + \tau h_2)}{\tau}\Big\|_2 \\ &\leq \|h_1 - h_2\|_2 \quad \forall\, h_1, h_2 \in L^2(\Omega). \end{aligned}$$

(ii) The desired results follow by very similar arguments to those used in part (i). From the global Lipschitz continuity with constant 1 of $\max : L^2(\Omega) \to L^2(\Omega)$ and from (5.19), we infer that $\max : L^\infty(0, T; L^2(\Omega)) \to L^\infty(0, T; L^2(\Omega))$ is well defined and Lipschitz continuous with constant 1. We now prove the directional differentiability at $y \in L^\infty(0, T; L^2(\Omega))$ in direction $h \in L^\infty(0, T; L^2(\Omega))$. We proceed as in part (i). From the directional differentiability of $\max : L^2(\Omega) \to L^2(\Omega)$ we obtain the convergence

$$\Big\|\frac{\max(y(t) + \tau h(t)) - \max(y(t))}{\tau} - \max{}'(y(t); h(t))\Big\|_2 \overset{\tau \searrow 0}{\longrightarrow} 0 \quad \text{f.a.a. } t \in (0, T). \tag{5.22}$$

On the other side, by employing e.g. (5.17) and (5.21), we arrive at

$$\left\| \max{}'(y(t); h(t)) \right\|_2 \leq \|h(t)\|_2 \text{ f.a.a. } t \in (0,T). \tag{5.23}$$

This leads for any $\tau \neq 0$ to

$$\left\| \frac{\max(y(t) + \tau h(t)) - \max(y(t))}{\tau} - \max{}'(y(t); h(t)) \right\|_2 \leq 2\,\|h(t)\|_2 \text{ f.a.a. } t \in (0,T), \tag{5.24}$$

where we also used the Lipschitz continuity with constant 1 of $\max : L^2(\Omega) \to L^2(\Omega)$. The measurability of

$$t \mapsto \left\| \tfrac{\max(y(t)+\tau h(t)) - \max(y(t))}{\tau} - \max{}'(y(t); h(t)) \right\|_2$$

for any $\tau \neq 0$ follows with the exactly same arguments from part (i), in combination with e.g. [71, Lemma 1.7]. In view of (5.22) and (5.24), Lebesgue's dominated convergence theorem yields now that, for any $\varrho \in [1,\infty)$, $\max : L^\infty(0,T; L^2(\Omega)) \to L^\varrho(0,T; L^2(\Omega))$ is directionally differentiable. The directional derivative is given by

$$\max{}'(y; h) = \max{}'(y(\cdot); h(\cdot)) \quad \text{a.e. in } (0,T). \tag{5.25}$$

and from (5.23) we deduce that $t \mapsto \max{}'(y;h)(t) \in L^\infty(0,T; L^2(\Omega))$, in view of $h \in L^\infty(0,T; L^2(\Omega))$. Finally, we notice that (5.18) follows from (5.25) and (5.17). The proof is now complete. □

5.4 Picard-Lindelöf's theorem in abstract function spaces

Lemma 5.7. *Let $r \in [1,\infty]$ be given. Moreover, let X be a Banach space and $z_0 \in X$. Suppose that the map $f : (0,T) \times X \to X$ satisfies*

(i) for any $z \in X$, the value $f(t,z) \in X$ is well defined f.a.a. $t \in (0,T)$,

(ii) the Nemytskii operator associated to f maps $L^r(0,T;X)$ to $L^r(0,T;X)$,

(iii) $f(t,\cdot)$ is globally Lipschitz continuous f.a.a. $t \in (0,T)$, with constant $L_f > 0$ independent of t.

Then, the initial value problem

$$\dot{z}(t) = f(t, z(t)) \quad \textit{f.a.a. } t \in (0,T), \quad z(0) = z_0 \tag{5.26}$$

admits a unique global solution $z \in W^{1,r}(0,T;X)$.

Proof. We intend to find (at first) local solutions for (5.26) by using similar arguments to those used for proving [14, Theorem 7.2.3]. That is, we solve (5.26) by Banach fixed-point theorem, which provides the existence of a unique local solution $z \in W^{1,r}(0,t;X) \hookrightarrow C([0,t];X)$, where t may be less than T. However, by considering (5.26) on (t,T) with initial value $z(t)$, one finds a new local solution. One proceeds in this way until a local solution on an interval containing the point T is achieved. The global solution is then obtained from the local ones. The time regularity of the local solutions ultimately transfers to the global solution of (5.26).

(I) Existence of local solutions

We begin by searching for fixed points for the mapping $L^r(0,t;X) \ni z \mapsto \mathcal{F}(z) : [0,t] \to X$, given by

$$\mathcal{F}(z)(\tau) := z_0 + \int_0^\tau f(s, z(s))\, ds \quad \forall\, \tau \in [0,t], \tag{5.27}$$

where $t \in (0,T]$ is to be determined so that the assumptions in Banach fixed-point theorem are satisfied. As it will turn out, a fixed point of $\mathcal{F}$ on $L^r(0,t;X)$ solves (5.26) on $(0,t)$.

We first show that $\mathcal{F}$ maps $L^r(0,t;X)$ to $W^{1,r}(0,t;X)$ for some arbitrary, but fixed $t \in (0,T]$. For $z \in L^r(0,t;X)$ we have, as a consequence of (ii), that $f(\cdot, z(\cdot)) \in L^r(0,t;X)$. In view of (5.27), we obtain from [81, Lemma 3.1.37] that $\tau \mapsto \mathcal{F}(z)(\tau) \in W^{1,r}(0,t;X)$. Hence, $\mathcal{F} : L^r(0,t;X) \to W^{1,r}(0,t;X)$ is indeed well defined. Note that due to $W^{1,r}(0,t;X) \hookrightarrow C([0,t];X)$, see e.g. [81, Theorem 3.1.41], $\mathcal{F}(z)$ is well defined in each $\tau \in [0,t]$. Moreover, as a result of [81, Lemma 3.1.37], it holds

$$\frac{d}{dt}\mathcal{F}(z)(\tau) = f(\tau, z(\tau)) \quad \text{f.a.a. } \tau \in (0,t). \tag{5.28}$$

We now search for $t > 0$ such that the function $\mathcal{F} : L^r(0,t;X) \to L^r(0,t;X)$ is a contraction, i.e., Lipschitz continuous with constant $K < 1$. Thanks to (5.27) and (iii) we have for all $z_1, z_2 \in L^r(0,t;X)$ the estimate

$$\begin{aligned} \|\mathcal{F}(z_1)(\tau) - \mathcal{F}(z_2)(\tau)\|_X &\le L_f \int_0^\tau \|z_1(s) - z_2(s)\|_X\, ds \\ &\le L_f \|z_1 - z_2\|_{L^1(0,t;X)} \\ &\le L_f\, t^{1-1/r} \|z_1 - z_2\|_{L^r(0,t;X)} \quad \text{for all } \tau \in [0,t], \end{aligned}$$

whence

$$\|\mathcal{F}(z_1) - \mathcal{F}(z_2)\|_{L^r(0,t;X)} \le L_f\, t \|z_1 - z_2\|_{L^r(0,t;X)}. \tag{5.29}$$

Notice that for the above estimate we employed the definition of the $L^r(0,t;X)$-norm. From (5.29) we read that $\mathcal{F} : L^r(0,t;X) \to L^r(0,t;X)$ is a contraction provided that $K := L_f\, t < 1$. That is why we choose to define

$$t := \begin{cases} T, & \text{if } L_f\, T < 1, \\ 1/(2L_f), & \text{otherwise.} \end{cases} \tag{5.30}$$

Now, by Banach fixed-point theorem, we can finally conclude that, for t as in (5.30), the equation

$$z = \mathcal{F}(z) \tag{5.31}$$

admits a unique solution in $L^r(0,t;X)$, which, for later purposes, we denote by z_1. Since $\mathcal{F}$ has range in $W^{1,r}(0,t;X)$, we obtain as an immediate consequence of (5.31) that $z_1 \in W^{1,r}(0,t;X)$. Moreover, (5.28) in combination with (5.31), and (5.27) yield

$$\dot{z}_1(\tau) = f(\tau, z_1(\tau)) \quad \text{f.a.a. } \tau \in (0,t), \quad z_1(0) = z_0. \tag{5.32}$$

Hence, we have proven that, indeed, the fix point of $\mathcal{F}$ on $L^r(0,t;X)$ solves (5.26) on $(0,t)$, where t is given by (5.30). Relying on [81, Corollary 3.1.39] we obtain that any solution in $W^{1,r}(0,t;X)$ of (5.32) solves (5.31), which in particular means that (5.32) is uniquely solvable. Altogether, we have shown that (5.26) admits a unique local solution in $W^{1,r}(0,t;X)$, where t is given by (5.30).

Thus, if $L_f T < 1$, this solution is in fact global and the proof is now complete. Otherwise, we further search for local solutions, this time for the initial value problem

$$\dot{z}(\tau) = f(\tau, z(\tau)) \quad \text{f.a.a. } \tau \in (t,T), \quad z(t) = z_1(t). \tag{5.33}$$

We proceed in the exact same way as above and find a unique $z_2 \in W^{1,r}(t,2t;X)$ which solves $z = \mathcal{F}(z)$, where $\mathcal{F} : L^r(t,2t;X) \to W^{1,r}(t,2t;X)$ must be accordingly redefined as

$$\mathcal{F}(z)(\tau) := z_1(t) + \int_t^\tau f(s, z(s))\, ds \quad \forall\, \tau \in [t, 2t].$$

Then, we deduce as above that (5.33) admits a unique local solution $z_2 \in W^{1,r}(t,2t;X)$. In the end, one finds for each $j = 1,...,n+1$, where $n = \lfloor 2\,T\,L_f \rfloor$, a unique $z_j \in W^{1,r}((j-1)t, jt;X)$, which satisfies

$$\dot{z}_j(\tau) = f(\tau, z_j(\tau)) \quad \text{f.a.a. } \tau \in ((j-1)t, jt), \quad z_j((j-1)t) = z_{j-1}((j-1)t). \tag{5.34}$$

Here we use the conventions $z_0(0) = z_0$ and $(n+1)t = T$. Of course, the number of initial value problems one has to solve, i.e., $n+1$, can be reduced by setting t larger in (5.30) (such that $L_f\, t < 1$).

(II) Existence of global solutions

In [14, Theorem 7.2.6] the global solvability of (5.26) is shown by constructing a (to $\|\cdot\|_{C([0,T];X)}$) equivalent norm such that $\mathcal{F}$ is a contraction on the space $C([0,T];X)$ endowed with the new norm. The latter one allows for deriving pointwise in time estimates, which combined with the Lipschitz continuity of f lead to a sharper estimate of the type (5.29). As we work with Lebesgue spaces, we cannot derive pointwise in time estimates in a similar way. That is why we construct a global solution of (5.26) by concatenating the local solutions found in the previous step. That is, we show that the global solution is the function $z : [0,T] \to X$ given by

$$z(\tau) := \sum_{j=1}^{n+1} \widetilde{z}_j(\tau) \quad \forall\, \tau \in [0,T], \tag{5.35}$$

where

$$\widetilde{z}_j := \begin{cases} z_j & \text{on } [(j-1)t, jt), \\ 0 & \text{elsewhere}, \end{cases} \quad \text{for } j = 1,...,n, \qquad \widetilde{z}_{n+1} := \begin{cases} z_{n+1} & \text{on } [nt, T], \\ 0 & \text{elsewhere}. \end{cases}$$

We first prove that $z \in W^{1,r}(0,T;X)$. To this end, let $\tau \in [0,T]$ be arbitrary, but fixed. This means that there exists $k \in \{1,...,n+1\}$ so that $\tau \in [(k-1)t, kt]$. From (5.35) and [81, Corollary 3.1.39] applied for $z_j \in W^{1,r}((j-1)t, jt; X)$, $j = k, ..., 1$, we have

$$\begin{aligned} z(\tau) = z_k(\tau) &= z_k((k-1)t) + \int_{(k-1)t}^{\tau} \dot{z}_k(s)\, ds \\ &= z_{k-1}((k-1)t) + \int_{(k-1)t}^{\tau} \dot{z}_k(s)\, ds \\ &= z_0 + \sum_{j=1}^{k-1} \int_{(j-1)t}^{jt} \dot{z}_j(s)\, ds + \int_{(k-1)t}^{\tau} \dot{z}_k(s)\, ds \\ &= z_0 + \int_0^{\tau} y(s)\, ds, \end{aligned}$$

where $y = \dot{z}_j$ a.e. in $((j-1)t, jt)$ for any $j = 1, ..., n+1$. Since $\dot{z}_j \in L^r((j-1)t, jt; X)$, the mapping $\tau \mapsto y(\tau)$ belongs to $L^r(0,T;X)$. This follows for example by defining y as the sum of the extensions of $\dot{z}_j$ on $(0,T)$ by zero, which means that y is sum of functions in $L^r(0,T;X)$. Now, [81, Lemma 3.1.37] in combination with the above identity ensures that $z \in W^{1,r}(0,T;X)$ with $\dot{z} = y$ a.e. in $(0,T)$. With (5.34) we then infer

$$\dot{z}(\tau) = \dot{z}_j = f(\tau, z_j(\tau)) \quad \text{a.e. in } ((j-1)t, jt),\ j = 1, ..., n+1,$$

i.e.,

$$\dot{z}(\tau) = f(\tau, z(\tau)) \quad \text{a.e. in } (0,T),$$

where we used (5.35). Therefrom, we also get $z(0) = z_0$. In other words, z given by (5.35) is a global solution of (5.26). In order to prove the uniqueness thereof, we argue as follows. Any global solution $\tilde{z}$ of (5.26) solves (5.32), i.e., $\tilde{z}|_{[0,t]} = z_1 \in W^{1,r}(0,t;X)$. In particular, $\tilde{z}(t) = z_1(t)$. Additionally, $\tilde{z}$ satisfies (5.33) on $(t, 2t)$, and thus, $\tilde{z}|_{[t,2t]} = z_2 \in W^{1,r}(t, 2t; X)$. By further arguing in this way, we obtain $\tilde{z}|_{[(j-1)t,jt]} = z_j \in W^{1,r}((j-1)t, jt; X)$ for all $j = 1, ..., n+1$ and the uniqueness of the local solutions yields the uniqueness of the global one. The proof is now complete. □

5.5 Miscellaneous

This section collects various general results, of which we often make use in the thesis.

Lemma 5.8. *Let X, Y, Z be Banach spaces and let $U \subset X$ be an open set. Moreover, let $f : U \times Y \to Z$ be directionally differentiable and Lipschitz continuous, i.e., there exists $L_f > 0$ so that*

$$\|f(x) - f(y)\|_Z \leq L_f \|x - y\|_{X \times Y} \quad \forall\, x, y \in U \times Y.$$

Then, for any $y \in U \times Y$, we have

$$\|f'(y; h)\|_Z \leq L_f \|h\|_{X \times Y} \quad \forall\, h \in X \times Y.$$

Proof. Let $y \in U \times Y$ and $h \in X \times Y$ be arbitrary, but fixed. Since $U \times Y$ is open, there exists $\tau > 0$ small enough such that $y + \tau h \in U \times Y$. By assumption, we then have

$$\|f(y + \tau h) - f(y)\|_Z \leq L_f \, \|\tau h\|_{X \times Y}. \tag{5.36}$$

Dividing by $\tau > 0$ and passing to the limit $\tau \searrow 0$ in (5.36) gives the claim. □

Lemma 5.9. *Let X be a Banach space. For any $t \in [0, T]$, we define the operator $\Theta_t : W^{1,1}(0, T; X) \to X$ as*

$$\Theta_t(\zeta) := \zeta(t).$$

Then, $\Theta_t \in \mathcal{L}(W^{1,1}(0, T; X), X)$ and there exists $C > 0$, independent of t, so that $\|\Theta_t\|_{\mathcal{L}(W^{1,1}(0,T;X),X)} \leq C$ for all $t \in [0, T]$.

Proof. Let $t \in [0, T]$ be arbitrary, but fixed. We notice that Θ_t is well defined, as a result of

$$W^{1,1}(0, T; X) \hookrightarrow C([0, T]; X), \tag{5.37}$$

see e.g. [81, Theorem 3.1.41]. We only address its boundedness, as the linearity is obvious. This follows easily, since from (5.37) we have for all $\zeta \in W^{1,1}(0, T; X)$

$$\|\Theta_t(\zeta)\|_X = \|\zeta(t)\|_X \leq \|\zeta\|_{C([0,T];X)} \leq C\|\zeta\|_{W^{1,1}(0,T;X)},$$

where $C > 0$ is the embedding constant, which depends only on T, see [81, (3.30)] for more details. Moreover, note that $\|\Theta_t\|_{\mathcal{L}(W^{1,1}(0,T;X),X)} \leq C$ for all $t \in [0, T]$. This completes the proof. □

Lemma 5.10. *Let $r \in [1, \infty]$ and $z \in W_0^{1,r}(0, T; X)$ be given, where X is a Banach space. If there exists a non-negative function $y \in L^r(0, T)$ and a constant $c \geq 0$ such that*

$$\|\dot{z}(t)\|_X \leq c\|z(t)\|_X + y(t) \quad \text{f.a.a. } t \in (0, T), \tag{5.38}$$

then

$$\|z\|_{W^{1,r}(0,T;X)} \leq K\|y\|_{L^r(0,T)},$$

with a positive constant $K = K(c, r, T)$ if $r < \infty$ and $K = K(c, T)$ if $r = \infty$. As a consequence, Poincaré-Friedrich's inequality in abstract function spaces holds true, i.e.,

$$\|z\|_{L^r(0,T;X)} \leq K\|\dot{z}\|_{L^r(0,T;X)} \quad \forall\, z \in W_0^{1,r}(0, T; X), \tag{5.39}$$

where $K = K(r, T) > 0$ is a constant (independent of r if $r = \infty$).

Proof. By integrating (5.38) we arrive at

$$\|z(t)\|_X \leq \int_0^t \|\dot{z}(s)\|_X \, ds \leq c \int_0^t \|z(s)\|_X \, ds + \int_0^t y(s) \, ds \quad \forall\, t \in [0, T],$$

where for the first inequality we employed [81, Corollary 3.1.39], $z(0) = 0$ and the embedding $W^{1,r}(0,T;X) \hookrightarrow C([0,T];X)$. Since $z \in C([0,T];X)$ and $[0,T] \ni t \mapsto \int_0^t y(s)\, ds \in [0,\infty)$ is nondecreasing, Gronwall's lemma yields

$$\|z(t)\|_X \leq \exp(c\,t) \int_0^t y(s)\, ds \leq C\|y\|_{L^r(0,T)} \quad \forall\, t \in [0,T].$$

In particular, $\|z\|_{L^r(0,T;X)} \leq C\|y\|_{L^r(0,T)}$. Notice that $C = \exp(cT)T^{(1-1/r)}$, whence $C = \exp(cT)T$ if $r = \infty$. By means of (5.38) we now have

$$\|\dot{z}\|_{L^r(0,T;X)} \leq c\|z\|_{L^r(0,T;X)} + \|y\|_{L^r(0,T)} \leq K(c,C)\|y\|_{L^r(0,T)},$$

which gives the first claim. Poincaré-Friedrich's inequality follows immediately by considering the special case $y := \|\dot{z}(\cdot)\|_X$ and by setting $c := 0$ in (5.38). The proof is now complete. □

Lemma 5.11. *Let X, Y be Banach spaces and let U be a nonempty open set in X. Moreover, let $x : [0,T] \to X$ be a Bochner measurable function, such that $x(t) \in U$ f.a.a. $t \in (0,T)$ and let $f : U \to Y$ be a continuous function. Then, the mapping $t \mapsto f(x(t)) \in Y$ is Bochner measurable.*

Proof. In view of the Bochner measurability of x, there exists a sequence of simple functions $\{x_n\}$ such that

$$x_n(t) \to x(t) \;\text{ f.a.a. } t \in (0,T) \quad \text{as } n \to \infty.$$

Since U is open and $x(t) \in U$ f.a.a. $t \in (0,T)$, there exists at almost all $t \in (0,T)$ some $m = m(t) \in \mathbb{N}$ large enough such that

$$x_n(t) \in U \quad \text{for all } n \geq m(t). \tag{5.40}$$

Applying the continuity of $f : U \to Y$ gives in turn

$$f(x_n(t)) \to f(x(t)) \;\text{ f.a.a. } t \in (0,T) \quad \text{as } n \to \infty. \tag{5.41}$$

Let now $n \in \mathbb{N}$ be arbitrary, but fixed. Since x_n is a simple function, it holds

$$x_n(t) = \textstyle\sum_{i=1}^{k_n} \chi_{B_{n,i}}(t)\, x_{n,i} \quad \text{f.a.a. } t \in (0,T),$$

with Lebesgue measurable subsets $B_{n,i} \subset [0,T]$ and $x_{n,i} \in X$, $i = 1, ..., k_n$. W.l.o.g. we assume that $\{B_{n,i}\}_i$ is a disjoint partition of $[0,T]$. Since $U \neq \emptyset$, we can fix $u \in U$. Then, the function $f_n : [0,T] \to Y$ given by

$$f_n(t) := \textstyle\sum_{i=1}^{k_n} \chi_{B_{n,i}}(t)\, \mathcal{F}(x_{n,i}),$$

where $\mathcal{F} : X \to Y$ satisfies $\mathcal{F}(z) = f(z)$ if $z \in U$ and $\mathcal{F}(z) = f(u)$ otherwise, is well defined and a simple function as well. In particular,

$$f_n(t) = f(x_n(t)) \text{ if } n \geq m(t), \;\text{ f.a.a. } t \in (0,T),$$

in view of (5.40). Note that here it was crucial that $\{B_{n,i}\}_i$ is a disjoint partition of $[0,T]$. We can now deduce from (5.41) the Bochner measurability of $f : [0,T] \to Y$. The proof is now complete. $\square$

Lemma 5.12. *Let X, Y, Z be Banach spaces and $\mathfrak{U} \subset L^\infty(0,T;X)$, $U \subset Y$ be open sets. Moreover, let $\mathfrak{F} : \mathfrak{U} \to L^\infty(0,T;Y)$ and $\mathcal{F} : U \to Z$ be two given mappings. Suppose that $\mathcal{F}$ is Lipschitz continuous and directionally differentiable. Further, let $x \in \mathfrak{U}$, $\delta x \in L^\infty(0,T;X)$ be given and choose $\varepsilon_1 > 0$ such that $B_{L^\infty(0,T;X)}(x, \varepsilon_1\|\delta x\|_{L^\infty(0,T;X)}) \subset \mathfrak{U}$. Assume that*

$$\mathfrak{F}\big(B_{L^\infty(0,T;X)}(x, \varepsilon_1\|\delta x\|_{L^\infty(0,T;X)})\big)(t) \in U \text{ f.a.a. } t \in (0,T).$$

Moreover, let $\delta y \in L^\infty(0,T;Y)$ be given and assume that there exists $\varepsilon_2 > 0$, independent of t, such that $B_Y(\mathfrak{F}(x)(t), \varepsilon_2\|\delta y\|_{L^\infty(0,T;Y)}) \subset U$ f.a.a. $t \in (0,T)$. Then, for any $\tau \in (0, \min\{\varepsilon_1, \varepsilon_2\})$, we have

$$\begin{aligned}&\Big\|\frac{\mathcal{F}\big(\mathfrak{F}(x+\tau\delta x)(t)\big) - \mathcal{F}\big(\mathfrak{F}(x)(t)\big)}{\tau} - \mathcal{F}'\big(\mathfrak{F}(x)(t);\delta y(t)\big)\Big\|_Z \\ &\le L_{\mathcal{F}}\Big\|\frac{\mathfrak{F}(x+\tau\delta x)(t) - \mathfrak{F}(x)(t)}{\tau} - \delta y(t)\Big\|_Y + R_{\mathcal{F}}(t,\tau) \quad \text{f.a.a. } t \in (0,T),\end{aligned}$$

where $L_{\mathcal{F}} > 0$ is the Lipschitz constant of $\mathcal{F}$ and $R_{\mathcal{F}} : (0,T) \times (0,1) \to \mathbb{R}^+$ satisfies

$$R_{\mathcal{F}}(\cdot,\tau) \to 0 \text{ in } L^\varrho(0,T) \quad \text{as } \tau \searrow 0,$$

for any $\varrho \in [1,\infty)$.

Proof. First notice that due to $\tau \in (0, \min\{\varepsilon_1, \varepsilon_2\})$ we have

$$x + \tau\delta x \in B_{L^\infty(0,T;X)}(x, \varepsilon_1\|\delta x\|_{L^\infty(0,T;X)}) \subset \mathfrak{U}$$

and from $\mathfrak{F}\big(B_{L^\infty(0,T;X)}(x, \varepsilon_1\|\delta x\|_{L^\infty(0,T;X)})\big)(t) \in U$ f.a.a. $t \in (0,T)$ we deduce $\mathfrak{F}(x+\tau\delta x)(t), \mathfrak{F}(x)(t) \in U$ f.a.a. $t \in (0,T)$. Moreover, the assumption on ε_2 ensures that $\mathfrak{F}(x)(t) + \tau\delta y(t) \in U$ f.a.a. $t \in (0,T)$. Therefore, all the quantities we deal with in the next estimate are well defined. Notice that the fact that $\varepsilon_2 > 0$ is independent of t implies that τ does not depend on t, which is crucial for proving the convergence in the second part of the proof. Now, from the Lipschitz continuity of $\mathcal{F}$ we obtain for almost all $t \in (0,T)$ the estimate

$$\begin{aligned}&\Big\|\frac{\mathcal{F}\big(\mathfrak{F}(x+\tau\delta x)(t)\big) - \mathcal{F}\big(\mathfrak{F}(x)(t)\big)}{\tau} - \mathcal{F}'\big(\mathfrak{F}(x)(t);\delta y(t)\big)\Big\|_Z \\ &\le \Big\|\frac{\mathcal{F}\big(\mathfrak{F}(x+\tau\delta x)(t)\big) - \mathcal{F}\big(\mathfrak{F}(x)(t)+\tau\delta y(t)\big)}{\tau}\Big\|_Z \\ &\quad + \Big\|\frac{\mathcal{F}\big(\mathfrak{F}(x)(t)+\tau\delta y(t)\big) - \mathcal{F}\big(\mathfrak{F}(x)(t)\big)}{\tau} - \mathcal{F}'\big(\mathfrak{F}(x)(t);\delta y(t)\big)\Big\|_Z \\ &\le L_{\mathcal{F}}\Big\|\frac{\mathfrak{F}(x+\tau\delta x)(t) - \mathfrak{F}(x)(t)}{\tau} - \delta y(t)\Big\|_Y + R_{\mathcal{F}}(t,\tau),\end{aligned}$$

where $R_{\mathcal{F}}(t,\cdot):(0,1)\to\mathbb{R}^+$ is given by

$$R_{\mathcal{F}}(t,\tau)=\begin{cases}\left\|\frac{\mathcal{F}(\mathfrak{F}(x)(t)+\tau\delta y(t))-\mathcal{F}(\mathfrak{F}(x)(t))}{\tau}-\mathcal{F}'\big(\mathfrak{F}(x)(t);\delta y(t)\big)\right\|_Z, & \text{if } \tau\in(0,\min\{\varepsilon_1,\varepsilon_2\}),\\ 0, & \text{otherwise,}\end{cases}$$

at almost all $t\in(0,T)$.

It now remains to prove the convergence of $R_{\mathcal{F}}$. This is shown by means of Lebesgue's dominated convergence theorem. To this end, first note that the mapping

$$t\mapsto\frac{\mathcal{F}(\mathfrak{F}(x)(t)+\tau\delta y(t))-\mathcal{F}(\mathfrak{F}(x)(t))}{\tau}\in Z$$

is Bochner measurable at all $\tau\in(0,\min\{\varepsilon_1,\varepsilon_2\})$, in view of Lemma 5.11. As pointwise (a.e. in $(0,T)$) limit of Bochner measurable functions, the directional derivative $t\mapsto\mathcal{F}'\big(\mathfrak{F}(x)(t);\delta y(t)\big)\in Z$ is Bochner measurable as well, cf. [81, Corollary 3.1.5]. Hence, as a result of e.g. [71, Lemma 1.7], $R_{\mathcal{F}}(\cdot,\tau)$ is measurable for all $\tau\in(0,1)$. Further, the directional differentiability of $\mathcal{F}$ gives in turn

$$R_{\mathcal{F}}(t,\tau)\to 0 \;\text{ as } \tau\searrow 0 \quad \text{f.a.a. } t\in(0,T).$$

By employing the Lipschitz continuity of $\mathcal{F}$ and by applying Lemma 5.8 for $f:U\times\{0\}\to Z,\ f(\cdot,0):=\mathcal{F}$, we arrive at

$$R_{\mathcal{F}}(t,\tau)\le 2L_{\mathcal{F}}\|\delta y(t)\|_Y \quad \text{f.a.a. } t\in(0,T),\ \forall\,\tau\in(0,1).$$

Since $\delta y\in L^\infty(0,T;Y)$, Lebesgue's dominated convergence theorem implies that

$$R_{\mathcal{F}}(\cdot,\tau)\to 0 \text{ in } L^\varrho(0,T) \quad \text{as } \tau\searrow 0,$$

for any $\varrho\in[1,\infty)$. This completes the proof. □

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