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Preprint 2018-01 February 2018

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Some central limit theorems for random walks associated with hypergeometric functions of type BC

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Abstract

The spherical functions of the noncompact Grassmann manifolds $G_{p,q}(\mathbb{F}) = G/K$ over $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with rank $q \geq 1$ and dimension parameter $p > q$ can be seen as Heckman-Opdam hypergeometric functions of type BC, when the double coset space $G//K$ is identified with some Weyl chamber $C_q^B \subset \mathbb{R}^q$ of type B. The associated double coset hypergroups on C_q^B may be embedded into a continuous family of commutative hypergroups $(C_q^B, *_p)$ with $p \in [2q-1, \infty[$ associated with these hypergeometric functions by a result of Rösler. Several limit theorems for random walks associated with these hypergroups were recently derived by the second author. We here present further limit theorems in particular for the case where the time parameter as well as p tend to ∞. For integers p, these results admit interpretations for group-invariant random walks on the Grassmannians G/K.

Key words: Hypergeometric functions associated with root systems, Heckman-Opdam theory, non-compact Grassmann manifolds, spherical functions, random walks on symmetric spaces, random walks on hypergroups, moment functions, central limit theorems, dimension to infinity.
AMS subject classification (2000): 60B15, 43A62, 60F05, 43A90, 33C67.

1 Introduction

In this paper we present several central limit theorems (CLTs) for group invariant random walks on the non-compact Grassmann manifolds $G_{p,q}(\mathbb{F}) = G/K$ over the (skew-)fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. We state these results via corresponding CLTs on the associated double coset spaces $G//K$ which can be identified with the Weyl chambers $C_q^B \subset \mathbb{R}^q$ of type B. The associated spherical functions, regarded as functions on C_q^B, are then hypergeometric functions of type BC, and it turns out that the CLTs can be stated more generally for certain classes of Markov chains on C_q^B whose transition probabilities are related with hypergeometric functions of type BC for a wider range of parameters than just for the group parameters.

Let us describe more details of the general setting. The Heckman-Opdam theory of hypergeometric functions associated with root systems generalizes the theory of spherical functions on Riemannian symmetric spaces; see [H], [HS] and [O] for the general theory, and [R2], [RKV], [RV], [Sch], [NPP] for some recent developments. In this paper we are mainly interested in the type BC, but we shall need also need some facts on the A-case as limit case; see [RKV], [RV] for these limits. We recapitulate that for the root system A_{q-1}, $q \geq 2$, the theory is connected with the groups $G := GL(q, \mathbb{F})$ with maximal compact subgroups $K := U(q, \mathbb{F})$, and for the root system BC_q, $q \geq 1$, with the non-compact Grassmann manifolds $\mathcal{G}_{p,q}(\mathbb{F}) := G/K$ with $p > q$, where depending on $\mathbb{F}$, the group G is

*This author has been supported by the Deutsche Forschungsgemeinschaft (DFG) via RTG 2131 *High-dimensional Phenomena in Probability - Fluctuations and Discontinuity.*

one of the indefinite orthogonal, unitary or symplectic groups $SO_0(q,p)$, $SU(q,p)$ or $Sp(q,p)$ with $K = SO(q)\times SO(p)$, $S(U(q)\times U(p))$ or $Sp(q)\times Sp(p)$, as maximal compact subgroup.

In all cases, the K-spherical functions on G (i.e., the nontrivial, K-biinvariant, multiplicative continuous functions on G) can be seen as as nontrivial, multiplicative continuous functions on the double coset space $G//K$ where $G//K$ carries the corresponding double coset convolution and commutative double coset hypergroup structure. The KAK-decomposition of G shows that $G//K$ may be identified with the Weyl chambers

$$C_q^A := \{x = (x_1, \cdots, x_q) \in \mathbb{R}^q :\ x_1 \geq x_2 \geq \cdots \geq x_q\}$$

of type A and

$$C_q^B := \{x = (x_1, \cdots, x_q) \in \mathbb{R}^q :\ x_1 \geq x_2 \geq \cdots \geq x_q \geq 0\}$$

of type B respectively. This identification is based on a exponential mapping $x \mapsto a_x \in G$ from the Weyl chamber to a system of representatives a_x of the double cosets in G with

$$a_x := e^{\underline{x}} \tag{1.1}$$

for $x \in C_q^A$ in the A-case, and

$$a_x := \begin{pmatrix} \cosh \underline{x} & \sinh \underline{x} & 0 \\ \sinh \underline{x} & \cosh \underline{x} & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix} \tag{1.2}$$

for $x \in C_q^B$ in the BC-case with the diagonal matrix notation

$$e^{\underline{x}} := \operatorname{diag}(e^{x_1}, \ldots, e^{x_q}),\ \cosh \underline{x} = \operatorname{diag}(\cosh x_1, \ldots, \cosh x_q),\ \sinh \underline{x} = \operatorname{diag}(\cosh x_1, \ldots, \cosh x_q).$$

We identify $G//K$ with C_q^A or C_q^B from now on. We also fix q and, in the BC-case, $p > q$.

For the spherical functions we follow [HS] and denote the Heckman-Opdam hypergeometric functions associated with the root systems

$$2 \cdot A_{q-1} = \{\pm 2(e_i - e_j) : 1 \leq i < j \leq q\} \subset \mathbb{R}^q$$

and

$$2 \cdot BC_q = \{\pm 2e_i, \pm 4e_i, \pm 2e_i \pm 2e_j : 1 \leq i < j \leq q\} \subset \mathbb{R}^q$$

by $F_A(\lambda, k; t)$ and $F_{BC}(\lambda, k; x)$ respectively with spectral variable $\lambda \in \mathbb{C}^q$ and multiplicity parameter(s) k. Here, $e_1, \ldots, e_q$ are the unit vectors in $\mathbb{R}^q$. The factor 2 in both root systems comes from the known connections of the Heckman-Opdam theory to spherical functions on symmetric spaces in [HS] and references there. In the A_{q-1}-case, the spherical functions on $G//K \simeq C_q^A$ are then given by

$$\varphi_\lambda^A(a_x) := \varphi_\lambda^A(x) := e^{i\cdot\langle x-\pi(x),\lambda\rangle} \cdot F_A(i\pi(\lambda), d/2; \pi(x)) \qquad (x \in \mathbb{R}^q,\ \lambda \in \mathbb{C}^q) \tag{1.3}$$

with multiplicity $k = d/2$ where

$$d := \dim_{\mathbb{R}} \mathbb{F} \in \{1,2,4\} \qquad \text{for} \qquad \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H},$$

and where

$$\pi : \mathbb{R}^q \to \mathbb{R}_0^q := \{t \in \mathbb{R}^q :\ x_1 + \ldots + x_q = 0\}$$

is the orthogonal projection w.r.t. the standard scalar product as in Eq. (6.7) of [RKV] and a_t is identified with x. In the BC-case, the spherical functions on $G//K \simeq C_q^B$ are given by

$$\varphi_\lambda^p(a_x) := \varphi_\lambda^p(x) := F_{BC}(i\lambda, k_p; x) \qquad (x \in \mathbb{R}^q,\ \lambda \in \mathbb{C}^q) \tag{1.4}$$

with multiplicity

$$k_p = (d(p-q)/2, (d-1)/2, d/2) \subset \mathbb{R}^3$$

corresponding to the roots $\pm 2e_i$, $\pm 4e_i$ and $2(\pm e_i \pm e_j)$ where again a_x is identified with x.

In the BC-case, the associated double coset convolutions $*_p$ of measures on C_q^B are written down explicitly in [R2] for $p \geq 2q$ such that these convolutions and the associated product formulas for the associated hypergeometric functions F_{BC} above can be extended to $p \in [2q-1, \infty[$ by analytic continuation. These convolutions $*_p$ on the space $\mathcal{M}(C_q^B)$ of all bounded regular Borel measures on C_q^B are associative, commutative, and probability-preserving, and they generate commutative hypergroups $(C_q^B, *_p)$ in the sense of Dunkl, Jewett, and Spector with $0 \in C_q^B$ as identity by [R2]. For hypergroups we generally refer to [J] and [BH]. The nontrivial multiplicative continuous functions of these commutative hypergroups $(C_q^B, *_p)$ are precisely the functions φ_λ^p with $\lambda \in \mathbb{C}^q$ by [R2]. This means that for all $x, y \in C_q^B$ and $\lambda \in \mathbb{C}^q$,

$$\varphi_\lambda^p(x)\varphi_\lambda^p(y) = \int_{C_q^B} \varphi_\lambda^p(t)\, d(\delta_x *_p \delta_y)(t)$$

where the probability measures $\delta_x *_p \delta_y \in \mathcal{M}^1(C_q^B)$ with compact support are given by

$$(\delta_x *_p \delta_y)(f) = \frac{1}{\kappa_p} \int_{B_q} \int_{U(q,\mathbb{F})} f\Big(\text{arcosh}\,(\sigma_{sing}(\sinh \underline{x}\, w\, \sinh \underline{y} \,+\, \cosh \underline{x}\, v\, \cosh \underline{y}))\Big)\, dv\, dm_p(w) \quad (1.5)$$

for $f \in C(C_q^B)$. Here, dv means integration w.r.t. the normalized Haar measure on $U(q,\mathbb{F})$, B_q is the matrix ball

$$B_q := \{w \in M_q(\mathbb{F}) : \ w^*w \leq I_q\},$$

and $dm_p(w)$ is the probability measure

$$dm_p(w) := \frac{1}{\kappa_p} \Delta(I - w^*w)^{d(p/2+1/2-q)-1}\, dw \quad \in \mathcal{M}^1(B_q) \qquad (1.6)$$

where dw is the Lebesgue measure on the ball B_q, and the normalization $\kappa_p > 0$ is chosen such that $dm_p(w)$ is a probability measure. For $p = 2q-1$ there is a corresponding degenerated formula where $m_p \in \mathcal{M}^1(B_q)$ becomes singular; see Section 3 of [R1] for details.

For fixed parameters $p \in [2q-1, \infty[$ and $d = 1, 2, 4$ we now consider random walks on the hypergroups $(C_q^B, *_p)$ as follows: Fix a probability measure $\nu \in \mathcal{M}^1(C_q^B)$, and consider a time-homogeneous Markov process $(\tilde{S}_k^p)_{k\geq 0}$ on C_q^B with start at the hypergroup identity $0 \in C_q^B$ and with the transition probability

$$P(\tilde{S}_{k+1}^p \in A \,|\, \tilde{S}_k^p = x) = (\delta_x *_p \nu)(A) \qquad (x \in C_q^B,\ A \subset C_q^B \quad \text{a Borel set}).$$

Such Markov processes are called random walks on the hypergroup $(C_q^B, *_p)$ associated with the measure ν. Notice that we here use p as a superscript, as this p may be variable below. The fixed parameters q and d are suppressed.

We shall present mainly two different types of CLTs for $(\tilde{S}_k^p)_{k\geq 0}$.

For the first type we start with some probability measure ν having classical second moments. For each constant $c \in [0,1]$ we consider the compression mapping $D_c(x) := cx$ on C_q^B as well as the compressed probability measures $\nu_c := D_c(\nu) \in \mathcal{M}^1(C_q^B)$ and the associated random walks $(S_k^{(p,c)})_{k\geq 0}$. We shall prove in Section 4 that $S_n^{(p,n^{-1/2})}$ converges for $n \to \infty$ in distribution to some kind of "Gaussian" measure $\gamma_t \in \mathcal{M}^1(C_q^B)$ which depends on p where the time parameter $t \geq 0$ can be computed via second moment of ν. Triangular CLTs of this type are well-known in probability theory on groups and hypergroups. We here in particular refer to [BH] and references there for several results in this direction for Sturm-Liouville hypergroups on $[0, \infty[$. Moreover, for integers $p \geq 2q$, this result is more or less a known CLT for biinvariant random walks on noncompact Grassmannians; see e.g. [G1], [G2], [Te1], [Te2], [Ri].

For the second CLT we study the random walks $(\tilde{S}_k^p)_{k\geq 0}$ for a given fixed probability measure $\nu \in \mathcal{M}^1(C_q^B)$ where the time k as well as the dimension parameter p tend to infinity in some coupled way. It turns out that under suitable moment conditions on ν and for any sequence $(p_n)_n \subset [2q, \infty[$ with

$p_n \to \infty$, there are normalizing vectors $m(n) \in \mathbb{R}^q$ such that $(S_n^{p_n} - m(n))/\sqrt{n}$ tends in distribution to some classical q-dimensional normal distribution $N(0, \Sigma^2)$ where the norming vectors $m(n)$ and the covariance matrix Σ^2 are explicitly known and depend ν. For $q = 1$, CLTs of this kind were given in [Gr1] and [V1] by completely different methods. Both proofs for $q = 1$ however are based on the fact that for $p \to \infty$, the hypergroup structures $(C_1^B = [0, \infty[, *_p)$ converge to some commutative semigroup structure on $C_1^B = [0, \infty[$ which is isomorphic with the additive semigroup $([0, \infty[, +)$. This observation finally shows that for large p, $(S_n^{p_n})_n$ behaves like a sum of iid random variables which then leads to the CLT. For $q \geq 2$, the situation is much more involved as here for $p \to \infty$, the hypergroup structures $(C_q^B, *_p)$ converge to the double coset structures $G//K$ in the case A_{q-1} in some way, where the dimension parameter $d = 1, 2, 4$ remains unchanged; see [RKV] and [RV] for the details. As for $q \geq 2$, this limit structure is more complicated than for $q = 1$, the details of the CLT and its proof in Section 3 will be more involved than in [Gr1] and [V1]. In fact, we will need stronger conditions either on the moments of ν or on the rate of convergence of $(p_n)_n$ to ∞ than in [Gr1]; see Theorems 4.1, 4.3 below. We remark that the CLTs in [Gr1], [V1], and here for the non-compact Grassmannians are related to other CLTs for radial random walks on Euclidean spaces of large dimensions in [Gr2] and references cited there. We also point out that our CLTs for $p \to \infty$ are closely related to some CLT in the case A_{q-1} in [V2] which depends heavily on the concept of moment functions on commutative hypergroups; see [BH] and [Z1] for the general background. In fact, we shall need these moment functions for the BC-hypergroups $(C_q^B, *_p)$ as well as for the limit cases associated with the case A_{q-1}. These moment function will be essential to describe the norming vectors $m(n)$ and the covariance matrix Σ^2 above. We shall collect several results on these functions in the next section. We point out that these results are mainly needed for the CLTs of Section 3, but not for those in Section 4. We also remark that our CLTs for $p \to \infty$ are related to the research in [B] on the limit behaviour of Brownian motions on hyperbolic spaces and noncompact Grassmannians when the dimension tends to infinity.

2 Modified moments

Generally, examples of moment functions on a commutative hypergroup can be obtained as partial derivatives of the multiplicative functions of the hypergroup w.r.t. the spectral variables at the identity character; see [BH]. To obtain explicit formulas for these moment functions for our particular examples on Weyl chambers, we start with explicit integral representations of the multiplicative functions in [RV] which are consequences of the well-known Harish-Chandra integral representation of spherical functions.

We start with some notations from matrix analysis; we here usually refer to the monograph [HJ]. For a Hermitian matrix $A = (a_{ij})_{i,j=1,\ldots,q}$ over $\mathbb{F}$ we denote by $\Delta(A)$ the determinant of A, and by $\Delta_r(A) = \det((a_{ij})_{1 \leq i,j \leq r})$ the r-th principal minor of A for $r = 1, \ldots, , q$. For $\mathbb{F} = \mathbb{H}$, these determinants are taken in the sense of Dieudonné, i.e. $\det(A) = (\det_{\mathbb{C}}(A))^{1/2}$, when A is considered as a complex matrix. For each positive Hermitian $q \times q$-matrix A and $\lambda \in \mathbb{C}^q$ we consider the power function

$$\Delta_\lambda(A) := \Delta_1(A)^{\lambda_1 - \lambda_2} \cdot \ldots \cdot \Delta_{q-1}(A)^{\lambda_{q-1} - \lambda_q} \cdot \Delta_q(A)^{\lambda_q}. \tag{2.1}$$

We shall also need the singular values $\sigma_1(a) \geq \sigma_2(a) \geq \ldots \geq \sigma_q(a)$ of a $q \times q$-matrix a which are ordered by size and which are the ordered eigenvalues of $a^* a$. Finally, for $x \in C_q^B$, $u \in U_q(\mathbb{F})$, and $w \in B_q$, we define

$$g(x, u, w) := u^*(\cosh \underline{x} + \sinh \underline{x} \cdot w)(\cosh \underline{x} + \sinh \underline{x} \cdot w)^* u. \tag{2.2}$$

We recapitulate the following facts; see Lemmas 4.10 and 4.8 of [RV]:

2.1 Lemma. (1) *Consider the probability measures m_p from (1.6). Then for each $n \in \mathbb{N}$ there exists a constant $C := C(q, n, \mathbb{F})$ such that all $p \geqslant 2q$,*

$$\int_{B_q} \frac{\sigma_1(w)^{2n}}{\Delta(I - w^* w)^{2n}} dm_p(w) \leq \frac{C}{p^n}. \tag{2.3}$$

(2) *Let* $x \in C_q^B, w \in B_q, u \in U(q,\mathbb{F})$ *and* $r = 1, ..., q$. *Then*

$$\frac{\Delta_r(g(x,u,w))}{\Delta_r(g(x,u,0))} \in [(1-\tilde{x}\sigma_1(w))^{2r}, (1+\tilde{x}\sigma_1(w))^{2r}] \quad \textit{with} \quad \tilde{x} := \min(x_1, 1).$$

We now recapitulate the moment functions in the A-case and then in BC-case from [V2].

2.2 Definition. The spherical functions of type A in (1.3) satisfy

$$\varphi_\lambda^A(x) = \int_{U(q,\mathbb{F})} \Delta_{(i\lambda-\rho^A)/2}\big(u^{-1}e^{2\underline{x}}\, u\big)\, du \qquad (x \in C_q^A) \tag{2.4}$$

with the half sum of positive roots

$$\rho^A := (\rho_1^A, \ldots, \rho_q^A) \in C_q^A \qquad \text{with} \qquad \rho_l^A := \frac{d}{2}(q+1-2l) \qquad (l = 1, \ldots, q); \tag{2.5}$$

see Section 3 of [RV]. Eq. (2.4) in particular yields that $\varphi_{-i\rho^A}^A \equiv 1$, and that for $\lambda \in \mathbb{R}^n$ and $x \in C_q^A$, we have $|\varphi_{\lambda-i\rho^A}^A(x)| \le 1$.

We now follow [V2]. For multiindices $l = (l_1, \ldots, l_q) \in \mathbb{N}_0^q$ we define the moment functions

$$\begin{aligned} m_l^A(x) :=& \frac{\partial^{|l|}}{\partial\lambda^l}\varphi_{-i\rho^A-i\lambda}^A(x)\Big|_{\lambda=0} := \frac{\partial^{|l|}}{(\partial\lambda_1)^{l_1}\cdots(\partial\lambda_n)^{l_q}}\varphi_{-i\rho^A-i\lambda}^A(x)\Big|_{\lambda=0} \\ =& \frac{1}{2^{|l|}}\int_K (\ln\Delta_1(u^{-1}e^{2\underline{x}}\, u))^{l_1} \cdot \left(\ln\left(\frac{\Delta_2(u^{-1}e^{2\underline{x}}\, u)}{\Delta_1(u^{-1}e^{2\underline{x}}\, u)}\right)\right)^{l_2} \cdots \left(\ln\left(\frac{\Delta_q(u^{-1}e^{2\underline{x}}\, u)}{\Delta_{q-1}(u^{-1}e^{2\underline{t}}\, u)}\right)\right)^{l_q} du \end{aligned} \tag{2.6}$$

of order $|l| := l_1 + \cdots + l_q$ for $t \in C_q^A$. Notice that the last equality in (2.6) follows from (2.4) by interchanging integration and derivatives. We denote the j-th unit vector by $e_j \in \mathbb{Z}_+^q$ and the moment functions of order 1 and 2 by m_{e_j} and $m_{e_j+e_k}$ $(j,k = 1,..,q)$. The q moment functions of first order lead to the vector-valued moment function

$$m_{\mathbf{1}}^A(x) := (m_{e_1}^A(x), \ldots, m_{e_q}^A(x)) \tag{2.7}$$

of first order. Moreover, the moment functions of second order can be grouped by

$$m_{\mathbf{2}}^A(x) := \begin{pmatrix} m_{2e_1}^A(x) & \cdots & m_{e_1+e_q}^A(x) \\ \vdots & & \vdots \\ m_{e_q+e_1}^A(x) & \cdots & m_{2e_q}^A(x) \end{pmatrix} \qquad \text{for} \quad x \in C_q^A.$$

We now form the $q \times q$-matrices $\Sigma^A(x) := m_{\mathbf{2}}^A(x) - m_{\mathbf{1}}^A(x)^x \cdot m_{\mathbf{1}}^A(x)$.

These moment functions have the following basic properties; see Section 2 of [V2]:

2.3 Lemma. (1) *There is a constant* $C = C(q)$ *such that for all* $x \in C_q^A$, $\|m_{\mathbf{1}}^A(x) - x\| \le C$.

(2) *For each* $t \in C_q^A$, $\Sigma^A(x)$ *is positive semidefinite.*

(3) *For* $x = c \cdot (1, \ldots, 1) \in C_q^A$ *with* $c \in \mathbb{R}$, $\Sigma^A(x) = 0$. *For all other* $x \in C_q^A$, $\Sigma^A(x)$ *has rank* $q-1$.

(4) *All second moment functions* $m_{e_i+e_j}^A(x)$ *are growing at most quadratically, and* $m_{2e_1}^A(x)$ *and* $m_{2e_q}^A(x)$ *are in fact growing quadratically.*

(5) *There exists a constant* $C = C(p)$ *such that for all* $x \in C_q^A$ *and* $\lambda \in \mathbb{R}^q$,

$$|\varphi_{-i\rho^A-\lambda}^A(x) - e^{i\langle\lambda, m_{\mathbf{1}}^A(x)\rangle}| \le C||\lambda||^2.$$

We now consider a probability measure $\nu \in \mathcal{M}^1(C_q^A)$. For $k \in \mathbb{N}$ we say that ν admits k-th moments of type A if for all $l \in \mathbb{N}_0^q$ with $|l| \le k$ the moment condition $m_l^A \in L^1(C_q^A, \nu)$ holds.
We then call $m_l^A(\nu) := \int_{C_q^A} m_l^A(x) d\nu(x)$ the l-th multivariate moment of ν. The vector

$$m_{\mathbf{1}}^A(\nu) := \int_{C_q^A} m_{\mathbf{1}}(x)\, d\nu(x) \in C_q^A \subset \mathbb{R}^q$$

is called the dispersion of ν. We also form the modified symmetric $q \times q$-covariance matrix

$$\Sigma^A(\nu) := \int_G m_{\mathbf{2}}\, d\nu \;-\; m_{\mathbf{1}}^A(\nu)^t \cdot m_{\mathbf{1}}^A(\nu).$$

We are interested in the A-case only as a limit of the BC-case for $p \to \infty$. For this we need an additional transformation

$$T : C_q^B \to C_q^B \subset C_q^A, \quad x = (x_1, ..., x_q) \mapsto \ln\cosh x := (\ln\cosh x_1, ..., \ln\cosh x_q) \tag{2.8}$$

cf. [RKV], [RV]. We define the modified moment functions $\tilde{m}_l(x) := m_l^A(T(x))$ which admit modified integral representations similar to (2.6). Moreover, for $\nu \in \mathcal{M}^1(C_q^B)$ we consider the image measure $T(\nu) \in \mathcal{M}^1(C_q^B) \subset \mathcal{M}^1(C_q^A)$. As $|x - \ln\cosh x| \le \ln 2$ for all $x \in [0, \infty[$ by an elementary calculation, we see that for all multiindices l, the l-th moment of type A of ν exists if and only if the l-th moment of type A of $T(\nu)$ exists. We put $\tilde{m}_l(\nu) := m_l^A(T(\nu))$ and $\tilde{\Sigma}(\nu) := \Sigma^A(T(\nu))$.

We next turn to the BC-case.

2.4 Definition. For all $p > 2q - 1$, $x \in C_q^B$, and $\lambda \in \mathbb{C}^q$, the functions in (1.4) satisfy

$$\varphi_\lambda^p(x) = \int_{B_q} \int_{U(q,\mathbb{F})} \Delta_{(i\lambda-\rho)/2}(g(x,u,w))\, du\, dm_p(w) \tag{2.9}$$

with the power function Δ_λ from (2.1), the half sum of positive roots

$$\rho = \rho(p) = \sum_{i=1}^q \big(\frac{d}{2}(p + q + 2 - 2i) - 1\big) e_i\,, \tag{2.10}$$

g as above, and with $m_p(w) \in \mathcal{M}^1(B_q)$ from (1.6); see [RV]. As in [RV] we define the moment functions for $l = (l_1, \dots, l_q) \in \mathbb{N}_0^q$ by:

$$\begin{aligned} m_l^p(x) :=& \frac{\partial^{|l|}}{\partial\lambda^l} \varphi^p_{-i\rho^{BC}-i\lambda}(x)\Big|_{\lambda=0} := \frac{\partial^{|l|}}{(\partial\lambda_1)^{l_1}\cdots(\partial\lambda_q)^{l_q}} \varphi^p_{-i\rho^{BC}-i\lambda}(x)\Big|_{\lambda=0} \\ =& \frac{1}{2^{|l|}} \int_{B_q}\int_{U(q,\mathbb{F})} (\ln\Delta_1(g(x,u,w)))^{l_1} \cdot \left(\ln\frac{\Delta_2(g(x,u,w))}{\Delta_1(g(x,u,w))}\right)^{l_2} \cdots \left(\ln\frac{\Delta_q(g(x,u,w))}{\Delta_{q-1}(g(x,u,w))}\right)^{l_q} du\, dm_p(w) \end{aligned} \tag{2.11}$$

for $x \in C_q^B$. We also form the vector-valued first moment function $m_{\mathbf{1}}^p$, the matrix-valued second moment function $m_{\mathbf{2}}^p$, as well as $\Sigma^p(x) := m_{\mathbf{2}}^p(x) - m_{\mathbf{1}}^p(x)^t \cdot m_{\mathbf{1}}^p(x)$ as above.

We have the following basic properties; see Section 3 of [V2]:

2.5 Lemma. (1) *There is a constant $C = C(p,q)$ such that for all $x \in C_q^B$,*

$$\|m_{\mathbf{1}}^p(x) - x\| \le C.$$

(2) *For each $x \in C_q^B$, $\Sigma^p(x)$ is positive semidefinite.*

(3) *$\Sigma^p(0) = 0$, and for $x \in C_q^B \setminus \{0\}$, $\Sigma^p(x)$ has full rank q.*

(4) *All second moment functions* $m^p_{e_j+e_l}(x)$ *are growing at most quadratically, and* $m^p_{2e_1}$ *is growing quadratically.*

(5) *There exists a constant* $C = C(p,q)$ *such that for all* $x \in C_q^B$ *and* $\lambda \in \mathbb{R}^q$,

$$|\varphi^p_{-i\rho-\lambda}(x) - e^{i\langle\lambda, m_1^p(x)\rangle}| \le C||\lambda||_2^2.$$

Similarly to the A-case, we also define multivariate l-th moments, dispersions, and covariance matrices of type BC(p) for measures $\nu \in \mathcal{M}^1(C_q^B)$.

We next derive estimates for $|\tilde{m}_l(\nu) - m_l^p(\nu)|$ for $l \in \mathbb{N}_0^q$ and large p under the assumption that these moments exist. For this we first show that for a given $\nu \in \mathcal{M}^1(C_q^q)$ the existence of moments of some maximal order is independent from taking classical moments, moments of type A, or moments of type BC. For our purpose it will be sufficient to restrict to the case when $|l|$ is even.
Let $k \in \mathbb{N}_0$ and $\nu \in \mathcal{M}^1(C_q^q)$. We then say that ν admits finite A-type moments of order at most $2k$ if

$$\tilde{m}_{2k\cdot e_1}, ..., \tilde{m}_{2k\cdot e_q} \in L^1(C_q^B, \nu).$$

Indeed, it follows immediately from the definition of moment functions in (2.6) and Hölder's inequality, that in this case all moments of order at most $2k$ are ν-integrable. Similarly, if

$$m^p_{2k\cdot e_1}, ..., m^p_{2k\cdot e_q} \in L^1(C_q^B, \nu)$$

then we say that ν admits finite BC(p)-type moments of order at most $2k$.

2.6 Proposition. *For* $k \in \mathbb{N}$ *and* $\nu \in \mathcal{M}^1(C_q^B)$ *the following statements are equivalent:*
(1) ν *admits all classical moments of order at most* $2k$*, i.e.* $\int_{C_q^B} x_1^{l_1} \cdots x_q^{l_q} d\nu(t) < \infty$ *for all* $l = (l_1, ..., l_q) \in \mathbb{N}_0^q$ *with* $|l| \le 2k$.
(2) ν *admits all moments of type A of order at most* $2k$.
(3) $T(\nu)$ *admits all moments of type A of order at most* $2k$.
(4) For each $p \ge 2q-1$, ν *admits all moments of type BC(p) of order at most* $2k$.

Proof. To show (1)⇒(2) it is sufficient to prove that $m^A_{2k\cdot e_1}, ..., m^A_{2k\cdot e_q} \in L^1(C_q^B, \nu)$. From (2.6) we have

$$m^A_{2k\cdot e_j}(\nu) = \frac{1}{2^{2k}} \int_{C_q^B} \int_{U(q,\mathbb{F})} \left(\ln \Delta_{j+1}(u^* e^{2\underline{x}} u) - \ln \Delta_j(u^* e^{2\underline{x}} u)\right)^{2k} du\, d\nu(x).$$

We now recall from Lemma 4.2 [V2] that $jx_q \le \ln \Delta_j(u^* e^{2\underline{x}} u) \le jx_1$ for $u \in U(q,\mathbb{F}), x \in C_q^B$, and $j = 1, ..., q$. Therefore, from elementary inequalities we obtain that

$$m^A_{2k\cdot e_j}(\nu) \le \frac{1}{2^{2k}} \int_{C_q^B} |(j(x_1 - x_q) + x_q|^{2k} \mathrm{d}\nu(x) < \infty. \tag{2.12}$$

To prove (2)⇒(1) it is sufficient to show that $\int_{C_q^B} x_1^{2k}\, d\nu(x) < \infty$. It can be easily seen that for every $u \in U(q,\mathbb{F})$ there exist coefficients $c_i(u) \ge 0$ for $i = 1, ...q$ with $\sum_{i=1}^q c_i(u) = 1$ such that

$$\Delta_1(u^* e^{2\underline{x}} u) = \sum_{i=1}^q c_i(u) e^{2x_i} \ge c_1(u) e^{2x_1}.$$

Thus, using the elementary inequality $2^{2k}(a^{2k}+b^{2k}) \ge (a+b)^{2k}$ for $a = \ln(c_1(u)e^{2x_1})$ and $b = -\ln c_1(u)$ we have

$$\begin{aligned}\int_{U(q,\mathbb{F})} \int_{C_q^B} (\ln \Delta_1(u^* e^{2\underline{x}} u))^{2k}\, du\, d\nu(x) &\ge \int_{U(q,\mathbb{F})} \int_{C_q^B} (\ln(c_1(u) e^{2x_1}))^{2k}\, du\, d\nu(x)\\ &\ge -\int_{U(q,\mathbb{F})} (|\ln c_1(u)|)^{2k}\, du + \int_{C_q^B} x_1^{2k}\, d\nu(x).\end{aligned}$$

Now, Lemma 5.1 and Proposition 4.9 of [V2] ensure that $\int_{U(q,\mathbb{F})}(|\ln c_1(u)|)^{2k}\,du$ is finite. Hence we have $\int_{C_q^B} x_1^{2k}\,d\nu(x) < \infty$ as desired.
The equivalence of (2) and (3) follows from

$$\frac{1}{4}u^* e^{2\underline{x}}u \le u^*(\cosh \underline{x})^2 u \le \frac{1}{2}u^* e^{2\underline{x}}u$$

which implies that

$$|\ln \Delta_j(u^*(\cosh \underline{x})^2 u) - \ln \Delta_j(u^* e^{2\underline{x}}u)| \le \ln 4.$$

To prove (3)⇒ (4) we recall from Lemma 6.4 in [V2] that

$$|\ln \Delta_j g(x,u,w) - \ln \Delta_j(u^*(\cosh \underline{x})u)| \le 2j\cdot \max(|\ln(1-\sigma_1(w))|, \ln(\sigma_1(w)+1)) := H_j(w). \quad (2.13)$$

It can be easily seen that $\int_{B_q} \ln(1+\sigma_1(w))^{2k}\mathrm{d}m_p(w)$ is finite.

Moreover, as $1 \ge \sigma_1(w) \ge .... \ge \sigma_q(w) \ge 0$ for $w \in B_q$ we have

$$\frac{1}{1-\sigma_1(w)} \le \frac{2}{1-\sigma_1(w)^2} \le 2\prod_{r=1}^{q}\frac{1}{1-\sigma_r(w)^2} \le \frac{2}{\Delta(I-w^*w)}. \quad (2.14)$$

Now, from Lemma 2.1 and (2.14) together with the elementary inequality

$$|\ln(1+z)| \leqslant \frac{|z|}{1-|z|} \text{ for } |z|<1 \quad (2.15)$$

we obtain that

$$\int_{B_q} |\ln(1-\sigma_1(w))|^{2k}\mathrm{d}m_p(w) \le 2^{2k}\int_{B_q}\sigma_1(w)^{2k}\cdot\Delta(I-w^*w)^{-2k}\mathrm{d}m_p(w) < \infty. \quad (2.16)$$

Hence, $\int_{B_q}|H_j(q)|^{2k}dm_p(w) < \infty$ for $j=1,..,q$. Therefore, using the elementary inequality $3^{2k}(a^{2k}+b^{2k}+c^{2k}) \ge (a+b+c)^{2k}$ we have

$$\begin{aligned} m^p_{2k\cdot e_j}(\nu) \le \left(\frac{3}{2}\right)^{2k}\int_{B_q\times U(q,\mathbb{F})\times C_q^B} &\Big(|\ln \Delta_{j+1}g(x,u,w) - \ln\Delta_{j+1}(u^*(\cosh\underline{x})u)|^{2k} + \\ &+ |\ln\Delta_{j+1}(u^*(\cosh\underline{x})u) - \ln\Delta_j(u^*(\cosh\underline{x})u)|^{2k} + \\ &+ |\ln\Delta_j g(x,u,w) - \ln\Delta_j(u^*(\cosh\underline{x})u)|^{2k}\Big)\mathrm{d}m_p(w)\,\mathrm{d}u\,\mathrm{d}\nu(x). \end{aligned} \quad (2.17)$$

We see that the right hand side of (2.17) is finite, from (2.13), (2.16) and the assumption that $m^A_{2k\cdot e_j}(\nu)$ is finite.

Finally, the converse statement (4)⇒(3) follows analogously from

$$\begin{aligned} m^A_{2k\cdot e_j}(\nu) \le \left(\frac{3}{2}\right)^{2k}\int_{B_q\times U(q,\mathbb{F})\times C_q^B} &[|\ln\Delta_{j+1}(u^*(\cosh\underline{x})u) - \ln\Delta_{j+1}g(x,u,w)|^{2k} \\ &+ |\ln\Delta_{j+1}g(x,u,w) - \ln\Delta_j g(x,u,w)|^{2k} \\ &+ |\ln\Delta_j(u^*(\cosh\underline{x})u) - \ln\Delta_j g(x,u,w)|^{2k}]\mathrm{d}m_p(w)\mathrm{d}u\mathrm{d}\nu(x). \end{aligned} \quad (2.18)$$

□

We now turn to the main result of the section:

2.7 Proposition. *Let $l=(l_1,....,l_q)\in\mathbb{N}_0^q$ with $|l|\ge 3$ and $\nu\in\mathcal{M}(C_q^B)$. Assume that ν admits finite moments of order $4(|l|-2)$. Then, there exists a constant $C := C(|l|,q,\nu)$ such that*

$$|\tilde{m}_l(\nu) - m_l^p(\nu)| \leqslant \frac{C}{\sqrt{p}}. \quad (2.19)$$

Proof. We consider the $|l|$ factors of the integrand in the integral representations (2.11) of the moment functions m_l^p and the modified version of (2.6) for $\tilde{m}_l$. For $i = 1, 2, ..., |l|$ these factors have the form:

$$f_i(x, u, w) := \ln \Delta_r(g(x, u, w)) - \ln \Delta_{r-1}(g(x, u, w)),$$
$$\tilde{f}_i(x, u, w) := \ln \Delta_r(g(x, u, 0)) - \ln \Delta_{r-1}(g(x, u, 0))$$

with the convention $\Delta_0 \equiv 1$ where $r \in \{1, ..., q\}$ is the smallest integer with $i \leq l_1 + ... + l_r$.
Then, from Lemma 2.1(2) and (2.15) for all $i = 1, ..., |l|, x \in C_q^B, u \in U(q, \mathbb{F}), w \in B_q$ we obtain that

$$|f_i(x, u, w) - \tilde{f}_i(x, u, w)| \leq 2 \max_{r=1,\dots,q} |\ln \Delta_r(g(x, u, w)) - \ln \Delta_r(g(x, u, 0))|$$
$$\leqslant 4q \cdot \frac{\tilde{x}\sigma_1(w)}{1 - \tilde{x}\sigma_1(w)} \leqslant 4q\tilde{x}\frac{\sigma_1(w)}{1 - \sigma_1(w)}$$

where $\tilde{x} = \min\{1, x\}$. Thus, by (2.14) we have

$$|f_i(x, u, w) - \tilde{f}_i(x, u, w)| \leq 8q\tilde{x}\frac{\sigma_1(w)}{\Delta(I - w^*w)}.$$

Now, notice that

$$|\tilde{m}_l(\nu) - m_l^p(\nu)| = \left| \frac{1}{2^{|l|}} \int_{B_q \times U(q,\mathbb{F}) \times C_q^B} \left(\prod_{i=1}^{|l|} f_i(x, u, w) - \prod_{i=1}^{|l|} \tilde{f}_i(x, u, w) \right) du dm_p(w) d\nu(t) \right| \tag{2.20}$$

Therefore, by a telescopic sum,

$$|\tilde{m}_l(\nu) - m_l^p(\nu)| =$$
$$= \Big| \frac{1}{2^{|l|}} \sum_{i=1}^{|l|} \int_{B_q \times U(q,\mathbb{F}) \times C_q^B} \Big((f_i(x, u, w) - \tilde{f}_i(x, u, w)) \times$$
$$\prod_{j=i+1}^{|l|} f_j(x, u, w) \prod_{k=1}^{i} \tilde{f}_k(x, u, w) \Big) du dm_p(w) d\nu(x) \Big|$$
$$\leq \frac{1}{2^{|l|}} \sum_{i=1}^{|l|} \int_{B_q \times U(q,\mathbb{F}) \times C_q^B} \Big| (f_i(x, u, w) - \tilde{f}_i(x, u, w)) \times$$
$$\prod_{j=i+1}^{|l|} f_j(x, u, w) \prod_{k=1}^{i} \tilde{f}_k(x, u, w) \Big| du dm_p(w) d\nu(x) \tag{2.21}$$

We estimate the summands of the expression of the last formula of (2.21) in two ways:
Summands for $i = 1$ and $|l|$:

From Cauchy-Schwarz inequality, (2.21) and Lemma 2.1 we obtain that

$$\int_{B_q\times U(q,\mathbb{F})\times C_q^B}\left|(f_1(x,u,w)-\tilde{f}_1(x,u,w))\prod_{j=2}^{|l|}f_j(x,u,w)\right|dudm_p(w)d\nu(x)$$
$$\le\left(\int_{B_q\times U(q,\mathbb{F})\times C_q^B}|f_i(x,u,w)-\tilde{f}_i(x,u,w)|^2dudm_p(w)d\nu(t)\right)^{1/2}\times$$
$$\times\left(\int_{B_q\times U_0(q,\mathbb{F})\times C_q^B}\prod_{j=2}^{|l|}f_j(x,u,w)^2dudm_p(w)d\nu(x)\right)^{1/2}$$
$$\le M_1\cdot 8q\left(\int_{B_q}\frac{\sigma_1(w)^2}{\Delta(I-w^*w)^2}dm_p(w)\right)^{1/2}$$
$$\le M_1\cdot\frac{C}{\sqrt{p}} \tag{2.22}$$

where

$$M_1:=M_1(\nu,|l|,q)=8q\cdot\max_{r\in\mathbb{N}_0^q,|r|\le 2(|l|-1)}\max\{\tilde{m}_r(\nu),m_r^p(\nu)\}$$

which is finite by initial assumption and Proposition 2.6. Similarly, we obtain same upper bound for the $|l|$'s summand in (2.21).
Now, let $i=2,...,q-1$. Here, we apply Hölder's inequality twice and obtain with the same arguments as above that

$$\left|\int_{B_q\times U_0(q,\mathbb{F})\times C_q^B}\Big((f_i(x,u,w)-\tilde{f}_i(x,u,w))\prod_{j=i+1}^{|l|}f_j(x,u,w)\prod_{k=1}^{i-1}\tilde{f}_k(x,u,w)\Big)dudm_p(w)d\nu(x)\right|$$
$$\le\left(\int_{B_q\times U_0(q,\mathbb{F})\times C_q^B}|(f_i(x,u,w)-\tilde{f}_i(x,u,w)|^2dudm_p(w)d\nu(t)\right)^{1/2}$$
$$\times\left(\int_{B_q\times U_0(q,\mathbb{F})\times C_q^B}\prod_{j=i+1}^{|l|}|f_j(x,u,w)|^4dudm_p(w)d\nu(x)\right)^{1/4}$$
$$\times\left(\int_{B_q\times U_0(q,\mathbb{F})}\prod_{k=1}^{i-1}|\tilde{f}_k(x,u,w)|^4dudm_p(w)d\nu(x)\right)^{1/4}$$
$$\le M_2\cdot\frac{C}{\sqrt{p}} \tag{2.23}$$

where

$$M_2:=M_2(\nu,|l|,q)=8q\cdot\max_{r\in\mathbb{N}_0^q,|r|\le 4(|l|-2)}\max\{\tilde{m}_r(\nu),m_r^p(\nu)\}$$

which is again finite by our assumption and Proposition 2.6. Thus, the estimates (2.22) and (2.23) give the desired assertion. □

3 Spherical Fourier transform

In this section we collect some well-known methods and facts about the spherical Fourier transform of type A and BC. We start with the identification of all multiplicative functions and of the dual space

in accordance with [R2] and [NPP] for $p \geq 2q-1$ in the BC-case.
The set of all continuous multiplicative functions

$$\chi(C_q^B, *_p) := \{f : C_q^B \to \mathbb{C} : f \text{ continuous}, \int_{C_q^B} d(\delta_x *_p \delta_y) = f(x)f(y)\}$$

is given by $\{\varphi_\lambda^p : \lambda \in \mathbb{C}^q\}$. Moreover, the set $\chi_b(C_q^B, *_p)$ of bounded functions in $\chi(C_q^B, *_p)$ is equal to $\{\varphi_\lambda^p : \Im\lambda \in co(W_q \cdot \rho)\}$ where co denotes the convex hull, and W_q^B the Weyl group of type B_q acting on $\mathbb{C}^q$. The dual space

$$(C_q^B, *_p)^\wedge := \{f \in \chi_b(C_q^B, *_p), f(x^-) = \overline{f(x)}\}$$

is $\{\varphi_\lambda^p : \lambda \in C_q^B \text{ or } \lambda \in i \cdot co(W_q^B \cdot \rho)\}$. Finally, the support of Plancherel measure is the set $\{\varphi_\lambda^p : \lambda \in C_q^B\}$.

3.1 Definition. Let $\nu \in \mathcal{M}^1(C_q^B)$. The BC-type spherical (or hypergroup) Fourier transform is given by

$$\mathcal{F}_{BC}^p(\nu)(\lambda) := \int_{C_q^B} \varphi_\lambda^p(x) d\nu(x)$$

for $\lambda \in \{\lambda \in \mathbb{C}^q : \Im\lambda \in co(W_q^B \cdot \rho)\}$.

We now give some estimates on spherical functions and Fourier transforms from [V2].

3.2 Lemma. *For all $x \in C_q^B$, $\lambda \in \mathbb{R}^q$, and $l \in \mathbb{N}_0^q$,*

$$\left| \frac{\partial^{|l|}}{\partial \lambda^l} \varphi_{\lambda - i\rho}^p(x) \right| \leqslant m_l^p(x)$$

3.3 Lemma. *Let $k \in \mathbb{N}_0$ and assume that $\nu \in \mathcal{M}^1(C_q^B)$ admits finite k-th modified moments. Then, for all $\lambda \in \mathbb{C}^q$ with $\Im\lambda \in co(W_q^B \cdot \rho)$, $\mathcal{F}_{BC}^p(\nu)(\cdot)$ is k-times continuously differentiable, and for all $l \in \mathbb{N}_0^n$ with $|l| \leqslant k$,*

$$\frac{\partial^{|l|}}{\partial \lambda^l} \mathcal{F}_{BC}^p(\nu)(\lambda) = \int_{C_q^B} \frac{\partial^{|l|}}{\partial \lambda^l} \varphi_\lambda^p(x) d\nu(x). \tag{3.1}$$

In particular,

$$\frac{\partial^{|l|}}{\partial \lambda^l} \mathcal{F}_{BC}(\nu)(-i\rho) = \int_{C_q^B} m_l^p(x) d\nu(x). \tag{3.2}$$

3.4 Remark. There are corresponding results to the Lemmas 3.2 and 3.3 for the A-case with the corresponding moment functions m_l^A for $l \in \mathbb{N}_0^q$ and the Fourier transform $\mathcal{F}_A$ and $\nu \in \mathcal{M}^1(C_q^A)$; see Lemmas 6.1, 6.2 in [V2].

4 Central limit theorems for growing parameters

In this section we derive two CLTs for random walks when the time and the dimension parameter p tend to infinity. The statements of both CLTs are similar, but the assumptions on the moments and the relation between the time and p are different. We first present a CLT where we assume some restriction on $(p_n)_{n\geq 1}$:

4.1 Theorem. *Let $(p_n)_{n\geq 1} \subset [2q-1, \infty[$ be an increasing sequence with $\lim_{n\to\infty} n/p_n = 0$. Let $\nu \in \mathcal{M}^1(C_q^B)$ be with $\nu \neq \delta_0$ and second moments. Consider the associated random walks $(S_n^p)_{n\geqslant 0}$ on C_q^B for $p \geq 2q-1$. Then*

$$\frac{S_n^{p_n} - n \cdot \tilde{m}_{\mathbf{1}}(\nu)}{\sqrt{n}}$$

converges in distribution to $\mathcal{N}(0, \tilde{\Sigma}(\nu))$.

Proof. We know from Lemma 4.2(2) of [RV] that there exists a constant $C > 0$ such that for all $p > 2q-1, x \in C_q^B, \lambda \in \mathbb{R}^q$,

$$|\varphi^p_{\lambda - i\rho}(x) - \varphi^A_{\lambda - i\rho^A}(\ln\cosh x)| \leqslant C \cdot \frac{\|\lambda\|_1 \cdot \tilde{x}}{p^{1/2}}$$

where $\|\lambda\|_1 := |\lambda_1| + \ldots \ |\lambda_q|$ and $\tilde{x} := min(x_1, 1) \geqslant 0$. Hence, denoting the half sums of positive roots of type BC associated with p_n as described in (2.10) by $\rho(n) := \rho^{BC}(p_n)$, for all $\nu \in \mathcal{M}^1(C_q^B)$, we get

$$\left| \int_{C_q^B} \varphi^{p_n}_{\lambda - i\rho(n)}(x) d\nu(x) - \int_{C_q^B} \varphi^A_{\lambda - i\rho^A}(\ln\cosh x) d\nu(x) \right| \leqslant C \cdot \frac{\|\lambda\|_1}{\sqrt{p_n}}. \tag{4.1}$$

Let $\nu^{(n,p)} \in \mathcal{M}^1(C_q^B)$ be the law of S_n^p. Then, $T(S_n^{p_n})$ has the distribution $T(\nu^{(n,p_n)})$ whose A-type spherical Fourier transform satisfies

$$\mathcal{F}_A(T(\nu^{(n,p_n)}))(\lambda - i\rho^A) = \int_{C_q^A} \varphi^A_{\lambda - i\rho^A}(x) dT(\nu^{(n,p_n)})(x) = \int_{C_q^B} \varphi^A_{\lambda - i\rho^A}(\ln\cosh x) d\nu^{(n,p_n)}(x) \tag{4.2}$$

for $\lambda \in \mathbb{R}^q$. Furthermore, by plugging $\nu^{(n,p_n)}$ into (4.1) we get

$$\begin{aligned}
\mathcal{F}_A(T(\nu^{(n,p_n)}))(\lambda - i\rho^A) &= \int_{C_q^B} \varphi^{p_n}_{\lambda - i\rho^A} d\nu^{(n,p_n)}(t) + O(\frac{\|\lambda\|_1}{p_n^{1/2}}) \\
&= \left(\int_{C_q^B} \varphi^{p_n}_{\lambda - i\rho^A} d\nu(x) \right)^n + O(\frac{\|\lambda\|_1}{p_n^{1/2}}) \\
&= \left(\mathcal{F}_A(T(\nu))(\lambda - i\rho^A) + O(\frac{\|\lambda\|_1}{p_n^{1/2}}) \right)^n + O(\frac{\|\lambda\|_1}{p_n^{1/2}}).
\end{aligned}$$

Using the the initial moment assumption and Lemma 2.6 we see that the first and second modified moments $\tilde{m}_{\mathbf{1}}$ and $\tilde{m}_{\mathbf{2}}$ exist. Moreover, all entries of the modified covariance matrix

$$\tilde{\Sigma}(\nu) = \tilde{m}_{\mathbf{2}}(\nu) - \tilde{m}_{\mathbf{1}}(\nu)^t \cdot \tilde{m}_{\mathbf{1}}(\nu)$$

are finite.
By Lemma 3.3, the Taylor expansion of $\mathcal{F}_A(T(\nu))(\lambda - i\rho^A)$ for $|\lambda| \to 0$ is given by

$$\mathcal{F}_A(T(\nu))(\lambda - i\rho^A) = 1 - i\langle \lambda, \tilde{m}_{\mathbf{1}}(\nu) \rangle - \lambda \tilde{m}_{\mathbf{2}}(\nu) \lambda^t + o(|\lambda|^2).$$

Using the initial assumption that $O(1/\sqrt{np_n}) = o(1/n)$ we obtain

$$\begin{aligned}
E(\varphi^A_{\lambda/\sqrt{n} - i\rho^A}&(T(S_n^{p_n}))e^{i\langle \lambda, \sqrt{n}\tilde{m}_{\mathbf{1}}(\nu)\rangle} = \mathcal{F}_A(T(\nu^{(n,p_n)}))(\lambda/\sqrt{n} - i\rho^A) \cdot e^{i\langle \lambda, \sqrt{n}\tilde{m}_{\mathbf{1}}(\nu)\rangle} \\
&= \left[\left(\mathcal{F}_A(T(\nu))(\lambda - i\rho^A) + O(\frac{\|\lambda\|_1}{\sqrt{np_n}}) \right)^n + O(\frac{\|\lambda\|_1}{\sqrt{np_n}}) \right] \cdot e^{i\langle \lambda, \frac{\tilde{m}_{\mathbf{1}}(\nu)}{\sqrt{n}} \rangle n} \\
&= \left[\left(1 - \frac{i\langle \lambda, \tilde{m}_{\mathbf{1}}(\nu)\rangle}{\sqrt{n}} - \frac{\lambda \tilde{m}_{\mathbf{2}}(\nu)\lambda^t}{2n} + o(\frac{1}{n}) \right) \times \right. \\
&\qquad \left. \times \left(1 + \frac{i\langle \lambda, \tilde{m}_{\mathbf{1}}(\nu)\rangle}{\sqrt{n}} - \frac{\langle \lambda, \tilde{m}_{\mathbf{1}}(\nu)\rangle^2}{2n} + o(\frac{1}{n}) \right) \right]^n \\
&= \left(1 - \frac{\lambda \tilde{\Sigma}(\nu) \lambda^t}{2n} + o(\frac{1}{n}) \right)^n.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} E(\varphi^A_{\lambda/\sqrt{n} - i\rho^A}(T(S_n^{p_n})) \cdot \exp(i\langle \lambda, \tilde{m}_{\mathbf{1}}(\nu)\rangle \sqrt{n})) = \exp(-\lambda \tilde{\Sigma}(\nu)\lambda^t/2). \tag{4.3}$$

On the other hand, from Lemma 2.3(5) we have

$$\lim_{n\to\infty} E(\varphi^A_{\lambda/\sqrt{n}-i\rho^A}(T(S_n^{p_n})) - \exp(-i\langle\lambda, \tilde{m}_{\mathbf{1}}(S_n^{p_n})\rangle/\sqrt{n})) = 0. \tag{4.4}$$

(4.3) and (4.4) and the fact that $|e^{i\langle\lambda,\sqrt{n}\tilde{m}_{\mathbf{1}}(\nu)\rangle}| \leqslant 1$ together yield that for all $\lambda \in \mathbb{R}^q$,

$$\lim_{n\to\infty} \exp(-i\langle\lambda, (\tilde{m}_{\mathbf{1}}(S_n^{p_n}) - n\cdot\tilde{m}_{\mathbf{1}}(\nu))\rangle/\sqrt{n}) = \exp(-\lambda\tilde{\Sigma}(\nu)\lambda^t/2).$$

Levy's continuity theorem for the classical q-dimensional Fourier transform implies that $(\tilde{m}_{\mathbf{1}}(S_n^{p_n}) - n\cdot\tilde{m}_{\mathbf{1}}(\nu)\rangle)/\sqrt{n}$ tends to the normal distribution $\mathcal{N}(0,\tilde{\Sigma}(\nu))$.
Now, Lemma 2.3(2) implies that $(T(S_n^{p_n}) - n\cdot\tilde{m}_{\mathbf{1}}(\nu)\rangle)/\sqrt{n}$ also converges to $\mathcal{N}(0,\tilde{\Sigma}(\nu))$.
Moreover, since $\lim_{x\to\infty}(x-\ln\cosh x) = \ln 2$, we see that $(\ln\cosh(S_n^{p_n}) - S_n^{p_n})/\sqrt{n} \to 0$, which implies that $(S_n^{p_n} - n\tilde{m}_{\mathbf{1}}(\nu))/\sqrt{n} \to N(0,\tilde{\Sigma}(\nu))$ as desired. ☐

4.2 Remark. For the rank one case $q = 1$ the preceding CLT was derived in [Gr1] with different techniques under weaker assumptions, namely without the restriction $n/p_n \to 0$ as $n\to\infty$. The proof in [Gr1] relies on the convergence of the moment functions

$$(m_1^p(x))^2 - m_2^p(x) \to 0 \tag{4.5}$$

on $[0,\infty[$ for $p\to\infty$. However, for $q \geq 2$ this convergence is no longer available.

We next try to get rid of the restriction $n/p_n \to 0$. We shall achieve this by assuming the existence of fourth moments in addition.

4.3 Theorem. *Let $(p_n)_{n\geq 1}$ be an increasing sequence with $p_1 \geqslant 2q-1$ and $\lim_{n\to\infty} p_n = \infty$. Let $\nu \in \mathcal{M}^1(C_q^B)$ with $\nu \neq \delta_0$ and with fourth moments. Consider the associated random walks $(S_n^p)_{n\geqslant 0}$ on C_q^B for $p \geq 2q-1$. Then*

$$\frac{S_n^{p_n} - nm_{\mathbf{1}}^{p_n}(\nu)}{\sqrt{n}}$$

converges in distribution to $\mathcal{N}(0,\tilde{\Sigma}(\nu))$.

Proof. We first notice that by Taylor's theorem and Proposition 2.7 for all $p \geqslant 2q-1$,

$$\begin{aligned}\left|E(\varphi^p_{\lambda/\sqrt{n}-i\rho}(S_n^p)) - \left(1 - \frac{i\langle\lambda, m_{\mathbf{1}}^p(\nu)\rangle}{\sqrt{n}} - \frac{\lambda m_{\mathbf{2}}^p(\nu)\lambda^t}{2n}\right)\right| &\leq \sum_{l\in\mathbb{N}^q,|l|=3} m_l^p(\nu)\frac{\lambda_1^{l_1}\ldots\lambda_q^{l_q}}{l_1!\ldots l_q!}\\ &\leq \frac{1}{n^{3/2}}\sum_{l\in\mathbb{N}^q,|l|=3}(\tilde{m}_l(\nu) + C/\sqrt{p})\frac{\lambda_1^{l_1}\ldots\lambda_q^{l_q}}{l_1!\ldots l_q!}\\ &\leq K_1\frac{\|\lambda\|_\infty^3}{n^{3/2}}\end{aligned} \tag{4.6}$$

for some constant $K_1 > 0$ which is independent of p. Analogously, for all $p \geqslant 2q-1$,

$$\left|e^{i\langle\lambda,\sqrt{n}m_{\mathbf{1}}^p(\nu)\rangle} - \left(1 + \frac{i\langle\lambda, m_{\mathbf{1}}^p(\nu)\rangle}{\sqrt{n}} - \frac{\langle\lambda, m_{\mathbf{1}}^p(\nu)\rangle^2}{2n}\right)\right| \leqslant K_2\frac{\|\lambda\|_\infty^3}{n^{3/2}} \tag{4.7}$$

for some $K_2 > 0$ independent of p.
Using estimates (4.6) and (4.7) we now follow similar paths as in the proof of Theorem 4.1. We however use the BC-type Fourier transform and BC-moments instead of objects of type A, and then

approximate A-type moments by BC-type moments using Proposition 2.7. Now, we have

$$\begin{aligned}
E(\varphi^{p_n}_{\lambda/\sqrt{n}-i\rho(n)}(S^{p_n}_n))e^{i\langle\lambda,\sqrt{n}m^{p_n}_{\mathbf{1}}(\nu)\rangle} &= \mathcal{F}^{p_n}_{BC}(\nu^{(n,p_n)})(\lambda/\sqrt{n}-i\rho(n))\cdot e^{i\langle\lambda,\sqrt{n}m^{p_n}_{\mathbf{1}}(\nu)\rangle}\\
&= \Bigg[\Big(1-\frac{i\langle\lambda,m^{p_n}_{\mathbf{1}}(\nu)\rangle}{\sqrt{n}}-\frac{\lambda m^{p_n}_{\mathbf{2}}(\nu)\lambda^t}{2n}+o(\frac{1}{n})\Big)\times\\
&\qquad\times\Big(1+\frac{i\langle\lambda,m^{p_n}_{\mathbf{1}}(\nu)\rangle}{\sqrt{n}}-\frac{\langle\lambda,m^{p_n}_{\mathbf{1}}(\nu)\rangle^2}{2n}+o(\frac{1}{n})\Big)\Bigg]^n\\
&= \Big(1-\frac{\lambda\Sigma^{p_n}(\nu)\lambda^t}{2n}+o(\frac{1}{n})\Big)^n
\end{aligned}$$

From Lemma 2.7 we also obtain that

$$|\lambda\Sigma^{p_n}(\nu)\lambda^t-\lambda\tilde{\Sigma}(\nu)\lambda^t| = O(\frac{|\lambda|^2}{\sqrt{p_n}})$$

for $p_n\to\infty$. Therefore, we have

$$\begin{aligned}
\lim_{n\to\infty}E(\varphi^{p_n}_{\lambda/\sqrt{n}-i\rho(n)}(S^{p_n}_n))e^{i\langle\lambda,\sqrt{n}m^{p_n}_{\mathbf{1}}(\nu)\rangle} &= \lim_{n\to\infty}\Big(1-\frac{\lambda\tilde{\Sigma}(\nu)\lambda^t}{2n}+\frac{\lambda(\Sigma^{p_n}(\nu)-\tilde{\Sigma(\nu)})\lambda^t}{2n}+o(\frac{1}{n})\Big)^n\\
&= \exp(-\lambda\tilde{\Sigma}(\nu)\lambda^t/2)
\end{aligned}$$

On the other hand from the Lemma 2.5(5) we have

$$\lim_{n\to\infty}E(\varphi^{p_n}_{\lambda/\sqrt{n}-i\rho(n)}(S^{p_n}_n)-\exp(-i\langle\lambda,m^{p_n}_{\mathbf{1}}(S^{p_n}_n)\rangle/\sqrt{n}))=0. \tag{4.8}$$

The rest of the proof is now analogous to that of Theorem 4.1. □

5 A central limit theorem with fixed p with inner normalization

In this section we present some CLT for some fixed p. We consider the following setting: Fix some nontrivial probability measure $\nu\in\mathcal{M}^1(C^B_q)$ with some moment condition and for $d\in]0,1]$ consider the component-wise compression map $D_d: x\mapsto d\cdot x$ on C^B_q as well as compressed measure $\nu_d := D_d(\nu)\in\mathcal{M}^1(C^B_q)$. For given ν and d we consider the random walk $(S^{(p,d)}_n)_{n\geqslant0}$ associated with ν_d. We investigate the limiting behavior of $(S^{(p,n^{-1/2})}_n)_{n\geqslant1}$. This case can be seen as CLT with inner standardization in contrast to the case with $(S^p_n)_{n\geq0}$ in Section 3 where we consider CLT with outer standardization $n^{1/2}$. These two CLTs exhibit different limiting procedures. The limit theorem for $(S^{(p,n^{-1/2})}_n)_{n\geqslant1}$ in the rank 1 case was studied by Zeuner [Z1]. In the group cases, this CLT is related with the CLTs in [G1], [G2], [Te1], [Te2], [Ri].

5.1 Definition. Let $p\geq 2q-1$ and $t\geq0$. A probability measure $\gamma_t=\gamma_t(p)\in\mathcal{M}^1(C^B_q)$ is called *BC(p)-Gaussian* with time parameter t and shape parameter p if

$$\mathcal{F}^p_{BC}(\gamma_t)(\lambda) = exp(\frac{-t(\lambda_1^2+...+\lambda_q^2+\|\rho\|_2^2)}{2})$$

for all $\lambda\in C^B_q\cup i\cdot co(W^B_q\cdot\rho)\subset\mathbb{C}^q$.

We notice that by injectivity of the hypergroup Fourier transform (see [J]), the measures γ_t are determined uniquely and that they form a weakly continuous convolution semigroup $(\gamma_t)_{t\geq0}$, i.e. for all $s,t\geq0$ we have $\gamma_s *_p\gamma_t=\gamma_{s+t}$ and $\gamma_0=\delta_0$. The existence of the measures γ_t for $t>0$ is not quite obvious at the beginning, but we shall see from the proof of he following CLT that γ_t indeed exists.

5.2 Theorem. *Let $\nu \in \mathcal{M}^1(C_q^B)$ with $\nu \neq \delta_0$ and with finite second moments. Let*

$$t := \frac{1}{qd}\int_{C_q^B} \|x\|_2^2 d\nu(x).$$

Then, $(S_n^{(p,n^{-1/2})})_{n\geq 1}$ tends in distribution for $n \to \infty$ to $\gamma_{\frac{t}{p}}$.

For the proof we need some information on φ_λ^p:

5.3 Lemma. *Let $p \in [2q-1, \infty[$ be fixed. Then:*

(1) *For all $i, j = 1, 2, ..., q$ with $i \neq j$ and all $\lambda \in \mathbb{C}^q$,*

$$\frac{\partial}{\partial x_i}\varphi_\lambda^p(0) = 0 \text{ and } \frac{\partial^2}{\partial x_i \partial x_j}\varphi_\lambda^p(0) = 0 \tag{5.1}$$

(2) *For all $i = 1, 2, ..., q,$ and $\lambda \in C_q^B \cup i \cdot co(W_q \cdot \rho)$,*

$$\frac{\partial^2}{\partial x_i^2}\varphi_\lambda^p(0) = -\frac{(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{pqd} < 0.$$

Proof. The spherical functions $\varphi_\lambda^p(x)$ are invariant under the action of the Weyl group of of type BC w.r.t. x. Therefore, $\varphi_\lambda^p(x_1, .., x_q)$ is even in each x_i, which leads to (1). Moreover, as $\varphi_\lambda^p(x_1, ..., x_q)$ is invariant under the permutations of x_i, $\frac{\partial^2}{\partial x_i^2}\varphi_\lambda^p(0)$ is independent of i. To complete the proof of (2), we recall from Eq. (1.2.6) in [HS] that for all $\lambda \in \mathbb{C}^q$ the hypergeometric function $F_{BC}(\lambda, k_p, \cdot)$ is the unique solution to the eigenvalue problem

$$Lf = -(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)f \tag{5.2}$$

for $x \in int(C_q^B) = \{x \in C_q^B : x_1 > x_2 > ... > x_q > 0\}$ with $f(0) = 1$ where the differential operator L is defined as

$$\begin{aligned} L := \sum_{1\leq i\leq q} &\left[\frac{\partial_i^2}{\partial x_i^2} + (2k_1 \coth(x_i) + 4k_2 \coth(2x_i))\frac{\partial_i}{\partial x_i}\right] \\ &+ 2k_3 \sum_{1\leq i<j\leq q}\left[\coth(x_i + x_j)\left(\frac{\partial_i}{\partial x_i} + \frac{\partial_j}{\partial x_j}\right) + \coth(x_i - x_j)\left(\frac{\partial_i}{\partial x_i} - \frac{\partial_j}{\partial x_j}\right)\right]. \end{aligned} \tag{5.3}$$

Notice here that the factors $2, 4, 2$ of the multiplicities k_1, k_2, k_3 respectively, originate from the directional derivatives w.r.t the roots in Eq. (1.2.6) in [HS].
Now, using part (1), $\varphi_\lambda^p(x) = F_{BC}(i\lambda, k_p, x)$, and the Taylor expansion of coth around 0, we have

$$\begin{aligned} -(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)\varphi_\lambda^p(0) &= \lim_{\|x\|\to 0} L\varphi_\lambda^p(x) \\ &= (q + 2qk_1 + 4qk_2 + 2q(q-1)k_3)\left.\frac{\partial_1^2}{\partial x_1^2}\varphi_\lambda^p(x)\right|_{x=0} \\ &= pqd \cdot \left.\frac{\partial_1^2}{\partial x_1^2}\varphi_\lambda^p(x)\right|_{x=0} \end{aligned}$$

for all $\lambda \in \mathbb{C}^q$. Finally, as $co(W_q^B \cdot \rho)$ is contained in $\{x \in \mathbb{R}^q : \|x\|_2 \leq \|\rho\|_2\}$, the final statement of (2) is also clear. □

Proof of Theorem 5.2. Lemma 5.3 and $\varphi_\lambda^p(x) \le 1$ for $x \in C_q^B$ ensure that there exists $c > 0$ such that

$$1 - c(x_1^2 + x_2^2 + ... + x_q^2) \leqslant \varphi_\lambda^p(x) \text{ for all } x \in C_q^B.$$

Consequently by Taylor expansion,

$$n \left| \varphi_\lambda^p(\frac{x}{\sqrt{n}}) - 1 + \frac{\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2}{2pqd} \cdot \frac{\|x\|_2^2}{n} \right| \le C\|x\|_2^2$$

for some constant $C > 0$ where $\|x\|_2^2$ is integrable w.r.t ν by our assumption. Thus, dominated convergence theorems yields that

$$\lim_{n\to\infty} n \int_{C_q^B} \left(\varphi_\lambda^p(\frac{x}{\sqrt{n}}) - 1 + \frac{(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{2pqd} \cdot \frac{\|x\|_2^2}{n} \right) d\nu(x) = 0.$$

Rewriting this relation as

$$\int_{C_q^B} \varphi_\lambda^p(\frac{x}{\sqrt{n}}) d\nu(x) = 1 - \frac{1}{n} \frac{(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{2pqd} \cdot \int_{C_q^B} \|x\|_2^2 d\nu(x) + o(\frac{1}{n})$$

we obtain

$$\begin{aligned} \mathcal{F}_{BC}^p(\mathbb{P}_{S_n^{(p,n^{-1/2})}})(\lambda) &= \int_{C_q^B} \varphi_\lambda^p(\frac{x}{\sqrt{n}}) d\nu^{(n)}(x) = \left[\int_{C_q^B} \varphi_\lambda^p(\frac{x}{\sqrt{n}}) d\nu(x) \right]^n \\ &= \left(1 - \frac{1}{n} \cdot \frac{(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{2pqd} \int_{C_q^B} \|x\|_2^2 d\nu(x) + o(\frac{1}{n}) \right)^n \end{aligned}$$

which implies

$$\begin{aligned} \lim_{n\to\infty} \mathcal{F}_{BC}^p(\mathbb{P}_{S_n^{(p,n^{-1/2})}})(\lambda) &= \exp\left(-\frac{(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{2pqd} \cdot \int_{C_q^B} \|x\|_2^2 d\nu(x) \right) \\ &= \exp\left(-\frac{t(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{2p} \right) \end{aligned}$$

for all $\lambda \in \mathbb{R}^q \cup i \cdot co(W_q^B \cdot \rho)$. Hence, by the weak version of Levy's continuity theorem for commutative hypergroups (see Theorem 4.2.11 in [BH]) there exists a bounded positive measure in $\mu \in \mathcal{M}_b^+(C_q^B)$ with

$$\mathcal{F}_{BC}^p(\mu)(\lambda) = \exp\left(-\frac{t(\lambda_1^2 + ... + \lambda_q^2 + \|\rho\|_2^2)}{2p} \right) \tag{5.4}$$

for all $\lambda \in \mathbb{R}^q$, and $(\mathbb{P}_{S_n^{n-1/2}})_{n\ge1}$ converges to μ vaguely.
Notice that the right hand side of (5.4) is obviously analytic for $\lambda \in \mathbb{C}^q$. Moreover, the left hand side is holomorphic for λ in the open set $I := \{a + ib \in \mathbb{C}^q : a \in \mathbb{R}^q, b \in Int(co(W_q^B \cdot \rho))\} \subset \mathbb{C}^q$. This follows from the fact that $\varphi_\lambda^p(\cdot)$ is holomorphic for $\lambda \in I$, that $|\varphi_\lambda^p(x)| \le 1$ for all $x \in C_q^B$ and $\lambda \in I$, and from some well-known theorem on the holomorphy of parameter integrals (which is a consequence of the theorems of Fubini and Morera). We thus conclude that equality (5.4) holds for all $\lambda \in \bar{I}$. Therefore, we have $\mathcal{F}_{BC}^p(\mu)(-i\rho) = 1$, i.e. the limiting positive measure μ is indeed a probability measure. This implies that $(\mathbb{P}_{S_n^{(p,n^{-1/2})}})_{n\ge1}$ converges weakly to $\mu = \gamma_{\frac{t}{p}}$ as desired. □

5.4 Remark. The considerations in the above proof yield that the probability measures γ_t in Definition 5.1 above indeed exist.

6 A law of large numbers for inner normalizations and growing parameters

We present a further limit theorem for $(S_n^{(p,n^{-1/2})})_{n\geq 1}$ when p and n go ∞ in a coupled way. It will turn out that then, under some canonical norming, the limiting distribution is a point measure, i.e., we obtain a weak law of large numbers:

6.1 Theorem. *Let $\nu \in \mathcal{M}^1(C_q^B)$ with $\nu \neq \delta_0$ and finite second moments. Let $t := \frac{1}{qd}\int_{C_q^B} \|x\|_2^2 d\nu(x)$ be as in Theorem 5.2 and $(p_n)_{n\geq 1} \subset [2q-1,\infty[$ be increasing with $\lim_{n\to\infty} n/p_n = 0$. Then, $(S_n^{(p_n,n^{-1/2})})_{n\geq 1}$ tends in distribution for $n\to\infty$ to the constant*

$$\ln\left(e^t + \sqrt{(e^t)^2 - 1}\right)\cdot(1,\ldots,1).$$

For the proof of theorem we first recapitulate the Taylor expansion for $\varphi_\lambda^A(x)$ at $x=0$ from [Gr1]:

6.2 Lemma. *For $\|x\|_2 \to 0$,*

$$\varphi_\lambda^A(x) = 1 + \frac{1}{q}(\lambda_1+\lambda_2+...+\lambda_q)\sum_{k=1}^q x_k + R_\lambda(x)$$

with

$$R_\lambda(x) = \sum_\alpha f_\alpha(\lambda)P_\alpha(x)$$

where the $P_\alpha(x)$ are symmetric polynomials in $x_1,...,x_q$ which are homogeneous of order ≥ 2.

We also need the following fact:

6.3 Lemma. *For $p \geq 2q-1$, the half sum $\rho = \rho^{BC}(p)$ satisfies the condition $\rho^A - \rho \in co(W_q^B\cdot\rho)$, where W_q^B is the Weyl group of type B_q.*

Proof. Denote $\hat\rho := (\rho_q, \rho_{q-1}..., \rho_1)$. Then, obviously , $-\rho, -\hat\rho \in W_q^B\cdot\rho$. On the other hand we have

$$\rho^A - \rho = \left(\frac{d}{2}(p+1)-1\right)(1,....,1) = \frac{1}{2}(-\rho-\hat\rho).$$

This proves the result. □

6.4 Proposition. *Let ν, t and $(p_n)_{n\geq 1}$ be defined as in Theorem 6.1. Consider the half sum of positive roots $\rho(n) := \rho^{BC}(p_n)$ of type BC associated with the parameters p_n as described in (2.10). Then, for all $\lambda\in\mathbb{C}^q$ with $\Im\lambda = \rho^A$,*

$$\int_{C_q^B}\varphi^{p_n}_{\lambda-i\rho(n)}(\frac{x}{\sqrt{n}})d\nu(x) = 1 + \frac{t}{n}\cdot\sum_{k=1}^q(\lambda_k - i\rho_k^A) + o(1/n) \text{ as } n\to\infty. \tag{6.1}$$

Proof. Lemma 6.2 and the Taylor expansion $\ln\cosh x = x^2 + O(x^4)$ show that for all $\lambda\in\mathbb{C}^q$ with such that $\Im\lambda \in co(W_q^A\cdot\rho^A)$

$$\varphi_\lambda^A(\ln\cosh\frac{x}{\sqrt{n}}) = 1 + \sum_{i=1}^q \lambda_i\frac{\|x\|_2^2}{nq} + R_\lambda(\frac{\|x\|^2}{n}) \tag{6.2}$$

for $n\to\infty$. On the other hand, Theorem 4.2(2) in [RV] states that

$$|\varphi^p_{\lambda-i\rho(n)}(\frac{x}{\sqrt{n}}) - \varphi^A_{\lambda-i\rho_A}(\ln\cosh\frac{x}{\sqrt{n}})| \leq C\cdot\frac{\|\lambda\|_1\cdot\min(1,x_1/\sqrt{n})}{\sqrt{p}} \tag{6.3}$$

for all $\lambda \in \mathbb{C}^q$ such that $\Im\lambda - \rho(n) \in co(W_q^B \cdot \rho(n))$. Notice that the analysis of the proof of Theorem 4.2(2) in [RV] shows that (6.3) is in fact precisely valid for

$$\lambda \in \{\lambda \in \mathbb{C}^q : \Im\lambda - \rho(n) \in co(W_q^B \cdot \rho(n)) \text{ and } \Im\lambda - \rho^A \in co(W_q^A \cdot \rho^A)\}.$$

If we combine (6.2) and (6.3) and use the Lemma 6.3 we see that as $p_n/n \to \infty$

$$\left| \varphi^{p_n}_{\lambda - i\rho(n)}(\frac{x}{\sqrt{n}}) - 1 - \sum_{k=1}^{q}(\lambda_k - i\rho_k^A)\frac{\|x\|_2^2}{qn} \right| = o(\frac{\|x\|_2^2}{n}) \quad \text{for all } \lambda \in \mathbb{C}^q \text{ with } \Im\lambda = \rho^A \tag{6.4}$$

which, by integrating w.r.t ν yields the result.

□

Proof of the Theorem 6.1. Let $\nu^{(n,p_n)}$ be the n-fold $*_{p_n}$ convolution power of ν. The Proposition 6.4 shows that for all $\lambda \in \mathbb{C}^q$ with $\Im\lambda = \rho^A$

$$\begin{aligned}\lim_{n\to\infty} \int_{C_q^B} \varphi^{p_n}_{\lambda - i\rho(n)}(\frac{x}{\sqrt{n}}) d\nu^{(n,p_n)}(x) &= \lim_{n\to\infty} \left(\int_{C_q^B} \varphi^{p_n}_{\lambda - i\rho(n)}(\frac{x}{\sqrt{n}}) d\nu(x) \right)^n \\ &= \lim_{n\to\infty} \left(1 + \frac{t}{n} \cdot \sum_{k=1}^{q}(\lambda_k - i\rho_k^A) + o(1/n) \right)^n \\ &= e^{t \cdot \sum_{k=1}^{q}(\lambda_k - i\rho_k^A)}.\end{aligned}$$

Thus, using (6.3) we have that

$$\begin{aligned}\lim_{n\to\infty} \mathcal{F}^A(\mathbb{P}_{T(S_n^{(p_n,n^{-1/2})})})(\lambda - i\rho^A) &= \lim_{n\to\infty} \int_{C_q^B} \varphi^A_{\lambda - i\rho^A}(\ln\cosh \frac{x}{\sqrt{n}}) d\nu^{(n,p_n)}(x) \\ &= \lim_{n\to\infty} \int_{C_q^B} \varphi^{p_n}_{\lambda - i\rho(n)}(\frac{x}{\sqrt{n}}) d\nu^{(n,p_n)}(x) \\ &= e^{t \cdot \sum_{k=1}^{q}(\lambda_k - i\rho_k^A)}\end{aligned}$$

for all $\lambda \in \mathbb{C}^q$ with $\Im\lambda = \rho^A$. By making substitution $\lambda \mapsto \lambda + i\rho^A$ above, we get

$$\lim_{n\to\infty} \mathcal{F}^A(\mathbb{P}_{T(S_n^{(p_n,n^{-1/2})})})(\lambda) = e^{t \cdot \sum_{k=1}^{q} \lambda_k} \tag{6.5}$$

for all $\lambda \in \mathbb{R}^q$. On the other hand from (2.4) we can easily see that

$$\begin{aligned}e^{t \cdot \sum_{k=1}^{q} \lambda_k} &= \varphi_\lambda^A(t(1, ..., 1)) \\ &= \mathcal{F}^A(\delta_{t(1,....,1)})(\lambda)\end{aligned}$$

for all $\lambda \in \mathbb{C}^q$ with $\Im\lambda \in co(W_q^A \cdot \rho^A)$. Since, the equality (6.5) is satisfied on $\mathbb{R}^q$, i.e support of the Plancherel measure, from Levy continuity theorem for commutative hypergroups (see Theorem 4.2.11 in [BH]) it follows that $\mathbb{P}_{T(S_n^{(p_n,n^{-1/2})})}$ converges weakly to the Dirac point measure $\delta_{t(1,...,1)}$. Now, since T^{-1} is a continuous function, from continuous mapping theorem we conclude that $\mathbb{P}_{S_n^{(p_n,n^{-1/2})}}$ converges weakly to

$$T^{-1}(\delta_{t\cdot(e_1,...,e_q)}) = \delta_{\ln\left(e^t + \sqrt{e^{2t}-1}\right)\cdot(1,...,1)}$$

as desired.

□

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