Functional central limit theorems for multivariate Bessel processes in the freezing regime

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Preprint 2019-01 January 2019

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FUNCTIONAL CENTRAL LIMIT THEOREMS FOR MULTIVARIATE BESSEL PROCESSES IN THE FREEZING REGIME

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Abstract. Multivariate Bessel processes $(X_{t,k})_{t\geq 0}$ describe interacting particle systems of Calogero-Moser-Sutherland type and are related with β-Hermite and β-Laguerre ensembles. They depend on a root system and a multiplicity k which corresponds to the parameter β in random matrix theory. In the recent years, several limit theorems were derived for $k \to \infty$ with fixed $t > 0$ and fixed starting point. Only recently, Andraus and Voit used the stochastic differential equations of $(X_{t,k})_{t\geq 0}$ to derive limit theorems for $k \to \infty$ with starting points of the form $\sqrt{k} \cdot x$ with x in the interior of the corresponding Weyl chambers. Here we provide associated functional central limit theorems which are locally uniform in t. The Gaussian limiting processes admit explicit representations in terms of matrix exponentials and the solutions of the associated deterministic dynamical systems.

1. Introduction

From a stochastic point of view, integrable interacting particle systems of Calogero-Moser-Sutherland type on the real line $\mathbb{R}$ with N particles are multivariate Bessel processes on appropriate closed Weyl chambers in $\mathbb{R}^N$. These processes are also often called Dunkl-Bessel, or radial Dunkl processes. They are time-homogeneous diffusion processes and can be described either by the generators of their transition semigroups and their explicit transition probabilities or as solution of the associated stochastic differential equations (SDEs); see [CGY, GY, R1, R2, RV1, RV2, DV, A]. In general, multivariate Bessel processes $(X_{t,k})_{t\geq 0}$ are described via root systems, a possibly multidimensional multiplicity parameter k and by their starting points $X_{0,k} := x$. From a physical point of view, the multiplicies k are coupling constants which describe the strength of interaction of the particles (and, sometimes, of the particles with some fixed wall). We restrict our attention to the root systems A_{N-1}, B_N, and D_N on $\mathbb{R}^N$ as these cover the most relevant cases in view of interacting particle systems and random matrix theory. We thus omit the dihedral cases on $\mathbb{R}^2$ in [Dem], the finitely many exceptional cases, as well as the obvious direct product situations, where all results follow from the corresponding results of the independent components.

Date: January 23, 2019.

2010 *Mathematics Subject Classification.* Primary 60F15; Secondary 60F05, 60J60, 60B20, 60H20, 70F10, 82C22, 33C67 .

Key words and phrases. Interacting particle systems, Calogero-Moser-Sutherland models, functional central limit theorems, zeros of Hermite polynomials, zeros of Laguerre polynomials, Hermite ensembles, Laguerre ensembles, Dyson Brownian motion.

We briefly recapitulate the most important cases A_{N-1} and B_N. For A_{N-1}, we have a multiplicity $k \in]0,\infty[$, the processes live on the closed Weyl chamber

$$C_N^A := \{x \in \mathbb{R}^N : \quad x_1 \ge x_2 \ge \ldots \ge x_N\},$$

and the generator of the transition semigroup is

$$Lf := \frac{1}{2}\Delta f + k\sum_{i=1}^N \Big(\sum_{j\ne i} \frac{1}{x_i - x_j}\Big)\frac{\partial}{\partial x_i} f, \tag{1.1}$$

where we assume reflecting boundaries, i.e., the domain of L is

$$D(L) := \{f|_{C_N^A} : \ f \in C^{(2)}(\mathbb{R}^N), \ \ f \ \text{ invariant under all coordinate permutations}\}.$$

For B_N, we have the multiplicity $k = (k_1, k_2) \in]0,\infty[^2$, the processes live on

$$C_N^B := \{x \in \mathbb{R}^N : \quad x_1 \ge x_2 \ge \ldots \ge x_N \ge 0\},$$

and the generator of the transition semigroup is

$$Lf := \frac{1}{2}\Delta f + k_2\sum_{i=1}^N \sum_{j\ne i}\Big(\frac{1}{x_i - x_j} + \frac{1}{x_i + x_j}\Big)\frac{\partial}{\partial x_i} f \quad + k_1\sum_{i=1}^N \frac{1}{x_i}\frac{\partial}{\partial x_i} f, \tag{1.2}$$

where we again assume reflecting boundaries, i.e., L has the domain

$$D(L) := \{f|_{C_N^B} : \ f \in C^{(2)}(\mathbb{R}^N), \ \ f \ \text{ invariant under all permutations and sign changes of all coordinates}\}.$$

By [R1, R2, RV1, RV2], the transition probabilities of arbitrary Bessel processes have the form

$$K_t(x, A) = c_k \int_A \frac{1}{t^{\gamma_k + N/2}} e^{-(\|x\|^2+\|y\|^2)/(2t)} J_k\Big(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\Big) \cdot w_k(y)\, dy \tag{1.3}$$

for $t > 0$, $x \in C_N$, and $A \subset C_N$ a Borel set. For the root systems A_{N-1} and B_N, we here have the weight functions w_k of the form

$$w_k^A(x) := \prod_{i<j}(x_i - x_j)^{2k}, \qquad w_k^B(x) := \prod_{i<j}(x_i^2 - x_j^2)^{2k_2} \cdot \prod_{i=1}^N x_i^{2k_1}, \tag{1.4}$$

and

$$\gamma_k^A(k) = kN(N-1)/2, \qquad \gamma_k^B(k_1, k_2) = k_2 N(N-1) + k_1 N \tag{1.5}$$

respectively. In all cases, w_k is homogeneous of degree $2\gamma_k$. Furthermore, $c_k > 0$ is a known normalization constant, and J_k is a multivariate Bessel function of type A_{N-1} or B_N with multiplicities k or (k_1, k_2) respectively which is analytic on $\mathbb{C}^N \times \mathbb{C}^N$ with $J_k(x, y) > 0$ for $x, y \in \mathbb{R}^N$. Moreover, $J_k(x, y) = J_k(y, x)$ and $J_k(0, y) = 1$ for all $x, y \in \mathbb{C}^N$. For this and further informations we refer e.g. to [R1, R2]. We notice that if we start the process from 0, then $X_{t,k}$ has the Lebesgue density

$$\frac{c_k}{t^{\gamma+N/2}} e^{-\|y\|^2/(2t)} \cdot w_k(y)\, dy \tag{1.6}$$

on C_N for $t > 0$.

Eq. (1.3) shows that for the root systems A_{N-1}, B_N, the multivariate Bessel processes are related to random matrix theory, as here for the starting point $X_{0,k} := 0 \in \mathbb{R}^N$, the distributions of $X_{t,k}$ $(t > 0)$ are just (up to scaling and with a proper choice of the parameters) the distributions of the ordered eigenvalues of β-Hermite and β-Laguerre ensembles respectively which were introduced by Dumitriu and

Edelman [DE1, DE2] in the context of their tridiagonal random matrix models. We also point out that for other starting points, more complicated tridiagonal random matrix models for the distributions of $X_{t,k}$ were studied recently; see [AG, HP].

We are interested in limit theorems for $(X_{t,k})_{t\geq 0}$ when one or several components of k tend to ∞ in a coupled way (we here briefly write $k \to \infty$ by misuse of notation). This means that (parts of) the coupling constants become large, i.e., from a physics point of view, that we have freezing. Here, $k \to \infty$ means that the drift terms become large compared with the diffusive Brownian part. Moreover, in the random matrix models mentioned above this roughly means $\beta \to \infty$ as in [DE2]. Most of these limit theorems for $k \to \infty$ were derived in the last years for $(X_{t,k})_{t\geq 0}$ when $t > 0$ and the starting points $x \in \mathbb{R}^N$ of the processes are fixed; see [AKM1, AKM2, AM, V] where for the special case $x = 0$ the results fit to the results of [DE2] for $\beta \to \infty$ in tridiagonal random matrix theory.

In the present paper we follow [AV1] and consider the processes $(X_{t,k})_{t\geq 0}$ given as solutions of the the stochastic differential equation of the form

$$dX_{t,k} = dB_t + \frac{1}{2}(\nabla(\ln w_k))(X_{t,k})\, dt$$

with starting points $X_{0,k}$ of the form $\sqrt{k}\cdot x$ with x in the interior of the corresponding Weyl chambers and an N-dimensional Brownian motion $(B_t)_{t\geq 0}$. In this case, the renormalized processes $(X_{t,k}/\sqrt{k})_{t\geq 0}$ start in x and satisfy SDEs with fixed drift parts, where the diffusive Brownian part tends to 0 for $k \to \infty$. In [AV1], several limit theorems were derived in this case where most of these limit theorems may be seen as strong laws of large numbers which are locally uniform in t. Moreover, only in one special limit case, a corresponding central limit theorem was derived in [AV1]. In the present paper we present an approach to functional central limit theorems which again are a.s. locally uniform in t. Our approach works for all root systems and all limits which appear under the label $k \to \infty$ in our focus.

Let us describe the main results more closely: As mentioned above, the renormalized processes $(X_{t,k}/\sqrt{k})_{t\geq 0}$ satisfy SDEs where the Brownian parts disappear for $k \to \infty$. We denote the solution of the associated deterministic limit differential equation system with initial condition x by $\phi := (\phi(t,x))_{t\geq 0}$. Using this ϕ, we shall derive an explicit Gaussian diffusion process $W := (W_t)_{t\geq 0}$ such that

$$\sqrt{k}\Big(\frac{X_{t,k}}{\sqrt{k}} - \phi(t,x)\Big) \longrightarrow W_t \tag{1.7}$$

for $k \to \infty$ locally uniformly in t with the rate $O(1/\sqrt{k})$ a.s..

Clearly, ϕ plays an essential role in this limit theorem. Unfortunately, we are not able to write down ϕ explicitly for arbitrary root systems and arbitrary starting points x in the interior of the associated Weyl chamber. On the other hand, we shall collect a lot of informations about ϕ. For instance, we show that in all cases, $\|\phi(t,x)\|_2 = \sqrt{Ct + \|x\|_2^2}$ with some known constant $C > 0$. This implies that we may decompose the dynamical system into an easy radial part and a difficult part on a sphere. Moreover, depending on the root system, we have a particular solution

$$\phi(t,x) = \sqrt{Ct + \|x\|_2^2} \cdot x_0 \tag{1.8}$$

with a particular vector x_0 on the unit sphere and the constant C as above. For the root systems A_{N-1}, B_N and D_N (and a particular meaning of $k \to \infty$), the components of x_0 consist of the ordered zeros of the Hermite polynomial H_N or some Laguerre polynomial L_N of order N respectively up to scaling. In fact, this particular vector x_0 already appeared in [AKM1, AKM2, AM, AV1] where the processes with fixed starting point were studied. We shall see in all cases that the stationary solution x_0 of the spherical part is attractive, i.e., the particular solutions in (1.8) are attracting for large t in some way. In this special important case of starting points, we shall compute more specific details about the limit process W like the covariance matrices of W_t for $t > 0$.

Notice that the cases in Sections 2-7 below, the central limit theorems for starting points of the form $c \cdot x_0$ with $c > 0$ hold also formally for $c = 0$, i.e., for the fixed starting point 0 on the boundary of the Weyl chamber by [V]. At a first glance, this kind of continuity seems to be natural. On the other hand, this is by no way obvious, as the generators and the SDEs become highly singular in 0.

This paper is organized as follows. In the next two sections we study Bessel processes of type A_{N-1} when the one-dimensional multiplicity k tends to ∞. In particular in Section 2 we discuss details of the solutions ϕ of the limit dynamical systems as these solutions appear in all stochastic limit theorems in a central way. Section 3 is devoted to a functional CLT for the Bessel processes.

In Sections 4 and 5 we then study the root systems B_N with multiplicities of the form $k = (k_1, k_2) = (\nu \cdot \beta, \beta)$ where $\nu > 0$ is fixed and β tends to ∞. Depending on ν, here the zeros of the Laguerre polynomial $L_N^{\nu-1}$ play a prominent role. As in Sections 2 and 3, we first study the solutions of the limit dynamical systems in Section 4 and present then a functional CLT in Section 5. In Section 6 we consider the root systems D_N with $k \in [0, \infty[$ for ∞. The case D_N is then used in Section 7 to settle also the limits $k = (k_1, k_2)$ for $k_1 \geq 0$ fixed and $k_2 \to \infty$ in the B_N-case. As for the root systems B_N the case $k = (k_1, k_2)$ for $k_2 \geq 0$ fixed and $k_1 \to \infty$ was already treated in [AV1], we skip this case here. In Section 8 we consider an extension to the multivariate Bessel processes by adding an additional drift term of the form $-\lambda X_{t,k}$, $\lambda \in \mathbb{R}$ and derive the corresponding limit theorems. For $\lambda > 0$ the resulting process is ergodic and mean reverting, which also determines the long-term behaviour of the limiting Gaussian process. In the one-dimensional case with $\lambda > 0$ the squared process is the Cox-Ingersoll-Ross process, widely used in mathematical finance. As many results and their proofs for this extension are completely analogous to the results in the preceding sections, we there only presents some major results and skip some proofs.

We here finally recapitulate that for all root systems, the Bessel processes $(X_{t,k})_{t\geq 0}$ may be seen as unique solutions of associated SDEs by the following theorem. This result is part of Lemma 3.4, Corollary 6.6, and Proposition 6.8 of [CGY], where the proofs of the first statements contain some gaps which can be closed with the results in [Sh]. We are grateful to P. Graczyk to a hint to this gap.

Theorem 1.1. *Assume that all multiplicities k are positive. Then, for each starting point $x \in C_N$ in the closed Weyl chamber and each $t > 0$, the Bessel process $(X_{t,k})_{t\geq 0}$ satisfies*

$$E\Big(\int_0^t \nabla(\ln w_k)(X_{s,k})\, ds\Big) < \infty,$$

and the initial value problem

$$X_{0,k} = x, \qquad dX_{t,k} = dB_t + \frac{1}{2}(\nabla(\ln w_k))(X_{t,k})\, dt \tag{1.9}$$

with an N-dimensional Brownian motion $(B_t)_{t\geq 0}$ has a unique (strong) solution $(X_{t,k})_{t\geq 0}$. This solution is a Bessel process as above.

Moreover, if all components of k are at least $1/2$, and if x is in the interior of C_N, then $(X_{t,k})_{t\geq 0}$ lives on the interior on C_N, i.e. $(X_{t,k})_{t\geq 0}$ does not hit the boundary a.s..

2. The dynamical system in the limit for the root system A_{N-1}

We first study Bessel processes and their limits for the root systems A_{N-1} as $k\to\infty$. We first recapitulate some results [AKM1] and [AV1]. The SDE (1.9) for Bessel processes $(X_{t,k})_{t\geq 0}$ of type A_{N-1} reads as

$$dX^i_{t,k} = dB^i_t + k\sum_{j\neq i}\frac{1}{X^i_{t,k}-X^j_{t,k}}dt \qquad (i=1,\ldots,N). \tag{2.1}$$

with an N-dimensional Brownian motion $(B^1_t,\ldots,B^N_t)_{t\geq 0}$. The renormalized processes $(\tilde X_{t,k} := X_{t,k}/\sqrt{k})_{t\geq 0}$ satisfy

$$d\tilde X^i_{t,k} = \frac{1}{\sqrt{k}}dB^i_t + \sum_{j\neq i}\frac{1}{\tilde X^i_{t,k}-\tilde X^j_{t,k}}dt \qquad (i=1,\ldots,N). \tag{2.2}$$

The solutions of this SDE are closely related to the following deterministic limit $k=\infty$; see Lemma 2.1 of [AV1]:

Lemma 2.1. *For $\epsilon>0$ consider the open subset $U_\epsilon := \{x\in C_N^A :\ d(x,\partial C_N^A)>\epsilon\}$ (where $\mathbb{R}^N$ carries the usual Euclidean distance). Then the function*

$$H: U_\epsilon\to\mathbb{R}^N,\quad x\mapsto\Big(\sum_{j\neq 1}\frac{1}{x_1-x_j},\ldots,\sum_{j\neq N}\frac{1}{x_N-x_j}\Big)$$

is Lipschitz continuous on U_ϵ with Lipschitz constant $L_\epsilon>0$, and for each starting point $x_0\in U_\epsilon$, the solution $\phi(t,x_0)$ of the dynamical system $\frac{dx}{dt}(t)=H(x(t))$ satisfies $\phi(t,x_0)\in U_\epsilon$ for all $t\geq 0$.

For certain starting points, the ODEs of Lemma 2.1 have simple solutions which can be expressed via the zeros of the N-th Hermite polynomial H_N, where we assume that the $(H_N)_{N\geq 0}$ are orthogonal w.r.t. the density e^{-x^2} as in [S]. We recapitulate from [AKM1] (or Section 6.7 of [S]):

Lemma 2.2. *For $y\in C_N^A$, the following statements are equivalent:*

1. *The function $2\sum_{i,j:i<j}\ln(x_i-x_j)-\|x\|^2/2$ is maximal at $y\in C_N^A$;*
2. *For $i=1,\ldots,N$: $\frac{1}{2}y_i=\sum_{j:j\neq i}\frac{1}{y_i-y_j}$;*
3. *The vector*

$$z:=(z_1,\ldots,z_N):=(y_1/\sqrt{2},\ldots,y_N/\sqrt{2})$$

consists of the ordered zeroes of H_N.

Lemma 2.2 leads to the following solution of the ODEs of Lemma 2.1; cf. [AV1]:

Corollary 2.3. *For the vector z as above and each $c>0$, a solution of the ODE in Lemma 2.1 is given by $\phi(t,c\cdot z)=\sqrt{2t+c^2}\cdot z$.*

We next show that the solution in Corollary 2.3 attracts all other solutions. For this we recapitulate that by [RV1], for a Bessel process $(X_{t,k})_{t\geq 0}$ of type A, $(\|X_{t,k}\|)_{t\geq 0}$ is a one-dimensional Bessel process on $[0,\infty[$ with multiplicity parameter $(kN+1)(N-1)/2$. This corresponds to the following property of ϕ, where we denote by $\dot\phi$ the derivative with respect to the first component:

$$\begin{aligned}\frac{d}{dt}\|\phi(t,x)\|^2 &= 2\sum_{i=1}^N \phi_i(t,x)\cdot\dot\phi_i(t,x)\\ &= 2\sum_{i,j=1,\ldots,N,i\neq j}\frac{\phi_i(t,x)}{\phi_i(t,x)-\phi_j(t,x)} \;=\; N(N-1).\end{aligned} \tag{2.3}$$

As $\|\phi(0,x)\|^2=\|x\|^2$, we see that for all $t\geq 0$ and all x,

$$\|\phi(t,x)\|^2 = N(N-1)t+\|x\|^2. \tag{2.4}$$

This is the first step for the following stability result:

Lemma 2.4. *For each initial value x in the interior of C_N^A, the solution ϕ of the ODE in Lemma 2.1 has the form*

$$\phi(t,x)=\sqrt{N(N-1)t+\|x\|^2}\cdot\phi_0(t,x) \qquad (t\geq 0)$$

where for all t,x, the function ϕ_0 satisfies

$$\|\phi_0(t,x)\|=1 \qquad \textit{and} \qquad \lim_{t\to\infty}\phi_0(t,x)=\sqrt{\frac{2}{N(N-1)}}\cdot z$$

with the vector z of Lemma 2.2.

Proof. Using (2.4), we define

$$\phi_0(t,x):=(\phi_{0,1}(t,x),\ldots,\phi_{0,N}(t,x)):=\frac{1}{\sqrt{N(N-1)t+\|x\|^2}}\cdot\phi(t,x)=\frac{\phi(t,x)}{\|\phi(t,x)\|} \tag{2.5}$$

with $\|\phi_0(t,x)\|=1$. The ODE in Lemma 2.1 for ϕ implies that

$$\begin{aligned}\frac{d}{dt}(\phi_{0,i}(t,x)) &= \frac{\dot\phi_i(t,x)}{\sqrt{N(N-1)t+\|x\|^2}}-\frac{N(N-1)\cdot\phi_i(t,x)}{2(N(N-1)t+\|x\|^2)^{3/2}}\\ &= \frac{1}{N(N-1)t+\|x\|^2}\left(\sum_{j\neq i}\frac{\sqrt{N(N-1)t+\|x\|^2}}{\phi_i(t,x)-\phi_j(t,x)}-\frac{N(N-1)}{2}\phi_{0,i}(t,x)\right)\\ &= \frac{1}{N(N-1)t+\|x\|^2}\left(\sum_{j\neq i}\frac{1}{\phi_{0,i}(t,x)-\phi_{0,j}(t,x)}-\frac{N(N-1)}{2}\phi_{0,i}(t,x)\right).\end{aligned}$$

Therefore,

$$\psi(t,x):=\phi_0\Big(\frac{N(N-1)}{2}t^2+\|x\|^2t,x\Big) \quad (t\geq 0)$$

satisfies

$$\dot\psi_i(t,x)=\sum_{j\neq i}\frac{1}{\psi_i(t,x)-\psi_j(t,x)}-\frac{N(N-1)}{2}\psi_i(t,x) \quad (i=1,\ldots,N) \tag{2.6}$$

with $\psi(0,x) = \phi_0(0,x) = x/\|x\|$. The ODE (2.6) is a gradient sytem $\dot{\psi} = (\nabla u)(\psi)$ with

$$u(y) := 2 \sum_{i,j=1,\ldots,N, i<j} \ln(y_i - y_j) - \|y\|^2 \cdot N(N-1)/4.$$

Lemma 2.2 ensures that u admits a unique local maximum on C_N^A, that this maximum is a global one, and that it located at

$$\sqrt{\frac{2}{N(N-1)}} \cdot z$$

where, by (D.22) of [AKM1], this vector has $\|.\|_2$-norm 1. We conclude from Section 9.4 of [HS] on gradient systems that this point is an asymptotically stable equilibrium point of the ODE (2.6). This and (2.5) now lead to the claim. □

We are not able to determine ϕ explicitly for arbitrary starting points and N. On the other hand, (2.3) is a special case of the observation that for each symmetric polynomial p in N variables, $t \mapsto p(\phi(t))$ is a polynomial in t which can be computed explicitly. For instance, it follows immediately from the definition of ϕ that $\frac{d}{dt} \sum_{k=1}^N \phi_k(t,x) = 0$ and thus

$$\sum_{k=1}^N \phi_k(t,x) = \sum_{k=1}^N x_k \qquad \text{for} \quad t \geq 0. \tag{2.7}$$

To derive a result for general symmetric polynomials, we use the elementary symmetric polynomials $e_k := e_k^N$ in N variables which are characterized by

$$\prod_{k=1}^N (z - x_k) = \sum_{k=0}^N (-1)^{N-k} e_{N-k}(x) z^k \qquad (z \in \mathbb{C},\ x = (x_1, \ldots, x_n)). \tag{2.8}$$

In particular, $e_0 = 1$, $e_1(x) = \sum_{k=1}^N x_k, \ldots, e_N(x) = \prod_{k=1}^N x_k$. As each symmetric polynomial in N variables is a polynomial in $e_1, \ldots, e_N$ by a classical result, the following lemma shows that all symmetric polynomials of ϕ are polynomials in t.

Lemma 2.5. *For each initial value x in the interior of C_N^A, consider the solution $\phi(t,x)$ of the ODE in Lemma 2.1. Then, for $k = 0, \ldots, N$, $t \mapsto e_k(\phi(t,x))$ is a polynomial in t of degree (at most) $\lfloor \frac{k}{2} \rfloor$ where the leading coefficient of order $\lfloor \frac{k}{2} \rfloor$ is given by*

$$\frac{(-1)^l \cdot N!}{2^l \cdot l!(N-2l)!} \quad (k = 2l \leq N) \quad \textit{and} \quad \frac{(-1)^l \cdot (N-1)!}{2^l \cdot l!(N-2l-1)!} \cdot \sum_{j=1}^N x_j \quad (k = 2l+1 \leq N).$$

Proof. The statement is clear for $k = 0, 1$. For $k \geq 2$ we use induction on k. For this we denote the elementary symmetric polynomial in R variables of order $k \leq R$ by e_k^R, and for a non-empty set $S \subset \{1, \ldots, N\}$, the vector $\phi_S(t,x) \in \mathbb{R}^{|S|}$ is the vector with the coordinates $\phi_i(t,x)$ for $i \in S$ in the natural ordering on S. With these notations we have for $k \geq 2$ that

$$\frac{d}{dt} e_k(\phi(t,x)) = \sum_{j=1}^N \frac{d\phi_j(t,x)}{dt} \cdot e_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}(t,x)).$$

Therefore, by the differential equation for ϕ,

$$\frac{d}{dt}e_k(\phi(t,x)) = \sum_{j=1}^{N}\sum_{i:i\neq j}\frac{e_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}(t,x))}{\phi_j(t,x)-\phi_i(t,x)} \tag{2.9}$$

$$= \frac{1}{2}\sum_{i,j=1,\ldots,n;i\neq j}\frac{e_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}(t,x)) - e_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{i\}}(t,x))}{\phi_j(t,x)-\phi_i(t,x)}.$$

Moreover, simple combinatorial computations yield for $i \neq j$

$$e_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}) - e_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{i\}}) = (\phi_i - \phi_j)e_{k-2}^{N-2}(\phi_{\{1,\ldots,N\}\setminus\{i,j\}}) \tag{2.10}$$

and

$$\sum_{i,j=1,\ldots,N;i\neq j} e_{k-2}^{N-2}(\phi_{\{1,\ldots,N\}\setminus\{i,j\}}) = (N-k+2)(N-k+1)e_{k-2}^{N}(\phi). \tag{2.11}$$

Therefore, by (2.9)-(2.11),

$$\frac{d}{dt}e_k(\phi(t,x)) = -\frac{1}{2}(N-k+2)(N-k+1)e_{k-2}^{N}(\phi(t,x)). \tag{2.12}$$

This recurrence relation and the known cases $k = 0, 1$ now easily lead to the claim. □

Remark 2.6. The differential equation for ϕ immediately implies that for each $r \in \mathbb{R}$ and x in the interior of C_N^A,

$$\frac{d}{dt}\Big(\phi(t, x + r\cdot(1,\ldots,1)) - \phi(t,x)\Big) = 0$$

and thus

$$\phi(t, x + r\cdot(1,\ldots,1)) = \phi(t,x) + r\cdot(1,\ldots,1). \tag{2.13}$$

This implies that we may assume $\sum_{j=1}^N x_j = 0$ without loss of generality for our initial conditions. If we do so, the degrees of the polynomials $t \mapsto e_k(\phi(t,x))$ for odd k can be chosen to be even smaller.

Lemma 2.5 and Eq. (2.8) can be used to compute $\phi(t,x)$ explicitly in the interior of C_N^A. First, one has to determine the polynomials $e_k(\phi(t,x))$ $(k = 1,\ldots,N)$. In a second step, one has to determine the ordered, different zeros of the polynomials in (2.8) from the coefficients of the polynomials which is a diffeomorphism and the inverse of a simple explicit polynomial map. We present an example.

Example 2.7. Let $N = 3$. Choose the starting point $x \in C_3^A$ with $x_1+x_2+x_3 = 0$ according to Remark 2.6. We here obtain

$$e_1(\phi(t,x)) = 0, \quad e_2(\phi(t,x)) = -3t + e_2(x), \quad e_3(\phi(t,x)) = e_3(x)$$

and thus

$$\prod_{k=1}^{3}(z - \phi_k(t,x)) = z^3 + (e_2(x) - 3t)z - e_3(x). \tag{2.14}$$

We now apply Cardano's formula in the casus irreducibilis. We first observe that the existence of 3 real zeros implies $3t - e_2(x) > 0$, and we have the solutions

$$\phi_k(t,x) = \sqrt{4t - \frac{4}{3}e_2(x)}\cdot\cos\Big(\frac{1}{3}\arccos\Big(\frac{\sqrt{27}}{2}\frac{e_3(x)}{(3t-e_2(x))^{3/2}}\Big) + \frac{2}{3}(1-k)\pi\Big)$$

for $k = 1, 2, 3$. The correct ordering $\phi_1(t,x) \geq \phi_2(t,x) \geq \phi_3(t,x)$ here follows easily from the case $t \to \infty$ in which case we have

$$\begin{aligned}\phi(t,x) &= \sqrt{4t - \frac{4}{3}e_2(x)} \cdot \Big((\sqrt{3/4}, 0, -\sqrt{3/4}) + O(t^{-3/2}) \Big) \\ &= \sqrt{6t - 2e_2(x)} \cdot (\sqrt{1/2}, 0, -\sqrt{1/2}) + O(t^{-1}) \quad =: \quad \tilde{\phi}(t) + O(t^{-1})\end{aligned}$$

with a solution $\tilde{\phi}$ of our differential equation of the type of Corollary 2.3. This improves Lemma 2.4 in a quantitative way.

We also remark that the discriminant of the polynomial (2.14) is given by

$$\Delta := e_3(x)^2/4 + (e_2(x) - 3t)^3/27.$$

By Cardano's formula, $\Delta = 0$ holds if and only if we have multiple (real) zeros in (2.14), i.e., a point on the boundary of C_3^A. Hence, if we formally start our solution at time $t = 0$ at some x on the boundary of C_3^A with $x_1 + x_2 + x_3 = 0$, then ϕ can be written down for all $t \geq 0$ such that $\phi(t,x)$ is in the interior for all $t > 0$.

We finally point out that with Remark 2.6 we can generalize all results to arbitrary starting points in C_3^A.

3. A functional central limit theorem for the root system A_{N-1}

In this section we use the solutions ϕ of the ODE in Lemma 2.1 in order to derive limit theorems for Bessel processes of type A_{N-1}. We have the following strong limit law; see Theorem 2.4 of [AV1].

Theorem 3.1. *Let x be a point in the interior of C_N^A, and let $y \in \mathbb{R}^N$. Let $k_0 \geq 1/2$ such that $\sqrt{k} \cdot x + y$ is in the interior of C_N^A for $k \geq k_0$.*

For $k \geq k_0$ consider the Bessel processes $(X_{t,k})_{t \geq 0}$ of type A_{N-1} starting at $\sqrt{k} \cdot x + y$. Then, for all $t > 0$,

$$\sup_{0 \leq s \leq t, k \geq k_0} \|X_{s,k} - \sqrt{k}\phi(s,x)\| < \infty \tag{3.1}$$

almost surely. In particular, locally uniformly in t a.s.,

$$X_{t,k}/\sqrt{k} \to \phi(t,x) \qquad \text{for} \qquad k \to \infty.$$

We now turn to an associated functional central limit theorem which makes the difference $X_{t,k} - \sqrt{k}\phi(t,x)$ in (3.1) more precise. For this we again fix a point x in the interior of C_N^A as before and consider the associated solution $t \mapsto \phi(t,x)$ $(t \geq 0)$ of the ODE in Lemma 2.1. We also introduce the N-dimensional process $(W_t)_{t \geq 0}$ which is the unique solution of the inhomogeneous linear SDE

$$dW_t^i = dB_t^i + \sum_{j \neq i} \frac{W_t^j - W_t^i}{(\phi_i(t,x) - \phi_j(t,x))^2} dt \qquad (i = 1, \ldots, N). \tag{3.2}$$

with initial condition $W_0 = 0$; notice that here the denominator is $\neq 0$ for all $t > 0$. The SDE (3.2) may be written in matrix notation as

$$dW_t = dB_t + A(t,x)W_t dt \tag{3.3}$$

with the matrices $A(t,x) \in \mathbb{R}^{N \times N}$ with

$$A(t,x)_{i,j} := \frac{1}{(\phi_i(t,x) - \phi_j(t,x))^2}, \quad A(t,x)_{i,i} := -\sum_{j \neq i} \frac{1}{(\phi_i(t,x) - \phi_j(t,x))^2}$$

for $i,j=1,\ldots,N$, $i\neq j$. The process $(W_t)_{t\geq 0}$ admits the explicit representation in terms of matrix-valued exponentials

$$W_t = e^{\int_0^t A(s,x)ds}\int_0^t e^{-\int_0^s A(u,x)du}dB_s \quad (t\geq 0). \tag{3.4}$$

This process is obviously Gaussian, and we shall describe it more closely below. It is related to the Bessel processes $(X_{t,k})_{t\geq 0}$ by the following functional CLT:

Theorem 3.2. *Let x be a point in the interior of C_N^A and let $y\in\mathbb{R}^N$. Let $k_0\geq 1/2$ such that $\sqrt{k}\cdot x+y$ is in the interior of C_N^A for $k\geq k_0$.*

For $k\geq k_0$ consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type A_{N-1} starting at $\sqrt{k}\cdot x+y$. Then, for all $t>0$,

$$\sup_{0\leq s\leq t,k\geq k_0}\sqrt{k}\cdot\|X_{s,k}-\sqrt{k}\phi(s,x)-W_s\|<\infty \tag{3.5}$$

almost surely. This means that $\sqrt{k}(\frac{X_{t,k}}{\sqrt{k}}-\phi(t,x))\longrightarrow W_t$ for $k\to\infty$ locally uniformly in t almost surely with rate $O(1/\sqrt{k})$.

Proof. For $k\geq k_0$ consider the processes

$$(R_{t,k}:=X_{t,k}-\sqrt{k}\phi(t,x)-W_t)_{t\geq 0}$$

on $\mathbb{R}^N$. Then $R_{0,k}=0$, and, by the SDEs (3.2) and (2.1) and the ODE for ϕ in Lemma 2.1,

$$\begin{aligned}R_{t,k}^i = k\int_0^t\sum_{j\neq i}\Bigg(&\frac{1}{X_{s,k}^i-X_{s,k}^j}-\frac{1}{\sqrt{k}(\phi_i(s,x)-\phi_j(s,x))}\\ &-\frac{W_s^j-W_s^i}{\big(\sqrt{k}(\phi_i(s,x)-\phi_j(s,x))\big)^2}\Bigg)ds\end{aligned}$$

for $i=1,\ldots,N$. We now use Taylor expansion for the function $1/x$ with Lagrange remainder around some point $x_0\neq 0$, i.e.,

$$\frac{1}{x}=\frac{1}{x_0}-\frac{x-x_0}{x_0^2}+\frac{(x-x_0)^2}{\tilde{x}^3}$$

with some $\tilde{x}$ between $x\neq 0$ and $x_0\neq 0$ where x,x_0 have the same sign. Taking

$$x=X_{s,k}^i-X_{s,k}^j \quad\text{and}\quad x_0=\sqrt{k}(\phi_i(s,x)-\phi_j(s,x)),$$

we arrive at

$$\begin{aligned}R_{t,k}^i &= -\int_0^t\Bigg(\sum_{j\neq i}\frac{(X_{s,k}^i-\sqrt{k}\phi_i(s,x)-W_s^i)-(X_{s,k}^j-\sqrt{k}\phi_j(s,x)-W_s^j)}{(\phi_i(s,x)-\phi_j(s,x))^2}+H_{s,k}^i\Bigg)ds\\ &= -\int_0^t\Bigg(\sum_{j\neq i}\frac{R_{s,k}^i-R_{s,k}^j}{(\phi_i(s,x)-\phi_j(s,x))^2}+H_{s,k}^i\Bigg)ds\end{aligned}$$

with the error terms

$$H_{s,k}^i = k\sum_{j\neq i}\frac{\Big((X_{s,k}^i-\sqrt{k}\phi_i(s,x))-(X_{s,k}^j-\sqrt{k}\phi_j(s,x))\Big)^2}{\Big(\sqrt{k}(\phi_i(s,x)-\phi_j(s,x))+D_{i,j}(s)\Big)^3}$$

where, by the Lagrange remainder,

$$|D_{i,j}(s)| \le |(X_{s,k}^j - \sqrt{k}\phi_j(s,x)) - (X_{s,k}^j - \sqrt{k}\phi_j(s,x))|.$$

By Theorem 3.1, this can be bounded by some a.s. finite random variable D independent of i,j, $s \in [0,t]$, and $k \ge k_0$ where D depends on x,y,t. Therefore, for all $i = 1,\dots,N$,

$$|H_{s,k}^i| \le \frac{1}{\sqrt{k}} H \qquad \text{for} \quad k \ge k_0,\ s \in [0,t]$$

with some a.s. finite random variable H. In summary,

$$R_{t,k} = -\int_0^t (A(s,x)R_{s,k} + H_{s,k})\,ds, \quad R_{0,k} = 0$$

and thus, for suitable norms and all $u \in [0,t]$,

$$\|R_{u,k}\| \le A\int_0^u \|R_{s,k}\|\,ds + \frac{t\cdot\|H\|}{\sqrt{k}}$$

with $A := \sup_{s\in[0,t]} \|A(s,x)\| < \infty$. Hence, by the classical lemma of Gronwall,

$$\|R_{u,k}\| \le \frac{t\|H\|}{\sqrt{k}} e^{tA}$$

for all $u \in [0,t]$. This yields the claim. □

Remark 3.3. The Bessel processes $(X_{t,k})_{t\ge0}$ of type A admit some algebraic properties which are related with corresponding algebraic properties of ϕ and the matrix function $A(t,x)$. We discuss some of them:

(1) $(X_{t,k})_{t\ge0}$ has the same scaling as Brownian motions, i.e., for all $r > 0$, the diffusion $(\frac{1}{r}X_{r^2t,k})_{t\ge0}$ is also a Bessel process of type A with the same k where clearly the starting point is changed. The corresponding relations for ϕ and A are

$$\phi(r^2t, rx) = r\cdot\phi(t,x), \quad A(r^2t, rx) = \frac{1}{r^2}A(t,x) \quad \text{for} \quad r>0,\ t\ge 0.$$

Moreover, if we consider $(W_t)_{t\ge0}$ from (3.2), then for $r > 0$, $(\frac{1}{r}W_{r^2t})_{t\ge0}$ is also a process of this type where x is replaced by rx in Eqs. (3.2)–(3.4).

(2) By (2.1), the center of gravity

$$\Big(\overline{X_{t,k}} := \frac{1}{N}(X_{t,k}^1 + \ldots + X_{t,k}^N) = \frac{1}{N}(B_t^1 + \ldots + B_t^N)\Big)_{t\ge0}$$

is a Brownian motion up to scaling. For ϕ and $(W_t)_{t\ge0}$ this means that

$$\sum_{i=1}^N \phi_i(t,x) = x_1 + \ldots + x_N \quad \text{and} \quad W_t^1 + \ldots + W_t^N = B_t^1 + \ldots + B_t^N$$

for $t \ge 0$. Moreover, this reflects the fact that the sums over all rows and columns of $A(t,x)$ are equal to 0. In particular, $A(t,x)$ is always singular.

(3) Let $(\widehat{X_{t,k}})_{t\ge0}$ be the orthogonal projection of the Bessel process $(X_{t,k})_{t\ge0}$ on $\mathbb{R}^N$ to the orthogonal complement $(1,\dots,1)^\perp \subset \mathbb{R}^N$ of the vector $(1,\dots,1)$; notice that we here write row vectors for simplicity of notation. $(\widehat{X_{t,k}})_{t\ge0}$ is again a diffusion which lives on this $(N-1)$-dimensional subspace which is stochastically independent of the center-of-gravity-process $(\overline{X_{t,k}})_{t\ge0}$ on $\mathbb{R}\cdot(1,\dots,1)$. As the latter one is quite simple (see (2)), the main difficulties

of the particle process $(X_{t,k})_{t\geq 0}$ are contained the $(N-1)$-dimensional process $(\widehat{X_{t,k}})_{t\geq 0}$. On the level of ϕ and A, we have the relations

$$\phi(t, x + r\cdot(1,\ldots,1)) = \phi(t,x) + r\cdot(1,\ldots,1)$$

and $A(t, x + r\cdot(1,\ldots,1)) = A(t,x)$ for $r \in \mathbb{R}$.

Remark 3.4. Similarly as in Theorem 3.2 we may deduce functional central limit theorems for powers $p \in \mathbb{N}$ of Bessel processes (which are taken in all coordinates). The most prominent examples are the squared Bessel process for $p = 2$. We have

$$\sqrt{k}((\frac{X_{t,k}}{\sqrt{k}})^p - \phi^p(t,x)) \longrightarrow W_{t,p} \tag{3.6}$$

for $k \to \infty$ locally uniformly in t almost surely with limiting processes which are given as solution of

$$dW_{t,p} = p\phi^{p-1}(t,x)dB_t + pA_p(t,x)W_{t,p}dt, \quad W_{0,p} = 0 \tag{3.7}$$

with the matrices $A_p(t,x) \in \mathbb{R}^{N\times N}$ with

$$A_p(t,x)_{i,j} := \frac{\phi_i^{p-1}(t,x)}{(\phi_i(t,x) - \phi_j(t,x))^2},$$

$$A_p(t,x)_{i,i} := -\sum_{j\neq i} \frac{(p-2)\phi_i^{p-1}(t,x) - (p-1)\phi_i^{p-2}(t,x)\phi_j(t,x)}{(\phi_i(t,x) - \phi_j(t,x))^2}$$

for $i,j = 1,\ldots,N$, $i \neq j$.

Remark 3.5. Assume now that N is odd, and that $(X_{t,k})_{t\geq 0}$ starts at $\sqrt{k}cz$ in the interior of C_N^A. In order to study the $\frac{N-1}{2}+1$-th component, we notice that $z_{\frac{N-1}{2}+1} = 0$ and that hence for $p \geq 2$ we have $W_t^{\frac{N-1}{2}+1} = 0$ which implies a degenerate normal limiting distribution. This suggests that for the $\frac{N-1}{2}+1$-th component we have a faster rate of convergence than $\sqrt{k}$. Indeed from Theorem 3.2 we see that as $k \to \infty$

$$X_{t,k}^{\frac{N-1}{2}+1} \xrightarrow{d} \mathcal{N}$$

for some normal random variable $\mathcal{N}$, hence $(X_{t,k}^{\frac{N-1}{2}+1})^p \xrightarrow{d} \mathcal{N}^p$ without normalizing sequence.

We finally calculate the covariance matrix of W_t for the special solution ϕ of Corollary 2.3 explicitly. For this we introduce the matrix $A \in \mathbb{R}^{N\times N}$ with

$$A_{i,j} := \frac{1}{(z_i - z_j)^2}, \quad A_{i,i} := -\sum_{j\neq i} \frac{1}{(z_i - z_j)^2} \tag{3.8}$$

for $i,j = 1,\ldots,N$, $i \neq j$ and the vector z as in 2.2. Moreover, let E be the N-dimensional unit matrix. It is shown in [AV2] that the matrix $E - A$ has the eigenvalues

$$1, 2, \ldots, N. \tag{3.9}$$

The eigenvectors are also known by [AV2], but more complicated. We omit details here. With these notations we have:

Lemma 3.6. *Assume that $(X_{t,k})_{t\geq 0}$ starts in the interior of C_N^A in $\sqrt{k}\cdot cz+y$ with $y\in\mathbb{R}$, z as in 2.2 and $c>0$. Then the covariance matrices $\Sigma_t\in\mathbb{R}^{N\times N}$ for $t>0$ of the limit process $(W_t)_{t\geq 0}$ are given by*

$$\Sigma_t=(t+\frac{c^2}{2})(E-A)^{-1}(E-e^{(\ln\frac{c^2}{2t+c^2})(E-A)}).$$

with eigenvalues $\lambda_k^A(t,c)=\frac{1}{2k}\frac{(2t+c^2)^k-c^{2k}}{(2t+c^2)^{k-1}}$ $(k=1,\ldots,N)$. In particular, $\lambda_1^A(t,c)=t$.

Proof. For the special case $\phi(s,cz)=\sqrt{2s+c^2}z$, the matrix function $A(s,cz)$ has the simple time-dependence $A(s,cz)=\frac{1}{2s+c^2}A$. Hence we obtain for the process

$$W_t=e^{(\ln(2t+c^2)-\ln c^2)A/2}\int_0^t e^{(-\ln(2s+c^2)+\ln c^2)A/2}dB_s\quad(t\geq 0).$$

Since A is real and symmetric and taking (3.9) into account, we may write A as $A=UDU^t$ with an orthogonal matrix U and with the diagonal matrix

$$D=diag(d_1,\ldots d_N):=diag(0,-1,-2,\ldots,-N+1).$$

This leads to

$$W_t=U\int_0^t diag\left(\Big(\frac{2t+c^2}{2s+c^2}\Big)^{\frac{d_1}{2}},\ldots,\Big(\frac{2t+c^2}{2s+c^2}\Big)^{\frac{d_N}{2}}\right)d\tilde{B}_sU^t$$

with the rotated Brownian motion $(\tilde{B}_t:=U^tB_tU)_{t\geq 0}$. This, the Itô-isometry, and $d_i/2\neq 1$ for all i yield

$$\begin{aligned}\Sigma_t &= U\cdot\int_0^t diag\left(\Big(\frac{2t+c^2}{2s+c^2}\Big)^{\frac{d_1}{2}},\ldots,\Big(\frac{2t+c^2}{2s+c^2}\Big)^{\frac{d_N}{2}}\right)^2 ds\cdot U^t\\
&= U\cdot\int_0^t diag\left(\Big(\frac{2t+c^2}{2s+c^2}\Big)^{d_1},\ldots,\Big(\frac{2t+c^2}{2s+c^2}\Big)^{d_N}\right)ds\cdot U^t\\
&= U\cdot diag\left(\frac{1}{2(1-d_1)}(2t+c^2-c^{2(1-d_1)}(2t+c^2)^{d_1}),\ldots,\right.\\
&\qquad\left.\frac{1}{2(1-d_N)}(2t+c^2-c^{2(1-d_N)}(2t+c^2)^{d_N})\right)\cdot U^t.\end{aligned}$$

Combining

$$(U(E-D)^{-1}U^t)^{-1}=U(E-D)U^t=E-UDU^t=E-A$$

with a reformulation of the i-th entry in the diagonal matrix

$$\frac{1}{2(1-d_i)}(2t+c^2-c^{2(1-d_i)}(2t+c^2)^{d_i})=\frac{1}{1-d_i}(t+\frac{c^2}{2})(1-e^{(1-d_i)\ln\frac{c^2}{2t+c^2}})$$

we obtain by functional calculus that

$$\begin{aligned}\Sigma_t &= (t+\frac{c^2}{2})(E-A)^{-1}\left(E-U\cdot diag\left(e^{(1-d_1)\ln\frac{c^2}{2t+c^2}},\ldots,e^{(1-d_N)\ln\frac{c^2}{2t+c^2}}\right)U^t\right)\\
&= (t+\frac{c^2}{2})(E-A)^{-1}(E-e^{(\ln\frac{c^2}{2t+c^2})(E-A)})\end{aligned}$$

which yields the desired form of the covariance matrix. □

Remark 3.7. (1) A key role plays the matrix $(E-A)^{-1}$ which is the covariance matrix which appeared in [V] for the case when starting $(X_{t,k})_{t\geq 0}$ in zero and in [DE2] in the context of asymptotics for eigenvalues of Hermite ensembles. Note that though we assume $c>0$ we may formally set $c=0$ and obtain $\Sigma_t = t(E-A)^{-1}$ as in [V]. Since $\lim_{t\to\infty}\Sigma_t/t = (E-A)^{-1}$, we obtain asymptotically in t the same result as starting in zero independent of the actual starting point cz.

(2) As noted in [DE2] $(E-A)^{-1}$ is the Hessian of the potential from part (1) of Lemma 2.2 evaluated at the maximizer. Hence the limiting result for $k,t\to\infty$ may be interpreted as a natural first order approximation induced by the underlying dynamical system.

(3) Inserting the definition of the matrix exponential in the representation of Σ_t yields $\Sigma_t = (t+\frac{c^2}{2})[-\ln\frac{c^2}{2t+c^2}E - \frac{(\ln\frac{c^2}{2t+c^2})^2}{2}(E-A)-\cdots]$. Hence we see that the diagonal elements have a different behaviour in t as the remaining entries of the covariance matrix.

(4) As the matrices in Lemma 3.6 satisfy $\Sigma_0 = 0$, the results of Lemma 3.6 are also valid for $t=0$.

(5) We point out that the methods of the proof of Lemma 3.6 also lead to explicit formulas for the covariances of arbitrary components at different times for the process $(W_t)_{t\geq 0}$.

Remark 3.8. Note that in principle also the eigenvalues of the the general $A(t,x)$ could be calculated, since the characteristic polynomials are again symmetric polynomials of the entries, which results in quotients of symmetric polynomials in ϕ.

4. The dynamical system in the limit for the root system B_N

In this and the next section we turn to Bessel processes for the root systems B_N with multiplicities $k=(k_1,k_2)=(\nu\cdot\beta,\beta)$ with $\nu>0$ fixed and $\beta\to\infty$. For this we first recapitulate some facts from [AKM2] and [AV1].

We first notice that the SDE (1.9) for Bessel processes of type B_N reads as

$$dX_{t,k}^i = dB_t^i + \beta\sum_{j\neq i}\Big(\frac{1}{X_{t,k}^i - X_{t,k}^j} + \frac{1}{X_{t,k}^i + X_{t,k}^j}\Big)dt + \frac{\nu\cdot\beta}{X_{t,k}^i}dt \tag{4.1}$$

for $i=1,\ldots,N$ with an N-dimensional Brownian motion $(B_t^1,\ldots,B_t^N)_{t\geq 0}$. The renormalized processes $(\tilde X_{t,k} := X_{t,k}/\sqrt{\beta})_{t\geq 0}$ satisfy

$$d\tilde X_{t,k}^i = \frac{1}{\sqrt{\beta}}dB_t^i + \sum_{j\neq i}\Big(\frac{1}{\tilde X_{t,k}^i - \tilde X_{t,k}^j} + \frac{1}{\tilde X_{t,k}^i + \tilde X_{t,k}^j}\Big)dt + \frac{\nu}{\tilde X_{t,k}^i}dt \tag{4.2}$$

for $i=1,\ldots,N$. These processes are related with the deterministic limit case $\beta=\infty$; see Lemma 3.1 of [AV1]:

Lemma 4.1. *Let $\nu>0$. For $\epsilon>0$ consider the open subset*

$$U_\epsilon := \{x\in C_N^B :\ x_N > \frac{\epsilon\nu}{N-1},\quad \text{and}\quad x_i - x_{i+1} > \epsilon \quad \text{for}\quad i=1,\ldots,N-1\}$$

of C_N^B. Then the function

$$H : U_\epsilon \to \mathbb{R}^N, \quad x \mapsto \begin{pmatrix} \sum_{j\neq 1}\left(\frac{1}{x_1-x_j}+\frac{1}{x_1+x_j}\right)+\frac{\nu}{x_1} \\ \vdots \\ \sum_{j\neq N}\left(\frac{1}{x_N-x_j}+\frac{1}{x_N+x_j}\right)+\frac{\nu}{x_N} \end{pmatrix}$$

is Lipschitz continuous. Moreover, for each starting point $x_0 \in U_\epsilon$, the solution $\phi(t,x_0)$ of $\frac{dx}{dt}(t) = H(x(t))$ satisfies $\phi(t,x_0) \in U_\epsilon$ for $t \geq 0$.

We discuss the general solutions $\phi(t,x)$ below. For certain starting points, ϕ can be determined in terms of zeros of Laguerre polynomials. For this we recapitulate that for $\alpha > 0$ the Laguerre polynomials $(L_n^{(\alpha)})_{n\geq 0}$ are orthogonal w.r.t. the density $e^{-x}\cdot x^\alpha$. We need the following characterization of the zeros of $L_N^{(\nu-1)}$; see [AKM1], or Section 6.7 of [S], or, in the present notation, [AV1]:

Lemma 4.2. *Let $\nu > 0$. For $y \in C_N^B$, the following statements are equivalent:*

(1) *The function*

$$W(x) := 2\sum_{i<j} \ln(x_i^2 - x_j^2) + 2\nu \sum_i \ln x_i - \|x\|^2/2$$

is maximal at $y \in C_N^B$;

(2) *For $i = 1,\ldots,N$,*

$$\frac{1}{2}y_i = \sum_{j:j\neq i} \frac{2y_i}{y_i^2 - y_j^2} + \frac{\nu}{y_i} = \sum_{j:j\neq i}\Big(\frac{1}{y_i-y_j}+\frac{1}{y_i+y_j}\Big)+\frac{\nu}{y_i};$$

(3) *If $z_1^{(\nu-1)},\ldots,z_N^{(\nu-1)}$ are the ordered zeros of $L_N^{(\nu-1)}$, then*

$$2z := 2(z_1^{(\nu-1)},\ldots,z_N^{(\nu-1)}) = (y_1^2,\ldots,y_N^2). \tag{4.3}$$

Lemma 4.2 leads to the following solutions of the ODE of Lemma 4.1; cf. [AV1]:

Corollary 4.3. *Let $\nu > 0$ and $y \in C_N^B$ the vector in Eq. (4.3). Then for each $c > 0$, a solution of the dynamical system in Lemma 4.1 is given by $\phi(t, c\cdot y) = \sqrt{t+c^2}\cdot y$.*

We next show that the solution in Corollary 4.3 attracts all other solutions in some way similar to the A-case in Lemma 2.4. For this we observe that by the ODE for ϕ,

$$\begin{aligned}\frac{d}{dt}\|\phi(t,x)\|^2 &= 2\sum_{i=1}^N \phi_i(t,x)\cdot\dot{\phi}_i(t,x)\\ &= 2\sum_{i,j=1,\ldots,N,i\neq j}\Big(\frac{\phi_i(t,x)}{\phi_i(t,x)-\phi_j(t,x)}+\frac{\phi_i(t,x)}{\phi_i(t,x)+\phi_j(t,x)}\Big)+2\nu N\\ &= 2N(N+\nu-1). \end{aligned}\tag{4.4}$$

As $\|\phi(0,x)\|^2 = \|x\|^2$, we obtain for $t \geq 0$ that

$$\|\phi(t,x)\|^2 = 2N(N+\nu-1)t + \|x\|^2. \tag{4.5}$$

This is the first step for the following stability result:

Lemma 4.4. *For each starting point x in the interior of C_N^B, the solution ϕ of the ODE in Lemma 4.1 has the form*

$$\phi(t,x) = \sqrt{2N(N+\nu-1)t + \|x\|^2}\cdot\phi_0(t,x) \qquad (t \geq 0)$$

where ϕ_0 satisfies

$$\|\phi_0(t,x)\| = 1 \qquad \text{and} \qquad \lim_{t\to\infty}\phi_0(t,x) = \frac{2}{N(N-1)}z$$

with the vector z of Lemma 4.2(3).

Proof. The proof is similar to that of Lemma 2.4. Using (4.5), we define

$$\phi_0(t,x) := \frac{1}{\sqrt{2N(N+\nu-1)t + \|x\|^2}}\cdot\phi(t,x) = \frac{\phi(t,x)}{\|\phi(t,x)\|} \tag{4.6}$$

with $\|\phi_0(t,x)\| = 1$. The ODE in Lemma 4.1 implies that

$$\begin{aligned}
\frac{d}{dt}(\phi_{0,i}(t,x)) &= \frac{\dot{\phi}_i(t,x)}{\sqrt{2N(N+\nu-1)t + \|x\|^2}} - \frac{N(N+\nu-1)\cdot\phi_i(t,x)}{(2N(N+\nu-1)t+\|x\|^2)^{3/2}}\\
&= \frac{1}{2N(N+\nu-1)t+\|x\|^2}\Biggl(\sum_{j\neq i}\frac{1}{\phi_{0,i}(t,x)-\phi_{0,j}(t,x)}+\\
&\quad+\sum_{j\neq i}\frac{1}{\phi_{0,i}(t,x)+\phi_{0,j}(t,x)}+\frac{\nu}{\phi_{0,i}(t,x)} - N(N+\nu-1)\phi_{0,i}(t,x)\Biggr).
\end{aligned}$$

Hence, $\psi(t,x) := \phi_0\Big(N(N+\nu-1)t^2 + \|x\|^2 t, x\Big)$ for $t \geq 0$ satisfies

$$\begin{aligned}
\dot{\psi}_i(t,x) = &\sum_{j\neq i}\frac{1}{\psi_i(t,x)-\psi_j(t,x)} + \sum_{j\neq i}\frac{1}{\psi_i(t,x)+\psi_j(t,x)}\\
&+\frac{\nu}{\psi_i(t,x)} - N(N+\nu-1)\cdot\psi_i(t,x)
\end{aligned} \tag{4.7}$$

for $i = 1,\ldots,N$ with $\psi(0,x) = \phi_0(0,x) = x/\|x\|$. The ODE (4.7) is a gradient system $\dot{\psi} = (\nabla u)(\psi)$ with

$$u(y) := \sum_{i,j=1,\ldots,N,i<j}(\ln(y_i-y_j)+\ln(y_i+y_j)) + \nu\sum_{i=1}^N \ln y_i - \frac{N(N+\nu-1)}{2}\|y\|^2.$$

Lemma 4.2 ensures that u admits a unique local maximum on C_N^B, that this maximum is a global one, and that it located at y with

$$(y_1^2,\ldots,y_N^2) = \frac{1}{N(N+\nu-1)}(z_1^{(\nu-1)},\ldots,z_N^{(\nu-1)}).$$

We notice that $\|y\| = 1$; see e.g. (C.10) in [AKM2]. These observations and (4.6) now lead to the claim as in the proof of Lemma 2.4. □

We are not able to determine ϕ explicitly for arbitrary starting points and N. On the other hand, as for the root systems of type A in Section 2, (4.4) is a special case of more general observation. For this we again use the elementary symmetric polynomials $e_k := e_k^N$ in N variables and define

$$\tilde{e}_k(x) := \tilde{e}_k^N(x) := e_k(x_1^2,\ldots,x_N^2) \qquad (k = 0,\ldots,N).$$

As each symmetric polynomial in N variables is a polynomial in $e_1, \ldots, e_N$ by some classical result, we obtain from the following lemma that all symmetric polynomials in squares of the components of ϕ are polynomials in t.

Lemma 4.5. *For each x in the interior of C_N^B, consider the solution $\phi(t,x)$ of the ODE in Lemma 4.1. Then, for $k = 0, \ldots, N$, $t \mapsto \tilde{e}_k(\phi(t,x))$ is a polynomial in t of degree k with leading coefficient*

$$2^k(N+\nu-1)(N+\nu-2)\cdots(N+\nu-k)\cdot\binom{N}{k} \quad (k \le N).$$

Proof. The statement is trivial for $k=0$ and follows from (4.5) for $k=1$. For $k \ge 2$ we use induction on k. We use the notations $\phi_S(t,x)$ and e_k^R from the proof of Lemma 2.5. We put $\tilde{e}_k^R(x) := e_k^R(x_1^2, \ldots, x_R^2)$. We then have for $k \ge 2$ that

$$\frac{d}{dt}\tilde{e}_k(\phi(t,x)) = 2\sum_{j=1}^N \frac{d\phi_j(t,x)}{dt}\cdot\phi_j(t,x)\cdot\tilde{e}_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}(t,x)).$$

Therefore, by the differential equation for ϕ,

$$\begin{aligned}\frac{d}{dt}\tilde{e}_k(\phi(t,x)) &= 2\sum_{j=1}^N\left(2\sum_{i:i\ne j}\frac{\phi_j(t,x)^2}{\phi_j(t,x)^2-\phi_i(t,x)^2} + \nu\right)\tilde{e}_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}(t,x))\\ &= 2\sum_{i,j=1,\ldots,N;i\ne j}\frac{\phi_j(t,x)^2\tilde{e}_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}(t,x)) - \phi_i(t,x)^2\tilde{e}_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{i\}}(t,x))}{\phi_j(t,x)^2-\phi_i(t,x)^2}\\ &\quad + 2\nu\sum_{j=1}^N\tilde{e}_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}(t,x)).\end{aligned} \tag{4.8}$$

Moreover, simple combinatorial computations show that for $k \le N-1$,

$$\phi_j^2\tilde{e}_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}) - \phi_i^2\tilde{e}_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{i\}}) = (\phi_j^2-\phi_i^2)\tilde{e}_{k-1}^{N-2}(\phi_{\{1,\ldots,N\}\setminus\{i,j\}}) \tag{4.9}$$

and

$$\sum_{i,j=1,\ldots,N;i\ne j}\tilde{e}_{k-1}^{N-2}(\phi_{\{1,\ldots,N\}\setminus\{i,j\}}) = (N-k+1)(N-k)\tilde{e}_{k-1}^N(\phi). \tag{4.10}$$

Moreover,

$$\phi_j^2\tilde{e}_{N-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}) - \phi_i^2\tilde{e}_{N-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{i\}}) = 0. \tag{4.11}$$

Furthermore,

$$\sum_{j=1}^N\tilde{e}_{k-1}^{N-1}(\phi_{\{1,\ldots,N\}\setminus\{j\}}) = (N-k+1)\tilde{e}_{k-1}^N(\phi). \tag{4.12}$$

Therefore, by (4.8)-(4.12), for $k \le N$,

$$\frac{d}{dt}\tilde{e}_k(\phi(t,x)) = 2(N-k+1)(N-k+\nu)\tilde{e}_{k-1}^N(\phi(t,x)). \tag{4.13}$$

This recurrence relation and the known cases $k = 0, 1$ lead easily to the claim. □

Example 4.6. Let $\nu > 0$, $N = 2$, and $x \in C_2^B$. Then, by (4.4), (4.13), and the initial conditions,

$$(z-\phi_1(t,x)^2)(z-\phi_2(t,x)^2) = z^2 - (\phi_1(t,x)^2+\phi_2(t,x)^2)z + \phi_1(t,x)^2\phi_2(t,x)^2$$

with

$$\phi_1(t,x)^2+\phi_2(t,x)^2 = 4(1+\nu)t+\|x\|^2; \quad \phi_1(t,x)^2\phi_2(t,x)^2 = 4\nu(1+\nu)t^2+2\nu\|x\|^2t+x_1^2x_2^2.$$

This yields, since ϕ is non-negative,

$$\phi_1(t,x) = \left(\frac{1}{2}\left(4(1+\nu)t+\|x\|^2+\sqrt{16(1+\nu)t^2+8\|x\|^2t+(x_1^2-x_2^2)^2}\right)\right)^{1/2},$$

$$\phi_2(t,x) = \left(\frac{1}{2}\left(4(1+\nu)t+\|x\|^2-\sqrt{16(1+\nu)t^2+8\|x\|^2t+(x_1^2-x_2^2)^2}\right)\right)^{1/2}$$

This implies in particular that for $t\to\infty$,

$$\begin{aligned}\phi_1(t,x) &= \left(\frac{1}{2}(4(1+\nu)t+\|x\|^2)\right)^{1/2}\left(1+\sqrt{\frac{1}{1+\nu}+\frac{\frac{\nu}{1+\nu}\|x\|^4-4x_1^2x_2^2}{(4(1+\nu)t+\|x\|^2)^2}}\right)^{1/2} \\ &= \left(\frac{1}{2}(4(1+\nu)t+\|x\|^2)\right)^{1/2}\left(1+\sqrt{\frac{1}{1+\nu}}\right)^{1/2} + O(t^{-1/2}), \\ \phi_2(t,x) &= \left(\frac{1}{2}(4(1+\nu)t+\|x\|^2)\right)^{1/2}\left(1-\sqrt{\frac{1}{1+\nu}}\right)^{1/2} + O(t^{-1/2}),\end{aligned}$$

which is some quantitative version of Lemma 4.4.

We also observe that our solutions $\phi(t,x)$ also exist when we start at time $t=0$ at any point x on the boundary of C_2^B and that for these solutions, $\phi(t,x)$ is in the interior of C_2^B for all $t>0$.

5. A functional central limit theorem for the root system B_N

The solutions ϕ in the preceding section appear in the following SLLN for Bessel processes of type B_N; see Theorem 3.4 of [AV1].

Theorem 5.1. *Let $\nu>0$. Let x be a point in the interior of C_N^B, and let $y\in\mathbb{R}^N$. Let $\beta_0\ge 1/2$ with $\sqrt{\beta}\cdot x+y$ in the interior of C_N^B for $\beta\ge\beta_0$. For $\beta\ge\beta_0$, consider the Bessel processes $(X_{t,k})_{t\ge0}$ of type B with $k=(k_1,k_2)=(\beta\cdot\nu,\beta)$, which start in $\sqrt{\beta}\cdot x+y$. Then, for all $t>0$,*

$$\sup_{0\le s\le t,\beta\ge\beta_0}\|X_{s,k}-\sqrt{\beta}\phi(s,x)\|<\infty \qquad a.s..$$

In particular, $X_{t,(\nu\cdot\beta,\beta)}/\sqrt{\beta}\to\phi(t,x)$ for $\beta\to\infty$ locally uniformly in t a.s..

We now turn to an associated functional central limit theorem which makes the difference $X_{t,k}-\sqrt{\beta}\phi(t,x)$ more precise for ν fixed and $\beta\to\infty$. As in Section 2 we fix some x in the interior of C_N^B and consider the associated solution $t\mapsto\phi(t,x)$ $(t\ge0)$. We also introduce an N-dimensional process $(W_t)_{t\ge0}$ as the unique solution of the inhomogeneous linear SDE

$$\begin{aligned}dW_t^i = dB_t^i + \sum_{j\ne i}\left(\frac{W_t^j-W_t^i}{(\phi_i(t,x)-\phi_j(t,x))^2}-\frac{W_t^j+W_t^i}{(\phi_i(t,x)+\phi_j(t,x))^2}\right)dt \\ -\frac{\nu\cdot W_t^i}{\phi_i(t,x)^2}dt \qquad (5.1)\end{aligned}$$

for $i = 1, \ldots, N$ with initial condition $W_0 = 0$. Notice that here all denominators are $\neq 0$ for $t > 0$. The SDE (5.1) may be written in matrix notation as

$$dW_t = dB_t + A_\nu(t,x)W_t dt \tag{5.2}$$

with the matrices $A_\nu(t,x) \in \mathbb{R}^{N\times N}$ with

$$\begin{aligned} A_\nu(t,x)_{i,j} &:= \frac{1}{(\phi_i(t,x)-\phi_j(t,x))^2} - \frac{1}{(\phi_i(t,x)+\phi_j(t,x))^2} \quad (i \neq j), \\ A_\nu(t,x)_{i,i} &:= \sum_{j\neq i}\left(\frac{-1}{(\phi_i(t,x)-\phi_j(t,x))^2} - \frac{1}{(\phi_i(t,x)+\phi_j(t,x))^2}\right) - \frac{\nu}{\phi_i(t,x)^2} \end{aligned} \tag{5.3}$$

for $i,j = 1,\ldots,N$. The process $(W_t)_{t\geq 0}$ is given in terms of matrix-exponentials by

$$W_t = e^{\int_0^t A_\nu(s,x)ds}\int_0^t e^{-\int_0^s A_\nu(u,x)du}dB_s \quad (t \geq 0). \tag{5.4}$$

This process is obviously Gaussian; we describe it more closely below. It is related to the Bessel processes $(X_{t,k})_{t\geq 0}$ as follows:

Theorem 5.2. *Let $\nu > 0$. Let x be a point in the interior of C_N^B and let $y \in \mathbb{R}^N$. Let $\beta_0 \geq 1/2$ such that $\sqrt{\beta}\cdot x + y$ is in the interior of C_N^A for $\beta \geq \beta_0$.*

For $\beta \geq \beta_0$ consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type B_N with $k = (\nu\beta, \beta)$ starting at $\sqrt{\beta}\cdot x + y$. Then, for all $t > 0$,

$$\sup_{0\leq s\leq t, \beta\geq\beta_0} \sqrt{\beta}\cdot\|X_{s,k} - \sqrt{\beta}\phi(s,x) - W_s\| < \infty \qquad a.s., \tag{5.5}$$

i.e., $X_{s,k} - \sqrt{\beta}\phi(s,x) \longrightarrow W_t$ for $\beta \to \infty$ locally uniformly in t a.s. with rate $O(1/\sqrt{\beta})$.

Proof. For $\beta \geq \beta_0$ consider the processes

$$(R_{t,\beta} := X_{t,(\nu\beta,\beta)} - \sqrt{\beta}\phi(t,x) - W_t)_{t\geq 0}$$

on $\mathbb{R}^N$ with $R_{0,\beta} = 0$. Then by the SDEs (5.1), (4.1) and the ODE in Lemma 4.1,

$$\begin{aligned} R^i_{t,\beta} = & \\ = \beta\int_0^t \Bigg[& \sum_{j\neq i}\left(\frac{1}{X^i_{s,k} - X^j_{s,k}} - \frac{1}{\sqrt{\beta}(\phi_i(s,x)-\phi_j(s,x))} - \frac{W^j_s - W^i_s}{\left(\sqrt{\beta}(\phi_i(s,x)-\phi_j(s,x))\right)^2}\right) \\ & + \sum_{j\neq i}\left(\frac{1}{X^i_{s,k} + X^j_{s,k}} - \frac{1}{\sqrt{\beta}(\phi_i(s,x)+\phi_j(s,x))} + \frac{W^j_s + W^i_s}{\left(\sqrt{\beta}(\phi_i(s,x)+\phi_j(s,x))\right)^2}\right) \\ & + \nu\left(\frac{1}{X^i_{s,k}} - \frac{1}{\sqrt{\beta}\phi_i(s,x)} + \frac{W^i_s}{\left(\sqrt{\beta}\phi_i(s,x)\right)^2}\right)\Bigg] ds \end{aligned} \tag{5.6}$$

for $i = 1,\ldots,N$. We use Taylor expansion for $1/x$ with Lagrange remainder around some point $x_0 \neq 0$, i.e.,

$$\frac{1}{x} = \frac{1}{x_0} - \frac{x - x_0}{x_0^2} + \frac{(x-x_0)^2}{\tilde{x}^3}$$

with some $\tilde{x}$ between $x \neq 0$ and $x_0 \neq 0$ which have the same signs. Taking

$$x = X^i_{s,k} \pm X^j_{s,k}, \qquad x_0 = \sqrt{\beta}(\phi_i(s,x) \pm \phi_j(s,x)) \quad \text{and}$$
$$x = X^i_{s,k}, \qquad x_0 = \sqrt{\beta}\phi_i(s,x),$$

we get

$$\begin{aligned} R^i_{t,\beta} = -\int_0^t \Bigg(& \sum_{j\neq i} \frac{(X^i_{s,k} - \sqrt{\beta}\phi_i(s,x) - W^i_s) - (X^j_{s,k} - \sqrt{\beta}\phi_j(s,x) - W^j_s)}{(\phi_i(s,x) - \phi_j(s,x))^2} \\ & + \sum_{j\neq i} \frac{(X^i_{s,k} - \sqrt{\beta}\phi_i(s,x) - W^i_s) + (X^j_{s,k} - \sqrt{\beta}\phi_j(s,x) - W^j_s)}{(\phi_i(s,x) + \phi_j(s,x))^2} \\ & + \frac{X^i_{s,k} - \sqrt{\beta}\phi_i(s,x) - W^i_s}{\phi_i(s,x)^2} + +H^i_{s,\beta}\Bigg) ds \\ = -\int_0^t \Bigg(& \sum_{j\neq i} \frac{R^i_{s,\beta} - R^j_{s,\beta}}{(\phi_i(s,x) - \phi_j(s,x))^2} + \sum_{j\neq i} \frac{R^i_{s,k} + R^j_{s,k}}{(\phi_i(s,x) + \phi_j(s,x))^2} \\ & + \nu \frac{R^i_{s,\beta}}{\phi_i(s,x)^2} + H^i_{s,\beta}\Bigg) ds \end{aligned}$$

with the error terms

$$\begin{aligned} H^i_{s,\beta} = \beta \Bigg[& \sum_{j\neq i} \frac{\Big((X^i_{s,k} - \sqrt{\beta}\phi_i(s,x)) - (X^j_{s,k} - \sqrt{\beta}\phi_j(s,x))\Big)^2}{\Big(\sqrt{\beta}(\phi_i(s,x) - \phi_j(s,x)) + D^-_{i,j}(s)\Big)^3} \\ & + \sum_{j\neq i} \frac{\Big((X^i_{s,k} - \sqrt{\beta}\phi_i(s,x)) + (X^j_{s,k} - \sqrt{\beta}\phi_j(s,x))\Big)^2}{\Big(\sqrt{\beta}(\phi_i(s,x) + \phi_j(s,x)) + D^+_{i,j}(s)\Big)^3} \\ & + \nu \frac{\Big(X^j_{s,k} - \sqrt{\beta}\phi_j(s,x))\Big)^2}{\Big(\sqrt{\beta}\phi_i(s,x) + D_i(s)\Big)^3}\Bigg] \end{aligned}$$

where, by the Lagrange remainders in the 3 Taylor expansions above,

$$|D^{\pm}_{i,j}(s)| \leq |(X^i_{s,k} - \sqrt{\beta}\phi_i(s,x)) \pm (X^j_{s,k} - \sqrt{\beta}\phi_j(s,x))|$$

and

$$|D_i(s)| \leq |X^i_{s,k} - \sqrt{\beta}\phi_i(s,x)|.$$

By Theorem 5.1, the terms $|D^{\pm}_{i,j}|, |D_i|$ can be bounded by some a.s. finite random variable D independent of i,j, the sign, $s \in [0,t]$, and $\beta \geq \beta_0$ where D depends on x, y, t. Therefore, for all $i = 1, \ldots, N$,

$$|H^i_{s,\beta}| \leq \frac{1}{\sqrt{\beta}} H \qquad \text{for} \quad \beta \geq \beta_0, \ s \in [0,t]$$

with some a.s. finite random variable H. In summary,

$$R_{t,\beta} = -\int_0^t (A_\nu(s,x) R_{s,\beta} + H_{s,\beta})\, ds, \quad R_{0,k} = 0$$

and thus, for suitable norms and all $u \in [0,t]$,

$$\|R_{u,\beta}\| \le A \int_0^u \|R_{s,\beta}\|\, ds + \frac{t \cdot \|H\|}{\sqrt{\beta}}$$

with $A := \sup_{s\in[0,t]} \|A_\nu(s,x)\| < \infty$. The classical lemma of Gronwall now implies that

$$\|R_{u,\beta}\| \le \frac{t\|H\|}{\sqrt{\beta}} e^{tA}$$

for all $u \in [0,t]$. This yields the claim. □

Remark 5.3. The Bessel processes $(X_{t,k})_{t\ge0}$ of type B have the same space-time scaling as Brownian motions and the Bessel processes of type A in Remark 3.3(1), i.e., for $r > 0$, $(\frac{1}{r}X_{r^2t,k})_{t\ge0}$ is also a Bessel process of type B with the same k with modified starting points. The corresponding relations for ϕ and A are

$$\phi(r^2t, rx) = r \cdot \phi(t,x), \quad A(r^2t, rx) = \frac{1}{r^2}A(t,x) \quad \text{for} \quad r > 0,\ t \ge 0.$$

Moreover, for the solution $(W_t)_{t\ge0}$ of (3.2), $(\frac{1}{r}W_{r^2t})_{t\ge0}$ is also a process of this type where x has to be replaced by rx in Eqs. (5.1)–(5.3).

In the end of this section we again calculate the covariance matrix of W_t for the special solution ϕ of Corollary 4.3 explicitly. For this we introduce the matrices $A_\nu = (A_{\nu,i,j})_{i,j} \in \mathbb{R}^{N\times N}$ with

$$A_{\nu,i,j} := \frac{1}{(y_i - y_j)^2} - \frac{1}{(y_i + y_j)^2}, \quad A_{\nu,i,i} := \sum_{j\ne i}\Big(\frac{-1}{(y_i - y_j)^2} - \frac{1}{(y_i + y_j)^2}\Big) - \frac{\nu}{y_i^2} \tag{5.7}$$

for $i,j = 1,\ldots,N$, $i \ne j$ and the vector y as in 4.2. Moreover, E is the N-dimensional unit matrix. It is shown in [AV2] that $E - 2A_\nu$ has the eigenvalues

$$2, 4, \ldots, 2N, \tag{5.8}$$

independent of ν. The eigenvectors are also known by [AV2], but more complicated. We omit details here. With these notations we have:

Lemma 5.4. *Let $\nu > 0$. Assume that the Bessel process $(X_{t,k})_{t\ge0}$ of type B with $k = (\nu\beta, \beta)$ starts in the point $\sqrt{\beta}\cdot cy + w$ in the interior of C_N^B with $w \in \mathbb{R}$, y as in 4.2, and $c > 0$. Then, the covariance matrices $\Sigma_{\nu,t} \in \mathbb{R}^{N\times N}$ for $t > 0$ of the limit Gaussian process $(W_t)_{t\ge0}$ are given by*

$$\Sigma_{\nu,t} = (t + c^2)(E - 2A_\nu)^{-1}(E - e^{\ln \frac{c^2}{t+c^2}(E-2A_\nu)})$$

with eigenvalues $\lambda_k^B(t,c) = \frac{1}{2k}\frac{(t+c^2)^{2k} - c^{4k}}{(t+c^2)^{2k-1}}$ $(k = 1,\ldots,N)$.

Proof. For the special case $\phi(s, cy) = \sqrt{s + c^2}y$ with $c > 0$ and the vector y in 4.2, the matrix function $A_\nu(s, cz)$ has the form $A_\nu(s, cy) = \frac{1}{s+c^2}A_\nu$. Hence,

$$W_t = e^{(\ln(t+c^2) - \ln c^2)A_\nu} \int_0^t e^{(-\ln(s+c^2)+\ln c^2)A_\nu} dB_s \quad (t \ge 0).$$

Since A_ν is real and symmetric and taking (5.8) into account, we may write A_ν as $A_\nu = UDU^t$ with an orthogonal matrix U and with the diagonal matrix

$$D = diag(d_1, \ldots d_N) := diag(-1/2, -3/2, \ldots, (-2N+1)/2).$$

This leads to

$$W_t = U \int_0^t diag\Bigg(\Big(\frac{t+c^2}{s+c^2}\Big)^{d_1}, \ldots, \Big(\frac{t+c^2}{s+c^2}\Big)^{d_N} \Bigg) d\tilde{B}_s \cdot U^t$$

with the rotated Brownian motion $(\tilde{B}_t := U^t B_t U)_{t\geq 0}$. This, the Itô-isometry, and $2d_i \neq 1$ for all i yield

$$\begin{aligned}
\Sigma_t &= U \cdot \int_0^t diag\Bigg(\Big(\frac{t+c^2}{s+c^2}\Big)^{d_1}, \ldots, \Big(\frac{t+c^2}{s+c^2}\Big)^{d_N} \Bigg)^2 ds \cdot U^t \\
&= U \cdot \int_0^t diag\Bigg(\Big(\frac{t+c^2}{s+c^2}\Big)^{2d_1}, \ldots, \Big(\frac{t+c^2}{s+c^2}\Big)^{2d_N} \Bigg) ds \cdot U^t \\
&= U \cdot diag\Bigg(\frac{1}{1-2d_1}(t+c^2 - c^{2(1-2d_1)}(t+c^2)^{2d_1}), \ldots, \\
&\qquad \frac{1}{1-2d_N}(t+c^2 - c^{2(1-2d_N)}(t+c^2)^{2d_N}) \Bigg) \cdot U^t.
\end{aligned}$$

Combining

$$(U(E-2D)^{-1}U^t)^{-1} = E - 2 \cdot UDU^t = E - 2A_\nu$$

with

$$\frac{1}{1-2d_i}(t+c^2 - c^{2(1-2d_i)}(t+c^2)^{2d_i}) = \frac{1}{1-2d_i}(t+c^2)(1 - e^{(1-2d_i)\ln\frac{c^2}{t+c^2}})$$

we obtain by functional calculus that

$$\begin{aligned}
\Sigma_{\nu,t} &= (t+c^2)(E-2A_\nu)^{-1}\Bigg(U \cdot diag\Bigg(1 - e^{(1-2d_1)\ln\frac{c^2}{t+c^2}}, \ldots, 1 - e^{(1-2d_N)\ln\frac{c^2}{t+c^2}} \Bigg) U^t \Bigg) \\
&= (t+c^2)(E-2A_\nu)^{-1}(E - e^{\ln\frac{c^2}{t+c^2}(E-2A_\nu)})
\end{aligned}$$

which yields the desired form of the covariance matrix. □

Remark 5.5. The eigenvalues of Σ_t in the cases A_{2N-1} and B_N are related by

$$\lambda_i^B(t,c) = \lambda_{2i}^A(t, c\cdot\sqrt{2}) \qquad (i = 1, \ldots, N)$$

independent of ν. We have the impression that this is connected in some way with the fact that $H_{2N}(x) = const.(N) \cdot L_N^{[-1/2)}(x^2)$ for $\nu = 1/2$.

Please notice that all preceding results hold for $\nu > 0$. We show below that most results are also valid for $\nu = 0$, where however, some results will have a slightly modified form. These results are closely related to the root systems of type D.

6. A functional central limit theorem for the root system D_N

We now briefly study limit theorems for Bessel processes of type D_N. We recapitulate that the associated closed Weyl chamber is

$$C_N^D = \{x \in \mathbb{R}^N : \quad x_1 \geq \ldots \geq x_{N-1} \geq |x_N|\},$$

i.e., C_N^D is a doubling of C_N^B w.r.t. the last coordinate. We have a one-dimensional multiplicity $k \geq 0$, and the SDE (1.9) for the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D has the form

$$dX_{t,k}^i = dB_t^i + k\sum_{j\neq i}\Big(\frac{1}{X_{t,k}^i - X_{t,k}^j} + \frac{1}{X_{t,k}^i + X_{t,k}^j}\Big)dt \tag{6.1}$$

for $i = 1, \ldots, N$ with an N-dimensional Brownian motion $(B_t^1, \ldots, B_t^N)_{t\geq 0}$. The renormalized processes $(\tilde{X}_{t,k} := X_{t,k}/\sqrt{k})_{t\geq 0}$ satisfy

$$d\tilde{X}_{t,k}^i = \frac{1}{\sqrt{k}}dB_t^i + \sum_{j\neq i}\Big(\frac{1}{\tilde{X}_{t,k}^i - \tilde{X}_{t,k}^j} + \frac{1}{\tilde{X}_{t,k}^i + \tilde{X}_{t,k}^j}\Big)dt \quad (i = 1, \ldots, N). \tag{6.2}$$

These processes are closely related with the deterministic limit $k = \infty$. For this limit case we recapitulate the following obvious facts from Lemma 4.1 of [AV1]:

Lemma 6.1. *For $\epsilon > 0$ consider the open subsets $U_\epsilon := \{x \in C_N^D : d(x, \partial C_N^D) > \epsilon\}$. Then the function*

$$H : U_\epsilon \to \mathbb{R}^N, \quad x \mapsto \begin{pmatrix} \sum_{j\neq 1}\Big(\frac{1}{x_1 - x_j} + \frac{1}{x_1 + x_j}\Big) \\ \vdots \\ \sum_{j\neq N}\Big(\frac{1}{x_N - x_j} + \frac{1}{x_N + x_j}\Big) \end{pmatrix}$$

is Lipschitz continuous on U_ϵ, and for each starting point $x_0 \in U_\epsilon$, the solution $\phi(t, x_0)$ of $\frac{dx}{dt}(t) = H(x(t))$ satisfies $\phi(t, x_0) \in U_\epsilon$ for $t \geq 0$.

We now proceed as in Section 4 of [AV1]. Using the representation

$$L_N^{(\alpha)}(x) := \sum_{k=0}^{N}\binom{N+\alpha}{N-k}\frac{(-x)^k}{k!} \qquad (\alpha \in \mathbb{R},\ N \in \mathbb{N})$$

of the Laguerre polynomials according to (5.1.6) of Szegö [S], we form the polynomial $L_N^{(-1)}$ of order $N \geq 1$ where, by (5.2.1) of [S],

$$L_N^{(-1)}(x) = -\frac{x}{N}L_{N-1}^{(1)}(x). \tag{6.3}$$

Hence, by continuity, the equivalence of (2) and (3) of Lemma 4.2 remains valid for $\nu = 0$ with the N zeros $z_1 > \ldots > z_N = 0$ of $L_N^{(-1)}$. In summary, by [AV1]:

Lemma 6.2. *For $r \in C_N^D$, the following statements are equivalent:*

(1) *The function $W_D(y) := 2\sum_{i<j}\ln(y_i^2 - y_j^2) - \|y\|^2/2$ is maximal in $r \in C_N^B$;*

(2) *$r_N = 0$, and for $i = 1, \ldots, N-1$,*

$$4\sum_{j:j\neq i}\frac{1}{r_i^2 - r_j^2} = 1;$$

(3) *If $z_1^{(1)} > \ldots > z_{N-1}^{(1)} > 0$ are the $N-1$ ordered zeros of $L_{N-1}^{(1)}$, then*

$$2 \cdot (z_1^{(1)}, \ldots, z_{N-1}^{(1)}, 0) = (r_1^2, \ldots, r_N^2). \tag{6.4}$$

As in the B-case in Section 3, Lemma 6.2 leads to the following particular solutions of the ODE of Lemma 6.1; cf. [AV1]:

Corollary 6.3. *Let $r \in C_N^D$ the vector in Eq. (6.4). Then for each $c > 0$, a solution of the dynamical system in Lemma 6.1 is given by $\phi(t, c \cdot r) = \sqrt{t + c^2} \cdot r$.*

Again, the solutions in Corollary 6.3 are attracting. For this we again first observe that as in Section 3, for all x in the interior of C_N^D and $t \geq 0$,

$$\|\phi(t,x)\|^2 = 2N(N-1)t + \|x\|^2. \tag{6.5}$$

This is the first step for the following stability result which is completely analogous to Lemma 4.4; we thus omit the proof.

Lemma 6.4. *For each x in the interior of C_N^D, the solution ϕ of the ODE in Lemma 4.1 has the form*

$$\phi(t,x) = \sqrt{2N(N-1)t + \|x\|^2} \cdot \phi_0(t,x) \qquad (t \geq 0)$$

where ϕ_0 satisfies

$$\|\phi_0(t,x)\| = 1 \qquad \textit{and} \qquad \lim_{t\to\infty} \phi_0(t,x) = \frac{2}{N(N-1)} r$$

with the vector r of Lemma 6.2.

Beside the particular solutions of the ODE in Lemma 6.3 we have the following observations for further special solutions. This result fits with Eq. (6.3) for $L_N^{(-1)}$.

Lemma 6.5. *Let x be a point in the interior of C_N^D with $x_N = 0$. Then the associated solution of the ODE in Lemma 6.1 satisfies $\phi(t,x)_N = 0$ for all t, and the first $N-1$ components $(\phi(t,x)_1, \ldots, \phi(t,x)_{N-1})$ solve the ODE of the B-case in Lemma 4.1 with dimension $n-1$ and $\nu = 2$.*

Moreover, if $x_N > 0$ or < 0, then for all t, $\phi(t,x)_N > 0$ or < 0 respectively.

Proof. If $x_N = 0$, then by the ODE in 6.1, $\frac{d}{dt}\phi(t,x)_N = 0$. This shows the first statements. These statements and the fact that the curves $(\phi(t,x))_t$ are either equal or do not intersect then show the last statement. □

The solutions ϕ of the ODE in Lemma 4.1 appear in the following SLLN; see Theorem 5.5 of [AV1]:

Theorem 6.6. *Let x be a point in the interior of C_N^D, and $y \in \mathbb{R}^N$. Let $k \geq 1/2$ with $\sqrt{k} \cdot x + y$ in the interior of C_N^B for $k \geq k_0$. For $k \geq k_0$, consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D_N starting in $\sqrt{k} \cdot x + y$. Then, for all $t > 0$,*

$$\sup_{0\leq s\leq t, k\geq k_0} \|X_{s,k} - \sqrt{k}\phi(s,x)\| < \infty$$

almost surely. In particular,

$$X_{t,k}/\sqrt{k} \to \phi(t,x) \qquad \textit{for} \qquad k \to \infty$$

locally uniformly in t a.s..

We now turn to an associated functional CLT for $X_{t,k} - \sqrt{k}\phi(t,x)$. As in Section 3 we fix some x in the interior of C_N^D and consider the associated solution $t \mapsto \phi(t,x)$ $(t \geq 0)$. We also introduce an N-dimensional process $(W_t)_{t\geq 0}$ as the unique solution of the inhomogeneous linear SDE

$$dW_t^i = dB_t^i + \sum_{j\neq i} \left(\frac{W_t^j - W_t^i}{(\phi_i(t,x) - \phi_j(t,x))^2} - \frac{W_t^j + W_t^i}{(\phi_i(t,x) + \phi_j(t,x))^2} \right) dt \tag{6.6}$$

for $i = 1, \ldots, N$ with initial condition $W_0 = 0$. The SDE (6.6) may be written in matrix notation as

$$dW_t = dB_t + A(t,x)W_t dt \tag{6.7}$$

with the matrix $A(t,x) \in \mathbb{R}^{N\times N}$ with

$$A(t,x)_{i,j} := \frac{1}{(\phi_i(t,x)-\phi_j(t,x))^2} - \frac{1}{(\phi_i(t,x)+\phi_j(t,x))^2} \quad (i \neq j),$$
$$A(t,x)_{i,i} := \sum_{j\neq i}\left(\frac{-1}{(\phi_i(t,x)-\phi_j(t,x))^2} - \frac{1}{(\phi_i(t,x)+\phi_j(t,x))^2}\right) \tag{6.8}$$

for $i,j = 1,\ldots,N$. The process $(W_t)_{t\geq 0}$ is Gaussian and given by

$$W_t = e^{\int_0^t A(s,x)ds}\int_0^t e^{-\int_0^s A(u,x)du} dB_s \quad (t \geq 0). \tag{6.9}$$

It is related to the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D by the following result. As the proof is completely analogous to that of Theorem 5.2, we omit the proof.

Theorem 6.7. *Let x be a point in the interior of C_N^D and let $y \in \mathbb{R}^N$. Let $k_0 \geq 1/2$ such that $\sqrt{k}\cdot x + y$ is in the interior of C_N^D for $k \geq k_0$.*

For $k \geq k_0$ consider the Bessel processes $(X_{t,k})_{t\geq 0}$ starting at $\sqrt{k}\cdot x + y$. Then, for all $t > 0$,

$$\sup_{0\leq s\leq t, k\geq k_0} \sqrt{k}\cdot \|X_{s,k} - \sqrt{k}\phi(s,x) - W_s\| < \infty \qquad a.s., \tag{6.10}$$

i.e., $X_{s,k} - \sqrt{k}\phi(s,x) \longrightarrow W_s$ for $k \to \infty$ locally uniformly in s a.s. with rate $O(1/\sqrt{k})$.

Remark 6.8. Consider the Bessel processes $(X_{t,k})_{t\geq 0}$ of Theorem 6.7 which start in $\sqrt{k}\cdot x$ for x in the interior of C_N^D with $x_N = 0$. Then, by Lemma 6.5, $\phi(t,x)_N = 0$ for all $t \geq 0$, and the matrix function A from (6.8) satisfies $A(t,x)_{N,N} = 0$.

We next calculate the covariance matrix of W_t for the special solution ϕ of Corollary 6.3. For this we introduce the matrix $A \in \mathbb{R}^{N\times N}$ with

$$A_{i,j} := \frac{1}{(r_i - r_j)^2} - \frac{1}{(r_i + r_j)^2}, \quad A_{i,i} := \sum_{j\neq i}\Big(\frac{-1}{(r_i - r_j)^2} - \frac{1}{(r_i + r_j)^2}\Big) \tag{6.11}$$

for $i,j = 1,\ldots,N$, $i \neq j$ and the vector r as in 6.2. By [AV2], $E - 2A$ has the eigenvalues

$$2, 4, \ldots, 2N. \tag{6.12}$$

The eigenvectors are also known by [AV2]; we omit details here. With these notations we obtain the following result. As its proof is again analog to that of Lemma 5.4, we skip the proof.

Lemma 6.9. *Assume that the Bessel processes $(X_{t,k})_{t\geq 0}$ of type D_N start in the points $\sqrt{k}\cdot cr + w$ in the interior of C_N^D with $w \in \mathbb{R}^N$, r as in 6.2, and $c > 0$. Then, the covariance matrices $\Sigma_t \in \mathbb{R}^{N\times N}$ for $t > 0$ of the limit Gaussian process $(W_t)_{t\geq 0}$ are given by*

$$\Sigma_t = (t + c^2)(E - 2A)^{-1}(E - e^{\ln\frac{c^2}{t+c^2}(E-2A)}).$$

Remark 6.10. Let x be a point in the interior of C_N^D with $x_N = 0$. Then, by Lemma 6.5, the matrices $A(t,x)$ of Eq. (6.8) satisfy $A(t,x)_{N,j} = A(t,x)_{j,N} = 0$ for $j \neq N$ and $t > 0$. We thus conclude from Eq. (6.9) that for the centered Gaussian process $(W_t)_{t\geq 0}$ the N-th component $(W_t^{(N)})_{t\geq 0}$ is independent from $(W_t^{(1)}, \ldots, W_t^{(N-1)})_{t\geq 0}$. This situation appears in particular in the setting of Lemma 6.9.

7. Further limit theorems for the case B

The preceding results for the root systems of type D are closely related to limit results for Bessel processes of type B for the degenerated case $(k_1, k_2) = (0, k)$ for $k \to \infty$ which was excluded in Section 3.

To explain the connection we recapitulate some well-known facts; see [AV1], [V]. Let $(X_{t,k}^D)_{t\geq 0}$ be a Bessel process of type D with multiplicity $k \geq 0$ on the chamber C_N^D starting in some point x in the interior of C_N^D. It follows from the generator of the associated semigroup (or the associated SDE (6.1)) that the process $(X_{t,k}^B)_{t\geq 0}$ with

$$X_{t,k}^{B,i} := X_{t,k}^{D,i} \quad (i = 1, \ldots, N-1), \quad X_{t,k}^{B,N} := |X_{t,k}^{D,N}|$$

is a Bessel process of type B with the multiplicity $(k_1, k_2) := (0, k)$ and starting point $(x_1, \ldots, x_{N-1}, |x_N|) \in C_N^B$ with $x_1 > \ldots > x_{N-1} > |x_N| \geq 0$. Notice that $(X_{t,k}^B)_{t\geq 0}$ is a diffusion with reflecting boundary where in particular the boundary parts with the N-th coordinate equal to zero are attained.

We now translate the results of the preceding section. For this we consider the solutions $\phi(t, x)$ of the ODE in Lemma 6.1 in the following two particular cases:

(1) If x is in the interior of C_N^B, then $\phi(t, x)$ will be also in the interior of C_N^B for all $t \geq 0$.
(2) If $x \in C_N^B$ satisfies $x_1 > \ldots > x_{N-1} > x_N = 0$, then we have $\phi(t, x)_1 > \ldots > \phi(t, x)_{N-1} > \phi(t, x)_N = 0$ for all $t \geq 0$.

Case (2) appears in particular for $\phi(t, x) = \sqrt{t + c^2} \cdot r$ for $c > 0$ and the vector r from Lemma 6.2 with $r_N = 0$.

Theorem 6.6 now reads as follows for the B-case with $(k_1, k_2) = (0, k)$ for $k \to \infty$:

Theorem 7.1. *Let x be a point as described above in (1) or (2). For $k \geq 1/2$, consider the Bessel processes $(X_{t,(0,k)})_{t\geq 0}$ of type B_N starting in $\sqrt{k} \cdot x$. Then, for all $t > 0$,*

$$\sup_{0\leq s\leq t, k\geq 1/2} \|X_{s,k} - \sqrt{k}\phi(s, x)\| < \infty \qquad a.s..$$

We next consider the Gaussian processes $(W_t)_{t\geq 0}$ of Eq. (6.9). Theorem 6.7 now leads to functional CLTs where the cases (1) and (2) have to be treated separately for geometric reasons. For the case (1) we have the following result:

Theorem 7.2. *Let x be a point in the interior of C_N^B. For $k \geq 1/2$ consider Bessel processes $(X_{t,(0,k)})_{t\geq 0}$ of type B starting at $\sqrt{k} \cdot x$. Then, for all $t > 0$,*

$$\sup_{0\leq s\leq t, k\geq k_0} \sqrt{k} \cdot \|X_{s,(0,k)} - \sqrt{k}\phi(s, x) - W_s\| < \infty \qquad a.s.. \tag{7.1}$$

Proof. As x is in the interior of C_N^B, we obtain that for each $t > 0$ and almost all $\omega \in \Omega$, the path $(\sqrt{k}\phi(s, x) - W_s(\omega))_{s\in[0,t]}$ is arbitrarily far away from the boundary of C_N^B whenever k is sufficiently large. This, the connection between the D- and B-case, and Theorem 6.7 thus lead to the theorem. □

Theorem 7.2 may be seen as Theorem 5.2 for the degenerate case $\nu = 0$ in the notation there.

We next turn to case (2):

Theorem 7.3. *Let $x \in C_N^B$ with $x_1 > \ldots > x_{N-1} > x_N = 0$. For $k \geq 1/2$ consider Bessel processes $(X_{t,(0,k)})_{t\geq 0}$ of type B starting at $\sqrt{k} \cdot x$. Then, for the process $(\tilde{W}_t := (W_t^{(1)}, \ldots, W_t^{(N-1)}, |W_t^{(N)}|))_{t\geq 0}$, and all $t > 0$,*

$$\sup_{0\leq s\leq t, k\geq k_0} \sqrt{k} \cdot \|X_{s,(0,k)} - \sqrt{k}\phi(s,x) - \tilde{W}_s\| < \infty \qquad a.s.. \tag{7.2}$$

Proof. This follows immediately from Theorem 6.7, the connection between the D- and B-case, and from $||a| - |b|| \leq |a - b|$ for $a, b \in \mathbb{R}$. □

We finally notice that for the process $(\tilde{W}_t)_{t\geq 0}$, the first $N - 1$ components form a Gaussian process which is independent from $(|W_t^{(N)}|)_{t\geq 0}$ by Remark 6.10. The distributions of $|W_t^{(N)}|$ clearly are one-sided normal distributions.

8. Extensions to multi-dimensional Bessel processes with an additional Ornstein-Uhlenbeck component

In this section we will consider an extension of our previous models by adding an additional drift coefficient of the form $-\lambda x$, $\lambda \in \mathbb{R}$, i.e. a component as in a classical Ornstein-Uhlenbeck setting

$$dY_{t,k} = dB_t + (\frac{1}{2}(\nabla(\ln w_k))(Y_{t,k}) - \lambda Y_{t,k})\, dt.$$

If $\lambda > 0$, we obtain a mean reverting process with speed of mean-reversion λ, which is an ergodic process. For $\lambda \leq 0$ the process is non-ergodic. For $N = 1$ and $\lambda > 0$ the squared process is the well-known Cox-Ingersoll-Ross process, widely used in mathematical finance.

We derive the results for the root system A_{N-1} as the same technique also holds for the other root systems. We consider processes $(Y_{t,k})_{t\geq 0}$ of type A_{N-1} as solutions of

$$dY_{t,k}^i = dB_t^i + \left(k \sum_{j\neq i} \frac{1}{Y_{t,k}^i - Y_{t,k}^j} - \lambda Y_{t,k}^i \right) dt \qquad (i = 1, \ldots, N). \tag{8.1}$$

with an N-dimensional Brownian motion $(B_t^1, \ldots, B_t^N)_{t\geq 0}$. By applying Itô's formula together with a time-change argument we see that Y may be given as a space-time transformation of the original X (with $\lambda = 0$), namely

$$Y_{t,k} = e^{-\lambda t} X_{\frac{e^{2\lambda t}-1}{2\lambda},k}.$$

For a proof based on the generators cf. [RV1].

A similar relation also holds for the solutions of the associated deterministic dynamical systems.

Lemma 8.1. *Let $\phi(t,x)$ be a solution of the dynamical system $\frac{dx}{dt}(t) = H(x(t))$ with starting point x in the interior of C_N^A as in Lemma 2.1. Then*

$$\phi(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x)$$

is a solution of the dynamical system $\frac{dx}{dt}(t) = H(x(t)) - \lambda x(t)$ with starting point x for all $t \geq 0$.

Proof. The claim follows by direct calculation. We denote by $\dot{\phi}(t,x)$ the derivative with respect to the first argument. For $i \in \{1, \cdots N\}$ we obtain by the space-time homogeneity in Remark 3.3 that

$$\begin{aligned}\frac{\partial}{\partial t}\phi_i(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x) &= \frac{\partial}{\partial t}\left(e^{-\lambda t}\phi_i(\frac{e^{2\lambda t}-1}{2\lambda}, x)\right)\\ &= -\lambda e^{-\lambda t}\phi_i(\frac{e^{2\lambda t}-1}{2\lambda}, x) + e^{-\lambda t}\frac{\partial}{\partial t}\phi_i(\frac{e^{2\lambda t}-1}{2\lambda}, x)\\ &= -\lambda e^{-\lambda t}\phi_i(\frac{e^{2\lambda t}-1}{2\lambda}, x) + e^{\lambda t}\dot{\phi}_i(\frac{e^{2\lambda t}-1}{2\lambda}, x)\\ &= -\lambda\phi_i(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x)\\ &\quad + \sum_{j\neq i}\frac{1}{\phi_i(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x) - \phi_j(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x)},\end{aligned}$$

which yields the desired result. □

With the same technique as in Theorem 3.1 and Theorem 3.2 we may deduce a functional central limit theorem for $(Y_{t,k})_{t\geq 0}$, namely

$$\sqrt{k}(\frac{Y_{t,k}}{\sqrt{k}} - \phi(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x)) \longrightarrow W_t \tag{8.2}$$

for $k \to \infty$ locally uniformly in t almost surely with rate $O(1/\sqrt{k})$, where W is given by

$$dW_t^i = dB_t^i + \left(\sum_{j\neq i}\frac{W_t^j - W_t^i}{(\phi_i(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x) - \phi_j(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x))^2} - \lambda W_t^i\right)dt. \tag{8.3}$$

for $i = 1, \ldots, N$ with initial condition $W_0 = 0$. The SDE (8.3) may be written in matrix notation as

$$dW_t = dB_t + A^\lambda(t,x)W_t dt \tag{8.4}$$

with the matrices $A^\lambda(t,x) \in \mathbb{R}^{N\times N}$ with

$$A^\lambda(t,x)_{i,j} := \frac{1}{(\phi_i(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x) - \phi_j(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x))^2},$$

$$A^\lambda(t,x)_{i,i} := -\sum_{j\neq i}\frac{1}{(\phi_i(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x) - \phi_j(\frac{1-e^{-2\lambda t}}{2\lambda}, e^{-\lambda t}x))^2} - \lambda$$

for $i, j = 1, \ldots, N$, $i \neq j$. The process $(W_t)_{t\geq 0}$ admits the explicit representation in terms of matrix-valued exponentials

$$W_t = e^{\int_0^t A^\lambda(s,x)ds}\int_0^t e^{-\int_0^s A^\lambda(u,x)du}dB_s \quad (t \geq 0). \tag{8.5}$$

Note that due to the constant term in the diagonal of $A^\lambda(t,x)$ for $\lambda \neq 0$, we obtain a linear time-dependence in the exponential of the matrix exponential which dominates the long-term behaviour of the covariance matrix as we will see in the following special case.

Lemma 8.2. *Assume that $(Y_{t,k})_{t\geq 0}$ starts in the interior of C_N^A in $\sqrt{k}\cdot cz+y$ with $y\in\mathbb{R}$, z as in 2.2 and $c>0$. Then the covariance matrices $\Sigma_t^\lambda\in\mathbb{R}^{N\times N}$ for $t>0$ of the limit process $(W_t)_{t\geq 0}$ are given by*

$$\Sigma_t^\lambda=\frac{1+e^{-2\lambda t}(\lambda c^2-1)}{2\lambda}(E-A)^{-1}(E-e^{\ln\frac{\lambda c^2}{e^{2\lambda t}-1+\lambda c^2}(E-A)}),$$

where A is defined by (3.8).

Proof. For the special starting points cz we obtain the special solution

$$\begin{aligned}\phi(\frac{1-e^{-2\lambda s}}{2\lambda},e^{-\lambda s}cz) &= e^{-\lambda s}\phi(\frac{e^{2\lambda s}-1}{2\lambda},cz)\\ &= e^{-\lambda s}\sqrt{\frac{e^{2\lambda s}-1}{\lambda}+c^2z}.\end{aligned}$$

Hence the matrix function $A^\lambda(s,cz)$ has the simple form with the same time-dependence for each entry

$$A^\lambda(s,cz)=\frac{\lambda e^{2\lambda s}}{e^{2\lambda s}-1+\lambda c^2}A-diag(\lambda s,\cdots,\lambda s),$$

where A is given by (3.8). This yields the process

$$W_t=\int_0^t e^{(t-s)diag(-\lambda,\cdots,-\lambda)+\ln\left(\frac{e^{2\lambda t}-1+\lambda c^2}{e^{2\lambda s}-1+\lambda c^2}\right)A}dB_s\quad(t\geq 0).$$

Since A is real and symmetric and taking (3.9) into account, we may write $A^\lambda(s,cz)$ as $A^\lambda(s,cz)=UDU^t$ with an orthogonal matrix U and with the diagonal matrix

$$D=diag(\frac{\lambda e^{2\lambda s}}{e^{2\lambda s}-1+\lambda c^2}d_1-\lambda s,\ldots\frac{\lambda e^{2\lambda s}}{e^{2\lambda s}-1+\lambda c^2}d_N-\lambda s).$$

This leads to

$$W_t=U\int_0^t diag\left(\left(\frac{e^{2\lambda t}-1+\lambda c^2}{e^{2\lambda s}-1+\lambda c^2}\right)^{\frac{d_1}{2}}e^{-(t-s)\lambda},\ldots,\left(\frac{e^{2\lambda t}-1+\lambda c^2}{e^{2\lambda s}-1+\lambda c^2}\right)^{\frac{d_N}{2}}e^{-(t-s)\lambda}\right)d\tilde{B}_sU^t$$

with the rotated Brownian motion $(\tilde{B}_t:=U^tB_tU)_{t\geq 0}$. This, the Itô-isometry, and $d_i/2\neq 1$ for all i yield

$$\begin{aligned}&\Sigma_t^\lambda\\ &= U\cdot\int_0^t diag\left(\left(\frac{e^{2\lambda t}-1+\lambda c^2}{e^{2\lambda s}-1+\lambda c^2}\right)^{d_1}e^{-2(t-s)\lambda},\ldots,\left(\frac{e^{2\lambda t}-1+\lambda c^2}{e^{2\lambda s}-1+\lambda c^2}\right)^{d_N}e^{-2(t-s)\lambda}\right)ds\cdot U^t\\ &= U\cdot diag\left(\frac{e^{-2\lambda t}}{2\lambda}\left(\frac{1}{1-d_1}(e^{2\lambda t}-1+\lambda c^2)-\frac{1}{1-d_1}(\lambda c^2)^{1-d_1}(e^{2\lambda t}-1+\lambda c^2)^{d_1}\right),\ldots,\right.\\ &\qquad\left.\frac{e^{-2\lambda t}}{2\lambda}\left(\frac{1}{1-d_N}(e^{2\lambda t}-1+\lambda c^2)-\frac{1}{1-d_N}(\lambda c^2)^{1-d_N}(e^{2\lambda t}-1+\lambda c^2)^{d_N}\right)\right)\cdot U^t.\end{aligned}$$

With a similar calculation as in the proof of Lemma 3.6 we obtain

$$\Sigma_t^\lambda=\frac{e^{-\lambda 2t}}{2\lambda}(e^{2\lambda t}-1-\lambda c^2)(E-A)^{-1}(E-e^{\ln(\frac{\lambda c^2}{e^{2\lambda t}-1+\lambda c^2})(E-A)})$$

which yields the desired form of the covariance matrix. □

Remark 8.3. (1) Note that the long-term behaviour of the covariance matrix is inherited by the long-term behaviour of Y. In the ergodic case for Y, i.e. $\lambda > 0$, we obtain $\lim_{t\to\infty} \Sigma_t^\lambda = \frac{1}{2\lambda}(E-A)^{-1}$. For $\lambda < 0$ we need an exponential scaling

$$\lim_{t\to\infty} e^{2\lambda t}\Sigma_t^\lambda = \frac{\lambda c^2 - 1}{2\lambda}(E-A)^{-1}(E - e^{\ln(\frac{\lambda c^2}{\lambda c^2 - 1})(E-A)}).$$

(2) Note that we can also recover the formula for Σ_t^λ in terms of Σ_t^0 by replacing t with $\frac{1-e^{-2\lambda t}}{2\lambda}$ and c by $e^{-\lambda t}c$. This also transfers to the eigenvalues and leads to the largest eigenvalue $(1-e^{-2\lambda t})/(2\lambda)$.

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