Limit Theorems for Multipower Variations of Lévy Driven and Fractional-Lévy-Motion Driven Processes

Dissertation

Zur Erlangung des akademischen Grades
Doctor rerum naturalium
(Dr. rer. nat.)

vorgelegt
der Fakultät für Mathematik
der Technischen Universität Dortmund

von
M.Sc.-Math. Andre Pessik

März 2019

Dissertation

Limit Theorems for Multipower Variations of Lévy Driven and Fractional-Lévy-Motion Driven Processes

Fakultät für Mathematik
Technische Universität Dortmund

Erstgutachter: Prof. Dr. *Jeannette H. C. Woerner*

Zweitgutachter: Prof. Dr. *Herold Dehling*

Tag der mündlichen Prüfung: *28.05.2019*

Contents

Notations

$\mathbb{N}$	natural numbers, i.e. $\{1, 2, \dots\}$
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{R}^{\mathbb{N}}$	$\{(x_i)_{i\in\mathbb{N}} \mid x_i \in \mathbb{R} \text{ for } i \in \mathbb{N}\}$
$\emptyset$	empty set
$\mathbf{1}$	indicator function
$x \wedge y$	$\min(x, y)$ for $x, y \in \mathbb{R}$
$x \vee y$	$\max(x, y)$ for $x, y \in \mathbb{R}$
$(x)_+$ resp. $(x)_-$	$(x)_+ = x \vee 0$ resp. $(x)_- = x \wedge 0$ for $x \in \mathbb{R}$
$\lceil x \rceil$ resp. $\lfloor x \rfloor$	ceiling function resp. floor function
$\operatorname{sign}(x)$	sign function with $\operatorname{sign}(0) = 0$
Γ	gamma function
$\lvert A \rvert$ for a set A	number of elements in A
$f(x) = \mathcal{O}\big(g(x)\big)$ as $x \to x_0$	$\exists K > 0$ so that $\frac{\lvert f(x)\rvert}{\lvert g(x)\rvert} < K$ as $x \to x_0$
$f(x) = o\big(g(x)\big)$ as $x \to x_0$	$\frac{\lvert f(x)\rvert}{\lvert g(x)\rvert} \to 0$ as $x \to x_0$
$f \equiv g$	$f(x) = g(x)$ for all x
$f(x) \sim g(x)$ as $x \to x_0$	$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$
λ	Lebesgue measure
$\mu \otimes \nu$	product measure of μ and ν
$\mathbb{P}$	probability measure
$\mathbb{E}$	expectation
$C(A), C^0(A)$	$\{f : A \to \mathbb{R} \mid f \text{ continuous}\}$

$C_b(A)$	$\{f \in C(A) \mid f \text{ bounded}\}$
$C^k(A)$ for $k \in \mathbb{N} \cup \{\infty\}$	$\{f : A \to \mathbb{R} \mid f$ k-times continuous differentiable$\}$
$C_c^k(A)$ for $k \in \mathbb{N} \cup \{0, \infty\}$	$\{f \in C^k(A) \mid f$ has compact support on $A\}$
$(S, \mathcal{A}, \mu)$	measure space
$(\Omega, \mathcal{A}, \mathbb{P})$, $(\Omega, \mathcal{F}, \mathbb{P})$	probability space
$L^p(S) = L^p(S, \mathcal{A}, \mu)$ for $p > 0$	L^p space w.r.t. $(S, \mathcal{A}, \mu)$
$L^p(\Omega)$ for $p > 0$	$L^p(\Omega, \mathcal{A}, \mathbb{P})$
$L^p(A)$, A borel measurable	Lebesgue space w.r.t. A
$\|\|f\|\|_{L^p(S)}$	$\left(\int_S \|f(x)\|^p \mu(dx)\right)^{\frac{1}{p}}$ for $f \in L^p(S) = L^p(S, \mathcal{A}, \mu)$
$X \overset{d}{=} Y$	$X = Y$ in distribution
$X = Y$, $X < Y$, $X \le Y$ a.s.	$X = Y$, $X < Y$, $X \le Y$ almost surely
$X_n \xrightarrow{d} X$	X_n converges in distribution to X
$X_n \xrightarrow{L-s} X$	X_n converges stable in law to X, cf. Definition 1.4.1
$X_n \xrightarrow{\mathbb{P}} X$	X_n converges in probability to X
$X_n \xrightarrow{a.s.} X$, $X_n \longrightarrow X$ a.s.	X_n converges almost surely to X
$\triangle_{i,n}^k X$ resp. $\blacktriangle_{i,n}^k g(s)$	differential filters of X resp. g, cf. (1.3.1) and (1.3.2)
⓪,①, …, ⑥	assumptions, cf. Section 1.2.1
$V_n^{\langle M \rangle}$	Mth order (multi)power variation, cf. Definition 1.3.2
$k_\star, k^\star, a_1, \dots, a_M$	parameters of $V_n^{\langle M \rangle}$, cf. Definition 1.3.2

Introduction

Power, bipower and multipower variations stem from the concept of using quadratic variations and covariations, which for example are a central part of the Itô calculus, as estimators for the integrated volatility.

A general multipower variation $V_n^{\langle m \rangle}$, which for $m = 1$ is called power variation and for $m = 2$ is referred to as bipower variation, is an object of the following form

$$V_n^{\langle m \rangle} := \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{m} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j} \quad \text{with} \quad \triangle_{i,n}^{k} X = \sum_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} X_{\frac{i-j}{n}}$$

and is a natural generalisation of quadratic variations ($m = k_1 = 1, p_1 = 2$) and covariations ($m = 2, k_1 = k_2 = p_1 = p_2 = 1$).

The idea of multipower variations goes back to a series of papers by Barndorff-Nielsen and Shephard with regard to some problems in financial econometrics, cf. [**4**], [**5**], [**6**], [**7**] and [**8**]. Power variations and their generalisation the multipower variations are for example used to handle high frequency data in stochastic volatility models and provide model-free estimators for the volatility. Moreover, general multipower variations in contrast to simple power variations allow, in an underlying model, to separate the continuous components and the jump components.

The versatile applicability of power/multipower variations that stems from the property of being model-free, i.e. to be not bound to a specific model, makes the study of power/multipower variations to the subject of various articles, e.g. in the context of power/bipower variations of continuous semimartingales in [**3**], of bipower variations of semimartingales with a focus on finance in [**17**], of bipower variations of Gaussian processes with stationary increments in [**2**], of multipower variations in the setting of Lévy-type processes in [**29**] and of the robustness of multipower variation towards jumps in the setting of Brownian semimartingales in [**9**].

In this thesis, as the title suggests, we will look into the limiting behaviour of multipower variations of Lévy driven respectively fractional-Lévy-motion driven processes.

A Lévy driven process is a process given by the following representation

$$X_t = \int_{-\infty}^{\infty} g(t-s) - \tilde{g}(-s)\ dL_s \qquad (t \in \mathbb{R}),$$

where L is a two-sided Lévy process and $g, \tilde{g}$ are deterministic real valued functions. This process exists in the sense of [**23**] as an integral over a random measure which is associated to the underlying Lévy process L. Note that Lévy driven processes have stationary, but not necessarily independent, increments, infinitely divisible marginal distributions and a correlation structure that can be modified to suit our needs in modelling. These properties and the fact that the class of Lévy driven processes includes Gaussian processes, like the fractional Brownian motion, as well as non-Gaussian type processes makes Lévy driven processes a popular choice in recent financial modelling, e.g. modelling the prices of electricity as done in [**19**]. Furthermore, similarly to the fractional Brownian motion a Lévy driven process in the case of $g(s) = \tilde{g}(s) = (s)_+^{\alpha}$, where $\alpha > 0$, is referred to as a fractional Lévy motion.

By replacing in the above representation of the Lévy driven process the driving Lévy process L by a fractional Lévy motion we obtain the fractional-Lévy-motion driven process which for example in the case of the underlying Lévy process being a pure jump Lévy process with finite second moments was introduced in [**21**]. Since processes driven by the fractional Lévy motion posses similar properties to Lévy driven processes and do not require the driving process to have independent increments, they can be seen as a generalisation of Lévy driven processes and similarly be used in financial modelling.

The goal of this thesis is on the one hand to derive limit theorems for multipower variations of Lévy driven processes, while also studying convergence rates as well as the properties of the limiting objects, and on the other hand to produce similar results in the case of more general driving processes. Our results are based on the first-order asymptotics for power variations of Lévy driven processes in [**10**]. As in [**10**] we will assume the underlying Lévy process to be a symmetric pure-jump Lévy process and use a similar approach to extend the first-order asymptotics to the multipower variations case. By including specific properties of Lévy driven processes, which under suitable assumptions on the kernel functions $g, \tilde{g}$ stem from the driving Lévy process L, e.g. L^2-isometry or scaling property, we will also provide additional limit theorems as well as produce convergence rates that even in the case of power variations were not known. Moreover, note that we will also extend all of our results for Lévy driven processes, which are compatible with the definition of fractional-Lévy-motion driven processes in [**21**], to the setting of processes driven by the fractional Lévy motion.

The structure of this thesis is as follows. In the first chapter we will introduce the basic assumptions, notations, definitions, properties and tools that will be used throughout this work. The focus of the second chapter will be the extension of the stable convergence in law limit theorems for power variations in [**10**, Theorem 1.1 (i)] and [**11**, Theorem 1.2 (i)] to the multipower variations case. The third chapter provides a version for multipower variations of [**10**, Theorem 1.1 (ii)], which similarly to [**10**] is based on an ergodic argument. In the fourth chapter we will, while focusing on convergence rates, present the extension of [**10**, Theorem 1.1 (iii)] to the multipower variations case and include some new limiting results. The last chapter contains the extensions of the limit theorems of the preceding chapters to fractional-Lévy-motion driven processes. Moreover, note that the main part of the assumptions in the limit theorems in chapter two to five focuses on the kernel functions rather than on the underlying Lévy process. Therefore we will exemplary introduce in each of these chapters some kernel functions with which the associated Lévy driven processes and fractional-Lévy-motion driven processes will satisfy the respective assumptions.

Danksagung

An dieser Stelle ist es mir ein besonderes Anliegen allen Personen, die mir im Rahmen meiner Promotion und bei der Erstellung dieser Arbeit zur Seite standen, meinen herzlichen Dank auszusprechen. Danke.

Ich danke vor allem *Frau Prof. Dr. Jeannette H. C. Woerner*, die mich als meine Erstbetreuerin durch die gesamte Promotion geleitet hat. Ihre kompetente Betreuung verbunden mit nützlichen Vorschlägen und Ideen haben mir stets geholfen auftretende Schwierigkeiten zu umgehen und mündeten letzendlich in der Erstellung dieser Arbeit.

Ein besonderer Dank geht an meinen Zweitbetreuer *Herrn Prof. Dr. Herold Dehling*, der mir immer als zusätlicher Ansprechpartner mit hilfreichen Vorschlägen und Ideen zur Verfügung stand.

Ich bedanke mich auch bei meinen Kollegen an der *TU Dortmund* und den Mitgliedern des *Graduiertenkollegs (GRK 2131)* für das harmonische Miteinander sowie für die produktive Zusammenarbeit.

Anschließend sei der *Deutschen Forschungsgemeinschaft (DFG)* gedankt, die mir durch ihre Finanzierung über das *GRK 2131* eine Promotion unter optimalen Bedingungen ermöglicht hat.

CHAPTER 1

Preliminaries

1.1. Lévy Processes

In this section we will introduce Lévy processes as well as highlight some related terms and properties which will be of relevance for this work. Moreover, each of the two subsections is dedicated to give a short introduction to specific Lévy processes and their properties which we will need later.

DEFINITION 1.1.1. A real valued process $L = (L_t)_{t \geq 0}$ is called (one-sided) Lévy process if the following conditions are satisfied.

- $L_0 = 0$ almost surely.
- For each $n \in \mathbb{N}$ and any choice of $0 \leq t_0 < t_1 < \cdots < t_n$ the random variables

$$L_{t_0}, L_{t_1} - L_{t_0}, \ldots, L_{t_n} - L_{t_{n-1}}$$

  are independent *(independent increments)*.
- The distribution of $L_{t+s} - L_s$ does not depend on s for all $s, t \geq 0$ *(stationary increments)*.
- The process L is stochastic continuous, i.e. for each $t \geq 0$ and $\varepsilon > 0$ we have

$$\lim_{s \to t} \mathbb{P}\big(|L_s - L_t| > \varepsilon\big) = 0.$$

- The process L is pathwise càdlàg, i.e. L_t is almost surely right continuous for $t \geq 0$ and has almost surely left limits for $t > 0$.

Note that a Lévy process $L = (L_t)_{t \geq 0}$ is an infinitely divisible process, cf. e.g. [**25**, Theorem 8.1.], and can be characterised by its associated characteristic triplet (γ, σ^2, ν), since by the Lévy-Khinchin formula, cf. e.g. [**13**, Theorem 3.1], the Lévy process L has for each $t \geq 0$ the following characteristic function

$$E\big(\exp(iuL_t)\big) = e^{t\psi(u)} \quad \text{for} \quad u \in \mathbb{R} \tag{1.1.1}$$

with

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|\leq 1\}}\right)\nu(dx),$$

where $\gamma \in \mathbb{R}$ is the *drift parameter*, $\sigma^2 \geq 0$ is the *Brownian component* and ν is the *Lévy measure* of L, i.e. ν satisfies

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}} x^2 \wedge 1 \, \nu(dx) < \infty. \tag{1.1.2}$$

Throughout this work we will consider two-sided Lévy processes in $\mathbb{R}$ with characteristic triplet (γ, σ^2, ν), i.e.

$$L_t := \begin{cases} L_t^{\langle 1 \rangle}, & \text{for } t \geq 0 \\ -L_{(-t)-}^{\langle 2 \rangle}, & \text{for } t < 0 \end{cases},$$

where $L^{\langle 1 \rangle}$ and $L^{\langle 2 \rangle}$ are two independent copies of the above introduced one-sided Lévy process with characteristic triplet (γ, σ^2, ν).

Furthermore, we will assume the one/two-sided Lévy process to be without a Brownian component, i.e. to have the characteristic triplet $(\gamma, 0, \nu)$, and often refer to a one/two-sided Lévy process simply as Lévy process.

DEFINITION 1.1.2. Let L be a (two-sided) Lévy process in $\mathbb{R}$ with characteristic triplet (γ, σ^2, ν).

(i) The process L is said to be *symmetric* if and only if the distribution of L is invariant under multiplication with -1.

(ii) The parameter

$$\beta := \inf\left\{\tau \geq 0 : \int_{-1}^{1} |x|^\tau \nu(dx) < \infty\right\}$$

is referred to as *Blumenthal-Getoor index*.

REMARK 1.1.3. (*i*) By using the Lévy-Khinchin formula, cf. e.g. [**13**, Theorem 3.1], it is easy to see that a Lévy process L with characteristic triplet (γ, σ^2, ν) is symmetric if and only if we have $\gamma = 0$ and ν is symmetric, cf. [**25**, 18. Exercise 3: E 18.1. (i)].

(*ii*) For the Blumenthal-Getoor index β it follows directly from (1.1.2) that $\beta \in [0, 2]$. Furthermore, we have

$$\int_{-1}^{1} |x|^p \nu(dx) < \infty, \qquad \text{for all} \qquad p > \beta.$$

By using this result, one can show in a similar way to [**13**, proof of Proposition 3.11] that

$$\sum_{a \le s \le b} |\Delta L_s|^p < \infty \qquad \text{for all} \qquad p > \beta,$$

where $a, b \in \mathbb{R}$ with $a < b$ and $\Delta L_s = L_s - L_{s-}$ denotes the jumps of a Lévy process with Blumenthal-Getoor index β.

(*iii*) By combining (1.1.1) with the relation between characteristic functions and moments, cf. e.g. [**25**, Proposition 2.5 (*ix*) and (*x*)], we get that the Lévy process $L = (L_t)_{t \in \mathbb{R}}$ with characteristic triplet $(\gamma, 0, \nu)$ is *centred*, i.e. $\mathbb{E}(L_t) = 0$ for all $t \in \mathbb{R}$, and has finite second moments, i.e. $\mathbb{E}(L_t)^2 < \infty$ for all $t \in \mathbb{R}$, if and only if

$$\gamma = -\int_{|x|>1} x \; \nu(dx) \quad \text{and} \quad \int_{|x|>1} x^2 \; \nu(dx) < \infty.$$

1.1.1. The Compound Poisson Process.

The compound Poisson process is one of the easiest examples for a pure jump Lévy process, which stems from the fact that by [**13**, Proposition 3.3] a compound Poisson process is a Lévy process whose paths are piecewise constant functions.

Furthermore, the compound Poisson process is an essential part of the Lévy-Itô decomposition, cf. e.g. [**13**, Proposition 3.7], and can therefore in some cases be used in order to approximate more general Lévy processes, e.g. Lévy processes with the characteristic triplet $(0, 0, \nu)$ whose Lévy measure ν is symmetric.

Definition 1.1.4. A Lévy process $(L_t)_{t\geq 0}$ with the following characteristic function

$$E\big(\exp(iuL_t)\big) = \exp\left(t\lambda \int_{\mathbb{R}} e^{iux} - 1\ \eta(dx)\right) \quad \text{for all } t, u \in \mathbb{R},$$

where $\lambda > 0$ and η is a distribution with $\eta(\{0\}) = 0$, is referred to as a *compound Poisson process.*

Remark 1.1.5. (*i*) The characteristic triplet (γ, σ^2, ν) of a compound Poisson process has the following form

$$\gamma = \lambda \int_{\mathbb{R}} x\mathbf{1}_{\{|x|\leq 1\}}\ \eta(dx),\ \ \sigma^2 = 0 \text{ and } \nu = \lambda\eta.$$

(*ii*) The Blumenthal-Getoor index β of a compound Poisson process is 0 since η is a distribution, i.e.

$$\int_{\mathbb{R}} x^2 \wedge 1\ \nu(dx) \leq \nu(\mathbb{R}) = \lambda\eta(\mathbb{R}) < \infty.$$

(*iii*) A short and comprehensive overview about alternative but equivalent definitions and the properties of a compound Poisson process can for example be found in [**25**, Chapter 1.4] and [**13**, Chapter 3.2].

1.1.2. The Symmetric α-Stable Lévy Process.

Now we come to a short discussion about the symmetric α-stable Lévy process and some of its properties. Note that Remark 1.2.5 (*iii*) focuses on the existence and some of the properties of random variables that are driven by a symmetric α-stable Lévy process.

Definition 1.1.6. Let $\alpha \in (0, 2)$. A symmetric Lévy process $(L_t)_{t\in\mathbb{R}}$ is called *symmetric α-stable Lévy process* if it has the following characteristic function

$$E\big(\exp(iuL_t)\big) = \exp\left(i|t|\gamma u + |t| \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|\leq 1\}}\right)\ \nu(dx)\right) \ \text{ for } \ t, u \in \mathbb{R},$$

where

$$\gamma = \begin{cases} \int_{|x|\leq 1} x \ \nu(dx), & \text{for } \alpha < 1 \\ 0, & \text{for } \alpha = 1 \\ \int_{|x|>1} x \ \nu(dx), & \text{for } \alpha > 1 \end{cases}$$

and the Lévy measure ν is absolutely continuous and has the density $f(x) = \frac{C}{|x|^{1+\alpha}}$ for a constant $C > 0$ with respect to the Lebesgue measure.

PROPOSITION 1.1.7. *For $\alpha \in (0,2)$ let $L = (L_t)_{t\in\mathbb{R}}$ be a symmetric α-stable Lévy process. The characteristic function of L satisfies for all $u, t \in \mathbb{R}$ the following representation*

$$E\big(\exp(iuL_t)\big) = e^{-|t|K|u|^{\alpha}},$$

where $K > 0$ is a suitable constant that is referred to as scale parameter.

PROOF. For $u = 0$ and arbitrary $t \in \mathbb{R}$ we have $E\big(\exp(iuL_t)\big) = e^0 = e^{-|t|K|u|^{\alpha}}$. In the case of $u \neq 0$ the representation $d\nu(x) = f(x)dx$ yields $\gamma = 0$ and

$$\begin{aligned} &\int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbf{1}_{\{|x|\leq 1\}}\right) \ \nu(dx) \\ = \ &-|u|^{\alpha} \underbrace{\int_{-\varepsilon}^{\varepsilon} \left(e^{iy} - 1 - iy\right) \ \nu(dy)}_{\longrightarrow 0, \text{ as } \varepsilon\downarrow 0} - \ 2|u|^{\alpha} \underbrace{\int_{\varepsilon}^{\infty} \big(1 - \cos(y)\big) \ \nu(dy)}_{\longrightarrow K/2, \text{ as } \varepsilon\downarrow 0}, \end{aligned}$$

where in the above equality $\varepsilon \in (0,1)$ is arbitrary and we used the substitution $y = ux$.

Hence, we have

$$E\big(\exp(iuL_t)\big) = e^{-|t|K|u|^{\alpha}}$$

for all $u, t \in \mathbb{R}$.

□

REMARK 1.1.8. (*i*) A direct consequence of Proposition 1.1.7 is that the symmetric α-stable Lévy process $(L_t)_{t\in\mathbb{R}}$ is self similar with parameter $1/\alpha$, i.e.

$$L_{at} \overset{d}{=} a^{\frac{1}{\alpha}} L_t$$

for all $a > 0$ and $t \in \mathbb{R}$.

(ii) The Blumenthal-Getoor index β of a symmetric α-stable Lévy process is equal to the parameter α, since

$$\int_{|x|\leq 1} |x|^{\delta} \nu(dx) = C \int_{|x|\leq 1} |x|^{\delta-\alpha-1} dx \begin{cases} = \infty, & \text{for } \delta \leq \alpha \\ < \infty, & \text{for } \delta > \alpha \end{cases} .$$

(iii) A more detailed overview about α-stable and symmetric α-stable Lévy process can for example be found in [**24**] and [**16**, Chapter 1].

1.2. Driven Processes

This section consists of three subsections. The first subsection contains a set of assumptions of which some are required in the second subsection in order to define the Lévy driven process and the rest will be needed in the subsequent chapters.

In the second subsection we will introduce the Lévy driven processes, and in the third subsection we will define processes driven by a fractional Lévy motion.

1.2.1. Assumptions.

In the following set of assumptions ν is a Lévy measure and $g, \tilde{g}$ are two deterministic functions from $\mathbb{R}$ to $\mathbb{R}$.

⓪ $\alpha > 0$, $k \in \mathbb{N}$ and $\theta \in (0, 2]$.

① $\limsup_{t\to\infty} \nu(x : |x| \geq t) t^{\theta} < \infty$.

② $g(x) = \tilde{g}(x) = 0$ for all $x \in (-\infty, 0)$.

③ For all $t \geq 0$ we have that $g(t - .) - \tilde{g}(-.) \in L^{\theta}(\mathbb{R})$ is bounded on $\mathbb{R}$.

④ $g(t) \sim ct^{\alpha}$ for $t \downarrow 0$, where $c \neq 0$.

⑤ $g \in C^k((0, \infty))$ and there exists a $\delta > 0$ such that

  (i) $|g^{(k)}(t)| \leq Kt^{\alpha-k}$ for all $t \in (0, \delta)$,

  (ii) $g^{(k)} \in L^{\theta}((\delta, \infty))$,

  (iii) $|g^{(k)}|$ is decreasing on (δ, ∞),

(iv) $|g^{(1)}| \in L^{\theta}((\delta,\infty))$ is decreasing on (δ,∞).

⑥ $\int_{\delta}^{\infty} |g^{(k)}(s)|^{\theta} \log(1/|g^{(k)}(s)|)ds < \infty$.

REMARK 1.2.1. The assumptions ⓪ to ③ are essential for the construction/existence of the *Lévy driven process* $(X_t)_{t\geq 0}$ in Definition 1.2.4 and Remark 1.2.5. Whereas the assumptions ④ to ⑥ are of a technical nature, i.e. the assumption

- ④ allows us to handle the asymptotic behavior of X_t for $t \downarrow 0$.
- ⑤ enables us to apply and work with Taylor's theorem.
- ⑥ is needed to get ⑤ to work in some special cases.

The next lemma shows that the Lévy measure of a Lévy process with finite second moments always satisfies assumption ①.

LEMMA 1.2.2. *Let ν be the Lévy measure of a Lévy process then it holds that*

$$\int_{|x|\geq 1} x^2 \nu(dx) < \infty \Longrightarrow \limsup_{t\to\infty} \nu(x : |x| \geq t)t^{\theta} < \infty, \ \forall \theta \in (0,2].$$

PROOF. For each $\theta \in (0,2]$ we have

$$\begin{aligned}\limsup_{t\to\infty} \nu(x : |x| \geq t)t^{\theta} &= \inf_{n\in\mathbb{N}} \sup_{t\geq n} \nu(x : |x| \geq t)t^{\theta} = \inf_{n\in\mathbb{N}} \sup_{t\geq n} \int_{|x|\geq t} t^{\theta} \nu(dx) \\ &\leq \inf_{n\in\mathbb{N}} \sup_{t\geq n} \int_{|x|\geq 1} x^{\theta} \nu(dx) = \int_{|x|\geq 1} x^{\theta} \nu(dx) \\ &\leq \int_{|x|\geq 1} x^{2} \nu(dx) < \infty.\end{aligned}$$

□

Now we will use the following proposition in order to illustrate a few functions that satisfy some of the above assumptions.

PROPOSITION 1.2.3. (i) *Let $0 < \alpha < \widehat{\alpha}$, $k \in \mathbb{N}$, $\theta \in (0,2]$, $\widetilde{g} \equiv 0$ and*

$$g(x) = \frac{(x)_+^{\alpha}}{(x+1)^{\widehat{\alpha}}} \quad \textit{for} \quad x \in \mathbb{R}$$

then the assumptions ⓪,②,④ and in the case of $\alpha - \widehat{\alpha} < -1/\theta$ the assumptions ③ as well as ⑤ are satisfied, where in ④ we have $c = 1$ and in ⑤ the parameter $\delta > 0$ is sufficient large.

(*ii*) *Let* $\alpha, \widehat{\alpha} \in (0,\infty)$*,* $k \in \mathbb{N}$*,* $\theta \in (0,2]$*,* $\tilde{g} \equiv 0$ *and*

$$g(x) = e^{-\widehat{\alpha}x}(x)_+^\alpha \quad \textit{for} \quad x \in \mathbb{R}$$

then the assumptions ⓪,②,③,④ *and* ⑤ *are satisfied, where in* ④ *we have* $c = 1$ *and in* ⑤ *the parameter* $\delta > 0$ *is sufficient large.*

(*iii*) *Let* $\alpha > 0$*,* $k \in \mathbb{N}$*,* $\theta \in (0,2]$ *and*

$$g(x) = \tilde{g}(x) = (x)_+^\alpha \quad \textit{for} \quad x \in \mathbb{R}$$

then the assumptions ⓪,②,④ *and in the case of* $\alpha - 1 < -1/\theta$ *the assumptions* ③ *as well as* ⑤ *are satisfied, where in* ④ *we have* $c = 1$ *and in* ⑤ *the parameter* $\delta > 0$ *is arbitrary.*

Proof. It is evident that in (i), (ii) and (iii) the assumptions ⓪ and ② are satisfied. Moreover, in (i), (ii) and (iii) we have $g(x) = \phi(x)(x)_+^\alpha$ for $x \geq 0$ with $\phi(y) \to 1$ as $y \downarrow 0$, i.e. assumption ④ holds with $c = 1$.

In the setting of (iii) let $t \geq 0$ then we have $|g(t-x) - g(-x)| \leq (t-x)_+^\alpha + (-x)_+^\alpha$ for $x \geq 0$, $|g(t-x) - g(-x)| \leq (t-x)_+^\alpha$ for $x \in (-t-1, 0)$, by the mean value theorem $|g(t-x) - g(-x)| \leq |\xi|^{\alpha-1} t$ for $x \leq -t-1$ and some $\xi \in [-x, t-x]$ as well as

$$\begin{aligned}\int_{\mathbb{R}} |g(t-x) - g(-x)|^\theta dx &\leq \int_0^t (t-x)_+^{\alpha\theta} dx + \int_{-t-1}^0 |(t-x)^\alpha - (-x)^\alpha|^\theta dx \\ &\quad + t^\theta \int_{-\infty}^{-t-1} |x|^{(\alpha-1)\theta} dx \\ &< \infty \quad \text{for} \quad \alpha - 1 < -\frac{1}{\theta},\end{aligned}$$

i.e. in the case of $\alpha - 1 < -1/\theta$ the assumption ③ is satisfied.

Furthermore, in the setting of (iii) for each $n \in \mathbb{N}$ we have

$$g^{(n)}(x) = \left(\prod_{i=0}^{n-1} (\alpha - i)\right) x^{\alpha - n} \quad \text{for} \quad x > 0,$$

which yields that in the case of $\alpha - 1 < -1/\theta$ the assumption ⑤ is satisfied for all $\delta > 0$.

By using

$$\frac{(x)_+^\alpha}{(x+1)^{\widehat{\alpha}}} \le (x)_+^\alpha \mathbf{1}_{\{x\le 1\}} + (x+1)^{\alpha-\widehat{\alpha}} \mathbf{1}_{\{x>1\}}$$

in the setting of (i) and

$$e^{-\widehat{\alpha}x}(x)_+^\alpha \le (x)_+^\alpha \mathbf{1}_{\{x\le 1\}} + \underbrace{e^{-\varepsilon x}(x)_+^\alpha}_{\longrightarrow 0, \text{ as } x\to\infty} e^{-x(\widehat{\alpha}-\varepsilon)} \mathbf{1}_{\{x>1\}} \quad \text{with} \quad \varepsilon \in (0,\widehat{\alpha})$$

in the setting of (ii), we get that assumption ③ holds on the one hand in the setting of (ii) and on the other hand in the case of $\alpha - \widehat{\alpha} < -1/\theta$ in the setting of (i).

For each $n \in \mathbb{N}$ an application of the general Leibniz rule in the setting of (ii) results in

$$\left(\frac{d}{dx}\right)^n g(x) = \sum_{l=0}^{n} \underbrace{\binom{n}{l}\left(\prod_{i=0}^{l-1}(\alpha-i)\right)\left(\prod_{j=0}^{n-l-1}(-\widehat{\alpha})\right)}_{=:\Lambda_{l,n}} x^{\alpha-l} e^{-\widehat{\alpha}x} = \frac{x^\alpha}{e^{\widehat{\alpha}x}} \sum_{l=0}^{n} \Lambda_{l,n} x^{-l} \tag{1.2.1}$$

and in the setting of (i) yields

$$\begin{aligned}\left(\frac{d}{dx}\right)^n g(x) &= \sum_{l=0}^{n} \underbrace{\binom{n}{l}\left(\prod_{i=0}^{l-1}(\alpha-i)\right)\left(\prod_{j=0}^{n-l-1}(-\widehat{\alpha}-j)\right)}_{=:\mu_{l,n}} x^{\alpha-l}(x+1)^{-\widehat{\alpha}-(n-l)} \\ &= \frac{x^\alpha}{(x+1)^{\widehat{\alpha}+n}} \sum_{l=0}^{n} \mu_{l,n}\left(1+\frac{1}{x}\right)^l = \frac{x^\alpha}{(x+1)^{\widehat{\alpha}+n}} \sum_{l=0}^{n}\sum_{i=0}^{l} \binom{l}{i} \mu_{l,n} x^{-i} \\ &= \frac{x^\alpha}{(x+1)^{\widehat{\alpha}+n}} \sum_{i=0}^{n} \lambda_{i,n} x^{-i} \quad \text{for suitable} \quad \lambda_{1,n},\dots,\lambda_{n,n} \in \mathbb{R}, \end{aligned} \tag{1.2.2}$$

where the second last equality is a consequence of the binomial theorem.

In order to conclude this proof note that for a sufficient large $\delta > 0$ assumption ⑤ follows in the setting of (ii) from (1.2.1) and in the setting of (i) from a combination of (1.2.2) and $\alpha - \widehat{\alpha} < -1/\theta$. □

1.2.2. Lévy Driven Processes.

In this subsection we will define the Lévy driven processes by using the same assumptions as in [**10**] and then present some additional assumptions, under which the absolute moments of the Lévy driven processes have a handy representation, cf. Remark 1.2.5 (ii) and (iii).

DEFINITION 1.2.4. Let $L = (L_t)_{t\in\mathbb{R}}$ be a (two-sided) symmetric Lévy process without a Brownian component and $g, \tilde{g}$ two deterministic functions from $\mathbb{R}$ to $\mathbb{R}$ such that the assumptions ⓪ to ③ are satisfied.

Then the stochastic process, defined by

$$X_t := \int_{-\infty}^{t} g(t-s) - \tilde{g}(-s)\, dL_s \qquad (t \geq 0)$$

is referred to as a *Lévy driven process* and in the case of $\tilde{g} \equiv 0$ as a *Lévy driven moving averages process.*

REMARK 1.2.5. Let ν denote the Lévy measure of the Lévy process L.

(i) By [**23**, Definition 2.5 and Theorem 2.7], we know that for each $t \geq 0$ the stochastic integral X_t exists as a limit in probability of integrals with respect to the random measure

$$\Lambda\big((a,b]\big) = L_b - L_a \quad \text{for} \quad a,b \in \mathbb{R} \quad \text{with} \quad a < b$$

of deterministic simpel functions that almost surely approximate the function $g(t-.) - \tilde{g}(.)$. Moreover, the limit does not depend on the sequence of approximating simple functions.

In order to apply [**23**, Theorem 2.7] we need to verify three conditions, which in the setting of Definition 1.2.4 simplify to

$$\int_{-t}^{\infty}\int_{\mathbb{R}} \big|\big(g(t+s) - \tilde{g}(s)\big)x\big|^2 \wedge 1\; \nu(dx)ds < \infty \qquad \text{for} \qquad t \geq 0.$$

The condition above is satisfied under the assumptions ⓪ to ③ as shown in [**10**, (3.1)].

(ii) Let in addition to the assumptions in Definition 1.2.4 the following two assumptions be satisfied.

- $\int_{|x|>1} x^2 \nu(dx) < \infty$, i.e. $\mathbb{E}(L_t^2) < \infty$ for all $t \in \mathbb{R}$.
- $g(T - .) - \tilde{g}(-.) \in L^2(\mathbb{R})$ for a fixed $T \geq 0$.

Then by Remark 1.1.3 (i) and (iii) we know that [**21**, (2.7)] is satisfied and that the symmetric Lévy process L is centered. Hence, an application of [**21**, Proposition 2.1] with the function $f(t,s) := g(T-s) - \tilde{g}(-s)$, which does not depend on t, yields

$$\mathbb{E}(X_T^2) = \mathbb{E}(L_1^2) \cdot \|g(T - .) - \tilde{g}(-.)\|_{L^2(\mathbb{R})}^2$$

for a fixed $T \geq 0$.

(iii) For $\alpha \in (0,2)$ let $L = (L_t)_{t\in\mathbb{R}}$ be a symmetric α-stable Lévy process, cf. Definition 1.1.6. An application of the results in [**24**, Section 3.4] yields that for each $\phi \in L^\alpha(\mathbb{R})$ the random variable $X := \int_{\mathbb{R}} \phi(s)\, dL_s$ is well defined as a limit in probability of the sequence $\big(\int_{\mathbb{R}} \phi_n \, dM\big)_{n\in\mathbb{N}}$, where M is the to L corresponding α-stable random measure and $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of simple function approximating ϕ. The limit does not depend on the approximating sequence $(\phi_n)_{n\in\mathbb{N}}$.

Note that by combining the definition of the symmetric α-stable Lévy process in [**24**, (1.3.1) and above] with [**24**, Proposition 3.4.1], where the skewness intensity of the to L corresponding α-stable random measure is identical to 0, we get that X is a symmetric α-stable random variable with scale parameter $\|\phi\|_{L^\alpha(\mathbb{R})}$.

Furthermore, since we can write $X = \|\phi\|_{L^\alpha(\mathbb{R})} Y$, where Y is a symmetric α-stable random variable with scale parameter 1, we obtain by using [**24**, Property 1.2.16] for all $p < \alpha$ the following result

$$\mathbb{E}|X|^p = \|\phi\|_{L^\alpha(\mathbb{R})}^p \mathbb{E}|Y|^p = K \|\phi\|_{L^\alpha(\mathbb{R})}^p < \infty,$$

where the constant $K > 0$ does not depend on the function ϕ.

1.2.3. Processes Driven by a Fractional-Lévy-Motion.

By using an alternative approach than the one described in Remark 1.2.5 (i), it is possible to define a Lévy driven process under different assumptions than in Definition 1.2.4. Note that under some additional assumptions this two definitions of the Lévy driven process coincide.

Moreover, the fractional Lévy motion we get by this alternative approach allows us to define processes driven by the fractional Lévy motion, which posses a representation as Lévy driven processes.

Definition 1.2.6. Let $L = (L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process with characteristic triplet $(\gamma, 0, \nu)$. Suppose that $\gamma = -\int_{|x|>1} x\nu(dx)$ and $\int_{|x|\geq 1} x^2\nu(dx) < \infty$, i.e. by Remark 1.1.3 (iii) we have $\mathbb{E}(L_t) = 0$ and $\mathbb{E}(L_t)^2 < \infty$ for all $t \in \mathbb{R}$.

Then in the case of the function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(t,\cdot) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$ we refer to the process

$$X_t := \int_{\mathbb{R}} f(t,s)\, dL_s \qquad (t \in \mathbb{R})$$

as a *Lévy driven process.*

Furthermore, for each $d \in (0, 1/2)$ the *Lévy driven process* given by

$$M_t := M_{d,t} := \frac{1}{\Gamma(d+1)} \int_{-\infty}^{t} (t-s)_+^d - (-s)_+^d \, dL_s \qquad (t \in \mathbb{R})$$

is also referred to as a *fractional Lévy motion.*

Remark 1.2.7. Note that a symmetric Lévy process L with finite second moments and without a Brownian component satisfies the assumptions on the Lévy process in Definition 1.2.6, cf Remark 1.1.3 (i) and (iii).

By [**21**, Proposition 2.1 and Theorem 3.3], the above defined Lévy driven process $(X_t)_{t\in\mathbb{R}}$ and the fractional Lévy motion $(M_{d,t})_{t\in\mathbb{R}}$ exist as limits of approximating step functions in $L^2(\Omega)$ and satisfy

$$\mathbb{E}(X_t^2) = \mathbb{E}(L_1^2)\|f(t,\cdot)\|^2_{L^2(\mathbb{R})} \quad \text{for } t \in \mathbb{R}$$

respectively

$$\mathbb{E}(M_{d,t}^2) = \mathbb{E}(L_1^2) \int_{-\infty}^{t} \left(\frac{(t-s)_+^d - (-s)_+^d}{\Gamma(d+1)} \right)^2 ds \quad \text{for } t \in \mathbb{R}.$$

Moreover, by additionally assuming in the setting of Definition 1.2.6 that the driving Lévy process L is symmetric, that for all $s, t \in \mathbb{R}$ the kernel function f is given by $f(t,s) = g(t-s) - \tilde{g}(-s)$ as well as that $g, \tilde{g}$, and L satisfy the assumptions ⓪ to ③, the above defined Lévy driven process $(X_t)_{t\in\mathbb{R}}$ coincides for $t \geq 0$ with the Lévy driven process in Definition 1.2.4.

The following definition and three propositions are the main tools we need in order to construct processes driven by a fractional Lévy motion, which also posses the representation as Lévy driven processes.

PROPOSITION 1.2.8. *Let* $0 < d < 1/2$.

(i) *For* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ *the left- and right-sided Riemann-Liouville fractional integrals* $(I_-^d f)$ *and* $(I_+^d f)$*, which are given by*

$$(I_-^d f)(x) := \frac{1}{\Gamma(d)} \int_x^{\infty} f(y)(y-x)^{d-1} dy$$

and

$$(I_+^d f)(x) := \frac{1}{\Gamma(d)} \int_{-\infty}^{x} f(y)(x-y)^{d-1} dy,$$

exist for almost all $x \in \mathbb{R}$.

(ii) *The mapping* $\|\cdot\|_H : L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to [0,\infty)$ *given by*

$$\|g\|_H := \left(\int_{\mathbb{R}} (I_-^d g)^2(x) dx \right)^{\frac{1}{2}}$$

is a norm on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$*, which for a suitable constant* $K > 0$ *satisfies*

$$\|g\|_H \leq K\big(\|g\|_{L^1(\mathbb{R})} + \|g\|_{L^2(\mathbb{R})}\big)$$

for all $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

PROOF. For (i) see [**21**, below (5.41)] and for (ii) see [**21**, (5.44) and above]. □

DEFINITION 1.2.9. The set H is the completion of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with respect to the norm $\|\cdot\|_H$, cf. Proposition 1.2.8 (ii).

PROPOSITION 1.2.10. *Let $(M_{d,t})_{t\in\mathbb{R}}$ be a fractional Lévy motion as in Definition 1.2.6 and $h \in H$. Then there exists a sequence $(\phi_k)_{k\in\mathbb{N}} \subset H$ of the form*

$$\phi_k = \sum_{i=1}^{n_k} a_i^{\langle k\rangle} \mathbf{1}_{\left(s_i^{\langle k\rangle}, s_{i+1}^{\langle k\rangle}\right]},$$

where $n_k \in \mathbb{N}, a_i^{\langle k\rangle} \in \mathbb{R}$ and $-\infty < s_1^{\langle k\rangle} < \cdots < s_{n_k+1}^{\langle k\rangle} < \infty$ for $i = 1, \ldots, n_k$ and $k \in \mathbb{N}$, satisfying $\|\phi_k - h\|_H \overset{k\to\infty}{\longrightarrow} 0$ so that

$$\int_{\mathbb{R}} \phi_k(s)\ dM_{d,s} = \sum_{i=1}^{n_k} a_i^{\langle k\rangle}\left(M_{d,s_{i+1}} - M_{d,s_i}\right)$$

converges in $L^2(\Omega)$ as $k \to \infty$ towards a limit denoted by $\int_{\mathbb{R}} h(s)\ dM_{d,s}$. The limit $\int_{\mathbb{R}} h(s)\ dM_{d,s}$ is independent of the approximating sequence $(\phi_k)_{k\in\mathbb{N}}$.

PROOF. See [**21**, Theorem 5.3]. □

PROPOSITION 1.2.11. *Let $(M_{d,t})_{t\in\mathbb{R}}$ be a fractional Lévy motion as in Definition 1.2.6 and $h \in H$. Then in $L^2(\Omega)$ we have*

$$\int_{\mathbb{R}} h(s)\ dM_{d,s} = \int_{\mathbb{R}} \left(I_-^d h\right)(s)\ dL_s.$$

PROOF. See [**21**, Proposition 5.5]. □

Based on the results in Proposition 1.2.10 we will now define processes driven by the fractional Lévy motion, which by including the results of Proposition 1.2.11 posses a very handy representation as Lévy driven processes, cf. Remark 1.2.13.

DEFINITION 1.2.12. Let $(M_{d,t})_{t\in\mathbb{R}}$ be a fractional Lévy motion as in Definition 1.2.6. Then for $h \in H$ we refer to

$$Y_{d,t} := \int_{-\infty}^{\infty} h(t-s)\ dM_{d,s} \qquad (t \in \mathbb{R})$$

as a *fractional-Lévy-motion driven process.*

Remark 1.2.13. In the setting of Definition 1.2.12 let $h_t(s) := h(t-s)$ for $t, s \in \mathbb{R}$. By applying Proposition 1.2.10 as well as Proposition 1.2.11 for each $t \in \mathbb{R}$ with the function h_t, we get on the one hand that the fractional-Lévy-motion driven process $Y = (Y_{d,t})_{t\in\mathbb{R}}$ exists in the L^2-sense and on the other hand that Y has in $L^2(\Omega)$ the following representation

$$Y_{d,t} = \int_{-\infty}^{\infty} h(t-s)\ dM_{d,s} = \int_{\mathbb{R}} h_t(s)\ dM_{d,s} = \int_{\mathbb{R}} \left(I_-^d h_t\right)(s)\ dL_s$$

for each $t \in \mathbb{R}$.

Furthermore, by using the definitions of $I_\pm^d$, we obtain

$$\begin{aligned} Y_{d,t} &= \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_s^{\infty} h_t(y)(y-s)^{d-1} dy\ dL_s \\ &= \frac{1}{\Gamma(d)} \int_{\mathbb{R}} \int_{-\infty}^{t-s} h(x)(t-s-x)^{d-1} dx\ dL_s = \int_{\mathbb{R}} \left(I_+^d h\right)(t-s)\ dL_s \end{aligned} \tag{1.2.3}$$

for each $t \in \mathbb{R}$ in $L^2(\Omega)$, where in order to get the second equality we used the substitution $x = t - y$. Since an equality in the L^2-sense implies an almost sure equality, the equality in (1.2.3) holds almost surely.

1.3. Differential Filters and Multipower Variations

Let $X = (X_t)_{t\in\mathbb{R}}$ be a stochastic process and $g : \mathbb{R} \to \mathbb{R}$ a deterministic function. For all $k, n \in \mathbb{N}$ and $i \in \mathbb{R}$ we define in an iterative way the *kth order (linear) differential filter* of the process X by

$$\triangle_{i,n}^k X := \triangle_{i,n}^{k-1} X - \triangle_{i-1,n}^{k-1} X \text{ with } \triangle_{i,n}^0 X = X_{\frac{i}{n}} \tag{1.3.1}$$

and the *modified kth order (linear) differential filter* of the function g by

$$\blacktriangle_{i,n}^k g(s) := \blacktriangle_{i,n}^{k-1} g(s) - \blacktriangle_{i-1,n}^{k-1} g(s) \text{ with } \blacktriangle_{i,n}^0 g(s) = g\left(\frac{i}{n} - s\right). \tag{1.3.2}$$

The above filters can also be represented by

$$\triangle_{i,n}^k X = \sum_{j=0}^{k} (-1)^j \begin{pmatrix} k \\ j \end{pmatrix} X_{\frac{i-j}{n}} \tag{1.3.3}$$

and

$$\blacktriangle_{i,n}^{k} g(s) = \sum_{j=0}^{k} (-1)^{j} \begin{pmatrix} k \\ j \end{pmatrix} g\left(\frac{i-j}{n} - s\right). \tag{1.3.4}$$

EXAMPLE 1.3.1.

- $\triangle_{i,n}^{1} X = X_{\frac{i}{n}} - X_{\frac{i-1}{n}}$,
- $\triangle_{i,n}^{2} X = X_{\frac{i}{n}} - 2X_{\frac{i-1}{n}} + X_{\frac{i-2}{n}}$,
- $\triangle_{i,n}^{3} X = X_{\frac{i}{n}} - 3X_{\frac{i-1}{n}} + 3X_{\frac{i-2}{n}} - X_{\frac{i-3}{n}}$.

DEFINITION 1.3.2. For $m \in \mathbb{N}$ let $X = (X_t^{\langle 1 \rangle}, \dots, X_t^{\langle m \rangle})_{t \geq 0}$ be a $\mathbb{R}^m$ valued (stochastic) process, $a = (a_1, \dots, a_m) \in \mathbb{Z}^m$, $k = (k_1, \dots, k_m) \in \mathbb{N}^m$as well as $p = (p_1, \dots, p_m) \in (0, \infty)^m$. Moreover, let

$$k_\star := \max_{j=1,\dots,M} (k_j - a_j) \quad \text{and} \quad k^\star := - \max_{j=1,\dots,M} (a_j).$$

Then we refer to

$$V_n^{\langle m \rangle} := V_n^{\langle m \rangle}(X; a; k; p) := \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{m} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j}$$

as *mth order (multi)power variation of X.*

REMARK 1.3.3. Note that by setting $\rho := \max(\lceil p_1 \rceil, \dots, \lceil p_M \rceil)$, we can transform a mth order power variation into a $(m\rho)$th order power variation as follows

$$V_n^{\langle m \rangle} = \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{m} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j} = \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{m} \prod_{l=1}^{\rho} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{\frac{p_j}{\rho}} = V_n^{\langle m\rho \rangle},$$

where $0 < p_{(\cdot)}/\rho \leq 1$.

EXAMPLE 1.3.4. The (first order) Power Variation

$$V_n^{\langle 1 \rangle} = \sum_{i=k}^{n} \left| \triangle_{i,n}^{k} X \right|^{p}$$

includes things like

$$\sum_{i=1}^{n} \left| X_{\frac{i}{n}} - X_{\frac{i-1}{n}} \right|^2 \quad \text{and} \quad \sum_{i=2}^{n} \left| X_{\frac{i}{n}} - 2X_{\frac{i-1}{n}} + X_{\frac{i-2}{n}} \right|.$$

Moreover, the Bipower Variation

$$V_n^{\langle 2 \rangle} = \sum_{i=k_\star}^{n+k^\star} \left| \triangle_{i+a_x,n}^{k_x} X \right|^{p_x} \left| \triangle_{i+a_y,n}^{k_y} Y \right|^{p_y}$$

encompasses things like

$$\sum_{i=1}^{n-1} \left| X_{\frac{i+1}{n}} - X_{\frac{i}{n}} \right| \left| X_{\frac{i}{n}} - X_{\frac{i-1}{n}} \right| \quad \text{as well as} \quad \sum_{i=k_x \vee k_y}^{n} \left| \triangle_{i,n}^{k_x} X \right|^{p_x} \left| \triangle_{i,n}^{k_y} Y \right|^{p_y}$$

and the Tripower Variation

$$V_n^{\langle 3 \rangle} = \sum_{i=k_\star}^{n+k^\star} \left| \triangle_{i+a_x,n}^{k_x} X \right|^{p_x} \left| \triangle_{i+a_y,n}^{k_y} Y \right|^{p_y} \left| \triangle_{i+a_z,n}^{k_z} Z \right|^{p_z}$$

includes things like

$$\sum_{i=\max(k_x,k_y,k_z)+5}^{n-7} \left| \triangle_{i-5,n}^{k_x} X \right|^{p_x} \left| \triangle_{i,n}^{k_y} Y \right|^{p_y} \left| \triangle_{i+7,n}^{k_z} Z \right|^{p_z}.$$

1.4. Stable Convergence in Law

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We denote the convergence in distribution/law by $\xrightarrow{d}$, the convergence in probability by $\xrightarrow{\mathbb{P}}$ and the almost sure convergence by $\xrightarrow{a.s.}$.

Definition 1.4.1. Let $(Y_n)_{n\in\mathbb{N}}$ be a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and Y a random variable on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$. We refer to the convergence of Y_n to Y as *stable convergence in law* respectively *$\mathcal{F}$-stable convergence in law* and denote it by $Y_n \xrightarrow{L-s} Y$ if and only if for all $\mathcal{F}$-measurable random variables U we have $(Y_n, U) \xrightarrow{d} (Y, U)$.

As illustrated in [**1**] the stable convergence in law of a sequence $(Y_n)_{n\in\mathbb{N}}$ of random variables is rather a property of the respective sequence $(Y_n)_{n\in\mathbb{N}}$ than of the corresponding sequence of distribution functions, which marks the main difference between the convergence in distribution and the stable convergence in law.

There are some technical advantages the stable convergence in law has over the convergence in distribution, for instance that $Y_n \xrightarrow{L-s} Y$ and $X_n \xrightarrow{\mathbb{P}} X$ imply $(Y_n, X_n) \xrightarrow{L-s} (Y, X)$, cf. [**28**, Lemma 2.21]. Moreover, by merely considering the above mentioned technical advantage in combination with the fact that many known limit theorems are stable, i.e. they hold true with respect to stable convergence in law, we are able to obtain many new and interesting results, which makes stable convergence in law a nice tool to work with.

Note that there are other equivalent characterisations of the above defined stable convergence in law, cf. [**1**] or [**22**], and that the usefulness of the respective characterisations depends on the given situation, cf. e.g. [**1**, above Theorem 1].

REMARK 1.4.2. (*i*) The following two results can be easily derived from the above definition.

- $Y_n \xrightarrow{\mathbb{P}} Y$ implies $Y_n \xrightarrow{L-s} Y$ and $Y_n \xrightarrow{L-s} Y$ implies $Y_n \xrightarrow{d} Y$, i.e. the stable convergence in law is an intermediate convergence between the convergence in probability and the convergence in distribution.
- The continuous mapping theorem and Slutsky's theorem can be extended to hold true in the case of stable convergence in law.

(*ii*) The assumption that Y is a random variable on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$ is essential in the above definition of $\mathcal{F}$-stable convergence in law.

Note that in the case of $Y_n \xrightarrow{L-s} Y$, where Y is $\mathcal{F}$-measurable, we have $(Y_n, Y) \xrightarrow{d} (Y, Y)$, which by the continuous mapping theorem results in $|Y_n - Y| \xrightarrow{d} 0$ respectively $Y_n \xrightarrow{\mathbb{P}} Y$.

1.5. Some Useful Inequalities

In this section we will present some nonstandard inequalities that will be used throughout this work.

LEMMA 1.5.1. *Suppose that the function g satisfies the assumptions* ⓪,②,④ *and* ⑤. *Then for a fixed $\mu \geq 0$ there exists a finite positive constant K, so that for all $n \in \mathbb{N}$ and $i \in \mathbb{R}$ the following inequalities hold*

$$(i)\ |\blacktriangle_{i,n}^{k} g(x)| \leq K\left(\tfrac{i}{n} - x\right)^{\alpha},\ x \in \left[\tfrac{i-k}{n}, \tfrac{i}{n}\right] \cup \left[\tfrac{i}{n} - \mu, \tfrac{i}{n}\right],$$

$$(ii)\ |\blacktriangle_{i,n}^{k} g(x)| \leq Kn^{-k}\left(\tfrac{i-k}{n} - x\right)^{\alpha-k},\ x \in \left(\tfrac{i}{n} - \delta, \tfrac{i-k}{n}\right),$$

$$(iii)\ |\blacktriangle_{i,n}^{k} g(x)| \leq Kn^{-k}\left(\mathbf{1}_{[\frac{i-k}{n}-\delta, \frac{i}{n}-\delta]}(x) + g^{(k)}\left(\tfrac{i-k}{n} - x\right)\mathbf{1}_{(-\infty, \frac{i-k}{n}-\delta)}(x)\right),$$
$$x \in \left(-\infty, \tfrac{i}{n} - \delta\right].$$

PROOF. See [**10**, Proof of Lemma 3.1]. Moreover, note that the proof of the case $x \in \left[\frac{i}{n} - \mu, \frac{i}{n}\right]$ in (i) uses the same argumentation as [**10**, proof of (3.5)]. □

Now we come to a simple generalization of Hölder's inequality.

LEMMA 1.5.2. *Let $(S, \mathcal{A}, \mu)$ be a measure space and $m \in \mathbb{N}$. Suppose that for $j = 1, \ldots, m$ we have $r_j \in [1, \infty)$ and $f_j \in L^{r_j}(S)$. Then for $1/R := \sum_{j=1}^{m} 1/r_j$ we have*

$$\|\prod_{j=1}^{m} f_j\|_{L^R(S)} \leq \prod_{j=1}^{m} \|f_j\|_{L^{r_j}(S)}.$$

PROOF. By applying Hölder's inequality with the parameters $p = r_m/R$ and $q = p/(p-1) = r_m/(r_m - R)$, we get

$$\|\prod_{j=1}^{m} f_j\|_{L^R(S)} \leq \|\prod_{j=1}^{m-1} f_j\|_{L^{\frac{Rr_m}{r_m-R}}(S)} \|f_m\|_{L^{r_m}(S)}.$$

The rest follows by induction using the fact that $\sum_{j=1}^{m-1} \frac{1}{r_j} = \frac{1}{R} - \frac{1}{r_m} = \frac{r_m - R}{Rr_m}$. □

The following lemma and corollary are one of our main tools in this work and can be seen as a kind of generalized version of Minkowski's inequality.

LEMMA 1.5.3. *Let* $(S, \mathcal{A}, \mu)$ *be a measure space and* $m \in \mathbb{N}$. *Suppose that for* $j = 1, \ldots, m$ *we have* $p_j \in (0, \infty)$ *and* $\mathcal{A}$*-measurable functions* $f_j, \hat{f}_j : S \to \mathbb{R}$ *satisfying* $\prod_{j=1}^m |f_j|^{p_j}, \prod_{j=1}^m |\hat{f}_j|^{p_j} \in L^1(S, \mathcal{A}, \mu)$.

Then it holds that

$$\left| \left(\int_S \prod_{j=1}^m |f_j|^{p_j} d\mu \right)^{\frac{1}{r}} - \left(\int_S \prod_{j=1}^m |\hat{f}_j|^{p_j} d\mu \right)^{\frac{1}{r}} \right| \leq \sum_{k=1}^m \left[\int_S \left(\prod_{j=k+1}^m |f_j|^{p_j} \right) \left(\prod_{j=1}^{k-1} |\hat{f}_j|^{p_j} \right) |f_j - \hat{f}_j|^{p_k} d\mu \right]^{\frac{1}{r}},$$

where $r = \prod_{j=1}^m (1 \vee p_j)$.

PROOF. The above result is a consequence of a combination of the subadditivity of $x \mapsto |x|^\alpha$ for $\alpha \in (0, 1]$ and the Minkowski inequality. □

COROLLARY 1.5.4. *Let* S *be a finite set and* $m \in \mathbb{N}$. *Suppose that* $a_i^{\langle j \rangle}, \hat{a}_i^{\langle j \rangle} \in \mathbb{R}$ *and* $p_j \in (0, \infty)$ *for* $j = 1, \ldots, m$ *and* $i \in S$. *Then we have*

$$\left| \left(\sum_{i \in S} \prod_{j=1}^m |a_i^{\langle j \rangle}|^{p_j} \right)^{\frac{1}{r}} - \left(\sum_{i \in S} \prod_{j=1}^m |\hat{a}_i^{\langle j \rangle}|^{p_j} \right)^{\frac{1}{r}} \right| \leq \sum_{k=1}^m \left[\sum_{i \in S} \left(\prod_{j=k+1}^m |a_i^{\langle j \rangle}|^{p_j} \right) \left(\prod_{j=1}^{k-1} |\hat{a}_i^{\langle j \rangle}|^{p_j} \right) |a_i^{\langle k \rangle} - \hat{a}_i^{\langle k \rangle}|^{p_k} \right]^{\frac{1}{r}},$$

where $r = \prod_{j=1}^m (1 \vee p_j)$.

PROOF. An application of Lemma 1.5.3 concludes this proof. □

This last lemma deals with the preservation of convergence rates under a specific transformations of the convergent sequence and its limit.

LEMMA 1.5.5. *Suppose for* $i = 1, 2$ *that* $(a_n^{\langle i \rangle})_{n \in \mathbb{N}}$ *and* $(c_n)_{n \in \mathbb{N}}$ *are real valued sequences with*

$$\left| |a_n^{\langle 1 \rangle}|^{\frac{1}{r}} - |a_n^{\langle 2 \rangle}|^{\frac{1}{r}} \right| \leq |c_n| \quad \text{and} \quad a_n^{\langle i \rangle} \xrightarrow[n \to \infty]{} a^{\langle i \rangle},$$

where $r \geq 1$ *and* $a^{\langle i \rangle} \in \mathbb{R}$*. Then for a suitable constant* $K > 0$ *we have*

$$\left| |a_n^{\langle 1 \rangle}| - |a_n^{\langle 2 \rangle}| \right| \leq K |c_n|.$$

PROOF. By using the mean value theorem and the fact that for $i = 1, 2$ the sequences $(a_n^{\langle i \rangle})_{n \in \mathbb{N}}$ converge, we get

$$\left| |a_n^{\langle 1 \rangle}| - |a_n^{\langle 2 \rangle}| \right| \leq r \left(\max_{i=1,2} |a_n^{\langle i \rangle}|^{1/r} \right)^{r-1} \left| |a_n^{\langle 1 \rangle}|^{\frac{1}{r}} - |a_n^{\langle 2 \rangle}|^{\frac{1}{r}} \right| \leq K |c_n|.$$

□

CHAPTER 2

Stable Convergence in Law Limit Theorems for Multipower Variations

The goal of this chapter is to provide stable convergence in law limit theorems for multipower variation based on the results for power variations presented in [**10**, Theorem 1.1 (i)] and [**11**, Theorem 1.2 (i)].

In the first section we will proceed as in the proof of the results for power variations and focus on Lévy driven processes that are driven by a compound Poisson process which will in the setting of compound Poisson driven processes allow us, in a natural way, to extend the respective results for power variations to the multipower variations case.

Note that the extension of the results for multipower variations of compound Poisson driven processes to other driving Lévy processes will require us, in contrast to the power variations case, to make some additional assumption on the kernel functions and the driving Lévy process, and will be discussed in section two.

Moreover, note that the last section will contain all the technical auxiliary results, which we will use in this chapter.

In order to provide a clear and comprehending overview of the notations and definitions used in Theorem 2.1.1 as well as in Theorem 2.2.1, and therefore to improve the readability of these theorems, we will summarise and present the respective notations and definitions in the following details.

Details 2.0.1.

- $V_n^{\langle M \rangle} := V_n^{\langle M \rangle}(X; a; k; p)$ *(cf. Definition 1.3.2).*
- *For each* $t \in \mathbb{R}$ *the jumps of the Lévy process* $(L_s)_{s \in \mathbb{R}}$ *at time* t *are denoted by* ΔL_t*, where* $\Delta L_t := L_t - L_{t-}$ *with* $L_{t-} := \lim_{s \uparrow t, s<t} L_s$.

- $(T_m)_{m\in\mathbb{N}}$ *is a sequence of* $\mathbb{F}$*-stopping times, where* $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ *is the filtration generated by the Lévy process* $(L_t)_{t\geq 0}$*, that exhausts the jumps of* $(L_t)_{t\geq 0}$*, i.e.*
  - $\{T_m(\omega) : m \geq 1\} \cap [0,\infty) = \{t \geq 0 : \Delta L_t \neq 0\}$,
  - $T_n(\omega) \neq T_m(\omega)$ *for* $n \neq m$ *with* $T_n(\omega) < \infty$.
- $H_m^{\langle M\rangle} := \sum_{l=-\min(a_1,\ldots,a_M)}^{\infty} \prod_{j=1}^{M} |c_j \cdot h_j(l + a_j + U_m)|^{p_j}$*, where*
  - $h_j(x) := \sum_{r=0}^{k_j} (-1)^r \frac{k_j!}{r!(k_j-r)!}(x-r)_+^{\alpha_j}$ *for* $x \in \mathbb{R}$,
  - $(U_m)_{m\in\mathbb{N}}$ *is a sequence of independent and uniform* $[0,1]$*-distributed random variables that on the one hand lives on the probability space* $(\tilde{\Omega}, \widetilde{\mathcal{F}}, \tilde{\mathbb{P}})$*, which is an extension of the underlying probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, and on the other hand is independent of the* σ*-algebra* $\mathcal{F}$.
- $C^{\langle M\rangle} := \prod_{j=1}^{M} |C_j|^{p_j}$ *with* $C_j := c_j \prod_{r=0}^{k_j-1}(\alpha_j - r)$.
- $r := \prod_{j=1}^{M}(1 \vee p_j)$.
- $\tau := \min\limits_{j=1,\ldots,M} \frac{\tau_j(\theta_j, S_j p_j)}{S_j}$*, where*
  - $\tau_j(\theta_j, S_j p_j) := \begin{cases} -S_j p_j \left(1 - \frac{1}{\theta_j} - \frac{1}{S_j p_j}\right), & \textit{for } \theta_j \in (1,2] \\ 1, & \textit{for } \theta_j \in (0,1] \end{cases}$.

Now we will verify some properties of the random variables Z and $\widehat{Z}$, which will appear as limits in Theorem 2.1.1 and Theorem 2.2.1 below.

Proposition 2.0.2. *The random variables*

$$\widehat{Z} = C^{\langle M\rangle} \sum_{m:T_m\in[0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^{M} p_j} \text{ and } Z = \sum_{m:T_m\in[0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^{M} p_j} H_m^{\langle M\rangle}$$

in Theorem 2.1.1 and Theorem 2.2.1 are infinitely divisible.

Moreover, for $t \in \mathbb{R}$ *we have*

$$\mathbb{E}\left(e^{it\widehat{Z}}\right) = \exp\left(\int_{\mathbb{R}_0} \left(e^{itC^{\langle M\rangle}|x|^{\sum_{j=1}^{M} p_j}} - 1\right)\nu(dx)\right) \tag{2.0.1}$$

and

$$\mathbb{E}\left(e^{itZ}\right) = \exp\left(\int_{\mathbb{R}_0\times\mathbb{R}} \left(e^{ity|x|^{\sum_{j=1}^{M} p_j}} - 1\right)\nu(dx)\eta(dy)\right), \tag{2.0.2}$$

where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$, ν *is the Lévy measure of the Lévy process* $(L_t)_{t\geq 0}$ *and* η *is the distribution of* $H_1^{\langle M \rangle}$.

Proof. In this proof we will use the same argumentation as in [**10**, Remark 2.2.]. Note that in this proof we will denote the Dirac measure in x by δ_x, where $x \in \mathbb{R}^k$ with $k \in \mathbb{N}$.

Since $(T_m)_{m\geq 1}$ is a sequence of stopping times that exhausts the jumps of the Lévy process $(L_t)_{t\geq 0}$, cf. Details 2.0.1, an application of [**25**, Theorem 19.2. (i)] yields that

$$\Lambda := \sum_{m=1}^{\infty} \delta_{(T_m, \Delta L_{T_m})}$$

is a Poisson random measure on $[0,1] \times \mathbb{R}_0$ with mean measure $\lambda \otimes \nu$.

The Lévy process $(L_t)_{t\geq 0}$ and the sequence of stopping times $(T_m)_{m\geq 1}$ live on the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whereas due to the properties of $(U_m)_{m\in\mathbb{N}}$ in Details 2.0.1 the sequence $(H_m^{\langle M \rangle})_{m\in\mathbb{N}}$ is a sequence of η-distributed random variables that lives on an extension of the underlying probability space and is independent of the σ-algebra $\mathcal{F}$.

Because of the above properties of Λ and $(H_m^{\langle M \rangle})_{m\in\mathbb{N}}$ the assumptions in [**27**, Definition 35. in Chapter 3.9] are satisfied and we are therefore able to apply [**27**, Theorem 36. in Chapter 3.9] in order to get that

$$\Upsilon := \sum_{m=1}^{\infty} \delta_{(T_m, \Delta L_{T_m}, H_m^{\langle M \rangle})}$$

is a Poisson random measure on $[0,1] \times \mathbb{R}_0 \times \mathbb{R}$ with mean measure $\lambda \otimes \nu \otimes \eta$.

By using the Poisson random measures Λ and Υ, we get the integral representations of $\widehat{Z}$ and Z, namely

$$\widehat{Z} = C^{\langle M \rangle} \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} = \int_{[0,1]\times\mathbb{R}_0} \Big(C^{\langle M \rangle} |x|^{\sum_{j=1}^M p_j} \Big) \Lambda(ds, dx)$$

and

$$Z = \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} H_m^{\langle M \rangle} = \int_{[0,1]\times\mathbb{R}_0\times\mathbb{R}} \Big(|x|^{\sum_{j=1}^M p_j} y \Big) \Upsilon(ds, dx, dy).$$

We get (2.0.1) as well as (2.0.2), and therefore that the random variables $\widehat{Z}$ and Z are infinitely divisible, by combining the above integral representations with the standard calculus for integrals over Poisson random measures. □

2.1. The Driving Process is a Compound Poisson Process

In this section we will present and proof the multipower variations case of [**10**, Theorem 1.1 (i)] and [**11**, Theorem 1.2 (i)] in the setting of the driving process of the Lévy driven processes being a compound Poisson process.

Note that Theorem 2.1.1 (ii) below, i.e. the extension for multipower variations of [**11**, Theorem 1.2 (i)], also provides a convergence rate with respect to almost sure convergence, which even in the case of power variations is new.

THEOREM 2.1.1. *For each $j = 1, \ldots, M$ suppose that the kernel functions $g_j, \tilde{g}_j$ and the symmetric compound Poisson process $(L_t)_{t\in\mathbb{R}}$ satisfy the assumptions ⓪ to ⑤ with respect to the parameters $\alpha_j, c_j, k_j, \theta_j$ and in the case of $\theta_j = 1$ the assumption ⑥ as well. Moreover, for each $j = 1, \ldots, M$ assume $a_j \in \mathbb{Z}$, $p_j > 0$, $S_j \geq 1$ with $\sum_{i=1}^{M} 1/S_i = 1$ and set*

$$X_t^{\langle j\rangle} := \int_{-\infty}^{t} g_j(t-s) - \tilde{g}_j(-s)\, dL_s \quad \text{for} \quad t \geq 0.$$

Then by using the definitions and notations in Details 2.0.1 we get the following two results.

(i) *If $\alpha_j < k_j - 1/(S_j p_j)$ for all j then it holds that*

$$n^{\sum_{j=1}^{M}\alpha_j p_j} V_n^{\langle M\rangle} \xrightarrow[n\to\infty]{L-s} Z := \sum_{m:T_m\in[0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^{M} p_j} H_m^{\langle M\rangle}.$$

(ii) *Suppose that for each $j = 1, \ldots, M$ the function $f_j : [0,\infty) \to \mathbb{R}$ given by $f_j(t) = g_j(t)t^{-\alpha_j}$ for $t > 0$ satisfies $f_j \in C^{k_j}([0,\infty))$ and $f_j(0) = c_j$.*

If for all j we have $\alpha_j = k_j - 1/(S_j p_j)$ as well as $1/(S_j p_j) + 1/\theta_j > 1$ then we deduce that

$$\frac{n^{\sum_{j=1}^{M}\alpha_j p_j}}{\log(n)} V_n^{\langle M\rangle} \xrightarrow[n\to\infty]{a.s.} \widehat{Z} := C^{\langle M\rangle} \sum_{m:T_m\in[0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^{M} p_j}$$

$$\textit{with}\ \left|\frac{n^{\sum_{j=1}^{M}\alpha_j p_j}}{\log(n)}V_n^{\langle M\rangle}-\widehat{Z}\right|=\mathcal{O}\big(\log(n)\big)^{-\frac{\tau}{r}}\ \textit{a.s. as}\ n\to\infty.$$

The following remark contains some kernel functions for which Theorem 2.1.1 is applicable.

Remark 2.1.2. Note that under suitable assumptions on the parameters $\alpha_{(\cdot)}$, $k_{(\cdot)}$, $\theta_{(\cdot)}$ and $p_{(\cdot)}$ we can apply Theorem 2.1.1 with the kernel functions introduced in Proposition 1.2.3.

2.1.1. Proof of Theorem 2.1.1. Note that the last section of this chapter contains some of the technical results that we will use in this proof.

In order to ease our notations we will throughout this proof denote all positive constants by K, although they may change from line to line, and assume $K \in \mathbb{N}$. Furthermore, for $j = 1, \dots, M$ we will often write $(\cdot)$ instead of j respectively $\langle\cdot\rangle$ instead of $\langle j\rangle$.

We assume without loss of generality that almost surely we have $0 \le T_1$ and $T_i < T_{i+1}$ for $i \in \mathbb{N}$. Otherwise we replace in the following proof the sequence of stopping times $(T_m)_{m\in\mathbb{N}}$ by the sequence $(\tilde{T}_m)_{m\in\mathbb{N}}$, which is given by

$$\tilde{T}_{m+1} := \inf_{k\in\mathbb{N}}\Big\{T_k : T_k > \tilde{T}_m\Big\} \quad \text{for } m \in \mathbb{N} \quad \text{with} \quad \tilde{T}_1 \equiv 0,$$

and then change at the end of the proof the order of summation in Z and $\widehat{Z}$, cf. Theorem 2.1.1, from $m \in \mathbb{N} : \tilde{T}_m \in [0,1]$ to $m \in \mathbb{N} : T_m \in [0,1]$.

Let $n \in \mathbb{N}$ be sufficiently large then we fix $\varepsilon \in \big(\overline{K}_1/n, \overline{K}_2\big)$, where

$$\overline{K}_1 > 2\left(4+|k_\star|+|k^\star|+\sum_{j=1}^{M}\big(|k_j|+|a_j|\big)\right) \quad \text{and} \quad \overline{K}_2 < 2\min(\delta_1,\dots,\delta_M),$$

and define the set

$$\Omega_\varepsilon := \Big\{\omega\in\Omega : \text{for all } j\ge 1 \text{ with } T_j(\omega)\in[0,1] \text{ we have } |T_{j+1}(\omega)-T_j(\omega)|>\varepsilon \\ \text{and } \Delta L_s(\omega)=0 \text{ for all } s\in[-\varepsilon,\varepsilon]\cup[1-\varepsilon,1+\varepsilon]\Big\}.$$

Note that by construction we have

$$\mathbb{P}(\Omega_\varepsilon) \uparrow 1 \quad \text{as } \varepsilon \downarrow 0. \tag{2.1.1}$$

For $i = k_{(\cdot)}, \ldots, n$ we decompose

$$\triangle_{i,n}^{k_{(\cdot)}} X^{\langle\cdot\rangle} = \int_{-\infty}^{\frac{i}{n}} \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)}(s)\ dL_s = M_{i,n,\varepsilon}^{\langle\cdot\rangle} + R_{i,n,\varepsilon}^{\langle\cdot\rangle}, \tag{2.1.2}$$

where

$$M_{i,n,\varepsilon}^{\langle\cdot\rangle} := \int_{\frac{i}{n}-\frac{\varepsilon}{2}}^{\frac{i}{n}} \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)}(s)\ dL_s \quad \text{and} \quad R_{i,n,\varepsilon}^{\langle\cdot\rangle} := \int_{-\infty}^{\frac{i}{n}-\frac{\varepsilon}{2}} \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)}(s)\ dL_s.$$

For

$$\widetilde{V}_{n,\varepsilon} := \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| M_{i+a_j,n,\varepsilon}^{\langle j\rangle} \right|^{p_j} \tag{2.1.3}$$

we have on Ω_ε the following representation

$$\begin{aligned}
\widetilde{V}_{n,\varepsilon} &= \sum_{i=k_\star+a_1}^{n+k^\star+a_1} \left| M_{i,n,\varepsilon}^{\langle 1\rangle} \right|^{p_1} \prod_{j=2}^{M} \left| M_{i+a_j-a_1,n,\varepsilon}^{\langle j\rangle} \right|^{p_j} \\
&\overset{\star_1}{=} \sum_{m:T_m\in[0,1]} \ \sum_{i\in\{k_\star+a_1,\ldots,n+k^\star+a_1\}:T_m\in\left(\frac{i}{n}-\frac{\varepsilon}{2},\frac{i}{n}\right]} \left| M_{i,n,\varepsilon}^{\langle 1\rangle} \right|^{p_1} \prod_{j=2}^{M} \left| M_{i+a_j-a_1,n,\varepsilon}^{\langle j\rangle} \right|^{p_j} \\
&\overset{\star_2}{=} \sum_{m:T_m\in[0,1]} \ \sum_{l=0}^{\lfloor \varepsilon n/2\rfloor+v_m} \left| M_{i_m+l,n,\varepsilon}^{\langle 1\rangle} \right|^{p_1} \prod_{j=2}^{M} \left| M_{i_m+l+a_j-a_1,n,\varepsilon}^{\langle j\rangle} \right|^{p_j} \\
&= \sum_{m:T_m\in[0,1]} \ \sum_{l=-a_1}^{\lfloor \varepsilon n/2\rfloor+v_m-a_1} \prod_{j=1}^{M} \left| M_{i_m+l+a_j,n,\varepsilon}^{\langle j\rangle} \right|^{p_j} \\
&\overset{\star_3}{=} \sum_{m:T_m\in[0,1]} \ \sum_{l=-\min(a_1,\ldots,a_M)}^{\lfloor \varepsilon n/2\rfloor+v_m-\max(a_1,\ldots,a_M)} \prod_{j=1}^{M} \left| M_{i_m+l+a_j,n,\varepsilon}^{\langle j\rangle} \right|^{p_j} \\
&\overset{\star_4}{=} \sum_{m:T_m\in[0,1]} \ \sum_{l=-\underbrace{\min(a_1,\ldots,a_M)}_{=:A_1}}^{\lfloor \varepsilon n/2\rfloor+v_m-\overbrace{\max(a_1,\ldots,a_M)}^{=:A_2}} \prod_{j=1}^{M} \left| \Delta L_{T_m} \blacktriangle_{i_m+l+a_j,n}^{k_j} g_j(T_m) \right|^{p_j},
\end{aligned} \tag{2.1.4}$$

where we used Lemma 2.3.1 (ii) (first result) and Lemma 2.3.3 (i) (last result) in $\star_1$, Lemma 2.3.1 (ii) and Lemma 2.3.3 (ii) in $\star_2$, Lemma 2.3.3 (ii) and (iii) in $\star_3$ as well as Lemma 2.3.3 (i) and (ii) in $\star_4$.

To (i):

Since the assumptions ② and ④ allow us to write $g_{(\cdot)}(t) = (t)_+^{\alpha(\cdot)} f_{(\cdot)}(t)$, where the function $f_{(\cdot)} : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $f_{(\cdot)}(t) \to c_{(\cdot)}$ for $t \downarrow 0$, we get by denoting $\{nT_m\} = nT_m - (i_m - 1)$ the following representation

$$\begin{aligned}
& n^{\alpha(\cdot)} g_{(\cdot)}\left(\frac{i_m + a_{(\cdot)} + l - r}{n} - T_m\right) \\
= \quad & n^{\alpha(\cdot)}\left(\frac{i_m + a_{(\cdot)} + l - r}{n} - T_m\right)_+^{\alpha(\cdot)} f_{(\cdot)}\left(\frac{i_m + a_{(\cdot)} + l - r}{n} - T_m\right) \\
= \quad & \Big(a_{(\cdot)} + l - r + (i_m - nT_m)\Big)_+^{\alpha(\cdot)} f_{(\cdot)}\left(\frac{a_{(\cdot)} + l - r}{n} + \frac{1}{n}(i_m - nT_m)\right) \\
= \quad & \Big(a_{(\cdot)} + l - r + (1 - \{nT_m\})\Big)_+^{\alpha(\cdot)} f_{(\cdot)}\left(\frac{a_{(\cdot)} + l - r}{n} + \frac{1}{n}(1 - \{nT_m\})\right) \qquad (2.1.5)
\end{aligned}$$

for all $l \in \mathbb{Z}$ and $r = 0, 1, \ldots, k_{(\cdot)}$.

The above representation and a combination of Lemma 2.3.1 (i), $f_{(\cdot)}(t) \to c_{(\cdot)}$ for $t \downarrow 0$, the continuous mapping theorem and Slutsky's theorem yield for each $d \in \mathbb{N}$,

$$\begin{aligned}
& \left(\prod_{j=1}^{M} \left| \sum_{r=0}^{k_j} (-1)^r \binom{k_j}{r} n^{\alpha_j} g_j\left(\frac{i_m + a_j + l - r}{n} - T_m\right)\right|^{p_j}\right)_{|l|, m \le d} \\
\xrightarrow[n\to\infty]{L-s} \quad & \left(\prod_{j=1}^{M} \left| \sum_{r=0}^{k_j} (-1)^r \binom{k_j}{r} c_j \cdot \Big(a_j + l - r + (1 - \widetilde{U}_m)\Big)_+^{\alpha_j}\right|^{p_j}\right)_{|l|, m \le d},
\end{aligned}$$

where $(\widetilde{U}_m)_{m\in\mathbb{N}}$ is a sequence of independent and uniform $[0,1]$-distributed random variables that on the one hand lives on the probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, which is an extension of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and on the other hand is independent of the σ-algebra $\mathcal{F}$.

By setting $U_m = 1 - \widetilde{U}_m$, we get for each $d \in \mathbb{N}$,

$$\left(\prod_{j=1}^{M} \left| n^{\alpha_j} \blacktriangle_{i_m+l+a_j,n}^{k_j} g_j(T_m) \right|^{p_j} \right)_{|l|,m \le d} \xrightarrow[n\to\infty]{L-s} \left(\prod_{j=1}^{M} \left| c_j \cdot h_j(l + a_j + U_m) \right|^{p_j} \right)_{|l|,m \le d}. \tag{2.1.6}$$

Furthermore, for $d \in \mathbb{N}$ we have

$$\begin{aligned} V_{n,\varepsilon,d} &:= \sum_{m: m \le d, T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} \sum_{l=-A_1}^{\lfloor \varepsilon d/2 \rfloor + v_m - A_2} \prod_{j=1}^{M} \left| n^{\alpha_j} \blacktriangle_{i_m+l+a_j,n}^{k_j} g_j(T_m) \right|^{p_j} \\ &\xrightarrow[n\to\infty]{L-s} Z_d := \sum_{m: m \le d, T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} \sum_{l=-A_1}^{\lfloor \varepsilon d/2 \rfloor + v_m - A_2} \prod_{j=1}^{M} \left| c_j \cdot h_j(l + a_j + U_m) \right|^{p_j} \\ &\xrightarrow[d\to\infty]{a.s.} Z = \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} \sum_{l=-A_1}^{\infty} \prod_{j=1}^{M} \left| c_j \cdot h_j(l + a_j + U_m) \right|^{p_j}, \end{aligned} \tag{2.1.7}$$

where the stable convergence in law follows from (2.1.6) and the continuous mapping theorem, cf. [**28**, Lemma 2.20 and Lemma 2.21].

Note that by a successively application of the fact that $h_{(\cdot)}$ is bounded on every compact interval, Lemma 1.5.2, the inequality

$$\left| h_{(\cdot)}(s) \right| \le K \left| s - k_{(\cdot)} \right|^{\alpha_{(\cdot)} - k_{(\cdot)}} \text{ for } s \in (\delta_{(\cdot)} + k_{(\cdot)}, \infty),$$

which follows from an application of Lemma 1.5.1 (iii) by using

$$h_{(\cdot)}(s) = \blacktriangle_{0,1}^{k_{(\cdot)}} \tilde{h}_{(\cdot)}(-s) \quad \text{for} \quad s \in \mathbb{R} \quad \text{with} \quad \tilde{h}_{(\cdot)}(s) := (s)_+^{\alpha_{(\cdot)}}$$

in combination with $U_m \in [0,1]$ a.s. and the fact that $s \mapsto (s)^{\alpha_{(\cdot)} - k_{(\cdot)}}$ is monotone decreasing on $(0, \infty)$, we obtain

$$Z < \underbrace{K \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} \left[1 + \prod_{j=1}^{M} \left(\sum_{l=K}^{\infty} \left| l + a_j - k_j \right|^{\overbrace{S_j p_j (\alpha_j - k_j)}^{<-1}} \right)^{\frac{1}{S_j}} \right]}_{<\infty \text{ a.s.}}, \tag{2.1.8}$$

where the finiteness is a consequence of L being a compound Poisson process, i.e. $\sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} < \infty$ almost surely.

For $d, n \in \mathbb{N}$ with $n > d$ we have the following decomposition

$$\begin{aligned} V_{n,\varepsilon} &:= n^{\sum_{j=1}^M \alpha_j p_j} \sum_{m:T_m \in [0,1]} \sum_{l=-A_1}^{\lfloor \varepsilon n/2 \rfloor + v_m - A_2} \prod_{j=1}^M \left| \Delta L_{T_m} \blacktriangle_{i_m+l+a_j,n}^{k_j} g_j(T_m) \right|^{p_j} \\ &= V_{n,\varepsilon,d} + \widetilde{A}_{n,\varepsilon,d}^{\langle 1 \rangle} + \widetilde{A}_{n,\varepsilon,d}^{\langle 2 \rangle} \end{aligned} \tag{2.1.9}$$

with

$$\widetilde{A}_{n,\varepsilon,d}^{\langle 1 \rangle} := \sum_{m:m \le d, T_m \in [0,1]} \sum_{l=\lfloor \varepsilon d/2 \rfloor + v_m - A_2 + 1}^{\lfloor \varepsilon n/2 \rfloor + v_m - A_2} \prod_{j=1}^M \left| \Delta L_{T_m} n^{\alpha_j} \blacktriangle_{i_m+l+a_j,n}^{k_j} g_j(T_m) \right|^{p_j}$$

and

$$\widetilde{A}_{n,\varepsilon,d}^{\langle 2 \rangle} := \sum_{m:m > d, T_m \in [0,1]} \sum_{l=-A_1}^{\lfloor \varepsilon n/2 \rfloor + v_m - A_2} \prod_{j=1}^M \left| \Delta L_{T_m} n^{\alpha_j} \blacktriangle_{i_m+l+a_j,n}^{k_j} g_j(T_m) \right|^{p_j}.$$

By assuming d to be sufficiently large, so that $\lfloor \varepsilon d/2 \rfloor - A_2 > \overline{K}_1$, we get

$$\widetilde{A}_{n,\varepsilon,d}^{\langle i \rangle} \le \widetilde{C}_{\varepsilon,d}^{\langle i \rangle} < \infty \text{ a.s.} \tag{2.1.10}$$

for $i = 1, 2$ with

$$\widetilde{C}_{\varepsilon,d}^{\langle 1 \rangle} := K \sum_{m:T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} \prod_{j=1}^M \left(\sum_{l=\lfloor \varepsilon d/2 \rfloor - A_2}^{\infty} |l + a_j - k_j|^{S_j p_j (\alpha_j - k_j)} \right)^{\frac{1}{S_j}}$$

and

$$\widetilde{C}_{\varepsilon,d}^{\langle 2 \rangle} := K \sum_{m:m>d, T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} \left[1 + \prod_{j=1}^M \left(\sum_{l=K}^{\infty} |l + a_j - k_j|^{S_j p_j (\alpha_j - k_j)} \right)^{\frac{1}{S_j}} \right],$$

where (2.1.10) follows by a similar argumentation as in (2.1.8), in which we utilise the results for i_m, v_m and T_m in Lemma 2.3.1 in combination with (2.1.11) and (2.1.12) below.

Note that on the one hand for $l = -a_{(\cdot)}, \ldots, k_{(\cdot)} - a_{(\cdot)}$ we have

$$\begin{aligned} \left| \blacktriangle_{i_m+l+a_{(\cdot)},n}^{k_{(\cdot)}} g_{(\cdot)}(T_m) \right| &\le K \left| \frac{i_m + a_{(\cdot)} + l}{n} - T_m \right|^{\alpha_{(\cdot)}} \\ &= K n^{-\alpha_{(\cdot)}} |a_{(\cdot)} + l + \underbrace{(i_m - nT_m)}_{\in [0,1] \text{ a.s.}} |^{\alpha_{(\cdot)}}, \end{aligned} \tag{2.1.11}$$

which in the case of $l = -a_{(\cdot)}, \dots, k_{(\cdot)} - a_{(\cdot)} - 1$ follows from Lemma 1.5.1 (i) as well as

$$T_m \in \left(\frac{i_m - 1}{n}, \frac{i_m}{n} \right] \subset \left[\frac{i_m + l + a_{(\cdot)} - k_{(\cdot)}}{n}, \frac{i_m + l + a_{(\cdot)}}{n} \right]$$

and in the case of $l = k_{(\cdot)} - a_{(\cdot)}$ is a consequence of a combination of the equality above (2.1.5) as well as of the convergence $f_{(\cdot)}(t) \to c_{(\cdot)}$ as $t \downarrow 0$, and on the other hand for $l = k_{(\cdot)} - a_{(\cdot)} + 1, \dots, \lfloor \varepsilon n/2 \rfloor + v_m - A_2$ we have

$$\begin{aligned} \left| \blacktriangle^{k_{(\cdot)}}_{i_m + l + a_{(\cdot)}, n} g_{(\cdot)}(T_m) \right| &\leq K n^{-k_{(\cdot)}} \left| \frac{i_m + a_{(\cdot)} + l - k_{(\cdot)}}{n} - T_m \right|^{\alpha_{(\cdot)} - k_{(\cdot)}} \\ &= K n^{-\alpha_{(\cdot)}} |a_{(\cdot)} + l - k_{(\cdot)} + \underbrace{(i_m - nT_m)}_{\in [0,1] \text{ a.s.}} |^{\alpha_{(\cdot)} - k_{(\cdot)}}, \end{aligned} \tag{2.1.12}$$

which follows, since $\varepsilon < \overline{K}_2$, from

$$T_m \in \left(\frac{i_m + a_{(\cdot)} + l}{n} - \frac{\varepsilon}{2}, \frac{i_m}{n} \right] \subset \left(\frac{i_m + l + a_{(\cdot)}}{n} - \delta_{(\cdot)}, \frac{i_m + l + a_{(\cdot)} - k_{(\cdot)}}{n} \right)$$

and Lemma 1.5.1 (ii).

Since we have the convergences in (2.1.7) and for all $\tilde{\varepsilon} > 0$,

$$\begin{aligned} \lim_{d \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(|V_{n,\varepsilon} - V_{n,\varepsilon,d}| \geq \tilde{\varepsilon} \right) &\leq \lim_{d \to \infty} \limsup_{n \to \infty} \mathbb{P}\left(\widetilde{C}^{\langle 1 \rangle}_{\varepsilon,d} + \widetilde{C}^{\langle 2 \rangle}_{\varepsilon,d} \geq \tilde{\varepsilon} \right) \\ &= \lim_{d \to \infty} \mathbb{P}\left(\widetilde{C}^{\langle 1 \rangle}_{\varepsilon,d} + \widetilde{C}^{\langle 2 \rangle}_{\varepsilon,d} \geq \tilde{\varepsilon} \right) = 0, \end{aligned}$$

where (2.1.9) and (2.1.10) are responsible for the inequality and the fact that we have $\widetilde{C}^{\langle i \rangle}_{\varepsilon,d} \to 0$ a.s. for $i = 1, 2$ and $d \to \infty$ is responsible for the last equality, we are able to apply [**12**, Theorem 3.2], which in combination with the representation in (2.1.4) yields

$$V_{n,\varepsilon} = n^{\sum_{j=1}^M \alpha_j p_j} \widetilde{V}_{n,\varepsilon} = n^{\sum_{j=1}^M \alpha_j p_j} \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^M \left| M^{\langle j \rangle}_{i+a_j,n,\varepsilon} \right|^{p_j} \xrightarrow[n \to \infty]{L-s} Z \quad \text{on } \Omega_\varepsilon. \tag{2.1.13}$$

Moreover, for

$$V_n := n^{\sum_{j=1}^M \alpha_j p_j} V^{\langle M \rangle}_n = n^{\sum_{j=1}^M \alpha_j p_j} \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^M \left| \triangle^{k_j}_{i+a_j,n} X^{\langle j \rangle} \right|^{p_j}$$

and the parameter r as in Details 2.0.1 we get

$$\begin{aligned}
&\left|(V_n)^{\frac{1}{r}} - \left(V_{n,\varepsilon}\right)^{\frac{1}{r}}\right| \\
&\le n^{\frac{1}{r}\sum_{j=1}^{M}\alpha_j p_j} \sum_{k=1}^{M}\left[\sum_{i=k_\star}^{n+k^\star}\left(\prod_{j=k+1}^{M}\left|\,\triangle_{i+a_j,n}^{k_j} X^{\langle j\rangle}\right|^{p_j}\right)\left(\prod_{j=1}^{k-1}\left|M_{i+a_j,n,\varepsilon}^{\langle j\rangle}\right|^{p_j}\right)\left|R_{i+a_k,n,\varepsilon}^{\langle k\rangle}\right|^{p_k}\right]^{\frac{1}{r}} \\
&\le \sum_{k=1}^{M}\left[\left(\prod_{j=k+1}^{M}\left(n^{\alpha_j S_j p_j}\sum_{i=k_{(\cdot)}}^{n}\left|\,\triangle_{i,n}^{k_j} X^{\langle j\rangle}\right|^{S_j p_j}\right)^{\frac{1}{S_j r}}\right)\left(n^{\alpha_k S_k p_k}\sum_{i=k_{(\cdot)}}^{n}\left|R_{i,n,\varepsilon}^{\langle k\rangle}\right|^{S_k p_k}\right)^{\frac{1}{S_k r}}\right. \\
&\qquad\qquad \left.\left(\prod_{j=1}^{k-1}\left(n^{\alpha_j S_j p_j}\sum_{i=k_{(\cdot)}}^{n}\left|M_{i,n,\varepsilon}^{\langle j\rangle}\right|^{S_j p_j}\right)^{\frac{1}{S_j r}}\right)\right], \qquad (2.1.14)
\end{aligned}$$

where the first inequality follows from the decomposition in (2.1.2) and an application of Corollary 1.5.4 and the second inequality is a consequence of Lemma 1.5.2 in combination with $k_\star + a_{(\cdot)} \ge k_{(\cdot)}$ and $k^\star + a_{(\cdot)} \le 0$.

By combining Lemma 2.3.2 (i), (2.1.14), Slutsky's theorem and the continuous mapping theorem, we get

$$(V_n)^{\frac{1}{r}} - \left(V_{n,\varepsilon}\right)^{\frac{1}{r}} \xrightarrow[n\to\infty]{\mathbb{P}} 0 \quad \text{ on } \Omega_\varepsilon,$$

which by utilising (2.1.13), Slutsky's theorem, and the continuous mapping theorem yields

$$(V_n)^{\frac{1}{r}} = \left(V_{n,\varepsilon}\right)^{\frac{1}{r}} + (V_n)^{\frac{1}{r}} - \left(V_{n,\varepsilon}\right)^{\frac{1}{r}} \xrightarrow[n\to\infty]{L-s} Z^{\frac{1}{r}} \quad \text{ on } \Omega_\varepsilon$$

respectively

$$V_n \xrightarrow[n\to\infty]{L-s} Z \quad \text{ on } \Omega_\varepsilon. \qquad (2.1.15)$$

In order to extend the convergence in (2.1.15) to the whole set Ω, i.e. to a set with probability 1, it is sufficient to verify

$$\lim_{n\to\infty} a_{n,\bullet} = \lim_{n\to\infty}\lim_{\varepsilon\downarrow 0} a_{n,\varepsilon} = \lim_{\varepsilon\downarrow 0}\lim_{n\to\infty} a_{n,\varepsilon} = \lim_{\varepsilon\downarrow 0} a_{\bullet,\varepsilon} = a_{\bullet,\bullet}, \qquad (2.1.16)$$

where

$$a_{n,\varepsilon} = \mathbb{P}(\Omega_\varepsilon)^{-1} \int_{\Omega_\varepsilon} f(V_n, U) d\mathbb{P}, \qquad a_{n,\bullet} = \int_\Omega f(V_n, U) d\mathbb{P},$$
$$a_{\bullet,\varepsilon} = \mathbb{P}(\Omega_\varepsilon)^{-1} \int_{\Omega_\varepsilon} f(Z, U) d\mathbb{P} \qquad \text{and} \qquad a_{\bullet,\bullet} = \int_\Omega f(Z, U) d\mathbb{P}$$

for arbitrary $f \in C_b(\mathbb{R})$ and $\mathcal{F}$-measurable random variables U.

In (2.1.16) the first/last equality follows from (2.1.1) and the dominated convergence theorem, the forelast equality is an application of (2.1.15) and the swapping of limits is possible since on the one hand by (2.1.15) as well as the assumptions on ε above (2.1.1) we have

$$a_{n,\varepsilon} \underset{n\to\infty}{\longrightarrow} a_{\bullet,\varepsilon} \quad \text{for } \varepsilon \in \left(0, \overline{K}_2\right)$$

and on the other hand $a_{n,\varepsilon} \to a_{n,\bullet}$ uniformly as $\varepsilon \downarrow 0$, which is a consequence of (2.1.1) and

$$\begin{aligned} |a_{n,\varepsilon} - a_{n,\bullet}| &\le \left|\mathbb{P}(\Omega_\varepsilon)^{-1} - 1\right| \int_{\Omega_\varepsilon} |f(V_n, U)| d\mathbb{P} + \int_{\Omega\setminus\Omega_\varepsilon} |f(V_n, U)| d\mathbb{P} \\ &\le K\left(\left|\mathbb{P}(\Omega_\varepsilon)^{-1} - 1\right| + \mathbb{P}(\Omega \setminus \Omega_\varepsilon)\right). \end{aligned}$$

To (ii):

The combination of the representation (2.1.5), the continuity of $f_{(\cdot)}$ on $[0, \infty)$ and $\{nT_m\} \in [0,1]$ a.s. yields

$$\frac{n^{\alpha_{(\cdot)} S_{(\cdot)} p_{(\cdot)}}}{\log(n)} \sum_{l=-A_1}^{A_3} \left|\blacktriangle^{k_{(\cdot)}}_{i_m+l+a_{(\cdot)},n} g_{(\cdot)}(T_m)\right|^{S_{(\cdot)} p_{(\cdot)}} \le \frac{K}{\log(n)} \quad \text{a.s. as } n \to \infty$$

with $A_3 := \max(k_1, \ldots, k_M) - A_1$, which by an application of Lemma 1.5.2 implies

$$W_{n,\varepsilon,m} := \frac{n^{\sum_{j=1}^M \alpha_j p_j}}{\log(n)} \sum_{l=-A_1}^{A_3} \prod_{j=1}^M \left|\blacktriangle^{k_j}_{i_m+l+a_j,n} g_j(T_m)\right|^{p_j} \le \frac{K}{\log(n)} \quad \text{a.s.} \tag{2.1.17}$$

as $n \to \infty$.

Furthermore, by applying the same argumentation as in [**11**, (4.6) and (4.9)] using our notations, we get

$$\frac{1}{\log(n)}\sum_{l=k_{(\cdot)}+1}^{\lfloor \varepsilon n/2\rfloor+v_m}\left|n^{\alpha_{(\cdot)}}\blacktriangle_{i_m+l,n}^{k_{(\cdot)}}g_{(\cdot)}(T_m)-h_{(\cdot)}(l+\{nT_m\})\right|^{S_{(\cdot)}p_{(\cdot)}}\leq\frac{K}{\log(n)} \tag{2.1.18}$$

almost surely as $n\to\infty$ as well as

$$\frac{1}{\log(n)}\sum_{l=k_{(\cdot)}+1}^{\lfloor \varepsilon n/2\rfloor+v_m}\left|h_{(\cdot)}(l+\{nT_m\})-C_{(\cdot)}l^{\alpha_{(\cdot)}-k_{(\cdot)}}\right|^{S_{(\cdot)}p_{(\cdot)}}\leq\frac{K}{\log(n)} \tag{2.1.19}$$

almost surely as $n\to\infty$.

By using the asymptotic expansion for harmonic numbers in [**20**, Theorem 3.2(a)], i.e.

$$\sum_{l=1}^{n}\frac{1}{l}=\kappa+\log(n)+\mathcal{O}\left(\frac{1}{n}\right)\text{ as }n\to\infty,$$

where κ is the Euler-Mascheroni constant, and the fact that

$$\sum_{l=A_3+1}^{\lfloor \varepsilon n/2\rfloor+v_m-A_2}|l+A_2|^{-1}\leq\sum_{l=A_3+1}^{\lfloor \varepsilon n/2\rfloor+v_m-A_2}\prod_{j=1}^{M}|l+a_j|^{\overbrace{(\alpha_j-k_j)p_j}^{=-1/S_j}}\leq\sum_{l=A_3+1}^{\lfloor \varepsilon n/2\rfloor+v_m-A_2}|l+A_1|^{-1},$$

we get

$$\left|\frac{1}{\log(n)}\sum_{l=A_3+1}^{\lfloor \varepsilon n/2\rfloor+v_m-A_2}|l+a_{(\cdot)}|^{(\alpha_{(\cdot)}-k_{(\cdot)})S_{(\cdot)}p_{(\cdot)}}-1\right|\leq\frac{K}{\log(n)} \tag{2.1.20}$$

almost surely as $n\to\infty$ as well as

$$\left|\frac{1}{\log(n)}\sum_{l=A_3+1}^{\lfloor \varepsilon n/2\rfloor+v_m-A_2}\prod_{j=1}^{M}\left|C_j(l+a_j)^{\alpha_j-k_j}\right|^{p_j}-C^{\langle M\rangle}\right|\leq\frac{K}{\log(n)} \tag{2.1.21}$$

almost surely as $n\to\infty$, where $C^{\langle M\rangle}:=\prod_{j=1}^{M}|C_j|^{p_j}$.

From (2.1.18), (2.1.19), (2.1.20), and Corollary 1.5.4 we conclude

$$\frac{n^{\alpha_{(\cdot)}S_{(\cdot)}p_{(\cdot)}}}{\log(n)}\sum_{l=k_{(\cdot)}+1}^{\lfloor \varepsilon n/2\rfloor+v_m}\left|\blacktriangle_{i_m+l,n}^{k_{(\cdot)}}g_{(\cdot)}(T_m)\right|^{S_{(\cdot)}p_{(\cdot)}}\xrightarrow[n\to\infty]{a.s.}\left|C_{(\cdot)}\right|^{S_{(\cdot)}p_{(\cdot)}} \tag{2.1.22}$$

and

$$\frac{1}{\log(n)} \sum_{l=k_{(\cdot)}+1}^{\lfloor \varepsilon n/2 \rfloor + v_m} \left| h_{(\cdot)}(l + \{nT_m\}) \right|^{S_{(\cdot)} p_{(\cdot)}} \xrightarrow[n\to\infty]{a.s.} \left| C_{(\cdot)} \right|^{S_{(\cdot)} p_{(\cdot)}}. \tag{2.1.23}$$

Let

$$\widehat{W}_{n,\varepsilon,m} := \frac{1}{\log(n)} \sum_{l=A_3+1}^{\lfloor \varepsilon n/2 \rfloor + v_m - A_2} \prod_{j=1}^{M} \left| n^{\alpha_j} \blacktriangle_{i_m+l+a_j,n}^{k_j} g_j(T_m) \right|^{p_j},$$

$$\widetilde{W}_{n,\varepsilon,m} := \frac{1}{\log(n)} \sum_{l=A_3+1}^{\lfloor \varepsilon n/2 \rfloor + v_m - A_2} \prod_{j=1}^{M} \left| C_j (l + a_j)^{\alpha_j - k_j} \right|^{p_j}$$

and let the parameter r be as in Details 2.0.1 then we have

$$\left| (\widehat{W}_{n,\varepsilon,m})^{\frac{1}{r}} - (\widetilde{W}_{n,\varepsilon,m})^{\frac{1}{r}} \right| \leq K \left(\frac{1}{\log(n)} \right)^{\frac{1}{r} \frac{1}{\max(S_1,\ldots,S_M)}} \qquad \text{a.s.} \tag{2.1.24}$$

as $n \to \infty$, which follows by adding and subtracting inside of the absolute value in (2.1.24) the term

$$\left(\frac{1}{\log(n)} \sum_{l=A_3+1}^{\lfloor \varepsilon n/2 \rfloor + v_m - A_2} \left| h_{(\cdot)}(l + \{nT_m\}) \right|^{p_{(\cdot)}} \right)^{\frac{1}{r}},$$

by applying Corollary 1.5.4 as well as Lemma 1.5.2 and then by using, since we have $a_{(\cdot)} - A_2 \leq 0$ and $a_{(\cdot)} + A_3 \geq k_{(\cdot)}$, a combination of the convergences in (2.1.20), (2.1.22) and (2.1.23) as well as the convergence rates in (2.1.18) and (2.1.19).

Moreover, by combining (2.1.21), (2.1.24) and Lemma 1.5.5, we obtain

$$\left| \widehat{W}_{n,\varepsilon,m} - C^{\langle M \rangle} \right| \leq K \left(\frac{1}{\log(n)} \right)^{\frac{1}{r} \frac{1}{\max(S_1,\ldots,S_M)}} \qquad \text{a.s.} \quad \text{as } n \to \infty. \tag{2.1.25}$$

Since by (2.1.3) and (2.1.4), we have

$$\begin{aligned} \widehat{V}_{n,\varepsilon} := \frac{n^{\sum_{j=1}^{M} \alpha_j p_j}}{\log(n)} \widetilde{V}_{n,\varepsilon} &= \frac{n^{\sum_{j=1}^{M} \alpha_j p_j}}{\log(n)} \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| M_{i+a_j,n,\varepsilon}^{\langle j \rangle} \right|^{p_j} \\ &= \sum_{m: T_m \in [0,1]} \left| \Delta L_{T_m} \right|^{\sum_{j=1}^{M} p_j} (W_{n,\varepsilon,m} + \widehat{W}_{n,\varepsilon,m}) \quad \text{ on } \Omega_\varepsilon, \end{aligned}$$

we get for

$$\widehat{V}_n := \frac{n^{\sum_{j=1}^M \alpha_j p_j}}{\log(n)} V_n^{\langle M \rangle} = \frac{n^{\sum_{j=1}^M \alpha_j p_j}}{\log(n)} \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \Delta_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j}$$

and

$$\widehat{Z} = C^{\langle M \rangle} \sum_{m:T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j}$$

the following two convergence results

$$|\widehat{V}_n - \widehat{V}_{n,\varepsilon}| \leq \widehat{K} \left(\frac{1}{\log(n)} \right)^{\frac{\tau}{r}} \qquad \text{a.s. on } \Omega_\varepsilon \text{ with } \widehat{K} < \infty \text{ a.s.,} \tag{2.1.26}$$

$$|\widehat{V}_{n,\varepsilon} - \widehat{Z}| \leq K \underbrace{\sum_{m:T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j}}_{<\infty \text{ a.s.}} \left(\frac{1}{\log(n)} \right)^{\frac{1}{r} \frac{1}{\max(S_1,\dots,S_M)}} \qquad \text{a.s. on } \Omega_\varepsilon \tag{2.1.27}$$

as $n \to \infty$, where on the one hand (2.1.26) follows by using in the following order Corollary 1.5.4, Lemma 1.5.2 with the parameters $r_{(\cdot)} = S_{(\cdot)}$, the decomposition (2.1.2), the convergences in Lemma 2.3.2 (ii), which can be used since we have $k_\star + a_{(\cdot)} \geq k_{(\cdot)}$ and $k^\star + a_{(\cdot)} \leq 0$, as well as by a subsequent application of Lemma 1.5.5 and on the other hand (2.1.27) is a consequence of (2.1.17), (2.1.25) and the fact that, since L is a compound Poisson process, we have $\sum_{m:T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} < \infty$ almost surely.

Note that by the definition of τ in Details 2.0.1 we have

$$\tau = \min_{j=1,\dots,M} \frac{\tau_j(\theta_j, S_j p_j)}{S_j} \leq \min_{j=1,\dots,M} \frac{1}{S_j} = \frac{1}{\max(S_1,\dots,S_M)}, \tag{2.1.28}$$

where the inequality is due to the fact that in the case of $\theta_{(\cdot)} \in (0,1]$ we have $\tau_{(\cdot)}(\theta_{(\cdot)}, S_{(\cdot)} p_{(\cdot)}) = 1$ and in the case of $\theta_{(\cdot)} \in (1,2]$ we have

$$0 < S_{(\cdot)} p_{(\cdot)} \underbrace{\left(\frac{1}{\theta_{(\cdot)}} + \frac{1}{S_{(\cdot)} p_{(\cdot)}} - 1 \right)}_{>0} = \tau_{(\cdot)}(\theta_{(\cdot)}, S_{(\cdot)} p_{(\cdot)}) = 1 + S_{(\cdot)} p_{(\cdot)} \underbrace{\left(\frac{1}{\theta_{(\cdot)}} - 1 \right)}_{<0} < 1.$$

In order to end this proof we combine (2.1.26), (2.1.27) as well as (2.1.28), which imply that for each $\omega \in A := \lim_{\varepsilon \downarrow 0} \Omega_\varepsilon$, where by (2.1.1) we have $\mathbb{P}(A) = 1$, there exist $\varepsilon(\omega)$, $K(\omega)$, $n(\omega) > 0$ with $\omega \in \Omega_{\varepsilon(\omega)}$ and

$$|\widehat{V}_n(\omega) - \widehat{Z}(\omega)| \leq K(\omega)\left(\frac{1}{\log(n)}\right)^{\frac{\tau}{r}}$$

for all $n > n(\omega)$.

□

2.2. The Driving Process is a Lévy Processes with Finete Second Moments

In this section we will, in order to extend Theorem 2.1.1 to driving Lévy processes with finite second moments, combine the L^2-properties of Lévy driven processes, cf. Remark 1.2.5 (ii), with the ideas of the proof of [**10**, Theorem 1.1 (i)] and [**11**, Theorem 1.2 (i)].

THEOREM 2.2.1. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a symmetric Lévy process without a Brownian component, with a Lévy measure ν satisfying $\int_{|x|\geq 1} x^2 \nu(dx) < \infty$ and Blumenthal-Getoor index $\beta < 2$. For each $j = 1, \ldots, M$ suppose that the kernel functions $g_j, \tilde{g}_j$ as well as the Lévy process L satisfy the assumptions ⓪,②,③,④,⑤ with respect to the parameters $\alpha_j, c_j, k_j, \theta_j$ and in the case of $\theta_j = 1$ the assumption ⑥ as well. Moreover, for each $j = 1, \ldots, M$ set*

$$X_t^{\langle j\rangle} := \int_{-\infty}^{t} g_j(t-s) - \tilde{g}_j(-s)\ dL_s \quad \textit{for} \quad t \geq 0$$

and assume $a_j \in \mathbb{Z}$, $p_j \in (0,2)$ with $\sum_{i\in\{1,\ldots,M\}\setminus\{j\}} p_i < 2$, $S_j \geq 1$ with $\sum_{i=1}^{M} 1/S_i = 1$ as well as

$$\| \blacktriangle_{i,n}^{k_j} g_j\|_{L^2(\mathbb{R})}^2 \leq n^{-2\alpha_j - 1} K \tag{2.2.1}$$

for all sufficiently large $n \in \mathbb{N}$ and $i = k_j, \ldots, n$, where $K > 0$ is a suitable constant. Then by denoting $Q_j := 2/(2 - \sum_{i\in\{1,\ldots,M\}\setminus\{j\}} p_i)$ for each j as well as by using the definitions and notations in Details 2.0.1, we get the following two results.

(i) *If for each $j = 1, \dots, M$ we have $Q_j p_j > \beta$ and <u>either</u> $Q_j p_j \geq \min\{S_j p_j, 2\}$ and $\alpha_j < \min\{k_j - 1/(S_j p_j), k_j - 1/2\}$ <u>or</u> $\alpha_j < k_j - 1/p_j$ then it holds that*

$$n^{\sum_{j=1}^M \alpha_j p_j} V_n^{\langle M \rangle} \xrightarrow[n\to\infty]{L-s} Z := \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} H_m^{\langle M \rangle}.$$

(ii) *Suppose that for each $j = 1, \dots, M$ the function $f_j : [0, \infty) \to \mathbb{R}$ given by $f_j(t) = g_j(t) t^{-\alpha_j}$ for $t > 0$ satisfies $f_j \in C^{k_j}([0, \infty))$ and $f_j(0) = c_j$.*

If for all j we have $Q_j p_j > \beta$, $\alpha_j = k_j - 1/(S_j p_j)$, $1/(S_j p_j) + 1/\theta_j > 1$ and $\min\{Q_j p_j, 2\} \geq S_j p_j$ then we deduce that

$$\frac{n^{\sum_{j=1}^M \alpha_j p_j}}{\log(n)} V_n^{\langle M \rangle} \xrightarrow[n\to\infty]{\mathbb{P}} \widehat{Z} := C^{\langle M \rangle} \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j}.$$

In the following remark we will on the one hand focus on cases in which a kernel function satisfies (2.2.1) and on the other hand introduce some kernel functions for which Theorem 2.2.1 is applicable.

Remark 2.2.2. (i) For $\alpha \in (0, 1/2)$ set $g(s) := (s)_+^\alpha$, $s \in \mathbb{R}$. Then by [**14**, Proposition 2] we have $\blacktriangle_{i,n}^k g \in L^2(\mathbb{R})$ for all $i \in \mathbb{Z}$ and $n \in \mathbb{N}$. Moreover, by substitution, we obtain

$$\begin{aligned} \| \blacktriangle_{i,n}^k g \|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\frac{i}{n}} \Big| \sum_{j=0}^k (-1)^j \binom{k}{j} \Big(\frac{i-j}{n} - s \Big)_+^\alpha \Big|^2 ds \\ &= n^{-2\alpha-1} \int_{-\infty}^0 \Big| \underbrace{\sum_{j=0}^k (-1)^j \binom{k}{j} (-j-s)_+^\alpha}_{= \blacktriangle_{0,1}^k g(s)} \Big|^2 ds = \frac{K}{n^{2\alpha+1}}, \end{aligned}$$

where $K > 0$ is a suitable constant.

(ii) Let the function g be as in Lemma 1.5.1 with $g^{(k)} \in L^2((\delta, \infty))$ and denote all positive constants by K. Then for sufficiently large $n \in \mathbb{N}$ and $i = k, \dots, n$ an application of Lemma 1.5.1 yields

$$\int_{\frac{i-(k+1)}{n}}^{\frac{i}{n}} | \blacktriangle_{i,n}^k g(s)|^2 ds \leq n^{-2\alpha-1} K \int_{-(k+1)}^0 (-s)^{2\alpha} ds$$

$$\int_{\frac{i}{n}-\delta}^{\frac{i-(k+1)}{n}} |\blacktriangle_{i,n}^{k} g(s)|^2 ds \leq K n^{-2\alpha-1} \int_{k-n\delta}^{-1} (-s)^{2(\alpha-k)} ds,$$

$$\int_{\frac{i-k}{n}-\delta}^{\frac{i}{n}-\delta} |\blacktriangle_{i,n}^{k} g(s)|^2 ds \leq K n^{-2k-1}$$

as well as

$$\int_{-\infty}^{\frac{i-k}{n}-\delta} |\blacktriangle_{i,n}^{k} g(s)|^2 ds \leq K n^{-2k} \int_{-\infty}^{-\delta} \left| g^{(k)}(-s) \right|^2 ds.$$

By assuming $\alpha - k < -1/2$, the above inequalities in combination with assumption ② imply

$$\| \blacktriangle_{i,n}^{k} g \|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\frac{i}{n}} |\blacktriangle_{i,n}^{k} g(s)|^2 ds \leq K n^{-2\alpha-1},$$

where $n \in \mathbb{N}$ is sufficiently large and $i = k, \ldots, n$.

(iii) Note that Remark 2.2.2 (i) and (ii) allow us, under suitable assumptions on the parameters $\alpha_{(\cdot)}, k_{(\cdot)}, \theta_{(\cdot)}$ and $p_{(\cdot)}$, to apply Theorem 2.2.1 with the kernel functions introduced in Proposition 1.2.3.

Now we will show that in the case of the driving process being a symmetric α-stable Lévy process a similar argumentation as in the proof of Theorem 2.2.1 would not work.

Remark 2.2.3. In the following we will write $(\cdot)$ instead of j for $j = 1, \ldots, M$.

By taking a look at the proof of Theorem 2.2.1, one can easily get the idea to obtain a similar result to Theorem 2.2.1 in the case of $(L_t)_{t\in\mathbb{R}}$ being a symmetric β-stable Lévy process by modifying some of the assumptions on the parameters $p_{(\cdot)}$ and by using Remark 1.2.5 (iii).

The proof of such a modified result would require us, in order to use Remark 1.2.5 (iii), to assume

$$\| \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)} \|_{L^\beta(\mathbb{R})}^{\beta} \leq n^{-\alpha_{(\cdot)}\beta-\beta/\widehat{\beta}} K \tag{2.2.2}$$

instead of

$$\| \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)} \|_{L^2(\mathbb{R})}^2 \le n^{-2\alpha_{(\cdot)}-1} K$$

for all sufficiently large $n \in \mathbb{N}$ and $i = k_{(\cdot)}, \ldots, n$, where $K > 0$ is a suitable constant and $0 < \widehat{\beta} < \beta$.

In the following we will show that assumption (2.2.2) contradicts the assumptions ② and ④, i.e. under assumption (2.2.2) it becomes impossible to use Theorem 2.1.1 and therefore to proceed as in the proof of Theorem 2.2.1.

Note that by assumption ② and ④ we have

$$g_{(\cdot)}(s) = (s)_+^{\alpha_{(\cdot)}} f_{(\cdot)}(s)$$

for $s \in \mathbb{R}$ with $f_{(\cdot)}(t) \to c_{(\cdot)} \neq 0$ as $t \downarrow 0$.

Moreover, for all $\varepsilon \in (0,1)$, $n \in \mathbb{N}$ and $s \in (i/n - \varepsilon/n, i/n)$ the above representation in combination with (1.3.4) yields

$$\blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)}(s) = g_{(\cdot)}(i/n - s) = (i/n - s)_+^{\alpha_{(\cdot)}} f_{(\cdot)}(i/n - s).$$

Hence, by fixing a sufficiently small $\varepsilon \in (0,1)$ with $|f_{(\cdot)}(s)| \ge |c_{(\cdot)}|/2$ for all $s \in (0, \varepsilon)$, we get

$$\int_{\frac{i-\varepsilon}{n}}^{\frac{i}{n}} \left| \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)}(s) \right|^\beta ds = n^{-\alpha_{(\cdot)}\beta - 1} \int_0^\varepsilon \left| (s)_+^{\alpha_{(\cdot)}} f_{(\cdot)}(s/n) \right|^\beta ds \ge n^{-\alpha_{(\cdot)}\beta - 1} \frac{|c_{(\cdot)}/2|^\beta}{\alpha_{(\cdot)}\beta + 1} \varepsilon^{\alpha_{(\cdot)}\beta + 1}$$

for all $n \in \mathbb{N}$.

The above inequality and (2.2.2) lead to the following contradiction

$$0 < K \le n^{1-\beta/\widehat{\beta}} \underset{n\to\infty}{\longrightarrow} 0,$$

where $K > 0$ is a suitable constant.

2.2.1. Proof of Theorem 2.2.1. Note that in this proof we will use some of the technical results that can be found in the last section of this chapter.

In order to ease our notations we will throughout this proof denote all positive constants by K, although they may change from line to line. Furthermore, we will assume $n \in \mathbb{N}$ to be sufficiently large and for $j = 1, \dots, M$ we will often write $(\cdot)$ instead of j respectively $\langle\cdot\rangle$ instead of $\langle j \rangle$.

Note that the Lévy measure ν and the parameter $\theta_{(\cdot)}$ satisfy assumption ①, cf. Lemma 1.2.2.

For each $d \in \mathbb{N}$ the stochastic process $\widehat{L}(d)$ given by

$$\widehat{L}_t(d) - \widehat{L}_s(d) = \sum_{u \in (s,t]} \Delta L_u \mathbf{1}_{\{|\Delta L_u| > 1/d\}} \qquad \text{for } s < t$$

is a symmetric compound Poisson process and $(T_m(d))_{m \in \mathbb{N}}$ defined by

$$T_m(d) := \begin{cases} T_m & \text{if } |\Delta L_{T_m}| > 1/d \\ \infty & \text{else} \end{cases}$$

is a sequence of stopping times that exhausts the jumps of $(\widehat{L}_t(d))_{t \geq 0}$.

Since

$$T_m(d) \in [0,1] \iff T_m \in [0,1] \text{ and } |\Delta L_{T_m}| > \frac{1}{d}$$

we get in the setting of (i)

$$\begin{aligned} Z_d &= \sum_{m: T_m(d) \in [0,1]} |\Delta \widehat{L}_{T_m(d)}(d)|^{\sum_{j=1}^M p_j} H_m^{\langle M \rangle} \\ &= \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} \mathbf{1}_{\{|\Delta L_{T_m}| > 1/d\}} H_m^{\langle M \rangle} \\ \xrightarrow[d \to \infty]{a.s.} Z &= \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} H_m^{\langle M \rangle} \end{aligned} \tag{2.2.3}$$

and in the setting of (ii)

$$\begin{aligned} \widehat{Z}_d &= C^{\langle M \rangle} \sum_{m: T_m(d) \in [0,1]} |\Delta \widehat{L}_{T_m(d)}(d)|^{\sum_{j=1}^M p_j} \\ &= C^{\langle M \rangle} \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} \mathbf{1}_{\{|\Delta L_{T_m}| > 1/d\}} \\ \xrightarrow[d \to \infty]{a.s.} \widehat{Z} &= C^{\langle M \rangle} \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j}, \end{aligned} \tag{2.2.4}$$

where the convergences follow from the dominated convergence theorem and the fact that $Z, \widehat{Z} < \infty$ a.s., since $H_m^{\langle M \rangle} < K$ a.s. and $\sum_{j=1}^M p_j > \beta$.

Moreover, for each $d \in \mathbb{N}$ we define the compound Poisson driven process

$$\widehat{X}_t^{\langle \cdot \rangle}(d) := \int_{-\infty}^t g_{(\cdot)}(t-s) - \tilde{g}_{(\cdot)}(-s)\; d\widehat{L}_s(d), \qquad t \geq 0,$$

and the stochastic process

$$X^{\langle \cdot \rangle}(d) := X^{\langle \cdot \rangle} - \widehat{X}^{\langle \cdot \rangle}(d), \tag{2.2.5}$$

where as stated in [**10**, below (4.35)] the process $X^{\langle \cdot \rangle}(d)$ is of the form (2.3.2).

For $I := \{1, \dots, M\}$ and the parameter r as in Details 2.0.1 we have

$$\begin{aligned}
&\left| \left(\sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j} \right)^{\frac{1}{r}} - \left(\sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \triangle_{i+a_j,n}^{k_j} \widehat{X}^{\langle j \rangle}(d) \right|^{p_j} \right)^{\frac{1}{r}} \right| \\
&\leq K \sum_{J \subset I : |J| < M} \left(\sum_{i=k_\star}^{n+k^\star} \left(\prod_{j \in J} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j} \right) \left(\prod_{j \in I \setminus J} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle}(d) \right|^{p_j} \right) \right)^{\frac{1}{r}} \\
&\leq K \left(\sum_{J \subset I : |J| < M} \sum_{i=k_\star}^{n+k^\star} \left(\prod_{j \in J} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j} \right) \left(\prod_{j \in I \setminus J} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle}(d) \right|^{p_j} \right) \right)^{\frac{1}{r}},
\end{aligned} \tag{2.2.6}$$

where the first inequality is a result of multiple applications of Corollary 1.5.4 using (2.2.5) and the second inequality follows by Hölder's inequality.

Let

$$v_n := \begin{cases} n^{\sum_{j=1}^M \alpha_j p_j}, & \text{in the setting of } (i) \\ n^{\sum_{j=1}^M \alpha_j p_j} \big(\log(n) \big)^{-1}, & \text{in the setting of } (ii) \end{cases}.$$

By fixing for each $J \subset I$ with $|J| \leq M-1$ a $\kappa(J) \in I \setminus J$ and by using the decomposition

$$\frac{1}{1} = \sum_{j \in I} \frac{1}{r_j} \text{ with } r_j := \begin{cases} Q_j & \text{if } j = \kappa(J) \\ 2/p_j & \text{else} \end{cases} \tag{2.2.7}$$

as well as the identity

$$v_n = \prod_{j=1}^{M} |v_{j,n}|^{p_j} \text{ with } v_{j,n} := \begin{cases} n^{\alpha_j}, & \text{in the setting of } (i) \\ n^{\alpha_j}\big(\log(n)\big)^{-\frac{1}{r_j p_j}}, & \text{in the setting of } (ii) \end{cases},$$

we obtain the following inequalities.

$$\begin{aligned}
&\sum_{i=k_\star}^{n+k^\star} \mathbb{E}\Bigg(\Big(\prod_{j\in J}\Big|\underbrace{v_{j,n}\,\triangle_{i+a_j,n}^{k_j} X^{\langle j\rangle}}_{=:x_{i,j,n}}\Big|^{p_j}\Big)\Big(\prod_{j\in I\setminus J}\Big|\underbrace{v_{j,n}\,\triangle_{i+a_j,n}^{k_j} X^{\langle j\rangle}(d)}_{=:x_{i,j,n,d}}\Big|^{p_j}\Big)\Bigg)\\
&=\sum_{i=k_\star}^{n+k^\star}\Big\|\Big(\prod_{j\in J}|x_{i,j,n}|^{p_j}\Big)\Big(\prod_{j\in I\setminus J}|x_{i,j,n,d}|^{p_j}\Big)\Big\|_{L^1(\Omega)}\\
&\le K\sum_{i=k_\star}^{n+k^\star}\Big(\prod_{j\in J}\big\||x_{i,j,n}|^{p_j}\big\|_{L^{r_j}(\Omega)}\Big)\Big(\prod_{j\in I\setminus J}\big\||x_{i,j,n,d}|^{p_j}\big\|_{L^{r_j}(\Omega)}\Big)\\
&\le K\sum_{i=k_\star}^{n+k^\star}\Big(\prod_{j\in J} n^{-\frac{1}{r_j}}\Big)\Big(\prod_{j\in I\setminus J}\big\||x_{i,j,n,d}|^{r_jp_j}\big\|_{L^1(\Omega)}^{\frac{1}{r_j}}\Big)\\
&\le K\prod_{j\in I\setminus J}\Big(\sum_{i=k_\star}^{n+k^\star}\big\||x_{i,j,n,d}|^{r_jp_j}\big\|_{L^1(\Omega)}\Big)^{\frac{1}{r_j}}\\
&=K\prod_{j\in I\setminus J}\Big(|v_{j,n}|^{r_jp_j}\sum_{i=k_\star}^{n+k^\star}\mathbb{E}\Big(\Big|\triangle_{i+a_j,n}^{k_j} X^{\langle j\rangle}(d)\Big|^{r_jp_j}\Big)\Big)^{\frac{1}{r_j}},
\end{aligned} \tag{2.2.8}$$

where the first as well as the third inequality are a consequence of Lemma 1.5.2 and the second follows from (2.2.9) below.

Since on J we have $p_{(\cdot)}r_{(\cdot)} = 2$, an application of [**21**, Proposition 2.1] using

$$\triangle_{i+a_{(\cdot)},n}^{k_{(\cdot)}} X^{\langle\cdot\rangle} = \int_{-\infty}^{\infty} \blacktriangle_{i+a_{(\cdot)},n}^{k_{(\cdot)}} g_{(\cdot)}(s)\, dL_s$$

and the same argumentation as in Remark 1.2.5 (ii) yields

$$\begin{aligned}
\frac{\big\||x_{i,(\cdot),n}|^{p_{(\cdot)}}\big\|_{L^{r_{(\cdot)}}(\Omega)}^{r_{(\cdot)}}}{|v_{(\cdot),n}|^2} &= \mathbb{E}\Big(\triangle_{i+a_{(\cdot)},n}^{k_{(\cdot)}} X^{\langle\cdot\rangle}\Big)^2 = \mathbb{E}\,(L_1)^2\,\|\blacktriangle_{i+a_{(\cdot)},n}^{k_{(\cdot)}}\, g_{(\cdot)}\|_{L^2(\mathbb{R})}^2\\
&\le n^{-2\alpha_{(\cdot)}-1}K.
\end{aligned} \tag{2.2.9}$$

Now by choosing the parameter r as in Details 2.0.1 and by setting

$$V_n := v_n \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j} \quad \text{and} \quad V_{n,d} := v_n \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \triangle_{i+a_j,n}^{k_j} \widehat{X}^{\langle j \rangle}(d) \right|^{p_j},$$

we obtain

$$\lim_{d\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\left| (V_n)^{\frac{1}{r}} - (V_{n,d})^{\frac{1}{r}} \right| > \varepsilon \right) = 0 \tag{2.2.10}$$

for each $\varepsilon > 0$, where in the following specified order we applied (2.2.6), Markov's inequality, (2.2.8) and, since we have $k_\star + a_{(\cdot)} \geq k_{(\cdot)}$ as well as $k^\star + a_{(\cdot)} \leq 0$, a combination of Lemma 2.3.5 and Lemma 2.3.4.

Note that for each $d \in \mathbb{N}$ in the setting of (i) respectively (ii) we have

$$V_{n,d} \xrightarrow[n\to\infty]{L-s} Z_d \quad \text{respectively} \quad V_{n,d} \xrightarrow[n\to\infty]{a.s.} \widehat{Z}_d, \tag{2.2.11}$$

which is due to a combination of Lemma 2.3.5 and Theorem 2.1.1.

In the setting of (i) a combination of (2.2.11), (2.2.3), (2.2.10) and [**12**, Theorem 3.2] yields

$$(V_n)^{\frac{1}{r}} \xrightarrow[n\to\infty]{L-s} (Z)^{\frac{1}{r}} \quad \text{respectively} \quad V_n \xrightarrow[n\to\infty]{L-s} Z.$$

In the setting of (ii) an application of (2.2.11) and (2.2.4) results in

$$(V_{n,d})^{\frac{1}{r}} - (\widehat{Z})^{\frac{1}{r}} \xrightarrow[n\to\infty]{a.s.} (\widehat{Z}_d)^{\frac{1}{r}} - (\widehat{Z})^{\frac{1}{r}} \xrightarrow[d\to\infty]{a.s.} 0,$$

which in combination with (2.2.10) and [**12**, Theorem 3.2] implies that

$$(V_n)^{\frac{1}{r}} - (\widehat{Z})^{\frac{1}{r}} \xrightarrow[n\to\infty]{d} 0,$$

i.e.

$$(V_n)^{\frac{1}{r}} \xrightarrow[n\to\infty]{\mathbb{P}} (\widehat{Z})^{\frac{1}{r}} \quad \text{respectively} \quad V_n \xrightarrow[n\to\infty]{\mathbb{P}} \widehat{Z}.$$

□

2.3. Some technical auxiliary results

The purpose of this chapter is to provide a comprehensive overview of the technical auxiliary results that we have used in the presiding sections of this chapter.

LEMMA 2.3.1. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a (two-sided) compound Poisson process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(T_m)_{m\in\mathbb{N}}$ be a sequence of $\mathbb{F}$-stopping times, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is the filtration generated by $(L_t)_{t\geq 0}$, that satisfies $0 \leq T_1 < T_2 < \ldots$ and exhausts the jumps of $(L_t)_{t\geq 0}$, i.e.*

- $\{T_m(\omega) : m \geq 1\} \cap [0, \infty) = \{t \geq 0 : \Delta L_t \neq 0\}$,
- $T_n(\omega) \neq T_m(\omega)$ *for* $n \neq m$ *with* $T_n(\omega) < \infty$.

Furthermore, for $\varepsilon > 0$, $m, n \in \mathbb{N}$ and $\omega \in \Omega$ we denote

$$i_m := i_m(\omega, n) \in \mathbb{N} \cup \{0\} \text{ as a random index satisfying } T_m \in \left(\frac{i_m - 1}{n}, \frac{i_m}{n}\right],$$

$$\{nT_m\} := nT_m - (i_m - 1)$$

and

$$\Omega_\varepsilon := \Big\{\omega \in \Omega : \text{for all } j \geq 1 \text{ with } T_j(\omega) \in [0,1] \text{ we have } |T_{j+1}(\omega) - T_j(\omega)| > \varepsilon \text{ and } \Delta L_s(\omega) = 0 \text{ for all } s \in [-\varepsilon, \varepsilon] \cup [1-\varepsilon, 1+\varepsilon]\Big\},$$

where $\Delta L_t := L_t - L_{t-}$ with $L_{t-} := \lim_{s\uparrow t, s<t} L_s$ are the jumps of L at time t.

Then we obtain the following results.

(*i*) *For each $d \in \mathbb{N}$ we have*

$$\big(\{nT_m\}\big)_{m\in\mathbb{N}, m\leq d} \xrightarrow{L-s} \big(\widetilde{U}_m\big)_{m\in\mathbb{N}, m\leq d} \quad \text{as } n \to \infty,$$

where $(\widetilde{U}_m)_{m\in\mathbb{N}}$ is a sequence of independent and uniform $[0,1]$-distributed random variables that on the one hand lives on the probability space $(\tilde{\Omega}, \widetilde{\mathcal{F}}, \tilde{\mathbb{P}})$, which is an extension of the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and on the other hand is independent of the σ-algebra $\mathcal{F}$.

(*ii*) *Let* $i, n \in \mathbb{N}$, $\varepsilon > 4/n$. *Then on the set* Ω_ε *in the case of* $T_m \in [0,1]$ *we have*

$$T_m \in \left(\frac{i}{n} - \frac{\varepsilon}{2}, \frac{i}{n}\right] \Longrightarrow T_{\widetilde{m}} \notin \left(\frac{i}{n} - \frac{\varepsilon}{2}, \frac{i}{n}\right] \text{ for } m \neq \widetilde{m}$$

and

$$T_m \in \left(\frac{i}{n} - \frac{\varepsilon}{2}, \frac{i}{n}\right] \Longleftrightarrow i \in \left\{i_m, i_m + 1, \ldots, i_m + \left\lfloor \frac{\varepsilon n}{2} \right\rfloor + v_m\right\} \text{ with}$$

$$v_m := \begin{cases} 0 & \text{if } \frac{1}{n}\left(i_m + \left\lfloor \frac{\varepsilon n}{2} \right\rfloor\right) - \frac{\varepsilon}{2} < T_m \\ -1 & \text{else} \end{cases},$$

where $\lfloor \cdot \rfloor$ *denotes the floor function.*

Proof. *To (i):* See [**10**, (4.4)].

To (ii): The first statement is a consequence of the definition of the set $\widehat{\Omega}_\varepsilon$ and the equivalence can be seen as follows.

$\Longrightarrow$**:** In the case of $i < i_m$, i.e. $i \le i_m - 1$, we have

$$\left(\frac{i_m}{n} - \frac{1}{n}, \frac{i_m}{n}\right] \cap \left(\frac{i}{n} - \frac{\varepsilon}{2}, \frac{i}{n}\right] = \varnothing. \tag{2.3.1}$$

If $i > i_m + \lfloor \varepsilon n/2 \rfloor + v_m$, then in the case of $v_m = 0$ we have (2.3.1), since

$$\frac{i}{n} - \frac{\varepsilon}{2} \ge \frac{i_m}{n} + \overbrace{\frac{1 + v_m}{n}}^{=1/n} + \overbrace{\frac{\lfloor \varepsilon n/2 \rfloor}{n} - \frac{\varepsilon}{2}}^{\in(-1/n,0]} > \frac{i_m}{n},$$

and in the case of $v_m = -1$, i.e. $\frac{1}{n}\left(i_m + \left\lfloor \frac{\varepsilon n}{2} \right\rfloor\right) - \frac{\varepsilon}{2} \ge T_m$, we have

$$\frac{i}{n} - \frac{\varepsilon}{2} \ge \frac{i_m}{n} + \overbrace{\frac{1 + v_m}{n}}^{=0} + \frac{\lfloor \varepsilon n/2 \rfloor}{n} - \frac{\varepsilon}{2} \ge T_m \quad \Longrightarrow \quad T_m \notin \left(\frac{i}{n} - \frac{\varepsilon}{2}, \frac{i}{n}\right].$$

$\Longleftarrow$**:** It is sufficient to show for $j_m := i_m + \lfloor \varepsilon n/2 \rfloor + v_m$ that

$$T_m \in \left(\frac{j_m}{n} - \frac{\varepsilon}{2}, \frac{j_m}{n}\right],$$

since for each $i \in \mathbb{N}$ with $i_m \le i \le j_m$ we have

$$T_m \in \left(\frac{i_m}{n} - \frac{\varepsilon}{2}, \frac{i_m}{n}\right] \cap \left(\frac{j_m}{n} - \frac{\varepsilon}{2}, \frac{j_m}{n}\right] \subset \left(\frac{j_m}{n} - \frac{\varepsilon}{2}, \frac{i_m}{n}\right] \subset \left(\frac{i}{n} - \frac{\varepsilon}{2}, \frac{i}{n}\right],$$

which follows from $\varepsilon > 4/n$ as well as

$$T_m \in \left(\frac{i_m}{n} - \frac{1}{n}, \frac{i_m}{n}\right] \subset \left(\frac{i_m}{n} - \frac{\varepsilon}{2}, \frac{i_m}{n}\right].$$

In the case of $v_m = 0$, i.e. $\frac{1}{n}\left(i_m + \left\lfloor \frac{\varepsilon n}{2} \right\rfloor\right) - \frac{\varepsilon}{2} < T_m$, we have

$$\frac{i_m}{n} + \frac{v_m}{n} + \frac{\lfloor \varepsilon n/2 \rfloor}{n} - \frac{\varepsilon}{2} < T_m \leq \frac{i_m}{n} \leq \frac{i_m}{n} + \frac{v_m}{n} + \frac{\lfloor \varepsilon n/2 \rfloor}{n}$$

and in the case of $v_m = -1$, since $\varepsilon > 4/n$, we have

$$\frac{i_m}{n} + \frac{v_m}{n} + \frac{\lfloor \varepsilon n/2 \rfloor}{n} - \frac{\varepsilon}{2} \leq \frac{i_m}{n} - \frac{1}{n} < T_m \leq \frac{i_m}{n} \leq \frac{i_m}{n} + \frac{v_m}{n} + \frac{\lfloor \varepsilon n/2 \rfloor}{n}.$$

□

The following two lemmas, which play a crucial role in the proof of Theorem 2.1.1, focus on some properties of the Lévy driven process respectively of it's decomposition into a dominant part and a rest part.

LEMMA 2.3.2. *Suppose that the kernel functions $g, \tilde{g}$ and the symmetric compound Poisson process $(L_t)_{t\in\mathbb{R}}$ satisfy the assumptions ⓪ to ⑤ with respect to the parameters α, c, k, θ and in the case of $\theta = 1$ the assumption ⑥ as well. Moreover, let the set Ω_ε be as in Lemma 2.3.1. Then we have for $k, n \in \mathbb{N}$ with $n \geq k$, $\varepsilon > 4/n$,*

$$X_t := \int_{-\infty}^{t} g(t-s) - \tilde{g}(-s)\, dL_s,$$

$$M_{i,n,\varepsilon} := \int_{\frac{i}{n}-\frac{\varepsilon}{2}}^{\frac{i}{n}} \blacktriangle_{i,n}^{k} g(s)\, dL_s \quad \text{and} \quad R_{i,n,\varepsilon} := \int_{-\infty}^{\frac{i}{n}-\frac{\varepsilon}{2}} \blacktriangle_{i,n}^{k} g(s)\, dL_s$$

the following results.

(*i*) *If $\alpha < k - 1/p$ for some $p > 0$ then we have*

$$n^{\alpha p} \sum_{i=k}^{n} \left|R_{i,n,\varepsilon}\right|^p \xrightarrow[n\to\infty]{\mathbb{P}} 0,$$

$$n^{\alpha p} \sum_{i=k}^{n} \left|\Delta_{i,n}^{k} X\right|^p \xrightarrow[n\to\infty]{L-s} Y \quad \text{and} \quad n^{\alpha p} \sum_{i=k}^{n} \left|M_{i,n,\varepsilon}\right|^p \xrightarrow[n\to\infty]{L-s} Y$$

on Ω_ε, where Y is the random variable that we would get by applying Theorem 2.1.1 (i) for $M = 1$ and $a_1 = 0$.

(*ii*) *Assume that the function* $f : [0,\infty) \to \mathbb{R}$ *given by* $f(t) = g(t)t^{-\alpha}$ *for* $t > 0$ *satisfies* $f \in C^k([0,\infty))$ *with* $f(0) = c$*. If for some* $p > 0$ *we have* $\alpha = k - 1/p$ *and* $1/p + 1/\theta > 1$ *then we obtain*

$$\frac{n^{\alpha p}}{\log(n)} \sum_{i=k}^{n} |R_{i,n,\varepsilon}|^p = \mathcal{O}\left(\frac{1}{\log(n)}\right)^{\tau(\theta,p)} \qquad a.s. \text{ as } n \to \infty,$$

$$\frac{n^{\alpha p}}{\log(n)} \sum_{i=k}^{n} \left|\Delta_{i,n}^{k} X\right|^p \xrightarrow[n\to\infty]{a.s.} \widehat{Y} \qquad \text{and} \qquad \frac{n^{\alpha p}}{\log(n)} \sum_{i=k}^{n} |M_{i,n,\varepsilon}|^p \xrightarrow[n\to\infty]{a.s.} \widehat{Y}$$

on Ω_ε*, where*

$$\tau(\theta,p) := \begin{cases} -p\left(1 - \frac{1}{\theta} - \frac{1}{p}\right), & \text{for } \theta \in (1,2] \\ 1, & \text{for } \theta \in (0,1] \end{cases}$$

and $\widehat{Y}$ *is the random variable that we would get by applying Theorem 2.1.1* (*ii*) *for* $M = 1$ *and* $a_1 = 0$.

Proof. With $M_{i,n,\varepsilon/2}$ instead of $M_{i,n,\varepsilon}$ as well as with $R_{i,n,\varepsilon/2}$ instead of $R_{i,n,\varepsilon}$ an application of [**10**, proof of (4.1), (4.7) and (4.17)] yields (*i*) and an application of [**11**, proof of (4.2), (4.14) and (4.16)] yields (*ii*). □

Lemma 2.3.3. *Let the set* Ω_ε *and the sequence* $(T_m)_{m\in\mathbb{N}}$ *be as in Lemma 2.3.1. Moreover, suppose that* $(M_{l,n,\varepsilon})_{l\in\mathbb{Z}}$ *is defined as in Lemma 2.3.2, i.e. under the same assumptions on the kernel functions* $g, \tilde{g}$ *and the symmetric compound Poisson process* $(L_t)_{t\in\mathbb{R}}$*. Then for* $n \in \mathbb{N}$ *and* $\varepsilon > 4/n$ *we get on the set* Ω_ε *the following results.*

(*i*) *Let* $i \in \mathbb{N}$*. If* $T_m \in [0,1] \cap \left(\frac{i}{n} - \frac{\varepsilon}{2}, \frac{i}{n}\right]$ *then we deduce that*

$$M_{i,n,\varepsilon} = \Delta L_{T_m} \blacktriangle_{i,n}^{k} g(T_m)$$

and

$$i \in \left\{i_m, i_m + 1, \ldots, i_m + \left\lfloor \frac{\varepsilon n}{2} \right\rfloor + v_m\right\}.$$

Furthermore, we have $M_{i,n,\varepsilon} = 0$ *in the case of* $T_m \notin [0,1] \cap \left(\frac{i}{n} - \frac{\varepsilon}{2}, \frac{i}{n}\right]$ *for all* $m \in \mathbb{N}$.

(*ii*) *For* $l, r = 0, \ldots, \left\lfloor \frac{\varepsilon n}{2} \right\rfloor$ *we have* $M_{l+r,n,\varepsilon} = M_{-l,n,\varepsilon} = M_{n+l+r,n,\varepsilon} = M_{n-l,n,\varepsilon} = 0$.

(*iii*) *Let* $T_m \in [0,1]$ *then it holds that* $M_{i_m - l,n,\varepsilon} = M_{i_m + \lfloor \varepsilon n/2 \rfloor + v_m + l,n,\varepsilon} = 0$ *for* $l = 0, \ldots, \left\lfloor \frac{\varepsilon n}{2} \right\rfloor$.

Proof. Since the sequence $(T_m)_{m\in\mathbb{N}}$ exhausts the jumps of $(L_t)_{t\geq 0}$ a combination of the definitions of Ω_ε, $M_{i,n,\varepsilon}$ and Lemma 2.3.1 (*ii*) yields (*i*).

Moreover, we obtain (*ii*) and (*iii*) by combining the definitions of $M_{i,n,\varepsilon}$ and in the case of (*iii*) also the equivalence in Lemma 2.3.1 (*ii*) with the fact that on Ω_ε we have on the one hand $\Delta L_s = 0$ for all $s \in [-\varepsilon, \varepsilon] \cup [1-\varepsilon, 1+\varepsilon]$ and on the other hand $|T_m - T_{m-1}| > \varepsilon$ for $T_m, T_{m-1} \in [0,1]$ respectively $|T_{m+1} - T_m| > \varepsilon$ for $T_m, T_{m+1} \in [0,1]$. □

In order to handle in the proof of Theorem 2.2.1 the rest term, that appears while approximating a Lévy driven process with a compound-Poisson driven process, we will need the following lemma.

Lemma 2.3.4. *Let* $L = (L_t)_{t\in\mathbb{R}}$ *be a symmetric Lévy process without a Brownian component and with Blumenthal-Getoor index* $\beta < 2$. *Suppose that* L *as well as the kernel functions* $g, \tilde{g}$ *satisfy the assumptions* ⓪ *to* ⑤ *with respect to the parameters* α, c, k, θ *and in the case of* $\theta = 1$ *the assumption* ⑥ *as well.*

Moreover, let for each $d \in \mathbb{N}$ *the process* $X(d)$ *be given by*

$$X_t(d) := \int_{(-\infty,t)\times\left[-\frac{1}{d},\frac{1}{d}\right]} \Big(g(t-s) - \tilde{g}(-s)\Big) x \, N(ds,dx), \tag{2.3.2}$$

where $t \geq 0$ *and* N *is the to the Lévy process* L *corresponding Poisson random measure, i.e.* $N(A) := \#\{t : (t, \Delta L_t) \in A\}$ *for all measurable* $A \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$.

(*i*) *If* $\alpha < k - 1/p$ *and* $\beta < p$ *for some* $p > 0$ *then we have*

$$\lim_{d\to\infty} \limsup_{n\to\infty} n^{\alpha p} \sum_{i=k}^{n} \mathbb{E}\left(|\, \Delta_{i,n}^{k} X(d)|^p\right) = 0.$$

(*ii*) *Assume that the function* $f : [0,\infty) \to \mathbb{R}$ *given by* $f(t) = g(t)t^{-\alpha}$ *for* $t > 0$ *satisfies* $f \in C^k([0,\infty))$ *with* $f(0) = c$.

If for some $p > 0$ *we have* $\alpha = k - 1/p$ *and* $1/p + 1/\theta > 1$ *then it holds that*

$$\lim_{d\to\infty} \limsup_{n\to\infty} \frac{n^{\alpha p}}{\log(n)} \sum_{i=k}^{n} \mathbb{E}\left(| \triangle_{i,n}^{k} X(d)|^{p}\right) = 0.$$

PROOF. For each $d \in \mathbb{N}$ the process $X(d)$ is well-defined as stated in [**10**, (4.18)] and [**11**, (4.17)]. Furthermore, we get (i) by applying [**10**, proof of Lemma 4.2] and (ii) by applying [**11**, proof of Lemma 4.1]. □

Note that in the proof of Theorem 2.2.1 we need to use Theorem 2.1.1 and Lemma 2.3.4. In the following lemma we will show that under the assumptions on the parameters in Theorem 2.2.1 the assumptions on the parameters in Theorem 2.1.1 and Lemma 2.3.4 are satisfied.

LEMMA 2.3.5. *Let* $j = 1, \ldots, M$. *In the setting of* (i) *in Theorem 2.2.1 we have*

- $\alpha_j < k_j - 1/(S_j p_j)$,
- $Q_j p_j > \beta$ *and* $\alpha_j < k_j - 1/(Q_j p_j)$,
- $2 > \beta$ *and* $\alpha_j < k_j - 1/2$,

and in the setting of (ii) *in Theorem 2.2.1 we have*

- $Q_j p_j > \beta$ *and* $2 > \beta$,
- $\alpha_j < k_j - 1/(Q_j p_j)$ *if* $Q_j p_j \neq S_j p_j$,
- $\alpha_j < k_j - 1/2$ *if* $Q_j p_j \neq S_j p_j$,
- $\alpha_j = k_j - 1/(S_j p_j)$ *and* $1/\theta_j + 1/(S_j p_j) > 1$.

PROOF. In order to ease our notations we will for $j = 1, \ldots, M$ throughout this proof write $(\cdot)$ instead of j.

In both settings we have on the one hand by assumption $2 > \beta$ and on the other hand $Q_{(\cdot)} p_{(\cdot)} > \beta$ due to $p_{(\cdot)} \in (0, 2)$ and the definition of $Q_{(\cdot)}$.

In the setting of (i) we get

$$\alpha_{(\cdot)} < \min\left\{k_{(\cdot)} - \frac{1}{S_{(\cdot)} p_{(\cdot)}}, k_{(\cdot)} - \frac{1}{Q_{(\cdot)} p_{(\cdot)}}, k_{(\cdot)} - \frac{1}{2}\right\}$$

in the *'either'*-case by combining

$$\alpha_{(\cdot)} < \min\left\{k_{(\cdot)} - \frac{1}{S_{(\cdot)}p_{(\cdot)}}, k_{(\cdot)} - \frac{1}{2}\right\} \quad \text{with} \quad Q_{(\cdot)}p_{(\cdot)} \geq \min\{S_{(\cdot)}p_{(\cdot)}, 2\}$$

and in the *'or'*-case by combining

$$\alpha_{(\cdot)} < k_{(\cdot)} - \frac{1}{p_{(\cdot)}} \quad \text{with} \quad p_{(\cdot)} \leq \min\{S_{(\cdot)}p_{(\cdot)}, Q_{(\cdot)}p_{(\cdot)}, 2\}.$$

Note that in the setting of (ii) we have by assumption $\alpha_{(\cdot)} = k_{(\cdot)} - 1/(S_{(\cdot)}p_{(\cdot)})$ as well as $1/\theta_{(\cdot)} + 1/(S_{(\cdot)}p_{(\cdot)}) > 1$. Moreover, since in the setting of (ii) we have $\alpha_{(\cdot)} = k_{(\cdot)} - 1/(S_{(\cdot)}p_{(\cdot)})$, we get by using the assumption $\min\{Q_{(\cdot)}p_{(\cdot)}, 2\} \geq S_{(\cdot)}p_{(\cdot)}$ that $\alpha_{(\cdot)} < k_{(\cdot)} - 1/(Q_{(\cdot)}p_{(\cdot)})$ in the case of $Q_{(\cdot)}p_{(\cdot)} \neq S_{(\cdot)}p_{(\cdot)}$ and $\alpha_{(\cdot)} < k_{(\cdot)} - 1/2$ in the case of $2 \neq S_{(\cdot)}p_{(\cdot)}$. □

CHAPTER 3

A Consistency Theorem for Multipower Variations where the Lévy Driven Process is Driven by a Symmetric α-Stable Lévy Process

In this chapter we will use the various properties of symmetric α-stable Lévy processes respectively of processes driven by symmetric α-stable Lévy processes to derive a natural extension of the results for power variations in [**10**, Theorem 1.1 (ii)] to the multipower variations case. In our proof we will, in order to use Birkhoff's ergodic theorem, proceed similarly to [**10**].

Note that [**10**, Theorem 1.1 (ii)] was used by the respective authors to proof the second-order asymptotic results in [**10**, Theorem 1.2], regrettably in the case of multipower variations there arise problems that prevent us from obtaining a similar result while using their approach.

THEOREM 3.0.1. *Suppose that the kernel functions $g, \tilde{g}$ as well as the symmetric β-stable Lévy process $(L_t)_{t\in\mathbb{R}}$ with $\beta < 2$ and scale parameter $\sigma > 0$, i.e. $\mathbb{E}\big(\exp(iuL_1)\big) = \exp(-\sigma^\beta |u|^\beta)$ for all $u \in \mathbb{R}$, satisfy the assumptions ⓪ to ⑤ with respect to the parameters α, c, k, θ, where $\theta = \beta$. Moreover, set*

$$X_t := \int_{-\infty}^{t} g(t-s) - \tilde{g}(-s)\, dL_s \quad for \quad t \geq 0$$

and assume for $j = 1, \ldots, M$ that $a_j \in \mathbb{Z}$, $k_j \in \mathbb{N}$ with $k \leq k_j$ and $p_j \in (0, \beta)$.

If $\alpha < k - 1/\beta$ and $\sum_{j=1}^{M} p_j < \beta$ then we have

$$n^{-1+(\alpha+1/\beta)\sum_{j=1}^{M} p_j} V_n^{\langle M \rangle} \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}\left(\prod_{j=1}^{M} \left| \sum_{l=0}^{k_j-k} (-1)^l \begin{pmatrix} k_j - k \\ l \end{pmatrix} Y^{\infty}_{k_\star + a_j - l} \right|^{p_j} \right),$$

where $k_\star$ is as in Definition 1.3.2,

$$V_n^{\langle M\rangle} := \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \Delta_{i+a_j,n}^{k_j} X \right|^{p_j} \qquad \textit{(cf. Definition 1.3.2),}$$

and

$$Y_t^\infty := \int_{-\infty}^{\infty} c \sum_{l=0}^{k} (-1)^l \binom{k}{l} (t-s-l)_+^\alpha \; dL_s \quad \textit{for} \quad t \geq 0.$$

The next remark exemplarily illustrates some kernel functions for which the above theorem is applicable.

Remark 3.0.2. Note that Theorem 3.0.1 is applicable with the kernel functions presented in Proposition 1.2.3.

3.1. Proof of Theorem 3.0.1

Throughout this proof we will denote all positive constants by K, although they may change from line to line. Furthermore, for $j = 1, \ldots, M$ we will often write $(\cdot)$ instead of j and $\langle\cdot\rangle$ instead of $\langle j\rangle$.

Since the assumptions in [**10**, Theorem 1.1 (ii)] are satisfied, we define

$$Y_t^n := \int_{-\infty}^{\infty} n^\alpha \sum_{l=0}^{k} (-1)^l \binom{k}{l} g\left(\frac{t-s-l}{n}\right) \; dL_s \quad \text{for} \quad t \geq 0$$

and

$$Y_t^\infty := \int_{-\infty}^{\infty} c \sum_{l=0}^{k} (-1)^l \binom{k}{l} (t-s-l)_+^\alpha \; dL_s \quad \text{for} \quad t \geq 0$$

as in [**10**, (4.38)].

Note that by [**10**, (4.39)] we have

$$\left\{ n^{\alpha+1/\beta} \, \Delta_{i,n}^k X : i = k, \ldots, n \right\} \stackrel{d}{=} \left\{ Y_i^n : i = k, \ldots, n \right\} \tag{3.1.1}$$

for all $n \in \mathbb{N}$ with $n \geq k$.

By combining the two representations of $\triangle_{i,n}^{k_{(\cdot)}} X$, cf. (1.3.1) and (1.3.3), we obtain for $i = k_\star + a_{(\cdot)}, \dots, n + k^\star + a_{(\cdot)}$ the following representation

$$\triangle_{i,n}^{k_{(\cdot)}} X = \sum_{l=0}^{k_{(\cdot)}-k} (-1)^l \begin{pmatrix} k_{(\cdot)} - k \\ l \end{pmatrix} \triangle_{i-l,n}^{k} X = \sum_{l=0}^{k_{(\cdot)}-k} b_l^{\langle\cdot\rangle} \triangle_{i-l,n}^{k} X. \tag{3.1.2}$$

Now we define by using $\widehat{\beta} := \sum_{j=1}^M p_j < \beta$ the parameters $Q_{(\cdot)} := \widehat{\beta}/p_{(\cdot)}$, which by definition satisfy $Q_{(\cdot)} \geq 1$, $Q_{(\cdot)}p_{(\cdot)} < \beta$ and $\sum_{j=1}^M 1/Q_j = 1$.

An application of [**10**, (4.45),(4.46) and the convergence below (4.46)] with $p = Q_{(\cdot)}p_{(\cdot)} < \beta$ yields

$$\frac{1}{n}\sum_{i=k}^{n} \left|Y_i^n - Y_i^\infty\right|^{Q_{(\cdot)}p_{(\cdot)}} \xrightarrow[n\to\infty]{\mathbb{P}} 0, \tag{3.1.3}$$

$$\frac{1}{n}\sum_{i=k}^{n} \left|Y_i^n\right|^{Q_{(\cdot)}p_{(\cdot)}} \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}\left|Y_k^\infty\right|^{Q_{(\cdot)}p_{(\cdot)}} \text{ and } \frac{1}{n}\sum_{i=k}^{n} \left|Y_i^\infty\right|^{Q_{(\cdot)}p_{(\cdot)}} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\left|Y_k^\infty\right|^{Q_{(\cdot)}p_{(\cdot)}}, \tag{3.1.4}$$

where by [**10**, (4.46)] we have $\mathbb{E}|Y_k^\infty|^{Q_{(\cdot)}p_{(\cdot)}} < \infty$.

Furthermore, we have

$$\begin{aligned}\sum_{i=k_\star}^{n+k^\star} \Big| \sum_{l=0}^{k_{(\cdot)}-k} b_l^{\langle\cdot\rangle} Y_{i+a_{(\cdot)}-l}^{\bullet}\Big|^{Q_{(\cdot)}p_{(\cdot)}} &\leq \sum_{i=k_\star}^{n+k^\star} \sum_{l=0}^{k_{(\cdot)}-k} \Big(|k_{(\cdot)} - k + 1||b_l^{\langle\cdot\rangle}||Y_{i+a_{(\cdot)}-l}^{\bullet}|\Big)^{Q_{(\cdot)}p_{(\cdot)}} \\ &\leq K \sum_{l=0}^{k_{(\cdot)}-k} \sum_{\underbrace{i=k_\star+a_{(\cdot)}-l}_{\geq k}}^{\overbrace{n+k^\star+a_{(\cdot)}-l}^{\leq n}} |Y_i^\bullet|^{Q_{(\cdot)}p_{(\cdot)}} \leq K \sum_{i=k}^{n} |Y_i^\bullet|^{Q_{(\cdot)}p_{(\cdot)}}\end{aligned} \tag{3.1.5}$$

for $\bullet \in \{n, \infty\}$, and by a similar argumentation we obtain

$$\sum_{i=k_\star}^{n+k^\star} \Big| \sum_{l=0}^{k_{(\cdot)}-k} b_l^{\langle\cdot\rangle} \Big(Y_{i+a_{(\cdot)}-l}^{n} - Y_{i+a_{(\cdot)}-l}^{\infty}\Big)\Big|^{Q_{(\cdot)}p_{(\cdot)}} \leq K \sum_{i=k}^{n} |Y_i^n - Y_i^\infty|^{Q_{(\cdot)}p_{(\cdot)}}. \tag{3.1.6}$$

Let $r = \prod_{j=1}^M (1 \vee p_j)$ and

$$A_n := \left|\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \Big|\underbrace{\sum_{l=0}^{k_j-k} b_l^{\langle j\rangle} Y_{i+a_j-l}^{n}}_{=:B_i^{\langle j\rangle}}\Big|^{p_j}\right)^{\frac{1}{r}} - \left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \Big|\underbrace{\sum_{l=0}^{k_j-k} b_l^{\langle j\rangle} Y_{i+a_j-l}^{\infty}}_{=:\widetilde{B}_i^{\langle j\rangle}}\Big|^{p_j}\right)^{\frac{1}{r}}\right|$$

then we deduce that

$$\begin{aligned} A_n &\leq \sum_{d=1}^{M}\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\left(\prod_{j=d+1}^{M}\left|B_i^{\langle j\rangle}\right|^{p_j}\right)\left(\prod_{j=1}^{d-1}\left|\widetilde{B}_i^{\langle j\rangle}\right|^{p_j}\right)\left|B_i^{\langle d\rangle}-\widetilde{B}_i^{\langle d\rangle}\right|^{p_d}\right)^{\frac{1}{r}} \\ &\leq \sum_{d=1}^{M}\left(\prod_{j=d+1}^{M}\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\left|B_i^{\langle j\rangle}\right|^{Q_jp_j}\right)^{\frac{1}{Q_jr}}\right)\left(\prod_{j=1}^{d-1}\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\left|\widetilde{B}_i^{\langle j\rangle}\right|^{Q_jp_j}\right)^{\frac{1}{Q_jr}}\right) \\ &\qquad \left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\left|B_i^{\langle d\rangle}-\widetilde{B}_i^{\langle d\rangle}\right|^{Q_dp_d}\right)^{\frac{1}{Q_dr}}, \end{aligned} \tag{3.1.7}$$

where the first inequality follows by Corollary 1.5.4 and the second by Lemma 1.5.2.

By combining (3.1.5) to (3.1.7), we get

$$\begin{aligned} A_n &\leq K\sum_{d=1}^{M}\left(\prod_{j=d+1}^{M}\left(\frac{1}{n}\sum_{i=k}^{n}|Y_i^n|^{Q_jp_j}\right)^{\frac{1}{Q_jr}}\right)\left(\prod_{j=1}^{d-1}\left(\frac{1}{n}\sum_{i=k}^{n}|Y_i^\infty|^{Q_jp_j}\right)^{\frac{1}{Q_jr}}\right) \\ &\qquad \left(\frac{1}{n}\sum_{i=k}^{n}|Y_i^n-Y_i^\infty|^{Q_dp_d}\right)^{\frac{1}{Q_dr}} \\ &\xrightarrow[n\to\infty]{\mathbb{P}} 0, \end{aligned} \tag{3.1.8}$$

where the convergence is a consequence of an interplay of (3.1.3), (3.1.4), Slutsky's theorem and the continuous mapping theorem.

Now we define the function $F:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}$ by

$$(x_{k+i})_{i\in\mathbb{N}_0}\mapsto\prod_{j=1}^{M}\left|\sum_{l=0}^{k_j-k}b_l^{\langle j\rangle}x_{k_\star+a_j-l}\right|^{p_j},$$

where by Definition 1.3.2 we have $k_\star+a_{(\cdot)}-(k_{(\cdot)}-k)\geq k$, and denote by T the forward shift operator on $\mathbb{R}^{\mathbb{N}}$, i.e.

$$T:\mathbb{R}^{\mathbb{N}}\to\mathbb{R}^{\mathbb{N}}\quad\text{with}\quad(x_i)_{i\in\mathbb{N}_0}\mapsto(x_{1+i})_{i\in\mathbb{N}_0}.$$

Then for each $h\in\mathbb{N}_0$ we get

$$F\big(T^h(x_{k+i})_{i\in\mathbb{N}_0}\big)=F\big((x_{k+h+i})_{i\in\mathbb{N}_0}\big)=\prod_{j=1}^{M}\left|\sum_{l=0}^{k_j-k}b_l^{\langle j\rangle}x_{h+k_\star+a_j-l}\right|^{p_j}.$$

As stated in [**10**, below (4.45)] the process $(Y_t^\infty)_{t\geq k}$ is ergodic, i.e. an application of Birkhoff's ergodic theorem, cf. e.g. [**18**, Satz 20.14], results in

$$\underbrace{\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\prod_{j=1}^{M}\Big|\sum_{l=0}^{k_j-k} b_l^{\langle j\rangle} Y_{i+a_j-l}^\infty\Big|^{p_j}}_{=\sum_{h=0}^{n+k^\star-k_\star} F(T^h(Y_{k+i}^\infty)_{i\in\mathbb{N}_0})} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}\Bigg(\underbrace{\prod_{j=1}^{M}\Big|\sum_{l=0}^{k_j-k} b_l^{\langle j\rangle} Y_{k_\star+a_j-l}^\infty\Big|^{p_j}}_{=F((Y_{k+i}^\infty)_{i\in\mathbb{N}_0})}\Bigg) < \infty. \tag{3.1.9}$$

The finiteness of the right side in (3.1.9) follows from Lemma 1.5.2 and

$$\mathbb{E}\Big|\sum_{l=0}^{k_{(\cdot)}-k} b_l^{\langle\cdot\rangle} Y_{k_\star+a_{(\cdot)}-l}^\infty\Big|^{Q_{(\cdot)}p_{(\cdot)}} \leq K\sum_{l=0}^{k_{(\cdot)}-k}\mathbb{E}\Big|Y_{k_\star+a_{(\cdot)}-l}^\infty\Big|^{Q_{(\cdot)}p_{(\cdot)}} \leq K\mathbb{E}\Big|Y_k^\infty\Big|^{Q_{(\cdot)}p_{(\cdot)}} < \infty,$$

where for the last inequality we used the stationarity of the discrete time sequence $(Y_t^\infty)_{t\in\mathbb{N},t\geq k}$, cf. [**10**, below (4.45)].

We conclude this proof with

$$\begin{aligned} n^{-1+(\alpha+1/\beta)\sum_{j=1}^M p_j} V_n^{\langle M\rangle} &= \frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\prod_{j=1}^{M}\Big|\sum_{l=0}^{k_j-k} b_l^{\langle j\rangle} n^{\alpha+1/\beta}\,\Delta_{i+a_j-l,n}^k X\Big|^{p_j} \\ &\overset{d}{=} \frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\prod_{j=1}^{M}\Big|\sum_{l=0}^{k_j-k} b_l^{\langle j\rangle} Y_{i+a_j-l}^n\Big|^{p_j} \\ &\xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}\Bigg(\prod_{j=1}^{M}\Big|\sum_{l=0}^{k_j-k} b_l^{\langle j\rangle} Y_{k_\star+a_j-l}^\infty\Big|^{p_j}\Bigg), \end{aligned}$$

where for the first equality we have used (3.1.2), for the second (3.1.1) and for the convergence a combination of (3.1.8), (3.1.9), Slutsky's theorem as well as the continuous mapping theorem. □

CHAPTER 4

Limit Theorems and Convergence Rates for Multipower Variations

The pathwise properties of Lévy driven processes, cf. Lemma 4.5.3, will be the main tool of this chapter. By using this tool we will on the one hand extend the results for power variations in [**10**, Theorem 1.1 (iii)] to the multipower variations case and on the other hand derive, in section one to three, limit theorems that include convergence rates. In contrast to the previous chapters the Lévy driven processes of a multipower variation in this chapter can be driven by different Lévy processes.

Note that even in the case of power variations the results of this chapter, which provide convergence rates, were not known.

In the fourth section we will, by considering Lévy driven processes of the Ornstein-Uhlenbeck type, present a simplified version of the preceding results of this chapter.

Moreover, in order to improve the readability of this chapter and also to highlight our main results, we will present all the technical auxiliary results, which we will use in this chapter, in the last section.

4.1. The Driving Processes are Lévy Processes

Now we come to the before mentioned extension of [**10**, Theorem 1.1 (iii)], namely Theorem 4.1.1 (i). Furthermore, note that by a simple modification of Theorem 4.1.1 (i) we are able to eliminate a step in the respective proof and therefore able to obtain the convergence rate in Theorem 4.1.1 (ii).

THEOREM 4.1.1. *For each* $j = 1, \dots, M$ *let* $Q_j \geq 1$ *and let* $L^{\langle j \rangle} = (L_t^{\langle j \rangle})_{t \in \mathbb{R}}$ *be a symmetric Lévy process without a Brownian component and with a Blumenthal-Getoor index* $\beta_j < 2$. *Suppose for each* $j = 1, \dots, M$ *that* $L^{\langle j \rangle}$ *as well as the kernel* *functions* $g_j, \tilde{g}_j$ *satisfy the assumptions* ⓪,①,②,③,⑤ *with respect to the parameters* $\alpha_j, \kappa_j, \theta_j$*, where we have* $\alpha_j > \kappa_j - 1/(Q_j \vee \beta_j)$*, and in the case of* $\theta_j = Q_j$ *the assumption* ⑥ *as well. Moreover, for each* $j = 1, \ldots, M$ *assume that* $a_j \in \mathbb{Z}$*,* $p_j > 0$*,* $k_j \in \mathbb{N}$ *with* $k_j \leq \kappa_j$ *and set*

$$X_t^{\langle j \rangle} := \int_{-\infty}^{t} g_j(t-s) - \tilde{g}_j(-s)\, dL_s^{\langle j \rangle} \quad \textit{for} \quad t \geq 0.$$

Then by denoting

$$V_n^{\langle M \rangle} := V_n^{\langle M \rangle}(X; a; k; p) := \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \Delta_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j} \qquad \textit{(cf. Definition 1.3.2)}$$

as well as

$$F_{j,l}(t) := \left(\frac{d}{dt}\right)^{l} X_t^{\langle j \rangle} \; \textit{with} \; \left(\frac{d}{dt}\right)^{\kappa_j} X_t^{\langle j \rangle} = \int_{-\infty}^{t} g_j^{(\kappa_j)}(t-s)\, dL_s^{\langle j \rangle} \;\; \lambda \otimes \mathbb{P} - a.s.,$$

cf. Lemma 4.5.3 below, we obtain the following two results.

(*i*) *If for* $j = 1, \ldots, M$ *there are* $S_j \geq 1$ *satisfying* $\sum_{j=1}^{M} 1/S_j = 1$ *and* $S_j p_j \geq 1$ *then by assuming* $S_j p_j \leq Q_j$ *in the case of* $k_j = \kappa_j$ *we get*

$$n^{-1+\sum_{j=1}^{M} k_j p_j} V_n^{\langle M \rangle} \xrightarrow[n\to\infty]{a.s.} \int_0^1 \prod_{j=1}^{M} \left| F_{j,k_j}(t) \right|^{p_j} dt.$$

(*ii*) *If for* $j = 1, \ldots, M$ *we have* $k_j < \kappa_j$ *then by assuming* $p_j \leq Q_j$ *in the case of* $k_j + 1 = \kappa_j$ *we get*

$$\left| n^{-1+\sum_{j=1}^{M} k_j p_j} V_n^{\langle M \rangle} - \int_0^1 \prod_{j=1}^{M} \left| F_{j,k_j}(t) \right|^{p_j} dt \right| = \mathcal{O}\left(n^{-\frac{1}{r} \min(1, p_1, \ldots, p_M)} \right)$$

almost surely as $n \to \infty$*, where* $r = \prod_{j=1}^{M} (1 \vee p_j)$*.*

Proof. Note that in this proof we will use some of the technical results which will be presented in the last section of this chapter. Moreover, in order to ease our notations we will throughout this proof for $j = 1, \ldots, M$ often write $(\cdot)$ instead of j and $\langle \cdot \rangle$ instead of $\langle j \rangle$.

To (i): An application of Lemma 4.5.3 yields that the process $X^{\langle \cdot \rangle}$ is almost surely $\kappa_{(\cdot)}$-times absolutely continuous on the interval $[0,1]$ with

$$F_{(\cdot),k_{(\cdot)}} = \left(\frac{d}{dt}\right)^{k_{(\cdot)}} X^{\langle \cdot \rangle} \in L^{Q_{(\cdot)}}([0,1]) \cap L^{S_{(\cdot)}p_{(\cdot)}}([0,1]).$$

By setting $\phi = (\phi_1, \ldots, \phi_M)$ and $\psi_l = (\psi_{1,l}, \ldots, \psi_{M,l})$, where $\phi_{(\cdot)} := X^{\langle \cdot \rangle}$ and the sequence $(\psi_{(\cdot),l})_{l \in \mathbb{N}}$ is chosen as in Lemma 4.5.5, we obtain for

$$W_n(\xi) := n^{-1+\sum_{j=1}^M k_j p_j} \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^M | \triangle_{i+a_j,n}^{k_j} \xi_j |^{p_j}$$

and

$$I(\xi) := \int_0^1 \prod_{j=1}^M |\xi_j^{(k_j)}(s)|^{p_j} ds$$

the following chain of convergences

$$\lim_{n\to\infty} W_n(\phi) = \lim_{n\to\infty} \lim_{l\to\infty} W_n(\psi_l) = \lim_{l\to\infty} \lim_{n\to\infty} W_n(\psi_l) = \lim_{l\to\infty} I(\psi_l) = I(\phi) \quad \text{a.s.},$$

where the first and last equality, of which the last also requires the application of Lemma 1.5.3 as well as Lemma 1.5.2, follow from Lemma 4.5.5, the third equality is a consequence of Lemma 4.5.4 and the swapping of limits in the second equality is based on the fact that by Lemma 4.5.4 and Lemma 4.5.5 the sequence $W_n(\psi_l)$ converges with respect to n and uniformly with respect to l. Hence, we get (i).

To (ii): By applying Lemma 4.5.3, we get that the process $X^{\langle \cdot \rangle}$ is almost surely $(k_{(\cdot)} + 1)$-times absolutely continuous on the interval $[0,1]$ with

$$\left(\frac{d}{dt}\right)^{k_{(\cdot)}+1} X^{\langle \cdot \rangle} \in L^{\max(1,p_{(\cdot)})}([0,1]).$$

Therefore a pathwise application of Lemma 4.5.4 yields (ii). □

The subsequent remark will, while illustrating the interplay between the parameter assumptions in Theorem 4.1.1 and the fact that there are no restrictions on the order M of the multipower variation $V_n^{\langle M \rangle}$, provide an additional convergence rate in Theorem 4.1.1 (ii).

REMARK 4.1.2. Note that Theorem 4.1.1 can be applied with the parameter $r = 1$ by transforming, as done in Remark 1.3.3, the Mth order power variation $V_n^{\langle M \rangle}$ into a $(M\rho)$th order power variation $V_n^{\langle M\rho \rangle}$, where $\rho := \max(\lceil p_1 \rceil, \dots, \lceil p_M \rceil)$. The respective convergence rate would then become

$$\mathcal{O}\left(n^{-\min(1, p_1/\rho, \dots, p_M/\rho)}\right)$$

as $n \to \infty$.

4.2. The Driving Processes are Lévy Processes with Finete Second Moments

In this section we will combine the pathwise properties, cf. Lemma 4.5.3 below, and the L^2-property, cf. Remark 1.2.5 (ii), of Lévy driven processes in order to obtain convergence rates in probability and in L^1.

THEOREM 4.2.1. *For $j = 1, \dots, M$ let $L^{\langle j \rangle} = (L_t^{\langle j \rangle})_{t \in \mathbb{R}}$ be a symmetric Lévy process without a Brownian component, with Lévy measure ν_j and Blumenthal-Getoor index $\beta_j < 2$. Suppose for each $j = 1, \dots, M$ that $L^{\langle j \rangle}$ as well as the kernel functions $g_j, \tilde{g}_j$ satisfy*

- *the assumptions ⓪,②,③,⑤ with respect to the parameters $\alpha_j, \kappa_j, \theta_j$, where we have $\alpha_j > \kappa_j - 1/(Q_j \vee \beta_j)$, and in the case of $\theta_j = Q_j$ the assumption ⑥ as well,*
- $\| \blacktriangle_{k_j, N_j}^{k_j} g_j \|_{L^2(\mathbb{R})} < \infty$ *for some $N_j \in \mathbb{N}$ with $N_j \geq k_\star + a_j$,*
- $g_j^{(\kappa_j)} \in L^2((0, \infty))$ *and* $\int_{|x| \geq 1} x^2 \nu_j(dx) < \infty$

with $Q_j \geq 1$ and $\kappa_j = k_j + 1$ for some $k_j \in \mathbb{N}$.

Moreover, for $j = 1, \dots, M$ assume $a_j \in \mathbb{Z}$, $p_j > 0$ with $\sum_{l=1}^M p_l \leq 2$ and set

$$X_t^{\langle j \rangle} := \int_{-\infty}^t g_j(t-s) - \tilde{g}_j(-s) \, dL_s^{\langle j \rangle} \quad \textit{for} \quad t \geq 0.$$

Then it holds that

$$\mathbb{E}\left| \left(n^{-1 + \sum_{j=1}^M k_j p_j} V_n^{\langle M \rangle} \right)^{\frac{1}{r}} - \left(\int_0^1 \prod_{j=1}^M \left| F_{j, k_j}(t) \right|^{p_j} dt \right)^{\frac{1}{r}} \right| = \mathcal{O}\left(n^{-\frac{1}{r} \min(1, p_1, \dots, p_M)} \right)$$

as $n \to \infty$, *where*

$$V_n^{\langle M \rangle} := V_n^{\langle M \rangle}(X;a;k;p) := \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j} \qquad \text{(cf. Definition 1.3.2)},$$

$$F_{j,k_j}(t) := \left(\frac{d}{dt} \right)^{k_j} X_t^{\langle j \rangle} \qquad \text{(cf. Lemma 4.5.3 below)}$$

and $r = \prod_{j=1}^{M}(1 \vee p_j)$.

The following remark will allow us, among other things, to derive an additional result from Theorem 4.2.1.

REMARK 4.2.2. (i) In Theorem 4.2.1 we can choose the parameter $r = 1$ by transforming the Mth order power variation $V_n^{\langle M \rangle}$ into a $(M\rho)$th order power variation $V_n^{\langle M\rho \rangle}$ with $\rho := \max(\lceil p_1 \rceil, \ldots, \lceil p_M \rceil)$ as done in Remark 1.3.3. Then the respective convergence rate would become

$$\mathcal{O}\left(n^{-\min(1, p_1/\rho, \ldots, p_M/\rho)}\right)$$

as $n \to \infty$.

(ii) Let g and $\tilde{g}$ be two functions with $g \in L^2((0,\delta)) \cap C^k((0,\infty))$ for some $\delta > 0$ so that the assumptions ⓪,② and ⑤(iii) are satisfied. Moreover, assume $n \in \mathbb{N}$ to be sufficiently large, $i = k, \ldots, n$ as well as $g^{(k)} \in L^2((\delta,\infty))$ and denote all positive constants by K.

By substitution and the assumption ②, we obtain

$$\| \blacktriangle_{i,n}^{k} g \|_{L^2([\frac{i}{n}-\delta, \frac{i}{n}])} = \left(\int_{\frac{i}{n}-\delta}^{\frac{i}{n}} \left| \sum_{j=0}^{k} (-1)^j \binom{k}{j} g\left(\frac{i-j}{n} - s \right) \right|^2 ds \right)^{\frac{1}{2}}$$
$$\leq K \sum_{j=0}^{k} \left(\int_0^{\delta - \frac{j}{n}} |g(s)|^2 ds \right)^{\frac{1}{2}} < K.$$

Furthermore, an application of Lemma 1.5.1 (iii), which can be proven without using the assumptions ④, ⑤(i), ⑤(ii) and ⑤(iv), yields

$$\int_{\frac{i-k}{n}-\delta}^{\frac{i}{n}-\delta} | \blacktriangle_{i,n}^{k} g(s)|^2 ds \leq K n^{-2k} \int_{\frac{i-k}{n}-\delta}^{\frac{i}{n}-\delta} 1 ds \leq K n^{-2k-1}$$

as well as

$$\int_{-\infty}^{\frac{i-k}{n}-\delta} | \blacktriangle_{i,n}^{k} g(s)|^2 ds \leq Kn^{-2k} \int_{-\infty}^{-\delta} \left|g^{(k)}(-s)\right|^2 ds \leq Kn^{-2k}.$$

By combining the above inequalities with assumption ②, we get

$$|| \blacktriangle_{i,n}^{k} g||_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\frac{i}{n}} | \blacktriangle_{i,n}^{k} g(s)|^2 ds \leq K.$$

By combining Theorem 4.2.1 and Remark 4.2.2, we get the following corollary.

Corollary 4.2.3. *In the setting of Theorem 4.2.1 assume that all assumptions are satisfied so that Theorem 4.2.1 is applicable. Then for each* $\varepsilon > 0$ *we have*

$$\mathbb{P}\Big(\Big|(V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}}\Big| \geq \varepsilon\Big) = \mathcal{O}\left(n^{-\frac{1}{r}\min(1,p_1,\ldots,p_M)}\right)$$

respectively

$$\mathbb{P}\Big(\Big|V_n - F\Big| \geq \varepsilon\Big) = \mathcal{O}\left(n^{-\min(1,p_1/\rho,\ldots,p_M/\rho)}\right)$$

as $n \to \infty$, *where* $\rho := \max(\lceil p_1 \rceil, \ldots, \lceil p_M \rceil)$,

$$V_n := n^{-1+\sum_{j=1}^{M} k_j p_j} V_n^{\langle M \rangle} \quad \textit{and} \quad F := \int_0^1 \prod_{j=1}^{M} \left|F_{j,k_j}(t)\right|^{p_j} dt.$$

Proof. By Markov's inequality, we have

$$\mathbb{P}\Big(\Big|(V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}}\Big| \geq \varepsilon\Big) \leq \frac{1}{\varepsilon}\mathbb{E}\Big|(V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}}\Big|$$

for all $\varepsilon > 0$.

Hence, we obtain the first result by an application of Theorem 4.2.1 and the second by setting $r = 1$ as well as applying a combination of Remark 4.2.2 (i) and Theorem 4.2.1. □

4.2.1. Proof of Theorem 4.2.1. Note that the last section of this chapter contains some of the technical results that we will use in this proof.

To ease our notations we will throughout this proof denote all positive constants by K, although they may change from line to line, and for $j = 1, \ldots, M$ often write $(\cdot)$ instead of j respectively $\langle\cdot\rangle$ instead of $\langle j\rangle$. Furthermore, we will denote $\phi_{(\cdot)}(t) := X_t^{\langle\cdot\rangle}$ and adopt the notations used in the proof of Lemma 4.5.4.

By Lemma 1.2.2, we know that the Lévy measure $\nu_{(\cdot)}$ and the parameter $\theta_{(\cdot)}$ satisfy assumption ①.

An application of Lemma 4.5.3 yields that $\phi_{(\cdot)}$ is pathwise $(k_{(\cdot)}+1)$-times absolutely continuous on the interval $[0,1]$ with

$$\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(t) = \int_{-\infty}^{t} g_{(\cdot)}^{(k_{(\cdot)}+1)}(t-s)\, dL_s^{\langle\cdot\rangle} \quad \lambda\otimes\mathbb{P}\text{-a.s..}$$

Moreover, note that the process

$$Y_t^{\langle\cdot\rangle} = \int_{-\infty}^{\infty} g_{(\cdot)}^{(k_{(\cdot)}+1)}(t-s)\mathbf{1}_{(0,\infty)}(t-s)\, dL_s^{\langle\cdot\rangle}, \qquad t\in\mathbb{R},$$

exists in the L^2-sense, cf. [**21**, Proposition 2.1].

Since for each $t\in[0,1]$ we have $Y_t^{\langle\cdot\rangle} = \phi_{(\cdot)}^{(k_{(\cdot)}+1)}(t)$ a.s. an application of [**21**, Proposition 2.1] results in

$$\begin{aligned}\int_0^1 \mathbb{E}\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(t)\right|^2 dt =&\mathbb{E}\left(L_1^{\langle\cdot\rangle}\right)^2 \int_0^1\int_{-\infty}^{t}\left|g_{(\cdot)}^{(k_{(\cdot)}+1)}(t-s)\right|^2 ds\, dt\\ =&\mathbb{E}\left(L_1^{\langle\cdot\rangle}\right)^2 \int_0^1\int_0^{\infty}\left|g_{(\cdot)}^{(k_{(\cdot)}+1)}(s)\right|^2 ds\, dt \le K. \end{aligned}\tag{4.2.1}$$

The pathwise absolute continuity of $\phi_{(\cdot)}^{(k_{(\cdot)})}$ implies that for each $t\in[0,1]$ we have

$$\begin{aligned}\left|\phi_{(\cdot)}^{(k_{(\cdot)})}(t)\right| \le&\left|\phi_{(\cdot)}^{(k_{(\cdot)})}(t)-\phi_{(\cdot)}^{(k_{(\cdot)})}(\xi_{(\cdot)})\right| + \left|\phi_{(\cdot)}^{(k_{(\cdot)})}(\xi_{(\cdot)})\right|\\ \le&\left|\phi_{(\cdot)}^{(k_{(\cdot)})}(\xi_{(\cdot)})\right| + \int_0^1\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(s)\right| ds\\ \le& N_{(\cdot)}^{k_{(\cdot)}}\left|\,\triangle_{k_\star+a_{(\cdot)},N_{(\cdot)}}^{k_{(\cdot)}}\phi_{(\cdot)}\right| + K\int_0^1\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(s)\right| ds \text{ a.s.,}\end{aligned}\tag{4.2.2}$$

where $0\le\xi_{(\cdot)} = (k_\star+a_{(\cdot)}-k_{(\cdot)})/N_{(\cdot)} \le (k_\star+a_{(\cdot)})/N_{(\cdot)}\le 1$ and the last inequality is based on (4.5.4), cf. proof of Lemma 4.5.4.

Furthermore, by the same argumentation as in Remark 1.2.5 (ii), we have

$$\begin{aligned}\mathbb{E}\left|\,\triangle_{k_\star+a_{(\cdot)},N_{(\cdot)}}^{k_{(\cdot)}}\phi_{(\cdot)}\right|^2 \le&\mathbb{E}\left(L_1^{\langle\cdot\rangle}\right)^2 \|\,\blacktriangle_{k_\star+a_{(\cdot)},N_{(\cdot)}}^{k_{(\cdot)}} g_{(\cdot)}\|_{L^2(\mathbb{R})}^2\\ \le&\mathbb{E}\left(L_1^{\langle\cdot\rangle}\right)^2 \|\,\blacktriangle_{k_{(\cdot)},N_{(\cdot)}}^{k_{(\cdot)}} g_{(\cdot)}\|_{L^2(\mathbb{R})}^2 \le K.\end{aligned}\tag{4.2.3}$$

Let $q_{(\cdot)} := 2/p_{(\cdot)}$ and in the case of $\sum_{j=1}^M p_j < 2$ set $q_{M+1} := 2/(2 - \sum_{j=1}^M p_j)$. By (4.5.3), cf. proof of Lemma 4.5.4, we get

$$\begin{aligned}
&\mathbb{E}\left|\left(n^{-1+\sum_{j=1}^M k_j p_j}\right)^{\frac{1}{r}} A_n\right| \leq \sum_{l=1}^M \mathbb{E}\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\left(\prod_{j=1,\dots,M:j\neq l}\left|B_{i,n}^{\langle j\rangle}\right|^{p_j}\right)\left|R_{i,n}^{\langle l\rangle}\right|^{p_l}\right)^{\frac{1}{r}}\\
\leq &\sum_{l=1}^M\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\mathbb{E}\left(\left(\prod_{j=1,\dots,M:j\neq l}\left|B_{i,n}^{\langle j\rangle}\right|^{p_j}\right)\left|R_{i,n}^{\langle l\rangle}\right|^{p_l}\right)\right)^{\frac{1}{r}}\\
\leq &\sum_{l=1}^M\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\left(\prod_{j=1,\dots,M:j\neq l}\left(\mathbb{E}\left|B_{i,n}^{\langle j\rangle}\right|^2\right)^{\frac{1}{q_j}}\right)\left(\mathbb{E}\left|R_{i,n}^{\langle l\rangle}\right|^2\right)^{\frac{1}{q_l}}\right)^{\frac{1}{r}}\\
\leq &K\sum_{l=1}^M\left(\left(\prod_{j=1,\dots,M:j\neq l}\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\mathbb{E}\left|B_{i,n}^{\langle j\rangle}\right|^2\right)^{\frac{1}{q_j}}\right)\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\mathbb{E}\left|R_{i,n}^{\langle l\rangle}\right|^2\right)^{\frac{1}{q_l}}\right)^{\frac{1}{r}}\\
\leq &K\sum_{l=1}^M n^{-\frac{p_l}{r}} \leq Kn^{-\frac{1}{r}\min(p_1,\dots,p_M)},
\end{aligned}\tag{4.2.4}$$

where the second inequality in the case of $r > 1$ requires an application of Hölder's inequality, the third as well as the fourth inequality follow by Lemma 1.5.2 and the second last inequality is a consequence of (4.2.5) and (4.2.6) below.

Note that by the Cauchy-Schwarz inequality we have on the one hand

$$\begin{aligned}
&\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\mathbb{E}\left|R_{i,n}^{\langle\cdot\rangle}\right|^2 = K\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\mathbb{E}\left(\int_{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}}^{\frac{i+a_{(\cdot)}}{n}}\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(s)\right|ds\right)^2\\
\leq &Kn^{-2}\sum_{i=k_\star}^{n+k^\star}\mathbb{E}\int_{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}}^{\frac{i+a_{(\cdot)}}{n}}\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(s)\right|^2 ds \leq Kn^{-2}\mathbb{E}\left(\int_0^1\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(s)\right|^2 ds\right)\\
\leq &Kn^{-2}\int_0^1\mathbb{E}\left(\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(s)\right|^2\right)ds \leq Kn^{-2},
\end{aligned}\tag{4.2.5}$$

where the second inequality uses the properties of $a_{(\cdot)}$, $k_\star$ and $k^\star$, cf. Definition 1.3.2, the third is an application of Fubini's theorem, and the last follows from (4.2.1), and on the other hand

$$\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\mathbb{E}\left|B_{i,n}^{\langle\cdot\rangle}\right|^2 \leq \frac{K}{n}\sum_{i=k_\star}^{n+k^\star}\mathbb{E}\left|R_{i,n}^{\langle\cdot\rangle}\right|^2 + \frac{K}{n}\sum_{i=k_\star}^{n+k^\star}\mathbb{E}\left|\phi_{(\cdot)}^{(k_{(\cdot)})}\left(\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}\right)\right|^2 \leq K, \tag{4.2.6}$$

where the first summand can be handled by (4.2.5) and the second by applying in the following order (4.2.2), Cauchy-Schwarz inequality and then handling the resulting summands with (4.2.3) respectively a combination of Cauchy-Schwarz inequality, Fubini's theorem and (4.2.1).

In the case of $k^\star - k_\star + 1 \geq 0$ we obtain by using (4.5.10), cf. proof of Lemma 4.5.4, the following inequalities

$$\begin{aligned}
&\mathbb{E}\left|\left(\int_0^1 \prod_{j=1}^M \left|\phi_j^{(k_j)}(s)\right|^{p_j} ds\right)^{\frac{1}{r}} - \left(\int_0^1 T_n(s)ds\right)^{\frac{1}{r}}\right| \\
\leq &\sum_{l=1}^M \mathbb{E}\left(\int_0^1 \left(\prod_{j=1,\dots,M:j\neq l} \left|\widetilde{B}_n^{\langle j\rangle}(s)\right|^{p_j}\right)\left|\widetilde{R}_n^{\langle l\rangle}(s)\right|^{p_l} ds\right)^{\frac{1}{r}} \\
\leq &\sum_{l=1}^M \left(\mathbb{E}\left(\int_0^1 \left(\prod_{j=1,\dots,M:j\neq l} \left|\widetilde{B}_n^{\langle j\rangle}(s)\right|^{p_j}\right)\left|\widetilde{R}_n^{\langle l\rangle}(s)\right|^{p_l} ds\right)\right)^{\frac{1}{r}} \\
\leq K &\sum_{l=1}^M \left(\left(\prod_{j=1,\dots,M:j\neq l} \left(\int_0^1 \mathbb{E}\left|\widetilde{B}_n^{\langle j\rangle}(s)\right|^2 ds\right)^{\frac{1}{q_j}}\right)\left(\int_0^1 \mathbb{E}\left|\widetilde{R}_n^{\langle l\rangle}(s)\right|^2 ds\right)^{\frac{1}{q_l}}\right)^{\frac{1}{r}} \\
\leq K &\sum_{l=1}^M n^{-\frac{p_l}{r}} \leq K n^{-\frac{1}{r}\min(p_1,\dots,p_M)},
\end{aligned} \tag{4.2.7}$$

where the second inequality in the case of $r > 1$ follows by an application of Hölder's inequality, the third by Fubini's theorem in combination with two applications of Lemma 1.5.2 and the second last inequality by (4.2.8) and (4.2.9) below.

By a similar argumentation as in (4.2.5) and (4.2.6), we obtain in the case of $k^\star - k_\star + 1 \geq 0$ on the on hand

$$\begin{aligned}
&\int_0^1 \mathbb{E}\left|\widetilde{R}_n^{\langle\cdot\rangle}(s)\right|^2 ds = \sum_{i=k_\star}^{n+k_\star-1} \int_{\frac{i-k_\star}{n}}^{\frac{i-k_\star+1}{n}} \mathbb{E}\left(\int_{\frac{i+a(\cdot)-k(\cdot)}{n}\wedge\frac{i-k_\star}{n}}^{\frac{i+a(\cdot)-k(\cdot)}{n}\vee\frac{i-k_\star+1}{n}} \left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(y)\right|dy\right)^2 ds \\
\leq &K n^{-1} \sum_{i=k_\star}^{n+k_\star-1} \int_{\frac{i-k_\star}{n}}^{\frac{i-k_\star+1}{n}} \mathbb{E}\left(\int_{\frac{i+a(\cdot)-k(\cdot)}{n}\wedge\frac{i-k_\star}{n}}^{\frac{i+a(\cdot)-k(\cdot)}{n}\vee\frac{i-k_\star+1}{n}} \left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(y)\right|^2 dy\right) ds \\
\leq &K n^{-1} \sum_{i=k_\star}^{n+k_\star-1} \int_{\frac{i-k_\star}{n}}^{\frac{i-k_\star+1}{n}} \int_{\frac{i+a(\cdot)-k(\cdot)}{n}\wedge\frac{i-k_\star}{n}}^{\frac{i+a(\cdot)-k(\cdot)}{n}\vee\frac{i-k_\star+1}{n}} \mathbb{E}\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(y)\right|^2 dy\, ds
\end{aligned}$$

$$\leq K n^{-2} \int_0^1 \mathbb{E}\Big|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(y)\Big|^2 dy \leq K n^{-2}, \tag{4.2.8}$$

where for the first to fourth inequality we used respectively the Cauchy-Schwarz inequality, Fubini's theorem, the properties of $a_{(\cdot)}$, $k_\star$, $k^\star$ (cf. Definition 1.3.2) as well as (4.2.1), and on the other hand

$$\int_0^1 \mathbb{E}\Big|\widetilde{B}_n^{\langle\cdot\rangle}(s)\Big|^2 ds \leq K \int_0^1 \mathbb{E}\Big|\phi_{(\cdot)}^{(k_{(\cdot)})}(s)\Big|^2 ds + K \int_0^1 \mathbb{E}\Big|\widetilde{R}_n^{\langle\cdot\rangle}(s)\Big|^2 ds \leq K, \tag{4.2.9}$$

where the first inequality is generated by an application of the Cauchy-Schwarz inequality, the second summand can be bounded by (4.2.8) and the first can be handled by the same argumentation as the second summand in (4.2.6).

In the case of $k^\star - k_\star + 1 < 0$ we get by a similar argumentation as in (4.2.7) that

$$\mathbb{E}\left|\left(\int_0^{1+\frac{k^\star - k_\star + 1}{n}} \prod_{j=1}^M \Big|\phi_j^{(k_j)}(s)\Big|^{p_j} ds\right)^{\frac{1}{r}} - \left(\int_0^{1+\frac{k^\star - k_\star + 1}{n}} T_n(s) ds\right)^{\frac{1}{r}}\right| \leq K n^{-\frac{1}{r}\min(p_1,\ldots,p_M)}. \tag{4.2.10}$$

Moreover, we obtain (4.2.11) and (4.2.12) below as follows. By applying in the following order Hölder's inequality in the case of $r > 1$, Fubini's theorem in the case of (4.2.12), Lemma 1.5.2 and (4.2.2), we get the first inequality in (4.2.11) and (4.2.12). The respective second inequalities can be obtained by using the Cauchy-Schwarz inequality and then by handling the respective second summands with a combination of the Cauchy-Schwarz inequality and Fubini's theorem. The last inequality in (4.2.11) and (4.2.12) is a consequence of (4.2.1) and (4.2.3).

In the case of $k^\star - k_\star + 1 > 0$ we have

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=n+k_\star}^{n+k^\star} \prod_{j=1}^M \Big|\phi_j^{(k_j)}\left(\frac{i + a_j - k_j}{n}\right)\Big|^{p_j}\right)^{\frac{1}{r}}$$

$$\leq \left(\frac{K}{n}\sum_{i=n+k_\star}^{n+k^\star} \prod_{j=1}^M \left(\mathbb{E}\Big(\Big| \triangle_{k_\star + a_j, N_j}^{k_j} \phi_j\Big| + \int_0^1 \Big|\phi_j^{(k_j+1)}(s)\Big| ds\Big)^2\right)^{\frac{1}{q_j}}\right)^{\frac{1}{r}}$$

$$\leq \left(\frac{K}{n} \sum_{i=n+k_\star}^{n+k^\star} \prod_{j=1}^{M} \left(\mathbb{E} \Big| \triangle_{k_\star+a_j,N_j}^{k_j} \phi_j \Big|^2 + \int_0^1 \mathbb{E} \Big| \phi_j^{(k_j+1)}(s) \Big|^2 ds \right)^{\frac{1}{q_j}} \right)^{\frac{1}{r}} \leq K n^{-\frac{1}{r}} \quad (4.2.11)$$

and in the case of $k^\star - k_\star + 1 < 0$ we have

$$\mathbb{E} \left(\int_{\frac{n+k^\star-k_\star+1}{n}}^{1} \prod_{j=1}^{M} \Big| \phi_j^{(k_j)}(t) \Big|^{p_j} dt \right)^{\frac{1}{r}}$$

$$\leq \left(K \int_{\frac{n+k^\star-k_\star+1}{n}}^{1} \prod_{j=1}^{M} \left(\mathbb{E} \left(\Big| \triangle_{k_\star+a_j,N_j}^{k_j} \phi_j \Big| + \int_0^1 \Big| \phi_j^{(k_j+1)}(s) \Big| ds \right)^2 \right)^{\frac{1}{q_j}} dt \right)^{\frac{1}{r}}$$

$$\leq \left(K \int_{\frac{n+k^\star-k_\star+1}{n}}^{1} \prod_{j=1}^{M} \left(\mathbb{E} \Big| \triangle_{k_\star+a_j,N_j}^{k_j} \phi_j \Big|^2 + \int_0^1 \mathbb{E} \Big| \phi_j^{(k_j+1)}(s) \Big|^2 ds \right)^{\frac{1}{q_j}} dt \right)^{\frac{1}{r}} \leq K n^{-\frac{1}{r}}. \quad (4.2.12)$$

The subadditivity of the function $s \mapsto |s|^{\frac{1}{r}}$ as well as a combination of (4.2.4) with (4.2.7), (4.2.11), (4.2.10) and (4.2.12) result in

$$\mathbb{E} \left| \left(n^{-1+\sum_{j=1}^{M} k_j p_j} V_n^{\langle M \rangle} \right)^{\frac{1}{r}} - \left(\int_0^1 \prod_{j=1}^{M} \Big| F_{j,k_j}(t) \Big|^{p_j} dt \right)^{\frac{1}{r}} \right| = \mathcal{O}\left(n^{-\frac{1}{r} \min(1,p_1,\ldots,p_M)} \right)$$

as $n \to \infty$. □

4.3. The Driving Processes are Symmetric α-Stable Lévy Processes

Now we will focus on Lévy driven processes that are driven by symmetric α-stable Lévy processes. By proceeding as in section two, while using the scaling property instead of the L^2-property, cf. Remark 1.2.5 (ii) and (iii), we will similarly to section two obtain convergence rates in probability and in L^1.

THEOREM 4.3.1. *For each $j = 1, \ldots, M$ suppose that the kernel functions $g_j, \tilde{g}_j$ as well as the symmetric β_j-stable Lévy process $(L_t^{\langle j \rangle})_{t \in \mathbb{R}}$ with $1 < \beta_j < 2$ and scale parameter $\sigma_j > 0$, i.e. $\mathbb{E}\big(\exp(iuL_1^{\langle j \rangle}) \big) = \exp(-\sigma_j^{\beta_j} |u|^{\beta_j})$ for all $u \in \mathbb{R}$, satisfy*

- *the assumptions ⓪ to ③ and ⑤ with respect to the parameters $\alpha_j, \kappa_j, \theta_j$, where $\theta_j = \beta_j$ and $\alpha_j > \kappa_j - 1/(Q_j \vee \beta_j)$, and in the case of $\theta_j = Q_j$ the assumption ⑥ as well,*

- $g_j^{(\kappa_j)} \in L^{\beta_j}((0,\infty))$ *and* $\| \blacktriangle_{k_j,N_j}^{k_j} g_j\|_{L^{\beta_j}(\mathbb{R})} < \infty$ *for some* $N_j \in \mathbb{N}$ *with* $N_j \geq k_\star + a_j$,

where $Q_j \geq 1$ *and* $\kappa_j = k_j + 1$ *for some* $k_j \in \mathbb{N}$. *Moreover, for each* $j = 1, \dots, M$ *assume* $a_j \in \mathbb{Z}$ *and* $p_j > 0$ *with* $\sum_{l=1}^M p_l \leq \widehat{\beta}$, *where* $1 < \widehat{\beta} < \min(\beta_1, \dots, \beta_M)$, *and set*

$$X_t^{\langle j \rangle} := \int_{-\infty}^t g_j(t-s) - \tilde{g}_j(-s)\, dL_s^{\langle j \rangle} \quad \textit{for} \quad t \geq 0.$$

Then we have

$$\mathbb{P}\Big(\Big|(V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}}\Big| \geq \varepsilon\Big) \leq \frac{1}{\varepsilon}\mathbb{E}\Big|(V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}}\Big| = \mathcal{O}\left(n^{-\frac{1}{r}\min(1,p_1,\dots,p_M)}\right)$$

as well as

$$\mathbb{P}\Big(\Big|V_n - F\Big| \geq \varepsilon\Big) \leq \frac{1}{\varepsilon}\mathbb{E}\Big|V_n - F\Big| = \mathcal{O}\left(n^{-\min(1,p_1/\rho,\dots,p_M/\rho)}\right)$$

for each $\varepsilon > 0$ *as* $n \to \infty$, *where* $\rho = \max(1, \lceil p_1 \rceil, \dots, \lceil p_M \rceil)$, $r = \prod_{j=1}^M (1 \vee p_j)$,

$$V_n = n^{-1+\sum_{j=1}^M k_j p_j} V_n^{\langle M \rangle}(X; a; k; p) \quad \textit{(cf. Definition 1.3.2)},$$

and

$$F = \int_0^1 \prod_{j=1}^M \left| \left(\frac{d}{dt}\right)^{k_j} X_t^{\langle j \rangle} \right|^{p_j} dt$$

with

$$\left(\frac{d}{dt}\right)^{\kappa_j} X_t^{\langle j \rangle} = \int_{-\infty}^t g_j^{(\kappa_j)}(t-s)\, dL_s^{\langle j \rangle} \;\; \lambda \otimes \mathbb{P} - a.s..$$

Proof. This proof consists of three steps.

<u>*Step 1:*</u> In this step of the proof we will show

$$\mathbb{E}\Big|(V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}}\Big| = \mathcal{O}\left(n^{-\frac{1}{r}\min(1,p_1,\dots,p_M)}\right) \qquad \text{as } n \to \infty$$

by modifying the proof of Theorem 4.2.1. Therefore we will use the same notations as in the proof of Theorem 4.2.1.

Note that contrary to the proof of Theorem 4.2.1 we do not need to use Lemma 1.2.2 since the assumption ① is satisfied.

In the modification of the proof of Theorem 4.2.1 we will instead of (4.2.1) use

$$\int_0^1 \mathbb{E}\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(t)\right|^{\widehat{\beta}} dt = K \int_0^1 \left(\int_{-\infty}^t \left|g_{(\cdot)}^{(k_{(\cdot)}+1)}(t-s)\right|^{\beta_{(\cdot)}} ds\right)^{\frac{\widehat{\beta}}{\beta_{(\cdot)}}} dt \le K,$$

which follows by a similar argumentation as used in (4.2.1) from Remark 1.2.5 (iii) and $g_{(\cdot)}^{(k_{(\cdot)}+1)} \in L^{\beta_{(\cdot)}}((0,\infty))$, and instead of (4.2.3) we will use

$$\begin{aligned}\mathbb{E}\left| \triangle_{k_\star+a_{(\cdot)},N_{(\cdot)}}^{k_{(\cdot)}} \phi_{(\cdot)}\right|^{\widehat{\beta}} &= \mathbb{E}\left|\int_{-\infty}^{\infty} \blacktriangle_{k_\star+a_{(\cdot)},N_{(\cdot)}}^{k_{(\cdot)}} g_{(\cdot)}(s)\, dL_s^{\langle\cdot\rangle}\right|^{\widehat{\beta}} \\ &= K\left(\int_{\mathbb{R}} | \blacktriangle_{k_\star+a_{(\cdot)},N_{(\cdot)}}^{k_{(\cdot)}} g_{(\cdot)}(s)|^{\beta_{(\cdot)}} ds\right)^{\frac{\widehat{\beta}}{\beta_{(\cdot)}}} \le K,\end{aligned}$$

which is a consequence of the representation

$$\triangle_{k_\star+a_{(\cdot)},N_{(\cdot)}}^{k_{(\cdot)}} X^{\langle\cdot\rangle} = \int_{-\infty}^{\infty} \blacktriangle_{k_\star+a_{(\cdot)},N_{(\cdot)}}^{k_{(\cdot)}} g_{(\cdot)}(s)\, dL_s^{\langle\cdot\rangle},$$

Remark 1.2.5 (iii) and the fact that $\| \blacktriangle_{k_{(\cdot)},N_{(\cdot)}}^{k_{(\cdot)}} g_{(\cdot)}\|_{L^{\beta_{(\cdot)}}(\mathbb{R})}$ is bounded.

Furthermore, in this modification of the proof of Theorem 4.2.1 we will also redefine the parameters $q_{(\cdot)}$ and q_{M+1} as follows. Let $q_{(\cdot)} := \widehat{\beta}/p_{(\cdot)}$ and in the case of $\sum_{j=1}^M p_j < \widehat{\beta}$ let $q_{M+1} := \widehat{\beta}/(\widehat{\beta} - \sum_{j=1}^M p_j)$.

By using Hölder's inequality with the parameters $p = \widehat{\beta}$ and $\tilde{p} = \widehat{\beta}/(\widehat{\beta}-1)$ instead of the Cauchy-Schwarz inequality in the proof of (4.2.5), (4.2.6), (4.2.8) and (4.2.9), we obtain

$$\frac{1}{n}\sum_{i=k_\star}^{n+k^\star} \mathbb{E}\left|R_{i,n}^{\langle\cdot\rangle}\right|^{\widehat{\beta}} \le Kn^{-\widehat{\beta}} \quad \text{and} \quad \frac{1}{n}\sum_{i=k_\star}^{n+k^\star} \mathbb{E}\left|B_{i,n}^{\langle\cdot\rangle}\right|^{\widehat{\beta}} \le K \tag{4.3.1}$$

as well as

$$\int_0^1 \mathbb{E}\left|\widetilde{R}_n^{\langle\cdot\rangle}(s)\right|^{\widehat{\beta}} ds \le Kn^{-\widehat{\beta}} \quad \text{and} \quad \int_0^1 \mathbb{E}\left|\widetilde{B}_n^{\langle\cdot\rangle}(s)\right|^{\widehat{\beta}} ds \le K, \tag{4.3.2}$$

where in the proof of (4.2.6) and (4.2.9) we need to use the respective first result in (4.3.1) and (4.3.2) instead of (4.2.5) and (4.2.8).

We get

$$\mathbb{E}\left|\left(n^{-1+\sum_{j=1}^{M}k_j p_j}\right)^{\frac{1}{r}} A_n\right| \le K\sum_{l=1}^{M} n^{-\frac{1}{r}\frac{1}{q_l}\widehat{\beta}} \le Kn^{-\frac{1}{r}\min(p_1,\dots,p_M)} \tag{4.3.3}$$

and in the case of $k^\star - k_\star + 1 \ge 0$ we obtain

$$\mathbb{E}\left|\left(\int_0^1 \prod_{j=1}^{M}\left|\phi_j^{(k_j)}(s)\right|^{p_j} ds\right)^{\frac{1}{r}} - \left(\int_0^1 T_n(s)ds\right)^{\frac{1}{r}}\right| \le K\sum_{l=1}^{M} n^{-\frac{1}{r}\frac{1}{q_l}\widehat{\beta}}$$
$$\le Kn^{-\frac{1}{r}\min(p_1,\dots,p_M)} \tag{4.3.4}$$

by replacing (4.2.5) and (4.2.6) by (4.3.1) as well as (4.2.8) and (4.2.9) by (4.3.2) in the argumentation used to obtain (4.2.4) and (4.2.7).

In the case of $k^\star - k_\star + 1 < 0$ a similar argumentation as used to get (4.3.4) yields

$$\mathbb{E}\left|\left(\int_0^{1+\frac{k^\star-k_\star+1}{n}} \prod_{j=1}^{M}\left|\phi_j^{(k_j)}(s)\right|^{p_j} ds\right)^{\frac{1}{r}} - \left(\int_0^{1+\frac{k^\star-k_\star+1}{n}} T_n(s)ds\right)^{\frac{1}{r}}\right| \le Kn^{-\frac{1}{r}\min(p_1,\dots,p_M)}. \tag{4.3.5}$$

Moreover, in the case of $k^\star - k_\star + 1 > 0$ we have

$$\mathbb{E}\left(\frac{1}{n}\sum_{i=n+k_\star}^{n+k^\star}\prod_{j=1}^{M}\left|\phi_j^{(k_j)}\left(\frac{i+a_j-k_j}{n}\right)\right|^{p_j}\right)^{\frac{1}{r}} \le Kn^{-\frac{1}{r}} \tag{4.3.6}$$

and in the case of $k^\star - k_\star + 1 < 0$ we have

$$\mathbb{E}\left(\int_{\frac{n+k^\star-k_\star+1}{n}}^{1}\prod_{j=1}^{M}\left|\phi_j^{(k_j)}(t)\right|^{p_j} dt\right)^{\frac{1}{r}} \le Kn^{-\frac{1}{r}}, \tag{4.3.7}$$

which follows by the application of Hölder's inequality with the parameters $p = \widehat{\beta}$ and $\widetilde{p} = \widehat{\beta}/(\widehat{\beta}-1)$ instead of the Cauchy-Schwarz inequality in the proof of (4.2.11) and (4.2.12).

We conclude this part of the proof in a similar way as done in the proof of Theorem 4.2.1, i.e. by combining the subadditivity of the function $s \mapsto |s|^{\frac{1}{r}}$ with (4.3.3), (4.3.4), (4.3.5), (4.3.6) and (4.3.7).

Step 2: By first transforming the Mth order power variation $V_n^{\langle M \rangle}$ into a $(M\rho)$th order power variation $V_n^{\langle M\rho \rangle}$ with $\rho := \max(\lceil p_1 \rceil, \ldots, \lceil p_M \rceil)$ as done in Remark 1.3.3 and then by applying *Step 1* with $r = 1$, we obtain

$$\mathbb{E}\Big|V_n - F\Big| = \mathcal{O}\left(n^{-\min(1, p_1/\rho, \ldots, p_M/\rho)}\right) \quad \text{as } n \to \infty.$$

Step 3: We can conclude this proof by applying for each $\varepsilon > 0$ the Markov inequality on

$$\mathbb{P}\Big(\Big|\big(V_n\big)^{\frac{1}{r}} - \big(F\big)^{\frac{1}{r}}\Big| \geq \varepsilon\Big) \quad \text{as well as} \quad \mathbb{P}\Big(\Big|V_n - F\Big| \geq \varepsilon\Big)$$

and by using the results proven in *Step 1* and *Step 2*. □

4.4. Some Examples

In this section we will exemplary provide some kernel functions with which the preceding results of this chapter are applicable.

REMARK 4.4.1. Note that Theorem 4.1.1 is applicable with the kernel functions presented in Proposition 1.2.3 (i) and (ii). Furthermore, Theorem 4.2.1, Corollary 4.2.3 and Theorem 4.3.1 are applicable in the case of the kernel functions being as in Proposition 1.2.3 (ii).

Now we will illustrate the preceding results of this chapter by applying them in the setting of Ornstein-Uhlenbeck type Lévy driven processes.

PROPOSITION 4.4.2. *For each $j = 1, \ldots, M$ suppose that the symmetric Lévy process $L^{\langle j \rangle} = (L_t^{\langle j \rangle})_{t \in \mathbb{R}}$ with Lévy measure ν_j and Blumenthal-Getoor index $\beta_j < 2$ is without a Brownian component and satisfies assumption ① with respect to the parameter $\theta_j \in (0, 2]$.*

Moreover, for each $j = 1, \dots, M$ assume $\lambda_j, p_j > 0$, $k_j \in \mathbb{N}$, $a_j \in \mathbb{Z}$, $c_j \in \mathbb{R}$ and set

$$X_t^{\langle j \rangle} := \int_{-\infty}^{t} g_j(t-s)\, dL_s^{\langle j \rangle} \quad \text{for} \quad t \geq 0$$

with $g_j(s) = c_j e^{-\lambda_j s} \mathbf{1}_{[0,\infty)}(s)$.

By denoting $\rho := \max(\lceil p_1 \rceil, \dots, \lceil p_M \rceil)$, $r = \prod_{j=1}^{M}(1 \vee p_j)$,

$$V_n := n^{-1+\sum_{j=1}^{M} k_j p_j} V_n^{\langle M \rangle} \quad \text{and} \quad F := \int_0^1 \prod_{j=1}^{M} \left| F_{j,k_j}(t) \right|^{p_j} dt,$$

where

$$V_n^{\langle M \rangle} := V_n^{\langle M \rangle}(X; a; k; p) := \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j} \qquad \text{(cf. Definition 1.3.2)}$$

and

$$F_{j,k_j}(t) := \int_{-\infty}^{t} g_j^{(k_j)}(t-s)\, dL_s^{\langle j \rangle},$$

we obtain

$$\left| V_n - F \right| = \mathcal{O}\left(\min\left(n^{-\frac{1}{r}\min(1,p_1,\dots,p_M)}, n^{-\min(1,p_1/\rho,\dots,p_M/\rho)} \right) \right) \qquad \text{a.s. as } n \to \infty.$$

Furthermore, if we have <u>either</u> for all $j = 1, \dots, M$ that

- $\sum_{l=1}^{M} p_l \leq 2$ *and* $\int_{|x|\geq 1} x^2 \nu_j(dx) < \infty$,

<u>or</u> for all $j = 1, \dots, M$ that

- $1 < \sum_{i=1}^{M} p_i \leq \min(\beta_1, \dots, \beta_M)$, $\beta_j = \theta_j \in (1,2)$ *and* $(L_t^{\langle j \rangle})_{t \in \mathbb{R}}$ *is a symmetric β_j-stable Lévy process with scale parameter $\sigma_j > 0$,*

then for each $\varepsilon > 0$ we get

$$\mathbb{P}\left(\left| (V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}} \right| \geq \varepsilon \right) \leq \frac{1}{\varepsilon} \mathbb{E}\left| (V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}} \right| = \mathcal{O}\left(n^{-\frac{1}{r}\min(1,p_1,\dots,p_M)} \right) \quad \text{as} \quad n \to \infty$$

as well as

$$\mathbb{P}\left(\left| V_n - F \right| \geq \varepsilon \right) \leq \frac{1}{\varepsilon} \mathbb{E}\left| V_n - F \right| = \mathcal{O}\left(n^{-\min(1,p_1/\rho,\dots,p_M/\rho)} \right) \quad \text{as} \quad n \to \infty.$$

.

Proof. Throughout this proof we will denote all positive constants by K, although they may change from line to line, and for $j = 1, \dots, M$ we will write $(\cdot)$ instead of j respectively $\langle\cdot\rangle$ instead of $\langle j\rangle$.

The functions $g_{(\cdot)}, \tilde{g}_{(\cdot)}$ with $\tilde{g}_{(\cdot)} \equiv 0$ have the following properties.

- For $s < 0$ we have $g_{(\cdot)}(s) = \tilde{g}_{(\cdot)}(s) = 0$.
- For $\theta > 0$ and $t \in \mathbb{R}$ we have
$$\int_{\mathbb{R}} |g_{(\cdot)}(t-s) - \tilde{g}_{(\cdot)}(-s)|^\theta ds = \frac{|c_{(\cdot)}|^\theta}{\theta\lambda_{(\cdot)}} e^{-\theta\lambda_{(\cdot)}t}[e^{\theta\lambda_{(\cdot)}t} - 0] < \infty.$$
- For $t, s \in \mathbb{R}$ we have
$$|g_{(\cdot)}(t-s) - \tilde{g}_{(\cdot)}(-s)| = |c_{(\cdot)}|e^{-\lambda_{(\cdot)}t}e^{\lambda_{(\cdot)}s}\mathbf{1}_{(-\infty,t]}(s) \leq |c_{(\cdot)}|.$$
- For $s \in (0,\infty)$ and $k \in \mathbb{N}$ we have $g_{(\cdot)}^{(k)}(s) = (-\lambda_{(\cdot)})^k c_{(\cdot)} e^{-\lambda_{(\cdot)}s}$, i.e.
  - $g_{(\cdot)} \in C^\infty((0,\infty))$ and $g_{(\cdot)}^{(k)} \in L^\theta((0,\infty))$ for $\theta > 0$,
  - for each $k \in \mathbb{N}$ the function $|g_{(\cdot)}^{(k)}|$ is decreasing on $(0,\infty)$,
  - for each $k \in \mathbb{N}$ and $\alpha \leq k$ we have $|g_{(\cdot)}^{(k)}(t)| \leq Kt^{\alpha-k}$, $t \in (0,\infty)$,
  - for $\theta > 0$ it holds that
$$\begin{aligned}&\int_0^\infty |g_{(\cdot)}^{(k)}(s)|^\theta \log(1/|g_{(\cdot)}^{(k)}(s)|)ds \\ \leq &K\int_0^\infty se^{-\theta\lambda_{(\cdot)}s}\, ds + K\int_0^\infty e^{-\theta\lambda_{(\cdot)}s}\, ds < \infty.\end{aligned}$$

By the above properties, we know that $g_{(\cdot)}, \tilde{g}_{(\cdot)}, L^{\langle\cdot\rangle}$ satisfy the assumptions ⓪, ①,②,③,⑤ and ⑥ with respect to the parameters α, k, θ, where $k \in \mathbb{N}, \theta = \theta_{(\cdot)}$ and $\alpha \in (0,k)$.

Furthermore, since we have
$$g_{(\cdot)}^{(k_{(\cdot)})} \in L^{\beta_{(\cdot)}}((0,\infty)) \cap L^2((0,\infty))$$

and $g_{(\cdot)}, \tilde{g}_{(\cdot)}$ satisfy the assumptions ⓪,② and ⑤, we obtain for $n \geq k_{(\cdot)}$ and $i = k_{(\cdot)}, \dots, n$ the following two properties

$$\| \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)} \|_{L^2(\mathbb{R})}^2 = \int_{-\infty}^{\frac{i}{n}} | \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)}(s)|^2 ds \leq K,$$

$$\| \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)} \|_{L^{\beta_{(\cdot)}}(\mathbb{R})}^{\beta_{(\cdot)}} = \int_{-\infty}^{\frac{i}{n}} | \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)}(s)|^{\beta_{(\cdot)}} ds \leq K,$$

where the first property follows from Remark 4.2.2 (ii) and the second is a consequence of using Lemma 1.5.1 in a similar way to Remark 4.2.2 (ii).

By applying Lemma 4.5.3 with respect to the parameters $\alpha_{(\cdot)}, k_{(\cdot)}, \theta_{(\cdot)}$, where $\alpha_{(\cdot)} \in (k_{(\cdot)} - 1/2, k_{(\cdot)})$, and $p = 2$ we get

$$\left(\frac{d}{dt}\right)^{k_{(\cdot)}} X_t^{\langle\cdot\rangle} = \int_{-\infty}^{t} g_{(\cdot)}^{(k_{(\cdot)})}(t-s)\, dL_s^{\langle\cdot\rangle} \qquad \lambda \otimes \mathbb{P} - \text{almost surely.} \tag{4.4.1}$$

An application of Theorem 4.1.1 (ii), Remark 4.1.2, Theorem 4.2.1, Remark 4.2.2 (i), Corollary 4.2.3 and Theorem 4.3.1 with the parameters $Q_{(\cdot)} = p_{(\cdot)} + 1$, $\kappa_{(\cdot)} = k_{(\cdot)} + 1$, $\alpha_{(\cdot)} = \kappa_{(\cdot)} - 1/\big(2(Q_{(\cdot)} \vee \beta_{(\cdot)})\big)$, $\theta_{(\cdot)}$ in combination with the representation (4.4.1) concludes this proof. □

4.5. Some technical auxiliary results

This section contains all the technical auxiliary results that we have used in the presiding sections of this chapter.

Definition 4.5.1. Let $I \subset \mathbb{R}$ be an interval. A function $\phi : I \to \mathbb{R}$ is called *absolutely continuous* if and only if on the interval I the function ϕ is almost everywhere differentiable with respect to the Lebesgue measure, i.e. for $a, b \in I$ with $a < b$ we have

$$\phi(b) - \phi(a) = \int_a^b \phi^{(1)}(s) ds.$$

Moreover, the function ϕ is said to be *k-times absolutely continuous* if and only if the functions $\phi, \phi^{(1)}, \dots, \phi^{(k-1)}$ are absolutely continuous.

Now we will look in to the effect Taylor's theorem has on the (linear) differential filter of an absolutely continuous function.

LEMMA 4.5.2. *Let $k, n \in \mathbb{N}$ with $k \geq n$. Moreover, let $g : [0,1] \to \mathbb{R}$ be a k-times absolutely continuous function. Then for $i = k, \ldots, n$ we have on the one hand*

$$\triangle_{i,n}^{k} g = \sum_{l,r=0}^{k-1} \frac{\kappa_{l,r}}{n^l} \int_{\frac{i-(r+1)}{n}}^{\frac{i-r}{n}} g^{(k)}(s) \left(\frac{i-r}{n} - s \right)^{k-1-l} ds$$

and in the case of $k \geq 2$ on the other hand

$$\triangle_{i,n}^{k-1} g = g^{(k-1)} \left(\frac{i-(k-1)}{n} \right) \frac{1}{n^{k-1}} + \sum_{l,r=0}^{k-2} \frac{\tau_{l,r}}{n^l} \int_{\frac{i-(r+1)}{n}}^{\frac{i-r}{n}} g^{(k)}(s) \left(\frac{i-r}{n} - s \right)^{k-1-l} ds,$$

where $(\kappa_{l,r})_{l,r=0,\ldots,k-1}, (\tau_{l,r})_{l,r=0,\ldots,k-2} \subset \mathbb{R}$ are suitable constants that do not depend on the parameters i, n and the function g.

PROOF. In order to ease our notations let

$$R(g^{(k)}, m, x, a) = \int_{x-a}^{x} g^{(k)}(s) \frac{(x-s)^m}{m!} ds.$$

An application of Taylor's theorem yields

$$\begin{aligned} \triangle_{i,n}^{1} g &= g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right) \\ &= \sum_{l_1=1}^{k-1} \frac{1}{l_1!} g^{(l_1)} \left(\frac{i-1}{n}\right) n^{-l_1} + \underbrace{R\left(g^{(k)}, k-1, \frac{i}{n}, \frac{1}{n}\right)}_{=:r_1(g,i,n)}. \end{aligned} \tag{4.5.1}$$

In the case of $k \geq 2$ we get by induction and by using Taylor's theorem for $m = 1, \ldots, k-1$ the following result

$$\begin{aligned} &\triangle_{i,n}^{m+1} g = \triangle_{i,n}^{m} g - \triangle_{i-1,n}^{m} g \\ &= \sum_{l_1=1}^{k-1} \cdots \sum_{l_m=1}^{k-1-\sum_{r=1}^{m-1} l_r} \left(\prod_{r=1}^{m} \frac{n^{-l_r}}{l_r!} \right) \left[g^{(\sum_{r=1}^{m} l_r)} \left(\frac{i-m}{n} \right) - g^{(\sum_{r=1}^{m} l_r)} \left(\frac{i-1-m}{n} \right) \right] \\ &\quad + r_m(g,i,n) - r_m(g,i-1,n) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{l_1=1}^{k-1} \cdots \sum_{l_{m+1}=1}^{k-1-\sum_{r=1}^{m} l_r} \left(\prod_{r=1}^{m+1} \frac{1}{l_r!} n^{-l_r} \right) g^{(\sum_{r=1}^{m+1} l_r)} \left(\frac{i-(m+1)}{n} \right) \\
&\quad + \sum_{l_1=1}^{k-1} \cdots \sum_{l_m=1}^{k-1-\sum_{r=1}^{m-1} l_r} \left(\prod_{r=1}^{m} \frac{1}{l_r!} n^{-l_r} \right) R\left(g^{(k)}, k-1-\sum_{r=1}^{m} l_r, \frac{i-m}{n}, \frac{1}{n} \right) \\
&\quad + r_m(g,i,n) - r_m(g,i-1,n) \\
&= \sum_{l_1=1}^{k-1} \cdots \sum_{l_{m+1}=1}^{k-1-\sum_{r=1}^{m} l_r} \left(\prod_{r=1}^{m+1} \frac{1}{l_r!} n^{-l_r} \right) g^{(\sum_{r=1}^{m+1} l_r)} \left(\frac{i-(m+1)}{n} \right) + r_{m+1}(g,i,n). \qquad (4.5.2)
\end{aligned}
$$

Hence, by (4.5.1) and (4.5.2), we have on the one hand

$$
\Delta_{i,n}^{k} g = r_k(g,i,n) = \sum_{l,r=0}^{k-1} \frac{\kappa_{l,r}}{n^l} R\left(g^{(k)}, k-1-l, \frac{i-r}{n}, \frac{1}{n} \right)
$$

and in the case of $k \geq 2$ on the other hand

$$
\begin{aligned}
\Delta_{i,n}^{k-1} g &= g^{(k-1)} \left(\frac{i-(k-1)}{n} \right) \frac{1}{n^{k-1}} + r_{k-1}(g,i,n) \\
&= g^{(k-1)} \left(\frac{i-(k-1)}{n} \right) \frac{1}{n^{k-1}} + \sum_{l,r=0}^{k-2} \frac{\tau_{l,r}}{n^l} R\left(g^{(k)}, k-1-l, \frac{i-r}{n}, \frac{1}{n} \right).
\end{aligned}
$$

□

The following lemma will provide the pathwise properties of Lévy driven processes, which are the foundation of this chapter.

LEMMA 4.5.3. *Let $p \geq 1$ and let $L = (L_t)_{t \in \mathbb{R}}$ be a symmetric Lévy process without a Brownian component and with a Blumenthal-Getoor index $\beta < 2$. Suppose that L as well as the kernel functions $g, \tilde{g}$ satisfy the assumptions ⓪ to ③ and ⑤ with respect to the parameters α, k, θ and in the case of $p = \theta$ the assumption ⑥ as well. Then in the case of $\alpha > k - 1/(p \vee \beta)$ the Lévy driven process*

$$
X_t = \int_{-\infty}^{t} g(t-s) - \tilde{g}(-s) \, dL_s \qquad (t \geq 0)
$$

is almost surely k-times absolutely continuous on the interval $[0,1]$ with

$$
\left(\frac{d}{dt} \right)^k X \in L^p([0,1]) \quad a.s. \quad and \quad \left(\frac{d}{dt} \right)^k X_t = \int_{-\infty}^{t} g^{(k)}(t-s) \, dL_s \quad \lambda \otimes \mathbb{P} - a.s..
$$

PROOF. See [**10**, Lemma 4.3]. □

Note that the next lemma is the main tool of this chapter for obtaining convergence rates.

LEMMA 4.5.4. *For $j = 1, \dots, M$ let $k_j \in \mathbb{N}$, $a_j \in \mathbb{Z}$ and $p_j > 0$.*

Moreover, for each $j = 1, \dots, M$ suppose that $\phi_j : [0,1] \to \mathbb{R}$ is a (k_j+1)-times absolutely continuous function satisfying $\int_0^1 |\phi_j^{(k_j+1)}(s)|^{\max(p_j,1)} ds < \infty$.

Then it holds that

$$\left| n^{-1+\sum_{j=1}^M k_j p_j} V_n^{\langle M \rangle}(\phi; a; k; p) - \int_0^1 \prod_{j=1}^M \left| \phi_j^{(k_j)}(s) \right|^{p_j} ds \right| = \mathcal{O}\left(n^{-\frac{1}{r}\min(1,p_1,\dots,p_M)} \right)$$

as $n \to \infty$, where

$$V_n^{\langle M \rangle}(\phi; a; k; p) := \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^M \left| \triangle_{i+a_j,n}^{k_j} \phi_j \right|^{p_j} \qquad \text{(cf. Definition 1.3.2)}$$

and $r = \prod_{j=1}^M (1 \vee p_j)$.

PROOF. In order to ease our notations we will throughout this proof denote all positive constants by K, although they may change from line to line, and for $j = 1, \dots, M$ often write $(\cdot)$ instead of j respectively $\langle \cdot \rangle$ instead of $\langle j \rangle$.

Furthermore, we will assume $n \in \mathbb{N}$ to be sufficiently large, so that on the one hand we have $k_\star \le n + k^\star$ and on the other hand the inequalities below regarding the parameters $a_{(\cdot)}, k_{(\cdot)}, k_\star, k^\star$ are satisfied.

Let $S_{(\cdot)} = \max(1, 1/p_{(\cdot)})$, $\tilde{S}_{(\cdot)} \in (1, \infty]$ with $1/S_{(\cdot)} + 1/\tilde{S}_{(\cdot)} = 1$, $\pi = \min_{i=1,\dots,M}\{p_i\}$ and

$$A_n := \left| \left(\sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^M \left| \triangle_{i+a_j,n}^{k_j} \phi_j \right|^{p_j} \right)^{\frac{1}{r}} - \left(\sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^M \left| \phi_j^{(k_j)} \left(\frac{i + a_j - k_j}{n} \right) \frac{1}{n^{k_j}} \right|^{p_j} \right)^{\frac{1}{r}} \right|.$$

An application of Corollary 1.5.4 yields

$$\left(n^{-1+\sum_{j=1}^M k_j p_j} \right)^{\frac{1}{r}} A_n$$

$$\leq \sum_{l=1}^{M}\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\left(\prod_{j=l+1}^{M}\left|n^{k_j}\,\triangle_{i+a_j,n}^{k_j}\,\phi_j\right|^{p_j}\right)\left(\prod_{j=1}^{l-1}\left|\phi_j^{(k_j)}\left(\frac{i+a_j-k_j}{n}\right)\right|^{p_j}\right)\left|R_{i,n}^{\langle l\rangle}\right|^{p_l}\right)^{\frac{1}{r}}$$

$$\leq \sum_{l=1}^{M}\left(\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\left(\prod_{j=1,\dots,M:j\neq l}\left|B_{i,n}^{\langle j\rangle}\right|^{p_j}\right)\left|R_{i,n}^{\langle l\rangle}\right|^{p_l}\right)^{\frac{1}{r}}, \tag{4.5.3}$$

where based on Lemma 4.5.2 we used

$$\begin{aligned}&\left|n^{k_{(\cdot)}}\,\triangle_{i+a_{(\cdot)},n}^{k_{(\cdot)}}\,\phi_{(\cdot)}-\phi_{(\cdot)}^{(k_{(\cdot)})}\left(\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}\right)\right|\\ \leq &K\sum_{l,r=0}^{k_{(\cdot)}-1}n^{k_{(\cdot)}-l}\int_{\frac{i+a_{(\cdot)}-(r+1)}{n}}^{\frac{i+a_{(\cdot)}-r}{n}}\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(s)\right|\left(\frac{i+a_{(\cdot)}-r}{n}-s\right)^{k_{(\cdot)}-l}ds\\ \leq &K\int_{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}}^{\frac{i+a_{(\cdot)}}{n}}\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(s)\right|ds=:R_{i,n}^{\langle\cdot\rangle}\end{aligned} \tag{4.5.4}$$

and

$$B_{i,n}^{\langle\cdot\rangle}:=R_{i,n}^{\langle\cdot\rangle}+\left|\phi_{(\cdot)}^{(k_{(\cdot)})}\left(\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}\right)\right|.$$

For $i=k_\star,\dots,n+k^\star$ we have $0\leq (i+a_{(\cdot)}-k_{(\cdot)})/n<(i+a_{(\cdot)})/n\leq 1$, which in combination with the continuity of the function $\phi_{(\cdot)}^{(k_{(\cdot)})}$ on the interval $[0,1]$, $\int_0^1|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(s)|^{\max(p_{(\cdot)},1)}ds<\infty$ and Hölder's inequality implies

$$\sum_{i=k_\star}^{n+k^\star}\left|R_{i,n}^{\langle\cdot\rangle}\right|^{S_{(\cdot)}p_{(\cdot)}}\leq Kn^{1-S_{(\cdot)}p_{(\cdot)}}\int_0^1\left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(s)\right|^{\max(p_{(\cdot)},1)}ds\leq Kn^{1-S_{(\cdot)}p_{(\cdot)}} \tag{4.5.5}$$

and

$$B_{i,n}^{\langle\cdot\rangle}\leq K \quad \text{for} \quad i=k_\star,\dots,n+k^\star. \tag{4.5.6}$$

By using Hölder's inequality with the parameters $S_{(\cdot)}$ and $\tilde{S}_{(\cdot)}$ in (4.5.3) followed by an application of (4.5.5) and (4.5.6), we get

$$\left(n^{-1+\sum_{j=1}^{M}k_jp_j}\right)^{\frac{1}{r}}A_n\leq K\sum_{l=1}^{M}n^{\frac{1}{r}\left(\frac{1}{S_l}+\frac{1}{\tilde{S}_l}-1-p_l\right)}=\mathcal{O}\left(n^{-\frac{1}{r}\pi}\right)\text{ as } n\to\infty. \tag{4.5.7}$$

Now we consider the representation

$$\frac{1}{n}\sum_{i=k_\star}^{n+k^\star}\prod_{j=1}^{M}\left|\phi_j^{(k_j)}\left(\frac{i+a_j-k_j}{n}\right)\right|^{p_j} = \int_0^{\frac{n+k^\star-k_\star+1}{n}} T_n(s)ds,$$

where

$$T_n(s) := \sum_{i=k_\star}^{n+k^\star}\prod_{j=1}^{M}\Bigg|\underbrace{\phi_j^{(k_j)}\left(\frac{i+a_j-k_j}{n}\right)\mathbf{1}_{\left(\frac{i-k_\star}{n},\frac{i-k_\star+1}{n}\right]}(s)}_{=:\psi_j(s)}\Bigg|^{p_j}.$$

Note that by continuity of the function $\phi_{(\cdot)}^{(k_{(\cdot)})}$ on the interval $[0,1]$ we have

$$\frac{1}{n}\sum_{i=n+k_\star}^{n+k^\star}\prod_{j=1}^{M}\left|\phi_j^{(k_j)}\left(\frac{i+a_j-k_j}{n}\right)\right|^{p_j} = \mathcal{O}\left(\frac{1}{n}\right) \text{ as } n\to\infty \tag{4.5.8}$$

in the case of $k^\star - k_\star + 1 > 0$ and

$$\int_{\frac{n+k^\star-k_\star+1}{n}}^{1}\prod_{j=1}^{M}\left|\phi_j^{(k_j)}(s)\right|^{p_j}ds = \mathcal{O}\left(\frac{1}{n}\right) \text{ as } n\to\infty \tag{4.5.9}$$

in the case of $k^\star - k_\star + 1 < 0$.

Furthermore, in the case of $k^\star - k_\star + 1 \geq 0$ we obtain by using Lemma 1.5.3 the following inequalities

$$\begin{aligned}
&\left|\left(\int_0^1\prod_{j=1}^{M}\left|\phi_j^{(k_j)}(s)\right|^{p_j}ds\right)^{\frac{1}{r}} - \left(\int_0^1 T_n(s)ds\right)^{\frac{1}{r}}\right| \\
&\leq \sum_{l=1}^{M}\left(\int_0^1\left(\prod_{j=l+1}^{M}\left|\phi_j^{(k_j)}(s)\right|^{p_j}\right)\left(\prod_{j=1}^{l-1}\left|\psi_j(s)\right|^{p_j}\right)\left|\widetilde{R}_n^{\langle l\rangle}(s)\right|^{p_l}ds\right)^{\frac{1}{r}} \\
&\leq \sum_{l=1}^{M}\left(\int_0^1\left(\prod_{j=1,\dots,M:j\neq l}\left|\widetilde{B}_n^{\langle j\rangle}(s)\right|^{p_j}\right)\left|\widetilde{R}_n^{\langle l\rangle}(s)\right|^{p_l}ds\right)^{\frac{1}{r}},
\end{aligned} \tag{4.5.10}$$

where for $i = k_\star,\dots,n+k_\star-1$ and $s\in\left(\frac{i-k_\star}{n},\frac{i-k_\star+1}{n}\right]$ by Taylor's theorem we used

$$\left|\phi_{(\cdot)}^{(k_{(\cdot)})}(s)-\psi_{(\cdot)}(s)\right| = \left|\phi_{(\cdot)}^{(k_{(\cdot)})}(s)-\phi_{(\cdot)}^{(k_{(\cdot)})}\left(\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}\right)\right|$$

$$\leq \int_{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n} \wedge \frac{i-k_\star}{n}}^{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n} \vee \frac{i-k_\star+1}{n}} \left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(y)\right| dy =: \widetilde{R}_n^{\langle\cdot\rangle}(s)$$

and

$$\widetilde{B}_n^{\langle\cdot\rangle}(s) := R_n^{\langle\cdot\rangle}(s) + \left|\phi_{(\cdot)}^{(k_{(\cdot)})}(s)\right|.$$

Since in the case of $k^\star - k_\star + 1 \geq 0$ for $i = k_\star, \ldots, n + k_\star - 1$ we have

$$0 \leq \frac{i + a_{(\cdot)} - k_{(\cdot)}}{n} \wedge \frac{i - k_\star}{n}, \frac{i + a_{(\cdot)} - k_{(\cdot)}}{n} \vee \frac{i - k_\star + 1}{n} \leq 1,$$

we get by a similar argumentation as in (4.5.5) and (4.5.6) that $\widetilde{B}_n^{\langle\cdot\rangle}(s)$ is bounded on the interval $[0, 1]$ and

$$\begin{aligned}
\int_0^1 \left|\widetilde{R}_n^{\langle\cdot\rangle}(s)\right|^{S_{(\cdot)}p_{(\cdot)}} ds &= \sum_{i=k_\star}^{n+k_\star-1} \int_{\frac{i-k_\star}{n}}^{\frac{i-k_\star+1}{n}} \left|\widetilde{R}_n^{\langle\cdot\rangle}(s)\right|^{S_{(\cdot)}p_{(\cdot)}} ds \\
&\leq \frac{1}{n} \sum_{i=k_\star}^{n+k_\star-1} \left(\int_{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n} \wedge \frac{i-k_\star}{n}}^{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n} \vee \frac{i-k_\star+1}{n}} \left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(y)\right| dy \right)^{S_{(\cdot)}p_{(\cdot)}} \\
&\leq K n^{-S_{(\cdot)}p_{(\cdot)}} \int_0^1 \left|\phi_{(\cdot)}^{(k_{(\cdot)}+1)}(y)\right|^{\max(p_{(\cdot)},1)} dy \leq K n^{-S_{(\cdot)}p_{(\cdot)}}.
\end{aligned}$$

An application of Hölder's inequality with the parameters $S_{(\cdot)}$ and $\widetilde{S}_{(\cdot)}$ in (4.5.10) followed by a combination of the above results for $\widetilde{B}_n^{\langle\cdot\rangle}$ and $\int_0^1 \left|\widetilde{R}_n^{\langle\cdot\rangle}(s)\right|^{S_{(\cdot)}p_{(\cdot)}} ds$ leads in the case of $k^\star - k_\star + 1 \geq 0$ to the following convergence

$$\left| \left(\int_0^1 \prod_{j=1}^{M} \left|\phi_j^{(k_j)}(s)\right|^{p_j} ds \right)^{\frac{1}{r}} - \left(\int_0^1 T_n(s) ds \right)^{\frac{1}{r}} \right| = \mathcal{O}\left(n^{-\frac{1}{r}\pi}\right) \tag{4.5.11}$$

as $n \to \infty$.

In the case of $k^\star - k_\star + 1 < 0$ we get similarly

$$\left| \left(\int_0^{1+\frac{k^\star-k_\star+1}{n}} \prod_{j=1}^{M} \left|\phi_j^{(k_j)}(s)\right|^{p_j} ds \right)^{\frac{1}{r}} - \left(\int_0^{1+\frac{k^\star-k_\star+1}{n}} T_n(s) ds \right)^{\frac{1}{r}} \right| = \mathcal{O}\left(n^{-\frac{1}{r}\pi}\right) \tag{4.5.12}$$

as $n \to \infty$.

We can conclude this proof by using the subadditivity of the function $s \mapsto |s|^{\frac{1}{r}}$ followed by a combination of (4.5.7), (4.5.8), (4.5.11) in the case of $k^\star - k_\star + 1 \geq 0$, by a combination of (4.5.9), (4.5.12) in the case of $k^\star - k_\star + 1 < 0$ as well as by a subsequent application of Lemma 1.5.5. □

This last lemma will allow us to replace the multipower variations of pathwise absolutely continuous functions by the multipower variations of pathwise $C_c^\infty(\mathbb{R})$-functions, which for example is a crucial part in the proof of Theorem 4.1.1 (i).

LEMMA 4.5.5. *For $j = 1, \ldots, M$ let $k_j \in \mathbb{N}$, $a_j \in \mathbb{Z}$, $p_j > 0$ as well as $S_j \geq 1$ with $S_j p_j \geq 1$ and $\sum_{l=1}^M 1/S_l = 1$. Moreover, for each $j = 1, \ldots, M$ suppose that $\phi_j : [0,1] \to \mathbb{R}$ is a k_j-times absolutely continuous function satisfying $\int_0^1 |\phi_j^{(k_j)}(s)|^{S_j p_j} ds < \infty$. Then for each $j = 1, \ldots, M$ there exist a sequence of functions $(\psi_{j,l})_{l\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ satisfying $\int_0^1 |\phi_j^{(k_j)}(s) - \psi_{j,l}^{(k_j)}(s)|^{S_j p_j} ds \xrightarrow{l\to\infty} 0$ so that*

$$\left(n^{-1+\sum_{j=1}^M k_j p_j}\right)^{\frac{1}{r}} \left| \left(V_n(\phi_1, \ldots, \phi_M)\right)^{\frac{1}{r}} - \left(V_n(\psi_{1,l}, \ldots, \psi_{M,l})\right)^{\frac{1}{r}} \right|$$

converges uniformly to 0 as $l \to \infty$ for sufficiently large $n \in \mathbb{N}$, where

$$V_n(\zeta_1, \ldots, \zeta_M) := \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^M \left| \triangle_{i+a_j,n}^{k_j} \zeta_j \right|^{p_j} \qquad \text{(cf. Definition 1.3.2)}$$

and $r = \prod_{j=1}^M (1 \vee p_j)$.

PROOF. In order to ease our notations we will throughout this proof denote all positive constants by K, although they may change from line to line, and for $j = 1, \ldots, M$ we will often write $(\cdot)$ instead of j. Furthermore, we will assume $n \in \mathbb{N}$ to be sufficiently large, so that besides the other properties requiring a sufficiently large n below we have $k_\star \leq n + k^\star$.

By [**26**, Theorem 3.17] there exists a sequence $(\psi_{(\cdot),l})_{l\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ with

$$\sum_{d=0}^{k_{(\cdot)}} \|\phi_{(\cdot)}^{(d)} - \psi_{(\cdot),l}^{(d)}\|_{L^{S_{(\cdot)}p_{(\cdot)}}(0,1)} \xrightarrow{l\to\infty} 0. \tag{4.5.13}$$

Since for sufficiently large $n \in \mathbb{N}$ as well as $i = k_\star, \ldots, n + k^\star$ we have

$$0 \leq (i + a_{(\cdot)} - k_{(\cdot)})/n < (i + a_{(\cdot)})/n \leq 1,$$

an application for each $l \in \mathbb{N}$ of Lemma 4.5.2 yields on the one hand

$$\begin{aligned}
\left|\triangle^{k_{(\cdot)}}_{i+a_{(\cdot)},n}\phi_{(\cdot)} - \triangle^{k_{(\cdot)}}_{i+a_{(\cdot)},n}\psi_{(\cdot),l}\right| &= \left|\triangle^{k_{(\cdot)}}_{i+a_{(\cdot)},n}\left(\phi_{(\cdot)} - \psi_{(\cdot),l}\right)\right| \\
&\leq K \sum_{d_1,d_2=0}^{k_{(\cdot)}-1} \frac{1}{n^{d_1}} \int_{\frac{i+a_{(\cdot)}-(d_2+1)}{n}}^{\frac{i+a_{(\cdot)}-d_2}{n}} \left|\phi^{(k_{(\cdot)})}_{(\cdot)}(s) - \psi^{(k_{(\cdot)})}_{(\cdot),l}(s)\right| \left(\frac{i+a_{(\cdot)}-d_2}{n} - s\right)^{k_{(\cdot)}-1-d_1} ds \\
&\leq K n^{1-k_{(\cdot)}} \int_{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}}^{\frac{i+a_{(\cdot)}}{n}} \left|\phi^{(k_{(\cdot)})}_{(\cdot)}(s) - \psi^{(k_{(\cdot)})}_{(\cdot),l}(s)\right| ds \\
&\leq K n^{\frac{1}{S_{(\cdot)}p_{(\cdot)}}-k_{(\cdot)}} \left(\int_{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}}^{\frac{i+a_{(\cdot)}}{n}} \left|\phi^{(k_{(\cdot)})}_{(\cdot)}(s) - \psi^{(k_{(\cdot)})}_{(\cdot),l}(s)\right|^{S_{(\cdot)}p_{(\cdot)}} ds\right)^{\frac{1}{S_{(\cdot)}p_{(\cdot)}}}, \qquad (4.5.14)
\end{aligned}$$

where in the case of $S_{(\cdot)}p_{(\cdot)} > 1$ the last inequality follows by an application of Hölder's inequality with the parameters $q_{(\cdot)} = S_{(\cdot)}p_{(\cdot)}$ and $\tilde{q}_{(\cdot)} = S_{(\cdot)}p_{(\cdot)}/(S_{(\cdot)}p_{(\cdot)} - 1)$, and on the other hand by a similar argumentation as above

$$\left|\triangle^{k_{(\cdot)}}_{i+a_{(\cdot)},n}\phi_{(\cdot)}\right| \leq K n^{\frac{1}{S_{(\cdot)}p_{(\cdot)}}-k_{(\cdot)}} \left(\int_{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}}^{\frac{i+a_{(\cdot)}}{n}} \left|\phi^{(k_{(\cdot)})}_{(\cdot)}(s)\right|^{S_{(\cdot)}p_{(\cdot)}} ds\right)^{\frac{1}{S_{(\cdot)}p_{(\cdot)}}} \qquad (4.5.15)$$

as well as

$$\left|\triangle^{k_{(\cdot)}}_{i+a_{(\cdot)},n}\psi_{(\cdot),l}\right| \leq K n^{\frac{1}{S_{(\cdot)}p_{(\cdot)}}-k_{(\cdot)}} \left(\int_{\frac{i+a_{(\cdot)}-k_{(\cdot)}}{n}}^{\frac{i+a_{(\cdot)}}{n}} \left|\psi^{(k_{(\cdot)})}_{(\cdot),l}(s)\right|^{S_{(\cdot)}p_{(\cdot)}} ds\right)^{\frac{1}{S_{(\cdot)}p_{(\cdot)}}}. \qquad (4.5.16)$$

For

$$A_{l,n} := \left(n^{-1+\sum_{j=1}^{M} k_j p_j}\right)^{\frac{1}{r}} \left|\left(V_n(\phi_1,\ldots,\phi_M)\right)^{\frac{1}{r}} - \left(V_n(\psi_{1,l},\ldots,\psi_{M,l})\right)^{\frac{1}{r}}\right|$$

an application of Corollary 1.5.4 followed by Lemma 1.5.2 and a combination of (4.5.14) to (4.5.16) yields

$$\begin{aligned}
A_{l,n} \leq K \sum_{d=1}^{M} &\left(\prod_{j=d+1}^{M}\left(\sum_{i=k_\star}^{n+k^\star}\int_{\frac{i+a_j-k_j}{n}}^{\frac{i+a_j}{n}}\left|\phi_j^{(k_j)}(s)\right|^{S_jp_j} ds\right)^{\frac{1}{S_jr}}\right) \\
&\left(\prod_{j=1}^{d-1}\left(\sum_{i=k_\star}^{n+k^\star}\int_{\frac{i+a_j-k_j}{n}}^{\frac{i+a_j}{n}}\left|\psi_{j,l}^{(k_j)}(s)\right|^{S_jp_j} ds\right)^{\frac{1}{S_jr}}\right) \\
&\left(\sum_{i=k_\star}^{n+k^\star}\int_{\frac{i+a_d-k_d}{n}}^{\frac{i+a_d}{n}}\left|\phi_d^{(k_d)}(s) - \psi_{d,l}^{(k_d)}(s)\right|^{S_dp_d} ds\right)^{\frac{1}{S_dr}}
\end{aligned}$$

$$\leq K \sum_{d=1}^{M} \Bigg(\prod_{j=d+1}^{M} \Bigg(\int_0^1 \Big| \phi_j^{(k_j)}(s) \Big|^{S_j p_j} ds \Bigg)^{\frac{1}{S_j r}} \Bigg) \Bigg(\prod_{j=1}^{d-1} \Bigg(\int_0^1 \Big| \psi_{j,l}^{(k_j)}(s) \Big|^{S_j p_j} ds \Bigg)^{\frac{1}{S_j r}} \Bigg)$$

$$\Bigg(\int_0^1 \Big| \phi_d^{(k_d)}(s) - \psi_{d,l}^{(k_d)}(s) \Big|^{S_d p_d} ds \Bigg)^{\frac{1}{S_d r}},$$

where the last inequality follows from the fact that for sufficiently large $n \in \mathbb{N}$ and $i = k_\star, \ldots, n + k^\star$ we have $0 \leq (i + a_{(\cdot)} - k_{(\cdot)})/n < (i + a_{(\cdot)})/n \leq 1$.

The above inequality and (4.5.13) imply that

$$\Big(n^{-1+\sum_{j=1}^{M} k_j p_j} \Big)^{\frac{1}{r}} \Big| \Big(V_n(\phi_1, \ldots, \phi_M) \Big)^{\frac{1}{r}} - \Big(V_n(\psi_{1,l}, \ldots, \psi_{M,l}) \Big)^{\frac{1}{r}} \Big|$$

converges uniformly to 0 as $l \to \infty$ for sufficiently large $n \in \mathbb{N}$. □

CHAPTER 5

Limit Theorems and Convergence Rates for Multipower Variations of Fractional-Lévy-Motion Driven Processes

The focus of this chapter lies on the extension of the limit theorems for Lévy driven processes, which were presented in the preceding chapters, to processes driven by the fractional Lévy motion. Our main tool will be Remark 1.2.13 that allows us, by using the underlying Lévy process, to represent a process driven by the fractional Lévy motion as a Lévy driven process.

Note that the results of the preceding chapters where the driving process is a symmetric α-stable Lévy processes can not be extended by using Remark 1.2.13, since by construction the underlying Lévy process of a process driven by a fractional Lévy motion is required to be a pure jump Lévy process with finite second moments.

Moreover, in order to illustrate the applicability of the main results of this chapter, i.e. Theorem 5.0.2 and Theorem 5.0.4 below, and to show that for some specific kernel function many assumptions in these results become redundant, we will apply them in the setting of Ornstein-Uhlenbeck type processes that are driven by the fractional Lévy motion.

To provide a clear and comprehending overview of the notations and definitions used in Theorem 5.0.2 and in Proposition 5.0.6, we will summarise and present the respective notations and definitions in the following details.

Details 5.0.1.

- $V_n^{\langle M\rangle} := V_n^{\langle M\rangle}(Y;a;k;p) = \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \triangle_{i+a_j,n}^{k_j} Y^{\langle j\rangle} \right|^{p_j}$, *cf. Definition 1.3.2.*
- *For each* $t \in \mathbb{R}$ *the jumps of the Lévy process* $(L_s)_{s\in\mathbb{R}}$ *at time* t *are denoted by* ΔL_t*, where* $\Delta L_t := L_t - L_{t-}$ *with* $L_{t-} := \lim_{s\uparrow t, s<t} L_s$.

- $(T_m)_{m\in\mathbb{N}}$ *is a sequence of* $\mathbb{F}$*-stopping times, where* $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ *is the filtration generated by the Lévy process* $(L_t)_{t\geq 0}$*, that exhausts the jumps of* $(L_t)_{t\geq 0}$*, i.e.*
  - $\{T_m(\omega) : m \geq 1\} \cap [0,\infty) = \{t \geq 0 : \Delta L_t \neq 0\}$,
  - $T_n(\omega) \neq T_m(\omega)$ *for* $n \neq m$ *with* $T_n(\omega) < \infty$.
- $H_m^{\langle M\rangle} := \sum_{l=-\min(a_1,\dots,a_M)}^{\infty} \prod_{j=1}^{M} |c_j \cdot h_j(l + a_j + U_m)|^{p_j}$*, where*
  - $h_j(x) := \sum_{r=0}^{k_j} (-1)^r \frac{k_j!}{r!(k_j-r)!}(x-r)_+^{\alpha_j}$ *for* $x \in \mathbb{R}$,
  - $(U_m)_{m\in\mathbb{N}}$ *is a sequence of independent and uniform* $[0,1]$*-distributed random variables that on the one hand lives on the probability space* $(\tilde{\Omega}, \widetilde{\mathcal{F}}, \tilde{\mathbb{P}})$*, which is an extension of the underlying probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, and on the other hand is independent of the* σ*-algebra* $\mathcal{F}$.
- $C^{\langle M\rangle} := \prod_{j=1}^{M} |C_j|^{p_j}$ *with* $C_j := c_j \prod_{r=0}^{k_j-1}(\alpha_j - r)$.
- $r := \prod_{j=1}^{M}(1 \vee p_j)$.
- $\tau := \min\limits_{j=1,\dots,M} \frac{\tau_j(\theta_j, S_j p_j)}{S_j}$*, where*
  - $\tau_j(\theta_j, S_j p_j) := \begin{cases} -S_j p_j\left(1 - \frac{1}{\theta_j} - \frac{1}{S_j p_j}\right), & \text{for } \theta_j \in (1,2] \\ 1, & \text{for } \theta_j \in (0,1] \end{cases}$.

Now we come to a limit theorem for processes driven by the fractional Lévy motion, which is based on the results in Theorem 2.1.1 and Theorem 2.2.1.

THEOREM 5.0.2. *Let* $L = (L_t)_{t\in\mathbb{R}}$ *be a symmetric Lévy process without a Brownian component, with a Lévy measure* ν *satisfying* $\int_{|x|\geq 1} x^2 \nu(dx) < \infty$ *and Blumenthal-Getoor index* $\beta < 2$. *For each* $j = 1,\dots,M$ *let* $0 < d_j < 1/2$ *and* $\phi_j \in H$*, cf. Definition 1.2.9, and suppose that the functions* $g_j, \tilde{g}_j$ *given by* $g_j = (I_+^{d_j}\phi_j)$, $\tilde{g}_j \equiv 0$ *as well as the Lévy process* L *satisfy the assumptions* ⓪,②,③,④,⑤ *with respect to the parameters* $\alpha_j, c_j, k_j, \theta_j$ *and in the case of* $\theta_j = 1$ *the assumption* ⑥ *as well. Moreover, for each* $j = 1,\dots,M$ *assume* $a_j \in \mathbb{Z}$, $p_j > 0$, $S_j \geq 1$ *with* $\sum_{i=1}^{M} 1/S_i = 1$ *and set*

$$Y_t^{\langle j\rangle} := \int_{-\infty}^{\infty} \phi_j(t-s)\, dM_s^{\langle j\rangle} \quad \textit{for} \quad t \in \mathbb{R}$$

with

$$M_t^{\langle j\rangle} := \frac{1}{\Gamma(d_j+1)} \int_{-\infty}^{t} (t-s)_+^{d_j} - (-s)_+^{d_j} \, dL_s \quad for \quad t \in \mathbb{R}.$$

Then by assuming for each $j = 1, \dots, M$ *in* (ii) *and* (iv) *that*

- *the function* $f_j : [0,\infty) \to \mathbb{R}$ *given by* $f_j(t) = g_j(t)t^{-\alpha_j}$ *for* $t > 0$ *satisfies* $f_j \in C^{k_j}([0,\infty))$ *with* $f_j(0) = c_j$,

as well as in (iii) *and* (iv) *that*

- $\| \blacktriangle_{i,n}^{k_j} g_j \|_{L^2(\mathbb{R})}^2 \leq n^{-2\alpha_j - 1} K$ *for all sufficiently large* $n \in \mathbb{N}$ *and* $i = k_j, \dots, n$, *where* $K > 0$ *is a suitable constant,*

we get by using the definitions and notations in Details 5.0.1 the following results.

(i) *If* $\alpha_j < k_j - 1/(S_j p_j)$ *for all* j *and* <u>*either*</u> L *is a compound Poisson process* <u>*or*</u> $M = 1$ *with* $p_M > \beta$ *then we have*

$$n^{\sum_{j=1}^M \alpha_j p_j} V_n^{\langle M\rangle} \xrightarrow{L-s} Z = \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} H_m^{\langle M\rangle} \quad as \quad n \to \infty.$$

(ii) *If* $\alpha_j = k_j - 1/(S_j p_j)$ *as well as* $1/(S_j p_j) + 1/\theta_j > 1$ *for all* j *and* <u>*either*</u> L *is a compound Poisson process* <u>*or*</u> $M = 1$ *with* $p_M > \beta$ *then we obtain*

$$\frac{n^{\sum_{j=1}^M \alpha_j p_j}}{\log(n)} V_n^{\langle M\rangle} \xrightarrow{\mathbb{P}} \widehat{Z} = C^{\langle M\rangle} \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} \quad as \quad n \to \infty.$$

In the case of L *being a compound Poisson process we have instead of the convergence in probability an almost sure convergence with convergence rate* $\mathcal{O}\big(\log(n)\big)^{-\frac{\tau}{r}}$.

(iii) *For* $j = 1, \dots, M$ *suppose that* $p_j \in (0,2)$ *with* $\sum_{i \in \{1,\dots,M\}\setminus\{j\}} p_i < 2$ *and set* $Q_j := 2/(2 - \sum_{i \in \{1,\dots,M\}\setminus\{j\}} p_i)$. *If for each* $j = 1, \dots, M$ *we have* $Q_j p_j > \beta$ *and* <u>*either*</u> $\alpha_j < \min\{k_j - 1/(S_j p_j), k_j - 1/2\}$ *and* $Q_j p_j \geq \min\{S_j p_j, 2\}$ <u>*or*</u> $\alpha_j < k_j - 1/p_j$ *then it holds that*

$$n^{\sum_{j=1}^M \alpha_j p_j} V_n^{\langle M\rangle} \xrightarrow{L-s} Z = \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} H_m^{\langle M\rangle} \quad as \quad n \to \infty.$$

(iv) *For* $j = 1, \dots, M$ *suppose that* $p_j \in (0,2)$ *with* $\sum_{i \in \{1,\dots,M\}\setminus\{j\}} p_i < 2$ *and set* $Q_j := 2/(2 - \sum_{i \in \{1,\dots,M\}\setminus\{j\}} p_i)$. *If for each* $j = 1, \dots, M$ *we have* $Q_j p_j > \beta$, $\alpha_j = k_j - 1/(S_j p_j)$, $1/(S_j p_j) + 1/\theta_j > 1$ *and* $\min\{Q_j p_j, 2\} \geq S_j p_j$ *then we deduce that*

$$\frac{n^{\sum_{j=1}^M \alpha_j p_j}}{\log(n)} V_n^{\langle M \rangle} \xrightarrow{\mathbb{P}} \widehat{Z} = C^{\langle M \rangle} \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} \quad as \quad n \to \infty.$$

Proof. By Remark 1.2.13, we have the following identity

$$V_n^{\langle M \rangle} = \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \Delta_{i+a_j,n}^{k_j} Y^{\langle j \rangle} \right|^{p_j} = \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left| \Delta_{i+a_j,n}^{k_j} X^{\langle j \rangle} \right|^{p_j},$$

which holds almost surely and where

$$X_t^{\langle j \rangle} := \int_{-\infty}^{t} g_j(t-s) - \tilde{g}_j(-s) \, dL_s$$

for $t \geq 0$ and $j = 1, \dots, M$.

A direct consequence of Lemma 1.2.2 is that the Lévy measure ν and the parameter θ_j satisfy assumption (1) for $j = 1, \dots, M$.

The application of [**10**, Theorem 1.1 (i)] and [**11**, Theorem 1.2 (i)] yields (i) and (ii) in the case of $M = 1$. Furthermore, by applying Theorem 2.1.1 and Theorem 2.2.1, we obtain the rest of Theorem 5.0.2. □

The following proposition will provide some insights on the properties of the limiting random variables Z and $\widehat{Z}$ in the above theorem.

Proposition 5.0.3. *The random variables* Z *and* $\widehat{Z}$ *in Theorem 5.0.2 are infinitely divisible. Furthermore, the characteristic function of* $\widehat{Z}$ *is of the form (2.0.1) and the characteristic function of* Z *is of the form (2.0.2).*

Proof. See the proof of Proposition 2.0.2. □

By a similar argumentation as in the above theorem, we will now extend the results of Theorem 4.1.1, Remark 4.1.2, Theorem 4.2.1, Remark 4.2.2 (i) and Corollary 4.2.3 to processes driven by the fractional Lévy motion.

THEOREM 5.0.4. *For each* $j = 1, \dots, M$ *let* $L^{\langle j \rangle} = (L_t^{\langle j \rangle})_{t \in \mathbb{R}}$ *be a symmetric Lévy process without a Brownian component, with a Lévy measure* ν_j *satisfying* $\int_{|x| \geq 1} x^2 \nu_j(dx) < \infty$ *and Blumenthal-Getoor index* $\beta_j < 2$.

Moreover, for each $j = 1, \dots, M$ *choose* $Q_j \geq 1$, $0 < d_j < 1/2$ *and* $\phi_j \in H$, *cf. Definition 1.2.9, and suppose that the functions* $g_j, \tilde{g}_j$ *given by* $g_j = \left(I_+^{d_j} \phi_j\right)$, $\tilde{g}_j \equiv 0$ *as well as the Lévy process* $L^{\langle j \rangle}$ *satisfy the assumptions* ⓪,②,③,⑤ *with respect to the parameters* $\alpha_j, \kappa_j, \theta_j$, *where* $\alpha_j > \kappa_j - 1/(Q_j \vee \beta_j)$, *and in the case of* $\theta_j = Q_j$ *the assumption* ⑥ *as well.*

For $j = 1, \dots, M$ *assume* $a_j \in \mathbb{Z}$, $p_j > 0$, $k_j \in \mathbb{N}$ *with* $k_j \leq \kappa_j$ *and set*

$$Y_t^{\langle j \rangle} := \int_{-\infty}^{\infty} \phi_j(t-s) \, dM_s^{\langle j \rangle} \quad \text{for} \quad t \in \mathbb{R}$$

with

$$M_t^{\langle j \rangle} := \frac{1}{\Gamma(d_j + 1)} \int_{-\infty}^{t} (t-s)_+^{d_j} - (-s)_+^{d_j} \, dL_s^{\langle j \rangle} \quad \text{for} \quad t \in \mathbb{R}.$$

Then by using the parameters $\rho = \max(\lceil p_1 \rceil, \dots, \lceil p_M \rceil)$ *and* $r = \prod_{j=1}^{M} (1 \vee p_j)$, *by denoting*

$$V_n = n^{-1 + \sum_{j=1}^{M} k_j p_j} V_n^{\langle M \rangle}(Y; a; k; p) \quad \text{(cf. Definition 1.3.2)},$$

as well as by setting

$$F = \int_0^1 \prod_{j=1}^{M} \left| \left(\frac{d}{dt}\right)^{k_j} X_t^{\langle j \rangle} \right|^{p_j} dt \text{ with } X_t^{\langle j \rangle} = \int_{-\infty}^{t} g_j(t-s) - \tilde{g}_j(-s) \, dL_s^{\langle j \rangle},$$

where

$$\left(\frac{d}{dt}\right)^{\kappa_j} X_t^{\langle j \rangle} = \int_{-\infty}^{t} g_j^{(\kappa_j)}(t-s) \, dL_s^{\langle j \rangle} \;\; \lambda \otimes \mathbb{P} - a.s.,$$

we get the following results.

(*i*) *If for* $j = 1, \dots, M$ *there are* $S_j \geq 1$ *satisfying* $\sum_{j=1}^{M} 1/S_j = 1$ *and* $S_j p_j \geq 1$ *then by assuming* $S_j p_j \leq Q_j$ *in the case of* $k_j = \kappa_j$ *we obtain*

$$V_n \xrightarrow{a.s.} F \quad \text{as} \quad n \to \infty.$$

(ii) *If for* $j = 1, \dots, M$ *we have* $k_j < \kappa_j$ *then by assuming* $p_j \le Q_j$ *in the case of* $k_j + 1 = \kappa_j$ *it holds that*

$$\left|V_n - F\right| = \mathcal{O}\left(\min\left(n^{-\frac{1}{r}\min(1,p_1,\dots,p_M)}, n^{-\min(1,p_1/\rho,\dots,p_M/\rho)}\right)\right)$$

almost surely as $n \to \infty$.

(iii) *Suppose for* $j = 1, \dots, M$ *that* $\kappa_j = k_j + 1$, $g_j^{(\kappa_j)} \in L^2((0,\infty))$ *as well as* $\| \blacktriangle_{k_j,N_j}^{k_j} g_j \|_{L^2(\mathbb{R})} < \infty$ *for some* $N_j \in \mathbb{N}$ *with* $N_j \ge k_\star + a_j$.

Then in the case of $\sum_{l=1}^{M} p_l \le 2$ *we have*

$$\mathbb{P}\left(\left|(V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}}\right| \ge \varepsilon\right) \le \frac{1}{\varepsilon}\mathbb{E}\left|(V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}}\right| = \mathcal{O}\left(n^{-\frac{1}{r}\min(1,p_1,\dots,p_M)}\right)$$

as well as

$$\mathbb{P}\left(\left|V_n - F\right| \ge \varepsilon\right) \le \frac{1}{\varepsilon}\mathbb{E}\left|V_n - F\right| = \mathcal{O}\left(n^{-\min(1,p_1/\rho,\dots,p_M/\rho)}\right)$$

for each $\varepsilon > 0$ *as* $n \to \infty$.

Proof. By Remark 1.2.13, we have the following identity

$$V_n^{\langle M \rangle}(Y;a;k;p) = \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left|\triangle_{i+a_j,n}^{k_j} Y^{\langle j \rangle}\right|^{p_j} = \sum_{i=k_\star}^{n+k^\star} \prod_{j=1}^{M} \left|\triangle_{i+a_j,n}^{k_j} X^{\langle j \rangle}\right|^{p_j},$$

which holds almost surely.

For each $j = 1, \dots, M$ we know by Lemma 1.2.2 that the Lévy measure ν_j and the parameter θ_j satisfy assumption ①.

Hence, the application of Theorem 4.1.1, Remark 4.1.2, Theorem 4.2.1, Remark 4.2.2 (i) and Corollary 4.2.3 results in Theorem 5.0.4. □

In the following remark we will consider under which assumptions on the function $\phi \in H$, cf. Definition 1.2.9, the corresponding right-sided Riemann-Liouville fractional integral $\left(I_+^d \phi\right)$, cf. Proposition 1.2.8, satisfies the assumptions in Section 1.2.1.

REMARK 5.0.5. Let $0 < d < 1/2$ and denote all positive constants by K.

(i) By assuming for $\phi \in H$ that $\phi(x) = 0$ for $x < 0$, we get $g(x) = \left(I_+^d \phi\right)(x) = 0$ for $x < 0$, i.e. g and $\tilde{g} \equiv 0$ satisfy assumption ②.

(ii) For $\phi \in H$ and $t, s \in \mathbb{R}$ we obtain by using the substitution $z = t - y$ the following identity

$$\begin{aligned}\left(I_+^d \phi\right)(t-s) &= \frac{1}{\Gamma(d)} \int_{-\infty}^{t-s} \phi(y)(t-s-y)^{d-1} dy \\ &= \frac{1}{\Gamma(d)} \int_s^{\infty} \phi(t-z)(z-s)^{d-1} dz = \left(I_-^d \psi_t\right)(s)\end{aligned}$$

with $\psi_t(s) = \phi(t-s)$.

Since $\phi \in H$ implies $\psi_t \in H$ for all $t \in \mathbb{R}$ an application of Proposition 1.2.8 (ii) yields $\left(I_-^d \psi_t\right) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$.

(iii) Let $\phi \in H$ with $\phi(x) = 0$ for $x < 0$ and $|\phi(x)| \leq \mu e^{-\lambda x}$ for $x \geq 0$, where $\mu, \lambda > 0$. By (i) we know that $\left(I_+^d \phi\right)(x) = 0$ for $x < 0$.

Furthermore, for $x \geq 0$ we have

$$\left|\left(I_+^d \phi\right)(x)\right| \leq \frac{1}{\Gamma(d)} \int_0^{\infty} \left|\phi(x-z)\right| z^{d-1} dz \leq K e^{-\lambda x} \int_0^x e^{\lambda z} z^{d-1} dz \xrightarrow[x \to \infty]{} 0,$$

where the limit is a consequence of L'Hôpital's rule.

Hence, $\left(I_+^d \phi\right)$ is bounded on $\mathbb{R}$.

(iv) Set $\phi = \chi \mathbf{1}_{[0,\infty)}$, where $\chi \in C^k\left((-\varepsilon, \infty)\right)$ for some $\varepsilon > 0$ and $k \in \mathbb{N}$. Then for $x \geq 0$ we have

$$g(x) = \left(I_+^d \phi\right)(x) = \int_0^x \chi(x-y)\psi(y) dy \quad \text{with} \quad \psi(y) = \frac{y^{d-1}}{\Gamma(d)}.$$

For $x > 0$ an (iterative) application of Leibniz's integral rule, cf. e.g. [**15**], yields

$$\begin{aligned}g^{(k)}(x) &= \sum_{i=0}^{k-1} \chi^{(i)}(0)\psi^{(k-i-1)}(x) + \int_0^x \chi^{(k)}(x-y)\psi(y) dy \\ &= \sum_{i=0}^{k-1} \chi^{(i)}(0) x^{d+i-k} \frac{\prod_{r=0}^{k-i-2}(d-1-r)}{\Gamma(d)} + \int_0^x \chi^{(k)}(x-y)\frac{y^{d-1}}{\Gamma(d)} dy.\end{aligned}$$

Note that the function g satisfies assumption ⑤ (i) for all $\delta > 0$ and $\alpha \in [0, d]$, since for $x \geq 0$ we have

$$\left| \int_0^x \chi^{(k)}(x-y) \frac{y^{d-1}}{\Gamma(d)} dy \right| \leq \frac{\max_{s \in [0,x]} |\chi^{(k)}(s)|}{\Gamma(d+1)} x^d.$$

Furthermore, by additionally assuming that $\tilde{\chi} \in H$, where $\tilde{\chi} = \chi^{(k)} \mathbf{1}_{[0,\infty)}$, we can proceed as in (ii) in order to obtain that $(I_+^d \tilde{\chi}) \in L^2(\mathbb{R})$.

A combination of $(I_+^d \tilde{\chi}) \in L^2(\mathbb{R})$, the identity

$$(I_+^d \tilde{\chi})(x) = \int_0^x \chi^{(k)}(x-y) \psi(y) dy \text{ for } x \geq 0$$

and the fact that $\psi^{(i)} \in L^2((\delta, \infty))$ for all $i \in \mathbb{N} \cup \{0\}$ and $\delta > 0$, since we have $d - 1 < -1/2$, yields that g satisfies assumption ⑤ (ii) for $\theta = 2$ and an arbitrary $\delta > 0$.

As our first application of Remark 5.0.5 we will consider Ornstein-Uhlenbeck type processes in the setting of Theorem 5.0.2.

PROPOSITION 5.0.6. *Let $L = (L_t)_{t \in \mathbb{R}}$ be a symmetric Lévy process without a Brownian component, a Lévy measure ν satisfying $\int_{|x| \geq 1} x^2 \nu(dx) < \infty$ and Blumenthal-Getoor index $\beta < 2$. Moreover, for $j = 1, \ldots, M$ suppose that $\lambda_j, p_j > 0$, $S_j \geq 1$ with $\sum_{i=1}^M 1/S_i = 1$, $d_j \in (0, 1/2)$, $k_j \in \mathbb{N}$, $a_j \in \mathbb{Z}$, $\mu_j \in \mathbb{R}$ with $\mu_j \neq 0$ and set*

$$Y_t^{\langle j \rangle} := \int_{-\infty}^{\infty} \phi_j(t-s) \, dM_s^{\langle j \rangle} \quad \text{for} \quad t \in \mathbb{R}$$

with

$$M_t^{\langle j \rangle} := \frac{1}{\Gamma(d_j + 1)} \int_{-\infty}^t (t-s)_+^{d_j} - (-s)_+^{d_j} \, dL_s \quad \text{for} \quad t \in \mathbb{R}$$

and $\phi_j(s) = \mu_j e^{-\lambda_j s} \mathbf{1}_{[0,\infty)}(s)$.

Then by using the definitions and notations in Details 5.0.1 with $\alpha_j = d_j$ and $c_j = \mu_j / \Gamma(d_j + 1)$ for $j = 1, \ldots, M$ we get the following results.

(i) *Let <u>either</u> L be a compound Poisson process <u>or</u> $M = 1$ with $p_M > \beta$.*

If for each $j = 1, \dots, M$ we have $d_j < k_j - 1/(S_j p_j)$ then it holds that

$$n^{\sum_{j=1}^M d_j p_j} V_n^{\langle M \rangle} \xrightarrow{L-s} Z = \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} H_m^{\langle M \rangle} \quad as \quad n \to \infty.$$

(ii) *For $j = 1, \dots, M$ suppose that $p_j \in (0,2)$ with $\sum_{i \in \{1,\dots,M\} \setminus \{j\}} p_i < 2$ and set $Q_j := 2/(2 - \sum_{i \in \{1,\dots,M\} \setminus \{j\}} p_i)$. If for each $j = 1, \dots, M$ we have $Q_j p_j > \beta$ and <u>either</u> $d_j < \min\{k_j - 1/(S_j p_j), k_j - 1/2\}$ and $Q_j p_j \geq \min\{S_j p_j, 2\}$ <u>or</u> $d_j < k_j - 1/p_j$ then we obtain*

$$n^{\sum_{j=1}^M d_j p_j} V_n^{\langle M \rangle} \xrightarrow{L-s} Z = \sum_{m: T_m \in [0,1]} |\Delta L_{T_m}|^{\sum_{j=1}^M p_j} H_m^{\langle M \rangle} \quad as \quad n \to \infty.$$

PROOF. Throughout this proof we will for $j = 1, \dots, M$ write $(\cdot)$ instead of j. Let $g_{(\cdot)} = (I_+^{d_{(\cdot)}} \phi_{(\cdot)})$, $\tilde{g}_{(\cdot)} \equiv 0$, $\alpha_{(\cdot)} = d_{(\cdot)}$, $c_{(\cdot)} = \mu_{(\cdot)}/\Gamma(d_{(\cdot)} + 1)$ and $\theta_{(\cdot)} = 2$.

By Remark 5.0.5 (i) to (iii) and Lemma 1.2.2, the functions $g_{(\cdot)}, \tilde{g}_{(\cdot)}$ and the Lévy process L satisfy the assumptions ⓪,①,②,③ with respect to the parameters $\alpha_{(\cdot)}$, $k_{(\cdot)}$ and $\theta_{(\cdot)}$. Furthermore, an application of L'Hôpital's rule, where the derivative of $g_{(\cdot)}$ can be obtained by using Remark 5.0.5 (iv), yields that $g_{(\cdot)}$ satisfies assumption ④ with respect to the parameters $\alpha_{(\cdot)}$ and $c_{(\cdot)}$.

For arbitrary $k \in \mathbb{N}$ and $\delta > 0$ it follows directly from Remark 5.0.5 (iv) that $g_{(\cdot)} \in C^k((0,\infty))$, $g_{(\cdot)}^{(k)} \in L^2((\delta,\infty))$ and $|g_{(\cdot)}(t)| \leq K t^{\alpha_{(\cdot)} - k}$ for $t \in (0,\delta)$ and some constant $K > 0$.

Moreover, for $k \in \mathbb{N}$ and $t > 1$ an (iterative) application of integration by parts in combination with Remark 5.0.5 (iv) results in

$$\begin{aligned} \frac{g_{(\cdot)}^{(k)}(t)}{\mu_{(\cdot)}} &= (-1)^{k-1} \sum_{i=0}^{k-1} \lambda_{(\cdot)}^i t^{d+i-k} \frac{\prod_{r=0}^{k-i-2}(1 - d_{(\cdot)} + r)}{\Gamma(d_{(\cdot)})} + (-1)^k \lambda_{(\cdot)}^k \int_0^t e^{\lambda_{(\cdot)}(s-t)} \frac{s^{d_{(\cdot)}-1}}{\Gamma(d_{(\cdot)})} ds \\ &= (-1)^k e^{-\lambda_{(\cdot)} t} \Big(\frac{\lambda_{(\cdot)}^k}{\Gamma(d_{(\cdot)})} \int_0^1 e^{\lambda_{(\cdot)} s} s^{d_{(\cdot)}-1} ds - \sum_{i=0}^{k-1} \lambda_{(\cdot)}^i e^{\lambda_{(\cdot)}} \frac{\prod_{r=0}^{k-i-2}(1 - d_{(\cdot)} + r)}{\Gamma(d_{(\cdot)})} \\ &\qquad + \frac{\prod_{r=0}^{k-1}(1 - d_{(\cdot)} + r)}{\Gamma(d_{(\cdot)})} \int_1^t e^{\lambda_{(\cdot)} s} s^{d_{(\cdot)}-1-k} ds \Big) \end{aligned}$$

$$= (-1)^k e^{-\lambda_{(\cdot)}t}\Big(\underbrace{\widetilde{K} + \widehat{K}\int_1^t e^{\lambda_{(\cdot)}s} s^{d_{(\cdot)}-1-k} ds}_{\longrightarrow\infty \text{ as } t\to\infty}\Big), \tag{5.0.1}$$

where $\widetilde{K} \in \mathbb{R}$ as well as $\widehat{K} > 0$ are suitable constants.

Note that for $k \in \mathbb{N}$ and all sufficiently large $t > 1$ by (5.0.1) we have

$$|g_{(\cdot)}^{(k)}|(t) = \operatorname{sign}(\mu_{(\cdot)})(-1)^k g_{(\cdot)}^{(k)}(t)$$

and therefore also

$$\frac{d}{dt}|g_{(\cdot)}^{(k)}|(t) = \operatorname{sign}(\mu_{(\cdot)})(-1)^k g_{(\cdot)}^{(k+1)}(t) < 0.$$

Hence, for each $k \in \mathbb{N}$ there exists a $\delta > 0$ so that the function $|g_{(\cdot)}^{(k)}|$ is decreasing on (δ, ∞).

We can conclude this proof by applying Theorem 5.0.2 (i) and (iii) since the functions $g_{(\cdot)}, \tilde{g}_{(\cdot)}$ and the Lévy process L satisfy the assumptions ⓪ to ⑤ with respect to the parameters $\alpha_{(\cdot)}$, $c_{(\cdot)}$, $k_{(\cdot)}$,$\theta_{(\cdot)}$ and by Remark 2.2.2 (ii) we have

$$\| \blacktriangle_{i,n}^{k_{(\cdot)}} g_{(\cdot)}\|_{L^2(\mathbb{R})}^2 \leq K n^{-2\alpha_{(\cdot)}-1},$$

where $K > 0$ is a suitable constant, $n \in \mathbb{N}$ is sufficiently large and $i = k_{(\cdot)}, \ldots, n$. □

Now we will use the results of Remark 5.0.5 in order to illustrate Theorem 5.0.4, in the setting of some specific linear combinations of Ornstein-Uhlenbeck type processes.

Proposition 5.0.7. *For each $j = 1, \ldots, M$ let $L^{\langle j\rangle} = (L_t^{\langle j\rangle})_{t\in\mathbb{R}}$ be a symmetric Lévy process without a Brownian component, a Lévy measure ν_j satisfying $\int_{|x|\geq 1} x^2 \nu_j(dx) < \infty$ and Blumenthal-Getoor index $\beta_j < 2$.*

Moreover, for $j = 1, \ldots, M$ assum $p_j > 0$, $d_j \in (0, 1/2)$, $k_j, \kappa_j \in \mathbb{N}$ with $k_j \leq \kappa_j$, $a_j \in \mathbb{Z}$ and set

$$Y_t^{\langle j\rangle} := \int_{-\infty}^{\infty} \phi_j(t-s)\, dM_s^{\langle j\rangle}, \quad \textit{for} \quad t \in \mathbb{R}$$

with

$$M_t^{\langle j\rangle} := \frac{1}{\Gamma(d_j+1)} \int_{-\infty}^{t} (t-s)_+^{d_j} - (-s)_+^{d_j} \, dL_s^{\langle j\rangle} \quad \text{for} \quad t \in \mathbb{R}$$

and $\phi_j(s) = \chi_j(s)\mathbf{1}_{[0,\infty)}(s) = \sum_{l=1}^{\kappa_j+2} \mu_{j,l} e^{-\lambda_{j,l}s}\mathbf{1}_{[0,\infty)}(s)$, *where* $0 < \lambda_{j,1} < \cdots < \lambda_{j,\kappa_j+2}$ *and the parameters* $\mu_{j,1},\ldots,\mu_{j,\kappa_j+2} \in \mathbb{R}$ *are chosen so that we have* $\sum_{l=1}^{\kappa_j+2} \mu_{j,l}/\lambda_{j,l} \neq 0$ *as well as* $\chi_j(0) = \chi_j^{(1)}(0) = \cdots = \chi_j^{(\kappa_j)}(0) = 0$.

Then by setting

$$V_n = n^{-1+\sum_{j=1}^M k_j p_j} V_n^{\langle M\rangle}(Y;a;k;p) \quad (\text{cf. Definition 1.3.2}),$$

$$F = \int_0^1 \prod_{j=1}^{M} \left| \int_{-\infty}^{t} \left(\int_0^{t-s} \chi^{(k_j)}(t-s-u) \frac{u^{d_j-1}}{\Gamma(d_j)} du \right) dL_s^{\langle j\rangle} \right|^{p_j} dt$$

as well as $\rho = \max(\lceil p_1\rceil,\ldots,\lceil p_M\rceil)$ *and* $r = \prod_{j=1}^M (1 \vee p_j)$ *we get the following results.*

(*i*) *If for* $j = 1,\ldots,M$ *there are* $S_j \geq 1$ *satisfying* $\sum_{j=1}^M 1/S_j = 1$ *and we have* $S_j p_j \geq 1$ *then it holds that*

$$V_n \xrightarrow{a.s.} F \quad \text{as} \quad n \to \infty.$$

(*ii*) *If for* $j = 1,\ldots,M$ *we have* $k_j < \kappa_j$ *then we obtain*

$$\left|V_n - F\right| = \mathcal{O}\left(\min\left(n^{-\frac{1}{r}\min(1,p_1,\ldots,p_M)}, n^{-\min(1,p_1/\rho,\ldots,p_M/\rho)}\right)\right)$$

almost surely as $n \to \infty$.

(*iii*) *If* $\sum_{l=1}^M p_l \leq 2$ *and for* $j = 1,\ldots,M$ *we have* $\kappa_j = k_j + 1$ *then we deduce that*

$$\mathbb{P}\left(\left|(V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}}\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon}\mathbb{E}\left|(V_n)^{\frac{1}{r}} - (F)^{\frac{1}{r}}\right| = \mathcal{O}\left(n^{-\frac{1}{r}\min(1,p_1,\ldots,p_M)}\right)$$

as well as

$$\mathbb{P}\left(\left|V_n - F\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon}\mathbb{E}\left|V_n - F\right| = \mathcal{O}\left(n^{-\min(1,p_1/\rho,\ldots,p_M/\rho)}\right)$$

for each $\varepsilon > 0$ *as* $n \to \infty$.

PROOF. Throughout this proof we will for $j = 1,\ldots,M$ write $(\cdot)$ instead of j and $\langle\cdot\rangle$ instead of $\langle j\rangle$.

Let $g_{(\cdot)} = \big(I_+^{d_{(\cdot)}}\phi_{(\cdot)}\big)$, $\tilde{g}_{(\cdot)} \equiv 0$, $\alpha_{(\cdot)}(k) = d_{(\cdot)} + k$ for $k \in \mathbb{N} \cup \{0\}$, $\theta_{(\cdot)} = 2$ and choose $Q_{(\cdot)} > \max\{S_j p_{(\cdot)}, 2\}$.

By Remark 5.0.5 (i) to (iii) and Lemma 1.2.2, we know that the Lévy process $L^{\langle\cdot\rangle}$, the functions $g_{(\cdot)}, \tilde{g}_{(\cdot)}$ and the parameter $\theta_{(\cdot)}$ satisfy the assumptions ①,② and ③. Since by assumption ③, we have $g_{(\cdot)} \in L^2(\mathbb{R})$, we get $\| \blacktriangle^{k_{(\cdot)}}_{k_{(\cdot)},N_{(\cdot)}} g_{(\cdot)}\|_{L^2(\mathbb{R})} < \infty$ for some $N_{(\cdot)} \in \mathbb{N}$ with $N_{(\cdot)} \geq k_\star + a_{(\cdot)}$.

Moreover, by Remark 5.0.5 (iv), we know that for each $k = 0, 1, \ldots, \kappa_{(\cdot)} + 1$ we have on the one hand $g_{(\cdot)} \in C^{\kappa_{(\cdot)}+1}\big((0,\infty)\big)$ as well as $g^{(k)}_{(\cdot)} \in L^2\big((\delta,\infty)\big)$ for an arbitrary $\delta > 0$ and on the other hand

$$g^{(k)}_{(\cdot)}(t) = \underbrace{\sum_{i=0}^{k-1} \chi^{(i)}_{(\cdot)}(0) t^{d_{(\cdot)}+i-k} \frac{\prod_{r=0}^{k-i-2}(d_{(\cdot)} - 1 - r)}{\Gamma(d_{(\cdot)})}}_{=0} + \int_0^t \chi^{(k)}_{(\cdot)}(t-s)\frac{s^{d_{(\cdot)}-1}}{\Gamma(d_{(\cdot)})} ds$$

$$= \frac{(-1)^k}{\Gamma(d_{(\cdot)})} \sum_{l=1}^{\kappa_{(\cdot)}+2} \mu_{(\cdot),l}\lambda^k_{(\cdot),l} \int_0^t \underbrace{e^{-\lambda_{(\cdot),l}(t-s)}}_{\leq 1} s^{d_{(\cdot)}-1} ds \tag{5.0.2}$$

for $t > 0$.

For each $k = 0, 1, \ldots, \kappa_{(\cdot)}$ the representation in (5.0.2) respectively the representation above it allows us to obtain

$$|g^{(k)}_{(\cdot)}| \leq K t^{d_{(\cdot)}} = K t^{\alpha_{(\cdot)}(k)-k}$$

for $t \in (0,\delta)$, where $\delta > 0$ is arbitrary and $K > 0$ is a suitable constant, as well as by setting $\phi = \chi^{(k)}_{(\cdot)} \mathbf{1}_{[0,\infty)}$ to proceed as in Remark 5.0.5 (ii) in order to get $g^{(k)}_{(\cdot)} \in L^2\big((0,\infty)\big)$.

Furthermore, for $t > 1$ and $k = 1, 2, \ldots, \kappa_{(\cdot)} + 1$ an (iterative) application of integration by parts in (5.0.2) yields

$$g^{(k)}_{(\cdot)}(t) = \frac{(-1)^k}{\Gamma(d_{(\cdot)})} \sum_{l=1}^{\kappa_{(\cdot)}+2} \mu_{(\cdot),l}\lambda^k_{(\cdot),l} e^{-\lambda_{(\cdot),l}t} \int_0^1 e^{\lambda_{(\cdot),l}s} s^{d_{(\cdot)}-1} ds$$

$$+ (-1)^{k+1} \sum_{l=1}^{\kappa_{(\cdot)}+2} \mu_{(\cdot),l} e^{-\lambda_{(\cdot),l}t} \sum_{i=0}^{k-1} \lambda^i_{(\cdot),l} e^{\lambda_{(\cdot),l}} \frac{\prod_{r=0}^{k-i-2}(1 - d_{(\cdot)} + r)}{\Gamma(d_{(\cdot)})}$$

$$+ (-1)^k \frac{\prod_{r=0}^{k-1}(1-d_{(\cdot)}+r)}{\Gamma(d_{(\cdot)})} \sum_{l=1}^{\kappa_{(\cdot)}+2} \mu_{(\cdot),l} \int_1^t e^{\lambda_{(\cdot),l}(s-t)} s^{d_{(\cdot)}-1-k} ds$$

$$=(-1)^k \frac{\prod_{r=0}^{k-1}(1-d_{(\cdot)}+r)}{\Gamma(d_{(\cdot)})} \sum_{l=1}^{\kappa_{(\cdot)}+2} \mu_{(\cdot),l} \Big(\frac{1}{\lambda_{(\cdot),l}} + o(1)\Big) t^{d_{(\cdot)}-1-k} \text{ as } t \to \infty, \tag{5.0.3}$$

where in order to get the first equality we used $\chi_{(\cdot)}^{(i)}(0) = \sum_{l=1}^{\kappa_{(\cdot)}+2} \mu_{(\cdot),l}\lambda_{(\cdot),l}^i = 0$ for $i = 0, 1, \ldots, \kappa_{(\cdot)}$ and the second equality follows by combining L'Hôpital's rule with the fact that for $\lambda > 0$ we have $e^{-\lambda t} = o(1)t^{d_{(\cdot)}-1-k}$ as $t \to \infty$.

Note that by (5.0.3) for all sufficiently large $t > 1$ we have

$$|g_{(\cdot)}^{(k)}|(t) = \operatorname{sign}\left(\sum_{l=1}^{\kappa_{(\cdot)}+2} \frac{\mu_{(\cdot),l}}{\lambda_{(\cdot),l}}\right)(-1)^k g_{(\cdot)}^{(k)}(t) \quad \text{for} \quad k = 1, \ldots, \kappa_{(\cdot)} + 1$$

and therefore also

$$\frac{d}{dt}|g_{(\cdot)}^{(k)}|(t) = \operatorname{sign}\left(\sum_{l=1}^{\kappa_{(\cdot)}+2} \frac{\mu_{(\cdot),l}}{\lambda_{(\cdot),l}}\right)(-1)^k g_{(\cdot)}^{(k+1)}(t) < 0 \quad \text{for} \quad k = 1, \ldots, \kappa_{(\cdot)}.$$

Hence, for each $k = 1, \ldots, \kappa_{(\cdot)}$ there exists a $\delta > 0$, so that the function $|g_{(\cdot)}^{(k)}|$ is decreasing on (δ, ∞).

Since for each $k = 1, \ldots, \kappa_{(\cdot)}$ the functions $g_{(\cdot)}, \tilde{g}_{(\cdot)}$ and the Lévy process $L^{\langle\cdot\rangle}$ satisfy the assumptions ⓪,①,②,③ and ⑤ with respect to the parameters $\alpha_{(\cdot)}(k)$, k, $\theta_{(\cdot)}$ and we have $\alpha_{(\cdot)}(k) - k > -1/(Q_{(\cdot)} \vee \beta_{(\cdot)})$, we can conclude this proof by applying Lemma 4.5.3, which leads to

$$\left(\frac{d}{dt}\right)^{k_{(\cdot)}} X_t^{\langle\cdot\rangle} = \int_{-\infty}^t g_{(\cdot)}^{(k_{(\cdot)})}(t-s)\, dL_s^{\langle\cdot\rangle} \;\lambda \otimes \mathbb{P} - \text{a.s.},$$

as well as by applying Theorem 5.0.4. □

Bibliography

[1] D. J. Aldous and G. K. Eagleson. On mixing and stability of limit theorems. *The Annals of Probability*, 6(2):325–331, 1978.

[2] O. E. Barndorff-Nielsen, J. M. Corcuera, M. Podolskij, and J. H. C. Woerner. Bipower variation for Gaussian processes with stationary increments. *Journal of Applied Probability*, 46(1):132–150, 2009.

[3] O. E. Barndorff-Nielsen, S.-E. Graversen, J. Jacod, M. Podolskij, and N. Shephard. A central limit theorem for realised power and bipower variations of continuous semimartingales. *From stochastic calculus to mathematical finance: the Shiryaev Festschrift*, pages 33–68, 2006.

[4] O. E. Barndorff-Nielsen and N. Shephard. Econometric analysis of realized volatility and its use in estimating stochastic volatility models. *Journal of the Royal Statistical Society Series B*, 64(2):253–280, 2002.

[5] O. E. Barndorff-Nielsen and N. Shephard. Realized power variation and stochastic volatility models. *Bernoulli*, 9(2):243–265, 2003.

[6] O. E. Barndorff-Nielsen and N. Shephard. Econometric analysis of realized covariation: High frequency based covariance, regression, and correlation in financial economics. *Econometrica*, 72(3):885–925, 2004.

[7] O. E. Barndorff-Nielsen and N. Shephard. Power and bipower variation with stochastic volatility and jumps. *Journal of Financial Econometrics*, 2(1):1–37, 2004.

[8] O. E. Barndorff-Nielsen and N. Shephard. Econometrics of testing for jumps in financial economics using bipower variation. *Journal of Financial Econometrics*, 4(1):1–30, 2006.

[9] O. E. Barndorff-Nielsen, N. Shephard, and M. Winkel. Limit theorems for multipower variation in the presence of jumps. *Stochastic Processes and their Applications*, 116(5):796–806, 2006.

[10] A. Basse-O'Connor, R. Lachièze-Rey, and M. Podolskij. Power variation for a class of stationary increments Lévy driven moving averages. *Annals of Probability*, 45(6B):4477–4528, 2017.

[11] A. Basse-O'Connor and M. Podolskij. On critical cases in limit theory for stationary increments Lévy driven moving averages. *Stochastics: An International Journal of Probability and Stochastic Processes*, 89(1):360–383, 2017.

[12] P. Billingsley. *Convergence of Probability Measures (second edition)*. Wiley, 1999.

[13] R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman and Hall/CRC Financial Mathematics Series, 2003.

[14] S. Engelke and J. H. C. Woerner. A unifying approach to fractional Lévy processes. *Stochastics and Dynamics*, 13(02), 2013.

[15] H. Flanders. Differentiation under the integral sign. *The American Mathematical Monthly*, 80(6):615–627, 1973.

[16] S. Glaser. Limit theorems of the power variation of fractional Lévy processes. *This dissertation is available at: http://hdl.handle.net/2003/34245*, 2015.

[17] N. Janicke. Bipower-variation bei Finanzmarktdaten mit unregelmäßigen Beobachtungsabständen. *This dissertation is available at: https://ediss.uni-goettingen.de/handle/11858/00-1735-0000-0006-B3A2-2*, 2008.

[18] A. Klenke. *Wahrscheinlichkeitstheorie*. Springer, 2006.

[19] D. Kobe. Oszillierende Ornstein-Uhlenbeck Prozesse und Modellierung von Elektrizitätspreisen. *This dissertation is available at: http://hdl.handle.net/2003/35359*, 2016.

[20] T. M.Apostol. *Introduction to Analytic Number Theory*. Springer Verlag, 1976.

[21] T. Marquardt. Fractional Lévy processes with an application to long memory moving average processes. *Bernoulli*, 12(6):1099–1126, 2006.

[22] M. Podolskij and M. Vetter. Understanding limit theorems for semimartingales: A short survey. *Statistica Neerlandica*, 64(3):329–351, 2010.

[23] B. S. Rajput and J. Rosinski. Spectral representations of infinitely divisible processes. *Probability Theory and Related Fields*, 82(3):451–487, 1989.

[24] G. Samorodnitsky and M. S. Taqqu. *Stable non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman and Hall/CRC, 1994.

[25] K.-I. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.

[26] B. Schweizer. *Partielle Differentialgleichungen: Eine anwendungsorientierte Einführung*. Springer Spektrum, 2013.

[27] R. Serfozo. *Basics of Applied Stochastic Processes*. Springer, 2009.

[28] N. Thamrongrat. Stable convergence in statistical inference and numerical approximation of stochastic processes. *This dissertation is available at: http://www.ub.uni-heidelberg.de/archiv/21445*, 2016.

[29] J. H. C. Woerner. Inference in Lévy-type stochastic volatility models. *Advances in Applied Probability*, 39(2):531–549, 2007.