Stochastic mean curvature flow

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Stochastic mean curvature flow

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Abstract

In this thesis we study a stochastically perturbed mean curvature flow (SMCF). In case of graphs, existence of weak solutions has already been established for one-dimensional and two-dimensional periodic surfaces with a spatially homogeneous perturbation. We extend this result by proving existence of solutions in arbitrary dimensions perturbed by noise which is white in time and colored in space. In addition, we work with a stronger notion of solution which corresponds to strong solutions in the PDE sense.

For this, we give a new interpretation of graphical SMCF as a degenerate variational stochastic partial differential equation (SPDE) with compact embedding. In order to infer existence of an approximating sequence, we extend the theory of variational SPDEs such that we can treat SMCF within this framework. In order to pass to the limit with the approximating sequence, we prove new a-priori bounds for graphical SMCF.

With this a-priori bounds, we can characterize the large-time behavior of solutions in case of spatially homogeneous noise. In particular, we will prove that solutions become asymptotically constant in space and behave like the driving noise in time. This strengthens a previously established one-dimensional large-time result by extending it to higher dimensions and proving stronger convergence.

Furthermore, we propose a numerical scheme for graphical SMCF which employs the variational interpretation we have analyzed before. Using this scheme, we present Monte-Carlo simulations visualizing the energy estimates we have used in the analytic part of this thesis.

Moreover, we discuss the regularity and uniqueness of solutions and give conditional results for both. We complement the previous results, especially the existence result, by investigating how they extend to SMCF with respect to an anisotropic notion of curvature.

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Contents

1. Introduction

1.1. Motivation

The evolution of surfaces plays an important role in many applications. One of the most prominent evolution laws is the *mean curvature flow* (MCF), which is the simplest geometric evolution law as it stems from the minimization of the area of a surface. A family of surfaces evolves according to mean curvature flow if its normal velocity V is equal to its mean curvature H, i.e.

$$V = H.$$

Later, we will see that in terms of a local parametrization MCF is a parabolic equation with similarities to the heat equation. When the surface at time t can be written as the graph of a function $u(\cdot, t)$ then MCF is characterized by the PDE

$$\partial_t u = \sqrt{1 + |\nabla u|^2}\nabla\cdot\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \Delta u - \frac{\nabla u \cdot \mathrm{D}^2 u \nabla u}{1 + |\nabla u|^2}.$$

MCF as a model for moving surfaces is considered among others in material science, biology and physics (cf. [TCH92, OS88, ESS92, Bel13] and the references therein) as well as recently in the theory of image processing (cf. [CDR03] and the references therein). Furthermore, it also appears as the sharp interface limit of the Allen-Cahn equation which itself is motivated by applications in material sciences, but also is related to a model from biology, cf. [Fif79].

Due to its interesting properties, for example the interpretation as the gradient flow of the area functional, MCF receives particular interest from a mathematical point of view. We refer to the foundational work of Brakke [Bra78] and the monographs [Eck04, Man11, Bel13] for an introduction to mean curvature flow and an overview of the relevant literature. It is worth mentioning that mean curvature flow in general will develop singularities, beyond which a natural extension of the evolution is not unique.

In applications mean curvature flow often only appears after assuming several simplifications, for example ignoring thermal fluctuations. In order to account for the presence of possibly unspecific additional contributions it is reasonable to perturb mean curvature flow with a random forcing ξ. This leads to so-called *stochastic mean curvature flow* (SMCF), i.e.

$$V = H + \xi.$$

1. Introduction

Stochastic mean curvature flow was first introduced in the physics community by Kawasaki and Ohta [KO82]. They consider the Allen-Cahn equation as a model for phase separation. To incorporate microscopic processes, which are neglected in the standard Allen-Cahn model, they perturb the equation with additive space-time white noise. They formally observe that in the low temperature limit, which corresponds to the sharp interface limit, the interface moves according to stochastic mean curvature flow with space-time white noise, i.e. formally ξ has zero mean and satisfies

$$\mathbb{E}\xi(x,t)\xi(y,s) = \delta(x-y)\delta(t-s).$$

This observation raised a lot of interest and was later revisited by the math community, cf. [Fun99], [Web10], [HRW12], [HW15] and the references therein. Röger and Weber proposed a multiplicative perturbation of the Allen-Cahn model [RW13, HR18], which formally also yields stochastic mean curvature flow in the limit but behaves better than the additive perturbation.

Besides its motivation from an applied point of view, SMCF is also motivated from mathematical interest in understanding the influence of small perturbations on MCF. In spirit of this motivation, a stochastic selection principle was proven by Dirr, Luckhaus and Novaga [DLN01] and independently by Souganidis and Yip [SY04]. The non-uniqueness of the MCF beyond singularities can be fixed by considering small perturbations, which force the evolution to select one of the possible solutions.

A special case is the situation where the surfaces are given as the graph of a function $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$. Under this condition SMCF is characterized by

$$\partial_t u = \sqrt{1+|\nabla u|^2}\nabla\cdot\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + \sqrt{1+|\nabla u|^2}\xi. \tag{1.1}$$

For the unperturbed MCF, i.e. $\xi = 0$, this situation was considered by Ecker and Huisken [EH89] as a first extension of the established theory for MCF of compact hypersurfaces to non-compact situations. More importantly, graphs can in fact be used to approximate the generic situation of a properly embedded hypersurface. When $u^\varepsilon = \frac{U^\varepsilon}{\varepsilon}$ is a solution of graphical SMCF and U^ε converges in a certain sense to a function $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, then formally all level sets of f evolve according to SMCF. This leads to the so-called level set formulation of SMCF. The function f solves

$$\partial_t f = |\nabla f|\nabla\cdot\left(\frac{\nabla f}{|\nabla f|}\right) + |\nabla f|\xi.$$

Note that we loose one dimension, i.e. the level sets of f are $(n-1)$-dimensional hypersurfaces whereas the graph of u^ε is a n-dimensional hypersurface. This approximation scheme was first used for unperturbed MCF by Evans and Spruck [ES91] in order to construct a level set solution.

This motivates us to study the graphical SMCF (1.1) for arbitrary dimensions in this thesis.

1.2. State of the art

Stochastic mean curvature flow gives rise to a *stochastic partial differential equation* (SPDE). For an introduction to the theory of SPDEs we refer to [DPZ14].

It is important to understand the two related but different concepts of interpreting stochastic integrals, that is Itô integration and Stratonovich integration. The Itô integral is a martingale, in particular has zero expectation. This is not true for a stochastic integral interpreted in the sense of Stratonovich, but it has the advantage that the classical chain rule holds.

The general validity of the chain rule for the Stratonovich differential makes it favorable in geometric problems, as it allows for reparametrizations without introducing additional terms. Nevertheless, it is possible to translate between the Itô and Stratonovich integral, cf. Remark 4.20. In the following we will always assume that the noise is given as $\xi = \circ\,\mathrm{d}W$ with $W = W(x,t)$ a suitable Wiener process and $\circ\,\mathrm{d}W$ denoting its Stratonovich differential, cf. Definition 4.3 and Definition 4.19.

Stochastic mean curvature flow has been treated by different authors focusing on different properties and using slightly different formulations. Among the first works concerning SMCF is a work by Yip [Yip98], who allows certain smooth anisotropies and colored white noise, i.e. noise which is white in time but smooth in space. In order to approximate SMCF he proposes a time-stepping scheme consisting of two computations in each time step. First, as in the unperturbed situation the surface is updated using the well-known ATW scheme, cf. [ATW93] and Section 2.5.3. The second step introduces the perturbation using a random flow deforming the underlying space. It is proven that this scheme in the limit $\Delta t \to 0$ is tight and formally converges to SMCF. The regularity of the limit and the rigorous identification of its evolution law is an open problem.

For spatially homogeneous noise, i.e. $W = \alpha\beta$ with $\alpha^2 < 2$ and β a real-valued Brownian motion, and an additional additive deterministic forcing term one can prove short-time existence of smooth solutions of SMCF, cf. [DLN01]. In general the existence time will be random and not bounded away from 0. This corresponds to the fact that singularities can appear arbitrarily fast, cf. Section 3.4. In case of spatially homogeneous noise one can transform the SPDE that is fulfilled by the signed distance function to a PDE with rough coefficients and apply a purely deterministic theory to infer existence. As mentioned above, by sending $\alpha \to 0$ they infer a stochastic selection principle in situations where the unperturbed evolution is not unique. In addition, it is also proven that solutions are stable under regularization of the noise.

This is related to the stochastic viscosity theory proposed by Lions and Souganidis in [LS98a, LS98b, LS00a, LS00b]. An extrinsic definition of stochastic viscosity solutions is obtained by taking the limit of regularizing the noise. It should be noted that the stochastic viscosity theory is not widely adopted as the first works were mostly announcing and a rigorous treatment is still an active field of research with relations to rough path theory, cf. [MPS19] and [Sou19]. Nevertheless, Souganidis and Yip [SY04] use this notation to analyze the existence of solutions of the level set equation with spatially homogeneous noise and also prove a stochastic selection principle.

In the case where the hypersurfaces are given as the graphs of a periodic function $u : \mathbb{T}^n \times [0,T] \to \mathbb{R}$ over the flat torus $\mathbb{T}^n$, $n \in \mathbb{N}$, the corresponding equation (1.1) is a second-order SPDE, which has non-divergence form for $n \geq 2$ and is perturbed by multiplicative non-linear gradient-dependent noise. This makes (1.1) difficult to treat. For example there is no abstract theory guaranteeing the existence of solutions in arbitrary dimensions.

Only for $n = 1$, where MCF is referred to as curve shortening flow, the main part of (1.1) can be rewritten in order to infer a divergence-form equation

$$\partial_t u = \partial_x \left(\arctan(\partial_x u)\right) + \sqrt{1 + |\partial_x u|^2}\xi. \tag{1.2}$$

This structure is exploited by Es-Sarhir and von Renesse [ESvR12] in order to prove existence of the stochastic curve shortening flow for spatially homogeneous noise. In addition, they conjecture that one can extend their proof in order to obtain an existence result for colored noise, i.e. $\xi = \circ\, \mathrm{d}W$ with $W = W(x,t)$ a Wiener process which is sufficiently regular in x. Furthermore, in the spatially homogeneous noise case they show that at large times solutions of (1.2) become spatially constant and behave like the driving Brownian motion in time.

Building on the one-dimensional analysis a finite element approximation of stochastic curve shortening flow is proposed, cf. [FLP14]. In order to derive convergence of their scheme, the authors need H^2 regularity of solutions. Since this regularity is not known for (1.2) they add an additional viscosity term to the equation and consider for $\varepsilon > 0$ and α, β as above the viscous equation

$$\mathrm{d}u = \varepsilon \partial_x^2 u + \partial_x \left(\arctan(\partial_x u)\right) \mathrm{d}t + \alpha\sqrt{1 + |\partial_x u|^2} \circ \mathrm{d}\beta.$$

After proving convergence of their scheme, they employ it to simulate the large-time behavior of solutions, consider a Monte-Carlo simulation of the Dirichlet energy $t \mapsto \mathbb{E} \int |\nabla u(x,t)|^2 \,\mathrm{d}x$, and simulate stochastic curve shortening flow with colored and space-time white noise.

In the two-dimensional situation $n = 2$ graphical SMCF (1.1) was treated by Hofmanová, Röger and von Renesse [HRvR17]. The authors exploit the gradient-flow structure of unperturbed MCF and observe that a version of the energy-dissipation inequality

$$\mathbb{E}\int\sqrt{1+|\nabla u(x,T)|^2}\,\mathrm{d}x + \frac{1}{2}\mathbb{E}\int_0^T\!\!\int\left|\nabla\cdot\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right|^2(x,t)\sqrt{1+|\nabla u(x,t)|^2}\,\mathrm{d}x\,\mathrm{d}t \\ \leq \mathbb{E}\int\sqrt{1+|\nabla u_0(x)|^2}\,\mathrm{d}x$$

for the surface area is still valid in case of stochastic perturbations. In order to approximate (1.1) they propose for $\varepsilon > 0$, $\eta > 0$ and $\alpha^2 < 2$ the higher-order viscous equation

$$\mathrm{d}u = -\eta(-\Delta)^K u + \varepsilon\Delta u + \sqrt{1+|\nabla u|^2}\nabla\cdot\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\mathrm{d}t + \alpha\sqrt{1+|\nabla u|^2}\circ\mathrm{d}\beta. \tag{1.3}$$

As the term of highest order in (1.3) is linear and elliptic and the perturbation is of lower order, existence of mild solutions for this equation follows by the well-known fixed point iteration and semigroup theory.

Using energy estimates similar to the energy-dissipation inequality they derive bounds which are sufficiently uniform in η and ε in order to first pass to the limit $\eta \to 0$ and then to $\varepsilon \to 0$. In order to identify the limit for $\varepsilon \to 0$ compensated compactness and Young measure theory are involved. They conclude the existence of martingale solutions of (1.1) in a weak sense.

1.3. Results and structure

This work continues and extends the results of [HRvR17] in several aspects. In the following, we will summarize our main results in terms of informal theorems with a focus on readability. For each of these theorems we will refer to its rigorous formulation in the subsequent chapters.

First, we generalize the existence result from $n = 2$ to arbitrary dimensions $n \geq 1$. Note that considering the graphical situation is motivated by the fact that the level set equation can be approximated by the graphical equation. Due to the construction it is necessary to solve the graphical equation in one dimension higher than the level set equation. That means that the result of [HRvR17] can only be used to approximate SMCF of curves, whereas the physical relevant situation is obtained for two-dimensional surfaces in $\mathbb{R}^3$. Furthermore, our existence result also holds for colored noise. This is even new for the one-dimensional case where it has been conjectured in [ESvR12, Remark 3.4]. In particular, we will prove the following theorem.

Theorem 1.1 (cf. Theorem 5.5)**.** *Under suitable assumptions on the initial data $u_0 \in H^1(\mathbb{T}^n)$ and the coefficients $(\varphi_l)_{l\in\mathbb{N}} \subset C^\infty(\mathbb{T}^n)$ there is a global-in-time martingale solution u of* (1.1) *where the noise is given as $\xi(x,t) = \sum_{l\in\mathbb{N}} \varphi_l(x) \circ \mathrm{d}\beta^l(t)$ with i.i.d. real-valued Brownian motions $(\beta^l)_{l\in\mathbb{N}}$.*

For the proof of Theorem 1.1 we will show new a-priori estimates for energies of the form $\mathfrak{E}(t) = \mathbb{E}\int_{\mathbb{T}^n} f(\nabla u(x,t))\,\mathrm{d}x$ with properly chosen functions f, including for example the Dirichlet energy recovered for $f(p) = |p|^2$. From these bounds we will deduce $H^2(\mathbb{T}^n)$ regularity of solutions, which allows us to give a stronger notion of solution compared to [HRvR17].

So far, uniqueness of solutions for graphical SMCF is only known in the one-dimensional case using the particular divergence form of the equation, cf. [ESvR12]. We give a stability and weak-strong uniqueness result, which implies uniqueness of solutions under certain regularity assumptions for $n \geq 1$.

Theorem 1.2 (cf. Theorem 6.1)**.** *Let u_1 and u_2 be solutions of* (1.1) *with colored noise. When u_1, u_2 are sufficiently regular then they will deviate at most exponentially in time. If they coincide at time $t = 0$ they will coincide for all times $t > 0$.*

Whether all solutions are sufficiently regular in order to apply Theorem 1.2 is still an open problem. However, we can give first regularity results under the assumption that the noise is spatially homogeneous, i.e. $\xi = \alpha \circ \mathrm{d}\beta$ with $\alpha^2 < 2$. The energy estimates imply a bound for the Lipschitz constant of solutions.

Theorem 1.3 (cf. Theorem 7.3)**.** *For Lipschitz-continuous initial data u_0 a solution u of* (1.1) *with spatially homogeneous noise is Lipschitz continuous for all times and the Lipschitz constant does not increase over time.*

Besides its importance for the regularity theory of solutions, Theorem 1.3 can also be used to characterize the large-time behavior of solutions.

Theorem 1.4 (cf. Theorem 7.5)**.** *For Lipschitz-continuous initial data u_0 a solution u of* (1.1) *at large times becomes spatially homogeneous and behaves like the driving Brownian motion in time.*

Theorem 1.4 generalizes a similar result from the one-dimensional case to higher dimensions and improves the convergence in terms of a stronger topology. For smooth solutions with small Lipschitz constant we will also prove a temporal decay of higher derivatives.

Moreover, it turns out that our energy estimates are very robust and allow to extend the existence and large-time results to graphical SMCF with respect to smooth anisotropies with spatially homogeneous noise.

In order to prove the above theorems we use methods differing from those used in [HRvR17]. In particular, we will use a new interpretation of graphical SMCF

(1.1) as a (degenerate) variational SPDE. To our knowledge this interpretation is new even for the unperturbed model.

Variational SPDEs form a special class of SPDEs and correspond to what is known as weak solution theory for deterministic PDEs.

The theory of variational SPDEs was initiated by the pioneering works of Pardoux [Par75] and Viot [Vio76]. They develop in their respective theses two different theories for variational SPDEs, one relying on the monotonicity of the equation and the other on the compactness of a certain embedding. The approach of Pardoux yields the existence and uniqueness of strong solutions, whereas the theory of Viot only implies the existence of martingale solutions, i.e. solutions which are weak in a stochastic sense, cf. Definition 4.32. In both situations the existence proof relies on a generalized Itô formula adapted to the variational structure of the equation.

The approach of Pardoux was constantly continued [Par79, Par87, MV88] and later reviewed in the book by Prévôt and Röckner [PR07]. Viot's theory attracted less attention as it was mentioned in [MV88], which is besides [Par07] one of few works referring to it.

Nevertheless, in this work we will revisit Viot's theory of variational SPDEs with compact embedding and extend it such that SMCF in case of graphs can be treated within this theory. For a detailed discussion about the differences between our generalization and the results of [Vio76] and [Par07] we refer to Section 4.2 and Section 4.3.

In order to understand (1.1) as a variational SPDE it is convenient to consider the equation the gradient of a solution fulfills

$$\mathrm{d}\nabla u = \nabla\left(\sqrt{1+|\nabla u|^2}\nabla\cdot\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\mathrm{d}t + \sqrt{1+|\nabla u|^2}\circ \mathrm{d}W\right).$$

This equation naturally has divergence-form and therefore can be understood as an equation in a dual space, which corresponds to testing the equation with sufficiently regular functions and using integration by parts. That means, that it can be treated as a variational SPDE. It also is (degenerate) coercive, since the coercivity is equivalent to the availability of an estimate for the Dirichlet energy of solutions, which we have mentioned before. As the drift lacks monotonicity we are not in the situation of the variational theory of Pardoux [Par75] respectively Prévôt and Röckner [PR07]. Instead we have to work in the spirit of Viot's thesis [Vio76] and exploit the compactness of the embedding $H^2(\mathbb{T}^n) \subset H^1(\mathbb{T}^n)$.

A key ingredient of the variational approach is the approximation using a Galerkin scheme. We propose a numerical scheme for graphical SMCF with colored noise which mimics this discretization. As we are working in the Gelfand triple $H^2(\mathbb{T}^n) \subset H^1(\mathbb{T}^n) \simeq (H^1(\mathbb{T}^n))' \subset (H^2(\mathbb{T}^n))'$ this algorithm uses $H^2(\mathbb{T}^n)$-compatible finite elements, which makes it different from previously considered numerical schemes for MCF and SMCF.

The subsequent chapters are organized as follows. We start with two introductory chapters. In Chapter 2 we quickly introduce the unperturbed mean curvature flow with some fundamental results, focusing on those results that motivate the analysis in this work. Chapter 3 is devoted to a formal deduction of SMCF, the corresponding level set equation and the graphical equation. Furthermore, we derive evolution laws of certain differential geometric objects and give a first example how to derive energy bounds.

In Chapter 4 we fix the notation for infinite dimensional stochastic evolution equations and give a precise definition of Stratonovich integration in comparison with Itô integration. Furthermore we extend the results of [Par75] and [Vio76] about variational SPDEs. We apply this theory in Chapter 5 to prove Theorem 1.1, i.e. the existence of a martingale solution of (1.1) in the case of colored white noise. For this, we first consider a viscous approximation, deduce existence from an abstract existence result for variational SPDEs, prove uniform bounds for L^p norms of the gradient of solutions and finally pass to the limit with the viscous term. The uniform bounds allow to identify the limit as a solution of (1.1).

Chapter 6 is devoted to the proof of a precise formulation of the weak-strong uniqueness result Theorem 1.2 for the solutions we have constructed before. It is an open question whether solutions are always regular enough in order to deduce pathwise uniqueness.

In Chapter 7 we concentrate on the case of spatially homogeneous noise. In this situation we will prove a maximum principle for the gradient of solutions implying Theorem 1.3, which is not reasonable for colored noise. As a consequence we can prove the large time result Theorem 1.4. We also include the proof of a first result into the direction of higher regularity of solutions. In particular we can prove that the L^2 norm of higher derivatives decays in time under the assumption that the Lipschitz constant of the initial data is small and the solution is smooth.

In Chapter 8 we describe the above mentioned discretization of graphical SMCF which is motivated by the variational interpretation of (1.1). This algorithm is used for Monte-Carlo simulations of the various energies that have been considered in the existence proof.

We end with Chapter 9 where the existence proof from Chapter 5 is extended to smooth anisotropic geometries, with some shortcuts which are available due to the restriction to spatially homogeneous noise.

Note that some of the results presented in this work already have been published in a recent preprint [DHR19]. The introduction to variational SPDEs and the Itô formula from Section 4.2 are only slightly changed in comparison to the preprint, for example we give a more general formulation of the Itô formula as we allow Wiener processes with arbitrary covariance operator. The abstract existence theorem for variational SPDEs in Section 4.3 is also part of that preprint. Here, we deduce the existence theorem from a new abstract convergence result for

SPDEs and give a more elaborated version of the proof.

The preprint [DHR19] is only concerned with spatially homogeneous noise. The existence result for SMCF with colored noise in Chapter 5 is in the spirit of the proof for spatially homogeneous noise presented in the preprint, but in order to account for the more general situation where an a-priori bound for the Lipschitz constant is not available, we have to employ a new strategy to find suitable a-priori bounds for a solution. As a consequence, the existence result presented here is more general even in case of spatially homogeneous noise as the assumption on the initial data is less restrictive.

The maximum principle for the gradient as well as the large-time behavior of solutions in case of spatially homogeneous noise in Chapter 7 already appeared in [DHR19]. Here, we include a novel result on higher regularity of solutions.

2. Mean curvature flow

In this chapter we will review the theory of unperturbed mean curvature flow with particular focus on those aspects that motivate the approach we take to analyze stochastic mean curvature flow. The material in this chapter is mostly taken from the monographs [Bel13], [Eck04] and [Sin10].

Let $(\mathcal{M}_t)_{t\in[0,T]}$, $T > 0$, be a family of smooth properly embedded hypersurfaces in $\mathbb{R}^{n+1}$ given by smooth immersions $F_t : \mathcal{M} \to \mathcal{M}_t$ with respect to a n-dimensional smooth manifold $\mathcal{M}$ without boundary. For the differential geometry of hypersurfaces we refer to Appendix B.

We say that the family $(\mathcal{M}_t)$ evolves according to mean curvature flow when the normal velocity is equal to the mean curvature. This can be expressed in terms of a differential equation for the immersions

$$\partial_t F_t(x) = \vec{H}_t(x)\ \forall x \in \mathcal{M},\ 0 < t < T, \tag{2.1}$$

where $\vec{H}_t(x)$ is the mean curvature vector of $\mathcal{M}_t$ at $F_t(x)$.

In the following, we will denote by $\nabla_{\mathcal{M}_t}, \nabla^{\mathcal{M}_t}\cdot$ and $\Delta_{\mathcal{M}_t}$ the tangential gradient, the tangential divergence and the Laplace-Beltrami operator, respectively, with respect to the differential structure of $\mathcal{M}_t$. For a definition of these objects we again refer to Appendix B. The corresponding objects with respect to the Euclidean structure will be denoted without sub- and superscripts.

Since $\Delta_{\mathcal{M}_t} F_t = \vec{H}_t$ we can rewrite (2.1) as $\partial_t F_t(x) = \Delta_{\mathcal{M}_t} F_t(x)$ and thus emphasize the similarity to the heat equation. Note that the appearance of $\Delta_{\mathcal{M}_t}$ makes this equation non-linear, but one can indeed show that (2.1) is a parabolic equation. Therefore, under suitable assumptions on the initial manifold, one can deduce short-time existence of solutions, cf. [Sin10, Theorem 4.1], [EH91, Proposition 4.1] and the references therein.

Theorem 2.1. *Let $\mathcal{M}$ be a smooth and compact n-dimensional manifold without boundary and F_0 a smooth immersion. Then* (2.1) *has a unique smooth short-time solution with initial condition $\mathcal{M}_0 = F_0(\mathcal{M})$.*

In [EH91, Proposition 4.2] it is proven that Theorem 2.1 extends to non-compact complete manifolds with a uniform local Lipschitz condition and bounded curvature.

For a convex and compact hypersurface the mean curvature vector is pointing inwards, hence mean curvature flow forces such a surface to shrink. The easiest example is given by shrinking spheres.

Example 2.2. We are interested in the evolution by MCF of a n-dimensional sphere with radius $r_0 > 0$. As reference manifold we choose the unit sphere $\mathcal{M} = \mathbb{S}^n \subset \mathbb{R}^{n+1}$ together with the immersion $F_0(x) \coloneqq r_0 x$, $x \in \mathcal{M}$, and set $\mathcal{M}_0 \coloneqq F_0(\mathcal{M}) = r_0\mathbb{S}^n$.

Since solutions of the mean curvature flow are unique and the evolution is invariant under rotations, we infer that by evolving the sphere we only change its radius. Hence, the unique solution of (2.1) can be written as $F_t(x) = r(t)x$ with $r(0) = r_0$ and $\mathcal{M}_t = r(t)\mathbb{S}^n$. To determine $r(t)$ we note that

$$\vec{H}_t(x) = -\frac{n}{r(t)}x, \ x \in \mathbb{S}^n.$$

Plugging this into (2.1) yields

$$r'(t) = -\frac{n}{r(t)}$$

which has the unique solution $r(t) = \sqrt{r_0^2 - 2nt}$.

Therefore the sphere shrinks to a point at time $T = \frac{r_0^2}{2n}$.

It was proven by Huisken in [Hui84] that this behavior is shared among all uniformly convex initial hypersurfaces.

Theorem 2.3 ([Hui84, Theorem 1.1])**.** *Let $\mathcal{M}_0$ be a smooth, compact, uniformly convex, properly embedded n-dimensional hypersurface in $\mathbb{R}^{n+1}$. Then the maximal existence time $T > 0$ for solutions of* (2.1) *is finite and $\mathcal{M}_t$ converges to a single point for $t \to T$.*

For non-convex n-dimensional surfaces with $n \geq 2$ different singularities might appear. A prominent example is given by two-dimensional dumbbell-shaped surfaces, which we will construct in the next example following the presentation of [Eck04].

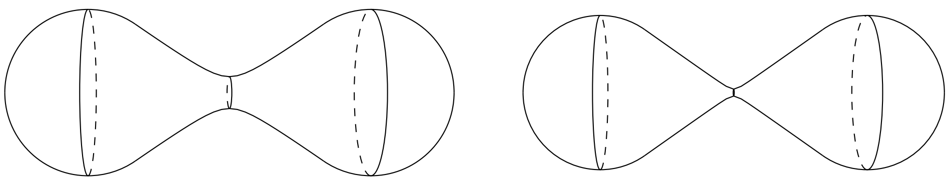

Figure 2.1.: Mean curvature flow of a dumbbell-shaped surface.

Example 2.4. For notational convenience we will write points $x \in \mathbb{R}^3$ as $x = (\hat{x}, z) \in \mathbb{R}^2 \times \mathbb{R}$. Let the hypersurface $\mathcal{M}_0 \subset \mathbb{R}^3$ be contained in a hyperboloid and contain two disjoint spheres in its interior, i.e.

$$\mathcal{M}_0 \subset \{x = (\hat{x}, z) \mid \gamma z^2 \geq |\hat{x}|^2 - \varepsilon^2\} \cap \{x \mid |x - x_0|^2 \geq r^2\} \cap \{x \mid |x + x_0|^2 \geq r^2\}$$

with parameters $\varepsilon > 0$, $\gamma \in (0,1)$, $x_0 = (0, z_0) \in \mathbb{R}^3$, $r > 0$. For a MCF $(\mathcal{M}_t)_t$ starting from $\mathcal{M}_0$ one can prove that a rescaled inclusion holds for $t > 0$

$$\mathcal{M}_t \subset \{x = (\hat{x}, z) \mid \gamma z^2 \geq |\hat{x}|^2 - \varepsilon^2 + 2(1-\gamma)t\} \cap \{x \mid |x \pm x_0|^2 \geq r^2 - 4t\}.$$

Note that the hyperboloid as well as the spheres shrink. An example is shown in Figure 2.1. For properly chosen parameters, the strongly curved neck dominates the evolution such that a pinch off occurs before the inscribed spheres shrink to a point. Beyond this singularity, a parametrization of the surface with respect to the initial surface is not possible.

2.1. Graphical case

A particular interesting situation is the case where the initial hypersurface can be written as the graph of a function u over some hyperplane in $\mathbb{R}^{n+1}$, which we without loss of generality always assume to be $\mathbb{R}^n \times \{0\}$. The mean curvature flow of the graph of u corresponds to the following partial differential equation for u

$$\partial_t u = \sqrt{1 + |\nabla u|^2}\nabla \cdot \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \Delta u - \frac{\nabla u \cdot \mathrm{D}^2 u \nabla u}{1 + |\nabla u|^2}. \tag{2.2}$$

For the derivation of this equation we refer to Section 3.3, where we consider the more general perturbed mean curvature flow, and Remark B.9, where the differential geometry of graphs is introduced.

The evolution of a graph is studied in [EH89]. A crucial argument in their analysis is the fact that the Lipschitz constant of u is preserved during the evolution. To prove this, they use a non-compact maximum principle which we will describe in Section 2.3. The approach we take in the subsequent chapters is similar. We refer to Example 3.12 for more details about the gradient bound.

Once this bound is established one can prove uniform estimates for the second fundamental form and its derivatives. Note that short-time existence for (2.2) follows with standard methods for parabolic equations. The uniform estimates allow one to extend the solution for arbitrary times and therefore one can infer the following existence result.

Theorem 2.5 ([EH89, Theorem 4.6])**.** *Let $\mathcal{M}_0$ be the graph of a Lipschitz-continuous function $u : \mathbb{R}^n \to \mathbb{R}$. Then there is a global-in-time solution $(\mathcal{M}_t)_{t>0}$ of* (2.1) *which is smooth for $t > 0$.*

Furthermore, one can characterize the large-time behavior of the solution $\mathcal{M}_t$: When the height of $\mathcal{M}_t$ stays bounded then $\mathcal{M}_t$ converges to a plane. Otherwise, after properly rescaling the manifolds, one can prove that asymptotically $\mathcal{M}_t$ behaves like an expanding selfsimilar solution of (2.1).

2.2. Gradient flow structure of mean curvature flow

An interesting fact about mean curvature flow is its formal interpretation as the gradient flow of the area functional. To recover this we first consider the first variation of certain integrals over a manifold $\mathcal{M}$.

Remark 2.6 (First variation of surface integral)**.** Let $\mathcal{M}$ be a smooth properly embedded hypersurface and $\Psi_\lambda \in C^\infty(\mathbb{R}^{n+1};\mathbb{R}^{n+1})$ diffeomorphisms for $\lambda > 0$ and $\Psi_0 = \mathrm{Id}$ such that $(\Psi_\lambda)_{\lambda \geq 0}$ is sufficiently smooth in λ. Furthermore, we define $X \coloneqq \partial_\lambda\Big|_{\lambda=0} \Psi_\lambda$ and $\mathcal{M}_\lambda \coloneqq \Psi_\lambda(\mathcal{M})$. For $f \in C_c^\infty(\mathbb{R}^{n+1})$ it holds that

$$\partial_\lambda\Big|_{\lambda=0} \int_{\mathcal{M}_\lambda} f = \int_{\mathcal{M}} f\nabla^{\mathcal{M}} \cdot X + \nabla f \cdot X = \int_{\mathcal{M}} \nabla^\perp f \cdot X - fX \cdot \vec{H},$$

with $\nabla^\perp f$ denoting the normal projection of the gradient of f, i.e. $\nabla^\perp f = \nabla f \cdot \nu\nu$.

Note that Remark 2.6 implies for a compact $\mathcal{M}$ and Ψ_λ as above that

$$\partial_\lambda\Big|_{\lambda=0} |\mathcal{M}_\lambda| = -\int_{\mathcal{M}} X \cdot \vec{H}.$$

Hence, after formally introducing the pseudo Riemannian structure on the space of all hypersurfaces with the tangent space at a fixed hypersurface consisting of all normal velocity fields together with the L^2 inner product as Riemannian metric, we infer that $-\vec{H}$ is the gradient of the area functional $\mathcal{M} \mapsto |\mathcal{M}|$ with respect to this particular Riemannian structure. In particular (2.1) is the gradient flow with respect to the area functional.

Since the induced metric on the space of all hypersurfaces is identically 0, cf. [MM06], this does not allow us to directly apply the theory of gradient flows to the mean curvature flow. Nevertheless, this interpretation motivates several results about mean curvature flow. For example the interpretation as a gradient flow yields the energy dissipation equality

$$\partial_t |\mathcal{M}_t| = -\int_{\mathcal{M}_t} |H|^2$$

which indeed can be verified for solutions of (2.1). During the next section there will be more results that exploit the formal gradient flow structure.

2.3. Monotonicity formula and estimates

Another important consequence of Remark 2.6 is Huisken's (weighted) monotonicity formula, c.f. [Eck04, Theorem 4.13].

Remark 2.7 (Weighted monotonicity formula)**.** Let $\Phi : \mathbb{R}^{n+1} \times (-\infty, 0) \to \mathbb{R}$,

$$\Phi(x,t) = \frac{1}{(-4\pi t)^{\frac{n}{2}}} \exp\left(\frac{|x|^2}{4t}\right)$$

be the n-dimensional backward heat kernel in $\mathbb{R}^{n+1}$ and $\Phi_{(x_0,t_0)}(x,t) \coloneqq \Phi(x - x_0, t - t_0)$ for $x_0 \in \mathbb{R}^{n+1}$ and $t_0 > 0$.

Furthermore, let $(\mathcal{M}_t)$ be a solution of (2.1) and $f : \mathbb{R}^{n+1} \times [0,\infty) \to \mathbb{R}$ be a sufficiently smooth function such that all the integrals involved are finite. Then it holds for all $t < t_0$ that

$$\begin{aligned} \partial_t \int_{\mathcal{M}_t} f\Phi_{(x_0,t_0)} &= \int_{\mathcal{M}_t} \left(\partial_t f + \vec{H} \cdot \nabla f - \Delta_{\mathcal{M}_t} f - \left| \vec{H} - \frac{\nabla^\perp \Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}} \right|^2 f \right) \Phi_{(x_0,t_0)} \\ &= \int_{\mathcal{M}_t} \left(\tfrac{\mathrm{d}}{\mathrm{d}t} f - \Delta_{\mathcal{M}_t} f - \left| \vec{H} - \frac{\nabla^\perp \Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}} \right|^2 f \right) \Phi_{(x_0,t_0)}, \end{aligned} \tag{2.3}$$

where we use the notation

$$\tfrac{\mathrm{d}}{\mathrm{d}t} f \coloneqq \tfrac{\mathrm{d}}{\mathrm{d}t} \left(f(F_t(x), t) \right) = \partial_t f + \vec{H} \cdot \nabla f.$$

The monotonicity formula (2.3) for $f = 1$ was first proven by Huisken in [Hui90]. He applied it to complement Theorem 2.3 and Example 2.4 by characterizing the singularities that might appear due to mean curvature flow for non-convex initial data.

The weighted monotonicity formula with arbitrary f can be used to derive a non-compact maximum principle for functions satisfying $\frac{\mathrm{d}}{\mathrm{d}t} f - \Delta_{\mathcal{M}_t} f \leq 0$, cf. [Eck04, Proposition 4.27]. We refer to Example 3.12 for more details.

2.4. Weak solutions

We have already seen in Theorem 2.3 and Example 2.4 that long-time existence of classical solutions of (2.1) can not be expected in the general case. As soon as singularities appear and the topology of the hypersurface changes there is no chance to have a classical solution. Nevertheless, there are several natural weak formulations of (2.1) trying to extend the notion of solution beyond singularities. As we only give a few examples we refer to [Bel13] for a more exhaustive overview over these theories.

One of these approaches was introduced in [Bra78] by Brakke. He considers the mean curvature flow in the setting of varifolds, cf. [Alm66]. This notion of solutions lacks uniqueness, but one can prove long-time existence and give partial regularity results. We refer to the recent book [Ton19] of Tonegawa which gives an introduction to the theory of Brakke's mean curvature flow.

2.4.1. Level set method

The level set method models the hypersurfaces $\mathcal{M}_t$, $t \geq 0$, as level sets of a time-dependent function $f = f(x,t)$. With the notation from Remark B.10 we infer that all level sets of f evolve according to mean curvature flow if f solves the partial differential equation

$$\partial_t f = |\nabla f| \nabla \cdot \left(\frac{\nabla f}{|\nabla f|} \right) = \Delta f - \frac{\nabla f \cdot \mathrm{D}^2 f \nabla f}{|\nabla f|^2}. \tag{2.4}$$

A detailed derivation of this equation can be found in Section 3.2.

Equation (2.4) is degenerate parabolic and not well-defined for $\nabla f = 0$. Nevertheless, in [CGG91] and at the same time in [ES91] this equation was treated within the framework of viscosity theory yielding a unique viscosity solution of (2.4). The downside of this notion of solution is the fact that the level sets of f can develop an non-empty interior and therefore loose their interpretation as surfaces. This corresponds to the non-uniqueness in the Brakke sense. For the level set method this leads to so-called *fattening* and was already described in [ES91].

In order to prove existence for (2.4) the equation is approximated in [ES91] for $\varepsilon > 0$ by the regularized equation

$$\partial_t f = \Delta f - \frac{\nabla f \cdot \mathrm{D}^2 f \nabla f}{\varepsilon^2 + |\nabla f|^2}. \tag{2.5}$$

Note that (2.5) can be rewritten into the graphical equation (2.2) by considering $u = \frac{1}{\varepsilon} f$. In order to pass to the limit $\varepsilon \to 0$ one has to infer estimates which are uniform in ε motivating a detailed study of the graphical case.

2.4.2. Diffuse interface approximation

In binary phase separation processes, MCF is often used to describe the motion of a sharp interface separating two coexisting phases. A different approach is to model the evolution not by a sharp interface but instead by a diffuse interface described by a smooth function $u : \mathbb{R}^{n+1} \to [-1,1]$ where values close to ± 1 model two different phases with a rapid change between these two values at the interface between the phases. The thickness of the interface is controlled by a small parameter $\varepsilon > 0$.

In this context the perimeter functional is approximated by the Van der Waals-Cahn-Hilliard energy

$$P_\varepsilon(u) := \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)$$

with a suitable double-well potential $W : \mathbb{R} \to \mathbb{R}$, for which $W(z) = \frac{1}{4}(1 - z^2)^2$ is a typical choice. It is a famous result of Modica and Mortola [MM77], [Mod87] that P_ε converges to a constant times the perimeter functional, where the notion of Γ-convergence with respect to the $L^1(\Omega)$-topology is used.

We have seen in Section 2.2 that the mean curvature flow is the formal L^2-gradient flow of the perimeter functional. Formally, a diffuse approximation of mean curvature flow can therefore be found by considering the L^2-gradient flow of the diffuse approximations P_ε of the perimeter functional. After properly rescaling in time we infer the Allen-Cahn equation

$$\varepsilon \partial_t u = \varepsilon \Delta u + \frac{1}{\varepsilon} W'(u). \tag{2.6}$$

This equation allows to handle diffuse interfaces with topological changes. When u^ε is a solution of (2.6) it is reasonable to think of the zero level set of the sharp interface limit $\lim_{\varepsilon \to 0} u^\varepsilon$ as a weak solution of mean curvature flow. For short times and with smooth initial data this notion of solution coincides with the classical mean curvature flow, cf. [dMS95]. When continuing the evolution beyond singularities it is proven in [Ilm93] and [ESS92] that under suitable assumptions solutions with respect to Brakke's notion and the level set notion can be recovered as sharp interface limits.

For more details we refer to [Bel13] and [Gar13].

2.5. Numerical schemes

Up to the time the first singularity appears, MCF (2.1) can be expressed as a parabolic PDE for the parametrization on a reference manifold $\mathcal{M}$. For this interpretation, a finite element scheme is described in [DDE05].

In the following we will focus on numerical schemes that simulate MCF beyond the first singularity.

2.5.1. Finite element discretization of level set equation by Deckelnick and Dziuk

In [DD03] a finite element discretization for the graphical case (2.2) and the level set equation (2.4) is proposed.

Starting with a bounded domain $\Omega \subset \mathbb{R}^n$ equation (2.2) is enriched with Dirichlet boundary conditions, which we for simplicity assume to be 0. (2.2) can equivalently be written as

$$\int_\Omega \frac{\partial_t u}{\sqrt{1+|\nabla u|^2}}\phi + \frac{\nabla u \cdot \nabla \phi}{\sqrt{1+|\nabla u|^2}} = 0 \ \forall \phi \in H_0^1(\Omega). \tag{2.7}$$

An important observation is the fact that the dissipation equality for the area is still encoded in (2.7): by choosing $\phi = \partial_t u$ we infer

$$\int_\Omega \frac{|\partial_t u|^2}{\sqrt{1+|\nabla u|^2}} + \partial_t \int_\Omega \sqrt{1+|\nabla u|^2} = 0.$$

Using piecewise polynomial finite elements in (2.7) leads to a semi-discrete problem which corresponds to a finite-dimensional nonlinear differential equation. The fact that the dissipation equality is true for the semi-discrete model implies long-time existence of solutions. By employing a semi-implicit time discretization a fully-discretized problem is deduced, for which convergence can be proven.

Once the graphical case is studied, an algorithm for the level set equation (2.4) is proposed which uses the regularization described in (2.5). For this algorithm convergence is proven in [DD03] for $n = 2$.

2.5.2. BMO scheme

A different algorithm was proposed by Bence, Merriman and Osher (BMO) in [MBO92], which is particularly easy to implement. Starting with $\mathcal{M}_0$ given as the boundary of a smooth set $E_0 \subset \mathbb{R}^{n+1}$, i.e. $\mathcal{M}_0 = \partial E_0$, and a time step size $\tau > 0$ one inductively constructs the approximating sequence $(\mathcal{M}_t^\tau)_{t\geq 0}$ through $\mathcal{M}_{k\tau}^\tau := \partial E_{k\tau}^\tau$ with $E_{k\tau}^\tau = \left\{u(\cdot,\tau) \geq \frac{1}{2}\right\}$ and u solving the heat equation

$$\begin{aligned} \partial_t u &= \Delta u \text{ in } \mathbb{R}^{n+1} \times (0,\infty) \\ u\Big|_{t=0} &= \chi_{E^\tau_{(k-1)\tau}} \text{ in } \mathbb{R}^{n+1} \end{aligned}$$

for $k \in \mathbb{N}$ with $E_0^\tau := E_0$ and constant interpolation in between.

In [MBO92] there is a heuristic argument why the diffusion of the characteristic function of a set on a small time scale behaves like the mean curvature flow. The rigorous convergence in the sense of viscosity solutions is analyzed in [Eva93]. In [LO19] it is shown that the BMO algorithm can be understood as a minimizing movement scheme in the sense of De Giorgi. Furthermore, a conditional convergence result is presented.

2.5.3. ATW algorithm

Another algorithm emphasizing the gradient flow character of mean curvature flow was described by Almgren, Taylor and Wang in [ATW93]. Again, with a time step size $\tau > 0$ and an initial hypersurface $\mathcal{M}_0 = \partial E_0$ it constructs a piecewise constant sequence $\mathcal{M}_t^\tau = \partial E_t^\tau$ by iteratively minimizing

$$E \mapsto |\partial E| + \frac{1}{\tau} \int_{E \Delta E_{t-h}^\tau} \operatorname{dist}(\cdot, \partial E_{t-h}^\tau).$$

In [LS95] a conditional convergence result for this scheme is proven.

2.5.4. Diffuse approximation

The diffuse approximation of MCF through the Allen-Cahn equation (2.6) can be used to implement numerical schemes to approximate MCF. For fixed thickness of the interface $\varepsilon > 0$ the Allen-Cahn equation is a parabolic equation of divergence form. The weak formulation of (2.6)

$$\int_\Omega \partial_t u \varphi + \int_\Omega \nabla u \cdot \nabla \varphi = \frac{1}{\varepsilon^2} \int_\Omega W'(u) \varphi \ \forall \varphi \in C_c^\infty(\Omega)$$

together with initial and boundary values for u allows for a natural finite element discretization, cf. [FP03, DDE05] and the references therein. Using arguments adapted to the special structure of the problem one can prove finite element error bounds of polynomial order in $\frac{1}{\varepsilon}$ and convergence of the scheme for fixed $\varepsilon > 0$ to the Allen-Cahn equation as well as for $\varepsilon \to 0$ convergence of the zero level set to MCF.

3. Derivation of SMCF and formal considerations

3.1. Parameterized SMCF

Similar to Chapter 2 we consider an evolution $(\mathcal{M}_t)_{t\in[0,T]}$, $T > 0$, of smooth properly embedded hypersurfaces in $\mathbb{R}^{n+1}$ given by smooth immersions $F_t : \mathcal{M} \to \mathcal{M}_t \subset \mathbb{R}^{n+1}$ with respect to a n-dimensional smooth manifold $\mathcal{M}$ without boundary.

We say that the family $(\mathcal{M}_t)_t$ evolves according to stochastic mean curvature flow if the immersions formally fulfill the stochastic differential equation

$$\mathrm{d}F_t(x) = \vec{H}_t(x)\,\mathrm{d}t + \sum_l \varphi_l(F_t(x))\nu_t(x) \circ \mathrm{d}\beta_t^l, \tag{3.1}$$

where $\vec{H}_t(x)$ is the mean curvature vector of $\mathcal{M}_t$ at $F_t(x)$ and $\nu_t(x)$ is a unit normal on $\mathcal{M}_t$ at $F_t(x)$. Furthermore $(W_t)_{t\geq 0} \coloneqq (\sum_{l\in\mathbb{N}} \varphi_l \beta_t^l)_{t\geq 0}$ denotes an infinite-dimensional Wiener process with $(\beta_t^l)_{t\geq 0}$ being independent real-valued Brownian motions and $\varphi_l : \mathbb{R}^{n+1} \to \mathbb{R}$ space-dependent functions, which we will specify later. The integration $\circ\,\mathrm{d}\beta_t^l$ should formally be understood as a Stratonovich integral. To abbreviate the notation we will also write $\circ\,\mathrm{d}W$ for the Stratonovich integral against $\sum_l \varphi_l \beta_t^l$ when there is no ambiguity for the argument of φ_l. In this chapter we will stay at a formal level, while rigorous arguments will follow in the subsequent chapters.

3.1.1. Formal derivation of evolution laws for geometric quantities

We would like to derive stochastic differential equations describing the evolution of geometric quantities in the situation where the immersions fulfills (3.1). We will use the fact that formally the chain rule applies to SPDEs with Stratonovich noise. The approach we take here is similar to the one in [Eck04, Appendix B] for deterministic MCF.

Let $\tilde{F} : \Omega \to \mathcal{M}$ be a smooth local parametrization of $\mathcal{M}$ for some open set $\Omega \subset \mathbb{R}^n$. Then, on a possibly smaller open set which we again denote by Ω, $F_t \circ \tilde{F} : \Omega \to \mathcal{M}_t$ is a smooth local parametrization of $\mathcal{M}_t$. Since the following arguments are purely local we can identify $F_t \circ \tilde{F}$ with F_t and think of it as a map

$F_t : \Omega \to \mathcal{M}_t$. Equation (3.1) still holds for F_t. In the following we will suppress the dependency of F_t and the geometric quantities on the time parameter t and simply write F for the parametrization.

For the notation we refer to Appendix B.

Remark 3.1 (Evolution of the metric)**.** We will start by deriving the evolution law of the metric g. Using $\nu \cdot \partial_j F = 0$ we infer

$$\begin{aligned} \mathrm{d}g_{ij} &= 2\partial_i \left(\circ \, \mathrm{d}F\right) \cdot \partial_j F \\ &= 2H\partial_i \nu \cdot \partial_j F \, \mathrm{d}t + 2\partial_i \nu \cdot \partial_j F \circ \mathrm{d}W \\ &= -2HA_{ij} \, \mathrm{d}t - 2A_{ij} \circ \mathrm{d}W, \end{aligned}$$

where H and A denote the scalar mean curvature and the second fundamental form, respectively.

For the inverse metric it holds that

$$\begin{aligned} \mathrm{d}g^{ij} &= -g^{-1}(\circ \, \mathrm{d}g)g^{-1} \\ &= 2HA^{ij} \, \mathrm{d}t + 2A^{ij} \circ \mathrm{d}W. \end{aligned}$$

Remark 3.2 (Evolution of the area element)**.** For the area element $\sqrt{g}$ we calculate

$$\begin{aligned} \mathrm{d}\sqrt{g} &= \frac{\sqrt{g}}{2} \operatorname{tr}\left(g^{-1} \circ \mathrm{d}g\right) \\ &= -\sqrt{g}H^2 \, \mathrm{d}t - \sqrt{g}H \circ \mathrm{d}W. \end{aligned}$$

Remark 3.3 (Evolution of the unit normal field)**.** For the unit normal field ν we have $\nu \cdot \partial_i F = 0$ for all i. Hence,

$$\begin{aligned} \circ \, \mathrm{d}\nu \cdot \partial_i F &= -\nu \cdot \partial_i \left(\circ \, \mathrm{d}F\right) \\ &= -\partial_i H \, \mathrm{d}t - \sum_l \partial_i \varphi_l \circ \mathrm{d}\beta^l \\ &= -\partial_i H \, \mathrm{d}t - \circ \, \mathrm{d}\partial_i W. \end{aligned}$$

Furthermore $|\nu| = 1$, hence $0 = \mathrm{d}|\nu|^2 = \nu \cdot \circ \, \mathrm{d}\nu$. Thus $\mathrm{d}\nu$ is a tangent vector, which is completely determined by $\circ \, \mathrm{d}\nu \cdot \partial_i F$:

$$\begin{aligned} \mathrm{d}\nu &= -\nabla_{\mathcal{M}_t} H \, \mathrm{d}t - \sum_l \nabla_{\mathcal{M}_t} \varphi_l \circ \mathrm{d}\beta^l \\ &= -\nabla_{\mathcal{M}_t} H \, \mathrm{d}t - \circ \, \mathrm{d}\nabla_{\mathcal{M}_t} W. \end{aligned}$$

Remark 3.4 (Rewriting the SMCF into Itô formulation)**.** Once we have derived the evolution law for the unit normal field ν we can rewrite (3.1) into its Itô formulation. For this, we need to compute the Itô-Stratonovich correction term, which we will introduce in detail in Definition 4.19. In Remark 4.20 we give a general rule how to find the correction term.

In the situation of (3.1) we need to compute the evolution law of $\varphi_l(F(x))\nu(x)$. In particular, we are only interested in that part of the evolution law that is given as a stochastic integral with respect to β^l. For a more detailed explanation we refer to Example 4.21, where we will revisit this particular example.

Using the classical chain rule, we find

$$\begin{aligned}
\mathrm{d}(\varphi_l(F(x))\nu(x)) &= \varphi_l \circ \mathrm{d}\nu + (\nabla\varphi_l \cdot (\circ\, \mathrm{d}F))\, \nu \\
&= \ldots \mathrm{d}t + \sum_{k\neq l} \ldots \circ \mathrm{d}\beta^k - \varphi_l \nabla_{\mathcal{M}_t}\varphi_l + \varphi_l \nabla\varphi_l \cdot \nu\nu \circ \mathrm{d}\beta^l \\
&= \ldots \mathrm{d}t + \sum_{k\neq l} \ldots \circ \mathrm{d}\beta^k - \varphi_l\, (\nabla\varphi_l)^T + \varphi_l\, (\nabla\varphi_l)^\perp \circ \mathrm{d}\beta^l,
\end{aligned}$$

where $(\nabla\varphi_l)^T$ is a different notation for the tangential gradient of φ_l and $(\nabla\varphi_l)^\perp$ denotes the normal projection of the gradient of φ_l.

This implies

$$\mathrm{d}F = \vec{H} - \frac{1}{2}\sum_l \left(\varphi_l\, (\nabla\varphi_l)^T - \varphi_l\, (\nabla\varphi_l)^\perp\right) \mathrm{d}t + \nu\, \mathrm{d}W. \tag{3.2}$$

Note that when $W = \alpha\beta$ with α constant and β a real-valued Brownian motion, which we will refer to as the case of spatially homogeneous noise, we infer

$$\mathrm{d}F = \vec{H}\, \mathrm{d}t + \alpha\nu \circ \mathrm{d}\beta = \vec{H}\, \mathrm{d}t + \alpha\nu\, \mathrm{d}\beta.$$

Remark 3.5 (About geometric SPDEs)**.** In case of deterministic evolution of surfaces, it is common to describe the evolution only in terms of the normal velocity of the surface. To make this precise, let $(F_t)_t$ be a family of smooth immersions of properly embedded hypersurfaces $(\mathcal{M}_t)_t$ in $\mathbb{R}^{n+1}$ with respect to a smooth manifold $\mathcal{M}$.

The normal velocity $V_t(x)$ of $\mathcal{M}_t$ in $F_t(x)$ is defined as

$$V_t(x) = \partial_t F_t(x) \cdot \nu_t(x).$$

A different definition of deterministic mean curvature flow is characterized by the scalar equation

$$V = H, \tag{3.3}$$

where we again suppress the dependence of V and H on x and t.

Note that (3.3) does not imply that $(F_t)_t$ solves the vector-valued equation

$$\partial_t F = \vec{H}, \tag{3.4}$$

but

$$\partial_t F = \vec{H} + (\partial_t F)^T,$$

where $(\partial_t F)^T$ denotes the tangential part of $\partial_t F$.

One can prove that (3.3) is up to tangential diffeomorphisms equivalent to (3.4), cf. [Eck04, Remark 2.2].

In case of stochastic mean curvature flow, the situation is slightly more involved. First we note that (3.1) is a vector-valued evolution law into normal direction. We have seen in the previous Remark 3.4 that this is not true for the equivalent Itô formulation (3.2), where the tangential term $(\nabla\varphi_l)^T$ appears.

This shows that a purely normal evolution of surfaces in the sense Stratonovich does not have to be a purely normal evolution in the sense of Itô and the other way around.

Nevertheless, when immersions $(\tilde{F}_t)_t$ as above are given with

$$\mathrm{d}\tilde{F} = \mu\,\mathrm{d}t + \sigma \circ \mathrm{d}W,$$

where μ and σ are sufficiently regular and have the right normal components $\mu \cdot \nu = H$ and $\sigma \cdot \nu = 1$, we can mimic the idea of the deterministic counterpart [Eck04, Remark 2.2] in order to formally infer a solution of the vector-valued parameterized SMCF (3.1).

For this, let $F_t(x) := \tilde{F}_t(\psi_t(x))$ with diffeomorphisms $\psi_t : \mathcal{M} \to \mathcal{M}$ which we have to specify. We compute

$$\mathrm{d}F = \mathrm{d}\tilde{F} + \mathrm{D}\tilde{F} \circ \mathrm{d}\psi.$$

As we want to solve (3.1), we are looking for ψ solving $\psi_0 = \mathrm{Id}_{\mathcal{M}}$ and

$$\mathrm{D}\tilde{F}_t(\psi_t(x)) \circ \mathrm{d}\psi_t(x) = -\mu_t^T(\psi_t(x))\,\mathrm{d}t - \sigma_t^T(\psi_t(x)) \circ \mathrm{d}W,$$

with μ^T and σ^T the tangential components of μ and σ. Since $\tilde{F}_t$ is an immersion, we infer that $\mathrm{D}\tilde{F}_t(\psi_t(x)) : T_{\psi_t(x)}\mathcal{M} \to T_{\tilde{F}_t(\psi_t(x))}\mathcal{M}_t$ is a bijection and therefore

$$\mathrm{d}\psi_t(x) = -\left(\mathrm{D}\tilde{F}_t(\psi_t(x))\right)^{-1} \mu_t^T(\psi_t(x))\,\mathrm{d}t - \left(\mathrm{D}\tilde{F}_t(\psi_t(x))\right)^{-1} \sigma_t^T(\psi_t(x)) \circ \mathrm{d}W.$$

Let

$$a(t,x) := -\left(\mathrm{D}\tilde{F}_t(x)\right)^{-1} \mu_t^T(x) \quad \text{and} \quad b(t,x) := -\left(\mathrm{D}\tilde{F}_t(x)\right)^{-1} \sigma_t^T(x).$$

With this notation the equation for $\psi_t(x)$ becomes

$$\mathrm{d}\psi_t(x) = a(t,\psi_t(x))\,\mathrm{d}t + b(t,\psi_t(x)) \circ \mathrm{d}W.$$

This is for fixed $x \in \mathcal{M}$ a finite-dimensional stochastic differential equation on the manifold $\mathcal{M}$. The fact that $a(t,\psi_t)$ and $b(t,\psi_t)$ are tangential and the perturbation is in the sense of Stratonovich implies that any solution $\psi_t(x)$ is indeed an element of $\mathcal{M}$. For the theory of SDEs on manifolds we refer to [Elw82] and [RW00]. Under suitable assumptions on μ, σ, $\tilde{F}$ and W that guarantee the existence of a solution ψ_t, we find that F is a solution of (3.1).

Remark 3.6 (Evolution of the second fundamental form)**.** For the second fundamental form we calculate

$$\begin{aligned}
\mathrm{d}A_{ij} &= \mathrm{d}(\partial_{ij}F\cdot\nu) = \partial_{ij}(\circ\,\mathrm{d}F)\cdot\nu + \partial_{ij}F\cdot\circ\,\mathrm{d}\nu\\
&= \partial_{ij}\vec{H}\cdot\nu - \partial_{ij}F\cdot\nabla_{\mathcal{M}_t}H\,\mathrm{d}t + \sum_l \partial_{ij}\left(\varphi_l\nu\right)\cdot\nu - \partial_{ij}F\cdot\nabla_{\mathcal{M}_t}\varphi_l\circ\mathrm{d}\beta^l\\
&= \partial_{ij}H + H\partial_{ij}\nu\cdot\nu - \partial_{ij}F\cdot\nabla_{\mathcal{M}_t}H\,\mathrm{d}t\\
&\quad + \sum_l \partial_{ij}\varphi_l + \varphi_l\partial_{ij}\nu\cdot\nu - \partial_{ij}F\cdot\nabla_{\mathcal{M}_t}\varphi_l\circ\mathrm{d}\beta^l\\
&= \nabla_i^{\mathcal{M}_t}\nabla_j^{\mathcal{M}_t}H + H\partial_{ij}\nu\cdot\nu\,\mathrm{d}t + \sum_l \nabla_i^{\mathcal{M}_t}\nabla_j^{\mathcal{M}_t}\varphi_l + \varphi_l\partial_{ij}\nu\cdot\nu\circ\mathrm{d}\beta^l\\
&= \nabla_i^{\mathcal{M}_t}\nabla_j^{\mathcal{M}_t}H - HA_{ik}A_j^k\,\mathrm{d}t + \sum_l \nabla_i^{\mathcal{M}_t}\nabla_j^{\mathcal{M}_t}\varphi_l - \varphi_l A_{ik}A_j^k\circ\mathrm{d}\beta^l\\
&= \nabla_i^{\mathcal{M}_t}\nabla_j^{\mathcal{M}_t}H - HA_{ik}A_j^k\,\mathrm{d}t + \circ\,\mathrm{d}\nabla_i^{\mathcal{M}_t}\nabla_j^{\mathcal{M}_t}W - A_{ik}A_j^k\circ\mathrm{d}W.
\end{aligned}$$

Furthermore we have

$$\begin{aligned}
\mathrm{d}A_j^i &= \circ\,\mathrm{d}g^{ik}A_{kj} + g^{ik}\circ\mathrm{d}A_{kj}\\
&= HA^{ik}A_{kj} + \nabla_{\mathcal{M}_t}^i\nabla_j^{\mathcal{M}_t}H\,\mathrm{d}t + \circ\,\mathrm{d}\nabla_{\mathcal{M}_t}^i\nabla_j^{\mathcal{M}_t}W + A^{ik}A_{kj}\circ dW
\end{aligned}$$

and

$$\mathrm{d}A^{ij} = 3HA^{ik}A_k^j + \nabla_{\mathcal{M}_t}^i\nabla_{\mathcal{M}_t}^j H\,\mathrm{d}t + \circ\,\mathrm{d}\nabla_{\mathcal{M}_t}^i\nabla_{\mathcal{M}_t}^j W + 3A^{ik}A_k^j\circ dW.$$

Similar to [Eck04, Appendix B] this implies

$$\mathrm{d}H = \mathrm{d}A_i^i = \Delta_{\mathcal{M}_t}H + H|A|^2\,\mathrm{d}t + \circ\,\mathrm{d}\Delta_{\mathcal{M}_t}W + |A|^2\circ\mathrm{d}W$$

and

$$\begin{aligned}
\mathrm{d}|A|^2 &= 2A_j^i\circ\mathrm{d}A_i^j\\
&= \Delta_{\mathcal{M}_t}|A|^2 + 2|A|^4 - 2|\nabla^{\mathcal{M}_t}A|^2\,\mathrm{d}t\\
&\quad + 2A_j^i\circ\mathrm{d}\nabla_{\mathcal{M}_t}^j\nabla_i^{\mathcal{M}_t}W + 2A_j^iA^{jk}A_{ki}\circ\mathrm{d}W.
\end{aligned}$$

3.2. Level set equation

We have seen in Section 2.4.1 that in the theory of deterministic mean curvature flow the level set method was introduced to consider topology changes. Similar to the deterministic theory we consider here a time-dependent function $f : \mathbb{R}^{n+1} \times [0,T] \to \mathbb{R}$ and ask under which conditions on f the level sets of f evolve according to (3.1). For the differential geometry of level sets we refer to Remark B.10.

Let $\lambda \in \mathbb{R}$ and $\mathcal{M}_0 := \{p \in \mathbb{R}^{n+1} \mid f(p,0) = \lambda\}$. Let $(\mathcal{M}_t)_t$ be an evolution of $\mathcal{M}_0$ with corresponding immersions $(F_t)_t$ with respect to the manifold $\mathcal{M}_0$. The manifold $\mathcal{M}_t$ and f should be compatible in the sense that $f(F_t(x),t) = \lambda$ for all $x \in \mathcal{M}_0$, $t \in [0,T]$. Formally differentiating this equation by using the chain rule yields

$$0 = \mathrm{d}f(F_t(x),t) + \nabla f(F_t(x),t) \cdot (\circ\, \mathrm{d}F_t(x))\,.$$

Hence,

$$\mathrm{d}f(F_t(x),t) = -\nabla f(F_t(x),t) \cdot \left(\vec{H}(x,t)\,\mathrm{d}t + \nu(x,t) \circ \mathrm{d}W(F_t(x),t)\right).$$

Using the same normal vector for $\mathcal{M}_t = \{p \in \mathbb{R}^{n+1} \mid f(p,t) = \lambda\}$ as in Remark B.10 we infer

$$\nu(x,t) = -\frac{\nabla f(F_t(x),t)}{|\nabla f(F_t(x),t)|} \qquad \text{and} \qquad H(x,t) = \nabla\cdot\left(\frac{\nabla f(\cdot,t)}{|\nabla f(\cdot,t)|}\right)(F_t(x)).$$

Hence,

$$\begin{aligned}\mathrm{d}f(F_t(x),t) = {}& |\nabla f(F_t(x),t)|\nabla\cdot\left(\frac{\nabla f(\cdot,t)}{|\nabla f(\cdot,t)|}\right)(F_t(x))\,\mathrm{d}t \\ &+ |\nabla f(F_t(x),t)| \circ \mathrm{d}W(F_t(x),t).\end{aligned}$$

And since this should hold for all level sets of f we infer that

$$\mathrm{d}f = |\nabla f|\nabla\cdot\left(\frac{\nabla f}{|\nabla f|}\right)\mathrm{d}t + |\nabla f| \circ \mathrm{d}W. \tag{3.5}$$

By expanding $W = \sum_l \varphi_l \beta^l$ we can rewrite (3.5) to

$$\mathrm{d}f(p,t) = |\nabla f(p,t)|\nabla\cdot\left(\frac{\nabla f}{|\nabla f|}\right)(p,t)\,\mathrm{d}t + \sum_l \varphi_l(p)|\nabla f(p,t)| \circ \mathrm{d}\beta^l.$$

We will call (3.5) the level set equation.

Note that one can not easily reconstruct a solution of the parameterized stochastic mean curvature flow (3.1) from the level set mean curvature flow.

Remark 3.7. The level set equation (3.5) can also be derived from the Itô formulation of the equation (3.2) for the immersions. With the notation from above and assuming that f is smooth in the spatial component, we compute the Taylor approximation

$$\begin{aligned} 0 &= f(F_t(x),t) - f(F_0(x),0) \\ &= f(F_0(x),t) - f(F_0(x),0) + \nabla f(F_0(x),t) \cdot \left(F_t(x) - F_0(x)\right) \\ &\quad + \frac{1}{2}\left(F_t(x) - F_0(x)\right) \cdot \mathrm{D}^2 f(F_0(x),t)\left(F_t(x) - F_0(x)\right) \\ &\quad + o(|F_t(x) - F_0(x)|^2). \end{aligned}$$

Hence

$$\begin{aligned} f(F_0(x),t) - f(F_0(x),0) &= -\nabla f(F_0(x),t) \cdot \left(F_t(x) - F_0(x)\right) \\ &\quad - \frac{1}{2}\left(F_t(x) - F_0(x)\right) \cdot \mathrm{D}^2 f(F_0(x),t)\left(F_t(x) - F_0(x)\right) \\ &\quad + o(|F_t(x) - F_0(x)|^2). \end{aligned} \tag{3.6}$$

Since we would like to have on the right hand side of the above equations only f evaluated at time 0 we rewrite it to

$$\begin{aligned} f(F_0(x),t) - f(F_0(x),0) = &- \left(\nabla f(F_0(x),t) - \nabla f(F_0(x),0)\right) \cdot \left(F_t(x) - F_0(x)\right) \\ &- \nabla f(F_0(x),0) \cdot \left(F_t(x) - F_0(x)\right) \\ &- \frac{1}{2}(F_t(x) - F_0(x)) \cdot \Big(\mathrm{D}^2 f(F_0(x),t) - \mathrm{D}^2 f(F_0(x),0)\Big)(F_t(x) - F_0(x)) \\ &- \frac{1}{2}\left(F_t(x) - F_0(x)\right) \cdot \mathrm{D}^2 f(F_0(x),0)\left(F_t(x) - F_0(x)\right) \\ &+ o(|F_t(x) - F_0(x)|^2). \end{aligned}$$

Now, note that by (3.6)

$$\begin{aligned} \nabla f(F_0(x),t) - \nabla f(F_0(x),0) &= -\mathrm{D}^2 f(F_0(x),t)\left(F_t(x) - F_0(x)\right) \\ &\quad - \left(\mathrm{D}\Big(F_t \circ F_0^{-1}\Big)^T (F_0(x)) - \mathrm{Id}\right) \nabla f(F_0(x),t) \\ &\quad + o(|F_t(x) - F_0(x)|) \\ &= -\mathrm{D}^2 f(F_0(x),t)\left(F_t(x) - F_0(x)\right) \\ &\quad - \left(\mathrm{D}\Big(F_t \circ F_0^{-1}\Big)^T (F_0(x)) - \mathrm{Id}\right) \nabla f(F_0(x),0) \\ &\quad + o(|F_t(x) - F_0(x)|). \end{aligned}$$

We therefore infer

$$
\begin{aligned}
f(F_0(x),t) - f(F_0(x),0) = {} & -\nabla f(F_0(x),0)\cdot\left(F_t(x)-F_0(x)\right)\\
& + \frac{1}{2}\left(F_t(x)-F_0(x)\right)\cdot \mathrm{D}^2 f(F_0(x),0)\left(F_t(x)-F_0(x)\right)\\
& + \nabla f(F_0(x),0)\cdot\Big(\mathrm{D}\Big(F_t\circ F_0^{-1}\Big)\left(F_0(x)\right)-\mathrm{Id}\Big)\left(F_t(x)-F_0(x)\right)\\
& + o(|F_t(x)-F_0(x)|^2).
\end{aligned}
$$

Hence, similarly to how the Itô formula is proven in the one-dimensional case, this implies

$$
\begin{aligned}
\mathrm{d}f(F_t(x),t) = {} & -\nabla f(F_t(x),t)\cdot\vec{H}(x,t)\\
& + \frac{1}{2}\nabla f(F_t(x),t)\cdot\sum_l \varphi_l(F_t(x))\left(\nabla^T\varphi_l(F_t(x)) - \nabla^{\perp}\varphi_l(F_t(x))\right)\\
& + \frac{1}{2}\sum_l \varphi_l^2(F_t(x))\nu(x,t)\cdot \mathrm{D}^2 f(F_t(x),t)\nu(x,t)\\
& + \nabla f(F_t(x),t)\cdot\sum_l \varphi_l(F_t(x))\,\mathrm{D}\Big|_{y=F_t(x)}\Big(\varphi_l(y)\nu(F_t^{-1}(y),t)\Big)\nu(x,t)\,\mathrm{d}t\\
& - \nabla f(F_t(x),t)\cdot\nu(x,t)\,\mathrm{d}W(F_t(x),t).
\end{aligned}
$$

Using the representation of ν and H in terms of f we infer

$$
\begin{aligned}
\mathrm{d}f = {} & |\nabla f|\nabla\cdot\left(\frac{\nabla f}{|\nabla f|}\right)\\
& - \frac{1}{2}\sum_l \varphi_l\nabla\varphi_l\cdot\nabla f\\
& + \frac{1}{2}\sum_l \varphi_l^2\frac{\nabla f\cdot \mathrm{D}^2 f\nabla f}{|\nabla f|^2}\\
& + \sum_l \varphi_l\nabla\varphi_l\cdot\nabla f\,\mathrm{d}t\\
& + |\nabla f|\,\mathrm{d}W.
\end{aligned}
$$

Hence, the Itô formulation of the level set equation is

$$
\begin{aligned}
\mathrm{d}f = {} & |\nabla f|\nabla\cdot\left(\frac{\nabla f}{|\nabla f|}\right) + \frac{1}{2}\sum_l\left(\varphi_l\nabla\varphi_l\cdot\nabla f + \varphi_l^2\frac{\nabla f\cdot\mathrm{D}^2 f\nabla f}{|\nabla f|^2}\right)\mathrm{d}t\\
& + |\nabla f|\,\mathrm{d}W.
\end{aligned}
\tag{3.7}
$$

One could also have started with (3.5) and deduce the Itô formulation (3.7) without referring to (3.2).

Remark 3.8. Note that (3.5) and (3.7) are invariant under reparametrizations. For this, let $\psi : \mathbb{R} \to \mathbb{R}$ be smooth with $\psi' > 0$ and $g \coloneqq \psi \circ f$ with f a solution of (3.5), which is equivalent to (3.7). Note that we assume that ψ is not only an injection but also does not change the orientation of the level sets. This is necessary, since the sign of the perturbation in (3.1) changes with the orientation.

Using the classical chain rule, we infer

$$\begin{aligned} \mathrm{d}g &= \psi'(f)|\nabla f|\nabla\cdot\left(\frac{\nabla f}{|\nabla f|}\right)\mathrm{d}t + \psi'(f)|\nabla f| \circ \mathrm{d}W \\ &= |\nabla g|\nabla\cdot\left(\frac{\nabla g}{|\nabla g|}\right)\mathrm{d}t + |\nabla g| \circ \mathrm{d}W, \end{aligned}$$

which is the Stratonovich formulation of the level set equation for g.

Thus, (3.7) as the Itô formulation of a Stratonovich SPDE that is invariant under reparametrizations itself has to be invariant. We can also give a direct proof of this invariance using the Itô formula, which we will introduce in Proposition 4.24. The Itô formula states that the evolution law of g in its Itô formulation consists of those terms that stem from the classical chain rule applied to (3.7) plus the additional term $\frac{1}{2}\psi''(g)\varphi_l^2|\nabla f|^2\,\mathrm{d}t$, i.e.

$$\begin{aligned} \mathrm{d}g = \psi'(f)&\left(|\nabla f|\nabla\cdot\left(\frac{\nabla f}{|\nabla f|}\right) + \frac{1}{2}\sum_l\left(\varphi_l\nabla\varphi_l\cdot\nabla f + \varphi_l^2\frac{\nabla f\cdot \mathrm{D}^2 f\nabla f}{|\nabla f|^2}\right)\right) \\ &+ \frac{1}{2}\sum_l \psi''(f)\varphi_l^2|\nabla f|^2\,\mathrm{d}t + \psi'(f)|\nabla f|\,\mathrm{d}W. \end{aligned}$$

Note that

$$\mathrm{D}^2 g = \psi'(f)\,\mathrm{D}^2 f + \psi''(f)\nabla f \otimes \nabla f,$$

where $\nabla f \otimes \nabla f \in \mathbb{R}^{n\times n}$ denotes the matrix $(\partial_i f \partial_j f)_{ij}$. Thus,

$$\frac{\nabla g\cdot \mathrm{D}^2 g\nabla g}{|\nabla g|^2} = \psi'(f)\frac{\nabla f\cdot \mathrm{D}^2 f\nabla f}{|\nabla f|^2} + \psi''(f)|\nabla f|^2.$$

This implies

$$\mathrm{d}g = |\nabla g|\nabla\cdot\left(\frac{\nabla g}{|\nabla g|}\right) + \frac{1}{2}\left(\varphi_l\nabla\varphi_l\cdot\nabla g + \varphi_l^2\frac{\nabla g\cdot \mathrm{D}^2 g\nabla g}{|\nabla g|^2}\right)\mathrm{d}t + |\nabla g|\,\mathrm{d}W$$

and g is a solution of (3.7).

3.3. Graphs

Similarly to the level set case one can consider the situation where the manifold is given as the graph of a function u.

For $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ we look for conditions on u such that the graphs of $u(\cdot, t)$ evolve according to stochastic mean curvature flow. Let $\mathcal{M} = \mathbb{R}^n$, $F_0(x) = (x, u(x, 0))$ and (F_t) be a solution of the equation (3.1) with respect to the manifold $\mathcal{M}$ and assume that $\mathcal{M}_t = \operatorname{graph} u(\cdot, t)$. To abbreviate the notation we will write $F_t(x) = (y_t(x), z_t(x)) \in \mathbb{R}^n \times \mathbb{R}$. Hence,

$$u(y_t(x), t) = z_t(x).$$

Formally applying the chain rule to this equation yields

$$\mathrm{d}u(y_t(x), t) + \nabla u(y_t(x), t) \cdot (\circ\, \mathrm{d}y_t(x)) = \mathrm{d}z_t(x).$$

With (3.1) this gives

$$\begin{aligned} \mathrm{d}u(y_t(x), t) = &-\nabla u(y_t(x), t) \cdot \left(\vec{H}^{1,\dots,n}(x, t)\,\mathrm{d}t + \nu^{1,\dots,n}(x, t) \circ \mathrm{d}W(F_t(x), t)\right) \\ &+ \vec{H}^{n+1}(x, t)\,\mathrm{d}t + \nu^{n+1}(x, t) \circ \mathrm{d}W(F_t(x), t). \end{aligned}$$

We will use the notation introduced in Remark B.9 with the same choice of a unit normal field at $\mathcal{M}_t$, in particular we will abbreviate

$$\mathbf{Q}(p) = \sqrt{1 + |p|^2} \qquad \text{and} \qquad \mathbf{v}(p) = \frac{p}{\mathbf{Q}(p)} \text{ for } p \in \mathbb{R}^n$$

such that

$$\nu(x, t) = \frac{1}{\mathbf{Q}(\nabla u(y_t(x), t))} \begin{pmatrix} -\nabla u(y_t(x), t) \\ 1 \end{pmatrix}.$$

With this choice we infer for the mean curvature

$$H(x, t) = \nabla\cdot(\mathbf{v}(\nabla u))\,(y_t(x), t)\,.$$

Hence,

$$\begin{aligned} \mathrm{d}u(y_t(x), t) &= \frac{1}{\mathbf{Q}(\nabla u(y_t(x), t))}\left(|\nabla u(y_t(x), t)|^2 + 1\right) \nabla\cdot(\mathbf{v}(\nabla u))\,(y_t(x), t)\,\mathrm{d}t \\ &\quad + \frac{1}{\mathbf{Q}(\nabla u(y_t(x), t))}\left(|\nabla u(y_t(x), t)|^2 + 1\right) \circ \mathrm{d}W(F_t(x), t) \\ &= \mathbf{Q}(\nabla u(y_t(x), t))\nabla\cdot(\mathbf{v}(\nabla u))\,(y_t(x), t)\,\mathrm{d}t \\ &\quad + \mathbf{Q}(\nabla u(y_t(x), t)) \circ \mathrm{d}W(F_t(x), t). \end{aligned}$$

Since this should hold for all $x \in \mathbb{R}^n$, we infer

$$\mathrm{d}u = \mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\,\mathrm{d}t + \sum_l \mathbf{Q}(\nabla u)\varphi_l(\cdot, u) \circ \mathrm{d}\beta^l. \tag{3.8}$$

Hence, the stochastic mean curvature flow of graphs is characterized by (3.8).

For a solution u of (3.8) one can consider the canonical graph parametrization given by $\mathcal{M} := \mathbb{R}^n$ and

$$F_t(x) := \begin{pmatrix} x \\ u(x,t) \end{pmatrix}, \ x \in \mathcal{M}. \tag{3.9}$$

For this parametrization it holds that

$$\mathrm{d}F_t(x) = \begin{pmatrix} 0 \\ \mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\,\mathrm{d}t + \sum_l \mathbf{Q}(\nabla u)\varphi_l(F_t) \circ \mathrm{d}\beta^l \end{pmatrix}.$$

Hence, F_t does not move into normal direction in its Stratonovich formulation and therefore is not a solution of (3.1). Nevertheless, as in Remark 3.5 the normal component of $\mathrm{d}F_t$ has the right structure

$$\begin{aligned} \circ\,\mathrm{d}F_t(x)\cdot\nu(x,t) &= \nabla\cdot\big(\mathbf{v}(\nabla u)\big)\,\mathrm{d}t + \sum_l \varphi_l(F_t)\circ \mathrm{d}\beta^l \\ &= H\,\mathrm{d}t + \circ\,\mathrm{d}W. \end{aligned}$$

3.4. Spheres

In this section we show how Example 2.2 extends to stochastic mean curvature flow. For this we consider a sphere $\mathcal{M}_0 = \mathcal{M} = \partial B(0, r_0) \subset \mathbb{R}^{n+1}$ with radius $r_0 > 0$ and $F_0 = \mathrm{Id}$. Furthermore we will assume that we have spatially homogeneous noise, i.e. the Wiener process W has the structure

$$W(x,t) = \alpha\beta(t)$$

with $\alpha \in \mathbb{R}$ and β a real-valued Brownian motion.

Because of the symmetry of (3.1) it is reasonable to assume that we can find a solution of (3.1) which has the form

$$F_t(x) = \frac{r(t)}{r_0}x$$

with a stochastic process r and $r(0) = r_0$. Plugging this Ansatz into (3.1) yields

$$\frac{\mathrm{d}r x}{r_0} = -\frac{n}{r_0 r}x\,\mathrm{d}t + \frac{1}{r_0}x \circ \mathrm{d}W.$$

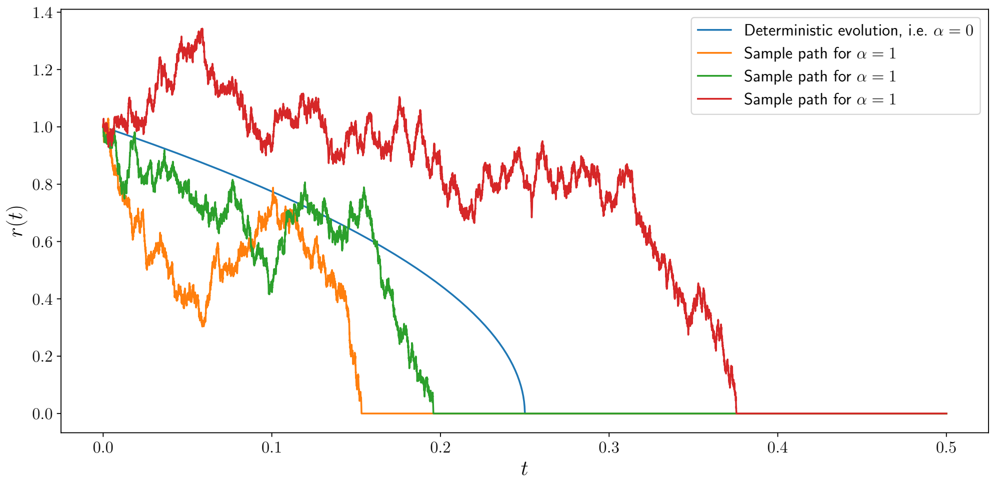

Figure 3.1.: Simulation of (3.10) using Euler-Maruyama scheme, cf. [KP92, 9.1], with $r_0 = 1$ and $n = 2$. Once $r(t_0) = 0$ is reached the evolution is continued by $r(t) = 0$ for $t \geq t_0$.

Hence, the radius of a sphere evolves according to the stochastic differential equation

$$\mathrm{d}r = -\frac{n}{r}\,\mathrm{d}t + \alpha \circ \mathrm{d}\beta = -\frac{n}{r}\,\mathrm{d}t + \alpha\,\mathrm{d}\beta. \tag{3.10}$$

Equation (3.10) defines a Bessel process r with negative index. For the theory of Bessel processes we refer to [GJY03] and the references therein. A simulation of (3.10) is shown in Figure 3.1 for $\alpha = 0$ and $\alpha = 1$.

3.5. Huisken's weighted monotonicity formula

In this section we want to derive a monotonicity formula for stochastic mean curvature flow which generalizes Huisken's weighted monotonicity formula from Section 2.3.

We will start be deriving a formula which holds for the most general form of (3.1) and later we will focus on the case of spatially homogeneous noise.

Remark 3.9 (Weak formulation of stochastic mean curvature flow)**.** Let $(\mathcal{M}_t)_t$ be a solution of stochastic mean curvature flow with corresponding immersions $(F_t)_t$ with respect to $\mathcal{M}$ solving (3.1).

Let $\rho : \mathbb{R}^{n+1} \times [0, \infty) \to \mathbb{R}$ be a time-dependent (deterministic) smooth test function. Integration over $\mathcal{M}_t$ can be expressed using a smooth local parametrization, cf. [Lee18, 2.31]. For simplicity we assume that we can find a global parametrization of $\mathcal{M}_t$ with respect to an open set $\Omega \subset \mathbb{R}^n$, which we will again denote by F_t as in Section 3.1.1. The general case can be recovered by using local parametrizations and a smooth partition of unity. We assume that all integrals appearing in the following computations are finite.

Using the evolution laws from Section 3.1.1 we infer

$$
\begin{aligned}
\mathrm{d}\int_{\mathcal{M}_t} \rho &= \mathrm{d}\Big(\int_\Omega \rho(F_t(x),t)\sqrt{g(x,t)}\,\mathrm{d}x\Big) \\
&= \int_{\mathcal{M}_t} \Big[\Big(\partial_t\rho + \nabla\rho\cdot\vec{H} - \rho H^2\Big)\,\mathrm{d}t + (\nabla\rho\cdot\nu - \rho H)\circ\mathrm{d}W\Big].
\end{aligned} \tag{3.11}
$$

Note that here and in the following we will compute integrals over the space variable x without explicitly stating that we are integrating over x. Furthermore we will continue to write $\mathrm{d}t$, $\mathrm{d}W$ and $\circ\,\mathrm{d}W$ as a shorthand notation for the deterministic integral, the Itô integral and the Stratonovich integral, respectively. In addition, we are always integrating all terms in front of these differentials as long as no other differential with respect to the time variable appears. In the above examples this means that the term

$$
\int_{\mathcal{M}_t} \partial_t\rho + \nabla\rho\cdot\vec{H} - \rho H^2\,\mathrm{d}t + \nabla\rho\cdot\nu - \rho H\circ\mathrm{d}W
$$

is an abbreviation for the process

$$
\begin{aligned}
T \mapsto \int_0^T &\left(\int_{\mathcal{M}_t} \partial_t\rho(F_t(x),t) + \nabla\rho(F_t(x),t)\cdot\vec{H}_t(x) - \rho(F_t(x),t)H_t^2(x)\,\mathrm{d}x\right)\mathrm{d}t \\
&+ \int_0^T \left(\int_{\mathcal{M}_t} \nabla\rho(F_t(x),t)\cdot\nu_t(x) - \rho(F_t(x),t)H_t(x)\,\mathrm{d}x\right)\circ\mathrm{d}W(t).
\end{aligned}
$$

For the Itô-Stratonovich correction of (3.11) we calculate

$$
\begin{aligned}
\mathrm{d}\Big(\int_{\mathcal{M}_t} (\nabla\rho\cdot\nu - \rho H)\varphi_l\Big) &= \ldots\,\mathrm{d}t + \sum_{k\neq l}\ldots\circ\mathrm{d}\beta^k \\
&+ \int_{\mathcal{M}_t} \Big(\nu\cdot\mathrm{D}^2\rho\nu\varphi_l - \nabla\rho\cdot\vec{H}\varphi_l + (\nabla\rho\cdot\nu - \rho H)\nabla\varphi_l\cdot\nu - \nabla\rho\cdot\nabla_{\mathcal{M}_t}\varphi_l\Big)\,\varphi_l \\
&+ \int_{\mathcal{M}_t} -(\nabla\rho\cdot\nu - \rho H)\,\varphi_l^2 H - \rho\varphi_l\Big(\Delta_{\mathcal{M}_t}\varphi_l + \varphi_l|A|^2\Big)\circ\mathrm{d}\beta^l.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathrm{d}\int_{\mathcal{M}_t} \rho = \int_{\mathcal{M}_t} \Bigg[\Big(& \partial_t \rho + (1-\varphi_l^2)\nabla\rho \cdot \vec{H} - \left(1 - \frac{\varphi_l^2}{2}\right)\rho H^2 \\
& + \frac{\varphi_l^2}{2}\nu \cdot \mathrm{D}^2\rho\nu + \frac{1}{2}\nabla^\perp \rho \cdot \nabla^\perp \varphi_l \varphi_l - \frac{1}{2}\rho \nabla\varphi_l \cdot \vec{H}\varphi_l - \frac{1}{2}\nabla\rho \cdot \nabla_{\mathcal{M}_t}\varphi_l\varphi_l \\
& - \frac{1}{2}\rho \Delta_{\mathcal{M}_t}\varphi_l\varphi_l - \frac{1}{2}\rho\varphi_l^2 |A|^2 \Big)\,\mathrm{d}t \\
& + \left(\tilde{\nabla}\rho \cdot \nu - \rho H\right)\mathrm{d}W \Bigg].
\end{aligned}
$$

By taking the expectation and formally using that the Itô integral is a martingale under appropriate assumptions, we infer

$$
\begin{aligned}
\mathbb{E}\int_{\mathcal{M}_T} \rho(\cdot, T) - \mathbb{E}\int_{\mathcal{M}_0}\rho(\cdot,0) = \mathbb{E}\int_0^T \int_{\mathcal{M}_t} \Bigg[& \partial_t \rho + (1-\varphi_l^2)\nabla\rho \cdot \vec{H} - \left(1 - \frac{\varphi_l^2}{2}\right)\rho H^2 \\
& + \frac{\varphi_l^2}{2}\nu \cdot \mathrm{D}^2\rho\nu + \frac{1}{2}\nabla^\perp\rho \cdot \nabla^\perp\varphi_l\varphi_l - \frac{1}{2}\rho\nabla\varphi_l \cdot \vec{H}\varphi_l - \frac{1}{2}\nabla_{\mathcal{M}_t}\rho \cdot \nabla_{\mathcal{M}_t}\varphi_l\varphi_l \\
& - \frac{1}{2}\rho\Delta_{\mathcal{M}_t}\varphi_l\varphi_l - \frac{1}{2}\rho\varphi_l^2|A|^2 \Bigg]\,\mathrm{d}t.
\end{aligned}
$$

Using

$$
\int_{\mathcal{M}_t} \nabla\rho \cdot \vec{H} = \int_{\mathcal{M}_t} -\nabla^{\mathcal{M}_t} \cdot \nabla\rho = \int_{\mathcal{M}_t} -\Delta\rho + \nu \cdot \mathrm{D}^2\rho\nu
$$

as well as

$$
\int_{\mathcal{M}_t} \rho\Delta_{\mathcal{M}_t}\varphi_l\varphi_l = -\int_{\mathcal{M}_t} \nabla_{\mathcal{M}_t}\rho \cdot \nabla_{\mathcal{M}_t}\varphi_l\varphi_l + \rho|\nabla_{\mathcal{M}_t}\varphi|^2
$$

we end up with

$$
\begin{aligned}
\mathbb{E}\int_{\mathcal{M}_T}\rho(\cdot,T) - \mathbb{E}\int_{\mathcal{M}_0}\rho(\cdot,0) = \mathbb{E}\int_0^T\int_{\mathcal{M}_t} \Bigg[& \partial_t\rho + \Delta\rho + 2\left(1 - \frac{\varphi_l^2}{2}\right)\nabla\rho \cdot \vec{H} \\
& - \left(1 - \frac{\varphi_l^2}{2}\right)\rho H^2 - \left(1 - \frac{\varphi_l^2}{2}\right)\nu \cdot \mathrm{D}^2\rho\nu + \frac{1}{2}\varphi_l\nabla^\perp\varphi_l \cdot \left(\nabla^\perp\rho - \rho\vec{H}\right) \\
& + \frac{1}{2}\rho|\nabla_{\mathcal{M}_t}\varphi_l|^2 - \frac{1}{2}\rho\varphi_l^2|A|^2 \Bigg]\,\mathrm{d}t,
\end{aligned}
$$

which can be further simplified to

$$\begin{aligned}\mathbb{E}\int_{\mathcal{M}_T}\rho(\cdot,T)-\mathbb{E}\int_{\mathcal{M}_0}\rho(\cdot,0)=\mathbb{E}\int_0^T\int_{\mathcal{M}_t}\Bigg[&\partial_t\rho+\Delta\rho+\left(1-\frac{\varphi_l^2}{2}\right)\frac{|\nabla^\perp\rho|^2}{\rho}\\&-\left(1-\frac{\varphi_l^2}{2}\right)\nu\cdot\mathrm{D}^2\rho\nu-\left(1-\frac{\varphi_l^2}{2}\right)\rho\left|\vec{H}-\frac{\nabla^\perp\rho}{\rho}+\frac{\varphi_l\nabla^\perp\varphi_l}{4-2\varphi_l^2}\right|^2\\&+\rho\frac{\varphi_l^2|\nabla^\perp\varphi_l|^2}{16-8\varphi_l^2}+\frac{1}{2}\rho|\nabla_{\mathcal{M}_t}\varphi_l|^2-\frac{1}{2}\rho\varphi_l^2|A|^2\Bigg]\,\mathrm{d}t.\end{aligned}$$

For positive ρ and $\sum_l\varphi_l^2\le 2$ the last four terms are either negative or can be controlled by Gronwall's inequality. The first four terms contain higher derivatives of ρ. Note that in Huisken's monotonicity formula for deterministic MFC the corresponding terms do not appear due to the particular choice of ρ. In order to generalize the monotonicity formula to SMCF, we are looking for ρ solving

$$\partial_t\rho+\Delta\rho+\left(1-\frac{\varphi_l^2}{2}\right)\frac{|\nabla^\perp\rho|^2}{\rho}-\left(1-\frac{\varphi_l^2}{2}\right)\nu\cdot\mathrm{D}^2\rho\nu=0.$$

Without assuming some special structure of the noise coefficients (φ_l) this equation in general only has $\rho\equiv$ const. as a solution. In the next remark, we will construct non-trivial solutions ρ under the assumption that the noise is spatially homogeneous.

Remark 3.10 (Monotonicity formula for spatially homogeneous noise)**.** Additionally to the assumptions of Remark 3.9 we will assume that the Wiener process is spatially homogeneous, i.e. $W=\alpha\beta$, with $\alpha^2<2$ and β a real-valued Brownian motion. Under this assumption the conclusion from Remark 3.9 reduces to

$$\begin{aligned}\mathbb{E}\int_{\mathcal{M}_T}\rho(\cdot,T)-\mathbb{E}\int_{\mathcal{M}_0}\rho(\cdot,0)=\mathbb{E}\int_0^T\int_{\mathcal{M}_t}\Bigg[&\partial_t\rho+\Delta\rho+\left(1-\frac{\alpha^2}{2}\right)\frac{|\nabla^\perp\rho|}{\rho}\\&-\left(1-\frac{\alpha^2}{2}\right)\nu\cdot\mathrm{D}^2\rho\nu-\left(1-\frac{\alpha^2}{2}\right)\rho\left|\vec{H}-\frac{\nabla^\perp\rho}{\rho}\right|^2-\frac{1}{2}\rho\alpha^2|A|^2\Bigg]\mathrm{d}t.\end{aligned}$$

To find a good candidate for ρ we set $\lambda=1-\frac{\alpha^2}{2}$ and make the Ansatz

$$\partial_t\rho(z,t)+\Delta\rho(z,t)+\lambda\frac{|\nabla\rho(z,t)\cdot v|^2}{\rho(z,t)}-\lambda v\cdot\mathrm{D}^2\rho(z,t)v=0 \tag{3.12}$$

which should hold for all $z\in\mathbb{R}^{n+1}$, $t\in[0,\infty)$ and $v\in\mathbb{R}^{n+1}$ with $|v|=1$. Note that in (3.12) the differential operators ∇, Δ and D^2 are with respect to z.

The Ansatz (3.12) implies that the value of $v \cdot \mathrm{D}^2\rho(z,t)v - \frac{|\nabla\rho(z,t)\cdot v|^2}{\rho(z,t)}$ does not depend on v, hence

$$\mathrm{D}^2\left(\log(\rho)\right)(z,t) = \gamma(t)\,\mathrm{Id}$$

with a function $\gamma : [0,\infty) \to \mathbb{R}$. We infer for ρ

$$\rho(z,t) = C(t)\exp\left(\frac{1}{2}\gamma(t)|z - z_0|^2\right)$$

with $z_0 \in \mathbb{R}^{n+1}$ and $C : [0,\infty) \to \mathbb{R}$. For C and γ we find with (3.12) the equations

$$\begin{aligned}\frac{1}{2}\gamma' + \gamma^2 &= 0 \text{ and}\\ \frac{C'}{C} + (n+1)\gamma - \lambda\gamma &= 0.\end{aligned}$$

Solving these equations yields two kinds of solutions, either

$$\rho(z,t) = \text{const.}$$

or for $z_0 \in \mathbb{R}^{n+1}$ and $t_0 > 0$

$$\rho(z,t) = c_0\rho_{z_0,t_0}(z,t)$$

with a constant $c_0 \in \mathbb{R}$ and

$$\rho_{z_0,t_0}(z,t) = \frac{1}{\left(4\pi(t_0 - t)\right)^{\frac{n+1-\lambda}{2}}}e^{-\frac{|z-z_0|^2}{4(t_0-t)}},\ z \in \mathbb{R}^{n+1}, t \in [0,t_0).$$

In the subsequent chapters we will only use the constant kernel as we are working on compact hypersurfaces and do not need the kernel to decay. In more general situations the kernels ρ_{z_0,t_0} might be useful to handle unbounded hypersurfaces.

Note that

$$\rho_{z_0,t_0}(z,t) = \left(4\pi(t_0 - t)\right)^{\frac{\lambda-1}{2}}\Phi_{z_0,t_0}(z,t)$$

with Φ_{z_0,t_0} as in Huisken's monotonicity formula for the unperturbed MCF a translation of the n-dimensional backward heat kernel in $\mathbb{R}^{n+1}$, cf. Remark 2.7.

Hence,

$$\mathbb{E}\mathcal{H}^n(\mathcal{M}_T) - \mathbb{E}\mathcal{H}^n(\mathcal{M}_0) + \mathbb{E}\int_0^T\int_{\mathcal{M}_t}\left(1 - \frac{\alpha^2}{2}\right)H^2 + \frac{1}{2}\alpha^2|A|^2\,\mathrm{d}t = 0$$

and for $T < t_0$

$$\mathbb{E}\int_{\mathcal{M}_T}\rho_{z_0,t_0}(\cdot,T) - \mathbb{E}\int_{\mathcal{M}_0}\rho_{z_0,t_0}(\cdot,0)$$
$$+\mathbb{E}\int_0^T\left[\int_{\mathcal{M}_t}\left(1-\frac{\alpha^2}{2}\right)\rho_{z_0,t_0}\left(H-\frac{(z_0-p)\cdot\nu}{2(t_0-t)}\right)^2+\frac{1}{2}\rho_{z_0,t_0}\alpha^2|A|^2\right]\mathrm{d}t = 0,$$

especially for $\alpha^2 \leq 2$

$$\mathbb{E}\mathcal{H}^n(\mathcal{M}_T) \leq \mathbb{E}\mathcal{H}^n(\mathcal{M}_0) \text{ and}$$
$$\mathbb{E}\int_{\mathcal{M}_T}\rho_{z_0,t_0}(\cdot,T) \leq \mathbb{E}\int_{\mathcal{M}_0}\rho_{z_0,t_0}(\cdot,0).$$

Remark 3.11 (Weighted monotonicity formula for spatially homogeneous noise)**.** In the situation of Remark 3.10 we consider a stochastic process $\psi \geq 0$ of functions on Ω. We assume that ψ has the decomposition

$$\mathrm{d}\psi(x,t) = \psi_\mu(x,t)\,\mathrm{d}t + \alpha\psi_\sigma(x,t)\circ\mathrm{d}\beta_t,$$

which can be equivalently written as

$$\mathrm{d}\psi = \psi_\mu + \frac{\alpha^2}{2}\psi_{\sigma\sigma}\,\mathrm{d}t + \alpha\psi_\sigma\,\mathrm{d}\beta_t.$$

Furthermore let $\rho \geq 0$ be one of the kernels found in Remark 3.10. We infer

$$\mathrm{d}\left(\int_{\mathcal{M}_t}\rho\psi\right) = \int_{\mathcal{M}_t}\psi\left(\partial_t\rho + \nabla\rho\cdot\vec{H} - \rho H^2\right) + \rho\psi_\mu\,\mathrm{d}t$$
$$+\alpha\psi\nabla\rho\cdot\nu - \alpha\rho\psi H + \alpha\rho\psi_\sigma\circ\mathrm{d}\beta$$

For the Itô-Stratonovich correction terms we calculate

$$\mathrm{d}\left(\int_{\mathcal{M}_t}\psi\nabla\rho\cdot\nu - \rho\psi H + \rho\psi_\sigma\right) = \ldots\,\mathrm{d}t$$
$$+\alpha\int_{\mathcal{M}_t}\psi\nu\cdot\mathrm{D}^2\rho\nu + 2\psi_\sigma\nabla\rho\cdot\nu - \psi\nabla\rho\cdot\vec{H} - \rho\psi_\sigma H - \rho\psi|A|^2 + \rho\psi_{\sigma\sigma}$$
$$-\alpha\int_{\mathcal{M}_t}\left(\psi\nabla\rho\cdot\nu - \rho\psi H + \rho\psi_\sigma\right)H\circ\mathrm{d}\beta.$$

Hence,

$$\mathrm{d}\left(\int_{\mathcal{M}_t}\rho\psi\right) = \int_{\mathcal{M}_t}\psi\left(\partial_t\rho + \left(1-\alpha^2\right)\nabla\rho\cdot\vec{H} - \left(1-\frac{\alpha^2}{2}\right)\rho H^2 + \frac{\alpha^2}{2}\nu\cdot\mathrm{D}^2\rho\nu\right)$$
$$+\rho\psi_\mu + \alpha^2\psi_\sigma\nabla\rho\cdot\nu - \alpha^2\rho\psi_\sigma H + \frac{\alpha^2}{2}\rho\psi_{\sigma\sigma} - \frac{\alpha^2}{2}\rho\psi|A|^2\,\mathrm{d}t$$
$$+\alpha\psi\nabla\rho\cdot\nu - \alpha\rho\psi H + \alpha\rho\psi_\sigma\,\mathrm{d}\beta.$$

We will continue only with the finite variation part of the above equation which we will denote by $\left(\int_{\mathcal{M}_t}\rho\psi\right)_\mu$, i.e.

$$\left(\int_{\mathcal{M}_t}\rho\psi\right)_\mu := \int_{\mathcal{M}_t}\psi\left(\partial_t\rho + \left(1-\alpha^2\right)\nabla\rho\cdot\vec{H} - \left(1-\frac{\alpha^2}{2}\right)\rho H^2 + \frac{\alpha^2}{2}\nu\cdot \mathrm{D}^2\rho\nu\right)$$
$$+\rho\psi_\mu + \alpha^2\psi_\sigma\nabla\rho\cdot\nu - \alpha^2\rho\psi_\sigma H + \frac{\alpha^2}{2}\rho\psi_{\sigma\sigma} - \frac{\alpha^2}{2}\rho\psi|A|^2.$$

By using the properties of ρ derived in Remark 3.10, in particular (3.12), we find

$$\left(\int_{\mathcal{M}_t}\rho\psi\right)_\mu = \int_{\mathcal{M}_t}\psi\left(-\Delta\rho - \nabla\rho\cdot\vec{H} + \nu\cdot \mathrm{D}^2\rho\nu - \left(1-\frac{\alpha^2}{2}\right)\rho\left(\vec{H} - \frac{\nabla^\perp\rho}{\rho}\right)^2\right)$$
$$+\rho\psi_\mu + \alpha^2\psi_\sigma\nabla\rho\cdot\nu - \alpha^2\rho\psi_\sigma H + \frac{\alpha^2}{2}\rho\psi_{\sigma\sigma} - \frac{\alpha^2}{2}\rho\psi|A|^2.$$

We integrate by parts and complete the square to infer

$$\begin{aligned}\left(\int_{\mathcal{M}_t}\rho\psi\right)_\mu &= \int_{\mathcal{M}_t} - \left(1-\frac{\alpha^2}{2}\right)\rho\psi\left(\vec{H} - \frac{\nabla^\perp\rho}{\rho}\right)^2 + \alpha^2\rho\psi_\sigma\nu\cdot\left(\frac{\nabla^\perp\rho}{\rho} - \vec{H}\right)\\ &\qquad + \rho\left(\psi_\mu - \Delta_{\mathcal{M}_t}\psi + \frac{\alpha^2}{2}\psi_{\sigma\sigma} - \frac{\alpha^2}{2}\psi|A|^2\right)\\ &= \int_{\mathcal{M}_t} - \left(1-\frac{\alpha^2}{2}\right)\rho\psi\left(\vec{H} - \frac{\nabla^\perp\rho}{\rho} + \frac{\alpha^2\psi_\sigma\nu}{(2-\alpha^2)\psi}\right)^2\\ &\qquad + \rho\left(\psi_\mu - \Delta_{\mathcal{M}_t}\psi + \frac{\alpha^2}{2}\psi_{\sigma\sigma} - \frac{\alpha^2}{2}\psi|A|^2 + \frac{\alpha^4\psi_\sigma^2}{(4-2\alpha^2)\psi}\right).\end{aligned}$$

Hence,

$$\mathbb{E}\int_{\mathcal{M}_t}\rho\psi$$

is decreasing if

$$\psi_\mu - \Delta_{\mathcal{M}_t}\psi + \frac{\alpha^2}{2}\psi_{\sigma\sigma} - \frac{\alpha^2}{2}\psi|A|^2 + \frac{\alpha^4\psi_\sigma^2}{(4-2\alpha^2)\psi} \le 0.$$

Example 3.12 (Gradient estimates in the graphical case)**.** Let us additionally assume that $\mathcal{M}_t$ is the graph of a function. We want to show that one can argue similar to [EH89, Section 3] to deduce bounds for the gradient.

Since $\mathcal{M}_t$ is a graph there is a $w \in \mathbb{R}^{n+1}$ such that $\nu \cdot w > 0$. From Remark 3.3 we deduce for $\psi = \frac{1}{\nu \cdot w}$ that

$$\mathrm{d}\psi = \psi^2 w \cdot \nabla_{\mathcal{M}_t} H \,\mathrm{d}t.$$

Note that ψ solves the same equation as in the deterministic case. Since $\partial_j \nu$ is a tangent vector it is completely determined by $A_{ij} = -\partial_i F \cdot \partial_j \nu$ and we infer

$$\partial_j \nu = -A_j^k \partial_k F$$

implying

$$\nabla^j_{\mathcal{M}_t} \nu = -A^{kj} \partial_k F.$$

Using the Codazzi equations, cf. [Lee18, Chapter 8], we infer for the Laplace-Beltrami operator applied component-wise to ν

$$\Delta_{\mathcal{M}_t} \nu = -\nabla_{\mathcal{M}_t} H - |A|^2 \nu.$$

Hence,

$$\begin{aligned} \Delta_{\mathcal{M}_t} \psi &= -\psi^2 \Delta_{\mathcal{M}_t} \nu \cdot w + 2\psi^3 \left| w \cdot \nabla_{\mathcal{M}_t} \nu \right|^2 \\ &= \psi^2 \nabla_{\mathcal{M}_t} H \cdot w + \psi |A|^2 + 2\psi^{-1} \left| \nabla_{\mathcal{M}_t} \psi \right|^2. \end{aligned}$$

This implies

$$\mathrm{d}\psi = \Delta_{\mathcal{M}_t} \psi - \psi |A|^2 - 2\psi^{-1} \left| \nabla_{\mathcal{M}_t} \psi \right|^2 \mathrm{d}t$$

and with the notation from above

$$\begin{aligned} \psi_\mu &= \Delta_{\mathcal{M}_t} \psi - \psi |A|^2 - 2\psi^{-1} |\nabla_{\mathcal{M}_t} \psi|^2 \le \Delta_{\mathcal{M}_t} \psi, \\ \psi_\sigma &= 0, \\ \psi_{\sigma\sigma} &= 0. \end{aligned}$$

Therefore $\mathbb{E} \int_{\mathcal{M}_t} \rho \psi$ is decreasing.

Let $\Psi = f(\psi)$ with a convex, increasing non-negative function f. It holds that

$$\begin{aligned} \Psi_\mu &= f' \psi_\mu = \Delta_{\mathcal{M}_t} \Psi - f' \psi |A|^2 - (2 f' \psi^{-1} + f'') |\nabla_{\mathcal{M}_t} \psi|^2 \le \Delta_{\mathcal{M}_t} \Psi, \\ \Psi_\sigma &= 0, \\ \Psi_{\sigma\sigma} &= 0. \end{aligned}$$

Hence $\mathbb{E} \int_{\mathcal{M}_t} \rho \Psi$ is also decreasing. Since this holds for all such f, we conclude that $\operatorname{esssup} \psi(\omega, x, t) \le \operatorname{esssup}_{\tilde{\omega}, \tilde{x}} \psi(\tilde{\omega}, \tilde{x}, 0)$. We will make this argument rigorous in Chapter 7 to deduce a maximum principle for the gradient in the graphical case.

4. Stochastic evolution equations in infinite dimensions

4.1. Stochastic integration

We present the theory of stochastic integration in infinite dimensional Hilbert spaces. Our main reference is [DPZ14].

Definition 4.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $I = [0, T]$ with $T < \infty$ or $I = [0, \infty)$. A system $(\mathcal{F}_t)_{t \in I}$ with $\mathcal{F}_t \subset \mathcal{F}$ is called a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ if $\mathcal{F}_t$ is a σ-algebra and $\mathcal{F}_t \subset \mathcal{F}_s$ for all $s, t \in I$ with $t \leq s$.

A filtration $(\mathcal{F}_t)_{t \in I}$ is called a normal filtration if

- it is complete with respect to $\mathcal{F}$, i.e. $A \in \mathcal{F}_0$ for all $A \in \mathcal{F}$ with $\mathbb{P}(A) = 0$ and
- it is right-continuous, i.e.

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s \ \forall t \in I, t < \sup I.$$

For a filtration $(\mathcal{F}_t)_{t \in I}$ we call $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbb{P})$ a stochastic basis.

Definition 4.2. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbb{P})$ be a stochastic basis and E a separable Banach space. An E-valued stochastic process $X = (X(t))_{t \in I}$ is called $(\mathcal{F}_t)_{t \in I}$-adapted if $X(t)$ is $\mathcal{F}_t$-measurable for all $t \in I$.

The process X is called predictable if $X : I \times \Omega \to E$ is measurable with respect to the predictable σ-algebra

$$\mathcal{P} := \sigma\left(\left\{(s,t] \times F \mid s, t \in I, s \leq t, F \in \mathcal{F}_s\right\} \cup \left\{\{0\} \times F \mid F \in \mathcal{F}_0\right\}\right).$$

In the following we will introduce the notion of a Wiener process on a Hilbert space U. For this, we will use the notion of trace class and Hilbert-Schmidt operators, for a definition of which we refer to Appendix A.1. For a self-adjoint non-negative trace class operator $Q \in L_1(U)$ we will denote by $\mathcal{N}(0, Q)$ the Gaussian measure on U with zero mean and covariance Q, cf. [DPZ14].

Definition 4.3. Let U be a separable Hilbert space, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $Q \in L_1(U)$ self-adjoint and non-negative. A U-valued stochastic process W is called a Wiener process with covariance operator Q if

- $W(0) = 0$,
- W has continuous trajectories in U,
- W has independent increments and
- $W(t) - W(s) \sim \mathcal{N}(0, (t-s)Q)$ for all $s, t \in I$, $s \leq t$.

For a given filtration $(\mathcal{F}_t)_{t \in I}$ we will call W an $(\mathcal{F}_t)_{t \in I}$-Wiener process with covariance operator Q if additionally

- $W(t)$ is $\mathcal{F}_t$-measurable for all $t \in I$,
- $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $s, t \in I$, $s \leq t$.

There is an often-used representation formula for Wiener processes, a detailed exposition of which can be found in [DPZ14, Section 4.1], especially [DPZ14, Propositions 4.3 and 4.7]), which we summarize as follows:

Proposition 4.4. *Let U be a separable Hilbert space, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbb{P})$ a stochastic basis and $Q \in L_1(U)$ self-adjoint and non-negative.*

A U-valued stochastic process W has a version which is a Wiener process on U with covariance operator Q if and only if there is an orthonormal system $(e_k)_{k \in \mathbb{I}}$ in U with a finite or countable index set $\mathbb{I}$ and independent real-valued Brownian motions $(\beta_k)_{k \in \mathbb{I}}$ such that $Q^{\frac{1}{2}} U \subset \overline{\operatorname{span}\left(\{e_k \mid k \in \mathbb{I}\}\right)}$ and

$$W(t) = \sum_{k \in \mathbb{I}} Q^{\frac{1}{2}} e_k \beta_k(t) \ \forall t \in I \tag{4.1}$$

with convergence in $L^2(\Omega; U)$. Furthermore, one can choose $(e_k)_{k \in \mathbb{I}}$ to be the eigenvectors of Q to the non-zero eigenvalues $(\lambda_k)_{k \in \mathbb{I}}$. Equation (4.1) then translates to

$$W(t) = \sum_{k \in \mathbb{I}} \sqrt{\lambda_k} e_k \beta_k(t) \ \forall t \in I.$$

Remark 4.5. Note that in the statement of Proposition 4.4 it is necessary to exclude the eigenvectors from $\ker Q$. For example $W = 0$ is a Wiener process on ℓ^2 with respect to any probability space $(\Omega, \mathcal{F}, \mathbb{P})$, but there are probability spaces which are too small to carry a real-valued Brownian motion.

Later on, in Proposition 4.9 it will turn out that it is natural to consider the space $U_0 = Q^{\frac{1}{2}} U$ with the induced scalar product

$$\langle x, y \rangle_{U_0} := \left\langle Q^{-\frac{1}{2}} x, Q^{-\frac{1}{2}} y \right\rangle_U, \ x, y \in U,$$

where $Q^{-\frac{1}{2}} : U_0 \to \left(\ker Q^{\frac{1}{2}}\right)^{\perp} = \overline{\mathcal{R}(Q^{\frac{1}{2}})} \subset U$ is the pseudo-inverse of $Q^{\frac{1}{2}} : U \to U_0$, cf. Definition A.9. Note that U_0 is a separable Hilbert space. The space U_0 is called the reproducing kernel space of W. If H is another separable Hilbert space, we will see that under some additional assumptions one can integrate processes Φ taking values in $L_2(U_0; H)$, which is less restrictive than assuming that Φ takes values in $L(U; H)$.[1]

Once this observation is made, one can consider a more general concept of Wiener processes without assuming that Q is a trace class operator as long as one can make sense of $T(W(t))$ for $T \in L_2(U_0; H)$.

Definition 4.6 (cf. [PR07, Section 2.5.1])**.** Let U be a separable Hilbert space, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ a stochastic basis and $Q \in L(U)$ self-adjoint and non-negative and define U_0 as above.

Let U_1 be another separable Hilbert space with a Hilbert-Schmidt embedding $J : U_0 \hookrightarrow U_1$. A Wiener process W on U_1 with covariance operator JJ^* is called a generalized Wiener process on U with covariance operator Q.

A generalized Wiener process W on U with covariance operator $Q = \mathrm{Id}$ is called cylindrical Wiener process.

The next proposition shows that the abstract notion of a generalized Wiener process can be equivalently stated with the help of a simple representation formula which is very similar to Proposition 4.4. Again, we refer to [DPZ14, Proposition 4.7 and the discussion thereafter]. As we gave a different but equivalent definition of a generalized Wiener process, we include a proof.

Proposition 4.7. *Let U be a separable Hilbert space, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ a stochastic basis and $Q \in L(U)$ self-adjoint and non-negative.*

W has a version which is a generalized Wiener process on U with covariance operator Q if and only if there is a separable Hilbert space U_1 with a Hilbert-Schmidt embedding $J : U_0 \hookrightarrow U_1$, an orthonormal system $(e_k)_{k\in\mathbb{I}}$ in U and independent real-valued Brownian motions $(\beta_k)_{k\in\mathbb{I}}$ such that $Q^{-\frac{1}{2}}U_0 \subset \overline{\operatorname{span}\left(\{e_k \mid k \in \mathbb{I}\}\right)}$ and

$$W(t) = \sum_{k\in\mathbb{I}} J\left(Q^{\frac{1}{2}}e_k\right)\beta_k(t) \ \mathbb{P}\text{-a.s. } \forall t \in I. \tag{4.2}$$

Proof. We will start by proving that (4.2) defines a generalized Wiener process on U with covariance operator Q. According to Definition 4.6 we have to show that W has a version which is a Wiener process on U_1 with covariance operator JJ^* in the sense of Definition 4.3.

To this end, let $(f_k)_{k\in\mathbb{N}}$ be an orthonormal basis of U_1 of eigenvectors of the self-adjoint trace-class operator $JJ^* : U_1 \to U_1$ with corresponding eigenvalues

[1] For $T \in L(U; H)$ one calculates $\|T\|^2_{L_2(U_0;H)} \le \|T\|^2_{L(U;H)}\|Q\|_{L_1(U)}$.

$(\lambda_k) \in \ell^1$. For $k \in \mathbb{N}$ with $\lambda_k \neq 0$ let

$$\tilde{\beta}_k := \frac{1}{\sqrt{\lambda_k}} \langle W(t), f_k \rangle_{U_1} = \frac{1}{\sqrt{\lambda_k}} \sum_{l \in \mathbb{I}} \left\langle JQ^{\frac{1}{2}} e_l, f_k \right\rangle_{U_1} \beta_l.$$

This defines a sequence of real-valued Brownian motions, which are independent because for $k \neq \tilde{k}$

$$\sum_{l \in \mathbb{I}} \left\langle JQ^{\frac{1}{2}} e_l, f_k \right\rangle_{U_1} \left\langle JQ^{\frac{1}{2}} e_l, f_{\tilde{k}} \right\rangle_{U_1} = \left\langle JQ^{\frac{1}{2}} Q^{-\frac{1}{2}} J^* f_k, f_{\tilde{k}} \right\rangle_{U_1} = \lambda_k \left\langle f_k, f_{\tilde{k}} \right\rangle_{U_1} = 0.$$

This definition at hand, (4.2) can be equivalently written as

$$W(t) = \sum_{\substack{k \in \mathbb{N} \\ \lambda_k \neq 0}} \sqrt{\lambda_k} f_k \tilde{\beta}_k(t).$$

Proposition 4.4 implies that W has a version which is a Wiener process on U_1 with covariance operator JJ^*.

For the opposite conclusion let us assume that W is a generalized Wiener process on U with covariance operator Q. By Definition 4.6 there is another separable Hilbert space U_1 with a Hilbert-Schmidt embedding $J : U_0 \hookrightarrow U_1$, such that W is a Wiener process on U_1 with covariance operator JJ^*.

We denote by $(\lambda_k)_{k \in \mathbb{I}} \in \ell^1$ the positive eigenvalues of JJ^* and by $(f_k)_{k \in \mathbb{I}} \subset U_1$ an orthonormal system of corresponding eigenvectors. Using the representation formula from Proposition 4.4, we find a sequence of independent real-valued Brownian motions $(\beta_k)_{k \in \mathbb{I}}$ such that

$$W(t) = \sum_{k \in \mathbb{I}} (JJ^*)^{\frac{1}{2}} f_k \beta_k(t) \ \mathbb{P}\text{-a.s. } \forall t \in I.$$

Let $g_k := \frac{1}{\sqrt{\lambda_k}} J^* f_k \in U_0$ for $k \in \mathbb{I}$. By definition $(g_k)_{k \in \mathbb{I}}$ is an orthonormal system in U_0. Because of the injectivity of J and the fact that $(f_k)_{k \in \mathbb{I}}$ has dense span in $\overline{JU_0}$, we know that $\operatorname{span}\{g_k \mid k \in \mathbb{I}\} = J^*(\operatorname{span}\{f_k \mid k \in \mathbb{I}\})$ is dense in U_0. Hence, $(g_k)_{k \in \mathbb{I}}$ is an orthonormal basis of U_0. Let $e_k := Q^{-\frac{1}{2}} g_k$ for $k \in \mathbb{I}$. Then again by construction $(e_k)_{k \in \mathbb{I}}$ is an orthonormal system in U with $Q^{-\frac{1}{2}} U_0 \subset \overline{\operatorname{span}\left(\{e_k \mid k \in \mathbb{I}\}\right)}$. Since $JQ^{\frac{1}{2}} e_k = \frac{1}{\sqrt{\lambda_k}} JJ^* f_k = \sqrt{\lambda_k} f_k$ we infer (4.2) and conclude the proof. $\square$

Remark 4.8. With the assumptions from Proposition 4.7 the representation formula (4.2) justifies to write

$$W(t) = \sum_{k \in \mathbb{N}} Q^{\frac{1}{2}} e_k \beta_k(t), \ t \in I$$

for a generalized Wiener process W on U, even though the sum only converges in the larger space U_1.

In the following proposition we will show that this generalization still allows us to make sense of $T(W(t))$ for $T \in L_2(U_0; H)$.

Proposition 4.9. *Let U and H be separable Hilbert spaces, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ a stochastic basis and W a generalized Wiener process on U with self-adjoint, non-negative covariance operator $Q \in L(U)$. Furthermore let U_0, U_1 and J as in Definition 4.6.*

For all $S \in L(U_1; H)$ it holds that

$$\|S(W(t))\|_{L^2(\Omega;H)} = \|S \circ J\|_{L_2(U_0;H)} \sqrt{t} \ \forall t \in I. \tag{4.3}$$

Furthermore $\{S \circ J \mid S \in L(U_1; H)\}$ is dense in $L_2(U_0; H)$, hence $TW := (T(W(t)))_{t\in I}$ is a well-defined H-valued stochastic process for all $T \in L_2(U_0; H)$. It holds that

$$T(W(t)) = \sum_{k\in\mathbb{N}} T\left(Q^{\frac{1}{2}} e_k\right) \beta_k(t) \ \mathbb{P}\text{-a.s. } \forall t \in I.$$

In particular the law of TW does not depend on the choice of U_1 and J.

Proof. Let

$$W(t) = \sum_{k\in\mathbb{N}} J\left(Q^{\frac{1}{2}} e_k\right) \beta_k(t) \mathbb{P}\text{-a.s. } \forall t \in I$$

be the representation of W from Proposition 4.7. We can assume that $(Q^{\frac{1}{2}} e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of U_0. For $S \in L(U_1; H)$ we infer for $t \in I$

$$S(W(t)) = \sum_{k\in\mathbb{N}} S \circ J\left(Q^{\frac{1}{2}} e_k\right) \beta_k(t)$$

and

$$\begin{aligned} \|S(W(t))\|^2_{L^2(\Omega;H)} &= t \sum_{k\in\mathbb{N}} \|S \circ J\left(Q^{\frac{1}{2}} e_k\right)\|^2_H \\ &= t\|S \circ J\|^2_{L_2(U_0;H)}. \end{aligned}$$

□

Once Proposition 4.9 is established, one can build up the theory of Itô integration. One starts by defining the integral of elementary processes, that is processes being piecewise constant in time and taking only a finite number of values in $L_2(U_0; H)$. Equation (4.3) extends to the so called Itô isometry for elementary processes between appropriate spaces. It is apparent that this definition then extends to a larger class of integrands using a density argument and gives rise to the so called

Itô integral. This procedure is explained in detail in [DPZ14, Section 4.2]. We have to fix some notation to summarize this result here.

For the remaining part of this section we will assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ is a stochastic basis with a normal filtration and U a separable Hilbert space. For the definition of a martingale we refer to [DPZ14].

Definition 4.10. Let $T \in I$. For a Banach space E we denote by $\mathcal{M}_T^2(E)$ the Banach space of all continuous square-integrable $(\mathcal{F}_t)$-adapted E-valued martingales M on $[0,T]$ with $M(0) = 0$ equipped with the norm

$$\|M\|^2_{\mathcal{M}_T^2(E)} \coloneqq \mathbb{E} \sup_{t\in[0,T]} \|M(t)\|_E^2.$$

Furthermore, for a generalized Wiener process W with covariance operator $Q \in L(U)$ and a separable Hilbert space H we denote by $\mathcal{N}_W^2(0,T) = \mathcal{N}_W^2(0,T; L_2(U_0;H))$ the space of all predictable $L_2(U_0;H)$-valued processes Φ for which the norm

$$\|\Phi\|^2_{\mathcal{N}_W^2(0,T)} \coloneqq \mathbb{E} \int_0^T \|\Phi(t)\|^2_{L_2(U_0;H)} \,\mathrm{d}t$$

is finite.

The space $\mathcal{N}_W(0,T) = \mathcal{N}_W(0,T; L_2(U_0;H))$ consists of all Φ as above satisfying the weaker bound

$$\mathbb{P}\left(\int_0^T \|\Phi(t)\|^2_{L_2(U_0;H)} \,\mathrm{d}t < \infty\right) = 1.$$

Remark 4.11. The spaces we have defined in Definition 4.10 will be of fundamental interest in the theory of Itô integration. We will see that the space $\mathcal{M}_T^2(H)$ contains all processes which can be written as the Itô integral of a process $\Phi \in \mathcal{N}_W^2(0,T; L_2(U_0;H))$.

The Itô integral of a process $\Phi \in \mathcal{N}_W(0,T; L_2(U_0;H))$ will still give rise to a continuous local martingale M, i.e. an $(\mathcal{F}_t)$-adapted continuous process such that there is a sequence of increasing stopping times $(\tau_k)_{k\in\mathbb{N}}$ with $\tau_k \to \infty$ a.s. and for all $k \in \mathbb{N}$ the stopped process $M^{\tau_k} \coloneqq (M_{\min(t,\tau_k)})_{t\in[0,T]}$ is a martingale. For the definition of a stopping time we refer to [DPZ14].

Definition 4.12 (Itô integral)**.** Let H be a separable Hilbert space and W a generalized Wiener process with covariance operator $Q \in L(U)$. There is an uniquely determined family $(\mathfrak{I}_T)_{T\in I}$ of linear isometries $\mathfrak{I}_T : \mathcal{N}_W^2(0,T) \to \mathcal{M}_T^2(H)$ with the following properties:

(i) for all $0 \le S \le T \in I$ it holds that $\mathfrak{I}_S$ and $\mathfrak{I}_T$ are compatible in the sense that for all $\Phi \in \mathcal{N}_W^2(0,T)$ and $t \in [0,S]$

$$\mathfrak{I}_T(\Phi)(t) = \mathfrak{I}_S(\Phi|_{[0,S]\times\Omega})(t),$$

allowing us to write $\mathfrak{I}_T(\Phi) =: (\int_0^t \Phi \,\mathrm{d}W)_{t\in[0,T]}$,

(ii) $\int_0^t \Phi \,\mathrm{d}W = \Phi W(t)$ for all $\Phi \in L_2(U_0; H)$ and

(iii) $\int_0^{t\wedge\tau} \Phi \,\mathrm{d}W = \int_0^t \chi_{[0,\tau]}\Phi \,\mathrm{d}W$ for all $\Phi \in \mathcal{N}_W^2(0,T)$ and stopping times $\tau \le T$.

For $\Phi \in \mathcal{N}_W^2(0,T)$ we call the process $\left(\int_0^t \Phi \,\mathrm{d}W\right)_{t\in[0,T]}$ the Itô integral of Φ with respect to W.

Remark 4.13. Note that the Itô integral is typically defined by its construction. However Definition 4.12 is well-suited to give a self-contained definition of the Itô integral directly summarizing some fundamental properties of it, without the need to rephrase the whole construction. The existence of the above notion of Itô integration is a consequence of [DPZ14, Section 4.2].

Furthermore, the definition gives a unique notion of Itô integration. This is due to the linearity of $\mathfrak{I}_T$ and (ii), because this yields the right notion for elementary processes and the class of elementary processes is dense in $\mathcal{N}_W^2(0,T)$, cf. [DPZ14, Proposition 4.22].

Part (iii) of Definition 4.12 allows us to carry out a localization procedure to extend Itô integration to the class $\mathcal{N}_W(0,T)$.

Definition 4.14. Let $T \in I$, H be a separable Hilbert space and W a generalized Wiener process with covariance operator $Q \in L(U)$. There is an unique extension of the Itô integral $\Phi \mapsto \left(\int_0^t \Phi \,\mathrm{d}W\right)_{t\in[0,T]}$ from the space $\mathcal{N}_W(0,T)$ into the space of all continuous local martingales, such that (iii) from Definition 4.12 holds for all $\Phi \in \mathcal{N}_W(0,T)$.

The next Proposition shows that the Itô integral commutes with linear operators, cf. [PR07, Lemma 2.4.1].

Proposition 4.15. *Let $T \in I$, $H, \tilde{H}$ be separable Hilbert spaces, W a generalized Wiener process with covariance operator $Q \in L(U)$ and $\Phi \in \mathcal{N}_W(0,T; L_2(U_0; H))$. For a linear operator $L \in L(H; \tilde{H})$ it holds that $L\Phi \in \mathcal{N}_W(0,T; L_2(U_0; \tilde{H}))$ and*

$$L\left(\int_0^T \Phi \,\mathrm{d}W\right) = \int_0^T L\Phi \,\mathrm{d}W \ \mathbb{P}\text{-a.s. in } \tilde{H}.$$

In [DPZ14, Section 3.4] the quadratic variation process of a Hilbert space valued martingale is introduced. For two Hilbert spaces H_1 and H_2 and $x \in H_1$, $y \in H_2$ we will denote by the tensor product $x \otimes y \in L(H_2; H_1)$ the linear operator $z \mapsto \langle z, y\rangle_{H_2} x$.

Definition 4.16 (Quadratic variation)**.** Let $T \in I$ and H_1, H_2 be separable Hilbert spaces. For $M \in \mathcal{M}_T^2(H_1)$ we will denote by $\langle\!\langle M \rangle\!\rangle$ the unique continuous and increasing $L_1(H_1)$-valued process starting in 0 such that

$$M \otimes M - \langle\!\langle M \rangle\!\rangle$$

is an $(\mathcal{F}_t)$-martingale. When $N \in \mathcal{M}_T^2(H_1)$ is another martingale, we will denote by

$$\langle\!\langle M, N \rangle\!\rangle := \frac{1}{4} \left(\langle\!\langle M + N \rangle\!\rangle - \langle\!\langle M - N \rangle\!\rangle \right)$$

the quadratic cross variation of M and N.

Using orthonormal bases $(e_k)_k$ and $(f_l)_l$ of H_1 and H_2, respectively, the above definition can be extended to define the quadratic cross variation of martingales $M \in \mathcal{M}_T^2(H_1), N \in \mathcal{M}_T^2(H_2)$ via

$$\langle\!\langle M, N \rangle\!\rangle := \sum_{k,l} \langle\!\langle \langle M, e_k \rangle_{H_1}, \langle N, f_l \rangle_{H_2} \rangle\!\rangle e_k \otimes f_l \in L_1(H_2; H_1).$$

Remark 4.17. One can extend the above notion of quadratic cross variation to so called semimartingales, for the definition of which we refer to [Mét82, 23.7 and 26.11].

We will only work with the following class of semimartingales: Let H be a separable Hilbert space, W a generalized Wiener process, X_0 an $\mathcal{F}_0$-measurable H-valued random variable, $\Phi \in \mathcal{N}_W(0, T)$ and φ a H-valued predictable process that is $\mathbb{P}$-a.s. Bochner integrable on $[0, T]$, i.e. $\int_0^T \|\varphi(t)\|_H \,\mathrm{d}t < \infty$ $\mathbb{P}$-a.s. Then the process X with

$$X(t) := X_0 + \int_0^t \varphi(s) \,\mathrm{d}s + \int_0^t \Phi(s) \,\mathrm{d}W(s), \ t \in [0, T], \tag{4.4}$$

is a semimartingale. We will often abbreviate (4.4) as

$$\mathrm{d}X = \varphi \,\mathrm{d}t + \Phi \,\mathrm{d}W.$$

The quadratic variation of X is given by

$$\langle\!\langle X \rangle\!\rangle = \langle\!\langle \int_0^{\cdot} \Phi \,\mathrm{d}W \rangle\!\rangle,$$

where the quadratic variation of the local martingale $\int_0^{\cdot} \Phi \,\mathrm{d}W$ is inferred using a localization procedure.

This definition at hand, we can state a more general version of [DPZ14, Theorem 4.27] which characterizes the quadratic variation of Itô integrals. Note that for $Q^{\frac{1}{2}} : U \to U_0$ we have $\left(Q^{\frac{1}{2}}\right)^* = Q^{-\frac{1}{2}} : U_0 \to U$ and $Q^{\frac{1}{2}} Q^{-\frac{1}{2}} = \mathrm{Id}_{U_0}$.

Proposition 4.18. *Let $T \in I$, H a separable Hilbert space, W a (generalized) Wiener process and $\Phi \in \mathcal{N}_W(0, T)$. Then*

$$\langle\!\langle \int_0^{\cdot} \Phi \,\mathrm{d}W \rangle\!\rangle_t = \int_0^t \Phi(s) \Phi^*(s) \,\mathrm{d}s \ \forall t \in [0, T].$$

Furthermore, the definition of the quadratic variation of a semimartingale allows us to define the Stratonovich integral by generalizing [Mét82, Chapter 5, E.8] to the infinite-dimensional case.

Definition 4.19. Let $T \in I$, H be a separable Hilbert space and W a generalized Wiener process with covariance operator $Q \in L(U)$. Furthermore, let $\Phi \in \mathcal{N}_W(0,T)$ be a semimartingale and $(Q^{\frac{1}{2}} g_l)_l$ a basis of U_0. The quadratic cross variation $\langle\!\langle \Phi, W \rangle\!\rangle$ is by definition an $L_1(U_1; L_2(U_0; H))$-valued process, but similar to the ideas of Proposition 4.9 can be treated as an $L_2(U_0; L_2(U_0; H))$-valued process.

The Stratonovich integral of Φ with respect to W is defined as

$$\int_0^T \Phi \circ \mathrm{d}W := \int_0^T \Phi \,\mathrm{d}W + \frac{1}{2}\left(\langle\!\langle \Phi, W \rangle\!\rangle_T Q^{\frac{1}{2}} g_l\right) Q^{\frac{1}{2}} g_l \in H.$$

We will refer to the term

$$\frac{1}{2}\left(\langle\!\langle \Phi, W \rangle\!\rangle_T Q^{\frac{1}{2}} g_l\right) Q^{\frac{1}{2}} g_l$$

as Itô-Stratonovich correction term, as it allows us to pass from the stochastic integral in the sense of Itô to the Stratonovich integral.

Remark 4.20. In the situation of Definition 4.19 let Φ be given by

$$\mathrm{d}\Phi = \mu \,\mathrm{d}t + \sigma \,\mathrm{d}W$$

with a predictable $L_2(U_0; H)$-valued process μ that is a.s. Bochner integrable and σ an $L_2(U_0; L_2(U_0; H))$-valued process integrable in the sense of Itô. Then

$$\langle\!\langle \Phi, W \rangle\!\rangle_T = \int_0^T \sigma(t) \,\mathrm{d}t.$$

Hence,

$$\int_0^T \Phi \circ \mathrm{d}W = \int_0^T \Phi \,\mathrm{d}W + \frac{1}{2}\sum_{l\in\mathbb{N}} \int_0^T \left(\sigma(t) Q^{\frac{1}{2}} g_l\right) Q^{\frac{1}{2}} g_l \,\mathrm{d}t.$$

Example 4.21. We revisit Remark 3.4 and apply the above considerations in order to find the Itô formulation of SMCF. For this we adapt the notation from Section 3.1, in particular let $(\beta^l)_{l\in\mathbb{N}}$ be independent real-valued Brownian motions. Furthermore, let U, H be separable Hilbert spaces. We assume that U is a function space of functions $\mathbb{R}^{n+1} \to \mathbb{R}$ containing the coefficients φ_l and H is a function space of functions $\mathcal{M} \to \mathbb{R}^{n+1}$ containing the immersions F_t.

The process $W := \sum_{l\in\mathbb{N}} \varphi_l \beta^l$ defines a Wiener process on U with covariance operator $Q := \sum_{l\in\mathbb{N}} \varphi_l \otimes \varphi_l$. For $t \in [0,T]$ we define the linear operator $\Phi(t) : U \to H$ through

$$\left(\Phi(t)g\right)(x) := g(F_t(x))\nu_t(x),\ g \in U.$$

With this notation the SPDE for the immersions $(F_t)_t$ (3.1) can be written as

$$\mathrm{d}F_t = \vec{H}\,\mathrm{d}t + \Phi \circ \mathrm{d}W.$$

In order to apply Remark 4.20 and rewrite this equation into its Itô formulation we need to compute the evolution of Φ. This is best done by considering the action of Φ on a function $g \in U$. Using the evolution of the normal field from Remark 3.3 we infer

$$(\mathrm{d}\Phi)g = \nabla g \cdot \vec{H}\nu - g\nabla_{\mathcal{M}_t} H\,\mathrm{d}t + \sum_{l\in\mathbb{N}} \left(\nabla g \cdot \nu\varphi_l \nu - g\nabla_{\mathcal{M}_t}\varphi_l\right) \circ \mathrm{d}\beta^l.$$

Let $\sigma(t) : U \to L(U;H)$ with

$$\Big(\left(\sigma(t)g_1\right) g_2\Big)(x) = \nabla g_2(F_t(x)) \cdot \nu_t(x) g_1(F_t(x))\nu_t(x) - g_2(F_t(x))\nabla_{\mathcal{M}_t} g_1(F_t(x))$$

for $g_1, g_2 \in U$. With this notation

$$\mathrm{d}\Phi = \ldots\mathrm{d}t + \sigma\,\mathrm{d}W.$$

We therefore conclude

$$\int_0^T \Phi \circ \mathrm{d}W = \int_0^T \Phi\,\mathrm{d}W + \frac{1}{2}\sum_{l\in\mathbb{N}} \int_0^T \Big(\varphi_l \left(\nabla\varphi_l\right)^{\perp} - \varphi_l \left(\nabla\varphi_l\right)^{T}\Big)\,\mathrm{d}t.$$

This implies the Itô formulation (3.2) of SMCF.

The Burkholder-Davis-Gundy inequality is an important inequality in the context of Hilbert space valued martingales and Itô integration, which we will use often throughout this thesis. We refer to [MR16] and the references therein.

Proposition 4.22 (Burkholder-Davis-Gundy)**.** *Let H be a separable Hilbert space. For $p > 0$ there are constants $c_p, C_p > 0$ such that for all H-valued continuous local martingales M with $M_0 = 0$ and $(\mathcal{F}_t)$-stopping times τ*

$$c_p\mathbb{E}\left(\mathrm{tr}\langle\!\langle M\rangle\!\rangle_\tau\right)^{\frac{p}{2}} \le \mathbb{E}\sup_{t\le\tau} \|M_t\|_H^p \le C_p\mathbb{E}\left(\mathrm{tr}\langle\!\langle M\rangle\!\rangle_\tau\right)^{\frac{p}{2}}.$$

When W is a (generalized) Wiener process and $\Phi \in \mathcal{N}_W(0,T)$ this reduces to

$$\begin{aligned} c_p\|\Phi\|^p_{L^p(\Omega;L^2(0,\tau;L_2(U_0;H)))} &\le \mathbb{E}\sup_{t\le\tau}\left\|\int_0^t \Phi(s)\,\mathrm{d}W(s)\right\|_H^p \\ &\le C_p\|\Phi\|^p_{L^p(\Omega;L^2(0,\tau;L_2(U_0;H)))}. \end{aligned}$$

Another important property of the Itô integral is its behavior under concatenation with sufficiently regular functions. The Itô formula is the substitute for the chain rule for differentiable functions. Before stating the Itô formula we introduce the notion of differentiable functions on a Banach space. For a more detailed exposition we refer to [BL00].

Definition 4.23. Let E, F be two Banach spaces, $U \subset E$ open and $f : U \to F$. We say that f is Gâteaux differentiable at $x_0 \in U$ if there is a bounded linear operator $\mathrm{D}f(x_0) : E \to F$ such that

$$\mathrm{D}f(x_0)\xi = \lim_{h\to 0} \frac{f(x_0 + h\xi) - f(x_0)}{h} \quad \forall \xi \in E.$$

We say that f is Fréchet differentiable at x_0 if there is a bounded linear operator $\mathrm{D}f(x_0) : E \to F$ such that

$$\lim_{\|\xi\|\to 0} \frac{f(x_0 + \xi) - f(x_0) - \mathrm{D}f(x_0)\xi}{\|\xi\|} = 0.$$

We say that f is continuous differentiable in U if f is Fréchet differentiable for all $x_0 \in U$ and the derivative $\mathrm{D}f : U \to L(E; F)$ is continuous. We denote by $C^1(U; F)$ the space of all continuous differentiable functions $f : U \to F$.

This definition naturally extends to higher derivatives, i.e. $f \in C^2(U; F)$ if $f \in C^1(U; F)$ and $\mathrm{D}f \in C^1(U; L(E; F))$. For $f \in C^2(U; F)$ we denote the second-order derivative by $\mathrm{D}^2 f : U \to L(E; L(E; F))$.

For $F = \mathbb{R}$ we write $C^k(U) \coloneqq C^k(U; \mathbb{R})$ for $k \in \mathbb{N}$.

A very classical version of the Itô formula is stated in [DPZ14, Theorem 4.32], which we cite here. Later, in Proposition 4.28, we will state a more general but less common version of the Itô formula in the context of variational SPDEs, which is a consequence of the classical one.

Proposition 4.24 (Itô formula)**.** *Let $T \in I$, H be a separable Hilbert space, W a (generalized) Wiener process, X_0 an $\mathcal{F}_0$-measurable H-valued random variable, $\Phi \in \mathcal{N}_W(0, T)$ and φ a H-valued predictable process, which is $\mathbb{P}$-a.s. Bochner integrable on $[0, T]$. Consider the continuous H-valued process*

$$X(t) \coloneqq X_0 + \int_0^t \varphi(s)\,\mathrm{d}s + \int_0^t \Phi(s)\,\mathrm{d}W(s), \ \ t \in [0, T].$$

For a function $F \in C^2([0, T] \times H)$ with F, $\partial_t F$, $\mathrm{D}F$ and $\mathrm{D}^2 F$ uniformly continuous on bounded subsets of $[0, T] \times H$ it holds that

$$\begin{aligned} \mathrm{d}F(t, X(t)) = {}& \partial_t F(t, X(t)) + \langle \mathrm{D}F(t, X(t)), \varphi(t)\rangle \\ &+ \frac{1}{2}\operatorname{tr}\left(\mathrm{D}^2 F(t, X(t))\Phi(t)\Phi^*(t)\right)\mathrm{d}t \\ &+ \langle \mathrm{D}F(t, X(t)), \Phi(t)\,\mathrm{d}W(t)\rangle . \end{aligned}$$

Next, we present a result called the Da Prato-Zabczyk factorization method, which we will us to prove Hölder regularity of stochastic integrals. A proof of it can be found in [Sei93].

Proposition 4.25 (Da Prato-Zabczyk factorization method)**.** *Let $T \in I$, H be a separable Hilbert space, W a (generalized) Wiener process, $p > 2$ and $\Phi \in \mathcal{N}_W^2(0,T)$ with $\mathbb{E}\int_0^T \|\Phi(t)\|_{L_2(U_0;H)}^p \,\mathrm{d}t < \infty$.*

For all $\lambda \in \left[0, \frac{1}{2} - \frac{1}{p}\right)$ there is a version of $\int_0^\cdot \Phi \,\mathrm{d}W$ which has λ-Hölder continuous paths in H and there is a constant C, which only depends on p, T, λ and U, such that

$$\mathbb{E}\left\|\int_0^\cdot \Phi \,\mathrm{d}W\right\|_{C^{0,\lambda}([0,T];H)}^p \leq C\mathbb{E}\int_0^T \|\Phi(t)\|_{L_2(U_0;H)}^p \,\mathrm{d}t.$$

4.2. Variational SPDEs under a compactness assumption

In this section we will consider infinite-dimensional stochastic differential equations with a variational structure and prove an Itô formula for this kind of equation. Let V, H be two separable Hilbert spaces with $V \subset H \simeq H' \subset V'$ and dense embeddings. For a Wiener process W on U and operators $A : V \to V'$ and $B : V \to L_2(U_0; H)$, which we do not assume to be linear, we will call

$$\mathrm{d}u = A(u)\,\mathrm{d}t + B(u)\,\mathrm{d}W \tag{4.5}$$

a variational SPDE. One of the difficulties that arise in the theory of variational SPDEs is the fact that the process u has to be V-valued in order to make sense of (4.5), whereas the right hand side of (4.5) in general only defines a V'-valued process.

The theory of variational SPDEs goes back to Pardoux [Par75] and Viot [Vio76]. We refer to [PR07] for an overview of the variational theory initiated by Pardoux.

An important tool in the study of variational SPDEs is an Itô formula, which adapts Proposition 4.24 to the special situation in order to infer the evolution of $F(u)$ for functions $F : H \to \mathbb{R}$, where Proposition 4.24 would only be applicable for functions $F : V' \to \mathbb{R}$. The Itô formula, which we state here, is related to the version in [Par75, II.II.§4] where more restrictive assumptions on F are made, see below.

During the whole section we will work with the following assumptions.

Assumptions 4.26. Let V and H be separable Hilbert spaces with $V \subset H \simeq H' \subset V'$ and V densely and compactly embedded in H. Furthermore, we will consider another separable Hilbert space U, which will be the space where a Wiener process is defined. For notational convenience we will restrict the presentation to the case of infinite-dimensional spaces, although finite-dimensional spaces could be treated as well.

Then we can find an orthonormal basis $(e^k)_{k\in\mathbb{N}}$ of H which is an orthogonal basis of V and we will use the abbreviation $\lambda_k := \|e^k\|_V^2$ for $k \in \mathbb{N}$. We will assume that the $(e^k)_k$ are arranged such that $(\lambda_k)_k$ is a non-decreasing sequence.

If not otherwise specified we will always assume that W is a (generalized) Wiener process on U with covariance operator $Q \in L(U)$. Furthermore, we will assume that W has the representation given by Proposition 4.7 and Remark 4.8, i.e.

$$W = \sum_{l\in\mathbb{N}} Q^{\frac{1}{2}} g_l \beta_l,$$

with $(g_l)_{l\in\mathbb{N}}$ an orthonormal system in U such that $Q^{\frac{1}{2}} g_l$ is a basis of U_0 and $(\beta_l)_{l\in\mathbb{N}}$ mutually independent real-valued $(\mathcal{F}_t)$-Brownian motions.

Remark 4.27 (Existence of $(e^k)_{k\in\mathbb{N}}$)**.** Let $J_V : V \to V'$ be the identification of V with its dual space V' via

$$\langle J_V x, y\rangle_{V',V} := \langle x, y\rangle_V\,,\ x, y \in V.$$

The restriction of the inverse of J_V to H together with the embedding of V into H gives $J_V^{-1}|_H : H \to H$ which is compact and self-adjoint. Hence we find an orthonormal basis $(e^k)_{k\in\mathbb{N}}$ of H of eigenvectors of J_V^{-1} with corresponding eigenvalues $\left(\frac{1}{\lambda_k}\right)_{k\in\mathbb{N}}$. For $k \in \mathbb{N}$ we have that $e^k = \lambda_k J_V^{-1} e^k \in V$ and

$$\left\langle e^k, e^l\right\rangle_V = \left\langle J_V e^k, e^l\right\rangle_{V',V} = \lambda_k \left\langle e^k, e^l\right\rangle_H = \lambda_k \delta_{k,l}\ \forall k, l \in \mathbb{N}.$$

Proposition 4.28 (Itô formula and continuity)**.** *Let $T > 0$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis with a normal filtration and W a (generalized) Wiener process on U. Furthermore let $u_0 \in L^2(\Omega; H)$ be $\mathcal{F}_0$-measurable and u, v, B be predictable processes with values in H, V' and $L_2(U_0; H)$, respectively, such that*

$$\begin{aligned} u &\in L^2(\Omega; L^2(0, T; V)),\\ v &\in L^2(\Omega; L^2(0, T; V')),\\ B &\in L^2(\Omega; L^2(0, T; L_2(U_0; H))), \end{aligned}$$

and

$$u(t) - u_0 = \int_0^t v(s)\,\mathrm{d}s + \int_0^t B(s)\,\mathrm{d}W(s) \quad \textit{in } V'\ \mathbb{P}\textit{-a.s.}\ \forall t \in [0, T]. \tag{4.6}$$

Then u has a version with continuous paths in H and for this version it holds that $u \in L^2(\Omega; C([0,T]; H))$ with

$$\|u(t)\|_H^2 - \|u_0\|_H^2 = \int_0^t 2\,\langle v(s), u(s)\rangle_{V',V} + \|B(s)\|^2_{L_2(U_0;H)}\,\mathrm{d}s \\ + 2\int_0^t \langle u(s), B(s)\,\mathrm{d}W(s)\rangle_H \quad \forall t \in [0,T].$$

Furthermore, if $F \in C^1(H)$ and the second Gâteaux derivative $\mathrm{D}^2 F : H \to L(H)$ exists with

- *F, $\mathrm{D}F$ and $\mathrm{D}^2 F$ bounded on bounded subsets of H,*
- *$\mathrm{D}^2 F : H \to L(H)$ continuous from the strong topology on H to the weak-$*$ topology on $L(H) = (L_1(H))'$ and*
- *$(\mathrm{D}F)|_V : V \to V$ continuous from the strong topology on V to the weak topology on V and growing at most linearly*

$$\|\,\mathrm{D}F(x)\|_V \le C\,(1 + \|x\|_V) \quad \forall x \in V,$$

then

$$F(u(t)) - F(u_0) = \int_0^t \langle v(s), \mathrm{D}F(u(s))\rangle_{V',V} \\ + \frac{1}{2}\,\mathrm{tr}\left[\mathrm{D}^2 F(u(s)) B(s)(B(s))^*\right]\mathrm{d}s \\ + \int_0^t \langle \mathrm{D}F(u(s)), B(s)\,\mathrm{d}W(s)\rangle_H \quad \mathbb{P}\text{-}a.s.\ \forall t \in [0,T].$$

Note that this generalization is similar to the result presented in [Par75, II.II.§4] where it was proven under slightly different assumptions. There, less is assumed about the spaces, i.e. the embedding $V \subset H$ is not assumed to be compact and V is assumed to be an arbitrary separable Banach space, which is uniformly smooth and convex, but F is assumed to be twice Fréchet differentiable. In our application, this assumption on F will not be fulfilled.

By analyzing the proof in [Par75, II.II.§4] one can see that this restrictive assumption on the differentiability of F can be relaxed. It is sufficient to assume the Gâteaux differentiability of $\mathrm{D}F : H \to H$ and the weak-$*$ continuity of $\mathrm{D}^2 F : H \to L(H)$, which ensures that the restriction $F|_V : V \to \mathbb{R}$ is twice Fréchet differentiable. Note that this Itô formula also appears in [Par79, I.3.2] and [Vio76, I.§1 Theorem 1.3] and with the weaker assumption on F in [Par07, Lemma 2.3.5] without proof.

Hence, for the readers convenience we include a different proof here, which makes use of the stronger assumptions on the spaces compared to [Par75] and adapts the proof of [DHV16, Proposition A.1], where H and V are assumed to be Sobolev spaces. In (4.7) below, we define a smoothing operator which in fact is the semigroup generated by $-J_V$ with $J_V : V \to V'$ given by the canonical identification of the Hilbert space V with its dual space. If $V = H_0^1(\mathbb{T}^n)$, $H = L^2(\mathbb{T}^n)$, then this is the classical heat semigroup.

Proof of Proposition 4.28. Step 1: Smoothing the solution.
For $\varepsilon > 0$ we consider the smoothing operator

$$\rho_\varepsilon : V' \to V, \qquad \rho_\varepsilon(v') := \sum_{k\in\mathbb{N}} \exp(-\varepsilon\lambda_k) \left\langle v', e^k \right\rangle_{V',V} e^k. \tag{4.7}$$

It is easy to verify that with $\varepsilon \to 0$ and $X \in \{V', H, V\}$

$$\rho_\varepsilon(x) \to x \text{ in } X \; \forall x \in X$$

and $\|\rho_\varepsilon\|_{L(X)} \leq 1$. Since $\rho_\varepsilon \in L(V'; V)$ we infer from Proposition 4.15 for all $t \in [0, T]$ that

$$\rho_\varepsilon(u(t)) - \rho_\varepsilon(u_0) = \int_0^t \rho_\varepsilon(v(s)) \,\mathrm{d}s + \int_0^t \rho_\varepsilon \circ B(s) \,\mathrm{d}W(s) \;\mathbb{P}\text{-a.s. in } V,$$

where $\rho_\varepsilon \circ B(s) : U_0 \to V$ denotes the composition of $B(s) : U_0 \to H$ with $\rho_\varepsilon : H \to V$.

In the following we will abbreviate $u_{0,\varepsilon} := \rho_\varepsilon(u_0)$, $u_\varepsilon(t) := \rho_\varepsilon(u(t))$, $v_\varepsilon(t) := \rho_\varepsilon(v(t))$ and $B_\varepsilon(t) := \rho_\varepsilon \circ B(t)$.

Step 2: u takes values in H.
The Itô formula from Proposition 4.24 for the function $\|\cdot\|_H^2$ implies

$$\begin{aligned} \|u_\varepsilon(t)\|_H^2 - \|u_{0,\varepsilon}\|_H^2 = &\int_0^t 2 \left\langle u_\varepsilon(s), v_\varepsilon(s) \right\rangle_H + \|B_\varepsilon(s)\|_{L_2(U_0;H)}^2 \,\mathrm{d}s \\ &+ 2 \int_0^t \left\langle u_\varepsilon(s), B_\varepsilon(s) \,\mathrm{d}W(s) \right\rangle_H \;\mathbb{P}\text{-a.s.} \end{aligned}$$

The Burkholder-Davis-Gundy inequality, cf. Proposition 4.22, $\|\rho_\varepsilon\|_{L(X)} \leq 1$ for all $X \in \{V', H, V\}$ and the Young inequality imply

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon(t)\|_H^2 &\leq C\|u_0\|_H^2 + C\|u\|_{L^2(\Omega;L^2(0,T;V))} \|v\|_{L^2(\Omega;L^2(0,T;V'))} \\ &\quad + C\|B\|_{L^2(\Omega;L^2(0,T;L_2(U_0;H)))}^2 + C\mathbb{E} \left(\int_0^T \|B(s)\|_{L_2(U_0;H)}^2 \|u_\varepsilon(s)\|_H^2 \,\mathrm{d}s \right)^{\frac{1}{2}} \\ &\leq C\|u_0\|_H^2 + C\|u\|_{L^2(\Omega;L^2(0,T;V))} \|v\|_{L^2(\Omega;L^2(0,T;V'))} \\ &\quad + C\|B\|_{L^2(\Omega;L^2(0,T;L_2(U_0;H)))}^2 + \frac{1}{2}\mathbb{E} \sup_{t\in[0,T]} \|u_\varepsilon(t)\|_H^2. \end{aligned}$$

Thus $\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_H^2$ is uniformly bounded in ε. With Fatou's Lemma we infer that

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|_H^2 &= \mathbb{E}\sup_{t\in[0,T]}\lim_{\varepsilon\to 0}\|u_\varepsilon(t)\|_H^2\\ &\le \mathbb{E}\liminf_{\varepsilon\to 0}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_H^2\\ &\le \liminf_{\varepsilon\to 0}\mathbb{E}\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_H^2<\infty.\end{aligned}\tag{4.8}$$

Step 3: Proving the Itô formula.
Now, in Lemma 4.29 below it is verified that $F|_V\in C^2(V)$ with $F|_V$, $\mathrm{D}(F|_V)$ and $\mathrm{D}^2(F|_V)$ uniformly continuous on bounded subsets of V. We apply the Itô formula from Proposition 4.24 to conclude that

$$\begin{aligned}F(u_\varepsilon(t))-F(u_{0,\varepsilon}) &= \int_0^t \langle v_\varepsilon(s),\mathrm{D}F(u_\varepsilon(s))\rangle_{V',V}\\ &\quad+\frac{1}{2}\int_0^t \mathrm{tr}\left[\mathrm{D}^2F(u_\varepsilon(s))B_\varepsilon(s)(B_\varepsilon(s))^*\right]\mathrm{d}s\\ &\quad+\int_0^t\langle \mathrm{D}F(u_\varepsilon(s)),B_\varepsilon(s)\,\mathrm{d}W(s)\rangle_H \quad \mathbb{P}\text{-a.s. } \forall t\in[0,T].\end{aligned}\tag{4.9}$$

Because of the assumptions on F and an infinite dimensional version of the dominated convergence theorem for stochastic integrals [Pro04, Theorem IV.32] we can pass to the limit $\varepsilon\to 0$ on both sides of this equation.

Hence, $F(u)$ has a continuous version for which

$$\begin{aligned}F(u(t))-F(u_0) &= \int_0^t \langle v(s),\mathrm{D}F(u(s))\rangle_{V',V}\,\mathrm{d}s\\ &\quad+\frac{1}{2}\int_0^t \mathrm{tr}\left[\mathrm{D}^2F(u(s))B(s)(B(s))^*\right]\mathrm{d}s\\ &\quad+\int_0^t\langle \mathrm{D}F(u(s)),B(s)\,\mathrm{d}W(s)\rangle_H \quad \forall t\in[0,T].\end{aligned}$$

Step 4: u has a continuous version.
We infer from the calculations above that there is a version of u such that $\|u\|_H^2$ is continuous and

$$\begin{aligned}\|u(t)\|_H^2-\|u_0\|_H^2 &= \int_0^t 2\langle v(s),u(s)\rangle_{V',V}+\|B(s)\|^2_{L_2(U_0;H)}\,\mathrm{d}s\\ &\quad+2\int_0^t\langle u(s),B(s)\,\mathrm{d}W(s)\rangle_H \quad \forall t\in[0,T].\end{aligned}$$

From (4.6) and (4.8) we infer that $u\in C([0,T];(H,w))$ a.s. It is well known (cf. [PR07, Theorem 4.2.5]) that this together with the continuity of $\|u\|_H^2$ implies that $u \in C([0,T];H)$ a.s. Since H is separable we can apply [DPZ14, Proposition 3.18] to conclude that $u : \Omega \to C([0,T];H)$ is measurable. This proves that $u \in L^2(\Omega; C([0,T];H))$. □

Lemma 4.29. *Under the assumptions of Proposition 4.28 we have $F|_V \in C^2(V)$ and $F|_V$, $\mathrm{D}F|_V : V \to V'$ and $\mathrm{D}^2F|_V : V \to L(V;V')$ are uniformly continuous on bounded subsets of V.*

Proof. We only have to prove the continuity of $\mathrm{D}^2F|_V : V \to L(V;V')$ and the uniform continuity on bounded subsets of V.

The compactness of the embeddings $V \subset H \simeq H' \subset V'$ implies that the embedding $L(H) \subset L(V;V')$ is compact. Thus, when $u_k \rightharpoonup u$ in V then $u_k \to u$ in H and by the assumptions from Proposition 4.28 we infer $\mathrm{D}^2F(u_k) \overset{*}{\rightharpoonup} \mathrm{D}^2F(u)$ in $L(H)$, hence $\mathrm{D}^2F(u_k) \to \mathrm{D}^2F(u)$ in $L(V;V')$. This proves that $\mathrm{D}^2F|_V : (V,w) \to L(V;V')$ is continuous.

Let $M \subset V$ be a bounded set in V, then M is precompact in H and therefore $F|_M : M \to \mathbb{R}$ and $\mathrm{D}F|_M : M \to H \subset V'$ are uniformly continuous. Furthermore M is precompact in (V,w) and therefore $\mathrm{D}^2F|_M : M \to L(V;V')$ is uniformly continuous. □

We will apply Proposition 4.28 to the appropriate spaces for the stochastic mean curvature flow, cf. Remark 5.1.

Corollary 4.30. *Let $T > 0$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis with a normal filtration and W a generalized Wiener process on U. Furthermore let $u_0 \in L^2(\Omega; H^1(\mathbb{T}^n))$ be $\mathcal{F}_0$-measurable and u, v, B be predictable processes with $u \in L^2(\Omega; L^2(0,T;H^2(\mathbb{T}^n)))$, $v \in L^2(\Omega; L^2(0,T;L^2(\mathbb{T}^n)))$ and $B \in L^2(\Omega; L^2(0,T;L_2(U_0;H^1(\mathbb{T}^n))))$ such that*

$$\mathrm{d}u = v\,\mathrm{d}t + B\,\mathrm{d}W \text{ in } L^2(\mathbb{T}^n). \tag{4.10}$$

Then u has a version with continuous paths in $H^1(\mathbb{T}^n)$ and for this version it holds that $u \in L^2(\Omega; C([0,T];H^1(\mathbb{T}^n)))$. If $F = F(z,p) \in C^2(\mathbb{R} \times \mathbb{R}^n)$ with $\partial_z^2 F$, $\partial_z \nabla_p F$ and $\mathrm{D}_p^2 F$ bounded then we have

$$\int_{\mathbb{T}^n} F(u(t), \nabla u(t))\,\mathrm{d}x - \int_{\mathbb{T}^n} F(u_0, \nabla u_0)\,\mathrm{d}x = \int_0^t \mu(s)\,\mathrm{d}s + \sum_{l\in\mathbb{N}} \int_0^t \sigma_l(s)\,\mathrm{d}\beta_l(s).$$

a.s. for all $t \in [0,T]$ with

$$\begin{aligned}\mu(s) = &\int_{\mathbb{T}^n} \Big(\partial_z F(u(s), \nabla u(s)) - \nabla\cdot\left(\nabla_p F(u(s), \nabla u(s))\right)\Big) v(s)\,\mathrm{d}x \\ &+ \frac{1}{2}\sum_{l\in\mathbb{N}} \int_{\mathbb{T}^n} \partial_{zz} F(u(s), \nabla u(s)) \left|B_l(s)\right|^2 \mathrm{d}x \\ &+ \frac{1}{2}\sum_{l\in\mathbb{N}} \int_{\mathbb{T}^n} \partial_z \nabla_p F(u(s), \nabla u(s)) \cdot \nabla \left(\left|B_l(s)\right|^2\right) \mathrm{d}x \\ &+ \frac{1}{2}\sum_{l\in\mathbb{N}} \int_{\mathbb{T}^n} \nabla B_l(s) \cdot \mathrm{D}_p^2 F(u(s), \nabla u(s)) \nabla B_l(s)\,\mathrm{d}x\end{aligned}$$

and for $l \in \mathbb{N}$

$$\sigma_l(s) = \int_{\mathbb{T}^n} \partial_z F(u(s), \nabla u(s)) B_l(s) + \nabla_p F(u(s), \nabla u(s)) \cdot \nabla B_l(s)\,\mathrm{d}x,$$

where $B_l \coloneqq BQ^{\frac{1}{2}} g_l \in L^2(\Omega; L^2(0,T; H^1(\mathbb{T}^n)))$. Note that the assumptions imply the integrability of the right hand side in the above expression.

Proof. We consider the spaces $V = H^2(\mathbb{T}^n)$ and $H = H^1(\mathbb{T}^n)$. To work in the framework from above we have to do the rather unusual identification of $w \in H^1(\mathbb{T}^n)$ with $J_H w \coloneqq -\Delta w + w \in H'$ where

$$\langle J_H w, \varphi\rangle_{H',H} = \langle w, \varphi\rangle_{H^1(\mathbb{T}^n)} = \int_{\mathbb{T}^n} \nabla w \cdot \nabla\varphi + w\varphi,\ \ \varphi \in H^1(\mathbb{T}^n).$$

Then

$$\begin{aligned}\mathrm{d}\langle J_H u, w\rangle_{H',H} &= \mathrm{d}\langle u, -\Delta w + w\rangle_{L^2(\mathbb{T}^n)} \\ &= \langle v, -\Delta w + w\rangle_{L^2(\mathbb{T}^n)}\,\mathrm{d}t \\ &\quad + \langle B\,\mathrm{d}W, -\Delta w + w\rangle_{L^2(\mathbb{T}^n)}\ \ \forall w \in H^2(\mathbb{T}^n),\end{aligned} \tag{4.11}$$

which is an equation for $J_H u$ in V'. We consider the function $G : H^1(\mathbb{T}^n) \to \mathbb{R}$ with

$$G(w) \coloneqq \int_{\mathbb{T}^n} F(w(x), \nabla w(x))\,\mathrm{d}x,\ \ w \in H^1(\mathbb{T}^n).$$

Since $F \in C^2$ it easy to check that $G \in C^1(H^1(\mathbb{T}^n))$ and that the second Gâteaux derivative $\mathrm{D}^2 G$ exists. We calculate for $w, \varphi, \psi \in H^1(\mathbb{T}^n)$

$$\begin{aligned}\langle \mathrm{D}G(w), \varphi\rangle_{H',H} &= \int_{\mathbb{T}^n} \partial_z F(w, \nabla w)\varphi + \nabla_p F(w, \nabla w) \cdot \nabla\varphi, \\ \left\langle \mathrm{D}^2 G(w)\varphi, \psi\right\rangle_{H',H} &= \int_{\mathbb{T}^n} \partial_{zz} F(w, \nabla w)\varphi\psi + \partial_z \nabla_p F(w, \nabla w) \cdot (\varphi\nabla\psi + \psi\nabla\varphi) \\ &\quad + \int_{\mathbb{T}^n} \nabla\varphi \cdot \mathrm{D}_p^2 F(w, \nabla w) \nabla\psi.\end{aligned}$$

We have that G and $\mathrm{D}G$ are bounded on bounded subsets of $H^1(\mathbb{T}^n)$ and that D^2G is bounded because of the bounds of the second derivatives of F. On bounded subsets of $L(H) = \big(L_1(H)\big)'$ the weak-$*$ topology is equivalent to the weak operator topology and therefore the continuity of $\mathrm{D}^2G : H \to (L(H), w^*)$ follows from the fact that for all $w_k \to w$ in $H^1(\mathbb{T}^n)$ and all $\varphi, \psi \in H^1(\mathbb{T}^n)$ we have by dominated convergence

$$\begin{aligned}
&\Big\langle \mathrm{D}^2G(w_k)\varphi, \psi \Big\rangle_{H',H} \\
&= \int_{\mathbb{T}^n} \partial_{zz}F(w_k, \nabla w_k)\varphi\psi + \partial_z\nabla_p F(w_k, \nabla w_k)\cdot(\varphi\nabla\psi + \psi\nabla\varphi) \\
&\to \int_{\mathbb{T}^n} \partial_{zz}F(w, \nabla w)\varphi\psi + \partial_z\nabla_p F(w, \nabla w)\cdot(\varphi\nabla\psi + \psi\nabla\varphi) \\
&= \Big\langle \mathrm{D}^2G(w)\varphi, \psi \Big\rangle_{H',H}.
\end{aligned}$$

For $w \in H^2(\mathbb{T}^n), \varphi \in H^1(\mathbb{T}^n)$ we have

$$\begin{aligned}
&\langle \mathrm{D}G(w), \varphi\rangle_{H',H} \\
&= \int_{\mathbb{T}^n} \Big(\partial_z F(w, \nabla w) - \partial_z\nabla_p F(w, \nabla w)\cdot\nabla w - \mathrm{D}_p^2F(w, \nabla w) : \mathrm{D}^2 w\Big)\,\varphi \\
&= \int_{\mathbb{T}^n} \Phi(w)\varphi,
\end{aligned}$$

with

$$\Phi(w) \coloneqq \partial_z F(w, \nabla w) - \partial_z\nabla_p F(w, \nabla w)\cdot\nabla w - \mathrm{D}_p^2F(w, \nabla w) : \mathrm{D}^2 w,$$

where $\mathrm{D}_p^2F(w, \nabla w) : \mathrm{D}^2w$ denotes the matrix scalar product, cf. Definition C.1. Because of the assumptions on F we find that $\Phi(w) \in L^2(\mathbb{T}^n)$ for $w \in H^2(\mathbb{T}^n)$ and $\Phi : H^2(\mathbb{T}^n) \to L^2(\mathbb{T}^n)$ is continuous. Since for the restriction of J_H to $H^2(\mathbb{T}^n)$ we have that $J_H|_{H^2(\mathbb{T}^n)} : H^2(\mathbb{T}^n) \to L^2(\mathbb{R}^n)$ is an isomorphism, we conclude $J_H^{-1}\circ\Phi(w) \in H^2(\mathbb{T}^n)$ and $J_H^{-1}\circ\Phi : H^2(\mathbb{T}^n) \to H^2(\mathbb{T}^n)$ is continuous with

$$\|J_H^{-1}\Phi(w)\|_{H^2(\mathbb{T}^n)} \le C\|\Phi(w)\|_{L^2(\mathbb{T}^n)} \le C(1 + \|w\|_{H^2(\mathbb{T}^n)}).$$

Note that for the application of Proposition 4.28 we shall have an equation for $\mathrm{d}u \cong \mathrm{d}J_H u$ in V', whereas (4.10) is an equation for $\mathrm{d}u$ in $L^2(\mathbb{T}^n)$. Therefore we have to use (4.11) to infer that a.s. for all $t \in [0, T]$

$$\begin{aligned}
&G(u(t)) - G(u_0) \\
&\quad= \int_0^t \Big\langle J_H v(s), J_H^{-1}\circ\Phi(u(s))\Big\rangle_{V',V} + \frac{1}{2}\operatorname{tr}\Big[\mathrm{D}^2G(u(s))B(s)(B(s))^*\Big]\,\mathrm{d}s \\
&\qquad+ \int_0^t \langle \mathrm{D}G(u(s)), B(s)\,\mathrm{d}W(s)\rangle_{H',H}.
\end{aligned}$$

Plugging in the definition of Φ as well as the expressions for D^2G and $\mathrm{D}G$ implies the result. □

4.3. Existence for variational SPDEs

We will adapt the approach of [Par07, Section 2.3.3] to prove existence of weak solutions for variational SPDEs

$$\begin{aligned} \mathrm{d}u &= A(u)\,\mathrm{d}t + B(u)\,\mathrm{d}W \\ u(0) &= u_0. \end{aligned} \tag{4.12}$$

This theory goes back to [Vio76], where an existence proof is given for certain operators A which have an additive structure and are linear in the highest order. In [Par07] this proof is generalized to a larger class of variational SPDEs, which is still not optimal and too restrictive in order to apply it to SMCF. We will give a detailed comparison of our result and the result in [Par07] after fixing the assumptions.

Without loss of generality we assume that W is a cylindrical Wiener process on a Hilbert space U. In addition to Assumptions 4.26 we will make the following assumptions.

Assumptions 4.31. Let $A : V \to V'$ and $B : V \to L_2(U; H)$. We will write $B^* : V \to L_2(H; U)$ for the adjoint operator $B^*(u) := \left(B(u)\right)^*$. We assume:

- **Coercivity:** There are constants $\alpha, C > 0$ such that for all $u \in V$

$$2\left\langle A(u), u\right\rangle_{V',V} + \|B(u)\|^2_{L_2(U_0;H)} \leq -\alpha\|u\|^2_V + C\left(1 + \|u\|^2_H\right). \tag{4.13}$$

- **Growth bounds:** There is a constant $C > 0$ and $\delta \in (0, 2]$ such that for all $u \in V$

$$\|A(u)\|^2_{V'} \leq C\left(1 + \|u\|^2_V\right), \tag{4.14}$$

$$\|B(u)\|^2_{L_2(U;H)} \leq C\left(1 + \|u\|^2_V\right) \text{ and} \tag{4.15}$$

$$\|B(u)\|^2_{L(U;V')} \leq C\left(1 + \|u\|^{2-\delta}_V + \|u\|^2_H\right). \tag{4.16}$$

- **Continuity:** $A : V \to V'$ is weak-weak-$*$ sequentially continuous, that means

$$u_k \rightharpoonup u \text{ in } V \;\Rightarrow\; A(u_k) \overset{*}{\rightharpoonup} A(u) \text{ in } V' \tag{4.17}$$

and $B^* : V \to L_2(H; U)$ is sequentially continuous from the weak topology on V to the strong operator topology on $L(H; U)$, that means

$$u_k \rightharpoonup u \text{ in } V \;\Rightarrow\; B^*(u_k)h \to B^*(u)h \text{ in } U\ \forall h \in H. \tag{4.18}$$

The assumptions (4.13), (4.14) and (4.17) are the same as in [Par07], whereas (4.18) is weaker. Furthermore we have replaced the sublinear growth bound from [Par07] for $B(u)$ by the weaker assumptions (4.15) and (4.16). These weaker assumptions are necessary to apply the theory to the stochastic mean curvature flow in Chapter 5. To prove this generalization we have to prove bounds for higher moments of the $\|\cdot\|_H$ norm of the approximations, whereas in the proof in [Par07] only the second moment of the $\|\cdot\|_H$ norm needed to be bounded. This will be done in Proposition 4.36 under the additional assumption that the corresponding higher moment of the $\|\cdot\|_H$ norm is bounded for the initial data. Similarly to the ideas of [HRvR17], we will use the Jakubowski-Skorokhod representation theorem [Jak97] for tight sequences in non-metric spaces to prove that our approximations converge on a different probability space. We will make use of similar arguments as in [BFHM18] to handle the unbounded time interval. Finally, we will show that this limit is a martingale solution of (4.12) using a general method of constructing martingale solutions without relying on any kind of martingale representation theorem, which was introduced in [BO07] and already used in [Ond10] and [HRvR17], among others.

Before stating the result, we define the different notions of solutions.

Definition 4.32.

(i) Let $I = [0,\infty)$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ be a stochastic basis with a normal filtration, W an $(\mathcal{F}_t)$-Wiener process on U and $u_0 \in L^2(\Omega; H)$ be $\mathcal{F}_0$-measurable. A predictable H-valued process u with $u \in L^2(\Omega; L^2(0,t;V))$ for all $t \in I$ is a strong solution of (4.12) with initial data u_0 if

$$\begin{aligned}\langle u(t), v\rangle_H - \langle u_0, v\rangle_H &= \int_0^t \langle A(u(s)), v\rangle_{V',V}\,\mathrm{d}s + \int_0^t \langle B(u(s))\,\mathrm{d}W(s), v\rangle_H \\ &= \int_0^t \langle A(u(s)), v\rangle_{V',V}\,\mathrm{d}s + \sum_{l\in\mathbb{N}} \int_0^t \langle B(u(s))g_l, v\rangle_H\,\mathrm{d}\beta_l(s)\end{aligned}$$

$\mathbb{P}$-a.s. for all $t \in I$ and $v \in V$.

(ii) Let Λ be a Borel probability measure on H with bounded second moments $\int_H \|z\|_H^2\,\mathrm{d}\Lambda(z) < \infty$. A martingale solution of (4.12) is given by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ together with W, u_0 and u such that (i) is satisfied and $\mathbb{P}\circ u_0^{-1} = \Lambda$.

Remark 4.33. Note that in Definition 4.32 for a martingale solution the stochastic basis is not prescribed but part of the solution. The concept of a martingale solution is a weak solution concept and therefore the term weak solution is a common synonym for it. In the context of SPDEs this notion is misleading as it might be mixed up with the concept of a weak solution in the deterministic PDE theory. We will therefore stick to the notion of a martingale solution even for finite-dimensional SDEs.

Our main result is:

Theorem 4.34. *Let $q > 2$ and Λ be a Borel probability measure on H with finite q-th moment*

$$\int_H \|z\|_H^q \, \mathrm{d}\Lambda(z) < \infty.$$

Then there is a martingale solution of (4.12) *with initial data Λ.*

To prove Theorem 4.34, we will use a standard Galerkin scheme (compare with [Par07, Chapter 2.3]) to prove that there is a martingale solution of (4.12) if the initial condition has bounded q-th moment in H for some $q > 2$. With the $(e^k)_{k\in\mathbb{N}}$ as in Assumptions 4.26 we will write

$$V_N \coloneqq \operatorname{span}\left(\{e^1, \dots, e^N\}\right), \ N \in \mathbb{N}.$$

Theorem 4.35. *Let $N \in \mathbb{N}$ and Λ be a Borel probability measure on H. Then there is a martingale solution of the finite-dimensional approximation of* (4.12).

That means, that there is a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty)}, \mathbb{P})$ with a normal filtration, $\beta_1, \dots, \beta_N$ mutually independent real-valued $(\mathcal{F}_t)$-Brownian motions and a predictable V_N-valued process u with $u \in L^2(\Omega; C([0,T]; V_N))$ for all $T > 0$ such that

$$\begin{aligned}\langle u(t), v\rangle_H &- \langle u(0), v\rangle_H \\ &= \int_0^t \langle A(u(s)), v\rangle_{V',V} \, \mathrm{d}s + \sum_{l=1}^N \int_0^t \langle B(u(s))g_l, v\rangle_H \, \mathrm{d}\beta_l(s)\end{aligned}$$

$\mathbb{P}$-a.s. for all $t \in [0,\infty)$ and $v \in V_N$, and

$$\mathbb{P} \circ u(0)^{-1} = \Lambda_N \coloneqq \Lambda \circ P_N^{-1},$$

where $P_N : H \to V_N$ is the orthogonal projection with respect to H.

Proof. We transform the equation into an N-dimensional stochastic differential equation for the $\mathbb{R}^N$-valued coefficients $a = (a_k)_{k=1,\dots,N}$ of $u = \sum_{k=1}^N a_k e^k$. For this let

$$\tilde{A} : \mathbb{R}^N \to \mathbb{R}^N$$
$$\left(\tilde{A}(a)\right)_k \coloneqq \left\langle A\left(\sum_{m=1}^N a_m e^m\right), e^k\right\rangle_{V',V}, \ k = 1, \dots, N, \ a \in \mathbb{R}^N$$

and

$$\tilde{B} : \mathbb{R}^N \to \mathbb{R}^{N\times N}$$
$$\left(\tilde{B}(a)\right)_l^k := \left\langle B\left(\sum_{m=1}^N a_m e^m\right) g_l, e^k \right\rangle_H, \ k,l = 1,\dots,N, \ a \in \mathbb{R}^N.$$

These mappings are continuous and grow at most linearly. Therefore we can apply a classical theorem for finite dimensional stochastic differential equations [HS12, Theorem 0.1] and [IW81, Theorem IV.2.4] to find a martingale solution of

$$\mathrm{d}a_k = \left(\tilde{A}(a)\right)_k \mathrm{d}t + \sum_{l=1}^N \left(\tilde{B}(a)\right)_l^k \mathrm{d}\beta_l, \ k = 1,\dots,N$$
$$\mathbb{P}\circ a(0)^{-1} = \tilde{\mathbb{P}}_N,$$

where

$$\tilde{\mathbb{P}}_N(M) := \Lambda_N\left(\left\{\sum_{k=1}^N a_k e^k \in H \,\middle|\, a \in M\right\}\right), \ M \subset \mathbb{R}^N.$$

Defining $u(t) = \sum_{k=1}^N a_k(t)e^k$ for $t \in [0,\infty)$ we find $u \in L^2(\Omega; C([0,T]; V_N))$ for all $T > 0$ with

$$\begin{aligned}\left\langle u(t), e^k\right\rangle_H - \left\langle u_0, e^k\right\rangle_H &= a_k(t) - a_k(0)\\ &= \int_0^t \left(\tilde{A}(a(s))\right)_k \mathrm{d}s + \sum_{l=1}^N \left(\tilde{B}(a(s))\right)_l^k \mathrm{d}\beta_l(s)\\ &= \int_0^t \left\langle A(u(s)), e^k\right\rangle_{V',V} \mathrm{d}s + \sum_{l=1}^N \int_0^t \left\langle B(u(s))g_l, e^k\right\rangle_H \mathrm{d}\beta_l(s)\end{aligned}$$

and

$$\mathbb{P}\circ u(0)^{-1} = \Lambda_N.$$

□

Proposition 4.36 (Estimates for the norm)**.** *Let $T > 0$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a stochastic basis with a normal filtration. Then there is a constant $C > 0$ that only depends on the constants from Assumptions 4.31, such that for all mutually independent real-valued $(\mathcal{F}_t)$-Brownian motions $(\beta_l)_{l\in\mathbb{N}}$, $N \in \mathbb{N}$ and all V_N-valued predictable processes $u \in L^2(\Omega; C([0,T]; V_N))$ with*

$$\begin{aligned}&\langle u(t), v\rangle_H - \langle u(0), v\rangle_H\\ &\quad= \int_0^t \langle A(u(s)), v\rangle_{V',V} \mathrm{d}s + \sum_{l=1}^N \int_0^t \langle B(u(s))g_l, v\rangle_H \mathrm{d}\beta_l(s)\end{aligned}$$

$\mathbb{P}$-a.s. for all $t \in [0,T]$ and $v \in V_N$, we have

$$\mathbb{E} \sup_{t\in[0,T]} \|u(t)\|_H^2 + \mathbb{E}\int_0^T \|u(t)\|_V^2 \, \mathrm{d}t \le Ce^{CT}\left(1 + \mathbb{E}\|u(0)\|_H^2\right).$$

Additionally, there is a $q_0 > 2$ such that $u(0) \in L^q(\Omega; H)$ for some $q \in (2, q_0)$ implies $u \in L^\infty(0,T; L^q(\Omega; H))$ with

$$\mathbb{E}\|u(t)\|_H^q \le e^{Ct}\left(1 + \mathbb{E}\|u(0)\|_H^q\right) \ \forall t \in [0,T].$$

Proof. From Proposition 4.28 we conclude that the following Itô formula holds for the norm of solutions

$$\begin{aligned} \mathrm{d}\|u\|_H^2 &= 2\left\langle A(u), u\right\rangle_{V',V} \mathrm{d}t + \sum_{k,l=1}^N \left\langle B(u)g_l, e^k\right\rangle_H^2 \mathrm{d}t + \sum_{l=1}^N 2\left\langle B(u)g_l, u\right\rangle_H \mathrm{d}\beta_l \\ &= 2\left\langle A(u), u\right\rangle_{V',V} \mathrm{d}t + \|B_N(u)\|_{L_2(U;H)}^2 \mathrm{d}t + 2\left\langle B_N(u)\,\mathrm{d}W, u\right\rangle_H, \end{aligned} \tag{4.19}$$

where $B_N : V \to L_2(U;H)$ is given by

$$\left\langle B_N(u)g_l, e^k\right\rangle_H := \begin{cases} \left\langle B(u)g_l, e^k\right\rangle_H & \text{if } k \le N \text{ and } l \le N, \\ 0 & \text{otherwise,} \end{cases} \quad k, l \in \mathbb{N},\ u \in V.$$

For $q \ge 1$ we use the Itô formula for real-valued semimartingales to deduce that

$$\begin{aligned} \mathrm{d}\left(1 + \|u\|_H^2\right)^q &= q\left(1 + \|u\|_H^2\right)^{q-1}\left(2\left\langle A(u), u\right\rangle_{V',V} + \|B_N(u)\|_{L_2(U;H)}^2\right)\mathrm{d}t \\ &\quad + 2q(q-1)\left(1 + \|u\|_H^2\right)^{q-2}\|B_N^*(u)u\|_U^2 \,\mathrm{d}t \\ &\quad + 2q\left(1 + \|u\|_H^2\right)^{q-1}\left\langle B_N(u)\,\mathrm{d}W, u\right\rangle_H. \end{aligned}$$

Since the Itô integral defines a local martingale and $1 + \|u\|_H^2 \ge 0$, we infer by taking the expectation for the stopped processes and Fatou's lemma that for all $t \in [0,T]$

$$\begin{aligned} &\mathbb{E}\left(1 + \|u(t)\|_H^2\right)^q - \mathbb{E}\left(1 + \|u(0)\|_H^2\right)^q \\ &\quad \le q\mathbb{E}\int_0^t \left(1 + \|u(s)\|_H^2\right)^{q-1}\left(2\left\langle A(u(s)), u(s)\right\rangle_{V',V} + \|B_N(u(s))\|_{L_2(U;H)}^2\right)\mathrm{d}s \\ &\qquad + 2q(q-1)\mathbb{E}\int_0^t \left(1 + \|u(s)\|_H^2\right)^{q-2}\|B_N^*(u(s))u(s)\|_U^2 \,\mathrm{d}s. \end{aligned}$$

Using the coercivity (4.13) and the growth bounds (4.15) we conclude for $q \in [1, 1+\varepsilon)$ with $\varepsilon < \frac{\alpha}{C}$ where C depends on the constants from (4.13) and (4.15), that

$$
\begin{aligned}
&\mathbb{E}\left(1+\|u(t)\|_H^2\right)^q - \mathbb{E}\left(1+\|u(0)\|_H^2\right)^q \\
&\quad\le q\mathbb{E}\int_0^t \left(1+\|u(s)\|_H^2\right)^{q-1}\left(-\alpha\|u(s)\|_V^2 + C\left(1+\|u(s)\|_H^2\right)\right)\mathrm{d}s \\
&\qquad + Cq(q-1)\mathbb{E}\int_0^t \left(1+\|u(s)\|_H^2\right)^{q-1}\|B_N^*(u(s))\|_{L_2(H;U)}^2\,\mathrm{d}s \\
&\quad\le -q\left(\alpha - C(q-1)\right)\mathbb{E}\int_0^t \left(1+\|u(s)\|_H^2\right)^{q-1}\|u(s)\|_V^2\,\mathrm{d}s \\
&\qquad + C\mathbb{E}\int_0^t \left(1+\|u(s)\|_H^2\right)^q \mathrm{d}s \\
&\quad\le C\mathbb{E}\int_0^t \left(1+\|u(s)\|_H^2\right)^q \mathrm{d}s \;\forall t\in[0,T]
\end{aligned}
\tag{4.20}
$$

and with a Gronwall argument

$$
\mathbb{E}\left(1+\|u(t)\|_H^2\right)^q \le e^{Ct}\mathbb{E}\left(1+\|u(0)\|_H^2\right)^q \;\;\forall t\in[0,T].
$$

This already implies that there is a constant $C>0$ such that

$$
\mathbb{E}\|u(t)\|_H^{2q} \le e^{Ct}\left(1+\mathbb{E}\|u(0)\|_H^{2q}\right)\;\;\forall t\in[0,T].
$$

Furthermore, we have for the stochastic integral in (4.19) using the Burkholder-Davis-Gundy inequality from Proposition 4.22 and the growth assumption (4.15)

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\left|2\int_0^t \left\langle B_N(u)\,\mathrm{d}W, u\right\rangle_H\right| &\le C\mathbb{E}\left[\int_0^T \|B_N^*(u(s))u(s)\|_U^2\,\mathrm{d}s\right]^{\frac{1}{2}} \\
&\le C\mathbb{E}\left[\sup_{t\in[0,T]}\|u(s)\|_H^2\left(1+\int_0^T\|u(s)\|_V^2\,\mathrm{d}s\right)\right]^{\frac{1}{2}} \\
&\le \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|u(s)\|_H^2 + C\left(1+\mathbb{E}\int_0^T\|u(s)\|_V^2\,\mathrm{d}s\right).
\end{aligned}
\tag{4.21}
$$

And from the last inequality in (4.20) for $q=1$ we infer

$$
\mathbb{E}\int_0^T \|u(s)\|_V^2\,\mathrm{d}s \le e^{CT}\left(1+\mathbb{E}\|u(0)\|_H^2\right),
$$

hence with (4.13) and (4.21)

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|_H^2 &\le \mathbb{E}\|u(0)\|_H^2 \\
&\quad + \mathbb{E}\sup_{t\in[0,T]}\int_0^t \left[2\left\langle A(u(\tau)),u(\tau)\right\rangle_{V',V} + \|B_N(u(\tau))\|_{L_2(U;H)}^2\right]\mathrm{d}\tau \\
&\quad + \mathbb{E}\sup_{t\in[0,T]} 2\int_0^t \left\langle B_N(u(\tau))\,\mathrm{d}W(\tau),u(\tau)\right\rangle_H \\
&\le \mathbb{E}\|u(0)\|_H^2 + \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|_H^2 + C\left(1+\mathbb{E}\int_0^T\|u(t)\|_V^2\,\mathrm{d}t\right) \\
&\le Ce^{CT}\left(1+\mathbb{E}\|u(0)\|_H^2\right) + \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|_H^2
\end{aligned}
$$

and therefore

$$
\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|_H^2 \le Ce^{CT}\left(1+\mathbb{E}\|u(0)\|_H^2\right).
$$

□

The proof of Theorem 4.34 will be a consequence of a more general result, which characterizes the limit of solutions of SPDEs. This result will also be applied in Chapter 5 in order to prove that there is a martingale solution of graphical SMCF.

For this we allow the coefficients to vary, as long as suitable assumptions similar to Assumptions 4.31 hold.

Theorem 4.37. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a fixed probability space. For $N\in\mathbb{N}$ let $(\mathcal{F}_t^N)_{t\in[0,\infty)}$ be a normal filtration and W^N a generalized $(\mathcal{F}_t^N)$-Wiener process on U with covariance operator $Q_N\in L(U)$ such that $Q_N\to Q$ in the strong operator topology (SOT) on $L(U)$. We denote the corresponding reproducing kernel spaces by $U_0^N := Q_N^{\frac{1}{2}}U$ and $U_0 := Q^{\frac{1}{2}}U$.*

In addition to Assumptions 4.26 we assume that $(A_N)_N,(B_N)_N$ are sequences of operators $A_N : V\to V'$ and $B_N : V\to L_2(U_0^N;H)$ with:

- ***Growth bounds:*** *There is a constant $C>0$ and $\delta\in(0,2]$ such that for all $N\in\mathbb{N}$ and $u\in V$*

$$\|A_N(u)\|_{V'}^2 \le C(1+\|u\|_V^2), \tag{4.22}$$

$$\|B_N(u)\|_{L_2(U_0^N;H)}^2 \le C(1+\|u\|_V^2) \text{ and} \tag{4.23}$$

$$\|B_N(u)\|_{L(U_0^N;V')}^2 \le C(1+\|u\|_V^{2-\delta}+\|u\|_H^2). \tag{4.24}$$

- ***Continuity:*** *There are $A : V \to V'$ and $B : V \to L_2(U_0; H)$ such that for all $u_N \rightharpoonup u$ in V it holds that*

$$A_N(u_N) \overset{*}{\rightharpoonup} A(u) \text{ in } V', \tag{4.25}$$

$$Q_N^{-\frac{1}{2}} B_N^*(u_N) \xrightarrow{SOT} Q^{-\frac{1}{2}} B^*(u) \text{ in } L(H; U). \tag{4.26}$$

Let u^N be a solution of

$$\mathrm{d}u^N = A_N(u^N)\,\mathrm{d}t + B_N(u^N)\,\mathrm{d}W^N, \tag{4.27}$$

with $\mathbb{P} \circ u^N(0)^{-1} \rightharpoonup \Lambda$ in the sense of probability laws on H (cf. [IW81, Definition 2.1]) such that for some $q > 2$ and all $T > 0$ it holds that $(u^N)_N$ is uniformly bounded in

$$L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega; L^2(0,T; V)) \cap L^\infty(0,T; L^q(\Omega; H)).$$

Then there is a martingale solution of

$$\mathrm{d}u = A(u)\,\mathrm{d}t + B(u)\,\mathrm{d}W \tag{4.28}$$

with initial data Λ, where W is a generalized Wiener process with covariance operator Q.

Before proving Theorem 4.37 we need the following Lemma, which provides convergence results for the deterministic operators involved.

Lemma 4.38. *Let the assumptions of Theorem 4.37 hold. Furthermore let $T > 0$ and*

$$\mathcal{X}_u := (L^2(0,T; V), w) \cap L^2(0,T; H) \cap C([0,T]; (H, w))$$

with $C([0,T]; (H,w))$ endowed with the compact-open topology.

Then for all $u_N \to u$ in $\mathcal{X}_u$ and $p < 2$ it holds that for all $v \in V$

$$\langle A_N(u_N), v \rangle_{V',V} \to \langle A(u), v \rangle_{V',V} \text{ in } L^p(0,T)$$

and

$$Q_N^{-\frac{1}{2}} B_N^*(u_N) v \to Q^{-\frac{1}{2}} B^*(u) v \text{ in } L^2(0,T; U).$$

Proof. Let $v \in V$ and $(u_N)_N \subset \mathcal{X}_u$ be a sequence with $u_N \to u$ in $\mathcal{X}_u$. For $M > 0$ we consider the functions

$$u_N^M(t) := \begin{cases} u_N(t) & \text{if } \|u_N(t)\|_V \le M, \\ u(t) & \text{otherwise} \end{cases}, \quad t \in [0,T], N \in \mathbb{N}.$$

Since $u \in L^2(0,T;V)$ we conclude that for almost every $t \in [0,T]$ the sequence $(u_N^M(t))_{N\in\mathbb{N}}$ is uniformly bounded in V. Furthermore we know for every $t \in [0,T]$ that $u_N^M(t) \rightharpoonup u(t)$ in H, because $u_N \to u$ in $\mathcal{X}_u$ implies

$$\left|\left\langle u_N^M(t) - u(t), h\right\rangle_H\right| \leq \left|\langle u_N(t) - u(t), h\rangle_H\right| \to 0 \text{ as } N \to \infty \; \forall h \in H.$$

Hence $u_N^M(t) \rightharpoonup u(t)$ in V for $N \to \infty$ for almost every $t \in [0,T]$.

The continuity assumptions (4.25) and (4.26) from Theorem 4.37 imply

$$\begin{aligned} \left\langle A_N(u_N^M(t)), v\right\rangle_{V',V} &\to \langle A(u(t)), v\rangle_{V',V} \qquad \text{and} \\ Q_N^{-\frac{1}{2}} B_N^*(u_N^M(t))v &\to Q^{-\frac{1}{2}} B^*(u(t))v \end{aligned}$$

for almost every $t \in [0,T]$. Furthermore, using the growth assumption (4.22) we get

$$\left|\left\langle A_N(u_N^M(t)), v\right\rangle_{V',V}\right|^2 \leq C\left(1 + \|u(t)\|_V^2 + \|u_N(t)\|_V^2\right)\|v\|_V^2.$$

Since the right hand side of the above inequality is bounded in $L^1(0,T)$, we infer with Vitali's convergence theorem for $p < 2$

$$\left\langle A_N(u_N^M), v\right\rangle_{V',V} \to \langle A(u), v\rangle_{V',V} \text{ in } L^p(0,T).$$

In order to infer a similar result for B, we use the growth bound (4.24), which by convergence also holds for B, and deduce

$$\begin{aligned} &\left\|Q_N^{-\frac{1}{2}} B_N^*(u_N^M(t))v - Q^{-\frac{1}{2}} B^*(u(t))v\right\|_U^2 \\ &\quad\leq C\left(\|B_N(u_N^M(t))\|_{L(U_0^N;V')}^2 + \|B(u(t))\|_{L(U_0;V')}^2\right)\|v\|_V^2 \\ &\quad\leq C\left(1 + \|u_N(t)\|_V^{2-\delta} + \|u(t)\|_V^{2-\delta} + \|u_N(t)\|_H^2 + \|u(t)\|_H^2\right)\|v\|_V^2. \end{aligned}$$

The right hand side is uniformly integrable, because $\|u_N\|_V^{2-\delta}$ is bounded in $L^{\frac{2}{2-\delta}}(0,T)$ and $\|u_N\|_H^2$ is convergent in $L^1(0,T)$. Therefore, by Vitali's convergence theorem

$$Q_N^{-\frac{1}{2}} B_N^*(u_N^M)v \to Q^{-\frac{1}{2}} B^*(u)v \text{ in } L^2(0,T;U).$$

Let $E_N^M \coloneqq \{t \in [0,T] \mid \|u_N(t)\|_V > M\}$ for $N \in \mathbb{N}$. For the measure of E_N^M we estimate

$$\left|E_N^M\right| \leq \int_0^T \frac{\|u_N(t)\|_V^2}{M^2}\,\mathrm{d}t \leq \frac{C}{M^2},$$

because $(u_N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^2(0,T;V)$.

As above one can conclude from the growth assumptions (4.22) and (4.24) and the fact that $\|u_N^M(t)\|_V \le \|u_N(t)\|_V + \|u(t)\|_V$ and $\|u_N^M(t)\|_H \le \|u_N(t)\|_H + \|u(t)\|_H$ that

$$\left|A_N(u_N)v - A_N(u_N^M)v\right|^p \text{ and } \left\|Q_N^{-\frac{1}{2}}B_N^*(u_N)v - Q_N^{-\frac{1}{2}}B_N^*(u_N^M)v\right\|_U^2$$

are uniformly integrable with respect to N and M. Hence,

$$\begin{aligned}\|A_N(u_N)v - A(u)v\|_{L^p(0,T)} \le{}& \|A_N(u_N)v - A_N(u_N^M)v\|_{L^p(E_N^M)} \\ &+ \|A_N(u_N^M)v - A(u)v\|_{L^p(0,T)}\end{aligned}$$

and

$$\begin{aligned}&\|Q_N^{-\frac{1}{2}}B_N^*(u_N)v - Q^{-\frac{1}{2}}B^*(u)v\|_{L^2(0,T;U)} \\ &\qquad\le \|Q_N^{-\frac{1}{2}}B_N^*(u_N)v - Q_N^{-\frac{1}{2}}B_N^*(u_N^M)v\|_{L^2(E_N^M;U)} \\ &\qquad\quad+ \|Q_N^{-\frac{1}{2}}B_N^*(u_N^M)v - Q^{-\frac{1}{2}}B^*(u)v\|_{L^2(0,T;U)}\end{aligned}$$

converge to 0 by first choosing M large such that the first terms on the right hand side become small and then choosing N large and using the convergences derived above. □

The proof of Theorem 4.37 will be carried out in several steps. We start by proving a weak Hölder continuity of the approximate solutions and the Wiener processes. With a compactness result of Simon [Sim87] we will conclude that the joint laws of the processes are tight, i.e. there is a sequence of increasing compact sets such that the probability of leaving these compact sets goes to zero, cf. [DPZ14]. The characterization of tight sequences given by Jakubowski [Jak97] yields the existence of a limit. With the convergence result Lemma 4.38 from above we will infer that this limit is a martingale solution of (4.28).

Proof of Theorem 4.37. Step 1: Uniform Hölder continuity of (u^N) *and* (W^N)*.* Note that by Hölder's inequality (u^N) is uniformly bounded in $L^\infty(0,T;L^{q'}(\Omega;H))$ for all $2 < q' \le q$. Hence, without loss of generality we might choose $q > 2$ sufficiently small during the next arguments.

Let Z be another separable Hilbert space with a Hilbert-Schmidt embedding $V' \subset Z$. Because (4.24) holds uniformly in N we have that $B_N(u^N)$ is uniformly bounded in $L^q(\Omega;L^q(0,T;L_2(U_0^N;Z)))$, since by the Hilbert-Schmidt embedding

$V' \subset Z$ it holds that $\|S\|_{L_2(U_0^N;Z)} \leq C\|S\|_{L(U_0^N;V')}$ for all $S \in L(U_0^N;V')$ and

$$\mathbb{E}\int_0^T \|B_N(u^N(t))\|^q_{L_2(U_0^N;Z)}\,\mathrm{d}t \leq C\mathbb{E}\int_0^T \|B_N(u^N(t))\|^q_{L(U_0^N;V')}\,\mathrm{d}t$$
$$\leq C\left(1+\mathbb{E}\int_0^T \|u^N(t)\|_V^2 + \sup_{t\in[0,T]}\mathbb{E}\|u^N(t)\|_H^q\right).$$

Note that $B_N(u^N) \in \mathcal{N}^2_{W^N}(0,T)$ for $N \in \mathbb{N}$. Therefore we can apply the factorization method stated in Proposition 4.25 to infer a uniform bound for the stochastic integral $\int_0^\cdot B_N(u^N(t))\,\mathrm{d}W_t^N \in L^2(\Omega; C^{0,\lambda}([0,T];Z))$ for some $\lambda > 0$. Furthermore, since (4.14) holds uniformly in N we get a uniform bound for $\int_0^\cdot A_N(u^N(t))\,\mathrm{d}t \in L^2(\Omega; C^{0,\frac{1}{2}}([0,T];V'))$. This implies by (4.27) a uniform bound for $u^N \in L^2(\Omega; C^{0,\lambda}([0,T];Z))$ for some $\lambda > 0$.

In order to infer the uniform Hölder continuity of the Wiener processes (W^N) we consider the separable Hilbert space U_1 given as the completion of U with respect to the scalar product $\langle g_{l_1}, g_{l_2}\rangle_{U_1} = a_{l_1}^2\delta_{l_1,l_2}$ for $l_1, l_2 \in \mathbb{N}$ and $(a_l)_{l\in\mathbb{N}} \in \ell^2$ a positive square-summable sequence. Then U is densely embedded in U_1 with a Hilbert-Schmidt embedding. The representation of W^N as in Proposition 4.7 converges in U_1. Hence, each W^N can be understood as a Wiener process on U_1 and the covariance operator fulfills $Q_N \in L_1(U_1)$. Since Q_N converges pointwise to Q, the Q_N are uniformly bounded in $L(U)$ by the uniform boundedness principle and therefore uniformly bounded in $L_1(U_1)$. The factorization method Proposition 4.25 for $W^N(t) = \int_0^t \Phi_N\,\mathrm{d}W^N$ where $\Phi_N : U_0^N \to U_1$ is the embedding of U_0^N into U_1 implies that for $\lambda \in (0,\frac{1}{2})$ the $(W^N)_N$ are uniformly bounded in $L^2(\Omega; C^{0,\lambda}([0,T];U_1))$.

Step 2: Tightness of the joint laws of $(u^N(0), u^N, W^N)$.
For $\lambda > 0$ the embeddings

$$C^{0,\lambda}([0,T];Z) \cap C([0,T];H) \to C([0,T];Z) \text{ and}$$
$$C^{0,\lambda}([0,T];Z) \cap L^2(0,T;V) \to L^2(0,T;H)$$

are compact because of the Ascoli theorem [Kel55, Theorem 7.17] and the characterization of compact sets in $L^2(0,T;H)$ in [Sim87, Theorem 5]. As on bounded subsets of H the weak topology (H,w) and the topology of Z coincide we also infer that the embedding

$$C^{0,\lambda}([0,T];Z) \cap C([0,T];H) \to C([0,T];(H,w))$$

is compact.

The embedding

$$C^{0,\lambda}([0,T];U_1) \to C([0,T];(U_1,w))$$

is also compact because of the Ascoli theorem [Kel55, Theorem 7.17]. Furthermore, the laws $\mathbb{P} \circ u^N(0)^{-1}$ are tight in H because of Prokhorov's theorem [IW81, Theorem 2.6].

Thus, the joint laws of $(u^N(0), u^N, W^N)$ are tight in $H \times \mathcal{X}_u^T \times \mathcal{X}_W^T$ with

$$\begin{aligned} \mathcal{X}_u^T &\coloneqq C([0,T];(H,w)) \cap L^2(0,T;H) \cap \Big(L^2(0,T;V),w\Big) \text{ and} \\ \mathcal{X}_W^T &\coloneqq C\left([0,T];(U_1,w)\right). \end{aligned}$$

This holds for all $T > 0$. Note that a set is compact in $\mathcal{X}_u \times \mathcal{X}_W$ with

$$\begin{aligned} \mathcal{X}_u &\coloneqq C_{\mathrm{loc}}([0,\infty);(H,w)) \cap L^2_{\mathrm{loc}}(0,\infty;H) \cap \Big(L^2_{\mathrm{loc}}(0,\infty;V),w\Big) \text{ and} \\ \mathcal{X}_W &\coloneqq C_{\mathrm{loc}}\left([0,\infty);(U_1,w)\right), \end{aligned}$$

where $C_{\mathrm{loc}}([0,\infty);(H,w))$ and $C_{\mathrm{loc}}([0,\infty);(U_1,w))$ are endowed with the compact-open topology, if and only if for all $T > 0$ the set (with all of its elements restricted to $[0,T]$) is compact in $\mathcal{X}_u^T \times \mathcal{X}_W^T$. We conclude similarly to [BFHM18, Proof of Proposition 4.3] that the joint laws of $(u^N(0), u^N, W^N)$ are tight in $H \times \mathcal{X}_u \times \mathcal{X}_W$.

Step 3: Existence of a.s. converging random variables on a different probability space.
Since the space $H \times \mathcal{X}_u \times \mathcal{X}_w$ satisfies a separability condition, see Lemma 4.39 below, we can apply the Jakubowski-Skorokhod representation theorem for tight sequences in nonmetric spaces [Jak97, Theorem 2]. We deduce the existence of a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, an strictly increasing sequence $(N_m)_{m\in\mathbb{N}} \subset \mathbb{N}$, $\mathcal{X}_u$-valued random variables $\tilde{u}^m$, $\tilde{u}$ and $\mathcal{X}_W$-valued random variables $\tilde{W}^m$, $\tilde{W}$ for $m \in \mathbb{N}$ such that

$$\begin{aligned} \tilde{u}^m(0) \to \tilde{u}(0) \ \tilde{\mathbb{P}}\text{-a.s. in } H, \\ \tilde{u}^m \to \tilde{u} \ \tilde{\mathbb{P}}\text{-a.s. in } \mathcal{X}_u, \\ \tilde{W}^m \to \tilde{W} \ \tilde{\mathbb{P}}\text{-a.s. in } \mathcal{X}_W \end{aligned}$$

and the joint law of $(\tilde{u}^m, \tilde{W}^m)$ coincides with the joint law of (u^{N_m}, W^{N_m}) for all $m \in \mathbb{N}$. To simplify the notation, we will assume that $N_m = m$ for $m \in \mathbb{N}$.

Step 4: Identification of the limit.
Let $(\mathcal{G}_t)_{t\in[0,\infty)}$ be the natural filtration of the process $(\tilde{u}, \tilde{W})$. That means $\mathcal{G}_t$ for $t \in [0,\infty)$ is the smallest σ-algebra such that $\tilde{u}(s) : \tilde{\Omega} \to H$ and $\tilde{W}(s) : \tilde{\Omega} \to U_1$ are measurable for all $s \in [0,t]$. The Pettis measurability theorem implies that for the Borel σ-algebras on H and U_1 we have $\mathcal{B}((H, \|\cdot\|_H)) = \mathcal{B}((H,w))$ and $\mathcal{B}((U_1, \|\cdot\|_{U_1})) = \mathcal{B}((U_1,w))$. Therefore $\mathcal{G}_t$ is contained in the σ-algebra generated by $\tilde{u}|_{[0,t]} : \tilde{\Omega} \to C([0,t];(H,w))$ and $\tilde{W}|_{[0,t]} : \tilde{\Omega} \to C([0,t];(U_1,w))$. Choosing dense subsets of $[0,t]$ and H respectively U_1, one can also show that $\tilde{u}|_{[0,t]}$ and $\tilde{W}|_{[0,t]}$ are measurable with respect to $\mathcal{G}_t$. Hence $\mathcal{G}_t$ is exactly the σ-algebra generated by $\tilde{u}|_{[0,t]}$ and $\tilde{W}|_{[0,t]}$.

Let $\mathcal{N} := \{M \in \tilde{\mathcal{F}} \mid \tilde{\mathbb{P}}(M) = 0\}$. We will consider the augmented filtration $(\tilde{\mathcal{F}}_t)_{t\in[0,\infty)}$ which is defined by

$$\tilde{\mathcal{F}}_t := \bigcap_{s>t} \sigma\left(\mathcal{G}_s \cup \mathcal{N}\right), \ t \in [0,\infty).$$

The augmented filtration $(\tilde{\mathcal{F}}_t)_t$ is a normal filtration. For $m \in \mathbb{N}$ we can do the same construction to define the natural filtration $(\mathcal{G}^m_t)_t$ and the augmented filtration $(\tilde{\mathcal{F}}^m_t)_t$ of $(\tilde{u}^m, \tilde{W}^m)$.

Let $t \in [0,\infty)$ and

$$\tilde{M}(t) := \tilde{u}(t) - \tilde{u}(0) - \int_0^t A(\tilde{u}(s))\,\mathrm{d}s, \tag{4.29}$$

$$\tilde{M}^m(t) := \tilde{u}^m(t) - \tilde{u}^m(0) - \int_0^t A_m(\tilde{u}^m(s))\,\mathrm{d}s, \tag{4.30}$$

$$M^m(t) := u^m(t) - u^m(0) - \int_0^t A_m(u^m(s))\,\mathrm{d}s = \int_0^t B_m(u^m(s))\,\mathrm{d}W^m(s). \tag{4.31}$$

For $s \in [0,t]$ let

$$\gamma : C([0,s];(H,w)) \times C([0,s];(U_1,w)) \to \mathbb{R}$$

be a bounded and continuous function. We will use the abbreviations

$$\begin{aligned} \gamma^m &:= \gamma\left(u^m|_{[0,s]}, W^m|_{[0,s]}\right), \\ \tilde{\gamma}^m &:= \gamma\left(\tilde{u}^m|_{[0,s]}, \tilde{W}^m|_{[0,s]}\right), \\ \tilde{\gamma} &:= \gamma\left(\tilde{u}|_{[0,s]}, \tilde{W}|_{[0,s]}\right) \end{aligned}$$

and for a time-dependent process X we write $X\Big|_s^t := X(t) - X(s)$.

Note that the joint law of $(\tilde{u}^m, \tilde{W}^m)$ coincides with the joint law of (u^m, W^m). The processes W^m and M^m are martingales and we can compute their quadratic variation using Proposition 4.18, which we encode in the next equations:

$$0 = \mathbb{E}\left(\gamma^m W^m\Big|_s^t\right) = \tilde{\mathbb{E}}\left(\tilde{\gamma}^m \tilde{W}^m\Big|_s^t\right) \text{ and} \tag{4.32}$$

$$(t-s)\,Q_m = \mathbb{E}\left(\gamma^m\left(W^m \otimes W^m\right)\Big|_s^t\right) = \tilde{\mathbb{E}}\left(\tilde{\gamma}^m\left(\tilde{W}^m \otimes \tilde{W}^m\right)\Big|_s^t\right) \tag{4.33}$$

as well as

$$0=\mathbb{E}\left(\gamma^m M^m\Big|_s^t\right)=\tilde{\mathbb{E}}\left(\tilde{\gamma}^m\tilde{M}^m\Big|_s^t\right)\quad\text{and}\tag{4.34}$$

$$\begin{aligned}0&=\mathbb{E}\left(\gamma^m\left(M^m\otimes M^m\right)\Big|_s^t-\gamma^m\int_s^t B_m(u^m(\sigma))B_m^*(u^m(\sigma))\,\mathrm{d}\sigma\right)\\&=\tilde{\mathbb{E}}\left(\tilde{\gamma}^m\left(\tilde{M}^m\otimes\tilde{M}^m\right)\Big|_s^t-\tilde{\gamma}^m\int_s^t B_m(\tilde{u}^m(\sigma))B_m^*(\tilde{u}^m(\sigma))\,\mathrm{d}\sigma\right).\end{aligned}\tag{4.35}$$

We also have to encode the fact that M^m can be written as an Itô integral driven by W^m. Together with the above equations this can be recovered from the following equation which holds in $L(U;H)$:

$$\begin{aligned}0&=\mathbb{E}\left(\gamma^m\left(M^m\otimes W^m\right)\Big|_s^t-\gamma^m\int_s^t B_m(u^m(\sigma))Q_m\,\mathrm{d}\sigma\right)\\&=\tilde{\mathbb{E}}\left(\tilde{\gamma}^m\left(\tilde{M}^m\otimes\tilde{W}^m\right)\Big|_s^t-\tilde{\gamma}^m\int_s^t B_m(\tilde{u}^m(\sigma))Q_m\,\mathrm{d}\sigma\right).\end{aligned}\tag{4.36}$$

In order to pass to the limit in the above equations we have to justify why we can interchange the integration with respect to the probability measure and the limit. To this end, we will repeatedly use the Vitali convergence theorem.

The Burkholder-Davis-Gundy inequality from Proposition 4.22 applied to W^m yields the uniform bound

$$\tilde{\mathbb{E}}\sup_{\sigma\in[0,t]}\|\tilde{W}^m(\sigma)\|_{U_1}^3=\mathbb{E}\sup_{\sigma\in[0,t]}\|W^m(\sigma)\|_{U_1}^3\le Ct^{\frac{3}{2}}$$

with a constant $C>0$ independent of m, since $Q_m\in L_1(U_1)$ is uniformly bounded in m. Therefore $\tilde{W}^m(\sigma)$ is uniformly integrable in $L^2(\Omega;U_1)$ for all $\sigma\in[0,t]$ and this implies that we can pass to the limit in (4.32) with weak convergence in U_1 and in (4.33) with convergence in the weak operator topology on $L(U_1)$ because for all $x,y\in U_1$ it holds that

$$\begin{aligned}(t-s)\left\langle Qx,y\right\rangle_{U_1}\leftarrow(t-s)\left\langle Q_m x,y\right\rangle_{U_1}&=\tilde{\mathbb{E}}\left(\tilde{\gamma}^m\left(\left\langle\tilde{W}^m,y\right\rangle_{U_1}\left\langle\tilde{W}^m,x\right\rangle_{U_1}\right)\Big|_s^t\right)\\&\to\tilde{\mathbb{E}}\left(\tilde{\gamma}\left(\left\langle\tilde{W},y\right\rangle_{U_1}\left\langle\tilde{W},x\right\rangle_{U_1}\right)\Big|_s^t\right)\\&=\left\langle\tilde{\mathbb{E}}\left(\tilde{\gamma}\left(\tilde{W}\otimes\tilde{W}\right)\Big|_s^t\right)x,y\right\rangle_{U_1}.\end{aligned}$$

The Kolmogorov continuity theorem [DPZ14, Theorem 3.3] implies that $\tilde{W}$ has a version with continuous paths in U_1, which we will again denote by $\tilde{W}$. Since the above argument holds for all γ, we conclude that $\tilde{W}$ is a square-integrable continuous $(\mathcal{G}_t)_t$-martingale with $(\mathcal{G}_t)_t$-quadratic variation in U given by

$$\langle\!\langle \tilde{W} \rangle\!\rangle_t = tQ. \tag{4.37}$$

Since $\tilde{W}$ is continuous, we infer that $\tilde{W}$ is a square-integrable $(\tilde{\mathcal{F}}_t)_t$-martingale and (4.37) also holds for the quadratic variation with respect to $(\tilde{\mathcal{F}}_t)_t$. By the Lévy martingale characterization [DPZ14, Theorem 4.6] we conclude that $\tilde{W}$ is a generalized $(\tilde{\mathcal{F}}_t)_t$-Wiener process on U with covariance operator Q.

From Lemma 4.38 we infer that $\tilde{M}^m(t) \overset{*}{\rightharpoonup} \tilde{M}(t)$ $\mathbb{P}$-a.s. in V'. In addition, the Burkholder-Davis-Gundy inequality from Proposition 4.22, the growth bound (4.24) and the assumed bounds for u imply for some $q > 2$ and all $v \in V$

$$\begin{aligned}
\tilde{\mathbb{E}} \sup_{\sigma \in [0,t]} \left| \left\langle \tilde{M}^m(t), v \right\rangle_{V',V} \right|^q &= \mathbb{E} \sup_{\sigma \in [0,t]} \left| \langle M^m(t), v \rangle_{V',V} \right|^q \\
&\leq C \mathbb{E} \left[\int_0^t \| B_m(u^m(s)) \|^2_{L(U_0^m; V')} \|v\|_V^2 \,\mathrm{d}s \right]^{\frac{q}{2}} \\
&\leq C \left(1 + \mathbb{E} \int_0^T \|u^m(s)\|_V^2 \,\mathrm{d}s + \sup_{s \in [0,T]} \mathbb{E} \|u^m(s)\|_H^q \right) \|v\|_V^q \\
&\leq C \|v\|_V^q
\end{aligned}$$

uniformly in m. Thus $\left\langle \tilde{M}^m(\sigma), v \right\rangle_{V',V}$ is uniformly integrable in $L^2(\Omega; \mathbb{R})$ with respect to $m \in \mathbb{N}$ and $\sigma \in [0,t]$ for all $v \in V$. The estimate also implies the uniform integrability of the integral terms in (4.35) and (4.36) in a weak sense. The pathwise convergence of these terms can be inferred from Lemma 4.38. We can pass to the limit with Vitali's convergence theorem in (4.32) with weak-$*$ convergence in V' and in (4.35) and (4.36) with convergence in the weak operator topology on $L(H)$ and $L(U; H)$, respectively.

We infer that $\tilde{M}$ is a V'-valued continuous square-integrable $(\mathcal{G}_t)_t$-martingale and because of the continuity also an $(\mathcal{F}_t)_t$-martingale. It holds for all $t \in [0, \infty)$ that

$$\tilde{M}(t) = \int_0^t B_m(\tilde{u}(s)) \,\mathrm{d}\tilde{W}(s) \ \mathbb{P}\text{-a.s. in } V'.$$

Furthermore we have

$$\Lambda \leftharpoondown \mathbb{P} \circ u^m(0)^{-1} = \tilde{\mathbb{P}} \circ \tilde{u}^m(0)^{-1} \rightharpoonup \tilde{\mathbb{P}} \circ \tilde{u}(0)^{-1},$$

which completes the proof. $\square$

Proof of Theorem 4.34. For $N \in \mathbb{N}$ let $V_N := \operatorname{span}(\{e^1, \dots, e^N\})$ and consider the V_N-valued process u^N from Theorem 4.35. The process u^N is a weak solution of the finite-dimensional approximation of (4.12) for a Wiener process W^N on U with covariance operator $Q_N : U \to U$, which is the orthogonal projection on $\operatorname{span}\{g_1, \dots, g_N\}$. Note that $Q_N \to \mathrm{Id}$ in the strong operator topology on $L(U)$. We can assume that the processes $(u^N)_{N\in\mathbb{N}}$ are defined on one common probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}([0,1]), \mathcal{L})$, because the proof of [HS12, Theorem 0.1] could be adapted to yield existence of weak solutions for the finite-dimensional approximation on this particular space (cf. [IW81, Theorem IV.2.3 and Theorem IV.2.4]). Furthermore we note that $\mathbb{P} \circ u^N(0)^{-1} = \Lambda_N = \Lambda \circ P_N^{-1} \rightharpoonup \Lambda$, since $P_N \to \mathrm{Id}$ in the strong operator topology.

We can apply Proposition 4.36 to infer that for all $T > 0$ the sequence $(u^N)_{N\in\mathbb{N}}$ is in $N \in \mathbb{N}$ uniformly bounded in

$$L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega; L^2(0,T; V)) \cap L^\infty(0,T; L^q(\Omega; H)).$$

for some $q > 2$.

Now, Theorem 4.37 yields the existence of a weak martingale solution of (4.12) with initial data Λ. □

Lemma 4.39. *Let U be a separable Hilbert space. Then the following spaces have the property, that there is a countable set of real-valued continuous functions on this space that separates points:*

- $C_{loc}([0,\infty); U)$,
- $L^2_{loc}(0,\infty; U)$,
- $(L^2_{loc}(0,\infty; U); w)$ *and*
- $C_{loc}([0,\infty); (U, w))$ *with the compact-open topology.*

Proof. Fix dense and countable subsets $Q \subset [0,\infty)$ and $V \subset U$. Consider the set of functions $\mathfrak{F} = \{u \mapsto \langle u(q), v\rangle_U \mid q \in Q, v \in V\}$. Then $\mathfrak{F}$ is a countable set of real-valued continuous functions on $C_{\mathrm{loc}}([0,\infty); (U,w))$ that separates points. Since $C_{\mathrm{loc}}([0,\infty); U) \subset C_{\mathrm{loc}}([0,\infty); (U,w))$ is continuously embedded, the functions in $\mathfrak{F}$ are also continuous on $C_{\mathrm{loc}}([0,\infty); U)$ and separate points.

The space $L^2(0,\infty; U)$ is a separable Hilbert space. Let $\mathfrak{G} \subset C_c(0,\infty; U) \subset \Big(L^2_{\mathrm{loc}}(0,\infty; U)\Big)'$ be a countable set which is dense in $L^2(0,\infty; U)$. Then $\mathfrak{G}$ is a set of continuous functions on $L^2_{\mathrm{loc}}(0,\infty; U)$ and $(L^2_{\mathrm{loc}}(0,T; U), w)$ that separates points on both spaces. □

5. Existence of solutions for graphical SMCF

This chapter is devoted to a rigorous formulation and proof of the existence result Theorem 1.1. Under suitable assumptions on the noise and the initial data we will prove for $n \in \mathbb{N}$ existence of global-in-time martingale solutions of stochastic mean curvature flow of graphs with colored noise, i.e. for all $t \in [0, \infty) =: I$ and $x \in \mathbb{T}^n$

$$\mathrm{d}u(x,t) = \mathbf{Q}(\nabla u(x,t))\nabla\cdot\left(\mathbf{v}(\nabla u(x,t))\right)\mathrm{d}t + \sum_{l\in\mathbb{N}} \mathbf{Q}(\nabla u(x,t))\varphi_l(x) \circ \mathrm{d}\beta^l(t), \quad (5.1)$$

where $\mathbf{Q}(p) = \sqrt{1+|p|^2}$ and $\mathbf{v}(p) = \frac{p}{\sqrt{1+|p|^2}}$ for $p \in \mathbb{R}^n$ as in Remark B.9. Note that we do not allow the noise coefficients $(\varphi_l)_{l\in\mathbb{N}}$ to depend on the height $u(x,t)$ of a point $(x, u(x,t))$ on the graph of $u(\cdot, t)$, because in general this kind of noise would not permit graphical solutions.

We will solve (5.1) with periodic boundary conditions. For this, let us introduce Sobolev spaces with respect to the flat torus $\mathbb{T}^n$.

Remark 5.1 (Periodic Sobolev spaces)**.** For $k \in \mathbb{N}$, $p \in [1, \infty)$ we will denote by $W^{k,p}(\mathbb{T}^n)$ the space of periodic Sobolev functions on the flat torus $\mathbb{T}^n$. This space can be obtained as the completion of the $[0,1]^n$-periodic $C^\infty(\mathbb{R}^n)$ functions with respect to the Sobolev norm $\|\cdot\|_{W^{k,p}([0,1]^n)}$. We will abbreviate $H^k(\mathbb{T}^n) := W^{k,2}(\mathbb{T}^n)$.

We will make the following assumptions:

Assumptions 5.2. If not otherwise specified, we will always assume that the coefficients are smooth, i.e. $\varphi_l \in C^\infty(\mathbb{T}^n)$ for $l \in \mathbb{N}$, and

- $\sum_{l\in\mathbb{N}} \varphi_l^2 < 2$ on $\mathbb{T}^n$ and
- $\sum_{l\in\mathbb{N}} \|\varphi_l^2\|_{C^2(\mathbb{T}^n)} < \infty$.

In the following, we will write $C = C_{\text{noise}}$ for any constant C that only depends on the assumptions on the coefficients $(\varphi_l)_{l\in\mathbb{N}}$, i.e. $\min_{x\in\mathbb{T}^n}\left(2 - \sum_{l\in\mathbb{N}} \varphi_l^2(x)\right) > 0$ and $\sum_{l\in\mathbb{N}} \|\varphi_l^2\|_{C^2(\mathbb{T}^n)}$.

For the driving processes, we will assume that $(\beta^l)_{l\in\mathbb{N}}$ is a family of independent real-valued standard Brownian motions.

Under the aforementioned assumptions, solutions of (5.1) are defined in the following way.

Remark 5.3 (Notion of solution)**.** Let $I = [0, T]$ with $T < \infty$ or $I = [0, \infty)$ and Λ a Borel probability measure on $H^1(\mathbb{T}^n)$ with bounded second moments, i.e.

$$\int_{H^1(\mathbb{T}^n)} \|z\|^2_{H^1(\mathbb{T}^n)} \, \mathrm{d}\Lambda(z) < \infty.$$

In accordance to Definition 4.32 we say that there is a martingale solution u of (5.1) on the time interval I, if there is a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in I}, \mathbb{P})$ with a normal filtration together with a family $(\beta^l)_{l\in\mathbb{N}}$ of independent real-valued standard $(\mathcal{F}_t)$-Brownian motions, an $\mathcal{F}_0$-measurable random variable $u_0 \in L^2(\Omega; H^1(\mathbb{T}^n))$ and a predictable $H^1(\mathbb{T}^n)$-valued process u with $u \in L^2(\Omega; L^2(0, t; H^2(\mathbb{T}^n)))$ for all $t \in I$ such that $\mathbb{P} \circ u_0^{-1} = \Lambda$ and

$$\begin{aligned} u(t) - u_0 = &\int_0^t \mathbf{Q}(\nabla u(s)) \nabla\cdot (\mathbf{v}(\nabla u(s))) \, \mathrm{d}s \\ &+ \sum_l \int_0^t \mathbf{Q}(\nabla u(s)) \varphi_l \circ \mathrm{d}\beta^l(s) \quad \mathbb{P}\text{-a.s. in } L^2(\mathbb{T}^n) \quad \forall t \in I. \end{aligned} \tag{5.2}$$

According to Definition 4.19 and Remark 4.20 we can compute the Itô-Stratonovich correction terms using the formula

$$\mathrm{d}\mathbf{Q}(\nabla u) = \ldots \mathrm{d}t + \sum_l \mathbf{v}(\nabla u) \cdot \nabla \left(\varphi_l \mathbf{Q}(\nabla u)\right) \circ \mathrm{d}\beta^l$$

and infer that (5.2) is equivalent to the Itô formulation

$$\begin{aligned} u(t) - u_0 = &\int_0^t \mathbf{Q}(\nabla u(s)) \nabla\cdot (\mathbf{v}(\nabla u(s))) + \frac{1}{2} \sum_l \varphi_l \mathbf{v}(\nabla u(s)) \cdot \nabla \left(\varphi_l \mathbf{Q}(\nabla u(s))\right) \mathrm{d}s \\ &+ \sum_l \int_0^t \mathbf{Q}(\nabla u(s)) \varphi_l \, \mathrm{d}\beta^l(s) \quad \mathbb{P}\text{-a.s. in } L^2(\mathbb{T}^n) \quad \forall t \in I. \end{aligned} \tag{5.3}$$

Note that the above equation is well-defined since for $u \in L^2(\Omega; L^2(0, t; H^2(\mathbb{T}^n)))$ it holds

$$\begin{aligned} \mathbf{Q}(\nabla u) \nabla\cdot (\mathbf{v}(\nabla u)) = \Delta u - \mathbf{v}(\nabla u) \cdot \mathrm{D}^2 u \mathbf{v}(\nabla u) &\in L^2(\Omega; L^2(0, t; L^2(\mathbb{T}^n))), \\ \sum_{l\in\mathbb{N}} \varphi_l \mathbf{v}(\nabla u) \cdot \nabla(\varphi_l \mathbf{Q}(\nabla u)) &\in L^2(\Omega; L^2(0, t; L^2(\mathbb{T}^n))) \text{ and} \\ (\varphi_l \mathbf{Q}(\nabla u))_{l\in\mathbb{N}} &\in L^2(\Omega; L^2(0, t; \ell^2(H^1(\mathbb{T}^n)))). \end{aligned}$$

Proposition 4.28 implies that a martingale solution u has a modification with continuous paths in $H^1(\mathbb{T}^n)$ and $u \in L^2(\Omega; C([0, t]; H^1(\mathbb{T}^n)))$ for all $t \in I$.

Remark 5.4 (Spatially homogeneous noise)**.** The spatially homogeneous noise case refers to $\varphi_1 \equiv \alpha \in (-\sqrt{2}, \sqrt{2})$ and $\varphi_l = 0$ for $l \geq 2$. Note that this particular situation is included in Assumptions 5.2. In this situation we will write $\beta \coloneqq \beta^1$.

The Itô formulation (5.3) becomes

$$\begin{aligned} \mathrm{d}u &= \mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big) + \frac{1}{2}\alpha^2\mathbf{v}(\nabla u)\cdot \mathrm{D}^2 u\mathbf{v}(\nabla u)\,\mathrm{d}t + \alpha\mathbf{Q}(\nabla u)\,\mathrm{d}\beta \\ &= \left(1 - \frac{\alpha^2}{2}\right)\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big) + \frac{\alpha^2}{2}\Delta u\,\mathrm{d}t + \alpha\mathbf{Q}(\nabla u)\,\mathrm{d}\beta. \end{aligned}$$

When the noise coefficient is in the admissible range $\alpha \in (-\sqrt{2}, \sqrt{2})$, the deterministic part of this equation is a convex combination of the mean curvature operator $\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)$ and the Laplace operator. In fact, $\left(1 - \frac{\alpha^2}{2}\right)\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big) + \frac{\alpha^2}{2}\Delta u$ is a strongly elliptic operator. But as the equation is perturbed by multiplicative gradient-dependent noise, this does not imply that (5.3) is a coercive equation in the sense of Assumptions 4.31.

In the next theorem we will state the existence result for graphical SMCF.

Theorem 5.5. *Let Λ be a Borel probability measure on $H^1(\mathbb{T}^n)$ with*

$$\int_{H^1(\mathbb{T}^n)} \|z\|^2_{H^1(\mathbb{T}^n)}\,\mathrm{d}\Lambda(z) < \infty \quad \text{and} \quad \int_{H^1(\mathbb{T}^n)} \|\nabla z\|^6_{L^6(\mathbb{T}^n)}\,\mathrm{d}\Lambda(z) < \infty.$$

Then, under Assumptions 5.2, there is a martingale solution u of (5.1) *for $I = [0,\infty)$ with initial data Λ.*

As mentioned above, equation (5.1) is not a coercive variational SPDE in the sense of Assumptions 4.31 preventing to directly infer Theorem 5.5 using a Galerkin scheme. Instead, we employ a different method that is based on a-priori bounds, which we will derive in Section 5.2 using the abstract Itô formula for variational SPDEs from Proposition 4.28.

The a-priori bounds imply that any solution of (5.1) with sufficiently regular initial data does not explode. In order to use this result for an existence proof, we need to construct a suitable sequence of functions $(u^\varepsilon)_{\varepsilon>0}$ such that this sequence approximates (5.1) and preserves the a-priori bounds. Once such a sequence is constructed, we can use the a-priori bounds to apply a stochastic compactness principle and deduce the existence of a limit of u^ε for $\varepsilon \to 0$. In the last step we will show that the a-priori bounds imply that the limit is sufficiently regular and the convergence is strong enough in order to identify the limit as a solution of (5.1).

We will construct the approximating sequence by solving for $\varepsilon > 0$ the so-called viscous equation

$$\mathrm{d}u = \varepsilon\Delta u + \mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\,\mathrm{d}t + \sum_l \mathbf{Q}(\nabla u)\varphi_l \circ \mathrm{d}\beta^l, \tag{5.4}$$

which differs from (5.1) by the additional viscosity term $\varepsilon\Delta u$. We will show in Section 5.1 that the viscous equation (5.4) is a coercive variational SPDE and therefore existence can be deduced from Theorem 4.34.

Before going into the details, we want to show the computations that assure the a-priori bounds for solutions of (5.1).

Remark 5.6 (A-priori bounds)**.** We want to consider the slightly more general case of stochastic mean curvature flow in which the noise coefficients might depend on the height of the function. To this end let u be a solution of

$$\mathrm{d}u = \mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))\,\mathrm{d}t + \sum_l \eta_l(\cdot, u)\mathbf{Q}(\nabla u) \circ \mathrm{d}\beta^l$$

where the coefficients $\eta_l = \eta_l(x, z)$ are assumed to be smooth. We recover our standard equation (5.1) for $\eta_l(x, z) = \varphi_l(x)$.

In the following we will denote by $\nabla_x\eta$ the gradient of η with respect to the x variable and by $\partial_z\eta$ the derivative with respect to the z variable. For the gradient of the composed function $\eta(\cdot, u)$ we will write $\nabla\eta = \nabla(\eta(\cdot, u)) = \nabla_x\eta + \partial_z\eta\nabla u$.

For a function $f : \mathbb{R}^n \to [0, \infty)$ we want to analyze the time evolution of the corresponding gradient-dependent energy

$$\mathfrak{E}(u) := \int_{\mathbb{T}^n} f(\nabla u).$$

Formally the chain rule applies when using the Stratonovich formulation of stochastic integration. In our case this yields

$$\mathrm{d}\mathfrak{E}(u) = \int_{\mathbb{T}^n} \nabla f(\nabla u) \cdot \nabla \left(\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))\,\mathrm{d}t + \sum_l \eta_l\mathbf{Q}(\nabla u) \circ \mathrm{d}\beta^l \right).$$

In particular, we are interested in the evolution of the expectation $\mathbb{E}\mathfrak{E}(u)$. Note that, at least under suitable assumptions, a stochastic integral in the sense of Itô does not contribute to the expectation. In order to use this fact, we rewrite the above evolution law of $\mathfrak{E}(u)$ into its Itô formulation. For this we refer to Remark 4.20 and compute

$$\begin{aligned}
\mathrm{d}\Big(\nabla f(\nabla u) \cdot \nabla(\eta_l\mathbf{Q}(\nabla u))\Big) &= \nabla(\eta_l\mathbf{Q}(\nabla u)) \cdot \mathrm{D}^2 f(\nabla u)\nabla(\circ\,\mathrm{d}u) \\
&\quad + \nabla f(\nabla u) \cdot \nabla(\partial_z\eta_l\mathbf{Q}(\nabla u) \circ \mathrm{d}u + \eta_l\mathbf{v}(\nabla u) \cdot \nabla(\circ\,\mathrm{d}u)) \\
&= \ldots\,\mathrm{d}t + \sum_{m\neq l} \ldots \circ \mathrm{d}\beta^m + \Big[\nabla(\eta_l\mathbf{Q}(\nabla u)) \cdot \mathrm{D}^2 f(\nabla u)\nabla(\eta_l\mathbf{Q}(\nabla u)) \\
&\quad + \nabla f(\nabla u) \cdot \nabla\left(\eta_l\partial_z\eta_l\mathbf{Q}(\nabla u)^2 + \eta_l\mathbf{v}(\nabla u) \cdot \nabla\left(\eta_l\mathbf{Q}(\nabla u)\right)\right)\Big] \circ \mathrm{d}\beta^l.
\end{aligned}$$

Hence, $\mathbb{E}\mathfrak{E}(u(T)) - \mathbb{E}\mathfrak{E}(u(0)) = \mathbb{E}\int_0^T \mu(t)\,\mathrm{d}t$ with

$$\begin{aligned}\mu = \int_{\mathbb{T}^n} &\nabla f(\nabla u)\cdot\nabla\Big(\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\Big)\\ &+\frac{1}{2}\sum_l \nabla f(\nabla u)\cdot\nabla\Big(\eta_l\partial_z\eta_l\mathbf{Q}(\nabla u)^2+\eta_l\mathbf{v}(\nabla u)\cdot\nabla\left(\eta_l\mathbf{Q}(\nabla u)\right)\Big)\\ &+\frac{1}{2}\sum_l \nabla(\eta_l\mathbf{Q}(\nabla u))\cdot\mathrm{D}^2f(\nabla u)\nabla(\eta_l\mathbf{Q}(\nabla u)).\end{aligned}$$

Note that in deducing the expression for μ and $\mathbb{E}\mathfrak{E}(u(t))$ we have not been rigorous as we have not proven that the chain rule holds for the Stratonovich differential in our situation. Furthermore, we can not justify that the assumptions of Remark 4.20 are fulfilled in order to compute the Itô-Stratonovich correction terms. We will fix these issues later by using the Itô formulation (5.3) of graphical SMCF and using the Itô formula from Corollary 4.30, which holds for a solution of (5.3) in the sense of Remark 5.3.

In addition, we have not been rigorous why the Itô integral can be neglected when taking the expectation. Because of the integrability of the integrand we can only infer by Definition 4.14 that the Itô integral is a local martingale.

It is worth mentioning, that the following manipulations of μ are rigorous for all $u \in H^2(\mathbb{T}^n)$ for sufficiently smooth f, i.e. $f \in C^2(\mathbb{R}^n)$ with bounded second derivatives, and sufficiently regular noise coefficients $(\eta_l)_{l\in\mathbb{N}}$. In particular height-independent noise coefficients $\eta_l(x,z) = \varphi_l(x)$ with $(\varphi_l)_{l\in\mathbb{N}}$ as in Assumptions 5.2 are admissible.

For notational convenience we will use the Einstein summation convention, i.e. we will sum over all indices appearing twice. For example, we will abbreviate $\Psi := 1 - \frac{\eta_l^2}{2} = 1 - \sum_{l\in\mathbb{N}}\frac{\eta_l^2}{2}$.

Note that

$$\begin{aligned}&\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)+\frac{1}{2}\eta_l\mathbf{v}(\nabla u)\cdot\nabla(\eta_l\mathbf{Q}(\nabla u))\\ &\qquad\qquad = \Psi\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)+\frac{1}{2}\eta_l^2\Delta u+\frac{1}{2}\eta_l\nabla u\cdot\nabla\eta_l.\end{aligned}$$

Hence, with the notation

$$\begin{aligned}\mu_{\mathrm{mcf}} &:= \int_{\mathbb{T}^n}\Psi\nabla f(\nabla u)\cdot\nabla\Big(\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\Big)+f(\nabla u)\partial_z\Psi\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big),\\ \mu_{\mathrm{noise}} &:= \int_{\mathbb{T}^n}\nabla f(\nabla u)\cdot\nabla\Big(\eta_l^2\Delta u+\eta_l\partial_z\eta_l\mathbf{Q}(\nabla u)^2+\eta_l\nabla u\cdot\nabla\eta_l\Big)\\ &\qquad+2\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\left(\nabla f(\nabla u)\cdot\nabla\Psi - f(\nabla u)\partial_z\Psi\right)\\ &\qquad+\nabla(\eta_l\mathbf{Q}(\nabla u))\cdot\mathrm{D}^2f(\nabla u)\nabla(\eta_l\mathbf{Q}(\nabla u))\end{aligned}$$

we have

$$\mu = \mu_{\mathrm{mcf}} + \frac{1}{2}\mu_{\mathrm{noise}}.$$

The choice of μ_{mcf} is motivated by the fact that it is the directional derivative of $u \mapsto \int_{\mathbb{T}^n} \Psi f(\nabla u)$ into the direction $\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)$. Therefore, it has the same structure as the time derivative of $\int_{\mathbb{T}^n} \Psi f(\nabla u)$ for solutions of the unperturbed mean curvature flow of graphs. This interpretation gives us the hint to handle μ_{mcf} similarly to the way time-dependent test functions are handled in the deterministic case, c.f. [Eck04, Proposition 4.6]. To this end, we will rewrite μ_{mcf} and also μ_{noise} using integration by parts. Note that there will be no boundary terms appearing because of the periodicity of u.

Motivated by Remark 3.11 we will artificially add and subtract the term $f(\nabla u)\Psi|\nabla\cdot\big(\mathbf{v}(\nabla u)\big)|^2$ to μ_{mcf} and integrate by parts, i.e.

$$\begin{aligned}
\mu_{\mathrm{mcf}} = \int_{\mathbb{T}^n} & -f(\nabla u)\Psi|\nabla\cdot\big(\mathbf{v}(\nabla u)\big)|^2 + \Psi\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\nabla\cdot\big(f(\nabla u)\mathbf{v}(\nabla u)\big) \\
& - \Psi\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\mathbf{v}(\nabla u)\cdot \mathrm{D}^2 u\nabla f(\nabla u) \\
& - \Psi\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\nabla\cdot\big(\mathbf{Q}(\nabla u)\nabla f(\nabla u)\big) \\
& + \Psi\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\nabla f(\nabla u)\cdot \mathrm{D}^2 u\mathbf{v}(\nabla u) \\
& - \mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\nabla f(\nabla u)\cdot\nabla\Psi + f(\nabla u)\partial_z\Psi\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big) \\
= \int_{\mathbb{T}^n} & -f(\nabla u)\Psi|\nabla\cdot\big(\mathbf{v}(\nabla u)\big)|^2 \\
& + \Psi\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\nabla\cdot\big(f(\nabla u)\mathbf{v}(\nabla u) - \mathbf{Q}(\nabla u)\nabla f(\nabla u)\big) \\
& - \mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\nabla f(\nabla u)\cdot\nabla\Psi + f(\nabla u)\partial_z\Psi\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big).
\end{aligned}$$

We will rewrite the second term by integration by parts

$$\begin{aligned}
\mu_{\mathrm{mcf}} = \int_{\mathbb{T}^n} & -f(\nabla u)\Psi|\nabla\cdot\big(\mathbf{v}(\nabla u)\big)|^2 \\
& - \nabla\cdot\big(\mathbf{v}(\nabla u)\big)\nabla\Psi\cdot\big(f(\nabla u)\mathbf{v}(\nabla u) - \mathbf{Q}(\nabla u)\nabla f(\nabla u)\big) \\
& + \nabla\Psi\cdot\mathrm{D}\big(\mathbf{v}(\nabla u)\big)\,\big(f(\nabla u)\mathbf{v}(\nabla u) - \mathbf{Q}(\nabla u)\nabla f(\nabla u)\big) \\
& + \Psi\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \mathrm{D}\big(f(\nabla u)\mathbf{v}(\nabla u) - \mathbf{Q}(\nabla u)\nabla f(\nabla u)\big) \\
& - \mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\nabla f(\nabla u)\cdot\nabla\Psi + f(\nabla u)\partial_z\Psi\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)
\end{aligned}$$

and note that two terms cancel out

$$\begin{aligned}
\mu_{\mathrm{mcf}} = \int_{\mathbb{T}^n} & -f(\nabla u)\Psi|\nabla\cdot\big(\mathbf{v}(\nabla u)\big)|^2 - f(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\nabla\Psi\cdot\mathbf{v}(\nabla u) \\
& + \nabla\Psi\cdot\mathrm{D}\big(\mathbf{v}(\nabla u)\big)\,\big(f(\nabla u)\mathbf{v}(\nabla u) - \mathbf{Q}(\nabla u)\nabla f(\nabla u)\big) \\
& + \Psi\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \Big(f(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) - \mathbf{Q}(\nabla u)\,\mathrm{D}^2 f(\nabla u)\,\mathrm{D}^2 u\Big) \\
& + f(\nabla u)\partial_z\Psi\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big).
\end{aligned}$$

Now, after one more integration by parts in the second line and completing the square, we infer

$$\begin{aligned}
\mu_{\mathrm{mcf}} = \int_{\mathbb{T}^n} & -f(\nabla u)\Psi|\nabla\cdot(\mathbf{v}(\nabla u))|^2 - f(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))\nabla\Psi\cdot\mathbf{v}(\nabla u) \\
& + f(\nabla u)\Delta\Psi - f(\nabla u)\mathbf{v}(\nabla u)\cdot\mathrm{D}^2\Psi\mathbf{v}(\nabla u) \\
& - f(\nabla u)\nabla\Psi\cdot\mathbf{v}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u)) \\
& + \Psi\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \Big(f(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) - \mathbf{Q}(\nabla u)\,\mathrm{D}^2 f(\nabla u)\,\mathrm{D}^2 u\Big) \\
& + f(\nabla u)\partial_z\Psi\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u)) \\
= \int_{\mathbb{T}^n} & -f(\nabla u)\Psi\left|\nabla\cdot(\mathbf{v}(\nabla u)) + \frac{\nabla\Psi\cdot\mathbf{v}(\nabla u)}{\Psi}\right|^2 + f(\nabla u)\frac{\left|\nabla\Psi\cdot\mathbf{v}(\nabla u)\right|^2}{\Psi} \\
& + f(\nabla u)\Big(\Delta\Psi - \mathbf{v}(\nabla u)\cdot\mathrm{D}^2\Psi\mathbf{v}(\nabla u) + \partial_z\Psi\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))\Big) \\
& + \Psi\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \Big(f(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) - \mathbf{Q}(\nabla u)\,\mathrm{D}^2 f(\nabla u)\,\mathrm{D}^2 u\Big).
\end{aligned}$$

Later, we will show that f can be chosen such that the last term is non-positive.

Let us repeat the definition of μ_{noise}:

$$\begin{aligned}
\mu_{\mathrm{noise}} = \int_{\mathbb{T}^n} & \nabla f(\nabla u)\cdot\nabla\left(\eta_l^2\Delta u + \eta_l\partial_z\eta_l\mathbf{Q}(\nabla u)^2 + \eta_l\nabla u\cdot\nabla\eta_l\right) \\
& + 2\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))\left(\nabla f(\nabla u)\cdot\nabla\Psi - f(\nabla u)\partial_z\Psi\right) \\
& + \nabla(\eta_l\mathbf{Q}(\nabla u))\cdot\mathrm{D}^2 f(\nabla u)\nabla(\eta_l\mathbf{Q}(\nabla u)).
\end{aligned}$$

By combining terms in the first line we infer

$$\begin{aligned}
\mu_{\mathrm{noise}} = \int_{\mathbb{T}^n} & \nabla f(\nabla u)\cdot\nabla\left(\nabla\cdot\left(\eta_l^2\nabla u\right) - \eta_l\nabla u\cdot\nabla\eta_l + \eta_l\partial_z\eta_l\mathbf{Q}(\nabla u)^2\right) \\
& + 2\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))\left(\nabla f(\nabla u)\cdot\nabla\Psi - f(\nabla u)\partial_z\Psi\right) \\
& + \nabla(\eta_l\mathbf{Q}(\nabla u))\cdot\mathrm{D}^2 f(\nabla u)\nabla(\eta_l\mathbf{Q}(\nabla u)).
\end{aligned}$$

We rewrite this expression by integration by parts and note that two terms cancel out, i.e.

$$\begin{aligned}
\mu_{\mathrm{noise}} = \int_{\mathbb{T}^n} & -\eta_l^2\,\mathrm{D}^2 f(\nabla u)\,\mathrm{D}^2 u:\mathrm{D}^2 u - \nabla f(\nabla u)\cdot\nabla\left(\eta_l\nabla u\cdot\nabla\eta_l - \eta_l\partial_z\eta_l\mathbf{Q}(\nabla u)^2\right) \\
& + 2\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))\left(\nabla f(\nabla u)\cdot\nabla\Psi - f(\nabla u)\partial_z\Psi\right) \\
& + \eta_l^2\mathbf{v}(\nabla u)\cdot\mathrm{D}^2 u\,\mathrm{D}^2 f(\nabla u)\,\mathrm{D}^2 u\mathbf{v}(\nabla u) + \mathbf{Q}(\nabla u)^2\nabla\eta_l\cdot\mathrm{D}^2 f(\nabla u)\nabla\eta_l.
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbf{Q}(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) &= \mathrm{D}^2 u - \mathbf{v}(\nabla u)\otimes\mathrm{D}^2 u\mathbf{v}(\nabla u) \text{ and} \\
\nabla u\cdot\nabla\eta_l &= \nabla u\cdot\nabla_x\eta_l + |\nabla u|^2\partial_z\eta_l,
\end{aligned}$$

hence

$$\begin{aligned}\mu_{\text{noise}} = \int_{\mathbb{T}^n} & -\eta_l^2 \mathbf{Q}(\nabla u)\, \mathrm{D}^2 f(\nabla u)\, \mathrm{D}^2 u : \mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T \\ & - \nabla f(\nabla u) \cdot \nabla \left(\eta_l \nabla u \cdot \nabla_x \eta_l - \eta_l \partial_z \eta_l\right) \\ & + 2\mathbf{Q}(\nabla u) \nabla\cdot \big(\mathbf{v}(\nabla u)\big) \big(\nabla f(\nabla u) \cdot \nabla \Psi - f(\nabla u) \partial_z \Psi\big) \\ & + \mathbf{Q}(\nabla u)^2 \nabla \eta_l \cdot \mathrm{D}^2 f(\nabla u) \nabla \eta_l.\end{aligned}$$

The second term can be further simplified by

$$\begin{aligned}& - \int_{\mathbb{T}^n} \nabla f(\nabla u) \cdot \nabla \left(\eta_l \nabla u \cdot \nabla_x \eta_l - \eta_l \partial_z \eta_l\right) \\ & = \int_{\mathbb{T}^n} -\nabla(f(\nabla u)) \cdot (\eta_l \nabla_x \eta_l) - \nabla u \cdot \mathrm{D}(\eta_l \nabla_x \eta_l) \nabla f(\nabla u) + \nabla f(\nabla u) \cdot \nabla(\eta_l \partial_z \eta_l) \\ & = \int_{\mathbb{T}^n} f(\nabla u) \nabla\cdot (\eta_l \nabla_x \eta_l) - \nabla u \cdot \mathrm{D}(\eta_l \nabla_x \eta_l) \nabla f(\nabla u) + \nabla f(\nabla u) \cdot \nabla(\eta_l \partial_z \eta_l).\end{aligned}$$

By $\Psi = 1 - \frac{\eta_l^2}{2}$ we infer

$$\begin{aligned}\mu_{\text{noise}} = \int_{\mathbb{T}^n} & -\eta_l^2 \mathbf{Q}(\nabla u)\, \mathrm{D}^2 f(\nabla u)\, \mathrm{D}^2 u : \mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T - f(\nabla u) \nabla\cdot \nabla_x \Psi \\ & + \nabla u \cdot \mathrm{D} \nabla_x \Psi \nabla f(\nabla u) - \nabla f(\nabla u) \cdot \nabla \partial_z \Psi \\ & + 2\mathbf{Q}(\nabla u) \nabla\cdot \big(\mathbf{v}(\nabla u)\big) \big(\nabla f(\nabla u) \cdot \nabla \Psi - f(\nabla u) \partial_z \Psi\big) \\ & + \mathbf{Q}(\nabla u)^2 \nabla \eta_l \cdot \mathrm{D}^2 f(\nabla u) \nabla \eta_l.\end{aligned}$$

Combining the computations for μ_{mcf} and μ_{noise} we infer

$$\begin{aligned}\mu = \int_{\mathbb{T}^n} & -f(\nabla u) \Psi \left| \nabla\cdot \big(\mathbf{v}(\nabla u)\big) + \frac{\nabla \Psi \cdot \mathbf{v}(\nabla u)}{\Psi} \right|^2 \\ & + \mathbf{Q}(\nabla u) \nabla\cdot \big(\mathbf{v}(\nabla u)\big) \nabla f(\nabla u) \cdot \nabla \Psi + f(\nabla u) \frac{\left|\nabla \Psi \cdot \mathbf{v}(\nabla u)\right|^2}{\Psi} \\ & + \mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \Big(\Psi f(\nabla u)\, \mathrm{D}\big(\mathbf{v}(\nabla u)\big) - \mathbf{Q}(\nabla u)\, \mathrm{D}^2 f(\nabla u)\, \mathrm{D}^2 u\Big) \\ & + f(\nabla u) \left(\Delta \Psi - \mathbf{v}(\nabla u) \cdot \mathrm{D}^2 \Psi \mathbf{v}(\nabla u) - \frac{1}{2} \nabla\cdot \nabla_x \Psi\right) \\ & + \frac{1}{2} \nabla f(\nabla u) \cdot \Big(\mathrm{D}^T \nabla_x \Psi \nabla u - \nabla \partial_z \Psi\Big) + \frac{1}{2} \mathbf{Q}(\nabla u)^2 \nabla \eta_l \cdot \mathrm{D}^2 f(\nabla u) \nabla \eta_l.\end{aligned}$$

Completing the square gives

$$\begin{aligned}\mu = \int_{\mathbb{T}^n} &-f(\nabla u)\Psi \left| \nabla\cdot\big(\mathbf{v}(\nabla u)\big) + \frac{\nabla\Psi\cdot\Big(f(\nabla u)\cdot\mathbf{v}(\nabla u) - \frac{1}{2}\mathbf{Q}(\nabla u)\nabla f(\nabla u)\Big)}{f(\nabla u)\Psi}\right|^2 \\ &+ \mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \Big(\Psi f(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) - \mathbf{Q}(\nabla u)\,\mathrm{D}^2 f(\nabla u)\,\mathrm{D}^2 u\Big) \\ &+ \frac{\Big|\nabla\Psi\cdot\Big(f(\nabla u)\mathbf{v}(\nabla u) - \frac{1}{2}\mathbf{Q}(\nabla u)\nabla f(\nabla u)\Big)\Big|^2}{f(\nabla u)\Psi} \\ &+ f(\nabla u)\left(\Delta\Psi - \mathbf{v}(\nabla u)\cdot\mathrm{D}^2\Psi\mathbf{v}(\nabla u) - \frac{1}{2}\nabla\cdot\nabla_x\Psi\right) \\ &+ \frac{1}{2}\nabla f(\nabla u)\cdot\Big(\mathrm{D}^T\,\nabla_x\Psi\nabla u - \nabla\partial_z\Psi\Big) + \frac{1}{2}\mathbf{Q}(\nabla u)^2\nabla\eta_l\cdot\mathrm{D}^2 f(\nabla u)\nabla\eta_l.\end{aligned}$$

Note that the first term is non-negative. The second term can be written as

$$\begin{aligned}\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T &: \Big(\Psi f(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) - \mathbf{Q}(\nabla u)\,\mathrm{D}^2 f(\nabla u)\,\mathrm{D}^2 u\Big) \\ &= \frac{\mathrm{D}^2 u A(\nabla u) : B(\nabla u)\,\mathrm{D}^2 u}{\mathbf{Q}(\nabla u)^2}\end{aligned}$$

with

$$\begin{aligned}A(\nabla u) &= \mathrm{Id} - \mathbf{v}(\nabla u)\otimes\mathbf{v}(\nabla u), \\ B(\nabla u) &= \Psi f(\nabla u)\,\big(\mathrm{Id} - \mathbf{v}(\nabla u)\otimes\mathbf{v}(\nabla u)\big) - \mathbf{Q}(\nabla u)^2\,\mathrm{D}^2 f(\nabla u).\end{aligned}$$

The matrix A is non-negative and under suitable assumptions on f the matrix B will be non-positive. Together with Lemma C.2 this implies that the whole second term is non-positive.

In the general case the other terms might contain second order derivatives of u for which we do not know if they are controlled by the good terms. But when we consider the situation where the noise coefficients do not depend on the height, i.e. $\eta_l(x,z) = \varphi_l(x)$, then the above expression for μ reduces to

$$\begin{aligned}\mu = \int_{\mathbb{T}^n} &-f(\nabla u)\Psi \left| \nabla\cdot\big(\mathbf{v}(\nabla u)\big) + \frac{\nabla\Psi\cdot\Big(f(\nabla u)\cdot\mathbf{v}(\nabla u) - \frac{1}{2}\mathbf{Q}(\nabla u)\nabla f(\nabla u)\Big)}{f(\nabla u)\Psi}\right|^2 \\ &+ \mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \Big(\Psi f(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) - \mathbf{Q}(\nabla u)\,\mathrm{D}^2 f(\nabla u)\,\mathrm{D}^2 u\Big) \\ &+ \frac{\Big|\nabla\Psi\cdot\Big(f(\nabla u)\mathbf{v}(\nabla u) - \frac{1}{2}\mathbf{Q}(\nabla u)\nabla f(\nabla u)\Big)\Big|^2}{f(\nabla u)\Psi} \\ &+ f(\nabla u)\left(\frac{1}{2}\Delta\Psi - \mathbf{v}(\nabla u)\cdot\mathrm{D}^2\Psi\mathbf{v}(\nabla u)\right) \\ &+ \frac{1}{2}\nabla f(\nabla u)\cdot\mathrm{D}^2\Psi\nabla u + \frac{1}{2}\mathbf{Q}(\nabla u)^2\nabla\varphi_l\cdot\mathrm{D}^2 f(\nabla u)\nabla\varphi_l.\end{aligned} \tag{5.5}$$

In this equation, the third to sixth term do not contain second order derivatives of u.

In order to derive a-priori bounds using the representation (5.5) of μ we will choose f such that the first two terms are non-positive and the other terms can be controlled by $\mathfrak{E}(u)$. An a-priori bound for $\mathfrak{E}(u)$ is then a consequence of the Gronwall lemma.

As mentioned before, in order to employ the a-priori bounds we will construct an approximating sequence (u^ε) solving for $\varepsilon > 0$ the viscous equation (5.4). That means, that we are constructing a solution of (5.1) utilizing the vanishing viscosity method, which is the foundation of the deterministic viscosity theory for first-order PDEs. It therefore has to be emphasized that the notion of solution from Remark 5.3 does not coincide with the notion of stochastic viscosity solutions in the sense of [LS98a].

5.1. Existence of viscous approximation

In this section we will show how to infer existence for the viscous equation (5.4) by interpreting it as a variational SPDE in the sense of Section 4.3.

Theorem 5.7. *Let $\varepsilon > 0$, $q > 2$ and Λ be a Borel probability measure on $H^1(\mathbb{T}^n)$ with*

$$\int_{H^1(\mathbb{T}^n)} \|z\|^2_{H^1(\mathbb{T}^n)} \,\mathrm{d}\Lambda(z) < \infty \text{ and } \int_{H^1(\mathbb{T}^n)} \|\nabla z\|^q_{L^2(\mathbb{T}^n)} \,\mathrm{d}\Lambda(z) < \infty.$$

Then, under Assumptions 5.2, there is a martingale solution u of (5.4) for $I = [0, \infty)$ with initial data Λ.

Proof. We intend to apply Theorem 4.34 in order to obtain a martingale solution to the equation the gradient ∇u fulfills for u satisfying (5.4), which in turn gives rise to a martingale solution to (5.4) itself. To this end, we will work with the spaces

$$\begin{aligned} V &\coloneqq \{\nabla u \mid u \in H^2(\mathbb{T}^n)\} \text{ with } \|\nabla u\|_V \coloneqq \|\nabla u\|_{H^1(\mathbb{T}^n;\mathbb{R}^n)}, \\ H &\coloneqq \{\nabla u \mid u \in H^1(\mathbb{T}^n)\} \text{ with } \|\nabla u\|_H \coloneqq \|\nabla u\|_{L^2(\mathbb{T}^n;\mathbb{R}^n)} \text{ and} \\ U &\coloneqq \ell^2. \end{aligned}$$

We have that $V \subset H$ densely and compactly. Furthermore, we can identify $L_2(U; H) = \ell^2(H)$.

Using the Einstein summation convention we define the operators

$$A_\varepsilon : V \to V'$$

$$\begin{aligned}
&\langle A_\varepsilon(\nabla u), \nabla w\rangle_{V',V} \\
&\quad \coloneqq - \int_{\mathbb{T}^n} \left(\varepsilon \Delta u + \mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big) + \frac{1}{2}\varphi_l \mathbf{v}(\nabla u) \cdot \nabla\left(\varphi_l \mathbf{Q}(\nabla u)\right)\right) \Delta w \\
&\quad = - \int_{\mathbb{T}^n} \left(\varepsilon \Delta u + \left(1 - \frac{\varphi_l^2}{2}\right) \mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big) + \frac{\varphi_l^2}{2}\Delta u + \frac{1}{4}\nabla(\varphi_l^2)\cdot\nabla u\right) \Delta w
\end{aligned}$$

and

$$\begin{aligned}
B = (B_l)_{l\in\mathbb{N}} &: V \to \ell^2(H) \\
B_l(\nabla u) \coloneqq \nabla\left(\varphi_l \mathbf{Q}(\nabla u)\right) &= \varphi_l \,\mathrm{D}^2 u \mathbf{v}(\nabla u) + \mathbf{Q}(\nabla u)\nabla\varphi_l.
\end{aligned}$$

We verify that the Assumptions 4.31 are fulfilled:

- **Coercivity:** The coercivity assumption is an assumption on the terms appearing in the time evolution of $\|\nabla u\|_H^2$ for a solution ∇u. Since $\|\nabla u\|_H^2$ and $\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u)^2$ only differ by a constant, the term

  $$\begin{aligned}
  &2\left\langle A_\varepsilon(\nabla u), \nabla u\right\rangle_{V',V} + \sum_l \|B_l(\nabla u)\|_H^2 \\
  &\quad = \int_{\mathbb{T}^n} -2\Bigg[\varepsilon\Delta u + \left(1 - \frac{\varphi_l^2}{2}\right)\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big) \\
  &\qquad\qquad + \frac{\varphi_l^2}{2}\Delta u + \frac{1}{4}\nabla(\varphi_l^2)\cdot\nabla u\Bigg]\Delta u \\
  &\qquad + \varphi_l^2 |\,\mathrm{D}^2 u \mathbf{v}(\nabla u)|^2 + \mathbf{Q}(\nabla u)^2 |\nabla\varphi_l|^2 + 2\mathbf{Q}(\nabla u)\varphi_l \nabla\varphi_l \cdot \mathrm{D}^2 u \mathbf{v}(\nabla u)
  \end{aligned}$$

  is up to the first part, which stems from the viscous approximation, exactly the term we have rewritten in Remark 5.6 for $f(\nabla u) = \mathbf{Q}(\nabla u)^2$. Repeating the computations yields

  $$\begin{aligned}
  &2\left\langle A_\varepsilon(\nabla u), \nabla u\right\rangle_{V',V} + \sum_l \|B_l(\nabla u)\|_H^2 \\
  &\quad = \int_{\mathbb{T}^n} -2\varepsilon(\Delta u)^2 - \left(1 - \frac{\varphi_l^2}{2}\right) |\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)|^2 \\
  &\qquad + \mathbf{Q}(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \left(\left(1 - \frac{\varphi_l^2}{2}\right)\mathbf{Q}(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) - 2\,\mathrm{D}^2 u\right) \\
  &\qquad - \frac{1}{4}\mathbf{Q}(\nabla u)^2 \Delta(\varphi_l^2) + \mathbf{Q}(\nabla u)^2 |\nabla\varphi_l|^2.
  \end{aligned}$$

Note that by Lemma C.2

$$
\begin{aligned}
\mathbf{Q}(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T &: \left(\left(1-\frac{\varphi_l^2}{2}\right)\mathbf{Q}(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) - 2\,\mathrm{D}^2 u\right)\\
&= -\,\mathrm{D}^2 u\,\big(\mathrm{Id} - \mathbf{v}(\nabla u)\otimes\mathbf{v}(\nabla u)\big)\\
&\qquad : \left(\left(1+\frac{\varphi_l^2}{2}\right)\mathrm{Id} + \left(1-\frac{\varphi_l^2}{2}\right)\mathbf{v}(\nabla u)\otimes\mathbf{v}(\nabla u)\right)\mathrm{D}^2 u\\
&\le 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&2\,\langle A_\varepsilon(\nabla u), \nabla u\rangle_{V',V} + \sum_l \|B_l(\nabla u)\|_H^2\\
&\quad\le -2\varepsilon\|\nabla u\|_V^2 + C\left(1+\|\nabla u\|_H^2\right)
\end{aligned}
$$

with a constant $C = C_{\text{noise}}$ that only depends on the coefficients $(\varphi_l)_{l\in\mathbb{N}}$, see Assumptions 5.2. This proves that equation (5.4) is coercive in the sense of (4.13).

- **Growth bounds:** There is a constant $C = C_{\text{noise}}$ such that

$$
\begin{aligned}
\|A_\varepsilon(\nabla u)\|_{V'}^2 &\le \left\|\varepsilon\Delta u + \left(1-\frac{\varphi_l^2}{2}\right)\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\right.\\
&\qquad\left. + \frac{\varphi_l^2}{2}\Delta u + \frac{1}{4}\nabla(\varphi_l^2)\cdot\nabla u\right\|_{L^2(\mathbb{T}^n)}^2\\
&\le C(1+\|\nabla u\|_{H^1(\mathbb{T}^n;\mathbb{R}^n)}^2),\\
\|B(\nabla u)\|_{\ell^2(H)}^2 &= \|\nabla(\varphi_l\mathbf{Q}(\nabla u))\|_{L^2(\mathbb{T}^n;\mathbb{R}^n)}^2 \le C(1+\|\nabla u\|_{H^1(\mathbb{T}^n;\mathbb{R}^n)}^2),\\
\|B(\nabla u)\|_{\ell^2(V')}^2 &\le \|\varphi_l\mathbf{Q}(\nabla u)\|_{L^2(\mathbb{T}^n)}^2 \le C(1+\|\nabla u\|_{L^2(\mathbb{T}^n;\mathbb{R}^n)}^2).
\end{aligned}
$$

- **Continuity:** When $\nabla u_k \rightharpoonup \nabla u$ in V, then $\nabla u_k \to \nabla u$ in H and therefore

$$
\begin{aligned}
\mathbf{v}(\nabla u_k)\cdot\mathrm{D}^2 u_k\mathbf{v}(\nabla u_k) &= \mathbf{v}(\nabla u_k)\otimes\mathbf{v}(\nabla u_k) : \mathrm{D}^2 u_k\\
&\rightharpoonup \mathbf{v}(\nabla u)\otimes\mathbf{v}(\nabla u) : \mathrm{D}^2 u \text{ in } L^1(\mathbb{T}^n)\\
&= \mathbf{v}(\nabla u)\cdot\mathrm{D}^2 u\mathbf{v}(\nabla u)
\end{aligned}
$$

and since $|\mathbf{v}(\nabla u_k)| \le 1$ also

$$
\mathbf{v}(\nabla u_k)\cdot\mathrm{D}^2 u_k\mathbf{v}(\nabla u_k) \rightharpoonup \mathbf{v}(\nabla u)\cdot\mathrm{D}^2 u\mathbf{v}(\nabla u) \text{ in } L^2(\mathbb{T}^n).
$$

The other terms in the definition of $A_\varepsilon(u_k)$ are linear in u_k, hence

$$A_\varepsilon(u_k) \overset{*}{\rightharpoonup} A_\varepsilon(u) \text{ in } V'.$$

Similarly,

$$\begin{aligned} B_l(\nabla u_k) &= \varphi_l \, \mathrm{D}^2 u_k \mathbf{v}(\nabla u_k) + \mathbf{Q}(\nabla u_k)\nabla\varphi_l \\ &\rightharpoonup \varphi_l \, \mathrm{D}^2 u \mathbf{v}(\nabla u) + \mathbf{Q}(\nabla u)\nabla\varphi_l = B_l(u) \text{ in } L^2(\mathbb{T}^n;\mathbb{R}^n). \end{aligned}$$

Hence, for all $\nabla w \in L^2(\mathbb{T}^n;\mathbb{R}^n)$ by dominated convergence

$$\Big(\langle B_l(u_k), \nabla w\rangle_H\Big)_{l\in\mathbb{N}} \to \Big(\langle B_l(u), \nabla w\rangle_H\Big)_{l\in\mathbb{N}} \text{ in } \ell^2.$$

Now, from Theorem 4.34 we can conclude that there is a martingale solution ∇u of

$$\begin{aligned} \mathrm{d}\nabla u &= \nabla\Big(\varepsilon\Delta u + \mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u)) + \frac{1}{2}\varphi_l \mathbf{v}(\nabla u)\cdot\nabla(\varphi_l\mathbf{Q}(\nabla u))\Big)\,\mathrm{d}t \\ &\quad + \sum_l \nabla\left(\varphi_l\mathbf{Q}(\nabla u)\right)\mathrm{d}\beta^l \\ &= \nabla\Big(\varepsilon\Delta u + \mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))\Big)\,\mathrm{d}t + \sum_l \nabla\left(\varphi_l\mathbf{Q}(\nabla u)\right)\circ\mathrm{d}\beta^l \text{ in } V' \end{aligned} \tag{5.6}$$

with initial data $\Lambda\circ\nabla^{-1}$.

Next, we will show that (5.6) is not only fulfilled in V' but also in $\Big(H^1(\mathbb{T}^n;\mathbb{R}^n)\Big)'$, hence weak in the PDE sense. For an arbitrary $\psi \in H^1(\mathbb{T}^n;\mathbb{R}^n)$ we take the Helmholtz decomposition $\psi = \nabla w + \phi$ with $w \in H^2(\mathbb{T}^n)$ and $\phi \in H^1(\mathbb{T}^n;\mathbb{R}^n)$ with $\nabla\cdot\phi = 0$ and since both sides of the equation for ∇u are orthogonal to divergence-free vector fields, we have for all $t \in [0,\infty)$

$$\begin{aligned} \int_{\mathbb{T}^n} (\nabla u(t) - \nabla u(0))\cdot\psi &= \int_0^t \Big\langle \nabla\Big(\mathbf{Q}(\nabla u(s))\nabla\cdot(\mathbf{v}(\nabla u(s)))\Big), \psi\Big\rangle_{(H^1)',H^1} \mathrm{d}s \\ &\quad + \sum_l \int_0^t \int_{\mathbb{T}^n} \nabla\left(\varphi_l\mathbf{Q}(\nabla u(s))\right)\cdot\psi\circ\mathrm{d}\beta^l(s) \end{aligned}$$

and therefore the equation for ∇u is also fulfilled in $\Big(H^1(\mathbb{T}^n;\mathbb{R}^n)\Big)'$.

By eventually enriching the stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in I},\mathbb{P})$ and using disintegration theory, we find an $\mathcal{F}_0$-measurable $H^1(\mathbb{T}^n)$-valued random variable u_0 with $\nabla u_0 = \nabla u(0)$ a.s. and $\Lambda = \mathbb{P}\circ u_0^{-1}$. Now, define for $t\in[0,\infty)$

$$\tilde{u}(t) \coloneqq u_0 + \int_0^t \mathbf{Q}(\nabla u(s))\nabla\cdot(\mathbf{v}(\nabla u(s)))\,\mathrm{d}s + \sum_l \int_0^t \varphi_l\mathbf{Q}(\nabla u(s))\circ\mathrm{d}\beta^l(s). \tag{5.7}$$

Note that by assumption $u_0 \in L^2(\Omega; L^2(\mathbb{T}^n))$ and also for $T \in [0,\infty)$

$$t \mapsto \int_0^t \mathbf{Q}(\nabla u(s))\nabla\cdot(\mathbf{v}(\nabla u(s)))\,\mathrm{d}s \in L^2(\Omega; L^2(0,T; L^2(\mathbb{T}^n))) \text{ and}$$
$$t \mapsto \sum_l \int_0^t \varphi_l \mathbf{Q}(\nabla u(s)) \circ \mathrm{d}\beta^l(s) \in L^2(\Omega; L^2(0,T; L^2(\mathbb{T}^n))).$$

Hence, $\tilde{u} \in L^2(\Omega; L^2(0,T; L^2(\mathbb{T}^n)))$. Furthermore

$$\nabla\tilde{u}(t) = \nabla u_0 + \int_0^t \nabla\left(\mathbf{Q}(\nabla u(s))\nabla\cdot(\mathbf{v}(\nabla u(s)))\right)\mathrm{d}s$$
$$+ \int_0^t \nabla\left(\varphi_l \mathbf{Q}(\nabla u(s))\right) \circ \mathrm{d}\beta^l(s) = \nabla u(t) \ \forall t \in [0,\infty) \ \mathbb{P}\text{-a.s.}$$

and by (5.7) $\tilde{u}$ is a martingale solution of (5.4). □

5.2. Itô formula for solutions and a-priori estimates

In this section we will derive the evolution laws of certain energies, motivated by the formal computations from Remark 5.6. With this evolution laws at hand, we will prove a-priori estimates for solutions of (5.1) and (5.4). It will turn out that these estimates yield uniform bounds in $\varepsilon > 0$ and also hold in the limit for $\varepsilon = 0$.

Note that a martingale solution of either (5.1) or (5.4) is a strong solution with respect to the appropriate choices of probability space, Wiener process and initial data, see Definition 4.32. The following results, in which we characterize the properties of a strong solution, therefore also apply to martingale solutions in the above sense.

5.2.1. Preliminary results

We will start by giving a motivating example which shows that the perimeter of solutions is controlled.

Example 5.8. We know that the deterministic mean curvature flow can be understood as the gradient flow of the perimeter functional. Hence, the natural dissipation inequality implies that $\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u)$ is decreasing for solutions of the deterministic mean curvature flow.

We will show that this property is still valid for stochastic mean curvature flow with spatially homogeneous noise. In the case of colored noise this is not true, but at least we have a control of the perimeter.

Note that the following computations are similar to Remark 5.6. In particular, we do not give rigorous arguments why the chain rule holds, how to pass from the Stratonovich to the Itô formulation and why the Itô integral does not contribute to the expectation. This problems will be addressed in Lemma 5.9 and Lemma 5.10 below.

Formally, for a solution u of (5.1) it holds that

$$\mathrm{d}\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u) = \int_{\mathbb{T}^n} \mathbf{v}(\nabla u) \cdot \nabla \Big(\mathbf{Q}(\nabla u) \nabla\cdot \big(\mathbf{v}(\nabla u)\big) \,\mathrm{d}t + \varphi_l \mathbf{Q}(\nabla u) \circ \mathrm{d}\beta^l \Big) ,$$

which is by Remark 4.20 formally equivalent to the Itô formulation

$$\begin{aligned}\mathrm{d}\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u) = \int_{\mathbb{T}^n} &\mathbf{v}(\nabla u) \cdot \nabla \left(\mathbf{Q}(\nabla u) \nabla\cdot \big(\mathbf{v}(\nabla u)\big) + \frac{1}{2}\varphi_l \mathbf{v}(\nabla u) \cdot \nabla(\varphi_l \mathbf{Q}(\nabla u)) \right) \\ &+ \frac{1}{2} \nabla(\varphi_l \mathbf{Q}(\nabla u)) \cdot \frac{\mathrm{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)}{\mathbf{Q}(\nabla u)} \nabla(\varphi_l \mathbf{Q}(\nabla u)) \,\mathrm{d}t \\ &+ \mathbf{v}(\nabla u) \cdot \nabla \left(\varphi_l \mathbf{Q}(\nabla u) \right) \mathrm{d}\beta^l .\end{aligned}$$

We will manipulate this term similar to Remark 5.6. In this particular situation, terms can be combined differently in order to infer a more accessible expression.

Using integration by parts we infer

$$\begin{aligned}\mathrm{d}\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u) = \int_{\mathbb{T}^n} &-\left(1 - \frac{\varphi_l^2}{2}\right) \mathbf{Q}(\nabla u) |\nabla\cdot \big(\mathbf{v}(\nabla u)\big)|^2 \\ &+ \mathbf{v}(\nabla u) \cdot \nabla \left(\frac{\varphi_l^2}{2} \Delta u + \frac{\varphi_l}{2} \nabla u \cdot \nabla \varphi_l \right) \\ &+ \frac{1}{2} \nabla(\varphi_l \mathbf{Q}(\nabla u)) \cdot \frac{\mathrm{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)}{\mathbf{Q}(\nabla u)} \nabla(\varphi_l \mathbf{Q}(\nabla u)) \,\mathrm{d}t \\ &+ \mathbf{v}(\nabla u) \cdot \nabla \left(\varphi_l \mathbf{Q}(\nabla u) \right) \mathrm{d}\beta^l .\end{aligned}$$

Note that

$$\frac{\varphi_l^2}{2} \Delta u + \frac{\varphi_l}{2} \nabla u \cdot \nabla \varphi_l = \nabla\cdot \left(\frac{\varphi_l^2}{2} \nabla u \right) - \frac{1}{4} \nabla u \cdot \nabla(\varphi_l^2).$$

Hence,

$$\begin{aligned}\mathrm{d}\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u) = \int_{\mathbb{T}^n} &-\left(1 - \frac{\varphi_l^2}{2}\right) \mathbf{Q}(\nabla u) |\nabla\cdot \big(\mathbf{v}(\nabla u)\big)|^2 \\ &- \frac{\varphi_l^2}{2} \mathbf{Q}(\nabla u) \,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \mathrm{D}\big(\mathbf{v}(\nabla u)\big) \\ &- \frac{1}{4} \mathbf{v}(\nabla u) \cdot \nabla(\nabla u \cdot \nabla(\varphi_l^2)) \\ &+ \frac{\mathbf{Q}(\nabla u)}{2} \nabla \varphi_l \cdot \big(\mathrm{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)\big) \nabla \varphi_l \,\mathrm{d}t \\ &+ \mathbf{v}(\nabla u) \cdot \nabla \left(\varphi_l \mathbf{Q}(\nabla u) \right) \mathrm{d}\beta^l .\end{aligned}$$

This can be further simplified to

$$
\begin{aligned}
\mathrm{d}\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u) = \int_{\mathbb{T}^n} & -\left(1-\frac{\varphi_l^2}{2}\right)\mathbf{Q}(\nabla u)|\nabla\cdot(\mathbf{v}(\nabla u))|^2 \\
& -\frac{\varphi_l^2}{2}\mathbf{Q}(\nabla u)\,\mathrm{D}(\mathbf{v}(\nabla u))^T : \mathrm{D}(\mathbf{v}(\nabla u)) \\
& +\frac{1}{4}\mathbf{Q}(\nabla u)\left(\Delta(\varphi_l^2)-\mathbf{v}(\nabla u)\cdot \mathrm{D}^2(\varphi_l^2)\mathbf{v}(\nabla u)\right) \\
& +\frac{1}{2}\mathbf{Q}(\nabla u)\nabla\varphi_l\cdot(\mathrm{Id}-\mathbf{v}(\nabla u)\otimes\mathbf{v}(\nabla u))\,\nabla\varphi_l\,\mathrm{d}t \\
& +\mathbf{v}(\nabla u)\cdot\nabla\left(\varphi_l\mathbf{Q}(\nabla u)\right)\mathrm{d}\beta^l.
\end{aligned}
$$

Now, we recover the deterministic case by setting $\varphi_l = 0$. That is,

$$
\partial_t \int_{\mathbb{T}^n}\mathbf{Q}(\nabla u) = -\int_{\mathbb{T}^n}\mathbf{Q}(\nabla u)|\nabla\cdot(\mathbf{v}(\nabla u))|^2 \leq 0.
$$

Since the stochastic integral is a (local) martingale, we formally infer for spatially homogeneous noise $\varphi_1(x) = \alpha \in (-\sqrt{2},\sqrt{2})$ and $\varphi_l = 0$ for $l \neq 1$ by Lemma C.2

$$
\begin{aligned}
\partial_t\left(\mathbb{E}\int_{\mathbb{T}^n}\mathbf{Q}(\nabla u)\right) &= -\left(1-\frac{\alpha^2}{2}\right)\mathbb{E}\int_{\mathbb{T}^n}\mathbf{Q}(\nabla u)|\nabla\cdot(\mathbf{v}(\nabla u))|^2 \\
&\quad -\frac{\alpha^2}{2}\mathbb{E}\int_{\mathbb{T}^n}\mathbf{Q}(\nabla u)\,\mathrm{D}(\mathbf{v}(\nabla u))^T : \mathrm{D}(\mathbf{v}(\nabla u)) \\
&\leq 0.
\end{aligned}
$$

In the general case we infer

$$
\begin{aligned}
\partial_t\left(\mathbb{E}\int_{\mathbb{T}^n}\mathbf{Q}(\nabla u)\right) \leq \mathbb{E}\int_{\mathbb{T}^n} & \Bigg[-\left(1-\frac{\varphi_l^2}{2}\right)\mathbf{Q}(\nabla u)|\nabla\cdot(\mathbf{v}(\nabla u))|^2 \\
&\quad -\frac{\varphi_l^2}{2}\mathbf{Q}(\nabla u)\,\mathrm{D}(\mathbf{v}(\nabla u))^T : \mathrm{D}(\mathbf{v}(\nabla u))\Bigg] \\
& + C_{\text{noise}}\mathbb{E}\int_{\mathbb{T}^n}\mathbf{Q}(\nabla u)
\end{aligned}
$$

and with a Gronwall argument we deduce that $\mathbb{E}\int_{\mathbb{T}^n}\mathbf{Q}(\nabla u)$ grows at most exponentially in time.

In geometric terms this last inequality yields for a constant $C = C_{\text{noise}} > 0$

$$
\begin{aligned}
\mathbb{E}\mathcal{H}^n(\mathcal{M}_t) + \mathbb{E}\int_0^t\int_{\mathcal{M}_s}\left[\left(1-\frac{\varphi_l^2}{2}\right)H^2(s)+\frac{\varphi_l^2}{2}|A(s)|^2\right]\mathrm{d}\mathcal{H}^n\,\mathrm{d}s \\
\leq e^{Ct}\mathbb{E}\mathcal{H}^n(\mathcal{M}_0),
\end{aligned} \tag{5.8}
$$

where $\mathcal{M}_t = \operatorname{graph} u(t)$, $H(t)$ is the mean curvature and $|A(t)|$ the length of the second fundamental form of $\mathcal{M}_t$ for $t \geq 0$.

Although (5.8) is the natural energy estimate for stochastic mean curvature flow, we will not use this estimate since stronger estimates are needed in order to pass to the limit $\varepsilon \to 0$ in the viscous equation (5.4). Nevertheless, we will argue similarly as in Example 5.8 to derive these stronger estimates.

In the next two Lemmas, we will justify the computations from Remark 5.6 under suitable assumptions on the energy. The abstract Itô formula Corollary 4.30 for variational SPDEs applies to our situation and yields the next Lemma.

Lemma 5.9. *Let $\varepsilon \geq 0$ and u be a strong solution of* (5.4) *on the time interval I. For a function $f \in C^2(\mathbb{R}^n)$ we consider the energy*

$$\mathfrak{E}(t) := \int_{\mathbb{T}^n} f(\nabla u(t)), \ t \in I.$$

If the second derivatives of f are bounded then

$$\begin{aligned}
\mathrm{d}\mathfrak{E} = \int_{\mathbb{T}^n} & - \varepsilon \, \mathrm{D}^2 f(\nabla u) \, \mathrm{D}^2 u : \mathrm{D}^2 u \\
& - \left(1 - \frac{\varphi_l^2}{2}\right) f(\nabla u) \left| \nabla \cdot (\mathbf{v}(\nabla u)) - \frac{\nabla(\varphi_l^2) \cdot \left(f\mathbf{v} - \frac{1}{2}\mathbf{Q}\nabla f\right)(\nabla u)}{2 f(\nabla u) \left(1 - \frac{\varphi_l^2}{2}\right)} \right|^2 \\
& + \mathrm{D}(\mathbf{v}(\nabla u))^T : \left(\left(1 - \frac{\varphi_l^2}{2}\right) f(\nabla u) \, \mathrm{D}(\mathbf{v}(\nabla u)) - \mathbf{Q}(\nabla u) \, \mathrm{D}^2 f(\nabla u) \, \mathrm{D}^2 u \right) \\
& + \frac{\left| \nabla(\varphi_l^2) \cdot \left(f(\nabla u)\mathbf{v}(\nabla u) - \frac{1}{2}\mathbf{Q}(\nabla u)\nabla f(\nabla u) \right) \right|^2}{4 f(\nabla u) \left(1 - \frac{\varphi_l^2}{2}\right)} \\
& - \frac{1}{2} f(\nabla u) \left(\frac{1}{2}\Delta(\varphi_l^2) - \mathbf{v}(\nabla u) \cdot \mathrm{D}^2(\varphi_l^2)\mathbf{v}(\nabla u) \right) \\
& - \frac{1}{4} \nabla f(\nabla u) \cdot \mathrm{D}^2(\varphi_l^2) \nabla u + \frac{1}{2}\mathbf{Q}(\nabla u)^2 \nabla \varphi_l \cdot \mathrm{D}^2 f(\nabla u) \nabla \varphi_l \, \mathrm{d}t \\
& + \nabla f \cdot \nabla \left(\varphi_l \mathbf{Q}(\nabla u)\right) \mathrm{d}\beta^l.
\end{aligned}$$

Note that the integral on the right hand side is finite.

Proof. We can apply Corollary 4.30 to infer

$$\begin{aligned}
\mathrm{d}\mathfrak{E} = \int_{\mathbb{T}^n} & - \nabla \cdot (\nabla f(\nabla u)) \left(\varepsilon \Delta u + \mathbf{Q}(\nabla u) \nabla \cdot (\mathbf{v}(\nabla u)) + \frac{1}{2}\varphi_l \mathbf{v}(\nabla u) \cdot \nabla \left(\varphi_l \mathbf{Q}(\nabla u)\right) \right) \\
& + \frac{1}{2} \nabla(\varphi_l \mathbf{Q}(\nabla u)) \cdot \mathrm{D}^2 f(\nabla u) \nabla(\varphi_l \mathbf{Q}(\nabla u)) \, \mathrm{d}t \\
& + \nabla f(\nabla u) \cdot \nabla(\varphi_l \mathbf{Q}(\nabla u)) \, \mathrm{d}\beta^l.
\end{aligned}$$

With the justification of this evolution law at hand, we can use the rigorous manipulations of the deterministic term from Remark 5.6 to deduce the claimed expression for $\mathrm{d}\mathfrak{E}$.

Since f grows at most quadratically and ∇f grows at most linearly and $u \in L^2(\Omega; L^2(0,t; H^2(\mathbb{T}^n)))$ for all $t \in I$, we infer that the integral on the right hand side is finite. □

By specializing Lemma 5.9 to the case $f(\nabla u) = g(\mathbf{Q}(\nabla u))$ we find, under suitable assumptions on g, a whole family of energies which can be controlled during the stochastic mean curvature flow. Note that by Remark 5.3 a solution u of (5.4) has continuous paths in $H^1(\mathbb{T}^n)$, i.e. $u \in L^2(\Omega; C([0,t]; H^1(\mathbb{T}^n)))$ for all $t \in I$. Therefore $u(0) \in H^1(\mathbb{T}^n)$ is well-defined, in particular for $\varepsilon = 0$.

Lemma 5.10. *Let $\varepsilon \geq 0$ and u be a strong solution of (5.4) on the time interval I. Furthermore let $g \in C^2(\mathbb{R}_{\geq 1})$ with*

- *g is increasing and convex, i.e. $g', g'' \geq 0$,*
- *there is a constant $p \geq 2$ such that $z \mapsto \frac{g'(z)}{z^{p-1}}$ is decreasing and*
- *$0 < \frac{g'(1)}{p} \leq g(1) \leq g'(1)$.*

Then we can estimate the energy

$$\mathfrak{E}(t) := \int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))),\ t \in I$$

by

$$\begin{aligned}\mathbb{E}\mathfrak{E}(t) + \mathbb{E}\int_0^t \int_{\mathbb{T}^n} \Big(\mathbf{Q}(\nabla u) g'(\mathbf{Q}(\nabla u)) - g(\mathbf{Q}(\nabla u))\Big) \mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \mathrm{D}\big(\mathbf{v}(\nabla u)\big)\, \mathrm{d}s \\ \leq e^{C(p^2+1)t}\mathbb{E}\mathfrak{E}(0) \qquad \forall t \in I\end{aligned} \tag{5.9}$$

with a constant $C = C_{noise}$ that only depends on the coefficients $(\varphi_l)_{l\in\mathbb{N}}$, see Assumptions 5.2. Furthermore, the integral on the left hand side of the above inequality is non-negative.

Proof. We will start by deriving some properties of the function g.

Since $\frac{g'(z)}{z^{p-1}}$ is decreasing we infer

$$g''(z)z - (p-1)g'(z) \leq 0 \qquad \forall z \in \mathbb{R}_{\geq 1}.$$

Furthermore, we have

$$\Big(g'(z)z - pg(z)\Big)' = g''(z)z - (p-1)g'(z) \leq 0$$

and $g'(1) - pg(1) \leq 0$. Hence,

$$g'(z)z - pg(z) \leq 0 \qquad \forall z \in \mathbb{R}_{\geq 1}.$$

Similarly, we have

$$(g'(z)z - g(z))' = g''(z)z \geq 0$$

and $g'(1) - g(1) \geq 0$. This implies

$$g'(z)z - g(z) \geq 0 \qquad \forall z \in \mathbb{R}_{\geq 1}.$$

We intend to apply Lemma 5.9 to the function $f(\nabla u) = g(\mathbf{Q}(\nabla u))$. We compute

$$\begin{aligned}
&\nabla f(\nabla u) = g'(\mathbf{Q}(\nabla u))\mathbf{v}(\nabla u) \text{ and} \\
&\mathrm{D}^2 f(\nabla u) = \frac{g'(\mathbf{Q}(\nabla u))}{\mathbf{Q}(\nabla u)} \operatorname{Id} - \left(\frac{g'(\mathbf{Q}(\nabla u))}{\mathbf{Q}(\nabla u)} - g'' \right) \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u).
\end{aligned}$$

Furthermore, we have

$$f(\nabla u)\mathbf{v}(\nabla u) - \frac{1}{2}\mathbf{Q}(\nabla u)\nabla f(\nabla u) = \Big(g(\mathbf{Q}(\nabla u)) - \frac{1}{2}\mathbf{Q}(\nabla u) g'(\mathbf{Q}(\nabla u)) \Big) \mathbf{v}(\nabla u)$$

and

$$\begin{aligned}
&\left(1 - \frac{\varphi_l^2}{2}\right) f(\nabla u)\, \mathrm{D}\big(\mathbf{v}(\nabla u)\big) - \mathbf{Q}(\nabla u)\, \mathrm{D}^2 f(\nabla u)\, \mathrm{D}^2 u \\
&\qquad = \left(\left(1 - \frac{\varphi_l^2}{2}\right) g(\mathbf{Q}(\nabla u)) - \mathbf{Q}(\nabla u) g'(\mathbf{Q}(\nabla u)) \right) \mathrm{D}\big(\mathbf{v}(\nabla u)\big) \\
&\qquad\quad - g''(\mathbf{Q}(\nabla u))\mathbf{v}(\nabla u) \otimes \mathrm{D}^2 u \mathbf{v}(\nabla u).
\end{aligned}$$

From the fact that

$$\operatorname{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u) \geq 0$$

we infer $\mathrm{D}^2 f(\nabla u) \geq 0$ and with Lemma C.2

$$\mathrm{D}^2 f(\nabla u)\, \mathrm{D}^2 u : \mathrm{D}^2 u \geq 0.$$

Lemma C.2 also implies

$$\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \mathrm{D}\big(\mathbf{v}(\nabla u)\big) \geq 0.$$

Hence, we have already proven that the integral on the left hand side of (5.9) is non-negative.

To verify the assumptions of Lemma 5.9 we first have to additionally assume that g'' is bounded. Then g' grows at most linearly and $\mathrm{D}^2 f$ is bounded. The above calculations and Lemma 5.9 imply

$$\begin{aligned}
\mathrm{d}\mathfrak{E} \leq \int_{\mathbb{T}^n} & \Big(g(\mathbf{Q}(\nabla u)) - \mathbf{Q}(\nabla u) g'(\mathbf{Q}(\nabla u)) \Big) \, \mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \mathrm{D}\big(\mathbf{v}(\nabla u)\big) \\
& + \frac{\Big(g(\mathbf{Q}(\nabla u)) - \frac{1}{2}\mathbf{Q}(\nabla u) g'(\mathbf{Q}(\nabla u)) \Big)^2 \left| \nabla(\varphi_l^2) \cdot \mathbf{v}(\nabla u) \right|^2}{4 g(\mathbf{Q}(\nabla u)) \Big(1 - \frac{\varphi_l^2}{2} \Big)} \\
& - \frac{1}{4} g(\mathbf{Q}(\nabla u)) \Delta(\varphi_l^2) \\
& + \frac{1}{2} \Big(g(\mathbf{Q}(\nabla u)) - \frac{1}{2} \mathbf{Q}(\nabla u) g'(\mathbf{Q}(\nabla u)) \Big) \, \mathbf{v}(\nabla u) \cdot \mathrm{D}^2(\varphi_l^2) \mathbf{v}(\nabla u) \\
& + \frac{1}{2} \mathbf{Q}(\nabla u) g'(\mathbf{Q}(\nabla u)) |\nabla \varphi_l|^2 \\
& - \frac{1}{2} \Big(g'(\mathbf{Q}(\nabla u)) \mathbf{Q}(\nabla u) - \mathbf{Q}(\nabla u)^2 g''(\mathbf{Q}(\nabla u)) \Big) \left| \nabla \varphi_l \cdot \mathbf{v}(\nabla u) \right|^2 \mathrm{d}t \\
& + g'(\mathbf{Q}(\nabla u)) \mathbf{v}(\nabla u) \cdot \nabla \left(\varphi_l \mathbf{Q}(\nabla u) \right) \mathrm{d}\beta^l \\
\leq \int_{\mathbb{T}^n} & \Bigg[\Big(g(\mathbf{Q}(\nabla u)) - \mathbf{Q}(\nabla u) g'(\mathbf{Q}(\nabla u)) \Big) \, \mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \mathrm{D}\big(\mathbf{v}(\nabla u)\big) \\
& \quad + C(p^2+1) g(\mathbf{Q}(\nabla u)) \Bigg] \mathrm{d}t \\
& + \int_{\mathbb{T}^n} g'(\mathbf{Q}(\nabla u)) \mathbf{v}(\nabla u) \cdot \nabla \left(\varphi_l \mathbf{Q}(\nabla u) \right) \mathrm{d}\beta^l,
\end{aligned}$$

with a constant $C = C_{\text{noise}}$. In particular, we have used that

$$\frac{\left| \nabla(\varphi_l^2) \cdot \mathbf{v}(\nabla u) \right|^2}{1 - \frac{\varphi_l^2}{2}} \leq \frac{\Big(\sum_{l \in \mathbb{N}} \|\varphi_l^2\|_{C^1(\mathbb{T}^n)} \Big)^2}{\min\limits_{\mathbb{T}^n} \Big(1 - \sum_{l \in \mathbb{N}} \frac{\varphi_l^2}{2} \Big)} \leq C_{\text{noise}}.$$

The stochastic integral defines a local martingale according to Definition 4.14, therefore there is an increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times with $\tau_n \to \sup I$ for $n \to \infty$, such that for all $t \in I$

$$\begin{aligned}
\mathbb{E}\mathfrak{E}(t \wedge \tau_n) + \mathbb{E} \int_0^{t \wedge \tau_n} & \Big(\mathbf{Q}(\nabla u) g'(\mathbf{Q}(\nabla u)) - g(\mathbf{Q}(\nabla u)) \Big) \, \mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \mathrm{D}\big(\mathbf{v}(\nabla u)\big) \, \mathrm{d}s \\
& \leq \mathbb{E}\mathfrak{E}(0) + C(p^2+1) \mathbb{E} \int_0^{t \wedge \tau_n} \mathfrak{E}(s) \, \mathrm{d}s.
\end{aligned}$$

And by pathwise continuity of u in $H^1(\mathbb{T}^n)$, Fatou's Lemma and monotone convergence we infer

$$\mathbb{E}\mathfrak{E}(t)+\mathbb{E}\int_0^t\Big(\mathbf{Q}(\nabla u)g'(\mathbf{Q}(\nabla u))-g(\mathbf{Q}(\nabla u))\Big)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T:\mathrm{D}\big(\mathbf{v}(\nabla u)\big)\,\mathrm{d}s$$
$$\leq \mathbb{E}\mathfrak{E}(0)+C(p^2+1)\mathbb{E}\int_0^t\mathfrak{E}(s)\,\mathrm{d}s.$$

Using Fubini's theorem we can interchange the time integral and the expectation, i.e.

$$\mathbb{E}\mathfrak{E}(t)+\mathbb{E}\int_0^t\Big(\mathbf{Q}(\nabla u)g'(\mathbf{Q}(\nabla u))-g(\mathbf{Q}(\nabla u))\Big)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T:\mathrm{D}\big(\mathbf{v}(\nabla u)\big)\,\mathrm{d}s$$
$$\leq \mathbb{E}\mathfrak{E}(0)+C(p^2+1)\int_0^t\mathbb{E}\mathfrak{E}(s)\,\mathrm{d}s.$$

The Gronwall Lemma implies

$$\mathbb{E}\mathfrak{E}(t)+\mathbb{E}\int_0^t\Big(\mathbf{Q}(\nabla u)g'(\mathbf{Q}(\nabla u))-g(\mathbf{Q}(\nabla u))\Big)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T:\mathrm{D}\big(\mathbf{v}(\nabla u)\big)\,\mathrm{d}s$$
$$\leq e^{C(p^2+1)t}\mathbb{E}\mathfrak{E}(0).$$

In the last step we need to get rid of the additional assumption that g'' is bounded. Therefore let g be an arbitrary function fulfilling the assumptions from Lemma 5.10.

Let $\eta\in C_c^\infty(\mathbb{R})$ be a smooth cut-off function, i.e. $0\leq\eta\leq 1$ and $\eta'(z)\leq 0$ for $z\geq 1$, with $\eta(z)=1$ for all $z\leq 1$. For $M>1$ we abbreviate $\eta_M(z):=\eta\left(\frac{z}{M}\right)$ and define $g_M\in C^2(\mathbb{R}_{\geq 1})$ through

$$g_M''(z):=g''(z)\eta_M(z),\ z\in\mathbb{R}_{\geq 1},\qquad g_M'(1)=g'(1),\qquad g_M(1)=g(1).$$

Then $g_M''\geq 0$ and $g_M'\geq g_M'(1)=g'(1)\geq 0$. Furthermore it holds for all $z\in\mathbb{R}_{\geq 1}$

$$\begin{aligned} g_M''(z)z-(p-1)g_M'(z) &= g''(z)\eta_M(z)z-(p-1)\left(g'(1)+\int_1^z g''(\xi)\eta_M(\xi)\,\mathrm{d}\xi\right)\\ &= g''(z)\eta_M(z)z-(p-1)\left(g'(z)\eta_M(z)-\int_1^z g'(\xi)\eta_M'(\xi)\,\mathrm{d}\xi\right)\\ &\leq 0. \end{aligned}$$

That means that $z\mapsto\frac{g_M'(z)}{z^{p-1}}$ is decreasing.

By construction the second derivative of g_M'' is bounded. Hence, we can apply the above calculations to the function g_M and infer

$$\mathbb{E}\mathfrak{E}_M(t)+\mathbb{E}\int_0^t\Big(\mathbf{Q}(\nabla u)g_M'(\mathbf{Q}(\nabla u))-g_M(\mathbf{Q}(\nabla u))\Big)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T:\mathrm{D}\big(\mathbf{v}(\nabla u)\big)\,\mathrm{d}s$$
$$\leq e^{C(p^2+1)t}\mathbb{E}\mathfrak{E}_M(0).$$

By monotone convergence all the integrals in the above inequality converge for $M \to \infty$ implying (5.9). ☐

As a consequence of Lemma 5.10 we can estimate the $L^p(\mathbb{T}^n)$ norm of $\mathbf{Q}(\nabla u)$ for solutions of (5.4).

Corollary 5.11. *Let $\varepsilon \geq 0$ and u be a strong solution of* (5.4) *on the time interval I. There is a constant $C = C_{noise}$ that only depends on the coefficients $(\varphi_l)_{l\in\mathbb{N}}$, see Assumptions 5.2, such that for $p \geq 2$*

$$\mathbb{E}\|\mathbf{Q}(\nabla u(t))\|^p_{L^p(\mathbb{T}^n)} + (p-1)\mathbb{E}\int_0^t\int_{\mathbb{T}^n}\mathbf{Q}(\nabla u(s))^p\,\mathrm{D}\big(\mathbf{v}(\nabla u(s))\big)^T:\mathrm{D}\big(\mathbf{v}(\nabla u(s))\big)\,\mathrm{d}s \\ \leq e^{C(p^2+1)t}\mathbb{E}\|\mathbf{Q}(\nabla u(0))\|^p_{L^p(\mathbb{T}^n)}\ \forall t \in I.$$

Proof. This result is a direct consequence of Lemma 5.10 for $g(z) = z^p$. ☐

5.2.2. Uniform bounds for the viscous approximation

Corollary 5.11 is of great importance, because it also bounds the Hessian of a solution in $L^2(\Omega; L^2(0,t;L^2(\mathbb{T}^n)))$, as we will show in the next theorem.

Theorem 5.12. *Let $\varepsilon \geq 0$ and u be a strong solution of* (5.4) *on the time interval I. Then for all $t \in I$*

$$\mathbb{E}\|\nabla u(t)\|^6_{L^6(\mathbb{T}^n)} + 5\mathbb{E}\int_0^t \|\,\mathrm{D}^2 u(s)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}s \\ \leq e^{Ct}\left(1 + \mathbb{E}\|\nabla u(0)\|^6_{L^6(\mathbb{T}^n)}\right)$$

with a constant $C = C_{noise}$.

Proof. The smallest eigenvalue of $A := \mathrm{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)$ is given by $1 - |\mathbf{v}(\nabla u)|^2 = \frac{1}{\mathbf{Q}(\nabla u)^2}$, hence by Lemma C.2

$$\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \mathrm{D}\big(\mathbf{v}(\nabla u)\big) = \frac{1}{\mathbf{Q}(\nabla u)^2}\,\mathrm{D}^2 u A : A\,\mathrm{D}^2 u \geq \frac{1}{\mathbf{Q}(\nabla u)^6}|\,\mathrm{D}^2 u|^2.$$

By Corollary 5.11 and $|\nabla u| \leq \mathbf{Q}(\nabla u) \leq |\nabla u| + 1$ we infer the claimed inequality. ☐

The next theorem deals with the continuity of solutions in $H^1(\mathbb{T}^n)$.

Theorem 5.13 ($C([0,T];H^1(\mathbb{T}^n))$ bounds)**.** *Let $\varepsilon \geq 0$ and u be a strong solution of* (5.4) *on the time interval $I = [0,T]$. Then*

$$\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|^2_{H^1(\mathbb{T}^n)} \leq C\left(1 + \mathbb{E}\|u(0)\|^2_{L^2(\mathbb{T}^n)} + \mathbb{E}\|\nabla u(0)\|^6_{L^6(\mathbb{T}^n)}\right)e^{CT}$$

with a constant $C = C_{noise}$.

Proof. We will start by deriving bounds for the gradient. Therefore, we choose $f(\nabla u) = \frac{1}{2}\mathbf{Q}(\nabla u)^2$ in Lemma 5.9. We infer similarly to the coercivity part of the proof of Theorem 5.7

$$\begin{aligned}
\mathrm{d}\int_{\mathbb{T}^n} |\nabla u|^2 &= \mathrm{d}\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u)^2 \\
&= \int_{\mathbb{T}^n} -2\varepsilon(\Delta u)^2 - \left(1 - \frac{\varphi_l^2}{2}\right) |\mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)|^2 \\
&\qquad + \mathbf{Q}(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \left(\left(1 - \frac{\varphi_l^2}{2}\right)\mathbf{Q}(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) - 2\,\mathrm{D}^2 u\right) \\
&\qquad - \frac{1}{4}\mathbf{Q}(\nabla u)^2\Delta(\varphi_l^2) + \mathbf{Q}(\nabla u)^2|\nabla\varphi_l|^2\,\mathrm{d}t \\
&\qquad - 2\varphi_l\mathbf{Q}(\nabla u)\Delta u\,\mathrm{d}\beta^l \\
&\le C\left(1 + \|\nabla u\|^2_{L^2(\mathbb{T}^n)}\right)\mathrm{d}t - 2\int_{\mathbb{T}^n} \varphi_l\mathbf{Q}(\nabla u)\Delta u\,\mathrm{d}\beta^l
\end{aligned}$$

with a constant $C = C_{\text{noise}}$. Using the Burkholder-Davis-Gundy inequality from Proposition 4.22 we infer for $\delta > 0$ and $T' \in [0,T]$

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T']}\left|\int_0^t\int_{\mathbb{T}^n}\varphi_l\mathbf{Q}(\nabla u(s))\Delta u(s)\,\mathrm{d}\beta^l(s)\right| \\
&\qquad\le \mathbb{E}\left(\int_0^{T'}\left|\int_{\mathbb{T}^n}\varphi_l\mathbf{Q}(\nabla u(t))\Delta u(t)\right|^2\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad\le C\mathbb{E}\left(\int_0^{T'}\|\mathbf{Q}(\nabla u(t))\|^2_{L^2(\mathbb{T}^n)}\|\Delta u(t)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\qquad\le \delta\mathbb{E}\sup_{t\in[0,T']}\|\mathbf{Q}(\nabla u(t))\|^2_{L^2(\mathbb{T}^n)} + \frac{C}{4\delta}\mathbb{E}\int_0^{T'}\|\Delta u(t)\|^2_{L^2(\mathbb{T}^n)}.
\end{aligned}$$

Hence, for $\delta = \frac{1}{4}$

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T']}\|\nabla u(t)\|^2_{L^2(\mathbb{T}^n)} &\le C\left(1 + \mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)} + \mathbb{E}\int_0^{T'}\|\Delta u(t)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t\right) \\
&\qquad + C\mathbb{E}\int_0^{T'}\|\nabla u(t)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t.
\end{aligned}$$

The Gronwall lemma implies

$$\mathbb{E}\sup_{t\in[0,T]}\|\nabla u(t)\|^2_{L^2(\mathbb{T}^n)} \le C\left(1 + \mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)} + \mathbb{E}\int_0^{T}\|\Delta u(t)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t\right)e^{CT}$$

and with Theorem 5.12

$$\mathbb{E} \sup_{t\in[0,T]} \|\nabla u(t)\|^2_{L^2(\mathbb{T}^n)} \le C\left(1+\mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)}+\mathbb{E}\|\nabla u(0)\|^6_{L^6(\mathbb{T}^n)}\right)e^{CT}.$$

To continue, we will apply the Itô formula Corollary 4.30 to derive

$$\begin{aligned}\frac{1}{2}\,\mathrm{d}\int_{\mathbb{T}^n} u^2 = \int_{\mathbb{T}^n} u\left(\varepsilon\Delta u+\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))+\frac{1}{2}\varphi_l\mathbf{v}(\nabla u)\cdot\nabla(\varphi_l\mathbf{Q}(\nabla u))\right)\mathrm{d}t\\ + u\varphi_l\mathbf{Q}(\nabla u)\,\mathrm{d}\beta^l.\end{aligned}$$

Similarly to the above estimates we infer

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T']}\|u(t)\|^2_{L^2(\mathbb{T}^n)} &\le \mathbb{E}\|u(0)\|^2_{L^2(\mathbb{T}^n)}+\mathbb{E}\int_0^{T'}\|u(t)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t\\ &\quad + C\mathbb{E}\int_0^{T'}\|\mathrm{D}^2u(t)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t + C\mathbb{E}\int_0^{T'}\|\mathbf{Q}(\nabla u(t))\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t.\end{aligned}$$

Again, the Gronwall lemma and Theorem 5.12 imply

$$\mathbb{E}\sup_{t\in[0,T]}\|u(t)\|^2_{L^2(\mathbb{T}^n)} \le C\left(1+\mathbb{E}\|u(0)\|^2_{L^2(\mathbb{T}^n)}+\mathbb{E}\|\nabla u(0)\|^6_{L^6(\mathbb{T}^n)}\right)e^{CT}.$$

□

5.3. Vanishing viscosity limit

The bounds from Section 5.2 at hand, we can apply an abstract result available in the context of variational SPDEs stated in Theorem 4.37 to pass to the limit $\varepsilon \to 0$ in (5.4) and infer the existence result Theorem 5.5.

Proof of Theorem 5.5. For $\varepsilon > 0$ let u^ε be a martingale solution of (5.4) with initial data Λ, which exists according to Theorem 5.7. We adapt the notation from the proof of Theorem 5.7. Let

$$\begin{aligned}&V := \{\nabla u \mid u\in H^2(\mathbb{T}^n)\} \text{ with } \|\nabla u\|_V := \|\nabla u\|_{H^1(\mathbb{T}^n;\mathbb{R}^n)},\\ &H := \{\nabla u \mid u\in H^1(\mathbb{T}^n)\} \text{ with } \|\nabla u\|_H := \|\nabla u\|_{L^2(\mathbb{T}^n;\mathbb{R}^n)} \text{ and}\\ &U := \ell^2.\end{aligned}$$

Furthermore let

$$A_\varepsilon : V \to V'$$

$$\begin{aligned}&\langle A_\varepsilon(\nabla u),\nabla w\rangle_{V',V}\\ &\quad := -\int_{\mathbb{T}^n}\left(\varepsilon\Delta u+\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))+\frac{1}{2}\varphi_l\mathbf{v}(\nabla u)\cdot\nabla\left(\varphi_l\mathbf{Q}(\nabla u)\right)\right)\Delta w\\ &\quad = -\int_{\mathbb{T}^n}\left(\varepsilon\Delta u+\left(1-\frac{\varphi_l^2}{2}\right)\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))+\frac{\varphi_l^2}{2}\Delta u+\frac{1}{4}\nabla(\varphi_l^2)\cdot\nabla u\right)\Delta w\end{aligned}$$

and

$$\begin{aligned} B = (B_l)_{l\in\mathbb{N}} &: V \to \ell^2(H) \\ B_l(\nabla u) &:= \nabla\left(\varphi_l \mathbf{Q}(\nabla u)\right) = \varphi_l \,\mathrm{D}^2 u \mathbf{v}(\nabla u) + \mathbf{Q}(\nabla u)\nabla\varphi_l. \end{aligned}$$

Since the solutions u^ε are constructed with [Jak97, Theorem 2] we can fix one probability space $(\Omega;\mathcal{F};\mathbb{P}) = ([0,1];\mathcal{B}([0,1]),\mathcal{L})$ such that for all $\varepsilon > 0$ we can find

- a normal filtration $(\mathcal{F}_t^\varepsilon)_{t\in[0,\infty)}$,
- a cylindrical Wiener process W^ε on U, i.e. a generalized Wiener process with covariance operator Id,
- a $(\mathcal{F}_t^\varepsilon)_t$-predictable process u^ε with $u^\varepsilon \in L^2(\Omega;L^2(0,T;H^2(\mathbb{T}^n)))$ for all $T\in[0,\infty)$

such that

$$\mathrm{d}\nabla u^\varepsilon = A_\varepsilon(\nabla u^\varepsilon)\,\mathrm{d}t + B(u)\,\mathrm{d}W^\varepsilon$$

and $\mathbb{P}\circ\left(u^\varepsilon(0)\right)^{-1} = \Lambda$.

From Theorem 5.12 and Theorem 5.13 we conclude that $(\nabla u^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in

$$L^2(\Omega;C([0,T];H)) \cap L^2(\Omega;L^2(0,T;V)) \cap L^\infty(0,T;L^6(\Omega;H)).$$

In the proof of Theorem 5.7 we have already verified that the growth bounds (4.14) and (4.16) hold uniformly in $\varepsilon > 0$. Furthermore the continuity assumption (4.18) is fulfilled.

Now, let

$$\begin{aligned} &A : V \to V' \\ &\langle A(\nabla u), \nabla w\rangle_{V',V} \\ &\quad := -\int_{\mathbb{T}^n} \left(\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u)) + \frac{1}{2}\varphi_l \mathbf{v}(\nabla u)\cdot\nabla\left(\varphi_l\mathbf{Q}(\nabla u)\right)\right)\Delta w \\ &\quad = -\int_{\mathbb{T}^n} \left(\left(1-\frac{\varphi_l^2}{2}\right)\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u)) + \frac{\varphi_l^2}{2}\Delta u + \frac{1}{4}\nabla(\varphi_l^2)\cdot\nabla u\right)\Delta w. \end{aligned}$$

The arguments in the proof of Theorem 5.7 show that A is weak-weak-$*$ sequentially continuous. Furthermore, for a weakly convergent series $\nabla w_\varepsilon \in V$ with $\nabla w_\varepsilon \rightharpoonup \nabla w$ in V it holds that

$$A_\varepsilon(\nabla w_\varepsilon) = \varepsilon\nabla\Delta w_\varepsilon + A(\nabla w_\varepsilon) \overset{*}{\rightharpoonup} A(\nabla w) \text{ in } V'.$$

Hence, from Theorem 4.37 we infer that there is a weak martingale solution ∇u of

$$\mathrm{d}\nabla u = A(\nabla u)\,\mathrm{d}t + B(\nabla u)\,\mathrm{d}W$$

with initial data $\Lambda \circ \nabla^{-1}$ and a cylindrical Wiener process W on U.

As in the proof of Theorem 5.7 this solution can be lifted to a martingale solution u of (5.1) with initial data Λ. $\square$

6. Uniqueness under a regularity assumption

In this chapter we will show that sufficiently regular solutions of SMCF deviate at most exponentially in time, cf. Theorem 1.2. For this, we have to assume that the solutions are more regular than we have proven in Chapter 5.

As a consequence, we will also infer a pathwise uniqueness result. We will refer to this result as a weak-strong uniqueness result, as it holds under the mild regularity assumption of Lipschitz continuity for one of the solutions and a strong regularity assumption for the other solution. In particular for spatially homogeneous noise, we will show in Theorem 7.3 that the assumption of Lipschitz continuity is always fulfilled, if the initial data is Lipschitz continuous.

Theorem 6.1 (Stability and pathwise uniqueness)**.** *For $T > 0$ let u_1, u_2 be strong solutions of* (5.1) *on a common probability space with respect to the same Wiener process. If Assumptions 5.2 hold and there is a constant $M > 0$ such that for all $t \in [0, T]$*

$$\|\nabla u_1(t)\|_{L^\infty(\mathbb{T}^n)} \leq M \qquad \textit{and} \qquad \|\,\mathrm{D}^2 u_2(t)\|_{L^\infty(\mathbb{T}^n)} \leq M \;\; a.s. \tag{6.1}$$

then there are constants $c_0 = c_0(M) > 0$ and $C = C(M, C_{noise}) < \infty$ such that for all $t \in [0, T]$

$$\begin{aligned}
&\mathbb{E}\|\nabla u_1(t) - \nabla u_2(t)\|^2_{L^2(\mathbb{T}^n)} + c_0\mathbb{E}\int_0^t \|\,\mathrm{D}^2 u_1(s) - \mathrm{D}^2 u_2(s)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}s \\
&\quad \leq e^{Ct}\mathbb{E}\|\nabla u_1(0) - \nabla u_2(0)\|^2_{L^2(\mathbb{T}^n)}.
\end{aligned}$$

If in addition $u_1(0) = u_2(0)$ a.s. then $u_1 = u_2$ a.s. in $C([0, T]; H^1(\mathbb{T}^n))$.

Proof. From Corollary 4.30 we deduce that

$$\frac{1}{2}\,\mathrm{d}\|\nabla u_1 - \nabla u_2\|^2_{L^2(\mathbb{T}^n)} = \mu\,\mathrm{d}t + \sum_{l\in\mathbb{N}} \sigma_l\,\mathrm{d}\beta^l$$

with

$$\begin{aligned}
\mu := \int_{\mathbb{T}^n} &-\Delta\,(u_1 - u_2)\Big(\mathbf{Q}(\nabla u_1)\nabla\cdot(\mathbf{v}(\nabla u_1)) - \mathbf{Q}(\nabla u_2)\nabla\cdot(\mathbf{v}(\nabla u_2))\Big) \\
&- \frac{1}{2}\varphi_l\Delta\,(u_1 - u_2)\Big(\mathbf{v}(\nabla u_1)\cdot\nabla\left(\varphi_l\mathbf{Q}(\nabla u_1)\right) - \mathbf{v}(\nabla u_2)\cdot\nabla\left(\varphi_l\mathbf{Q}(\nabla u_2)\right)\Big) \\
&+ \frac{1}{2}\Big|\nabla\left(\varphi_l(\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2))\right)\Big|^2
\end{aligned}$$

and

$$\sigma_l \coloneqq \int_{\mathbb{T}^n} \nabla\left(u_1 - u_2\right) \cdot \nabla\left(\varphi_l(\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2))\right),\ l \in \mathbb{N}.$$

Using integration by parts and the periodicity of u_1 and u_2 we infer

$$\begin{aligned}\mu = \int_{\mathbb{T}^n} &- \left|\mathrm{D}^2(u_1 - u_2)\right|^2 \\ &+ \left(1 - \frac{\varphi_l^2}{2}\right)\Delta(u_1 - u_2)\Big(\mathbf{v}(\nabla u_1)\cdot \mathrm{D}^2 u_1 \mathbf{v}(\nabla u_1) - \mathbf{v}(\nabla u_2)\cdot \mathrm{D}^2 u_2 \mathbf{v}(\nabla u_2)\Big) \\ &- \frac{1}{4}\Delta(u_1 - u_2)\nabla(\varphi_l^2)\cdot\nabla(u_1 - u_2) + \frac{1}{2}\varphi_l^2\left|\nabla(\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2))\right|^2 \\ &+ \frac{1}{2}(\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2))\nabla(\varphi_l^2)\cdot\nabla(\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2)) \\ &+ \frac{1}{2}(\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2))^2|\nabla\varphi_l|^2.\end{aligned}$$

We will continue with each term separately. Note that the first term is non-positive. We rewrite the second term and estimate

$$\begin{aligned}&\int_{\mathbb{T}^n}\left(1 - \frac{\varphi_l^2}{2}\right)\Delta(u_1 - u_2)\Big(\mathbf{v}(\nabla u_1)\cdot \mathrm{D}^2 u_1 \mathbf{v}(\nabla u_1) - \mathbf{v}(\nabla u_2)\cdot \mathrm{D}^2 u_2 \mathbf{v}(\nabla u_2)\Big) \\ &\quad = \int_{\mathbb{T}^n}\left(1 - \frac{\varphi_l^2}{2}\right)\Delta(u_1 - u_2)\mathbf{v}(\nabla u_1)\cdot \mathrm{D}^2(u_1 - u_2)\,\mathbf{v}(\nabla u_1) \\ &\qquad\quad + \left(1 - \frac{\varphi_l^2}{2}\right)\Delta(u_1 - u_2)\left(\mathbf{v}(\nabla u_1) + \mathbf{v}(\nabla u_2)\right)\cdot \mathrm{D}^2 u_2\left(\mathbf{v}(\nabla u_1) - \mathbf{v}(\nabla u_2)\right) \\ &\quad \le \int_{\mathbb{T}^n}\frac{1}{2}\left(1 - \frac{\varphi_l^2}{2}\right)\left(\Delta(u_1 - u_2)\right)^2 + \frac{1}{2}\left(1 - \frac{\varphi_l^2}{2}\right)\left|\mathrm{D}^2(u_1 - u_2)\right|^2\left|\mathbf{v}(\nabla u_1)\right|^2 \\ &\qquad\quad + \frac{\delta}{2}\left(1 - \frac{\varphi_l^2}{2}\right)\left(\Delta(u_1 - u_2)\right)^2 \\ &\qquad\quad + \frac{1}{\delta}\left(1 - \frac{\varphi_l^2}{2}\right)\|\mathrm{D}^2 u_2\|^2_{L^\infty(\mathbb{T}^n)}|\nabla u_1 - \nabla u_2|^2.\end{aligned}$$

Note that for a smooth function $\psi : \mathbb{T}^n \to \mathbb{R}$

$$\begin{aligned}\int_{\mathbb{T}^n}\psi\left(\Delta(u_1 - u_2)\right)^2 = \int_{\mathbb{T}^n}\psi&\left|\mathrm{D}^2(u_1 - u_2)\right|^2 - |\nabla(u_1 - u_2)|^2\Delta\psi \\ &+ \nabla(u_1 - u_2)\cdot \mathrm{D}^2\psi\nabla(u_1 - u_2).\end{aligned}$$

Hence, after choosing $\delta \leq \frac{1}{2(1+M^2)} \leq 1 - \|v(\nabla u_1)\|^2_{L^\infty(\mathbb{T}^n)} - \frac{1}{2(1+M^2)}$

$$
\begin{aligned}
&\int_{\mathbb{T}^n} \left(1 - \frac{\varphi_l^2}{2}\right) \Delta(u_1 - u_2) \Big(\mathbf{v}(\nabla u_1) \cdot \mathrm{D}^2 u_1 \mathbf{v}(\nabla u_1) - \mathbf{v}(\nabla u_2) \cdot \mathrm{D}^2 u_2 \mathbf{v}(\nabla u_2)\Big) \\
&\leq \int_{\mathbb{T}^n} \left(1 - \frac{1}{4(1+M^2)}\right) \left(1 - \frac{\varphi_l^2}{2}\right) \left|\mathrm{D}^2(u_1 - u_2)\right|^2 + C(C_{\mathrm{noise}}, M)|\nabla u_1 - \nabla u_2|^2.
\end{aligned}
$$

For the third term we have

$$
\begin{aligned}
&-\int_{\mathbb{T}^n} \Delta(u_1 - u_2) \nabla(\varphi_l^2) \cdot \nabla(u_1 - u_2) \\
&\qquad = \int_{\mathbb{T}^n} \nabla(u_1 - u_2) \cdot \mathrm{D}^2(\varphi_l^2) \nabla(u_1 - u_2) - \frac{1}{2} \Delta(\varphi_l^2) |\nabla(u_1 - u_2)|^2 \\
&\qquad \leq C_{\mathrm{noise}} \int_{\mathbb{T}^n} |\nabla u_1 - \nabla u_2|^2.
\end{aligned}
$$

The fourth term can be estimated by

$$
\begin{aligned}
&\int_{\mathbb{T}^n} \varphi_l^2 \left|\nabla(\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2))\right|^2 \\
&\qquad = \int_{\mathbb{T}^n} \varphi_l^2 \left|\mathrm{D}^2(u_1 - u_2) \mathbf{v}(\nabla u_1) + \mathrm{D}^2 u_2 \left(\mathbf{v}(\nabla u_1) - \mathbf{v}(\nabla u_2)\right)\right|^2 \\
&\qquad \leq \int_{\mathbb{T}^n} (1+\delta) \varphi_l^2 \|\mathbf{v}(\nabla u_1)\|^2_{L^\infty(\mathbb{T}^n)} \left|\mathrm{D}^2(u_1 - u_2)\right|^2 \\
&\qquad\qquad + \left(1 + \frac{1}{\delta}\right) \varphi_l^2 \|\,\mathrm{D}^2 u_2\|^2_{L^\infty(\mathbb{T}^n)} |\nabla u_1 - \nabla u_2|^2.
\end{aligned}
$$

By choosing $\delta \leq \frac{3}{4M^2} \leq \frac{1}{\|\mathbf{v}(\nabla u_1)\|^2_{L^\infty(\mathbb{T}^n)}} - 1 - \frac{1}{4M^2}$ we infer

$$
\begin{aligned}
&\int_{\mathbb{T}^n} \varphi_l^2 \left|\nabla(\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2))\right|^2 \\
&\qquad \leq \int_{\mathbb{T}^n} \left(1 - \frac{1}{4(1+M^2)}\right) \varphi_l^2 \left|\mathrm{D}^2(u_1 - u_2)\right|^2 + C(C_{\mathrm{noise}}, M)|\nabla u_1 - \nabla u_2|^2.
\end{aligned}
$$

Using the Lipschitz continuity of $p \mapsto \mathbf{Q}(p)$ we infer for the fifth and sixth term

$$
\begin{aligned}
&\int_{\mathbb{T}^n} \frac{1}{2} \nabla\Big((\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2))^2\Big) \cdot \nabla(\varphi_l^2) + (\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2))^2 |\nabla \varphi_l|^2 \\
&\qquad\qquad \leq \int_{\mathbb{T}^n} C_{\mathrm{noise}} (\mathbf{Q}(\nabla u_1) - \mathbf{Q}(\nabla u_2))^2 \leq \int_{\mathbb{T}^n} C_{\mathrm{noise}} |\nabla u_1 - \nabla u_2|^2.
\end{aligned}
$$

Combining all these estimate we infer

$$
\mu \leq -c_0 \int_{\mathbb{T}^n} \left|\mathrm{D}^2(u_1 - u_2)\right|^2 + C(C_{\mathrm{noise}}, M) \int_{\mathbb{T}^n} |\nabla u_1 - \nabla u_2|^2
$$

with $c_0 := \frac{1}{4(1+M^2)}$.

Since the Itô integral defines a local martingale and by Fatou's lemma, we infer for all $t \in [0,T]$

$$\begin{aligned}
&\mathbb{E}\|\nabla u_1(t) - \nabla u_2(t)\|^2_{L^2(\mathbb{T}^n)} + c_0\mathbb{E}\int_0^t \|\,\mathrm{D}^2 u_1(s) - \mathrm{D}^2 u_2(s)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}s \\
&\quad \le \mathbb{E}\|\nabla u_1(0) - \nabla u_2(0)\|^2_{L^2(\mathbb{T}^n)} + C\mathbb{E}\int_0^t \|\nabla u_1(s) - \nabla u_2(s)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}s.
\end{aligned}$$

The Gronwall lemma implies for all $t \in [0,T]$

$$\begin{aligned}
&\mathbb{E}\|\nabla u_1(t) - \nabla u_2(t)\|^2_{L^2(\mathbb{T}^n)} + c_0\mathbb{E}\int_0^t \|\,\mathrm{D}^2 u_1(s) - \mathrm{D}^2 u_2(s)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}s \\
&\quad \le e^{Ct}\mathbb{E}\|\nabla u_1(0) - \nabla u_2(0)\|^2_{L^2(\mathbb{T}^n)}.
\end{aligned}$$

Now, let us assume that $u_1(0) = u_2(0)$ a.s. From the above estimate we infer that for all $t \in [0,T]$ it holds that $\nabla u_1(t) = \nabla u_2(t)$ a.s. and the a.s. pathwise continuity of $\nabla u_1, \nabla u_2$ in $L^2(\mathbb{T}^n)$ implies $\nabla u_1 = \nabla u_2$ a.s. in $C([0,T]; L^2(\mathbb{T}^n))$.

Since u_1 and u_2 are solutions of (5.1) we also have for all $t \in [0,T]$

$$\begin{aligned}
u_1(t) &= u_1(0) + \int_0^t \mathbf{Q}(\nabla u_1(s))\nabla\cdot\big(\mathbf{v}(\nabla u_1(s))\big)\,\mathrm{d}s + \sum\int_0^t \varphi_l \mathbf{Q}(\nabla u_1(s)) \circ \mathrm{d}\beta^l(s) \\
&= u_2(0) + \int_0^t \mathbf{Q}(\nabla u_2(s))\nabla\cdot\big(\mathbf{v}(\nabla u_2(s))\big)\,\mathrm{d}s + \sum\int_0^t \varphi_l \mathbf{Q}(\nabla u_2(s)) \circ \mathrm{d}\beta^l(s) \\
&= u_2(t) \text{ a.s.}
\end{aligned}$$

□

Remark 6.2. It is an open problem whether the regularity assumptions (6.1) always hold for solutions of (5.1) at least under some additional conditions on the initial data.

Assuming this, the pathwise uniqueness derived in Theorem 6.1 can be used to infer the uniqueness of the law of the solution process. For this purpose one can use the same ideas as in the proof of the classical Yamada-Watanabe theorem [YW71]. Instead of going into the technical details here we refer to [RSZ08] where the Yamada-Watanabe theorem is proven in the setting of variational stochastic partial differential equations. Furthermore we refer to [KS88, 5.3.D] and [RY91, Theorem IX.1.7] for the proof of the Yamada-Watanabe theorem in the context of finite-dimensional stochastic differential equations.

The described strategy would also yield the existence of a strong solution of (5.1), i.e. a solution on a given probability space which has to be at least rich enough to define a Wiener process W and a random variable u_0 with the prescribed initial law, which is independent of W.

7. The case of spatially homogeneous noise

In this chapter we consider (5.1) with spatially homogeneous noise, i.e.

$$\mathrm{d}u = \mathbf{Q}(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\,\mathrm{d}t + \alpha\mathbf{Q}(\nabla u)\circ\mathrm{d}\beta \tag{7.1}$$

with $\alpha^2 \in [0,2)$ and a real-valued standard Brownian motion β.

Note that SMCF with spatially homogeneous noise has previously been considered among others by Dirr, Luckhaus and Novaga [DLN01] and Souganidis and Yip [SY04], both proving a stochastic selection principle in the limit $\alpha \to 0$. The case of graphs has been analyzed for $n = 1$ by Es-Sarhir and von Renesse [ESvR12] and for $n = 2$ by Hofmanová, Röger and von Renesse [HRvR17].

In Theorem 7.3 we will justify the argument from Example 3.12 and prove that solutions of (7.1) preserve the Lipschitz constant of the initial data. The analogue result for deterministic MCF of graphs was proven in [EH89]. As in the deterministic case, we will use the Lipschitz bound to deduce a large-time result in Theorem 7.5 that extends the large-time characterization for $n = 1$ from [ESvR12] by providing a stronger convergence for $T \to \infty$. Furthermore, the large-time results holds in arbitrary dimensions.

In addition, we can adapt the ideas of the a-priori bounds from Chapter 5 to derive estimates for higher-order derivatives.

7.1. Maximum principle for the gradient

Corollary 7.1. *Let u be a strong solution of* (7.1) *on the time interval I and $f \in C^2(\mathbb{R}^n)$ with bounded second-order derivatives. For the energy*

$$\mathfrak{E}(t) := \int_{\mathbb{T}^n} f(\nabla u(t)),\ t \in I$$

it holds that

$$\begin{aligned}\mathrm{d}\mathfrak{E} = \int_{\mathbb{T}^n}\Big[&-\Big(1-\frac{\alpha^2}{2}\Big)f(\nabla u)\,\Big|\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\Big|^2 \\ &+ \mathrm{D}\big(\mathbf{v}(\nabla u)\big)^T : \Big(\Big(1-\frac{\alpha^2}{2}\Big)f(\nabla u)\,\mathrm{D}\big(\mathbf{v}(\nabla u)\big) - \mathbf{Q}(\nabla u)\,\mathrm{D}^2 f(\nabla u)\,\mathrm{D}^2 u\Big)\mathrm{d}t \\ &- f(\nabla u)\nabla\cdot\big(\mathbf{v}(\nabla u)\big)\,\mathrm{d}\beta\Big].\end{aligned} \tag{7.2}$$

Proof. This is a direct consequence of Lemma 5.9 and the identity

$$\int_{\mathbb{T}^n} \nabla f(\nabla u) \cdot \nabla \left(\mathbf{Q}(\nabla u)\right) = \int_{\mathbb{T}^n} \nabla \left(f(\nabla u)\right) \cdot \mathbf{v}(\nabla u) = -\int_{\mathbb{T}^n} f(\nabla u) \nabla \cdot \big(\mathbf{v}(\nabla u)\big).$$

□

Corollary 7.2. *Let u be a strong solution of (7.1) on the time interval $I = [0,T]$ and $g \in C^2(\mathbb{R}_{\geq 1})$ such that g is non-negative, increasing and convex, i.e. $g, g', g'' \geq 0$, g'' is bounded and $g(1) \leq g'(1)$.*

Then for all $t \in I$ we have that

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) + \Big(1 - \frac{\alpha^2}{2}\Big)\mathbb{E}\int_0^t\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(s)))\left|\nabla\cdot\big(\mathbf{v}(\nabla u(s))\big)\right|^2 \mathrm{d}s \\
&\quad + \mathbb{E}\int_0^t\int_{\mathbb{T}^n}\Bigg[\Big(\mathbf{Q}(\nabla u(s)) g'(\mathbf{Q}(\nabla u(s))) - \Big(1 - \frac{\alpha^2}{2}\Big) g(\mathbf{Q}(\nabla u(s)))\Big) \\
&\qquad\qquad\qquad \cdot \mathrm{D}\big(\mathbf{v}(\nabla u(s))\big)^T : \mathrm{D}\big(\mathbf{v}(\nabla u(s))\big)\Bigg]\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^t\int_{\mathbb{T}^n} g''(\mathbf{Q}(\nabla u(s)))\Big(|\,\mathrm{D}^2 u(s)\mathbf{v}(\nabla u(s))|^2 - |\mathbf{v}(\nabla u(s))\cdot \mathrm{D}^2 u(s)\mathbf{v}(\nabla u(s))|^2\Big)\mathrm{d}s \\
&\quad \leq \mathbb{E}\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(0))),
\end{aligned}$$

with all integrals being positive. In particular, $\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u))$ is a non-negative supermartingale.

Furthermore, for a constant $C = C(\alpha) > 0$ we have that

$$\mathbb{E}\sup_{t\in[0,T]}\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) \leq C\mathbb{E}\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(0))).$$

Proof. The first part follows from Corollary 7.1 in the same way Lemma 5.10 follows from Lemma 5.9. Note that no Gronwall argument is needed since no lower-order term appears in (7.2).

For the second part we note that by Corollary 7.1

$$\mathbb{E}\sup_{t\in[0,T]}\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) \leq \mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(s)))\nabla\cdot\big(\mathbf{v}(\nabla u(s))\big)\,\mathrm{d}\beta(s)\right|$$

and because of the Burkholder-Davis-Gundy inequality from Proposition 4.22

$$
\begin{aligned}
&\mathbb{E} \sup_{t\in[0,T]} \int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) \\
&\leq C\mathbb{E}\left(\int_0^T \left(\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t)))\nabla\cdot\left(\mathbf{v}(\nabla u(t))\right)\right)^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq C\mathbb{E}\left(\int_0^T \left[\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) \int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) \left|\nabla\cdot\left(\mathbf{v}(\nabla u(t))\right)\right|^2\right] \mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\mathbb{E} \sup_{t\in[0,T]} \int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) + C\mathbb{E}\int_0^T \int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) \left|\nabla\cdot\left(\mathbf{v}(\nabla u(t))\right)\right|^2 \mathrm{d}t.
\end{aligned}
$$

This and the first part imply that

$$
\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} \int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) &\leq C\mathbb{E}\int_0^T \int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) \left|\nabla\cdot\left(\mathbf{v}(\nabla u(t))\right)\right|^2 \mathrm{d}t \\
&\leq C(\alpha)\mathbb{E}\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(0))).
\end{aligned}
$$

□

Corollary 7.2 at hand, we can prove the following rigorous formulation of Theorem 1.3.

Theorem 7.3. *Let u be a strong solution of* (7.1) *on the time interval $I = [0, T]$ with uniformly Lipschitz continuous initial data, i.e. $\|\nabla u(0)\|_{L^\infty(\mathbb{T}^n)} \leq L$ a.s. for a constant $L > 0$.*

Then it holds $\|\nabla u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^n))} \leq L$ a.s.

Proof. Let $g(w) \coloneqq (w - \sqrt{1 + L^2})_+^3 - (w - 1 - \sqrt{1 + L^2})_+^3$, $w \geq 1$. Then g is non-negative, increasing and convex with g'' bounded and $g(1) = g'(1) = 1$. We can apply Corollary 7.2 and deduce

$$
0 \leq \mathbb{E} \sup_{t\in[0,T]} \int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(t))) \leq C\mathbb{E}\int_{\mathbb{T}^n} g(\mathbf{Q}(\nabla u(0))) = 0.
$$

Thus $\|\mathbf{Q}(\nabla u(t))\|_{L^\infty(0,T;L^\infty(\mathbb{T}^n))} \leq \sqrt{1 + L^2}$ a.s., which implies the Lipschitz continuity of u. □

7.2. Large-time behavior

As mentioned above, this section is devoted to the proof of a large-time result that follows from Theorem 7.3. We will give a stronger characterization of the large-time behavior compared to [ESvR12]. This is available, since we can bound the Hessian of a solution in a appropriate norm.

Proposition 7.4. *Let u be a strong solution of* (7.1) *on the time interval $I = [0,\infty)$ with uniformly Lipschitz continuous initial data, i.e. $\|\nabla u(0)\|_{L^\infty(\mathbb{T}^n)} \le L$ a.s. for a constant $L > 0$.*

Then there are constants $c_0 = c_0(L)$ and $C = C(\alpha)$ such that

$$\mathbb{E} \sup_{s\in[0,t]} \|\nabla u(s)\|^2_{L^2(\mathbb{T}^n)} + c_0 \mathbb{E} \int_0^t \|\operatorname{D}^2 u(s)\|^2_{L^2(\mathbb{T}^n)} \,\mathrm{d}s \le C\mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)} \ \forall t \in I. \tag{7.3}$$

Proof. From Theorem 7.3 we deduce that $\|\nabla u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^n))} \le L$ a.s. Furthermore, for $g(w) = w^2$, $w \ge 1$, we infer from Corollary 7.2 for all $t \in [0,\infty)$

$$\begin{aligned}\mathbb{E}\|\nabla u(t)\|^2_{L^2(\mathbb{T}^n)} + \left(1 + \frac{\alpha^2}{2}\right) \mathbb{E} \int_0^t\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u(s))^2 \operatorname{D}\big(\mathbf{v}(\nabla u(s))\big)^T : \operatorname{D}\big(\mathbf{v}(\nabla u(s))\big) \,\mathrm{d}s \\ \le \mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)}\end{aligned}$$

and

$$\mathbb{E} \sup_{s\in[0,t]} \|\nabla u(s)\|^2_{L^2(\mathbb{T}^n)} \le C\mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)}.$$

Note that

$$\begin{aligned}\mathbf{Q}(\nabla u)^2 \operatorname{D}\big(\mathbf{v}(\nabla u)\big)^T : \operatorname{D}\big(\mathbf{v}(\nabla u)\big) = \operatorname{D}^2 u \left(\operatorname{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)\right) \\ : \left(\operatorname{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)\right) \operatorname{D}^2 u\end{aligned}$$

and for the smallest eigenvalue we have

$$\lambda_{\min} \left(\operatorname{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)\right) \ge 1 - |\mathbf{v}(\nabla u)|^2 = \frac{1}{\mathbf{Q}(\nabla u)^2} \ge \frac{1}{1+L^2} \text{ a.s.}$$

By Lemma C.2 we have

$$\mathbf{Q}(\nabla u)^2 \operatorname{D}\big(\mathbf{v}(\nabla u)\big)^T : \operatorname{D}\big(\mathbf{v}(\nabla u)\big) \ge \frac{1}{(1+L^2)^2} |\operatorname{D}^2 u|^2$$

and this proves (7.3). □

We will use the bound of the Hessian, to prove that solutions at large times become spatially homogeneous and behave like the driving Brownian motion in time, cf. Theorem 1.4.

Theorem 7.5. *Let u be a strong solution of* (7.1) *on the time interval $I = [0,\infty)$ with uniformly Lipschitz continuous initial data, i.e. $\|\nabla u(0)\|_{L^\infty(\mathbb{T}^n)} \le L$ a.s. for a constant $L > 0$.*

Then there is a real-valued random variable $\bar{u} \in L^1(\Omega)$ such that

$$\mathbb{E} \sup_{t\ge T} \|u(t) - \alpha\beta(t) - \bar{u}\|_{H^1(\mathbb{T}^n)} \to 0 \text{ for } T \to \infty.$$

Proof. The estimates in Theorem 7.3 and Proposition 7.4 imply that

$$\mathrm{D}^2 u \in L^2(\Omega; L^2(0,\infty; L^2(\mathbb{T}^n)))$$

and

$$\|\nabla u\|_{L^\infty(0,\infty;L^\infty(\mathbb{T}^n))} \leq L \text{ a.s.}$$

For the convergence as $T \to \infty$ we note that by Corollary 4.30

$$\begin{aligned}
\mathrm{d}\int_{\mathbb{T}^n} (u - \alpha\beta) &= \int_{\mathbb{T}^n} \mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u))\,\mathrm{d}t + \alpha\int_{\mathbb{T}^n} (\mathbf{Q}(\nabla u) - 1) \circ \mathrm{d}\beta \\
&= -\left(1 - \frac{\alpha^2}{2}\right)\int_{\mathbb{T}^n} \mathbf{v}(\nabla u)\cdot \mathrm{D}^2 u\mathbf{v}(\nabla u)\,\mathrm{d}t + \int_{\mathbb{T}^n} \alpha\,(\mathbf{Q}(\nabla u) - 1)\,\mathrm{d}\beta.
\end{aligned}$$

For the drift we estimate with a Poincaré inequality

$$\begin{aligned}
\left|\int_{\mathbb{T}^n} \mathbf{v}(\nabla u)\cdot \mathrm{D}^2 u\mathbf{v}(\nabla u)\right| &\leq \int_{\mathbb{T}^n} \left|\mathrm{D}^2 u\right| |\nabla u| \\
&\leq \|\mathrm{D}^2 u\|_{L^2(\mathbb{T}^n)}\|\nabla u\|_{L^2(\mathbb{T}^n)} \leq C\|\mathrm{D}^2 u\|^2_{L^2(\mathbb{T}^n)}.
\end{aligned}$$

Hence,

$$\mathbb{E}\left|\int_0^\infty\int_{\mathbb{T}^n} \mathbf{v}(\nabla u(t))\cdot \mathrm{D}^2 u(t)\mathbf{v}(\nabla u(t))\,\mathrm{d}t\right| \leq C\mathbb{E}\int_0^\infty \|\mathrm{D}^2 u(t)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t < \infty.$$

The estimate

$$\begin{aligned}
\mathbb{E}\int_0^\infty \left(\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u(t)) - 1\right)^2 \mathrm{d}t &\leq C\mathbb{E}\int_0^\infty \|\nabla u(t)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t \\
&\leq C\mathbb{E}\int_0^\infty \|\mathrm{D}^2 u(t)\|^2_{L^2(\mathbb{T}^n)} < \infty,
\end{aligned}$$

implies that the stochastic integral

$$\int_0^\infty\int_{\mathbb{T}^n} (\mathbf{Q}(\nabla u(t)) - 1)\,\mathrm{d}\beta(t)$$

is well-defined.

Let

$$\begin{aligned}
\bar{u} := \frac{1}{|\mathbb{T}^n|}\Bigg(\int_{\mathbb{T}^n} u_0 - \left(1 - \frac{\alpha^2}{2}\right)\int_0^\infty\int_{\mathbb{T}^n} \mathbf{v}(\nabla u(t))\cdot \mathrm{D}^2 u(t)\mathbf{v}(\nabla u(t))\,\mathrm{d}t \\
+ \alpha\int_0^\infty\int_{\mathbb{T}^n} (\mathbf{Q}(\nabla u(t)) - 1)\,\mathrm{d}\beta(t)\Bigg).
\end{aligned}$$

The estimates from above imply $\bar{u} \in L^1(\Omega)$.

The bound $\mathrm{D}^2 u \in L^2(\Omega; L^2(0,\infty; L^2(\mathbb{T}^n)))$ at hand we find a sequence $(t_k)_{k\in\mathbb{N}}$ of increasing times $t_k \to \infty$ such that $\mathbb{E}\| \mathrm{D}^2 u(t_k)\|^2_{L^2(\mathbb{T}^n)} \to 0$ for $k \to \infty$. We apply a Poincaré inequality to obtain

$$\|u(t) - \alpha\beta(t) - \bar{u}\|_{H^1(\mathbb{T}^n)} \leq C\left(\|\nabla u(t)\|_{L^2(\mathbb{T}^n)} + \left|\int_{\mathbb{T}^n} (u(t) - \alpha\beta(t) - \bar{u})\right|\right). \tag{7.4}$$

From Proposition 7.4 we infer

$$\mathbb{E}\sup_{t\geq t_k} \|\nabla u(t)\|^2_{L^2(\mathbb{T}^n)} \leq C\mathbb{E}\|\nabla u(t_k)\|^2_{L^2(\mathbb{T}^n)}.$$

Hence,

$$\begin{aligned}
\lim_{T\to\infty}\left(\mathbb{E}\sup_{t\geq T}\|\nabla u(t)\|_{L^2(\mathbb{T}^n)}\right)^2 &= \lim_{k\to\infty}\left(\mathbb{E}\sup_{t\geq t_k}\|\nabla u(t)\|_{L^2(\mathbb{T}^n)}\right)^2 \\
&\leq \lim_{k\to\infty}\mathbb{E}\sup_{t\geq t_k}\|\nabla u(t)\|^2_{L^2(\mathbb{T}^n)} \\
&\leq C\lim_{k\to\infty}\mathbb{E}\|\nabla u(t_k)\|^2_{L^2(\mathbb{T}^n)} \\
&\leq C\lim_{k\to\infty}\mathbb{E}\|\mathrm{D}^2 u(t_k)\|^2_{L^2(\mathbb{T}^n)} = 0.
\end{aligned}$$

For the second term in (7.4) we have with the Burkholder-Davis-Gundy inequality from Proposition 4.22 and the estimates from above

$$\begin{aligned}
&\mathbb{E}\sup_{t\geq T}\left|\int_{\mathbb{T}^n}(u(t) - \alpha\beta(t) - \bar{u})\right| \\
&\leq \left(1 - \frac{\alpha^2}{2}\right)\mathbb{E}\sup_{t\geq T}\left|\int_t^\infty\int_{\mathbb{T}^n}\mathbf{v}(\nabla u(s))\cdot \mathrm{D}^2 u(s)\mathbf{v}(\nabla u(s))\,\mathrm{d}s\right| \\
&\quad + \alpha\mathbb{E}\sup_{t\geq T}\left|\int_t^\infty\int_{\mathbb{T}^n}(\mathbf{Q}(\nabla u(s)) - 1)\,\mathrm{d}\beta(s)\right| \\
&\leq C\mathbb{E}\int_T^\infty \|\mathrm{D}^2 u(t)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t + C\left(\mathbb{E}\int_T^\infty\|\nabla u(t)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\to 0 \text{ for } T\to\infty.
\end{aligned}$$

□

From Theorem 7.5 we can deduce the next corollary which extends the one-dimensional result from [ESvR12, Theorem 4.2] to higher dimensions. Furthermore it improves the convergence in distribution in $C_{\mathrm{loc}}([0,\infty); L^2(\mathbb{T}^n))$ to convergence in $L^1(\Omega; C_b([0,\infty), H^1(\mathbb{T}^n)))$.

Corollary 7.6. *Under the same assumptions as in Theorem 7.5 we have for* $T \to \infty$

$$(u(T+t) - u(T))_{t \geq 0} - \alpha(\beta(T+t) - \beta(T))_{t \geq 0} \to 0$$

in $L^1(\Omega; C_b([0,\infty); H^1(\mathbb{T}^n)))$.

Proof. We estimate

$$\begin{aligned}\mathbb{E} \sup_{t \geq 0} &\| u(T+t) - u(T) - \alpha(\beta(T+t) - \beta(T)) \|_{H^1(\mathbb{T}^n)} \\ &\leq 2\mathbb{E} \sup_{t \geq T} \| u(t) - \alpha\beta(t) - \bar{u} \|_{H^1(\mathbb{T}^n)} \to 0.\end{aligned}$$

□

7.3. Higher regularity of solutions

In this section we will derive a-priori bounds for higher-derivatives of solutions of (5.1). These bounds will imply a decay in time formula for higher derivatives with the same asymptotics as for the deterministic heat equation. In doing so, we have to assume that the Lipschitz constant of the initial data is small. Furthermore, we have to assume that the solution is smooth, although the decay formula would make sense as long as the initial data has bounded second moments in $H^1(\mathbb{T}^n)$. This seems to be a technical restriction due to the lack of a suitable approximation of either the solution or the $L^2(\mathbb{T}^n)$ norm of the higher-derivatives. Note that neither taking difference quotients instead of derivatives nor mollifying the solution seem to work, as the interpolation inequalities used below rule out the difference quotients and the non-linear structure of the equation rules out the mollification approach.

Nevertheless, we think that the following estimates are still interesting enough to be included here and might be helpful for future research.

In what follows we will have to calculate higher derivatives of functions like $\mathbf{v}(\nabla u) \cdot \mathrm{D}^2 u \mathbf{v}(\nabla u)$. In order to handle these terms we will introduce some notation.

Remark 7.7 (Notation)**.** We recall the definitions $\mathbf{Q} : \mathbb{R}^n \to \mathbb{R}$ with $\mathbf{Q}(p) = \sqrt{1 + |p|^2}$ and $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ with $\mathbf{v}(p) = \frac{p}{\mathbf{Q}(p)}$ for $p \in \mathbb{R}^n$ from Remark B.9. Let $A : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ with $A(p) \coloneqq \mathbf{v}(p) \otimes \mathbf{v}(p) = \frac{p \otimes p}{1+|p|^2}$, $p \in \mathbb{R}^n$. It holds that $A, \mathbf{Q}, \mathbf{v} \in C^\infty$.

The k-th derivative $\mathrm{D}^k A(p)$ of A can be understood as a k-linear map $\mathrm{D}^k A(p) : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ for which we will write

$$\mathrm{D}^k A(p) \langle q_1, \ldots, q_k \rangle$$

where $q_1, \dots, q_k \in \mathbb{R}^n$ are the directions in which to evaluate the derivative $\mathrm{D}^k A(p)$. Similarly, the k-th derivatives of $\mathbf{Q}$ and $\mathbf{v}$ can be understood as k-linear maps.

Since $A, \mathbf{v} \in C^\infty$ we infer that for every bounded subset $B \subset \mathbb{R}^n$ there is a constant $C > 0$ such that

$$\left|\mathrm{D}^k A(p)\langle q_1, \dots, q_k\rangle\right| \le C \prod_{j=1}^k |q_j| \quad \text{and} \quad \left|\mathrm{D}^k \mathbf{v}(p)\langle q_1, \dots, q_k\rangle\right| \le C \prod_{j=1}^k |q_j|. \tag{7.5}$$

For a sufficiently regular function $g : \mathbb{R}^n \to \mathbb{R}$ we will use the multi-index notation for derivatives, i.e. for a multi-index $\mathbf{k} \in \mathbb{N}_0^n$ we write

$$\partial^{\mathbf{k}} g(x) \coloneqq \partial_{x_1}^{\mathbf{k}_1} \cdots \partial_{x_n}^{\mathbf{k}_n} g(x).$$

For $k \in \mathbb{N}$ we write

$$\left|\mathrm{D}^k g(x)\right|^2 \coloneqq \sum_{\substack{\mathbf{k} \in \mathbb{N}_0^n \\ |\mathbf{k}| = k}} |\partial^{\mathbf{k}} g(x)|^2,$$

which is consistent with the Euclidean and Frobenius norm for $k = 1$ and $k = 2$, respectively.

Lemma 7.8. *For a function $u \in H^{2+m}(\mathbb{T}^n)$ with $\nabla u \in L^\infty(\mathbb{T}^n)$ and $m \in \mathbb{N}_0$ it holds that $\mathbf{v}(\nabla u) \cdot \mathrm{D}^2 u \mathbf{v}(\nabla u) \in H^m(\mathbb{T}^n)$ and $\mathbf{Q}(\nabla u) \in H^{m+1}(\mathbb{T}^n)$.*

Furthermore for $k \le m$

$$\begin{aligned} \| \mathrm{D}^k \Big(\mathbf{v}(\nabla u) \cdot \mathrm{D}^2 u \mathbf{v}(\nabla u)\Big) \|_{L^2(\mathbb{T}^n)} &+ \| \mathrm{D}^{k+1} (\mathbf{Q}(\nabla u)) \|_{L^2(\mathbb{T}^n)} \\ &\le h(\|\nabla u\|_{L^\infty(\mathbb{T}^n)}) \| \mathrm{D}^{k+2} u\|_{L^2(\mathbb{T}^n)} \end{aligned}$$

for a continuous, increasing function h with $h(0) = 0$.

Proof. Let $\mathbf{k} \in \mathbb{N}_0^n$ be a multi-index with $|\mathbf{k}| = k \le m$. In the subsequent computations we will abbreviate $l = |\mathbf{l}|$ for all multi-indices $\mathbf{l}$ appearing. We will denote by $\mathbf{e}^{(1)}, \dots, \mathbf{e}^{(n)} \in \mathbb{N}_0^n$ the multi-indices corresponding to derivation into the direction of the coordinate axes. With the notation $A(\nabla u) = \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)$ we compute

$$\begin{aligned} &\partial^{\mathbf{k}} \Big(\mathbf{v}(\nabla u) \cdot \mathrm{D}^2 u \mathbf{v}(\nabla u)\Big) \\ &= \sum_{\substack{\mathbf{j} \le \mathbf{l} \le \mathbf{k} \\ \mathbf{i}^{(1)} + \dots + \mathbf{i}^{(j)} = \mathbf{l} - \mathbf{j}}} \partial^{\mathbf{k}-\mathbf{l}} \mathrm{D}^2 u : \mathrm{D}^j A(\nabla u) \left\langle \begin{matrix} \partial^{\mathbf{i}^{(1)} + \mathbf{e}^{(1)}} \nabla u, & \dots, & \partial^{\mathbf{i}^{(\mathbf{j}_1)} + \mathbf{e}^{(1)}} \nabla u, \\ \partial^{\mathbf{i}^{(\mathbf{j}_1+1)} + \mathbf{e}^{(2)}} \nabla u, & \dots, & \partial^{\mathbf{i}^{(\mathbf{j}_1+\mathbf{j}_2)} + \mathbf{e}^{(2)}} \nabla u, \\ \dots, & \dots, & \dots, \\ \partial^{\mathbf{i}^{(\mathbf{j}_1 + \dots + \mathbf{j}_{n-1}+1)} + \mathbf{e}^{(n)}} \nabla u, & \dots, & \partial^{\mathbf{i}^{(j)} + \mathbf{e}^{(n)}} \nabla u \end{matrix} \right\rangle \end{aligned}$$

and estimate with (7.5)

$$\begin{aligned}\left|\partial^{\mathbf{k}}\left(\mathbf{v}(\nabla u)\cdot \mathrm{D}^2 u\mathbf{v}(\nabla u)\right)\right| &\le \left|\partial^{\mathbf{k}}\,\mathrm{D}^2 u\right|\left|\mathbf{v}(\nabla u)\right|^2 \\ &\quad + C\sum_{\substack{\mathbf{j}\le\mathbf{l}\le\mathbf{k},\, j\ge 1\\ \mathbf{i}^{(1)}+\cdots+\mathbf{i}^{(j)}=\mathbf{l}-\mathbf{j}}}\left|\partial^{\mathbf{k}-\mathbf{l}}\,\mathrm{D}^2 u\right|\prod_{a=1}^{j}\left|\partial^{\mathbf{i}^{(a)}}\,\mathrm{D}^2 u\right| \\ &\le \left|\partial^{\mathbf{k}}\,\mathrm{D}^2 u\right|\left|\mathbf{v}(\nabla u)\right|^2 + C\sum_{\substack{1\le j\le k\\ l_0+\cdots+l_j=k+1}}\prod_{a=0}^{j}\left|\mathrm{D}^{l_a}\nabla u\right|\end{aligned}\tag{7.6}$$

with a constant $C > 0$ depending on $\|\nabla u\|_{L^\infty(\mathbb{T}^n)}$.

For $l = 0, \ldots, k+1$ choose

$$\frac{1}{p_l} := \frac{l}{2(k+1)} \quad \text{and} \quad \vartheta_l := \frac{l}{k+1}.$$

By definition $2 \le p_l \le \infty$ and $\vartheta_l \in [0,1]$ with

$$\frac{1}{p_l} = \frac{l}{n} + \vartheta_l\left(\frac{1}{2} - \frac{k+1}{n}\right).$$

The Gagliardo-Nirenberg interpolation inequality on the flat torus, cf. [Aub82, 3.70 Theorem], implies

$$\|\,\mathrm{D}^l\nabla u\|_{L^{p_l}(\mathbb{T}^n)} \le C\|\,\mathrm{D}^{k+1}\nabla u\|_{L^2(\mathbb{T}^n)}^{\vartheta_l}\|\nabla u\|_{L^\infty(\mathbb{T}^n)}^{1-\vartheta_l}.$$

Furthermore, for $1 \le l \le k$ and $l_0 + \cdots + l_j = k+1$ it holds

$$\frac{1}{p_{l_0}} + \cdots + \frac{1}{p_{l_j}} = \frac{l_0 + \cdots + l_k}{2(k+1)} = \frac{1}{2}.$$

The Hölder inequality implies

$$\begin{aligned}\left\|\prod_{a=0}^{j}\left|\mathrm{D}^{l_a}\nabla u\right|\right\|_{L^2(\mathbb{T}^n)} &\le \prod_{a=0}^{j}\left\|\mathrm{D}^{l_a}\nabla u\right\|_{L^{p_{l_a}}(\mathbb{T}^n)} \\ &\le C\|\,\mathrm{D}^{k+1}\nabla u\|_{L^2(\mathbb{T}^n)}^{\sum_{a=0}^{j}\vartheta_{l_a}}\|\nabla u\|_{L^\infty(\mathbb{T}^n)}^{\sum_{a=0}^{j}(1-\vartheta_{l_a})} \\ &\le C\|\,\mathrm{D}^{k+1}\nabla u\|_{L^2(\mathbb{T}^n)}\|\nabla u\|_{L^\infty(\mathbb{T}^n)}^{j}.\end{aligned}$$

Combining this inequality with (7.6) yields

$$\begin{aligned}\left\|\partial^{\mathbf{k}}\left(\mathbf{v}(\nabla u)\cdot \mathrm{D}^2 u\mathbf{v}(\nabla u)\right)\right\|_{L^2(\mathbb{T}^n)} &\le \|\mathbf{v}(\nabla u)\|^2_{L^\infty(\mathbb{T}^n)}\left\|\partial^{\mathbf{k}}\,\mathrm{D}^2 u\right\| \\ &\quad + C\left(\sum_{1\le j\le k}\|\nabla u\|^j_{L^\infty(\mathbb{T}^n)}\right)\|\,\mathrm{D}^{k+1}\nabla u\|_{L^2(\mathbb{T}^n)} \\ &\le h(\|\nabla u\|_{L^\infty(\mathbb{T}^n)})\|\,\mathrm{D}^{k+1}\nabla u\|_{L^2(\mathbb{T}^n)}\end{aligned}$$

with a continuous, increasing function h and $h(0)=0$. We have used $\left|\mathbf{v}(p)\right| \le \frac{|p|}{\sqrt{1+|p|^2}} \to 0$ for $|p|\to 0$.

Repeating these computations for all multi-indices $\mathbf{k}$ with $|\mathbf{k}| = k \le m$ yields $\mathbf{v}(\nabla u)\cdot \mathrm{D}^2 u\mathbf{v}(\nabla u) \in H^m(\mathbb{T}^n)$ with the claimed inequality.

Similarly, one proves that $\mathbf{Q}(\nabla u) \in H^{m+1}(\mathbb{T}^n)$. □

The next theorem generalizes Proposition 7.4 to higher derivatives and reformulates it for $m = 0$.

Theorem 7.9 (A-priori estimates for higher derivatives)**.** *Let u be a strong solution of (7.1) on the time interval $I = [0,T]$ with uniformly Lipschitz continuous initial data, i.e. $\|\nabla u(0)\|_{L^\infty(\mathbb{T}^n)} \le L$ a.s for a constant $L > 0$. Furthermore, assume that $u \in L^2(\Omega; L^2(0,T; H^{m+2}(\mathbb{T}^n)))$, $u(0) \in L^2(\Omega; H^{m+1}(\mathbb{T}^n))$ and $m \in \mathbb{N}_0$.*

Then there is a version of u with $u \in L^2(\Omega; C([0,T]; H^{m+1}(\mathbb{T}^n)))$ and if L is sufficiently small there are constants $c_0 = c_0(m) > 0$ and $C = C(m) > 0$ such that

$$\mathbb{E}\|\,\mathrm{D}^{m+1}u(t)\|^2_{L^2(\mathbb{T}^n)} + c_0\mathbb{E}\int_0^t \|\,\mathrm{D}^{m+2}u(s)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}s \le \mathbb{E}\|\,\mathrm{D}^{m+1}u(0)\|^2_{L^2(\mathbb{T}^n)}$$

and

$$\mathbb{E}\sup_{t\in[0,T]}\|\,\mathrm{D}^{m+1}u(t)\|^2_{L^2(\mathbb{T}^n)} \le C\mathbb{E}\|\,\mathrm{D}^{m+1}u(0)\|^2_{L^2(\mathbb{T}^n)}.$$

Proof. From Theorem 7.3 we infer

$$\|\nabla u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^n))} \le L \text{ a.s.}$$

Lemma 7.8 implies $\mathbf{Q}(\nabla u)\nabla\cdot(\mathbf{v}(\nabla u)) \in L^2(\Omega; L^2(0,T; H^m(\mathbb{T}^n)))$ and $\mathbf{Q}(\nabla u) \in L^2(\Omega; L^2(0,T; H^{m+1}(\mathbb{T}^n)))$. Therefore, we can apply Proposition 4.28 to deduce that u has a version with $u \in L^2(\Omega; C([0,T]; H^{m+1}(\mathbb{T}^n)))$ and for a multi-index

$\mathbf{k} \in \mathbb{N}_0^n$ with $|\mathbf{k}| = m$ we have

$$\begin{aligned}
\mathrm{d}\int_{\mathbb{T}^n} \left|\partial^{\mathbf{k}} \nabla u\right|^2 &= -2 \int_{\mathbb{T}^n} \partial^{\mathbf{k}} \Delta u \partial^{\mathbf{k}} \left(\Delta u + \left(1 - \frac{\alpha^2}{2}\right) \mathbf{v}(\nabla u) \cdot \mathrm{D}^2 u \mathbf{v}(\nabla u) \right) \\
&\quad + \alpha^2 \int_{\mathbb{T}^n} \left|\partial^{\mathbf{k}} \nabla \mathbf{Q}(\nabla u)\right|^2 \mathrm{d}t + 2\alpha \int_{\mathbb{T}^n} \partial^{\mathbf{k}} \nabla u \cdot \partial^{\mathbf{k}} \nabla \mathbf{Q}(\nabla u) \,\mathrm{d}\beta \\
&\coloneqq \mu \,\mathrm{d}t + \sigma \,\mathrm{d}\beta
\end{aligned}$$

with

$$\begin{aligned}
\mu &= -2 \int_{\mathbb{T}^n} \partial^{\mathbf{k}} \Delta u \partial^{\mathbf{k}} \left(\Delta u + \left(1 - \frac{\alpha^2}{2}\right) \mathbf{v}(\nabla u) \cdot \mathrm{D}^2 u \mathbf{v}(\nabla u) \right) \\
&\quad + \alpha^2 \int_{\mathbb{T}^n} \left|\partial^{\mathbf{k}} \nabla \mathbf{Q}(\nabla u)\right|^2 \text{ and} \\
\sigma &= 2\alpha \int_{\mathbb{T}^n} \partial^{\mathbf{k}} \nabla u \cdot \partial^{\mathbf{k}} \nabla \mathbf{Q}(\nabla u).
\end{aligned}$$

Integration by parts, the Hölder inequality and Lemma 7.8 imply

$$\begin{aligned}
\mu &\leq -2\|\partial^{\mathbf{k}} \,\mathrm{D}^2 u\|_{L^2(\mathbb{T}^n)}^2 + 2\left(1 - \frac{\alpha^2}{2}\right) \|\partial^{\mathbf{k}} \Delta u\|_{L^2(\mathbb{T}^n)} \|\mathbf{v}(\nabla u) \cdot \mathrm{D}^2 u \mathbf{v}(\nabla u)\|_{H^m(\mathbb{T}^n)} \\
&\quad + \alpha^2 \|\mathbf{Q}(\nabla u)\|_{H^{m+1}(\mathbb{T}^n)}^2 \\
&\leq -\|\partial^{\mathbf{k}} \,\mathrm{D}^2 u\|_{L^2(\mathbb{T}^n)}^2 + h^2(L) \|\,\mathrm{D}^{m+2} u\|_{L^2(\mathbb{T}^n)}^2.
\end{aligned}$$

Since the Itô integral $\sigma \,\mathrm{d}\beta$ is a local martingale, we can deduce by Fatou's Lemma for all $t \in [0, T]$

$$\begin{aligned}
\mathbb{E}\|\partial^{\mathbf{k}} \nabla u(t)\|_{L^2(\mathbb{T}^n)}^2 &\leq \mathbb{E}\|\partial^{\mathbf{k}} \nabla u(0)\|_{L^2(\mathbb{T}^n)}^2 - \mathbb{E} \int_0^t \|\partial^{\mathbf{k}} \,\mathrm{D}^2 u(s)\|_{L^2(\mathbb{T}^n)}^2 \,\mathrm{d}s \\
&\quad + h^2(L) \mathbb{E} \int_0^t \|\,\mathrm{D}^{m+2} u(s)\|_{L^2(\mathbb{T}^n)}^2 \,\mathrm{d}s.
\end{aligned}$$

We can repeat the above computations for all multi-indices $\mathbf{k}$ with $|\mathbf{k}| = m$. By continuity of h and $h(0) = 0$, there is a constant $c_0 > 0$, that depends on m, such that for L sufficiently small

$$\mathbb{E}\|\mathrm{D}^{m+1} u(t)\|_{L^2(\mathbb{T}^n)}^2 + c_0 \mathbb{E} \int_0^t \|\mathrm{D}^{m+2} u(s)\|_{L^2(\mathbb{T}^n)}^2 \,\mathrm{d}s \leq \mathbb{E}\|\mathrm{D}^{m+1} u(0)\|_{L^2(\mathbb{T}^n)}^2. \tag{7.7}$$

In addition, the Burkholder-Davis-Gundy inequality Proposition 4.22 implies for a fixed multi-index $\mathbf{k}$

$$\begin{aligned}
\mathbb{E} \sup_{s \in [0,t]} \left| \int_0^s \sigma(\tau) \,\mathrm{d}\beta(\tau) \right| &\leq C\alpha \mathbb{E} \left(\int_0^t \left(\int_{\mathbb{T}^n} \partial^{\mathbf{k}} \nabla u(s) \cdot \partial^{\mathbf{k}} \nabla \mathbf{Q}(\nabla u(s)) \right)^2 \mathrm{d}s \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \mathbb{E} \sup_{s \in [0,t]} \|\partial^{\mathbf{k}} \nabla u(s)\|_{L^2(\mathbb{T}^n)}^2 + C\mathbb{E} \int_0^t \|\partial^{\mathbf{k}} \nabla \mathbf{Q}(\nabla u(s))\|_{L^2(\mathbb{T}^n)}^2 \,\mathrm{d}s
\end{aligned}$$

and by Lemma 7.8 and (7.7)

$$\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s \sigma(\tau)\,\mathrm{d}\beta(\tau)\right| \le \frac{1}{2}\mathbb{E}\sup_{s\in[0,t]}\|\partial^{\mathbf{k}}\nabla u(s)\|^2_{L^2(\mathbb{T}^n)} + C\mathbb{E}\|\,\mathrm{D}^{m+1}u(0)\|^2_{L^2(\mathbb{T}^n)}.$$

We conclude

$$\mathbb{E}\sup_{s\in[0,t]}\|\,\mathrm{D}^{m+1}u(s)\|^2_{L^2(\mathbb{T}^n)} \le C\mathbb{E}\|\,\mathrm{D}^{m+1}u(0)\|^2_{L^2(\mathbb{T}^n)}.$$

□

Corollary 7.10 (Smoothing property)**.** *Under the same assumptions as Theorem 7.9 and L sufficiently small there is a constant $C = C(m) > 0$ such that for all $1 \le k \le m$*

$$\mathbb{E}\|\,\mathrm{D}^{k+1}u(t)\|^2_{L^2(\mathbb{T}^n)} \le Ct^{-k}\mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)}. \tag{7.8}$$

Proof. We will prove this result by induction over k.

Theorem 7.9 implies that $t \mapsto \mathbb{E}\|\,\mathrm{D}^2u(t)\|^2_{L^2(\mathbb{T}^n)}$ is decreasing and

$$c_0\mathbb{E}\int_0^t \|\,\mathrm{D}^2u(s)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}s \le \mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)}.$$

Therefore

$$\mathbb{E}\|\,\mathrm{D}^2u(t)\|^2_{L^2(\mathbb{T}^n)} \le \frac{1}{t}\mathbb{E}\int_0^t \|\,\mathrm{D}^2u(s)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}s \le \frac{1}{c_0t}\mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)}.$$

Now, let $1 \le k \le m-1$ and (7.8) hold for k. As $k+1 \le m$ we infer from Theorem 7.9 that $t \mapsto \mathbb{E}\|\,\mathrm{D}^{k+2}u(t)\|^2_{L^2(\mathbb{T}^n)}$ is decreasing and for all $0 \le s \le t \le T$

$$c_0\mathbb{E}\int_s^t \|\,\mathrm{D}^{k+2}u(\sigma)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}\sigma \le \mathbb{E}\|\,\mathrm{D}^{k+1}u(s)\|^2_{L^2(\mathbb{T}^n)}.$$

For $s = \frac{t}{2}$ we deduce by (7.8) that

$$\begin{aligned}\mathbb{E}\|\,\mathrm{D}^{k+2}u(t)\|^2_{L^2(\mathbb{T}^n)} &\le \frac{2}{t}\mathbb{E}\int_s^t \|\,\mathrm{D}^{k+2}u(\sigma)\|^2_{L^2(\mathbb{T}^n)}\,\mathrm{d}\sigma \\ &\le \frac{2}{c_0t}\mathbb{E}\|\,\mathrm{D}^{k+1}u(s)\|^2_{L^2(\mathbb{T}^n)} \\ &\le Ct^{-1}s^{-k}\mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)} \\ &= Ct^{-(k+1)}\mathbb{E}\|\nabla u(0)\|^2_{L^2(\mathbb{T}^n)}.\end{aligned}$$

□

8. Simulations

We describe a finite element discretization of (5.1) and use this for Monte-Carlo simulations of energies that appeared in the preceding chapters. The analysis presented in the preceding chapters motivates to use a variational formulation of (5.1) with respect to the scalar product of $H^1(\mathbb{T}^n)$, instead of the common choice of the $L^2(\mathbb{T}^n)$ scalar product. We refer to [FLP14] for a finite element scheme in the case of $n = 1$ which uses a variational formulation of (5.1) in $L^2(\mathbb{T}^n)$.

The numerical scheme at hand, we present Monte-Carlo simulations for certain energies which appeared during the existence and regularity proofs.

8.1. Discretization

For a given parameter $N \in \mathbb{N}$ we divide the unit cube $[0,1]^n$ into a uniform grid of N^n cubes being translations of $[0,h]^n$ with length $h \coloneqq \frac{1}{N}$. Let $V_N \subset H^2(\mathbb{T}^n)$ be the finite dimensional subspace of (periodic) piecewise polynomials, which we obtain by considering an n-dimensional analog of the Bogner-Fox-Schmit element [Bra07, 5.10]. That is on each cell, up to affine transformations, the space of polynomials spanned by $x \mapsto p_1(x_1) \cdot \dots \cdot p_n(x_n)$ with $p_1, \dots, p_n \in \{P_1, \dots, P_4\}$ and $P_i : [0,1] \to \mathbb{R}$ being the cubic polynomials uniquely determined by:

	$P_i(0)$	$P_i'(0)$	$P_i(1)$	$P_i'(1)$
$i = 0$	1	0	0	0
$i = 1$	0	1	0	0
$i = 2$	0	0	1	0
$i = 3$	0	0	0	1

On each cell there are 4^n local degrees of freedom and due to the periodicity and the continuous differentiability of functions in V_N it has the dimension $\dim V_N = (2N)^n$.

Instead of solving the SPDE (5.1) we look for a V_N-valued process u solving the stochastic ordinary differential equation (SODE)

$$\mathrm{d}u = \mathrm{proj}_{V_N} \Big(\mathbf{Q}(\nabla u) \nabla \cdot (\mathbf{v}(\nabla u)) \Big) \, \mathrm{d}t + \sum_{l=1}^{M} \mathrm{proj}_{V_N} \left(\mathbf{Q}(\nabla u) \varphi_l \right) \circ \mathrm{d}\beta^l \tag{8.1}$$

with another parameter $M \in \mathbb{N}$ and $\mathrm{proj}_{V_N} : H^1(\mathbb{T}^n) \to V_N$ being the orthogonal projection onto V_N with respect to the scalar product of $H^1(\mathbb{T}^n)$. Equation (8.1) is equivalent to

$$\mathrm{d}\int_{\mathbb{T}^n} uw + \nabla u \cdot \nabla w = \int_{\mathbb{T}^n} \Big[(w - \Delta w)\, \mathbf{Q}(\nabla u) \nabla\cdot (\mathbf{v}(\nabla u))\, \mathrm{d}t \\ + \sum_{l=1}^{M} (w - \Delta w)\, \mathbf{Q}(\nabla u^N) \varphi_l \circ \mathrm{d}\beta^l \Big] \ \forall w \in V_N$$

and has the Itô formulation

$$\mathrm{d}\int_{\mathbb{T}^n} uw + \nabla u \cdot \nabla w = \int_{\mathbb{T}^n} \Big[(w - \Delta w)\, F(\cdot, \nabla u, \mathrm{D}^2 u)\, \mathrm{d}t \\ + \sum_{l=1}^{M} (w - \Delta w)\, B_l(\cdot, \nabla u)\, \mathrm{d}\beta^l \Big] \tag{8.2}$$

with

$$F(x,p,A) := \operatorname{tr} A - \left(1 - \frac{\sum_{l=1}^M \varphi_l(x)^2}{2}\right) \mathbf{v}(p) \cdot A\mathbf{v}(p) + \frac{1}{2}\sum_{l=1}^{M} \varphi_l(x) \nabla\varphi_l(x) \cdot p,$$
$$B_l(x,p) := \mathbf{Q}(p)\varphi_l(x).$$

Now, let $w_1, \dots, w_{(2N)^n} \in V_N$ be the basis of V_N which we get by considering affine transformations of products of $P_1, \dots, P_4$. Furthermore, let

$$u(t) = \sum_{k=1}^{(2N)^n} \hat{u}_k(t) w_k.$$

In terms of the coefficients $\hat{u}(t)$ equation (8.2) becomes

$$\mathrm{d}\sum_{k=1}^{(2N)^n} A_{\tilde{k}k} \hat{u}_k(t) = \int_{\mathbb{T}^n} \Big[\left(w_{\tilde{k}} - \Delta w_{\tilde{k}}\right) F(\cdot, \nabla u, \mathrm{D}^2 u)\, \mathrm{d}t \\ + \sum_{l=1}^{M} \left(w_{\tilde{k}} - \Delta w_{\tilde{k}}\right) B_l(\cdot, \nabla u)\, \mathrm{d}\beta^l \Big]$$

for all $\tilde{k} = 1, \dots, (2N)^n$ with

$$A_{\tilde{k}k} := \int_{\mathbb{T}^n} w_k w_{\tilde{k}} + \nabla w_k \cdot \nabla w_{\tilde{k}}, \quad k, \tilde{k} = 1, \dots, (2N)^n.$$

We infer a fully-discrete explicit system with the Euler-Maruyama scheme, cf. [KP92, 9.1]. For this, let $\Delta t > 0$ and $t_j := j\Delta t$ for $j \in \mathbb{N}$. We approximate $u(t_j)$ through a V_N-valued random variable u^j with $u^j := \sum_{k=1}^{(2N)^n} \hat{u}_k^j w_k$ where we define the coefficients $\hat{u}^j = \left(\hat{u}_k^j\right)_k$ through

$$A\hat{u}^j - A\hat{u}^{j-1} = \hat{F}(\hat{u}^{j-1})\Delta t + \sum_{l=1}^{M} \hat{B}_l(\hat{u}^{j-1}) \left(\beta^l(t_j) - \beta^l(t_{j-1})\right), \quad j \geq 1, \tag{8.3}$$

where

$$\left(\hat{F}(\hat{u}^j)\right)_k := \int_{\mathbb{T}^n} (w_k - \Delta w_k)\, F(\cdot, \nabla u^j, \mathrm{D}^2 u^j),$$
$$\left(\hat{B}_l(\hat{u}^j)\right)_k := \int_{\mathbb{T}^n} (w_k - \Delta w_k)\, B_l(\cdot, \nabla u^j),\ k = 1, \dots, (2N)^n,\ l = 1, \dots, M.$$

Furthermore, we prescribe an initial condition for $\hat{u}^0$. In order to simulate (8.3) we restrict ourselves to a finite number of samples $P \in \mathbb{N}$ of the initial data and the increments of the Brownian motions.

To compute A, $\hat{F}$ and $\hat{B}$ we use on each cell a Gaussian quadrature formula with 4^n points corresponding to 4 points in each dimension. With this choice of degree of the quadrature formula the matrix A is computed exactly.

To solve the linear equation (8.3) in each timestep, we use the conjugate gradient method for small P. When the number of samples P becomes large, the explicit computation of A^{-1} is preferred.

The algorithm is implemented in Python using SciPy, cf. [VGO+19].

8.2. Monte-Carlo simulations

In this section we present simulations of $t \mapsto \mathbb{E}\mathfrak{E}(u(t))$ for a solution u of (5.1) for certain energies $\mathfrak{E}$. With the notation from above, we approximate the energy by

$$\mathbb{E}\mathfrak{E}(u(t_j)) \approx \frac{1}{P} \sum_{p=1}^{P} \mathfrak{E}(u^{j,p}),$$

where $u^{j,p}$ corresponds to the p-th sample of u^j, $p = 1, \dots, P$.

In all cases we assume $n = 2$. Furthermore, for $N \in \mathbb{N}$ we set $\Delta t := \frac{0.01}{N^2}$.

We will denote by $(g_k)_{k \in \mathbb{N}}$ the orthonormal basis of $L^2(\mathbb{T}^n)$ which can be canonically constructed from the basis

$$\{1, \sqrt{2}\cos(2\pi x), \sqrt{2}\sin(2\pi x), \sqrt{2}\cos(4\pi x), \sqrt{2}\sin(4\pi x), \dots\}$$

of $L^2(\mathbb{T}^1)$, such that $g_1 = 1$.

8.2.1. Monte-Carlo simulation for spatially homogeneous noise

To simulate the case of spatially homogeneous noise, described by (7.1), we set $M = 1$ and $\varphi_1 = \alpha = 1$. As initial data we choose $U_0 := \sum_{k=1}^{7^2} \lambda_k g_k$ with coefficients $\lambda_k \in (0, 1)$ which are randomly chosen but identical for all samples. The discrete initial data u^0 is inferred by projecting U_0 to V_N orthogonal with respect to the $L^2(\mathbb{T}^n)$ scalar product.

We have shown in Example 5.8 that the expectation of the area $\mathbb{E} \int_{\mathbb{T}^n} \mathbf{Q}(\nabla u(t))$ is decreasing in t. Figure 8.1 shows this decrease for $N = 32$ on the time interval

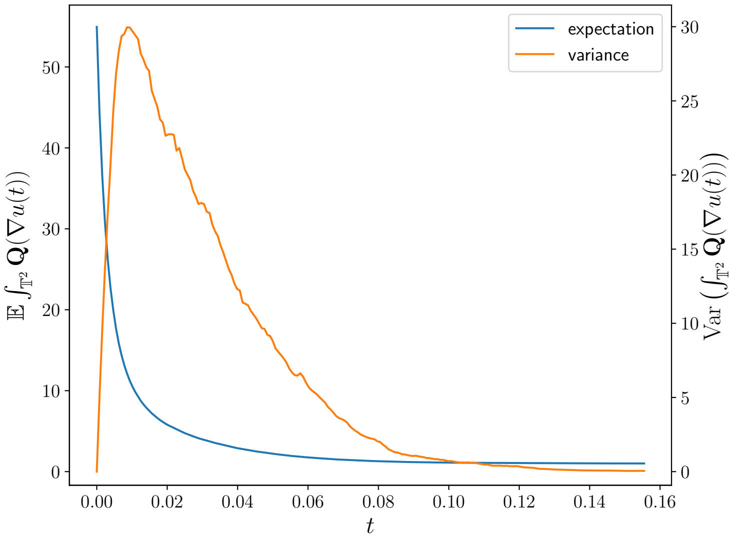

Figure 8.1.: Decay of expectation of area for $N = 32$, $M = 1$, $P = 4096$ and $\Delta t = \frac{0.01}{N^2}$.

$[0, 16000\Delta t] = [0, \frac{5}{32}]$, where the average is taken over $P = 4096$ trajectories. Furthermore, we recover the fact that $\mathbb{E} \int_{\mathbb{T}^n} \mathbf{Q}(\nabla u(t))$ converges to 1, which is a consequence of the large-time result Theorem 7.5. The variance of $\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u(t))$ increases at the beginning, but after some time it starts decreasing and converges to zero. Note that the plot of the variance is very rough, which indicates that the Monte-Carlo simulation has too few samples.

In Figure 8.2 the same energy is plotted with more samples on the shorter time interval $[0, \frac{5}{256}]$ for $N = 32$ and $P = 65536$ as well as $N = 64$ and $P = 12288$, showing the same behavior as before, but with a smoother evolution of the variance.

Of great importance for the existence proof in Chapter 5 is the bound in L^p of $\mathbf{Q}(\nabla u)$, c.f. Corollary 5.11, which is equivalent to a bound for the L^p norm of ∇u. In the case of spatially homogeneous noise, this bound generalizes to the maximum principle for the gradient of solutions, c.f. Theorem 7.3. Instead of simulating the L^∞ norm of solutions, which is numerically very unstable, in Figure 8.3 the L^p norms for $p \geq 2$ are shown. When the exponent p increases, a single outlier has more importance, hence the number of trajectories simulated P has to increase in order to get a smooth result. In order to keep the computation time acceptable we have to reduce the number of timesteps. Figure 8.3 also suggests that there is an acceleration of the evolution after some time. Furthermore, after this acceleration the evolution is more regular, for example the slope of

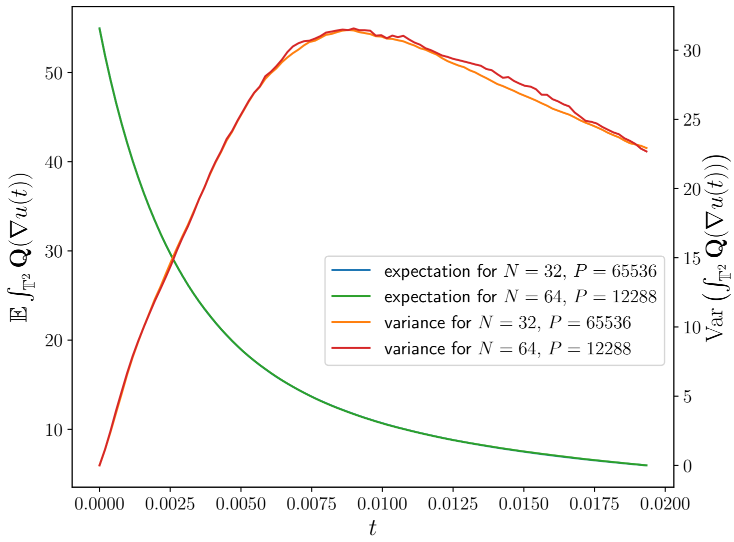

Figure 8.2.: Decay of expectation of area for different values of N and P with $M = 1$ and $\Delta t = \frac{0.01}{N^2}$.

$\log\left(\mathbb{E}\int_{\mathbb{T}^n} |\nabla u(t)|^p\right)^{\frac{1}{p}}$ becomes constant and Figure 8.4 shows that for $p = 2$ there is a rapid decrease of the variance of the evolution. Our interpretation is that the behavior of (7.1) changes when the gradient is "small". This is in accordance with the a-priori regularity results from Section 7.3, which only hold for small gradients.

8.2.2. Monte-Carlo simulation for colored noise

We consider (5.1) with the same initial-data as above for not necessarily spatially homogeneous noise. For the noise coefficients we choose $\varphi_l = 2^{-l+1} g_l$ with $l \in \mathbb{N}$. Since $\varphi_1 = 1$, for $M = 1$ we have the same evolution law as in the spatially homogeneous case from above. Note that (φ_l) fulfills Assumptions 5.2.

In Figure 8.5 the evolution of the area is plotted. In contrast to the spatially homogeneous case the evolution is slower and less regular, for example the variance is larger. Furthermore, the plotted expectation is rougher indicating that one needs more samples in the colored noise case in order to get a good estimate for the expectation. We have shown in Example 5.8 that the expectation of the area grows at most exponentially in time. The Monte-Carlo simulations suggests that the expectation of the area is decreasing, at least for large values. The same holds true for the L^p norms of the gradient, the evolution of which are shown in Figure 8.6. Nevertheless, we can not expect the area and the L^p norms to

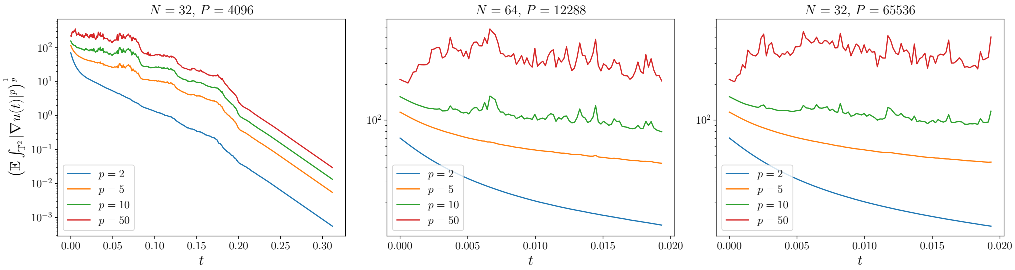

Figure 8.3.: Decay of $\left(\mathbb{E}\int_{\mathbb{T}^n} |\nabla u(t)|^p\right)^{\frac{1}{p}}$ for certain values of p with $M = 1$ and $\Delta t = \frac{0.01}{N^2}$.

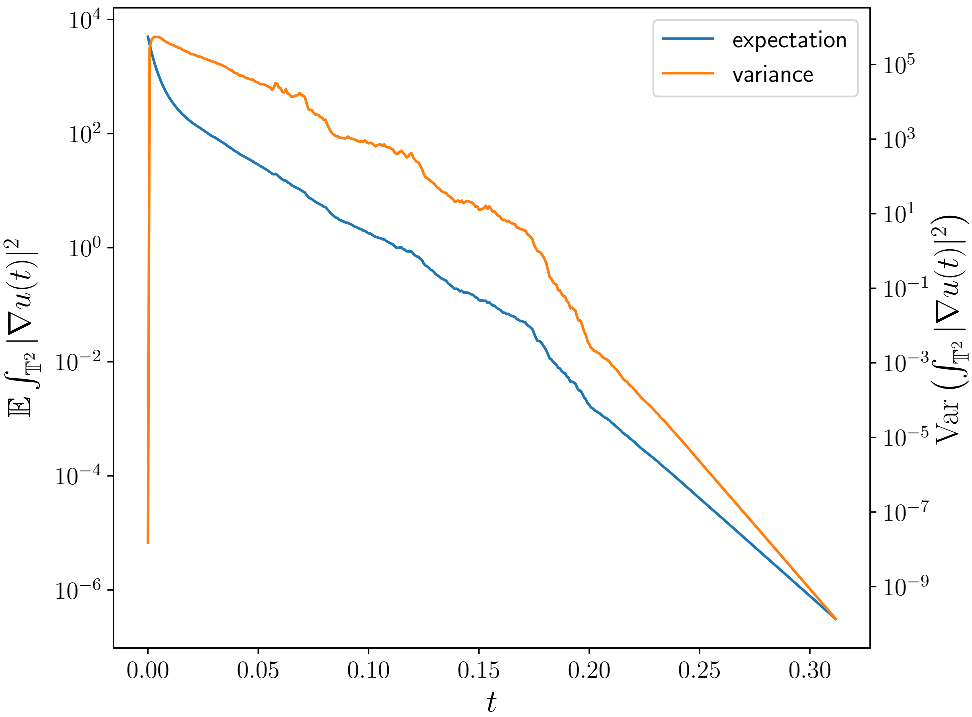

Figure 8.4.: Evolution of $\mathbb{E}\int_{\mathbb{T}^n} |\nabla u(t)|^2$ with $N = 32,\ M = 1,\ P = 4096$ and $\Delta t = \frac{0.01}{N^2}$.

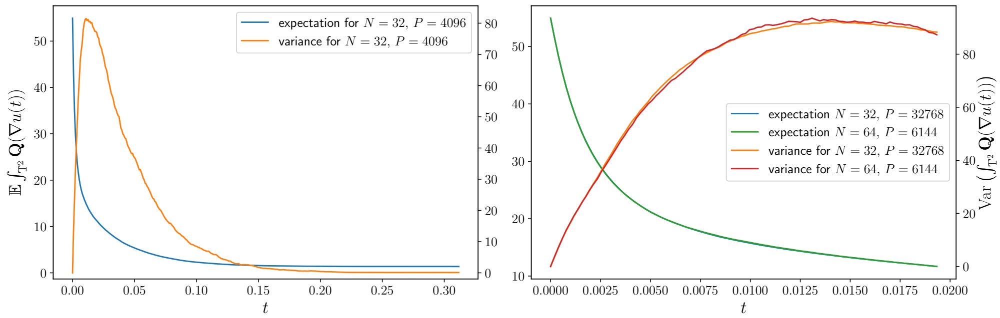

Figure 8.5.: Evolution of $\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u(t))$ for $M = 25$ and several values for N and P with $\Delta t = \frac{0.01}{N^2}$.

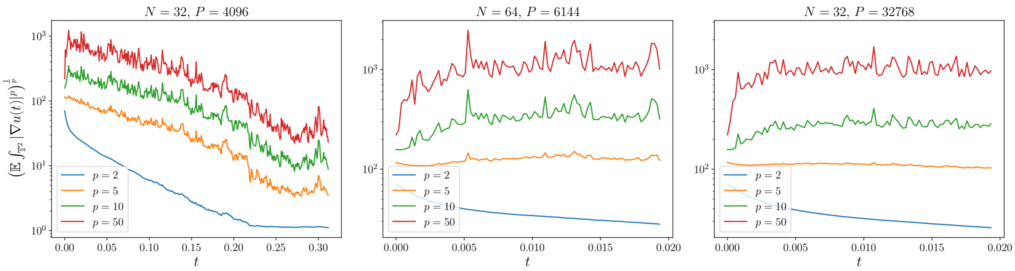

Figure 8.6.: Evolution of $\left(\mathbb{E} \int_{\mathbb{T}^n} |\nabla u(t)|^p\right)^{\frac{1}{p}}$ for certain values of p with $M = 25$ and $\Delta t = \frac{0.01}{N^2}$.

converge to 1 and 0, respectively, as the colored noise prevents constant solutions.

To analyze how the colored noise introduces spatial roughness, we have repeated the simulations with $U_0 = 0$ as initial data. Figure 8.7 shows that the expectation of the area increases with a nearly linear behavior near $t = 0$. There seems to be a saturation happening, which prevents the area to grow beyond a certain bound. Figure 8.8 supports the interpretation that the evolution is saturated after some time, as the L^p norms of the gradient of solutions for $p \gg 1$ increase but stay bounded.

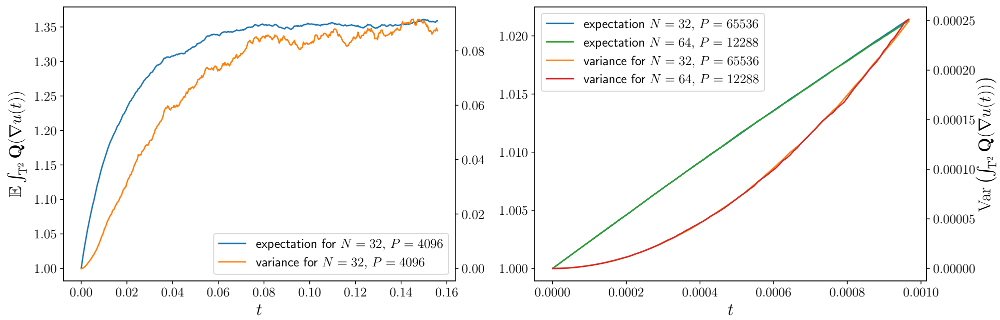

Figure 8.7.: Evolution of $\int_{\mathbb{T}^n} \mathbf{Q}(\nabla u(t))$ for $M = 25$ and several values for N and P with $\Delta t = \frac{0.01}{N^2}$ starting from $U_0 = 0$.

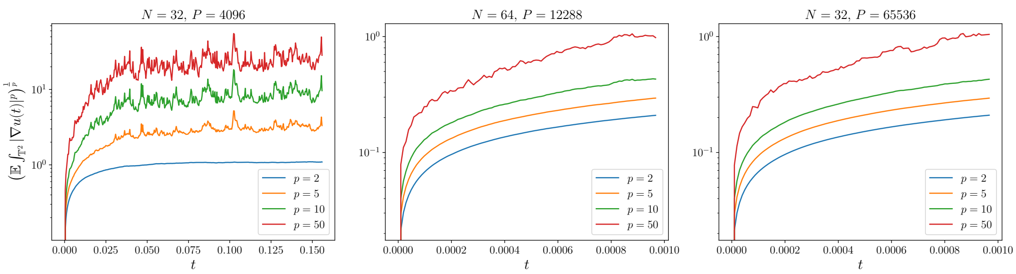

Figure 8.8.: Evolution of $\left(\mathbb{E} \int_{\mathbb{T}^n} |\nabla u(t)|^p\right)^{\frac{1}{p}}$ for certain values of p with $M = 25$ and $\Delta t = \frac{0.01}{N^2}$ starting from $U_0 = 0$.

9. Extension to anisotropic SMCF

In this chapter we will extend the previous considerations to the case of anisotropic SMCF. We will prove existence of solutions for graphs with spatially homogeneous noise. Since the calculations in the case of colored noise become very involved we restrict the whole presentation to spatially homogeneous noise.

9.1. Anisotropic mean curvature flow

In this section we introduce the notation for anisotropic geometries and state the anisotropic version of (3.1) on a formal level. We adapt the notation from [Bel04].

Definition 9.1. A convex function $\phi : \mathbb{R}^{n+1} \to [0, \infty)$ which is

(i) positively 1-homogeneous, i.e. $\phi(\lambda z) = \lambda\phi(z)$ for all $\lambda > 0$, $z \in \mathbb{R}^{n+1}$ and

(ii) coercive, i.e. there is a $c_0 > 0$ such that $\phi(z) \geq c_0|z|$ for all $z \in \mathbb{R}^{n+1}$

is called a Minkowski norm on $\mathbb{R}^{n+1}$. If additionally $\phi^2 \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and ϕ^2 is strictly convex then we will call ϕ a regular Minkowski norm.

Given a Minkowski norm ϕ we define the corresponding unit ball $B_\phi = \{z \in \mathbb{R}^{n+1} \mid \phi(z) \leq 1\}$, which is sometimes also called the Wulff shape of ϕ. Furthermore the dual norm ϕ° is given by

$$\phi^\circ(z^\circ) \coloneqq \sup\{z \cdot z^\circ \mid z \in B_\phi\}, \; z^\circ \in \mathbb{R}^{n+1}.$$

For a regular ϕ we introduce the duality map

$$T_\phi \coloneqq \phi\nabla\phi$$

which is the unique solution of

$$\phi^\circ(T_\phi(z))^2 = \phi(z)^2 = z \cdot T_\phi(z).$$

According to [Bel04, Definition 2.2] the regularity of a Minkowski norm ϕ can equivalently be characterized in terms of the unit ball B_ϕ. Since we could not find a good reference, we sketch the proof of the equivalence.

Remark 9.2 (Regular Minkowski norm)**.** A Minkowski norm ϕ is regular if and only if ∂B_ϕ is smooth and all principal curvatures are strictly positive.

Sketch of the proof. Let ϕ be a regular Minkowski norm and $\lambda > 0$ such that $\mathrm{D}^2\phi^2 \geq \lambda\,\mathrm{Id}$. Then ∂B_ϕ is smooth since $\mathcal{M} := \partial B_\phi$ is a compact level set of ϕ^2 which can be locally parameterized as the graph of a smooth function. From Remark B.10 we infer for the Weingarten map $s : T_z\mathcal{M} \to T_z\mathcal{M}$ for a tangent vector $X \in T_z\mathcal{M}$

$$s(X) \cdot X = \frac{X \cdot \mathrm{D}^2(\phi^2)X}{|\nabla(\phi^2)|} \geq \lambda \frac{|X|^2}{|\nabla(\phi^2)|}.$$

Therefore the principal curvatures of $\mathcal{M}$ are bounded from below by $\frac{\lambda}{|\nabla(\phi^2)|}$ and $\nabla(\phi^2)$ is bounded on $\mathcal{M}$ because of its continuity and the compactness of $\mathcal{M}$.

For the opposite conclusion we assume that $\mathcal{M} = \partial B_\phi$ is a smooth hypersurface with all of its principal curvatures bounded from below by a constant γ. With the same formula for the Weingarten map as above we infer for all tangent vectors $X \in T_z\mathcal{M}$

$$X \cdot \mathrm{D}^2(\phi^2)X \geq \gamma|\nabla(\phi^2)||X|^2.$$

In Lemma 9.5 we prove for $z \in \mathcal{M}$ that $\nabla(\phi^2)(z) \cdot z = 2\phi(z)\nabla\phi(z) \cdot z = 2\phi^2(z) = 2$. This implies $|\nabla(\phi^2)| \geq \frac{2}{|z|}$ and therefore for all $z \in \mathcal{M}$

$$X \cdot \mathrm{D}^2(\phi^2)(z)X \geq 2\frac{\gamma}{|z|}|X|^2.$$

Furthermore we know that $\mathrm{D}^2(\phi^2)(z)z = \nabla(\phi^2)(z) \perp T_z\mathcal{M}$ and $z \cdot \mathrm{D}^2(\phi^2)(z)z = 2\phi^2(z)$. Since $z \notin T_z\mathcal{M}$ we can conclude that $\mathrm{D}^2(\phi^2) \geq c\,\mathrm{Id}$ with a constant $c > 0$ that only depends on γ and c_0. ☐

In the subsequent part of this section we will always assume that ϕ is a regular Minkowski norm. Then the dual norm ϕ° is also regular, c.f. [Bel04, Section 2.2].

Definition 9.3 (Anisotropic area)**.** Let $\mathcal{M}$ be a smooth properly embedded hypersurface in $\mathbb{R}^{n+1}$. The anisotropic area of $\mathcal{M}$ with respect to ϕ is given by

$$|\mathcal{M}|_\phi := \int_{\mathcal{M}} \phi^\circ(\nu)\,\mathrm{d}\mathcal{H}^n.$$

In [BP96, Theorem 5.1] a formula for the first variation of the anisotropic area is proven. In order to state this formula we will introduce some notation generalizing the notion of a unit normal field and mean curvature to the anisotropic setting.

Definition 9.4. Let $\mathcal{M}$ be a smooth properly embedded hypersurface in $\mathbb{R}^{n+1}$ with a smooth local parametrization F. We denote by

$$\begin{aligned} \nu_\phi &:= \frac{\nu}{\phi^\circ(\nu)}, \\ n_\phi &:= T_{\phi^\circ}(\nu_\phi) = \nabla\phi^\circ(\nu) \text{ and} \\ H_\phi &:= -\nabla\cdot n_\phi = -g^{ij}\partial_i n_\phi \cdot \partial_j F \end{aligned}$$

the anisotropic unit normal field ν_ϕ, the Cahn-Hoffmann vector field n_ϕ and the anisotropic mean curvature H_ϕ, where (g^{ij}) denotes the inverse metric, cf. Definition B.2.

Before continuing with the first variation formula we summarize some properties of the objects introduced above which are basically consequences of the homogeneity of ϕ.

Lemma 9.5. *With the notation from above it holds that*

(i) $\nabla\phi$ *is positively* 0*-homogeneous with* $\nabla\phi(z)\cdot z = \phi(z)$ *for all* $z \in \mathbb{R}^{n+1}\setminus\{0\}$,

(ii) $\mathrm{D}^2\phi$ *is positively* (-1)*-homogeneous with* $\mathrm{D}^2\phi(z)\cdot z = 0$ *for all* $z \in \mathbb{R}^{n+1}\setminus\{0\}$,

(iii) $n_\phi \cdot \nu_\phi = 1$ *and*

(iv) $\nabla^{\mathcal{M}}\cdot n_\phi = \nabla\cdot n_\phi$.

Remark 9.6 (First variation of anistropic area)**.** Let $\mathcal{M}$ be a smooth properly embedded hypersurface in $\mathbb{R}^{n+1}$. For $\lambda > 0$ let $\Psi_\lambda : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be sufficiently regular diffeomorphisms with $\Psi_0 = \mathrm{Id}$. Let $\mathcal{M}_\lambda \coloneqq \Psi_\lambda(\mathcal{M})$ and $X \coloneqq \partial_\lambda\Big|_{\lambda=0}\Psi_\lambda$. Then

$$\partial_\lambda\Big|_{\lambda=0}|\mathcal{M}_\lambda|_\phi = \int_{\mathcal{M}} X\cdot\nu\nabla\cdot n_\phi = \int_{\mathcal{M}} \phi^\circ(\nu)H_\phi\nu_\phi\cdot X. \tag{9.1}$$

We recover (9.1) using a formal calculation. In the following computations all derivatives with respect to λ are evaluated at $\lambda = 0$. It holds that

$$\begin{aligned}\partial_\lambda|\mathcal{M}_\lambda|_\phi = \partial_\lambda\int_{\mathcal{M}_\lambda}\phi^\circ(\nu) &= \int_{\mathcal{M}}\phi^\circ\nabla^{\mathcal{M}}\cdot X + \partial_\lambda\nu\cdot\nabla\phi^\circ \\ &= \int_{\mathcal{M}}\phi^\circ(\nu)\nabla^{\mathcal{M}}\cdot X - \nu_i\nabla_{\mathcal{M}}X_i\cdot n_\phi.\end{aligned}$$

Note that

$$\begin{aligned}\int_{\mathcal{M}} -\nu_i\nabla_{\mathcal{M}}X_i\cdot n_\phi &= \int_{\mathcal{M}} -\nabla^{\mathcal{M}}\cdot\Big(\nu\cdot Xn_\phi\Big) + X_i\nabla_{\mathcal{M}}\nu_i\cdot n_\phi + \nu\cdot X\nabla^{\mathcal{M}}\cdot n_\phi \\ &= \int_{\mathcal{M}}\phi^\circ(\nu)\nu\cdot XH + X_i\nabla_{\mathcal{M}}\nu_i\cdot n_\phi + \nu\cdot X\nabla^{\mathcal{M}}\cdot n_\phi.\end{aligned}$$

In order to continue we need the symmetry of $(\nabla_{\mathcal{M}}\nu_i)_j$ in i and j, which is a consequence of

$$(\nabla_{\mathcal{M}}\nu_i)_j = g^{kl}\partial_l\nu_i\partial_kF_j = -A^{km}\partial_mF_i\partial_kF_j.$$

Hence

$$\int_{\mathcal{M}} -\nu_i \nabla_{\mathcal{M}} X_i \cdot n_\phi = \int_{\mathcal{M}} \phi^\circ(\nu)\nu \cdot XH + X \cdot \nabla_{\mathcal{M}}(\phi^\circ(\nu)) + \nu \cdot X \nabla^{\mathcal{M}} \cdot n_\phi$$
$$= \int_{\mathcal{M}} -\phi^\circ(\nu)\nabla^{\mathcal{M}} \cdot X + \nu \cdot X \nabla^{\mathcal{M}} \cdot n_\phi.$$

This implies

$$\partial_\lambda |\mathcal{M}_\lambda|_\phi = \int_{\mathcal{M}} X \cdot \nu \nabla^{\mathcal{M}} \cdot n_\phi = \int_{\mathcal{M}} X \cdot \nu \nabla \cdot n_\phi.$$

Formula (9.1) at hand, it is proven in [BP96, Proposition 5.1] that in a suitable sense the direction of maximal slope of the anisotropic area is given by the anisotropic mean curvature vector $H_\phi n_\phi$. This motivates to define the anisotropic stochastic mean curvature flow with the notation from Section 3.1 through the stochastic differential equation

$$\mathrm{d}F = H_\phi n_\phi \,\mathrm{d}t + n_\phi \circ \mathrm{d}W. \tag{9.2}$$

As mentioned above we will assume that W is spatially homogeneous, i.e. $W = \alpha\beta$ with a real-valued Brownian motion β and a constant α with $\alpha^2 < 2$.

Similar to Section 3.1.1 we will derive evolution laws for the geometric quantities.

Remark 9.7 (Evolution of geometric quantities)**.** For the metric tensor g we infer

$$\mathrm{d}g_{ij} = 2H_\phi \partial_i n_\phi \cdot \partial_j F + 2\partial_i H_\phi n_\phi \cdot \partial_j F \,\mathrm{d}t + 2\partial_i n_\phi \cdot \partial_j F \circ \mathrm{d}W.$$

We will use the notation

$$(A_\phi)_{ij} := -\partial_i n_\phi \cdot \partial_j F$$

for an anisotropic generalization of the second fundamental form. This notation is compatible with

$$H_\phi = -\nabla^{\mathcal{M}} \cdot n_\phi = g^{ij}(A_\phi)_{ij}.$$

With this notation the above evolution equation for the metric tensor can be rewritten to

$$\mathrm{d}g_{ij} = -2H_\phi (A_\phi)_{ij} + 2\partial_i H_\phi n_\phi \cdot \partial_j F \,\mathrm{d}t - 2(A_\phi)_{ij} \circ \mathrm{d}W.$$

For the inverse metric we infer

$$\mathrm{d}g^{ij} = 2H_\phi (A_\phi)^{ij} - 2g^{ij} \nabla^k H_\phi n_\phi \cdot \partial_j F \,\mathrm{d}t + 2(A_\phi)^{ij} \circ \mathrm{d}W.$$

For the volume element it holds

$$\mathrm{d}\sqrt{g} = -H_\phi^2\sqrt{g} + \nabla_{\mathcal{M}_t} H_\phi \cdot n_\phi \,\mathrm{d}t - H_\phi\sqrt{g} \circ \mathrm{d}W.$$

As in Remark 3.3 $\mathrm{d}\nu$ is a tangent vector and since $\partial_i n_\phi$ also is a tangent vector we have

$$\mathrm{d}\nu \cdot \partial_i F = -\phi^\circ(\nu)\partial_i H_\phi \,\mathrm{d}t.$$

This gives

$$\mathrm{d}\nu = -\phi^\circ(\nu)\nabla_{\mathcal{M}_t} H_\phi \,\mathrm{d}t$$

and for the Cahn-Hoffmann vector field

$$\mathrm{d}n_\phi = -\phi^\circ(\nu)\,\mathrm{D}^2\phi^\circ(\nu)\nabla_{\mathcal{M}_t} H_\phi \,\mathrm{d}t.$$

We conclude for the anisotropic area element

$$\mathrm{d}\big(\phi^\circ(\nu)\sqrt{g}\big) = -\phi^\circ(\nu)H_\phi^2\sqrt{g}\,\mathrm{d}t - \phi^\circ(\nu)H_\phi\sqrt{g} \circ \mathrm{d}W.$$

As we are interested in the evolution of $\mathbb{E}\int_{\mathcal{M}_t} \phi^\circ(\nu)$ we need to rewrite the above equation into its Itô formulation. In order to determine the Itô-Stratonovich correction terms we need the stochastic part of the evolution equation of the anisotropic mean curvature. We start by deriving the corresponding part of the evolution law for the anisotropic second fundamental form. It holds

$$\mathrm{d}(A_\phi)_{ij} = \ldots \mathrm{d}t - \partial_i n_\phi \cdot \partial_j n_\phi \circ \mathrm{d}W.$$

since $\partial_i n_\phi$ is a tangent vector and $\partial_i n_\phi \cdot \partial_j F = -(A_\phi)_{ij}$ we recover $\partial_i n_\phi = -(A_\phi)_i^k \partial_k F$ which gives

$$\mathrm{d}(A_\phi)_{ij} = \ldots \mathrm{d}t - (A_\phi)_{ik}(A_\phi)_j^k \circ \mathrm{d}W.$$

Hence

$$\begin{aligned}\mathrm{d}(A_\phi)_i^j &= \ldots \mathrm{d}t + 2(A_\phi)^{jk}(A_\phi)_{ki} - (A_\phi)^{jk}(A_\phi)_{ki} \circ \mathrm{d}W \\ &= \ldots \mathrm{d}t + (A_\phi)^{jk}(A_\phi)_{ki} \circ \mathrm{d}W.\end{aligned}$$

This implies for the anisotropic mean curvature

$$\mathrm{d}H_\phi = \ldots \mathrm{d}t + |A_\phi|^2 \circ \mathrm{d}W.$$

This evolution law at hand we can rewrite the evolution of the anisotropic area element into its Itô formulation

$$\mathrm{d}(\phi^\circ(\nu)\sqrt{g}) = -\left(1 - \frac{\alpha}{2}\right)\phi^\circ(\nu)H_\phi^2\sqrt{g} - \frac{\alpha^2}{2}\phi^\circ(\nu)|A_\phi|^2\sqrt{g}\,\mathrm{d}t - \phi^\circ(\nu)H_\phi\sqrt{g}\,\mathrm{d}W.$$

This implies

$$\mathbb{E}\int_{\mathcal{M}_T}\phi^\circ(\nu)+\mathbb{E}\int_0^T\int_{\mathcal{M}_t}\left(1-\frac{\alpha^2}{2}\right)\phi^\circ(\nu)H_\phi^2+\frac{\alpha^2}{2}\phi^\circ(\nu)|A_\phi|^2\,\mathrm{d}t \\ \leq \mathbb{E}\int_{\mathcal{M}_0}\phi^\circ(\nu) \tag{9.3}$$

which is the anisotropic version of Example 5.8.

We proceed as in the isotropic situation by deriving the level set formulation of (9.2).

Remark 9.8 (Level set formulation of (9.2))**.** We adopt the notation from Section 3.2. In particular we assume that $\mathcal{M}_t=\{p\in\mathbb{R}^{n+1}\mid f(p,t)=\lambda\}$ for some constant $\lambda\in\mathbb{R}$. This assumption yields

$$0=\mathrm{d}f(F_t(x),t)+\nabla f(F_t(x),t)\cdot n_\phi(x,t)H_\phi(x,t)\,\mathrm{d}t \\ +\nabla f(F_t(x),t)\cdot n_\phi(x,t)\circ\mathrm{d}W. \tag{9.4}$$

Due to the homogeneity of ϕ it is more convenient to choose the normal vector

$$\nu(x,t)=\frac{\nabla f(F_t(x),t)}{|\nabla f(F_t(x),t)|}$$

which has the opposite orientation as in Remark B.10. We infer because of the homogeneity of ϕ° and $\nabla\phi^\circ$

$$\begin{aligned}\nu_\phi(x,t)&=\frac{\nabla f(F_t(x),t)}{\phi^\circ(\nabla f(F_t(x),t))},\\ n_\phi(x,t)&=\nabla\phi^\circ(\nabla f(F_t(x),t))\text{ and}\\ H_\phi(x,t)&=-\nabla\cdot\left(\nabla\phi^\circ(\nabla f)\right)(F_t(x),t).\end{aligned}$$

Plugging this into (9.4) and using Lemma 9.5 gives

$$\begin{aligned}\mathrm{d}f(F_t(x),t)&=\nabla f(F_t(x),t)\cdot\nabla\phi^\circ(\nabla f(F_t(x),t))\nabla\cdot\left(\nabla\phi^\circ(\nabla f)\right)(F_t(x),t)\,\mathrm{d}t\\ &\quad-\nabla f(F_t(x),t)\cdot\nabla\phi^\circ(\nabla f(F_t(x),t))\circ\mathrm{d}W\\ &=\phi^\circ(\nabla f(F_t(x),t))\nabla\cdot\left(\nabla\phi^\circ(\nabla f)\right)(F_t(x),t)\,\mathrm{d}t\\ &\quad-\phi^\circ(\nabla f(F_t(x),t))\circ\mathrm{d}W.\end{aligned}$$

Since this should hold for all x and all level sets we can simplify it to

$$\mathrm{d}f=\phi^\circ(\nabla f)\nabla\cdot\left(\nabla\phi^\circ(\nabla f)\right)\mathrm{d}t-\phi^\circ(\nabla f)\circ\mathrm{d}W.$$

Note that $-W$ also is a Wiener process with the same distribution as W. Hence, we do not change the qualitative behavior of this equation by dropping the minus sign in front of the perturbation for notational convenience. We call

$$\mathrm{d}f = \phi^\circ(\nabla f)\nabla\cdot\big(\nabla\phi^\circ(\nabla f)\big)\,\mathrm{d}t + \phi^\circ(\nabla f)\circ\mathrm{d}W \tag{9.5}$$

the level set formulation of (9.2). Whenever we want to translate (9.5) back to a solution of (9.2) for a particular Wiener process, we have to compensate the different signs of the Wiener processes involved. Note that this difficulty appears here but not in Section 3.2, because here we have chosen the unit normal with the opposite orientation. In general, we have for a Minkowski norm ϕ that $\phi(x) \neq \phi(-x)$. Hence, the unit normal we have chosen here is the more convenient choice as otherwise there would be minus signs appearing in the arguments of the Minkowski norm and its derivatives in (9.5). Nevertheless, with the orientation from Section 3.2 we would not have to change the sign of the Wiener process.

Example 9.9 (Stochastic mean curvature flow of Wulff shape)**.** Using the definition of the duality maps T_ϕ and T_{ϕ° we infer that $\nabla\phi^\circ(\nabla\phi(z)) = \frac{z}{\phi(z)}$ for all $z \in \mathbb{R}^{n+1}\setminus\{0\}$. According to Remark 9.8 this implies for the Cahn-Hoffmann vector field and the anisotropic mean curvature of $r\partial B_\phi = \{z\in\mathbb{R}^{n+1} \mid \phi(z) = r\}$

$$n_\phi = \frac{z}{r} \quad \text{and} \quad H_\phi = -\nabla\cdot\big(\nabla\phi^\circ(\nabla\phi)\big) = -\frac{n}{r}.$$

Let $\mathcal{M} = \partial B_\phi$ and $F_t(x) = r(t)z$ for $z \in \mathcal{M}$ and $r(0) = r_0$. Then the evolution of $r(t)\partial B_\phi$ induced by (9.2) is characterized by

$$\mathrm{d}rz = -\frac{n}{r}z\,\mathrm{d}t + z\circ\mathrm{d}W \Leftrightarrow \mathrm{d}r = -\frac{n}{r}\,\mathrm{d}t + \alpha\,\mathrm{d}\beta.$$

Comparing this with (3.10) we find that the radius of the Wulff shape solves the same stochastic differential equation as the radius of a sphere in the isotropic setting.

Remark 9.10 (The graphical case)**.** Let us now assume that $\mathcal{M}_t$ is the graph of a function $u(\cdot,t)$. In order to avoid the issues arising from the choice of the orientation as in Remark 9.8 we derive the equation for u not from (9.2) but directly from the level set equation (9.5), which already accounts for it.

To this end, let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be given by

$$f(x,\xi) = u(x) - \xi.$$

The zero level set of f is the graph of u. Note that

$$\nabla f = \begin{pmatrix}\nabla u\\ -1\end{pmatrix}$$

and

$$\nu = \frac{1}{\sqrt{1+|\nabla u|^2}} \begin{pmatrix} \nabla u \\ -1 \end{pmatrix}.$$

We will use the notation introduced in Remark B.9, especially

$$\mathbf{Q}(p) = \sqrt{1+|p|^2} \qquad \text{and} \qquad \mathbf{v}(p) = \frac{p}{\mathbf{Q}(p)} \text{ for } p \in \mathbb{R}^n.$$

Furthermore, we define their anisotropic analogs

$$\mathbf{Q}_{\phi^\circ}(p) \coloneqq \phi^\circ(p,-1) \qquad \text{and} \qquad \mathbf{v}_{\phi^\circ}(p) \coloneqq \nabla \mathbf{Q}_{\phi^\circ}(p) = \nabla_p \phi^\circ(p,-1),$$

where we write $\nabla\phi^\circ = \left(\begin{smallmatrix} \nabla_p \phi^\circ \\ \partial_w \phi^\circ \end{smallmatrix}\right)$ with $\nabla_p \phi^\circ \in \mathbb{R}^n$ and $\partial_w \phi^\circ \in \mathbb{R}$. With this notation we infer

$$\begin{aligned} \nu_\phi &= \frac{1}{\mathbf{Q}_{\phi^\circ}(\nabla u)} \begin{pmatrix} \nabla u \\ -1 \end{pmatrix}, \\ n_\phi &= \begin{pmatrix} \mathbf{v}_{\phi^\circ}(\nabla u) \\ \partial_w \phi^\circ(\nabla u, -1) \end{pmatrix} \text{ and} \\ H_\phi &= \nabla\cdot\Big(\mathbf{v}_{\phi^\circ}(\nabla u)\Big). \end{aligned}$$

Plugging this into the level set equation (9.5) gives

$$\mathrm{d}u = \mathbf{Q}_{\phi^\circ}(\nabla u) \nabla\cdot\Big(\mathbf{v}_{\phi^\circ}(\nabla u)\Big)\,\mathrm{d}t + \mathbf{Q}_{\phi^\circ}(\nabla u) \circ \mathrm{d}W, \tag{9.6}$$

which therefore characterizes the anisotropic stochastic mean curvature flow of graphs.

Note that (9.6) has the same structure as (3.8).

9.2. Existence in the graphical case

In this section we will prove existence of solutions for anisotropic stochastic mean curvature flow under the assumption that the initial data is the graph of a function u. For notational convenience we will consider the evolution with respect to ϕ°, which corresponds to (9.6) with ϕ° replaced by ϕ, i.e.

$$\mathrm{d}u = \mathbf{Q}_{\phi}(\nabla u) \nabla\cdot\Big(\mathbf{v}_{\phi}(\nabla u)\Big)\,\mathrm{d}t + \mathbf{Q}_{\phi}(\nabla u) \circ \mathrm{d}W. \tag{9.7}$$

Assumptions 9.11. We assume that $\alpha \in \mathbb{R}$ with $\alpha^2 < 2$ and ϕ is a regular Minkowski norm which additionally fulfills

$$\mathrm{D}_p^2 \phi(p,w) \le 2\phi(p,w)\,\mathrm{Id} \;\; \forall p \in \mathbb{S}^{n-1}, w \in \mathbb{R}. \tag{9.8}$$

Furthermore, we will assume that $W = \alpha\beta$ with a real-valued Brownian motion β.

Remark 9.12. Note that (9.8) is a technical assumption which we need in order to prove that the Dirichlet energy $\mathbb{E} \int_{\mathbb{T}^n} |\nabla u|^2$ is decreasing for solutions of (9.7). Since the Dirichlet energy is not adapted to ϕ it is natural that some additional condition on ϕ comes in. This technicality is due to the fact that we will treat (9.7) as a variational SPDE in the setting of Section 4.2.

At the moment it is not clear how to get rid of this assumption. It would be natural to replace the variational theory which builds up on the weak formulation of (9.7) with respect to the isotropic L^2 norm of ∇u by a theory which uses a weak formulation which respects the anisotropy. However, we do not pursue this in this thesis.

By considering an Euclidean transformation induced by a symmetric positive definite matrix $A \in \mathbb{R}^{n\times n}$ one could weaken (9.8) to

$$\mathrm{D}_p^2\phi(p,w) \leq 2\phi(p,w)A \ \forall p \in \mathbb{R}^n \text{ with } p\cdot Ap = 1, w \in \mathbb{R}$$

and consider on $L^2(\mathbb{T}^n;\mathbb{R}^n)$ the scalar product $\langle \psi_1, \psi_2\rangle_A := \int_{\mathbb{T}^n} \psi_1 \cdot A\psi_2$, $\psi_1, \psi_2 \in L^2(\mathbb{T}^n;\mathbb{R}^n)$. We will not go into detail about this generalization.

Theorem 9.13. *Let Λ be a Borel probability measure on $H^1(\mathbb{T}^n)$ with*

$$\int_{H^1(\mathbb{T}^n)} \|z\|^2_{H^1(\mathbb{T}^n)} \,\mathrm{d}\Lambda(z) < \infty \text{ and } \|\nabla z\|_{L^\infty(\mathbb{T}^n;\mathbb{R}^n)} \leq L \ \Lambda\text{-a.s.}$$

for a constant $L > 0$. Then, under Assumptions 9.11, there is a martingale solution u of (9.7) for $I = [0,\infty)$ with initial data Λ.

In addition, the uniform bounds we will derive in this section also imply that the large-time result Theorem 7.5 also holds for solutions of (9.7).

Remark 9.14 (Viscous equation)**.** In order to prove Theorem 9.13 we proceed as in Chapter 5 and introduce for $\varepsilon > 0$ the viscous equation

$$\mathrm{d}u = \varepsilon\Delta u + \mathbf{Q}_\phi(\nabla u)\nabla\cdot\Big(\mathbf{v}_\phi(\nabla u)\Big)\,\mathrm{d}t + \mathbf{Q}_\phi(\nabla u)\circ \mathrm{d}W. \tag{9.9}$$

Since (9.9) has the same structure as (5.4) we can repeat the proof of Lemma 5.9 to deduce its anisotropic analog, which is the foundation for the existence proof.

Lemma 9.15. *Let $\varepsilon \geq 0$ and u be a strong solution of (9.9) on the time interval I. Let $f \in C^2(\mathbb{R}^n)$ with bounded second order derivatives. For the energy*

$$\mathfrak{E}(t) := \int_{\mathbb{T}^n} f(\nabla u(t)),\ t \in I$$

it holds that

$$\begin{aligned}
\mathrm{d}\mathfrak{E} = \int_{\mathbb{T}^n} &- \varepsilon\, \mathrm{D}^2 f(\nabla u)\, \mathrm{D}^2 u : \mathrm{D}^2 u - \left(1 - \frac{\alpha^2}{2}\right) f(\nabla u) \left|\nabla\cdot\Big(\mathbf{v}_\phi(\nabla u)\Big)\right| \\
&- \mathrm{D}^2 u\, \mathrm{D}_p^2 \phi(\nabla u, -1) : \Bigg[\mathbf{Q}_\phi(\nabla u)\, \mathrm{D}^2 f(\nabla u)\, \mathrm{D}^2 u \\
&\qquad\qquad - \left(1 - \frac{\alpha^2}{2}\right) f(\nabla u)\, \mathrm{D}_p^2 \phi(\nabla u, -1)\, \mathrm{D}^2 u\Bigg]\, \mathrm{d}t \\
&- f(\nabla u) \nabla\cdot \Big(\mathbf{v}_\phi(\nabla u)\Big)\, \mathrm{d}W.
\end{aligned}$$

A direct consequence of Lemma 9.15 is the following Lemma.

Lemma 9.16. *Let $\varepsilon \geq 0$ and u be a strong solution of* (9.9) *on the time interval I. Let $g \in C^2([0,\infty))$ be non-negative, increasing and convex with bounded second derivative and $g(0) = 0$. Then the energy*

$$\mathfrak{E}(t) := \int_{\mathbb{T}^n} g(\mathbf{Q}_\phi(\nabla u(t))),\ t \in I$$

is estimated by

$$\begin{aligned}
&\mathbb{E}\mathfrak{E}(t) + \left(1 - \frac{\alpha^2}{2}\right) \mathbb{E}\int_0^t\!\!\int_{\mathbb{T}^n} g \left|\nabla\cdot\Big(\mathbf{v}_\phi(\nabla u)\Big)\right| \mathrm{d}s \\
&\quad + \mathbb{E}\int_0^t\!\!\int_{\mathbb{T}^n} \mathbf{Q}_\phi(\nabla u) g'' \mathbf{v}_\phi(\nabla u) \cdot \mathrm{D}^2 u\, \mathrm{D}_p^2 \phi(\nabla u, -1)\, \mathrm{D}^2 u \mathbf{v}_\phi(\nabla u)\, \mathrm{d}s \\
&\quad + \mathbb{E}\int_0^t\!\!\int_{\mathbb{T}^n} \Big(\mathbf{Q}_\phi(\nabla u) g' - \Big(1 - \frac{\alpha^2}{2}\Big) g\Big)\, \mathrm{D}^2 u\, \mathrm{D}_p^2 \phi(\nabla u, -1) : \mathrm{D}_p^2 \phi(\nabla u, -1)\, \mathrm{D}^2 u\, \mathrm{d}s \\
&\leq \mathbb{E}\mathfrak{E}(0)
\end{aligned}$$

for all $t \in I$ with all integrals on the left side of the equation being non-negative and with g, g' and g'' evaluated at $\mathbf{Q}_\phi(\nabla u)$.

In order to interpret (9.9) as a variational SPDE in the sense of Assumptions 4.31, we will present a second application of Lemma 9.15 for $f(\nabla u) = |\nabla u|^2$. In this situation, we recover the coercivity assumption (4.13).

Proposition 9.17 (Coercivity of the viscous equation)**.** *Let $\varepsilon \geq 0$ and u be a strong solution of* (9.9) *on the time interval I. For the Dirichlet energy we have*

$$\begin{aligned}
\mathrm{d}\int_{\mathbb{T}^n} |\nabla u|^2 + 2\varepsilon \int_{\mathbb{T}^n} |\,\mathrm{D}^2 u|^2 + \left(1 - \frac{\alpha^2}{2}\right) \int_{\mathbb{T}^n} |\nabla u|^2 \left|\nabla\cdot\Big(\mathbf{v}_\phi(\nabla u)\Big)\right|^2 \mathrm{d}t \\
\leq - \int_{\mathbb{T}^n} |\nabla u|^2 \nabla\cdot \Big(\mathbf{v}_\phi(\nabla u)\Big)\, \mathrm{d}W.
\end{aligned}$$

Proof. We apply Lemma 9.15 to $f(\nabla u) = |\nabla u|^2$ and infer

$$\begin{aligned}
\mathrm{d}\int_{\mathbb{T}^n} |\nabla u|^2 = \int_{\mathbb{T}^n} & - 2\varepsilon |\,\mathrm{D}^2 u|^2 - \left(1 - \frac{\alpha^2}{2}\right) |\nabla u|^2 \left|\nabla\cdot \Big(\mathbf{v}_\phi(\nabla u)\Big)\right|^2 \\
& - \mathrm{D}^2 u\, \mathrm{D}_p^2 \phi(\nabla u, -1) \colon \Bigg[2\mathbf{Q}_\phi(\nabla u)\,\mathrm{Id} \\
& \qquad - \left(1 - \frac{\alpha^2}{2}\right) |\nabla u|^2\, \mathrm{D}_p^2 \phi(\nabla u, -1) \Bigg]\, \mathrm{D}^2 u \,\mathrm{d}t \\
& - |\nabla u|^2 \nabla\cdot \Big(\mathbf{v}_\phi(\nabla u)\Big)\, \mathrm{d}W.
\end{aligned}$$

Because of the convexity of ϕ we have $\mathrm{D}_p^2 \phi \geq 0$. Assumption (9.8) implies

$$\begin{aligned}
2\mathbf{Q}_\phi(p)\,\mathrm{Id} - \Big(1 - \frac{\alpha^2}{2}\Big) |p|^2\, \mathrm{D}_p^2 \phi(p, -1) &\geq |p| \bigg(2\phi\Big(\frac{p}{|p|}, -\frac{1}{|p|}\Big)\,\mathrm{Id} - \mathrm{D}_p^2 \phi\Big(\frac{p}{|p|}, -\frac{1}{|p|}\Big) \bigg) \\
&\geq 0.
\end{aligned}$$

By combining these estimates with Lemma C.2 we find the claimed inequality. □

The coercivity result Proposition 9.17 at hand, we can prove existence of martingale solutions of the viscous equation (9.9).

Theorem 9.18. *Let $\varepsilon > 0$, $q > 2$ and Λ be a Borel probability measure on $H^1(\mathbb{T}^n)$ with*

$$\int_{H^1(\mathbb{T}^n)} \|z\|^2_{H^1(\mathbb{T}^n)}\, \mathrm{d}\Lambda(z) < \infty \quad \textit{and} \quad \int_{H^1(\mathbb{T}^n)} \|\nabla z\|^q_{L^2(\mathbb{T}^n)}\, \mathrm{d}\Lambda(z) < \infty.$$

Then, under Assumptions 9.11, there is a martingale solution u of (9.9) *for $I = [0, \infty)$ with initial data Λ.*

Proof. The existence of martingale solutions can be concluded as in the proof of Theorem 5.7 by interpreting (9.9) as a variational equation for the gradient ∇u. Note that the coercivity of this equation is a consequence of the computations in Proposition 9.17. □

In order to pass to the limit $\varepsilon \to 0$ we deduce bounds for the gradient of solutions. We will start by stating a maximum principle for the gradient, which generalizes Theorem 7.3. Note that we measure the Lipschitz constant with respect to the Minkowski norm.

Theorem 9.19 (Maximum principle for the gradient)**.** *Let $\varepsilon \geq 0$ and u be a strong solution of* (9.9) *on the time interval $[0, T]$ with initial data u_0 which is* *uniformly Lipschitz continuous, i.e.* $\|\phi(\nabla u_0, -1)\|_{L^\infty(\mathbb{T}^n)} \leq L$ $\mathbb{P}$*-a.s. for a constant* $L > 0$.

Then the solution has the same Lipschitz constant for all times, i.e.

$$\|\phi(\nabla u, -1)\|_{L^\infty(0,T;L^\infty(\mathbb{T}^n))} \leq L \quad \mathbb{P}\text{-}a.s.$$

Proof. This is a direct consequence of Lemma 9.16 for a suitable approximation g of $z \mapsto (z-L)_+$ as in the proof of Theorem 7.3. □

As a consequence of the maximum principle for the gradient, we can prove a bound for the Hessian.

Proposition 9.20 (Hessian bound)**.** *Let* $\varepsilon \geq 0$ *and* u *be a strong solution of* (9.9) *on the time interval* $[0,T]$ *with initial data* u_0 *which is uniformly Lipschitz continuous, i.e.* $\|\phi(\nabla u_0, -1)\|_{L^\infty(\mathbb{T}^n)} \leq L$ $\mathbb{P}$*-a.s. for a constant* $L > 0$. *Then there is a constant* $c = c(L, \phi)$ *such that*

$$\mathbb{E}\int_{\mathbb{T}^n} \mathbf{Q}_\phi(\nabla u(t))^2 + c\mathbb{E}\int_0^t \int_{\mathbb{T}^n} |\,\mathrm{D}^2 u(s)|^2 \,\mathrm{d}s \leq \mathbb{E}\int_{\mathbb{T}^n} \mathbf{Q}_\phi(u_0) \;\forall t \in [0,T].$$

Proof. We apply Lemma 9.16 to $g(z) = z^2$ and infer

$$\begin{aligned}\mathbb{E}\int_{\mathbb{T}^n} \mathbf{Q}_\phi(\nabla u(t))^2 + \mathbb{E}\int_0^t \int_{\mathbb{T}^n} \mathbf{Q}_\phi(\nabla u(s))^2 \,\mathrm{D}\Big(\mathbf{v}_\phi(\nabla u(s))\Big) : \mathrm{D}\Big(\mathbf{v}_\phi(\nabla u(s))\Big)^T \,\mathrm{d}s \\ \leq \mathbb{E}\int_{\mathbb{T}^n} \mathbf{Q}_\phi(\nabla u(0))^2.\end{aligned} \tag{9.10}$$

Since ϕ is a regular anisotropy it holds for all $z, \zeta \in \mathbb{R}^{n+1}\setminus\{0\}$ that $\zeta \cdot \mathrm{D}^2\phi(z)\zeta > 0$ for $\zeta \notin \operatorname{span}\{z\}$. Otherwise there would be an $0 \neq X = \lambda z + \mu\zeta \in T_{\frac{z}{|z|}}\partial B_\phi \cap \operatorname{span}\{z, \zeta\}$ such that

$$\begin{aligned}\frac{1}{|z|} X \cdot \mathrm{D}^2\phi\left(\frac{z}{|z|}\right) X &= X \cdot \mathrm{D}^2\phi(z) X \\ &= \lambda^2 z \cdot \mathrm{D}^2\phi(z) z + 2\lambda\mu\zeta \cdot \mathrm{D}^2\phi(z) z + \mu^2 \zeta \cdot \mathrm{D}^2\phi(z)\zeta \\ &= 0\end{aligned}$$

since $\mathrm{D}^2\phi(z)$ is (-1)-homogeneous and $\mathrm{D}^2\phi(z)z = 0$ by Lemma 9.5, which contradicts the assumption that ∂B_ϕ has positive principal curvatures.

In particular $q \cdot \mathrm{D}_p^2\phi(p,-1)q = \binom{q}{0} \cdot \mathrm{D}^2\phi(p,-1)\binom{q}{0} > 0$ for all $p, q \in \mathbb{R}^n$ with $q \neq 0$. Hence, $\mathrm{D}_p^2\phi(p,-1)$ is positive definite. Furthermore, we infer from Theorem 9.19 that $\|\nabla u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^n))} \leq c(L)$ $\mathbb{P}$-a.s. The continuity of

$p \mapsto \mathrm{D}_p^2\phi(p,-1)$ implies that there is a constant c that depends on L and ϕ such that

$$\mathrm{D}_p^2\phi(\nabla u,-1) \geq c\,\mathrm{Id} \;\text{ a.s. in } \Omega \times [0,T] \times \mathbb{T}^n.$$

With Lemma C.2 we conclude that

$$\mathrm{D}\Big(\mathbf{v}_\phi(\nabla u)\Big) : \mathrm{D}\Big(\mathbf{v}_\phi(\nabla u)\Big)^T = \mathrm{D}_p^2\phi(\nabla u,-1)\,\mathrm{D}^2 u : \mathrm{D}^2 u\,\mathrm{D}_p^2\phi(\nabla u,-1) \geq c^2|\,\mathrm{D}^2 u|^2$$

almost surely in $\Omega \times [0,T] \times \mathbb{T}^n$. Plugging this inequality into (9.10) yields the asserted estimate. □

Proof of Theorem 9.13. With the existence result for the viscous equation, the maximum principle for the gradient and the Hessian bound at hand, cf. Theorem 9.18, Theorem 9.19 and Proposition 9.20, we can follow the lines of the proof of the isotropic result in Section 5.3 to conclude the existence of martingale solutions for (9.7). □

10. Outlook

In this chapter, we will give an outlook how the results of this thesis might be used or extended in future research. In particular, we will summarize the open problems that were mentioned during the previous chapters.

Remark 10.1 (SMCF for arbitrary hypersurfaces)**.** In this work, we mostly focus on the situation where the hypersurfaces are given as the graphs of a time-dependent function u. As mentioned before, the generic situation can be described by the level set equation (3.5). Formally, this equation can be approximated by the graphical case. For this, let $u^\varepsilon = \frac{U^\varepsilon}{\varepsilon}$ solve the graphical SMCF (3.8), i.e.

$$\partial_t U^\varepsilon = \sqrt{\varepsilon^2 + |\nabla U^\varepsilon|^2}\nabla\cdot\left(\frac{\nabla U^\varepsilon}{\sqrt{\varepsilon^2 + |\nabla U^\varepsilon|^2}}\right)\mathrm{d}t + \sqrt{\varepsilon^2 + |\nabla U^\varepsilon|^2}\circ \mathrm{d}W. \tag{10.1}$$

Our previous analysis of the graphical situation in particular yields existence of martingale solutions of (10.1) under appropriate assumptions on the noise and the initial data.

In order to pass to the limit $\varepsilon \to 0$ and prove that a limit $f := \lim_{\varepsilon\to 0} U^\varepsilon$ exists and solves (3.5) we need uniform estimate for U^ε in $\varepsilon > 0$. It is not clear, whether the estimates from Chapter 5 or in case of spatially homogeneous noise from Chapter 7 can be adapted in order to infer a sufficiently strong control of U^ε in $\varepsilon > 0$. Besides this, it is necessary to find the right notion of solution for the level set equation (3.5) such that it is compatible with this kind of approximation. In particular it is not clear if in analogy to the deterministic case, cf. [ES91], the theory of stochastic viscosity solutions [LS98a, LS98b, LS00a, LS00b] is well suited for this.

Remark 10.2 (Uniqueness and regularity)**.** In Chapter 6 we prove a conditional stability and uniqueness result under the assumption that solutions are sufficiently regular. In particular, we have to assume more regularity than the existence theorem in Chapter 5 guarantees.

The stability result is based on an estimate of the L^2 difference of the gradient of two solutions of (5.1). This Ansatz does not exploit the geometric origin of graphical SMCF. At the moment, it is not clear if an intrinsically geometric notion of stability is available and would yield a stronger result.

In the situation of compact properly embedded hypersurfaces moving according to SMCF, it is a natural question whether a pathwise inclusion principle holds.

In a strong formulation, this can be characterized by the pathwise growth of the distance between two solutions driven by the same Wiener process. The corresponding result is true for deterministic mean curvature flow, cf. [Bel13, Theorem 5.4]. In fact, it can also be easily verified in the situation of Section 3.4 for spherical solutions of SMCF with spatially homogeneous noise.

In Section 7.3 we have proven a first regularity result for graphical SMCF with spatially homogeneous noise. Besides holding interest on its own, proving higher regularity of solutions is a different attempt to strengthen the uniqueness result in Chapter 6. Nevertheless, it is an open problem whether we can prove enough regularity in order to apply the uniqueness result at least under appropriate assumptions on the initial data and the noise.

Remark 10.3 (Convergence of a numerical scheme)**.** In Chapter 8 we have proposed a numerical scheme for graphical SMCF. Since it is based on the same Galerkin approximation as in the existence proof for variational SPDEs from Section 4.3, we expect the scheme to converge for $\varepsilon > 0$ to a martingale solution of the viscous approximation (5.4).

The question of a rate of convergence as well as the convergence to a strong solution is strongly related to the uniqueness issue mentioned in Remark 10.2.

Furthermore, it is not clear if convergence can be proven not only for the viscous equation but also for graphical SMCF (5.1).

The regularity proven in Chapter 5 and Chapter 7 in case of spatially homogeneous noise goes beyond the regularity results that previously were available for $n = 1$, cf. [ESvR12]. Therefore, it is a natural question whether these results can be used in order to strengthen the convergence proven in [FLP14] for a numerical scheme in the one-dimensional case.

Remark 10.4 (Numerical observations in case of colored noise)**.** The Monte-Carlo simulations in Section 8.2.2 for SMCF with colored noise suggest some fine properties of certain energies, which we have not proven before. For example, the best analytic results from Chapter 5 imply that the expectation of the L^p norm of the gradient of a solution grows at most exponentially. Nevertheless, in the simulations we see that for spatially rough data, i.e. situations where the aforementioned norm is large, SMCF has a regularizing effect and the norm decreases.

It is not clear, if this is a numerical artifact which is due to the discretization and the particular choice of the noise coefficients or if it is a feature of SMCF with colored noise.

Related to this is the large-time behavior of solutions of SMCF with colored noise. The mean curvature flow has a regularizing effect while the colored noise introduces spatial roughness. Depending on the regularity of the solution, we expect one of those effects to dominate. Due to their interplay we expect that at large times the solution stays in a region where the effects are balanced.

Remark 10.5 (Different boundary conditions and unbounded domains)**.** In this work, we have considered graphical SMCF on the flat torus, i.e with periodic boundary conditions. In the deterministic situation, several works deal with Neumann and Dirichlet boundary conditions for graphical MCF, cf. [Ger76], [Hui89] and [MT17] and the references therein. In the presence of stochastic perturbations, we expect that the approach taken in this thesis can be used to prove existence of graphical SMCF on bounded convex domains with Neumann boundary data.

For a smooth function u on a bounded convex domain U with Neumann boundary condition $\nabla u \cdot \nu = 0$ on ∂U it holds that

$$\nabla u \cdot \mathrm{D}^2 u \nu = -\nabla u \cdot \mathrm{D}\nu \nabla u \leq 0.$$

The formal calculations regarding the energy estimates from Remark 5.6 can be repeated for non-periodic data with Neumann boundary condition. With the above inequality, the boundary terms formally have the right sign. Thus, we expect the energy estimates to hold in this situation, at least under suitable assumptions on the noise coefficients. For a similar strategy in the deterministic case we refer to [MT17].

For more general bounded domains, the boundary terms will not be neglectable. Instead, these terms might be estimated in the same way the boundary terms are handled in [Ger76] and [Hui89].

On unbounded domains, in particular for $U = \mathbb{R}^n$, it is not clear if existence of solutions of graphical SMCF can be proven with our approach. In this situation, the monotonicity formula derived in Section 3.5 with a non-constant kernel might be used in order to infer the integrability of certain terms with no sufficient spatial decay, cf. [EH89] for the deterministic situation.

Remark 10.6 (Technical assumption on anisotropy)**.** In Chapter 9 we prove existence of graphical SMCF with respect to an anisotropy ϕ. In order to use the same strategy as in the isotropic case, in particular the viscous approximation and the theory of variational SPDEs, we need to restrict to anisotropies satisfying a technical assumption. This assumption assures that the anisotropy is compatible with the Dirichlet energy. Furthermore, we have assumed that the noise is spatially homogeneous.

It is an open problem, whether one can remove the assumption on the anisotropy by choosing a different approximation technique. It is also not clear, if one can derive existence under the presence of colored noise.

Remark 10.7 (Use the variational framework for other equations)**.** The theory of variational SPDEs with a compact embedding, that we present in Section 4.2 and for which we prove existence in Section 4.3, extends the previously available theory. It is an interesting question, if it can be applied to different equations that previously could not be treated within the theory of variational SPDEs.

A. Some results about bounded linear operators

In this chapter we will state some results about bounded linear operators on Banach spaces. In some cases we will restrict the presentation to Hilbert spaces, if the general theory is not needed in this work.

A.1. Trace class and Hilbert-Schmidt operators

We will adapt the presentation of [DPZ14, Appendix B]. For proofs and a more detailed exposition see also [DS63, XI.6 and XI.9], [RS72, VI.6] and [Kab11, 12].

Definition A.1. Let H be a Hilbert space. For a non-negative operator $A \in L(H)$ and an orthonormal basis $(e_k)_{k \in \mathbb{I}}$ of H

$$\operatorname{tr} A := \sum_{k \in \mathbb{I}} \langle A e_k, e_k \rangle$$

is called the trace of A.

Remark A.2. In the situation of Definition A.1 $\operatorname{tr} A$ is well-defined, i.e. it does not depend on the choice of the orthonormal basis.

Definition A.3. Let H be a Hilbert space. An operator $A \in L(H)$ is called a trace class operator if $\operatorname{tr} |A| < \infty$.

We denote by

$$L_1(H) := \{A \in L(H) \mid \operatorname{tr} |A| < \infty\}$$

the set of all trace class operators and define

$$\|A\|_{L_1(H)} := \operatorname{tr} |A|, \ A \in L_1(H).$$

For an orthonormal basis $(e_k)_{k \in \mathbb{I}}$ of H we will call

$$\operatorname{tr} A := \sum_{k \in \mathbb{I}} \langle A e_k, e_k \rangle$$

the trace of $A \in L_1(H)$.

Remark A.4. Note that in the situation of Definition A.3 the operator $|A| = \sqrt{A^*A}$ is well-defined, cf. [RS72, Theorem VI.9].

Furthermore, for $A \in L_1(H)$ the trace $\operatorname{tr} A$ is well-defined and coincides for non-negative trace class operators with the trace defined in Definition A.1.

Theorem A.5 (cf. [RS72, Theorem VI.19] and [RS72, Theorem VI.20])**.** *If H is a Hilbert space then $L_1(H)$ is an ideal in $L(H)$ and $(L_1(H); \|\cdot\|_{L_1(H)})$ is a Banach space.*

Definition A.6. Let H_1 and H_2 be two Hilbert spaces. An operator $A \in L(H_1; H_2)$ is called a Hilbert-Schmidt operator if A^*A is a trace class operator on H_1.

We denote by

$$L_2(H_1; H_2) \coloneqq \{A \in L(H_1; H_2) \mid A^*A \in L_1(H_1)\}$$

the set of all Hilbert-Schmidt operators and define the scalar product

$$\langle A_1, A_2 \rangle_{L_2(H_1;H_2)} \coloneqq \operatorname{tr}\left(A_1^* A_2\right), \; A_1, A_2 \in L_2(H_1; H_2).$$

For the induced norm we write

$$\|A\|_{L_2(H_1;H_2)} = \sqrt{\operatorname{tr}\left(A^*A\right)}, \; A \in L_2(H_1; H_2).$$

Theorem A.7 (cf. [RS72, Theorem VI.22] and [RS72, Problem VI.48])**.** *If H_1 and H_2 are Hilbert spaces then $(L_2(H_1; H_2); \langle \cdot, \cdot \rangle_{L_2(H_1;H_2)})$ is a Hilbert space.*

Remark A.8. Note that the spaces introduced in Definition A.3 and Definition A.6 are special instances of the Schatten classes, cf. [Kab11, 12.5]. More precisely, for Hilbert spaces H_1 and H_2 it holds that

$$L_1(H_1) = \left\{ A \in L(H_1) \mid A \text{ is compact and } \sum_{k \in \mathbb{N}} s_k(A) < \infty \right\}$$

and

$$L_2(H_1; H_2) = \left\{ A \in L(H_1; H_2) \mid A \text{ is compact and } \sum_{k \in \mathbb{N}} s_k^2(A) < \infty \right\},$$

with $s_k(A)$ denoting the k-th singular value of a compact operator A.

Furthermore the trace class operators coincide with the nuclear operators in the context of Banach spaces.

A.2. Moore-Penrose inverse

Compare with [PR07, Appendix C].

Definition A.9. Let H_1, H_2 be Hilbert spaces and $T \in L(H_1; H_2)$. The Moore-Penrose inverse $T^{-1} : T(H_1) \to (\ker T)^{\perp}$ of the operator T is given by the inverse of $T|_{(\ker T)^{\perp}} : (\ker T)^{\perp} \to T(H_1)$.

Remark A.10. Note that in the setting of Definition A.9 the restriction of T to $(\ker T)^{\perp}$, i.e. $T|_{(\ker T)^{\perp}} : (\ker T)^{\perp} \to T(H_1)$, is bijective.

As a consequence of the open mapping theorem the Moore-Penrose inverse is a bounded linear operator.

Proposition A.11. *In the setting of Definition A.9 we have that:*

(i) $T^{-1}y = \operatorname{argmin}\{\|x\|_H \mid Tx = y\}$ *for all* $y \in T(H_1)$*,*

(ii) $TT^{-1}T = T$*,*

(iii) $T^{-1}TT^{-1} = T^{-1}$*,*

(iv) $\left(TT^{-1}\right)^* = TT^{-1}$*,*

(v) $\left(T^{-1}T\right)^* = T^{-1}T$*.*

B. Differential geometry of hypersurfaces

In this chapter we will introduce the terminology we will use to describe the differential geometry of hypersurfaces in $\mathbb{R}^{n+1}$. For a more detailed exposition of the concepts of Riemannian manifolds we refer to [Lee18], where we have taken most of the notation from. Furthermore, we would like to refer to [Eck04] for an exposition which focuses on the objects which are important for the mean curvature flow. At some points we will also use ideas from [Bel13] where the geometric objects are introduced and expressed in terms of the signed distance function.

During this chapter we will not give the optimal regularity assumptions and instead always assume that all objects are sufficiently smooth.

We assume that $\mathcal{M} \subset \mathbb{R}^{n+1}$ is a smooth properly embedded hypersurface. We will denote by $F : \Omega \to F(\Omega) \subset \mathcal{M}$ a smooth local parametrization for some open set $\Omega \subset \mathbb{R}^n$. We introduce the (local) geometric objects in terms of F, but up to their representation in local coordinates they are independent of the choice of F. If not otherwise specified $p = F(x)$ denotes a point on the manifold $\mathcal{M}$ for $x \in \Omega$.

Definition B.1 (Tangent space)**.** The tangent vectors $\partial_1 F(x), \ldots, \partial_n F(x)$ are linearly independent and form a basis of the n-dimensional tangent space $T_p\mathcal{M}$. The cotangent space $T_p^*\mathcal{M}$ is the dual space of the tangent space. The dual basis is given by the cotangent vectors $\mathrm{d}x^1, \ldots, \mathrm{d}x^n$, which are the differentials of $p = F(x^1, \ldots, x^n) \mapsto x^i$.

For a tangent vector $X \in T_p\mathcal{M}$ and a cotangent vector $Y \in T_p^*\mathcal{M}$ we write $X = X^i \partial_i F(x)$ and $Y = Y_i \, \mathrm{d}x^i$ using the Einstein summation convention. Since Y is a linear form on $T_p\mathcal{M}$, we can apply it to X and get $Y(X) = X^i Y_i$.

Definition B.2 (Metric and area element)**.** Since $\mathcal{M}$ is a submanifold of $\mathbb{R}^{n+1}$, the Euclidean structure of $\mathbb{R}^{n+1}$ induces a Riemannian metric g on $\mathcal{M}$. For two tangent vectors $X, Y \in T_p\mathcal{M} \subset \mathbb{R}^{n+1}$ we have

$$\langle X, Y \rangle_g = X \cdot Y = X^i Y^j \partial_i F \cdot \partial_j F =: g_{ij} X^i Y^j.$$

The matrix $(g_{ij})_{i,j=1,\ldots,n}$ is invertible and for the inverse matrix we write $(g^{ij}) = (g_{ij})^{-1}$. The area element is given by

$$\sqrt{g} := \sqrt{\det g}.$$

Note that $\langle X, \cdot \rangle_g$ defines a cotangent vector with components $\left(\langle X, \cdot \rangle_g\right)_i = g_{ij} X^j$.

For a vector in the ambient space $X \in \mathbb{R}^{n+1}$ we will denote by X^T its tangential part, i.e. X^T is the tangent vector uniquely determined by

$$\left\langle X^T, Y \right\rangle_g = X \cdot Y \ \forall Y \in T_p\mathcal{M}.$$

It can be expressed in terms of the metric g via

$$X^T = g^{ij} X \cdot \partial_i F \partial_j F.$$

Definition B.3 (Integration)**.** For a function $f : F(\Omega) \to \mathbb{R}$ under suitable assumptions the integral

$$\int_{\mathcal{M}} f \, \mathrm{d}\mathcal{H}^n = \int_{\Omega} \sqrt{g} f \circ F \, \mathrm{d}\mathcal{L}^n$$

is well-defined. Using a smooth partition of unity this definition can be extended to define the integral $\int_{\mathcal{M}} f \, \mathrm{d}\mathcal{H}^n$ for a function $f : \mathcal{M} \to \mathbb{R}$.

Definition B.4 (Tangential gradient)**.** For a smooth function $f : \mathcal{M} \to \mathbb{R}$ the differential $\mathrm{d}f_p \in T_p^*\mathcal{M}$ is a well-defined cotangent vector. The corresponding tangent vector

$$\nabla_{\mathcal{M}} f(p) = \nabla_{\mathcal{M}}^i f(p) \partial_i F(x) \coloneqq g^{ij} \, \mathrm{d}f_p(\partial_j F(x)) \partial_i F(x) = g^{ij} \partial_j \left(f \circ F\right)(x) \partial_i F(x)$$

is called the tangential gradient of f.

Definition B.5 (Covariant derivative and Christoffel symbols)**.** The Christoffel symbols Γ_{ij}^k are the coefficients of the tangential part of $\partial_{ij} F$, i.e.

$$\left(\partial_{ij} F\right)^T \eqqcolon \Gamma_{ij}^k \partial_k F.$$

In terms of the metric the Christoffel symbols can be expressed as

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}\right).$$

With help of the Christoffel symbols one can define the covariant derivative of higher-order tensor fields.

Let G be a smooth (k, l)-tensor field, i.e. a k-times contravariant and l-times covariant tensor field. Expressing G in a basis we have

$$G = G_{j_1, \dots, j_l}^{i_1, \dots, i_k} \partial_{i_1} F \otimes \ldots \otimes \partial_{i_k} F \otimes \mathrm{d}x^{j_1} \otimes \cdots \otimes \mathrm{d}x^{j_l}.$$

The covariant derivative of G is the $(k, l+1)$-tensor field given by

$$\nabla^{\mathcal{M}} G = \nabla^{\mathcal{M}}_m G \, \mathrm{d}x^m$$

with coefficients

$$\begin{aligned}\left(\nabla^{\mathcal{M}}_m G\right)^{i_1,\dots,i_k}_{j_1,\dots,j_l} &\coloneqq \nabla^{\mathcal{M}}_m G^{i_1,\dots,i_k}_{j_1,\dots,j_l} \\ &\coloneqq \partial_m \left(G^{i_1,\dots,i_k}_{j_1,\dots,j_l} \circ F\right) + \sum_{s=1}^{k} G^{i_1,\dots,p,\dots,i_k}_{j_1,\dots,j_l} \Gamma^{i_s}_{mp} - \sum_{s=1}^{l} G^{i_1,\dots,i_k}_{j_1,\dots,p,\dots,j_l} \Gamma^{p}_{mj_s}.\end{aligned}$$

Definition B.6 (Divergence and Laplacian)**.** Let X be a smooth vector field. The divergence of X in terms of the covariant derivative is given by

$$\nabla^{\mathcal{M}} \cdot X \coloneqq \nabla^{\mathcal{M}}_i X^i = \partial_i (X^i \circ F) + \Gamma^i_{ij} X^j.$$

It can also be expressed as

$$\nabla^{\mathcal{M}} \cdot X = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} X^i \circ F\right).$$

For a smooth function $f : \mathcal{M} \to \mathbb{R}$ the Laplacian of f is given by

$$\Delta_{\mathcal{M}} f \coloneqq \nabla^{\mathcal{M}} \cdot \nabla_{\mathcal{M}} f = \nabla^{\mathcal{M}}_i \left(\nabla^i_{\mathcal{M}} f\right).$$

There are several different expressions for the Laplacian of f, i.e.

$$\begin{aligned}\Delta_{\mathcal{M}} f &= g^{ij} \nabla^{\mathcal{M}}_i \nabla^{\mathcal{M}}_j f = \operatorname{tr}_g \nabla^{\mathcal{M}} \nabla^{\mathcal{M}} f = g^{ij} \left(\partial_{ij} (f \circ F) - \Gamma^k_{ij} \partial_k (f \circ F)\right) \\ &= \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j (f \circ F)\right).\end{aligned}$$

Definition B.7 (Unit normal field and second fundamental form)**.** Let ν be a smooth unit normal field, i.e. $|\nu(p)| = 1$ and $\nu(p) \perp T_p\mathcal{M}$ for all $p \in \mathcal{M}$.

The second fundamental form is a symmetric 2-covariant tensor field A, which describes the difference between differentiation in the ambient space and the covariant derivative, i.e.[1]

$$\begin{aligned}\partial_{ij} F &= (\partial_{ij} F)^T + (\partial_{ij} F)^{\perp} \\ &\eqqcolon \nabla^{\mathcal{M}}_i (\partial_j F) + A(\partial_i F, \partial_j F) \nu.\end{aligned}$$

Hence, the tensor can be represented by

$$A = A_{ij} \, \mathrm{d}x^i \otimes \mathrm{d}x^j$$

with

$$A_{ij} \coloneqq \partial_{ij} F \cdot \nu = -\partial_i F \cdot \partial_j \nu.$$

[1] Note that the second fundamental form changes sign, when ν changes sign. We have used a different sign convention as in [Eck04], but are in accordance with [Lee18] and [Bel13].

Definition B.8 (Shape operator, Weingarten map and curvature)**.** The shape operator or Weingarten map s is the $(1,1)$-tensor field corresponding to A by raising one index. That means that

$$s = A^i_j \partial_i F \otimes \mathrm{d}x^j$$

with $A^i_j = g^{ik} A_{kj}$.

The shape operator at a point p can be identified with the self-adjoint linear endomorphism on $T_p\mathcal{M}$ given by $X = X^i \partial_i F \mapsto A^i_j X^j \partial_i F$. The eigenvalues of s are called the principal curvatures $\kappa_1, \ldots, \kappa_n$. The mean curvature H is the sum of the principal curvatures

$$\begin{aligned} H &\coloneqq \kappa_1 + \ldots + \kappa_n \\ &= A^i_i = \operatorname{tr} s \\ &= g^{ij} A_{ij} = \operatorname{tr}_g A. \end{aligned}$$

The mean curvature vector is

$$\vec{H} = H\nu.$$

The mean curvature vector is independent of the choice of ν.

Note that $\Delta_{\mathcal{M}} F = \vec{H}$ component-wise.

Remark B.9 (Differential geometry of graphs)**.** We will specialize the above notation to the case that the hypersurface $\mathcal{M}$ is the graph of a function $u : \Omega \to \mathbb{R}$. The canonical global parametrization of $\mathcal{M}$ is given by

$$F(x) \coloneqq \begin{pmatrix} x \\ u(x) \end{pmatrix}, \ x \in \Omega.$$

For the metric tensor we calculate

$$g_{ij} = \delta_{ij} + \partial_i u \partial_j u,$$

which is $(g_{ij}) = \operatorname{Id} + \nabla u \otimes \nabla u$. In the following we will use the abbreviations

$$\begin{aligned} \mathbf{Q}(p) &\coloneqq \sqrt{1 + |p|^2} \text{ and} \\ \mathbf{v}(p) &\coloneqq \frac{p}{\sqrt{1+|p|^2}} = \frac{p}{\mathbf{Q}(p)} \end{aligned}$$

for $p \in \mathbb{R}^n$. With this notation we can write the inverse metric as

$$(g^{ij}) = \operatorname{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)$$

and the area element as

$$\sqrt{g} = \mathbf{Q}(\nabla u).$$

For the Christoffel symbols we infer

$$\begin{aligned}\Gamma_{ij}^{k} &= \frac{1}{2} g^{kl} \left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij} \right) \\ &= \frac{\partial_{ij} u \partial_k u}{\mathbf{Q}(\nabla u)^2}.\end{aligned}$$

Let $f : \Omega \to \mathbb{R}$ and $h : \Omega \to \mathbb{R}^{n+1}$. The tangential gradient of f is given by

$$\nabla_{\mathcal{M}} f = \begin{pmatrix} \left(\mathrm{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)\right) \nabla f \\ \frac{\nabla u \cdot \nabla f}{\mathbf{Q}(\nabla u)^2} \end{pmatrix},$$

where ∇ denotes the gradient with respect to the differential structure of Ω.

The divergence of h is given by

$$\begin{aligned}\nabla^{\mathcal{M}} \cdot h &= g^{ij} \partial_i h \cdot \partial_j F \\ &= \left(\mathrm{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)\right) : \mathrm{D} h_{1,\ldots,n} + \frac{1}{\mathbf{Q}(\nabla u)^2} \nabla u \cdot \nabla h_{n+1}.\end{aligned}$$

Hence,

$$\Delta_{\mathcal{M}} f = \left(\mathrm{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)\right) : \mathrm{D}^2 f - v \cdot \nabla f \nabla \cdot \left(\mathbf{v}(\nabla u)\right).$$

A smooth unit normal field is given by

$$\nu := \frac{1}{\mathbf{Q}(\nabla u)} \begin{pmatrix} -\nabla u \\ 1 \end{pmatrix}.$$

For this choice of a normal field the second fundamental form has the components

$$A_{ij} = \frac{\partial_{ij} u}{\mathbf{Q}(\nabla u)}$$

and the Weingarten maps has the components

$$\begin{aligned}A_j^i &= \frac{1}{\mathbf{Q}(\nabla u)} \left(\mathrm{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)\right)_{ik} \partial_{kj} u \\ &= \partial_j \left(\mathbf{v}_i(\nabla u)\right).\end{aligned}$$

Hence, the mean curvature is given by

$$H = \nabla \cdot \left(\mathbf{v}(\nabla u)\right).$$

Remark B.10 (Differential geometry of level sets)**.** Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function. For $\lambda \in \mathbb{R}$ we consider the level set $\mathcal{M} := \{f = \lambda\}$ and assume that $\nabla f \neq 0$ on $\mathcal{M}$. Then $\mathcal{M}$ is a properly embedded smooth hypersurface in $\mathbb{R}^{n+1}$ by [Lee18, Theorem A.24]. Since the gradient of f is non-vanishing and a rotation of $\mathcal{M}$ does not change its geometric properties we can assume that locally $\partial_{n+1} f \neq 0$ and therefore locally solve the equation $f(x, u(x)) = \lambda$. This yields a local graph parametrization of $\mathcal{M}$ by $F(x) = (x, u(x))$. We will denote the components of the gradient of f by $\nabla f = (\hat{\nabla} f, \partial_{n+1} f) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$. With this notation we infer

$$\nabla u = -\frac{1}{\partial_{n+1} f} \hat{\nabla} f. \tag{B.1}$$

In Remark B.9 we express the geometric objects related to $\mathcal{M}$ in terms of u. These expressions and (B.1) at hand we can characterize them in terms of f.

The components of the metric tensor with respect to the coordinates introduced by the local graph parametrization are given by

$$g_{ij} = \delta_{ij} + \partial_i u \partial_j u = \delta_{ij} + \frac{\partial_i f \partial_j f}{(\partial_{n+1} f)^2}.$$

For the area element and the inverse metric it holds that

$$\sqrt{g} = \mathbf{Q}(\nabla u) = \sqrt{1 + \frac{|\hat{\nabla} f|^2}{(\partial_{n+1} f)^2}} = \frac{1}{|\partial_{n+1} f|} \sqrt{1 + |\nabla f|^2}$$

and

$$(g^{ij}) = \operatorname{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u) = \operatorname{Id} - \frac{\hat{\nabla} f \otimes \hat{\nabla} f}{|\nabla f|^2}.$$

For a function $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ let $h(x) = \varphi(x, u(x))$. For the gradient of h we compute

$$\nabla h = \hat{\nabla} \varphi + \partial_{n+1} \varphi \nabla u = \hat{\nabla} \varphi - \frac{\partial_{n+1} \varphi}{\partial_{n+1} f} \hat{\nabla} f.$$

This implies for the tangential gradient of φ

$$\begin{aligned} \nabla_{\mathcal{M}} \varphi = \nabla_{\mathcal{M}} h &= \begin{pmatrix} (\operatorname{Id} - \mathbf{v}(\nabla u) \otimes \mathbf{v}(\nabla u)) \nabla h \\ \frac{\nabla u \cdot \nabla h}{\mathbf{Q}(\nabla u)^2} \end{pmatrix} \\ &= \begin{pmatrix} \left(\operatorname{Id} - \frac{\hat{\nabla} f \otimes \hat{\nabla} f}{|\nabla f|^2}\right) \nabla h \\ -\frac{\hat{\nabla} f \cdot \nabla h}{|\nabla f|^2} \partial_{n+1} f \end{pmatrix} = \left(\operatorname{Id} - \frac{\nabla f \otimes \nabla f}{|\nabla f|^2}\right) \nabla \varphi. \end{aligned}$$

Similarly, when $\psi : \mathbb{R}^{n+1} \to T_p\mathcal{M} \subset \mathbb{R}^{n+1}$ is a smooth vector field then

$$\nabla^{\mathcal{M}} \cdot \psi = \nabla \cdot \psi - \frac{\nabla f \cdot \mathrm{D}\psi \nabla f}{|\nabla f|^2}.$$

Combining these expressions we find for the Laplace-Beltrami operator

$$\begin{aligned}\Delta_{\mathcal{M}} \varphi &= \left(\mathrm{Id} - \frac{\nabla f \otimes \nabla f}{|\nabla f|^2} \right) : \left(\mathrm{D}^2 \varphi - \frac{\nabla f \cdot \nabla \varphi}{|\nabla f|^2} \mathrm{D}^2 f \right) \\ &= \left(\mathrm{Id} - \frac{\nabla f \otimes \nabla f}{|\nabla f|^2} \right) : \mathrm{D}^2 \varphi - \frac{\nabla f \cdot \nabla \varphi}{|\nabla f|} \nabla \cdot \left(\frac{\nabla f}{|\nabla f|} \right).\end{aligned}$$

Let

$$\nu := -\frac{\nabla f}{|\nabla f|} \qquad \text{and} \qquad \tilde{\nu} := \frac{1}{\mathbf{Q}(\nabla u)} \begin{pmatrix} -\nabla u \\ 1 \end{pmatrix}.$$

We infer from (B.1)

$$\tilde{\nu} = -\operatorname{sgn}(\partial_{n+1} f) \nu.$$

The unit normal fields ν and $\tilde{\nu}$ induce possibly different second fundamental forms

$$\tilde{A}_{ij} = \frac{\partial_{ij} u}{\mathbf{Q}(\nabla u)} \qquad \text{and} \qquad A_{ij} = -\operatorname{sgn}(\partial_{n+1} f) \tilde{A}_{ij}$$

with

$$\mathrm{D}^2 u = -\frac{\hat{\mathrm{D}^2} f}{\partial_{n+1} f} + \frac{\partial_{n+1} \hat{\nabla} f \otimes \hat{\nabla} f + \hat{\nabla} f \otimes \partial_{n+1} \hat{\nabla} f}{(\partial_{n+1} f)^2} - \frac{\partial_{n+1,n+1} f \hat{\nabla} f \otimes \hat{\nabla} f}{(\partial_{n+1} f)^3}$$

and $\frac{\operatorname{sgn}(\partial_{n+1} f)}{\mathbf{Q}(\nabla u)} = \frac{\partial_{n+1} f}{|\nabla f|}$. Hence,

$$(A_{ij}) = \frac{\hat{\mathrm{D}^2} f}{|\nabla f|} - \frac{\partial_{n+1} \hat{\nabla} f \otimes \hat{\nabla} f + \hat{\nabla} f \otimes \partial_{n+1} \hat{\nabla} f}{\partial_{n+1} f |\nabla f|} + \frac{\partial_{n+1,n+1} f \hat{\nabla} f \otimes \hat{\nabla} f}{(\partial_{n+1} f)^2 |\nabla f|}.$$

This expression becomes more convenient by considering its action on tangent vectors $X, Y \in T_pM \subset \mathbb{R}^{n+1}$. Let $X = X^i \partial_i F$ and $Y = Y^j \partial_j F$ be their representations with respect to the basis $(\partial_i F)_i$ of T_pM. Let $\hat{X} = (X^1, \ldots, X^n)$ and $\hat{Y} = (Y^1, \ldots, Y^n)$ be the coordinates of X and Y. Since $\partial_i F = \begin{pmatrix} e_i \\ -\frac{\partial_i f}{\partial_{n+1} f} \end{pmatrix}$ we have

$$X = \begin{pmatrix} \hat{X} \\ -\frac{\hat{X} \cdot \hat{\nabla} f}{\partial_{n+1} f} \end{pmatrix} \qquad \text{and} \qquad Y = \begin{pmatrix} \hat{Y} \\ -\frac{\hat{Y} \cdot \hat{\nabla} f}{\partial_{n+1} f} \end{pmatrix}.$$

This implies

$$A(X,Y) = X^i A_{ij} Y^j = \frac{X \cdot \mathrm{D}^2 f Y}{|\nabla f|}.$$

For the Weingarten map $s : T_p\mathcal{M} \to T_p\mathcal{M}$ we note that for $X \in T_p\mathcal{M}$ the image $s(X)$ is defined as the unique solution of

$$A(X,Y) = \langle s(X), Y \rangle_g \ \ \forall Y \in T_p\mathcal{M}.$$

Since g is the metric induced by the standard metric on $\mathbb{R}^{n+1}$ we infer

$$\frac{X \cdot \mathrm{D}^2 f Y}{|\nabla f|} = A(X,Y) = s(X) \cdot Y \ \forall Y \in T_p\mathcal{M}.$$

This implies that $s(X)$ is the orthogonal projection of $\frac{\mathrm{D}^2 f X}{|\nabla f|}$ onto the tangent space $T_p\mathcal{M}$. This projection can be expressed in terms of the unit normal

$$s(X) = \frac{1}{|\nabla f|}\left(\mathrm{Id} - \frac{\nabla f \otimes \nabla f}{|\nabla f|^2}\right)\mathrm{D}^2 f X = \mathrm{D}\left(\frac{\nabla f}{|\nabla f|}\right) X.$$

Note that $\mathrm{D}\left(\frac{\nabla f}{|\nabla f|}\right)$ has at least a one-dimensional kernel and n eigenvectors living in the tangent space. The corresponding eigenvalues are the principal curvatures of $\mathcal{M}$. The mean curvature is therefore given by

$$H = \nabla \cdot \left(\frac{\nabla f}{|\nabla f|}\right).$$

C. Matrix scalar product

Definition C.1. For two matrices $A, B \in \mathbb{R}^{n \times n}$ we define the matrix scalar product

$$A : B \coloneqq A_{ij} B_{ij}.$$

The Frobenius norm of A is given by

$$|A|^2 \coloneqq A : A.$$

If in addition matrices $C, D \in \mathbb{R}^{n \times n}$ are given then we give higher precedence to matrix multiplication than to the matrix scalar product, i.e.

$$AB : CD \coloneqq (AB) : (CD).$$

Lemma C.2. *Let $A, B, C \in \mathbb{R}^{n \times n}$ be symmetric matrices with $B, C \geq 0$. Then*

$$AB : A \geq \lambda_{min}(B)|A|^2 \qquad \text{and} \qquad AB : CA \geq \lambda_{min}(B)\lambda_{min}(C)|A|^2,$$

where $\lambda_{min}(M)$ denotes the smallest eigenvalue of a matrix $M \in \mathbb{R}^{n \times n}$.

Proof. Let $D, E \in \mathbb{R}^{n \times n}$ be symmetric square roots of B and C, i.e. $DD = B$ and $EE = C$. Furthermore let $D = \sum_k \sqrt{\lambda_k} e_k \otimes e_k$ with an orthonormal basis $(e_k)_k$ of eigenvalues of B. Then

$$AB : A = ADD : A = AD : AD = \lambda_k |Ae_k|^2 \geq \lambda_{\min}(B)|A|^2$$

and

$$AB : CA = ADD : EEA = |EAD|^2 \geq \lambda_{\min}(B)|EA|^2 \geq \lambda_{\min}(B)\lambda_{\min}(C)|A|^2.$$

$\square$

Bibliography

[Alm66] Frederick J. Almgren, Jr. *Plateau's problem: An invitation to varifold geometry.* W. A. Benjamin, Inc., New York-Amsterdam, 1966.

[ATW93] Fred Almgren, Jean E. Taylor, and Lihe Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31(2):387–438, 1993.

[Aub82] Thierry Aubin. *Nonlinear analysis on manifolds. Monge-Ampère equations*, volume 252 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, New York, 1982.

[Bel04] Giovanni Bellettini. Anisotropic and crystalline mean curvature flow. In *A sampler of Riemann-Finsler geometry*, volume 50 of *Math. Sci. Res. Inst. Publ.*, pages 49–82. Cambridge Univ. Press, Cambridge, 2004.

[Bel13] Giovanni Bellettini. *Lecture notes on mean curvature flow, barriers and singular perturbations*, volume 12 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)].* Edizioni della Normale, Pisa, 2013.

[BFHM18] Dominic Breit, Eduard Feireisl, Martina Hofmanová, and Bohdan Maslowski. Stationary solutions to the compressible navier–stokes system driven by stochastic forces. *Probability Theory and Related Fields*, Oct 2018.

[BL00] Yoav Benyamini and Joram Lindenstrauss. *Geometric nonlinear functional analysis. Vol. 1*, volume 48 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2000.

[BO07] Zdzisław Brzeźniak and Martin Ondreját. Strong solutions to stochastic wave equations with values in Riemannian manifolds. *J. Funct. Anal.*, 253(2):449–481, 2007.

[BP96] Giovanni Bellettini and Maurizio Paolini. Anisotropic motion by mean curvature in the context of Finsler geometry. *Hokkaido Math. J.*, 25(3):537–566, 1996.

[Bra78] Kenneth A. Brakke. *The motion of a surface by its mean curvature*, volume 20 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1978.

[Bra07] Dietrich Braess. *Finite elements*. Cambridge University Press, Cambridge, third edition, 2007. Theory, fast solvers, and applications in elasticity theory, Translated from the German by Larry L. Schumaker.

[CDR03] Ulrich Clarenz, Gerhard Dziuk, and Martin Rumpf. On generalized mean curvature flow in surface processing. In *Geometric analysis and nonlinear partial differential equations*, pages 217–248. Springer, Berlin, 2003.

[CGG91] Yun Gang Chen, Yoshikazu Giga, and Shun'ichi Goto. Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations. *J. Differential Geom.*, 33(3):749–786, 1991.

[DD03] Klaus Deckelnick and Gerhard Dziuk. *Numerical Approximation of Mean Curvature Flow of Graphs and Level Sets*, pages 53–87. Springer Berlin Heidelberg, Berlin, Heidelberg, 2003.

[DDE05] Klaus Deckelnick, Gerhard Dziuk, and Charles M. Elliott. Computation of geometric partial differential equations and mean curvature flow. *Acta Numer.*, 14:139–232, 2005.

[DHR19] Nils Dabrock, Martina Hofmanová, and Matthias Röger. Existence of martingale solutions and large-time behavior for a stochastic mean curvature flow of graphs. *arXiv preprint arXiv:1903.04785*, 2019.

[DHV16] Arnaud Debussche, Martina Hofmanová, and Julien Vovelle. Degenerate parabolic stochastic partial differential equations: quasilinear case. *Ann. Probab.*, 44(3):1916–1955, 2016.

[DLN01] Nicolas Dirr, Stephan Luckhaus, and Matteo Novaga. A stochastic selection principle in case of fattening for curvature flow. *Calculus of Variations and Partial Differential Equations*, 13(4):405–425, 2001.

[dMS95] Piero de Mottoni and Michelle Schatzman. Geometrical evolution of developed interfaces. *Trans. Amer. Math. Soc.*, 347(5):1533–1589, 1995.

[DPZ14] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2014.

[DS63] Nelson Dunford and Jacob T. Schwartz. *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space.* With the assistance of William G. Bade and Robert G. Bartle. Interscience Publishers John Wiley & Sons New York-London, 1963.

[Eck04] Klaus Ecker. *Regularity theory for mean curvature flow.* Progress in Nonlinear Differential Equations and their Applications, 57. Birkhäuser Boston Inc., Boston, MA, 2004.

[EH89] Klaus Ecker and Gerhard Huisken. Mean curvature evolution of entire graphs. *Annals of Mathematics*, 130(3):453–471, 1989.

[EH91] Klaus Ecker and Gerhard Huisken. Interior estimates for hypersurfaces moving by mean curvature. *Inventiones mathematicae*, 105(3):547–69, 1991.

[Elw82] K. D. Elworthy. *Stochastic differential equations on manifolds*, volume 70 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge-New York, 1982.

[ES91] Lawrence C. Evans and Joel Spruck. Motion of level sets by mean curvature. I. *J. Differential Geom.*, 33(3):635–681, 1991.

[ESS92] L. C. Evans, H. M. Soner, and P. E. Souganidis. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.*, 45(9):1097–1123, 1992.

[ESvR12] Abdelhadi Es-Sarhir and Max-K. von Renesse. Ergodicity of stochastic curve shortening flow in the plane. *SIAM J. Math. Anal.*, 44(1):224–244, 2012.

[Eva93] Lawrence C. Evans. Convergence of an algorithm for mean curvature motion. *Indiana Univ. Math. J.*, 42(2):533–557, 1993.

[Fif79] Paul C. Fife. *Mathematical aspects of reacting and diffusing systems*, volume 28 of *Lecture Notes in Biomathematics.* Springer-Verlag, Berlin-New York, 1979.

[FLP14] Xiaobing Feng, Yukun Li, and Andreas Prohl. Finite element approximations of the stochastic mean curvature flow of planar curves of graphs. *Stoch. Partial Differ. Equ. Anal. Comput.*, 2(1):54–83, 2014.

[FP03] Xiaobing Feng and Andreas Prohl. Numerical analysis of the Allen-Cahn equation and approximation for mean curvature flows. *Numer. Math.*, 94(1):33–65, 2003.

[Fun99] T. Funaki. Singular limit for stochastic reaction-diffusion equation and generation of random interfaces. *Acta Math. Sin. (Engl. Ser.)*, 15(3):407–438, 1999.

[Gar13] Harald Garcke. Curvature driven interface evolution. *Jahresber. Dtsch. Math.-Ver.*, 115(2):63–100, 2013.

[Ger76] Claus Gerhardt. Global regularity of the solutions to the capillarity problem. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 3(1):157–175, 1976.

[GJY03] Anja Göing-Jaeschke and Marc Yor. A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349, 2003.

[HR18] Martin Heida and Matthias Röger. Large deviation principle for a stochastic Allen-Cahn equation. *J. Theoret. Probab.*, 31(1):364–401, 2018.

[HRvR17] Martina Hofmanová, Matthias Röger, and Max-K. von Renesse. Weak solutions for a stochastic mean curvature flow of two-dimensional graphs. *Probab. Theory Related Fields*, 168(1-2):373–408, December 2017.

[HRW12] Martin Hairer, Marc D. Ryser, and Hendrik Weber. Triviality of the 2D stochastic Allen-Cahn equation. *Electron. J. Probab.*, 17:no. 39, 14, 2012.

[HS12] Martina Hofmanová and Jan Seidler. On weak solutions of stochastic differential equations. *Stoch. Anal. Appl.*, 30(1):100–121, 2012.

[Hui84] Gerhard Huisken. Flow by mean curvature of convex surfaces into spheres. *J. Differential Geom.*, 20(1):237–266, 1984.

[Hui89] Gerhard Huisken. Nonparametric mean curvature evolution with boundary conditions. *J. Differential Equations*, 77(2):369–378, 1989.

[Hui90] Gerhard Huisken. Asymptotic behavior for singularities of the mean curvature flow. *J. Differential Geom.*, 31(1):285–299, 1990.

[HW15] Martin Hairer and Hendrik Weber. Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions. *Ann. Fac. Sci. Toulouse Math. (6)*, 24(1):55–92, 2015.

[Ilm93] Tom Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.*, 38(2):417–461, 1993.

[IW81] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1981.

[Jak97] A. Jakubowski. The almost sure Skorokhod representation for subsequences in nonmetric spaces. *Teor. Veroyatnost. i Primenen.*, 42(1):209–216, 1997.

[Kab11] Winfried Kaballo. *Grundkurs Funktionalanalysis*. Heidelberg: Spektrum Akademischer Verlag, 2011.

[Kel55] John L. Kelley. *General topology*. D. Van Nostrand Company, Inc., Toronto-New York-London, 1955.

[KO82] Kyozi Kawasaki and Takao Ohta. Kinetic drumhead model of interface. i. *Progress of Theoretical Physics*, 67(1):147–163, 1982.

[KP92] Peter E. Kloeden and Eckhard Platen. *Numerical solution of stochastic differential equations*, volume 23 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1992.

[KS88] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988.

[Lee18] John M. Lee. *Introduction to Riemannian manifolds*, volume 176 of *Graduate Texts in Mathematics*. Springer, Cham, 2018. Second edition of [MR1468735].

[LO19] Tim Laux and Felix Otto. The thresholding scheme for mean curvature flow and de giorgi's ideas for minimizing movements, 2019.

[LS95] Stephan Luckhaus and Thomas Sturzenhecker. Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations*, 3(2):253–271, 1995.

[LS98a] Pierre-Louis Lions and Panagiotis E. Souganidis. Fully nonlinear stochastic partial differential equations. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 326(9):1085–1092, 1998.

[LS98b] Pierre-Louis Lions and Panagiotis E. Souganidis. Fully nonlinear stochastic partial differential equations: non-smooth equations and applications. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 327(8):735–741, 1998.

[LS00a] Pierre-Louis Lions and Panagiotis E. Souganidis. Fully nonlinear stochastic pde with semilinear stochastic dependence. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 331(8):617–624, 2000.

[LS00b] Pierre-Louis Lions and Panagiotis E. Souganidis. Uniqueness of weak solutions of fully nonlinear stochastic partial differential equations. *Comptes Rendus de l'Académie des Sciences - Series I - Mathematics*, 331(10):783–790, 2000.

[Man11] Carlo Mantegazza. *Lecture notes on mean curvature flow*, volume 290 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2011.

[MBO92] Barry Merriman, James Bence, and Stanley Osher. Diffusion generated motion by mean curvature. In Jean E. Taylor, editor, *Computational Crystal Growers Workshop*, Selected Lectures in Mathematics, pages 73—83. American Mathematical Society, Providence, RI, 1992.

[Mét82] Michel Métivier. *Semimartingales*, volume 2 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin-New York, 1982. A course on stochastic processes.

[MM77] Luciano Modica and Stefano Mortola. Un esempio di Γ^--convergenza. *Boll. Un. Mat. Ital. B (5)*, 14(1):285–299, 1977.

[MM06] Peter W. Michor and David Mumford. Riemannian geometries on spaces of plane curves. *J. Eur. Math. Soc. (JEMS)*, 8(1):1–48, 2006.

[Mod87] Luciano Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch. Rational Mech. Anal.*, 98(2):123–142, 1987.

[MPS19] Anis Matoussi, Dylan Possamaï, and Wissal Sabbagh. Probabilistic interpretation for solutions of fully nonlinear stochastic PDEs. *Probab. Theory Related Fields*, 174(1-2):177–233, 2019.

[MR16] Carlo Marinelli and Michael Röckner. On the maximal inequalities of Burkholder, Davis and Gundy. *Expo. Math.*, 34(1):1–26, 2016.

[MT17] Masashi Mizuno and Keisuke Takasao. Gradient estimates for mean curvature flow with Neumann boundary conditions. *NoDEA Nonlinear Differential Equations Appl.*, 24(4):Art. 32, 24, 2017.

[MV88] Michel Métivier and Michel Viot. On weak solutions of stochastic partial differential equations. In *Stochastic analysis (Paris, 1987)*, volume 1322 of *Lecture Notes in Math.*, pages 139–150. Springer, Berlin, 1988.

[Ond10] Martin Ondreját. Stochastic nonlinear wave equations in local Sobolev spaces. *Electron. J. Probab.*, 15:no. 33, 1041–1091, 2010.

[OS88] Stanley Osher and James A Sethian. Fronts propagating with curvature-dependent speed: Algorithms based on hamilton-jacobi formulations. *Journal of Computational Physics*, 79(1):12–49, 1988.

[Par75] Etienne Pardoux. *Equations aux dérivées partielles stochastiques non linéaires monotones*. PhD thesis, Paris XI, 1975.

[Par79] Etienne Pardoux. Stochastic partial differential equations and filtering of diffusion processes. *Stochastics*, 3(2):127–167, 1979.

[Par87] Etienne Pardoux. Two-sided stochastic calculus for spdes. In Giuseppe Da Prato and Luciano Tubaro, editors, *Stochastic Partial Differential Equations and Applications*, pages 200–207, Berlin, Heidelberg, 1987. Springer Berlin Heidelberg.

[Par07] Etienne Pardoux. Stochastic partial differential equations. Lectures given in Fudan University, 2007.

[PR07] C. Prévôt and M. Röckner. *A Concise Course on Stochastic Partial Differential Equations*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2007.

[Pro04] Philip E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability.

[RS72] Michael Reed and Barry Simon. *Methods of modern mathematical physics. I. Functional analysis*. Academic Press, New York-London, 1972.

[RSZ08] Michael Röckner, Byron Schmuland, and Xicheng Zhang. Yamada-watanabe theorem for stochastic evolution equations in infinite dimensions. *Condensed Matter Physics*, 11(2):247–259, 2008.

[RW00] L. C. G. Rogers and David Williams. *Diffusions, Markov processes, and martingales. Vol. 2*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[RW13] Matthias Röger and Hendrik Weber. Tightness for a stochastic Allen-Cahn equation. *Stoch. Partial Differ. Equ. Anal. Comput.*, 1(1):175–203, 2013.

[RY91] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1991.

[Sei93] Jan Seidler. Da Prato-Zabczyk's maximal inequality revisited. I. *Math. Bohem.*, 118(1):67–106, 1993.

[Sim87] Jacques Simon. Compact sets in the space $L^p(0,T;B)$. *Ann. Mat. Pura Appl. (4)*, 146:65–96, 1987.

[Sin10] Carlo Sinestrari. Formation of singularities in the mean curvature flow. In *Mean curvature flow and isoperimetric inequalities*, Adv. Courses Math. CRM Barcelona, pages 1–43. Birkhäuser Verlag, Basel, 2010.

[Sou19] Panagiotis E. Souganidis. *Pathwise Solutions for Fully Nonlinear First- and Second-Order Partial Differential Equations with Multiplicative Rough Time Dependence*, pages 75–220. Springer International Publishing, Cham, 2019.

[SY04] P. E. Souganidis and N. K. Yip. Uniqueness of motion by mean curvature perturbed by stochastic noise. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 21(1):1–23, 2004.

[TCH92] J.E. Taylor, J.W. Cahn, and C.A. Handwerker. Overview no. 98 i-geometric models of crystal growth. *Acta Metallurgica Et Materialia*, 40(7):1443–1474, 1992. cited By 247.

[Ton19] Yoshihiro Tonegawa. *Brakke's mean curvature flow*. SpringerBriefs in Mathematics. Springer, Singapore, 2019. An introduction.

[VGO+19] Pauli Virtanen, Ralf Gommers, Travis E. Oliphant, Matt Haberland, Tyler Reddy, David Cournapeau, Evgeni Burovski, Pearu Peterson, Warren Weckesser, Jonathan Bright, Stéfan J. van der Walt, Matthew Brett, Joshua Wilson, K. Jarrod Millman, Nikolay Mayorov, Andrew R. J. Nelson, Eric Jones, Robert Kern, Eric Larson, CJ Carey, İlhan Polat, Yu Feng, Eric W. Moore, Jake Vand erPlas, Denis Laxalde, Josef Perktold, Robert Cimrman, Ian Henriksen, E. A. Quintero, Charles R Harris, Anne M. Archibald, Antônio H. Ribeiro, Fabian Pedregosa, Paul van Mulbregt, and SciPy 1. 0 Contributors. SciPy 1.0–Fundamental Algorithms for Scientific Computing in Python. *arXiv e-prints*, page arXiv:1907.10121, Jul 2019.

[Vio76] Michel Viot. *Solutions faibles d'équations aux dérivées partielles stochastiques non linéaires.* PhD thesis, Paris VI, 1976.

[Web10] Hendrik Weber. On the short time asymptotic of the stochastic Allen-Cahn equation. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(4):965–975, 2010.

[Yip98] Nung Kwan Yip. Stochastic motion by mean curvature. *Archive for Rational Mechanics and Analysis*, 144(4):313–355, 1998.

[YW71] Toshio Yamada and Shinzo Watanabe. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.*, 11:155–167, 1971.