# On time-harmonic Maxwell's equations in periodic media

Dissertation

Zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt
der Fakultät für Mathematik
der Technischen Universität Dortmund

von

Maik Urban

Dortmund, November 2019

**Dissertation**

*On time-harmonic Maxwell's equations in periodic media*

Fakultät für Mathematik
Technische Universität Dortmund

Erstgutachter: Prof. Dr. Ben Schweizer

Zweitgutachter: Prof. Dr. Anne-Sophie Bonnet-Ben Dhia

Tag der mündlichen Prüfung: 10. Juni 2020

# Acknowledgements

I would like to express my sincere gratitude to my supervisor Professor Ben Schweizer for the continuous and patient support of my PhD study and related research. His passion for mathematics, his encouragement, and his attention to detail have always been inspiring and of great help to me. Without his generosity and patience in answering my numerous questions and his valuable advice, I would not have been able to write this thesis.

In addition, I am grateful to the members of Lehrstuhl I and the Biomathematics group at the TU Dortmund, including Professor Peter Bella, Andreas Rätz, Professor Matthias Röger and Saskia Stockhaus for interesting discussions and for creating an inspiring and motivating atmosphere.

Moreover, I would like to thank my colleagues Nils Dabrock, Elena El Behi-Gornostaeva, Peter Furlan, Stephan Hausberg, Lisa Helfmeier, Sascha Knüttel, Agnes Lamacz, Patrick Mrozek, Klaas-Hendrik Poelstra, Daniel Trampe, and Carsten Zwilling for all the mathematical and non-mathematical discussions, and exciting dart games.

I would also like to thank my parents Antje and Nils for their constant support: financial, emotional, and intellectual. They have created the means and established the path by which I could travel to this point. Last but not least, I want to thank my family and, particularly, my brothers Tobias and Dominik and my sister Anna for their constant support and advice through the past years.

This thesis is dedicated to my grandfather Klaus Grunitz who has always inspired me with his seemingly inexhaustible knowledge and his insatiable curiosity. Thank you!

# Contents

# Preface

Studying the propagation of waves through inhomogeneous media has a long history, during which a variety of inhomogeneities has been concerned. In this thesis we focus on the propagation of time-harmonic electromagnetic waves through periodic media; more specifically, two regimes of wave propagation are considered: in Part II, we assume that the period of the medium is much smaller than the wavelength of the wave, which amounts to homogenising time-harmonic Maxwell's equations. The propagation of electromagnetic waves in a bounded periodic waveguide is discussed in Part III, in which we assume the period of the medium and the wavelength of the electromagnetic wave to be of the same order. In Part I, we introduce notation and recall some results from functional analysis as well as statements on two-scale convergence, which will be used extensively. This part can be skipped by the more experienced reader, with occasional glances at specific topics if necessary.

**Short overview and orientation.** Part I is divided into three chapters.

In Chapter 1, we recall Banach's closed range theorem and state some properties of compact operators as well as of Fredholm operators on Banach spaces. Furthermore, we prove a variant of the Lax-Milgram lemma for Banach spaces. The results of this chapter are mainly used in Part III.

Function spaces for Maxwell's equations are considered in Chapter 2. More precisely, we introduce the spaces of $L^2$-vector fields for which the distributional curl or the distributional divergence are again $L^2$-mappings, and define the normal trace operator for bounded Lipschitz domains. We further define function spaces of periodic functions and prove a Helmholtz decomposition for $L^2$-vector fields on the three-dimensional flat torus $\mathbb{T}^3$.

Chapter 3 presents an overview of those results on two-scale convergence, which we use in Part II of this thesis; we recall, in particular, a compactness theorem and a statement on the connection between two-scale convergence and weak $L^2$-convergence.

In Part II we consider the time-harmonic Maxwell equations in a bounded subdomain of $\mathbb{R}^3$, which contains a periodic array of obstacles $\Sigma_\eta$. In recent years, there has been a great interest in constructing artificial materials, sometimes referred to as metamaterials, that exhibit astonishing optical properties. Such materials are usually constructed by choosing appropriate obstacles $\Sigma_\eta$ and considering electromagnetic waves with a wavelength that is much larger than the period of the array of obstacles.

Our intention in this second part of the thesis is not to propose a new microstructure leading to a metamaterial with unusual properties; instead, we clarify the connection between the topology of the microstructure and transmission properties of the metamaterial. Although very general microstructures are allowed, we stay in the framework of periodic homogenisation and use the tool of two-scale convergence. This allows us to follow a well-established scheme in order to homogenise Maxwell's equations: at first, we derive and analyse the cell problems for the two-scale limits $E_0$ and $H_0$ of the electric and magnetic field, respectively. We then use solutions of those cell problems to obtain the effective system.

As we consider very general microstructures, the analysis of the cell problems is non-trivial. Their solution spaces, for instance, depend on the topology of the microstructure. The characterisation of these solution spaces is possible using *k-loops* and the so-called *geometric average*—two tools, which were developed in [SU18] and [PSU19], respectively. In Chapter 4, both tools are defined and we discuss some of their properties. Moreover, we present the main ideas, developed in the articles [SU18], [PSU19] as well as in this thesis, by two microstructures.

Chapter 5 is devoted to the derivation of the effective Maxwell equations in the case of perfectly conducting microstructures $\Sigma_\eta$. This assumption on the conductivity ensures that the cell problems can directly be analysed using the geometric average and $k$-loops; see Lemmas 5.2 and 5.9. The characterisation of the solution spaces allows us to derive the effective Maxwell system—our main contribution in the second part of Chapter 5. It is noteworthy that the effective fields are *not* understood in the standard way as cell averages. Instead, we define them via the geometric average. In this way, we obtain an effective system that has the form of Maxwell's equations for a linear material; see Theorem 5.12. At the end of Chapter 5, we discuss our results using four microstructures.

The more realistic case of highly conductive microstructures is treated in Chapter 6. Although the cell problem for the two-scale limit $E_0$ of the electric field is identical to the one discussed in Chapter 5, the cell problem for $H_0$ is much more involved. Following an idea of Bouchitté, Bourel and Felbacq [BBF17], we derive a variational identity for $H_0$, which is equivalent to the cell problem. Using the geometric average, we can then characterise the solution space of this identity; see Proposition 6.9. Having a characterisation of the solution spaces of the cell problems allows us to obtain the effective Maxwell system. Similar to Chapter 5, we define the effective fields using the geometric average. In this way, we obtain the same effective system as in Chapter 5; see Theorem 6.10.

In Part III, we consider the time-harmonic Maxwell equations in an unbounded and periodically perforated waveguide with periodic boundary conditions on the lateral boundary. As the waveguide is unbounded, radiation conditions at infinity must additionally be imposed to ensure well-posedness of the problem. Let us stress that, in contrast to Part II, the wavelength of the electromagnetic wave and the distance between two holes in the waveguide are assumed to be of the same order. Establishing the existence of a solution to Maxwell's equations in the unbounded perforated waveguide is out of the scope of this thesis. To simplify the problem, we assume that the geometry of the waveguide allows us to reduce Maxwell's equations to a scalar Helmholtz equation. Following a recent idea of Schweizer [Sch19], we then show that there exists a solution to this Helmholtz equation in a bounded perforated waveguide that satisfies a special boundary condition, which we call a "radiation condition at a finite distance". More precisely, we consider a periodic array $\mathcal{O} \subset \mathbb{R} \times \mathbb{S}^1$

of holes and the bounded waveguide $\Omega_R := (-R, R) \times \mathbb{S}^1$. For fixed constants $R, L > 0$ and a right-hand side $h \in L^2(\Omega_{R+L})$, we seek a distributional solution $u \in H^1(\Omega_{R+L})$ to the Helmholtz equation $-\Delta u - k^2 u = h$ in the perforated waveguide $\Omega_R \cap \mathcal{O}^c$ such that $u$ restricted to the box $(-(R+L), -R) \times \mathbb{S}^1$ transports energy to the left and $u$ restricted to $(R, R+L) \times \mathbb{S}^1$ transports energy to the right.

A standard tool to establish the existence of a solution to a Helmholtz-like equation in an unbounded domain (with appropriate boundary conditions) is the so-called limiting absorption principle. This principle is usually based on an operator theoretic approach. As we seek a solution to the Helmholtz equation in the perforated waveguide that satisfies a “radiation condition at a finite distance”, it is convenient to formulate the problem using sesquilinear forms instead of operators. We therefore prove an abstract limiting absorption principle for sesquilinear forms, from which we deduce the existence of a solution to the Helmholtz equation in the perforated, bounded waveguide that satisfies the special boundary condition.

Chapter 7 starts with the derivation of the scalar Helmholtz equation from Maxwell’s equations and a discussion of our replacement of the radiation condition at infinity. We formulate the “radiation condition at a finite distance” with appropriate function spaces and sesquilinear forms.

Chapter 8 is devoted to an abstract limiting absorption principle for sesquilinear forms on Banach spaces; see Theorem 8.6. The proof of this principle is based on a Fredholm alternative for sesquilinear forms, which we discuss in Proposition 8.2. This chapter can be read independently of Chapters 7 and 9, since we consider abstract sesquilinear forms on Banach spaces.

In Chapter 9, we first establish the existence and uniqueness of solutions to a family of auxiliary problems, from which we deduce the existence of a solution the Helmholtz equation in the bounded perforated waveguide that satisfies the “radiation condition at a finite distance”; see Theorem 9.5. For this last step, we use the abstract limiting absorption principle from Chapter 8. This principle can only be applied if the Helmholtz equation has at most one solution. Using standard arguments, we show in Proposition 9.6 that this uniqueness assumption is satisfies except for an at most countable set of numbers $k^2 > 0$.

**Published articles.** Some parts of this thesis are based on the following articles and preprints by the author—obtained partially in collaboration with *Klaas Hendrik Poelstra* and *Ben Schweizer*:

[SU18] Ben Schweizer and Maik Urban. Effective Maxwell’s equations in general periodic microstructures. *Appl. Anal.*, 97(13):2210-2230, 2018

[SU19] Ben Schweizer and Maik Urban. On a limiting absorption principle for sesquilinear forms with an application to the Helmholtz equation in a waveguide. Accepted for the *proceedings volume of the Conference on Mathematics of Wave Phenomena 2018 at Karlsruhe*, Springer

[PSU19] Klaas H. Poelstra, Ben Schweizer and Maik Urban. The geometric average of curl-free fields in periodic geometries. *Technical Report*, *TU Dortmund*, submitted, 2019

In the beginning of sections and subsections, we indicate whether they contain material from the above articles. Results from the following article are related but not used in this thesis:

[OSUV18] Mario Ohlberger, Ben Schweizer, Maik Urban and Barbara Verfürth. Mathematical analysis of transmission properties of electromagnetic meta-materials. Accepted for *Netw. Heterog. Media*, 15(1) 2020, *arXiv:1809.08824*, 2018

**Notation and conventions.** Let us fix basic notation and conventions used in the rest of this thesis.

- Constants are denoted by $C$ and may change from line to line. If necessary, the dependence of parameters is mentioned in parentheses.
- We often consider families of functions that are indicated by a continuous parameter $\eta > 0$ or $\delta > 0$. Abusing notation, we still use the term "sequence" for these ordered families. If no ambiguity is possible, we do not relabel subsequences.
- We denote by $\mathbb{F}$ either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. If not explicitly stated, any vector space is assumed to be over the field $\mathbb{F}$.
- Let $V$ and $W$ be vector spaces over $\mathbb{F}$. A map $\ell\colon V \to W$ is *linear* if $\ell(\alpha u + v) = \alpha\ell(u) + \ell(v)$ and *anti-linear* if $\ell(\alpha u + v) = \overline{\alpha}\ell(u) + \ell(v)$ for all $u, v \in V$ and $\alpha \in \mathbb{F}$. For $\alpha \in \mathbb{R}$ we set $\overline{\alpha} = \alpha$. We call a map $\mathfrak{b}\colon V \times W \to \mathbb{F}$ a *bilinear form* if it is linear in each of its arguments, and a *sesquilinear form* if it is linear in the first and anti-linear in the second argument. If a statement is formulated for general $\mathbb{F}$, we always call a map $\mathfrak{b}\colon V \times W \to \mathbb{F}$ sesquilinear to emphasise that the complex case is included.
- Let $\Omega$ be an open subset of $\mathbb{R}^d$. We write $C^0(\Omega;\mathbb{F})$ for the space of continuous functions $\Omega \to \mathbb{F}$. For all $k \in \mathbb{N}$ with $k \geq 1$, we denote by $C^k(\Omega;\mathbb{F})$ the space of $k$-times continuously differentiable functions on $\Omega$. The space of smooth maps is given by $C^\infty(\Omega;\mathbb{F}) := \bigcap_{k\in\mathbb{N}} C^k(\Omega;\mathbb{F})$. For $k \in \mathbb{N}_0 \cup \{+\infty\}$ and $m \in \mathbb{N}$ with $m \geq 2$, we identify the space $C^k(\Omega;\mathbb{F}^m)$ with $C^k(\Omega;\mathbb{F})^m$. Moreover, $C^k_c(\Omega;\mathbb{F})$ denotes the space of $C^k(\Omega;\mathbb{F})$-functions that have compact support in $\Omega$. Let $\overline{\Omega}$ be the closure of $\Omega$. We denote by $C^k(\overline{\Omega};\mathbb{F})$ with $k \in \mathbb{N}_0 \cup \{+\infty\}$ the space of functions in $C^k(\Omega;\mathbb{F})$ which have a continuous extension to $\overline{\Omega}$ and for which all partial derivatives up to order $k$ have a continuous extension to $\overline{\Omega}$.
- Let $\Omega$ be an open subset of $\mathbb{R}^d$ with a Lipschitz boundary, and let $k \in \mathbb{N}_0$. We use Sobolev spaces on open subsets of $\mathbb{R}^d$ in the usual way; see for instance the book by Evans and Gariepy [EG15, Chapter 4]. We write $H^k(\Omega;\mathbb{F})$ for $W^{k,2}(\Omega;\mathbb{F})$. For $m \in \mathbb{N}$ with $m \geq 2$ we identify the spaces $H^k(\Omega;\mathbb{F}^m)$ and $H^k(\Omega;\mathbb{F})^m$.
- If $(X, \mathfrak{A}, \mu)$ is a measure space with $\mu(X) < \infty$ and $f \in L^1(X,\mu)$, then
$$\fint_X f := \fint_X f(x)\,\mathrm{d}x := \frac{1}{\mu(X)} \int_X f(x)\,\mathrm{d}x\,.$$

# Part I

# Preliminaries

*Bernard of Chartres used to say that we are like dwarfs sitting on the shoulders of giants so that we are able to see more and further than they, not indeed by the sharpness of our own vision or the height of our bodies, but because we are lifted up on high and raised aloft by the greatness of giants.*

—John of Salisbury, *Metalogicon*

CHAPTER 1

# Linear Operators and the Lax-Milgram lemma

This section is devoted to some well-known results from functional analysis, which we will use throughout this thesis.

We remind the reader that there are two natural dual spaces for a given normed complex vector space $V$:

$$V' := \Big\{ \ell \colon V \to \mathbb{F} \,\Big|\, \ell \text{ is linear and bounded} \Big\} \tag{1.0.1}$$

and

$$V^* := \Big\{ \ell \colon V \to \mathbb{F} \,\Big|\, \ell \text{ is anti-linear and bounded} \Big\} . \tag{1.0.2}$$

The map $V' \to V^*$, $\ell \mapsto \overline{\ell}$ is an isometric isomorphism between the normed spaces $V'$ and $V^*$. Note that both spaces are complex Banach spaces. Let us emphasise that the pairing $\langle \cdot , \cdot \rangle_{V',V} \colon V' \times V \to \mathbb{C}$ is a bilinear form whereas $\langle \cdot , \cdot \rangle_{V^*,V} \colon V^* \times V \to \mathbb{C}$ is a sesquilinear form.

## 1.1 Linear Operators

Let $V$ and $W$ be normed vector spaces over the same field. If $T \colon V \to W$ is linear and bounded, then its *dual* $T' \colon W' \to V'$ is a linear and bounded operator that is uniquely defined by the identity

$$\big\langle w', Tv \big\rangle_{W',W} = \big\langle T'w', v \big\rangle_{V',V} \quad \text{for all } v \in V \text{ and } w' \in W' . \tag{1.1.1}$$

If $V$ is a normed vector space and $U \subset V$ is a linear subspace, we define the *annihilator $U^\perp$ of $U$* as the set

$$U^\perp := \Big\{ v' \in V' \,\Big|\, \langle v', u \rangle_{V',V} = 0 \text{ for all } u \in U \Big\} . \tag{1.1.2}$$

We identify the annihilator $U^\perp$ of $U$ and the orthogonal complement of $U$ if $V$ is a Hilbert space. The *annihilator* ${}^\perp(U')$ of a linear subspace $U'$ of $V'$ is given as

$$^{\perp}(U') := \left\{ v \in V \,\middle|\, \langle u', v \rangle_{V',V} = 0 \text{ for all } u' \in U' \right\}. \tag{1.1.3}$$

Let $T\colon V \to W$ be a bounded linear operator between two normed spaces. Using the definition of the dual operator $T'$, a straightforward computation shows the identity

$$\ker T' = (\operatorname{im} T)^{\perp}. \tag{1.1.4}$$

Proposition 1.1. (Banach's closed range theorem) – *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two Banach spaces over the same field, and let $T\colon \mathfrak{X} \to \mathfrak{Y}$ be a linear and bounded operator. Then the following statements are equivalent:*

*(i) The range* $\operatorname{im} T$ *of $T$ is closed in $\mathfrak{Y}$.*

*(ii) The range* $\operatorname{im} T'$ *of $T'$ is closed in $\mathfrak{X}'$.*

*(iii)* $\operatorname{im} T = {}^{\perp}(\ker T')$.

*(iv)* $\operatorname{im} T' = (\ker T)^{\perp}$.

*Proof.* We refer to [Cia13, Theorem 5.11-5] for a proof of this result. □

We turn our focus now on two classes of linear operators, which play an essential role in Chapter 8.

## Compact operators

Let us recall that a subset $B$ of a normed space $V$ is *relatively compact in $V$* if its closure $\overline{B}$ is compact in $V$.

Definition 1.2. (Compact operator) – *Let $V$ and $W$ be two normed vector spaces over the same field. A linear operator $K\colon V \to W$ is* compact *if for all bounded sets $B \subset V$, the image $K(B)$ is relatively compact in $W$.*

Every compact operator is bounded. Indeed, as the closed unit ball $B_1$ in $V$ is bounded, $\overline{K(B_1)}$ is compact and thus bounded; in particular,

$$\sup_{\|v\| \le 1} \|Kv\| < \infty.$$

The next result gives an equivalent characterisation of compact mappings.

Proposition 1.3. – *Let $V$ and $W$ be two normed vector spaces of the same field, and let $K\colon V \to W$ be a linear operator. Then $K$ is compact if and only if $K$ maps every bounded sequence $(v_n)_n$ in $V$ onto a sequence $(Kv_n)_n$ that has a convergent subsequence.*

*Proof.* The proof of this result is straightforward; we refer to [Kre78, Theorem 8.1-3]. □

A basic property of a compact linear operator is that it transforms weakly convergent sequences into strongly convergent sequences.

Proposition 1.4. – *Let $V$ and $W$ be normed spaces over the same field and let $K\colon V \to W$ be a compact linear operator. Assume $(v_n)_n$ is a weakly convergent sequence in $V$ with weak limit $v \in V$. Then $(Kv_n)_n$ strongly converges to $Kv$ in $W$.*

*Proof.* We refer to [Kre78, Theorem 8.1-7] for a proof of this result. □

Given a linear operator on a reflexive Banach space, the following characterisation of compactness turns out to be useful.

Lemma 1.5. – *Let $\mathfrak{X}$ be a reflexive Banach space, and let $K\colon \mathfrak{X} \to \mathfrak{X}^*$ be a bounded linear operator. Then the following statements are equivalent.*

- *(i) The operator $K$ is compact.*
- *(ii) Every bounded sequence $(u_n)_n$ in $\mathfrak{X}$ has a subsequence $(u_n)_n$ that weakly converges in $\mathfrak{X}$ to some $u \in \mathfrak{X}$ such that*

$$\lim_{n\to\infty} \langle Ku_n, v_n\rangle_{\mathfrak{X}^*,\mathfrak{X}} = \langle Ku, v\rangle_{\mathfrak{X}^*,\mathfrak{X}} \tag{1.1.5}$$

*for every sequence $(v_n)_n$ in $\mathfrak{X}$ that weakly converges in $\mathfrak{X}$ to some $v \in \mathfrak{X}$.*

*Proof.* Assume $K$ is compact. Fix a bounded sequence $(u_n)_n$ in $\mathfrak{X}$. Due to Propositions 1.3 and 1.4 there exists a subsequence $(Ku_n)_n$ such that $Ku_n \to Ku$ in $\mathfrak{X}^*$ for some $u \in \mathfrak{X}$. Let $(v_n)_n$ be any sequence in $\mathfrak{X}$ with $v_n \to v$ weakly in $\mathfrak{X}$. Weakly convergent sequences are bounded, and thus

$$\begin{aligned} \left|\langle Ku_n, v_n\rangle_{\mathfrak{X}^*,\mathfrak{X}} - \langle Ku, v\rangle_{\mathfrak{X}^*,\mathfrak{X}}\right| &\le \left|\langle Ku_n - Ku, v_n\rangle_{\mathfrak{X}^*,\mathfrak{X}}\right| + \left|\langle Ku, v_n - v\rangle_{\mathfrak{X}^*,\mathfrak{X}}\right| \\ &\le C\left\|Ku_n - Ku\right\|_{\mathfrak{X}^*} + \left|\langle Ku, v - v_n\rangle_{\mathfrak{X}^*,\mathfrak{X}}\right|. \end{aligned}$$

Sending $n \to \infty$ yields the claim.

Assume *(ii)* holds and let $(u_n)_n$ be a bounded sequence in $\mathfrak{X}$. In view of Proposition 1.3, we need to show that $(Ku_n)_n$ admits a strongly convergent subsequence. As $(u_n)_n$ is bounded, we find a subsequence $(u_n)_n$ that weakly converges in $\mathfrak{X}$ to some element $u \in \mathfrak{X}$. We claim that $(Ku_n)_n$ strongly converges to $Ku$. Suppose the contrary: there exists $\varepsilon > 0$ and a subsequence $(Ku_n)_n$ such that $\|Ku_n - Ku\|_{\mathfrak{X}^*} > \varepsilon$ for all $n \in \mathbb{N}$. By definition of the dual norm, for each $n \in \mathbb{N}$ we find $v_n \in \mathfrak{X}$ with $\|v_n\|_{\mathfrak{X}} \le 1$ such that

$$\varepsilon < \|Ku_n - Ku\|_{\mathfrak{X}^*} \le \left|\langle Ku_n - Ku, v_n\rangle_{\mathfrak{X}^*,\mathfrak{X}}\right| + \frac{\varepsilon}{2}\,. \tag{1.1.6}$$

The sequence $(v_n)_n$ is bounded in $\mathfrak{X}$ and thus admits a weakly convergent subsequence $(v_n)_n$ with weak limit $v \in \mathfrak{X}$. This together with (1.1.5) implies

$$\lim_{n\to\infty} \left|\langle Ku_n - Ku, v_n\rangle_{\mathfrak{X}^*,\mathfrak{X}}\right| = 0\,.$$

Thus, sending $n \to \infty$ in (1.1.6) yields a contradiction. Consequently, $(Ku_n)_n$ converges to $Ku$ in $\mathfrak{X}$. As the sequence $(u_n)_n$ was chosen arbitrarily, the claim is proved. □

The next result clarifies the relationship between compact operators and their duals.

Proposition 1.6. – *Let $V$ and $W$ be two normed vector spaces over the same field, and let $K\colon V \to W$ be a linear operator. Then the following statements hold:*

*(i) If $K$ is compact, so is its adjoint $K'\colon W' \to V'$.*

*(ii) Assume $W$ is complete and $K$ is bounded. If $K'$ is compact, so is $K$.*

*Proof.* A proof of the first part can be found in [Kre78, Theorem 8.2-5]. For a proof of the second part, we refer to [Heu92, Satz 79.3]. □

The spectral properties of compact operators on an infinite dimensional Banach space resembles those of square matrices on finite dimensional vector spaces.

Proposition 1.7. – *Let $V$ be an infinite dimensional normed space, and let $K\colon V \to V$ be a compact operator. Then the spectrum $\sigma(K)$ of $K$ is an at most countable set and every nonzero $\lambda \in \sigma(K)$ is an eigenvalue of $K$.*

*Proof.* The statement that every nonzero $\lambda \in \sigma(K)$ is an eigenvalue of $K$ is proved in [Kre78, Theorem 8.6-4]. In [Kre78, Theorem 8.3-1] it is shown that the set of eigenvalues of $K$ is at most countable and the only point of accumulation is 0. □

### Fredholm operators

Given a vector space $V$ and a linear subspace $U \subset V$, the *codimension of $U$* is defined as

$$\operatorname{codim} U := \dim V/U\,.$$

If $W$ is an algebraic complement of $U$, that is, the vector space $V$ admits the decomposition $V = U \oplus W$, then the quotient map $\pi\colon W \to V/U$ is bijective and hence $\dim W = \operatorname{codim} U$. Let $T\colon E \to V$ be a linear map between two vector spaces over the same field. The *cokernel of $T$* is defined as

$$\operatorname{coker} T := V/\operatorname{im} T\,.$$

Clearly, $\dim \operatorname{coker} T = \operatorname{codim} \operatorname{im} T$.

In this section we restrict our attention to operators on Banach spaces.

Definition 1.8. (Fredholm operator and Fredholm index) – *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two Banach spaces over the same field. A bounded linear operator $T\colon \mathfrak{X} \to \mathfrak{Y}$ is a* Fredholm operator *if its kernel* $\ker T$ *and its cokernel* $\operatorname{coker} T$ *are finite dimensional vector spaces. If $T$ is a Fredholm operator, then the integer*

$$\operatorname{ind} T := \dim \ker T - \dim \operatorname{coker} T$$

*is the* Fredholm index *of $T$.*

The Fredholm index can equivalently be defined as $\operatorname{ind} T = \dim \ker T - \operatorname{codim} \operatorname{im} T$.

Remark 1. – *The definition of Fredholm operators is not consistent across the literatur. Operators* $T\colon \mathfrak{X} \to \mathfrak{Y}$ *with* $\dim\ker T < \infty$ *and* $\operatorname{codim}\operatorname{im} T < \infty$ *are sometimes called* Noetherian operators. *A* Fredholm operator *in this terminology is a Noetherian operator* $T\colon \mathfrak{X} \to \mathfrak{Y}$ *with* $\operatorname{ind} T = 0$.

Some authors assume in the definition of a Fredholm operator additionally that the range of the operator is closed. This leads to an equivalent definition, since every bounded linear operator with finite dimensional cokernel has a closed range. Let us demonstrate this fact:

Lemma 1.9. – *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be Banach spaces over the same field. If* $T\colon \mathfrak{X} \to \mathfrak{Y}$ *is a linear and bounded operator with* $\operatorname{codim}\operatorname{im} T < \infty$*, then the range* $\operatorname{im} T$ *is closed in* $\mathfrak{Y}$.

*Proof.* The kernel $\ker T$ is a closed subspace of $\mathfrak{X}$, since $T$ is bounded and linear; hence, the quotient space $\mathfrak{X}/\ker T$ is a Banach space. The induced operator

$$\tilde{T}\colon \mathfrak{X}/\ker T \to \mathfrak{Y}\,, \quad [x] \mapsto T(x)$$

is injective and satisfies $\operatorname{im}\tilde{T} = \operatorname{im} T$. As $\operatorname{codim}\operatorname{im} T < \infty$, there exists a linear subspace $W \subset \mathfrak{Y}$ with $\dim W < \infty$ such that the vector space $\mathfrak{Y}$ admits the direct decomposition $\mathfrak{Y} = \operatorname{im} T \oplus W$. Define the vector space $\hat{\mathfrak{X}} := (\mathfrak{X}/\ker T) \times W$. As $\mathfrak{X}/\ker T$ and $W$ are Banach spaces, so is $\hat{\mathfrak{X}}$ with respect to the norm $\|([x], w)\|_{\hat{\mathfrak{X}}} := \|[x]\|_{\mathfrak{X}/\ker T} + \|w\|_W$. The operator

$$\hat{T}\colon \hat{\mathfrak{X}} \to \mathfrak{Y}\,, \quad \hat{T}\big([x], w\big) := T(x) + w$$

is linear, bounded, and bijective. Thus, its inverse operator $\hat{T}^{-1}$ is bounded, since $\hat{\mathfrak{X}}$ and $\mathfrak{Y}$ are Banach spaces. As

$$\operatorname{im} T = \big(\hat{T}^{-1}\big)^{-1}\big((\mathfrak{X}/\ker T) \times \{0\}\big)$$

we find that $\operatorname{im} T$ is closed in $\mathfrak{Y}$. □

Proposition 1.6 discussed the relationship between compact operators and their duals; the next result clarifies the connection between Fredholm operators and their duals.

Lemma 1.10. – *Let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be Banach spaces over the same field, and let* $T\colon \mathfrak{X} \to \mathfrak{Y}$ *be a Fredholm operator. Then* $T'\colon \mathfrak{Y}' \to \mathfrak{X}'$ *is also a Fredholm operator with* $\operatorname{ind} T' = -\operatorname{ind} T$.

*Proof.* We recall the identity (1.1.4). It is straightforward to see that the vector spaces $(\operatorname{im} T)^\perp$ and $(\mathfrak{Y}/\operatorname{im} T)'$ are isomorphic; in particular, $(\operatorname{im} T)^\perp$ is a finite dimensional space, since $\operatorname{codim}\operatorname{im} T < \infty$. Combining this observation with (1.1.4) yields

$$\operatorname{ind} T = \dim\ker T - \operatorname{codim}\operatorname{im} T = \dim\ker T - \dim\ker T'\,. \tag{1.1.7}$$

By assumption, $\ker T$ is finite dimensional, and so is its dual space $(\ker T)'$. The mapping

$$\mathfrak{X}'/(\ker T)^\perp \to (\ker T)'\,, \qquad [x'] \mapsto x'|_{\ker T}$$

is a well-defined linear bijection; see [BK14, Proposition 3.51]. Hence, we may identify $\mathfrak{X}'/(\ker T)^\perp$ and $(\ker T)'$. Thanks to Lemma 1.9, the range $\operatorname{im} T$ of $T$ is closed in $\mathfrak{Y}$. Thus, applying Proposition 1.1, we find that

$$\dim(\ker T)' = \dim \mathfrak{X}'/(\ker T)^\perp = \dim \mathfrak{X}'/(\operatorname{im} T') = \dim \operatorname{coker}(T') \,. \tag{1.1.8}$$

Using that $\dim \ker T = \dim(\ker T)'$ and combining (1.1.8) with (1.1.7), we find that the dual operator $T'$ is a Fredholm operator with

$$\operatorname{ind} T = -\operatorname{ind} T' \,.$$

This proves the claim. □

Fredholm operators are stable under composition and we have the following result.

Lemma 1.11. – *Let $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ be Banach spaces over the same field. If $T\colon \mathfrak{X} \to \mathfrak{Y}$ and $S\colon \mathfrak{Y} \to \mathfrak{Z}$ are Fredholm operators, then so is $ST\colon \mathfrak{X} \to \mathfrak{Z}$ with*

$$\operatorname{ind}(ST) = \operatorname{ind} S + \operatorname{ind} T \,.$$

*Proof.* For a proof of this result we refer to [Tay11, Proposition 7.6]. □

An import class of Fredholm operators is given by compact perturbations of a fixed Fredholm operator.

Lemma 1.12. – *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces over the same field, and let $T\colon \mathfrak{X} \to \mathfrak{Y}$ be a Fredholm operator. If $K\colon \mathfrak{X} \to \mathfrak{Y}$ is a compact operator, then $T + K$ is a Fredholm operator with* $\operatorname{ind}(T + K) = \operatorname{ind} T$.

*Proof.* We refer to [Tay11, Corollaries 7.2 and 7.5]. □

## 1.2 The Lax-Milgram lemma for Banach spaces

Consider a real Hilbert space $\mathfrak{H}$. Each bounded bilinear form $\mathfrak{b}\colon \mathfrak{H} \times \mathfrak{H} \to \mathbb{R}$ induces a bounded linear operator $B\colon \mathfrak{H} \to \mathfrak{H}$, $u \mapsto Bu$, where $Bu \in \mathfrak{H}$ is uniquely determined by $\langle Bu, v\rangle_{\mathfrak{H}} = \mathfrak{b}(u, v)$ for all $v \in \mathfrak{H}$. Assuming the bilinear form $\mathfrak{b}$ is coercive—that is, there exists a constant $c_0 > 0$ such that

$$\mathfrak{b}(u, u) \geq c_0 \|u\|^2_{\mathfrak{H}} \quad \text{for all } u \in \mathfrak{H} \,, \tag{1.2.1}$$

Lax and Milgram [LM54] showed that the operator $B$ has a bounded inverse. This result had also been obtained by others, we mention Višik [Viš51].

This section is devoted to the generalisation of this result for sesquilinear forms $\mathfrak{b}\colon \mathfrak{X} \times V \to \mathbb{F}$, where $\mathfrak{X}$ is an arbitrary Banach space over $\mathbb{F}$ and $V$ is a reflexive normed space over the same field. Such generalisations are classical: Sauer [Sau66] established the result in the case $\mathfrak{X} = V$ is a reflexive Banach space and showed that reflexivity is necessary. We consider the more general case of a reflexive Banach space $V$ and an arbitrary Banach space $\mathfrak{X}$, and prove a variant of a result by Hayden [Hay68]. Of course, in this case, the coercivity estimate (1.2.1) has to be replaced appropriately.

We say a sesquilinear form $\mathfrak{b}\colon \mathfrak{X} \times V \to \mathbb{F}$ is $\mathfrak{X}$-*elliptic* if there exists a constant $c_0 > 0$ such that

$$\sup_{\substack{v \in V \\ \|v\|_V = 1}} \big|\mathfrak{b}(u,v)\big| \geq c_0 \|u\|_{\mathfrak{X}} \qquad \text{for all } u \in \mathfrak{X}\,. \tag{1.2.2}$$

If $\mathfrak{X} = V$ is a real Hilbert space, then $\mathfrak{b}$ is $\mathfrak{X}$-elliptic if it is coercive. Every $\mathfrak{X}$-*elliptic* sesquilinear form $\mathfrak{b}$ has the property that $u = 0$ if and only if $\mathfrak{b}(u,v) = 0$ for all $v \in V$. There might be, however, a non-trivial element $v \in V$ such that $\mathfrak{b}(u,v) = 0$ for all $u \in \mathfrak{X}$. A sesquilinear form $\mathfrak{b}\colon \mathfrak{X} \times V \to \mathbb{F}$ is *non-degenerate* if the following two requirements are satisfied:

(i) For every $u \in \mathfrak{X} \setminus \{0\}$ there exists $v \in V$ such that $\mathfrak{b}(u,v) \neq 0$;

(ii) For every $v \in V \setminus \{0\}$ there exists $u \in \mathfrak{X}$ such that $\mathfrak{b}(u,v) \neq 0$.

We are now in the position to state the generalisation of the Lax-Milgram lemma.

Proposition 1.13. – *Let $\mathfrak{X}$ be a Banach space over the field $\mathbb{F}$, and let $V$ be a reflexive normed space over the same field. Let $\mathfrak{b}\colon \mathfrak{X} \times V \to \mathbb{F}$ be a non-degenerate, bounded, and $\mathfrak{X}$-elliptic sesquilinear form. Then the following statements hold:*

*(i) For every $\ell \in V^*$ there exists a unique $u \in \mathfrak{X}$ such that*

$$\mathfrak{b}(u,\cdot) = \ell(\cdot)\,. \tag{1.2.3}$$

*Moreover, the unique solution $u \in \mathfrak{X}$ to* (1.2.3) *satisfies the estimate*

$$\|u\|_{\mathfrak{X}} \leq \frac{1}{c_0}\|\ell\|_{V^*}\,,$$

*where $c_0 > 0$ is the constant from* (1.2.2).

*(ii) The operator $B\colon \mathfrak{X} \to V^*$, $u \mapsto Bu$ with*

$$\langle Bu, v\rangle_{V^*,V} \coloneqq \mathfrak{b}(u,v) \qquad \textit{for all } v \in V$$

*is a Banach space isomorphism; that is, $B$ is a bijective, linear and bounded operator between Banach spaces.*

*Proof.* In order to show *(i)*, it suffices to prove statement *(ii)* and to show that $\|B^{-1}\| \leq c_0^{-1}$.

*Step 1.* The sesquilinear form $\mathfrak{b}$ is linear in its first argument and bounded, which implies linearity and boundedness of $B$. We claim that $B$ is an injective operator with closed range. Indeed, since $\mathfrak{b}$ is $\mathfrak{X}$-elliptic, we find that

$$\|Bu\|_{V^*} = \sup_{\substack{v \in V \\ \|v\|_V \leq 1}} \big|\langle Bu, v\rangle_{V^*,V}\big| \geq \sup_{\substack{v \in V \\ \|v\|_V = 1}} \big|\mathfrak{b}(u,v)\big| \geq c_0\|u\|_{\mathfrak{X}} \tag{1.2.4}$$

for all $u \in \mathfrak{X}$. From inequality (1.2.4) it is straightforward to conclude that $B$ has closed range and is injective. We are left to prove that $B$ is surjective.

*Step 2.* Suppose $B$ is not surjective. We then find $v_0^* \in V^*$ such that $v_0^* \notin \operatorname{im} B$. As $\operatorname{im} B$ is a closed subspace, the Hahn-Banach theorem implies the existence of a linear functional $\ell \in V^{**}$ such that $\ell(v_0^*) = 1$ and $\ell|_{\operatorname{im} B} = 0$. Using the reflexivity of $V$, we find a unique element $v_\ell \in V$ such that

$$\ell\big(v^*\big) = \overline{\langle v^*, v_\ell \rangle}_{V^*,V} \qquad \text{for all } v^* \in V^* \,. \tag{1.2.5}$$

For every $u \in \mathfrak{X}$,

$$\mathfrak{b}(u, v_\ell) = \langle Bu, v_\ell \rangle_{V^*,V} = \overline{\ell\big(Bu\big)} = 0 \,, \tag{1.2.6}$$

where we used that $\ell$ vanishes on $\operatorname{im} B$ to obtain the last equation. Since $\mathfrak{b}$ is non-degenerate, we deduce from (1.2.6) that $v_\ell = 0$. In view of (1.2.5), this is a contradiction to $\ell(v_0^*) = 1$.

We have shown that $B$ is a bijective, linear, and bounded operator; its inverse $B^{-1}$ is thus also bounded. Moreover, from (1.2.4) we deduce that $\|B^{-1}\| \leq c_0^{-1}$. □

Remark 2. – *In Proposition 1.13, we assumed that $V$ is a reflexive normed space, which already implies completeness. Indeed, $V$ is isomorphic to $V^{**}$, which as the dual space of $V^*$ is a Banach space. We write, however, "reflexive normed space" instead of "reflexive Banach space" to emphasise that completeness is not used in the proof of the statement.*

CHAPTER 2

# Function spaces for Maxwell's equations

## 2.1 Spaces related to divergence and curl

The analysis of Maxwell's equations requires special function spaces involving the divergence and curl of vector fields. Our objective in this chapter is to define these function spaces and to list some of their properties. We follow the book by Girault and Raviart [GR86]. Throughout this section, $\Omega \subset \mathbb{R}^3$ is bounded, open, and connected set with a Lipschitz boundary.

Studying cell problems, which naturally arise in the homogenization of time-harmonic Maxwell's equations, it is necessary to consider vector fields that are less regular than $H^1(\Omega; \mathbb{C}^3)$. Every vector field $u \in L^2(\Omega; \mathbb{F}^3)$ induces a distribution div $u\colon C_c^\infty(\Omega; \mathbb{F}) \to \mathbb{F}$ by

$$\left\langle \operatorname{div} u, \varphi \right\rangle_{C_c^\infty(\Omega;\mathbb{F})', C_c^\infty(\Omega;\mathbb{F})} := -\int_\Omega \left\langle u, \nabla\varphi \right\rangle .$$

If this distribution is induced by an $L^2(\Omega; \mathbb{F})$-function, we write div $u \in L^2(\Omega; \mathbb{F})$. The space

$$H_{\mathbb{F}}(\operatorname{div}, \Omega) := \left\{ u \in L^2(\Omega; \mathbb{F}^3) \;\middle|\; \operatorname{div} u \in L^2(\Omega; \mathbb{F}) \right\} \tag{2.1.1}$$

is a Hilbert space when equipped with the norm

$$\|u\|_{H_{\mathbb{F}}(\operatorname{div}, \Omega)} := \|u\|_{L^2(\Omega;\mathbb{F}^3)} + \|\operatorname{div} u\|_{L^2(\Omega;\mathbb{F})} .$$

We have the following density result.

Proposition 2.1. – *If $\Omega$ is an open and bounded subset of $\mathbb{R}^3$ with a Lipschitz boundary, then the space $C^\infty(\overline{\Omega}; \mathbb{F}^3)$ is dense in $H_{\mathbb{F}}(\operatorname{div}, \Omega)$.*

*Proof.* The statement is proved in [GR86] for $\mathbb{F} = \mathbb{R}$; see Theorem 2.4. The case $\mathbb{F} = \mathbb{C}$ follows easily, since every vector field $u \in H_{\mathbb{C}}(\operatorname{div}, \Omega)$ can be written as $u = \operatorname{Re}\{u\} + \operatorname{Im}\{u\}$ with $\operatorname{Re}\{u\}, \operatorname{Im}\{u\} \in H_{\mathbb{R}}(\operatorname{div}, \Omega)$. □

It is a classical result from the theory of Sobolev spaces that there exists a unique linear and bounded extension $\mathrm{tr}\colon H^1(\Omega;\mathbb{F}) \to H^{1/2}(\partial\Omega;\mathbb{F})$ of the linear map $C^\infty(\overline{\Omega};\mathbb{F}) \to C^0(\partial\Omega;\mathbb{F})$, $\varphi \mapsto \varphi|_{\partial\Omega}$; see, for instance, [GR86, Theorem 1.5]. Using the density result of Proposition 2.1, we can introduce the *normal trace* on $\partial\Omega$ of vector fields in $H_{\mathbb{F}}(\mathrm{div}\,,\Omega)$.

Theorem 2.2. (Normal trace) – *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^3$ with a Lipschitz boundary. Denote by $\nu$ the unit outward normal on $\partial\Omega$. The mapping $\mathrm{tr}_\nu\colon C^\infty(\overline{\Omega};\mathbb{F}^3) \to L^2(\partial\Omega;\mathbb{F})$, $u \mapsto \langle u|_{\partial\Omega}, \nu\rangle$ can be extended to a linear and bounded mapping $\mathrm{tr}_\nu\colon H_{\mathbb{F}}(\mathrm{div}\,,\Omega) \to H^{-1/2}(\partial\Omega;\mathbb{F})$. Moreover, Green's formula*

$$\int_\Omega \langle u, \nabla\varphi\rangle + \int_\Omega (\mathrm{div}\; u)\,\overline{\varphi} = \big\langle \mathrm{tr}_\nu(u), \mathrm{tr}(\varphi)\big\rangle_{H^{-1/2}(\partial\Omega),H^{1/2}(\partial\Omega)}$$

*holds for all $u \in H_{\mathbb{F}}(\mathrm{div}\,,\Omega)$ and $\varphi \in H^1(\Omega;\mathbb{F})$.*

*Proof.* The theorem is proved for $H_{\mathbb{R}}(\mathrm{div}\,,\Omega)$ in [GR86, Theorem 2.5]. The very same proof is also valid for the case $\mathbb{F} = \mathbb{C}$, since only the density of $C^\infty(\overline{\Omega};\mathbb{C}^3)$ in $H_{\mathbb{C}}(\mathrm{div}\,,\Omega)$ is used, which was proved in Proposition 2.1. □

Abusing notation, we shall always write $\langle u, \nu\rangle$ instead of $\mathrm{tr}_\nu(u)$; similarly, we simply write $\varphi$ instead of $\mathrm{tr}(\varphi)$ if no ambiguity is possible.

The time-harmonic Maxwell equations are equations for complex-valued vector fields $u\colon \Omega \to \mathbb{C}^3$. We shall therefore simply write $H(\mathrm{div}\,,\Omega)$ instead of $H_{\mathbb{C}}(\mathrm{div}\,,\Omega)$. The kernel of the normal trace operator is denoted by $H_0(\mathrm{div}\,,\Omega)$; that is,

$$H_0(\mathrm{div}\,,\Omega) \coloneqq \Big\{ u \in H_{\mathbb{C}}(\mathrm{div}\,,\Omega) \;\Big|\; \mathrm{tr}_\nu(u) = 0 \Big\}\,. \tag{2.1.2}$$

Similar to $H(\mathrm{div}\,,\Omega)$ we introduce the space

$$H(\mathrm{curl}\,,\Omega) \coloneqq \Big\{ u \in L^2(\Omega;\mathbb{C}^3) \;\Big|\; \mathrm{curl}\; u \in L^2(\Omega;\mathbb{C}^3) \Big\}\,, \tag{2.1.3}$$

which is a Hilbert space with respect to the norm

$$\|u\|_{H(\mathrm{curl}\,,\Omega)} \coloneqq \|u\|_{L^2(\Omega;\mathbb{C}^3)} + \|\mathrm{curl}\; u\|_{L^2(\Omega;\mathbb{C}^3)}\,.$$

The completion of $C_c^\infty(\Omega;\mathbb{C}^3)$ with respect to the norm $\|\cdot\|_{H(\mathrm{curl}\,,\Omega)}$ is denoted by $H_0(\mathrm{curl}\,,\Omega)$. Similar to $H_0(\mathrm{div}\,,\Omega)$, the space $H_0(\mathrm{curl}\,,\Omega)$ can also be characterised by the *tangential trace*. We do not need this notion and refer to [GR86, Section 2.3] for more details.

## 2.2 Spaces of periodic functions

The closed unit cube $[0,1]^3$ in $\mathbb{R}^3$ is denoted by $Y$; sometimes referred to as the *unit cell* or *periodicity cell*. Let $m \in \mathbb{N}$ with $m \geq 1$. Maps $f\colon \mathbb{R}^3 \to \mathbb{F}^m$ that satisfy

$$f(y + \mathrm{e}_i) = f(y) \quad \text{ for all } y \in \mathbb{R}^3 \text{ and } i \in \{1,\ldots,d\}$$

are called *$Y$-periodic*. For $k \geq 0$, we define

$$C^k_\sharp(Y;\mathbb{F}^m) \coloneqq \Big\{ f|_Y\colon Y \to \mathbb{F}^m \;\Big|\; f \in C^k(\mathbb{R}^3;\mathbb{F}^m) \text{ is } Y\text{-periodic} \Big\}\,.$$

The Sobolev space $H^k_\sharp(Y;\mathbb{F}^m)$ is defined as the completion of $C^\infty_\sharp(Y;\mathbb{F}^m)$ with respect to the $H^k$-norm.

On the unit cube $Y$, we can define an equivalence relation $\sim$ by identifying opposite faces—that is, $y \sim y'$ whenever $y - y' \in \mathbb{Z}^3$. The quotient $Y/\sim$ can be identified with the flat three-dimensional torus $\mathbb{T}^3$, and every $Y$-periodic map $Y \to \mathbb{F}^m$ can be viewed as a map $\mathbb{T}^3 \to \mathbb{F}^m$. We denote the canonical projection $Y \hookrightarrow \mathbb{T}^3$ by $\iota$ and the universal covering $\mathbb{R}^3 \to \mathbb{T}^3$ by $\pi$. Given a subset $U$ of $Y$, we introduce the "periodic continuation" of $U$ as the set

$$\tilde{U} \coloneqq \bigcup_{m\in\mathbb{Z}^3} (m+U)\,. \tag{2.2.1}$$

We note that $\iota(U) = \pi(U) \subset \mathbb{T}^3$ is open if and only if $\iota^{-1}\big(\iota(U)\big) \subset Y$ is open. Let us warn the reader that, in general, $\iota(U) \subset \mathbb{T}^3$ is *not* open although $U \subset Y$ is open. Indeed, for $\varepsilon \in (0,1/2)$, the set $U \coloneqq [0,\varepsilon)^3$ is open in $Y$ but $\iota(U) \subset \mathbb{T}^3$ is not open.

Definition 2.3. (Lipschitz domain on $\mathbb{T}^3$) – *Let $U \subset Y$ be a non-empty subset for which $\iota(U) \subset \mathbb{T}^3$ is open, and let $\tilde{U} \subset \mathbb{R}^3$ be the set defined in* (2.2.1). *We say $\iota(U)$ has a* Lipschitz boundary *if $\tilde{U}$ has a Lipschitz boundary. Moreover, $\iota(U)$ is a* Lipschitz domain *provided $\iota(U)$ is connected and has a Lipschitz boundary.*

Assume $U \subset Y$ is non-empty and let $\tilde{U}$ be the set defined in (2.2.1). Similar to $C^k_\sharp(Y;\mathbb{F}^m)$, we define the space

$$C^k_\sharp(U;\mathbb{F}^m) \coloneqq \Big\{ f|_U : U \to \mathbb{F}^m \;\Big|\; f \in C^k(\tilde{U};\mathbb{F}^m) \text{ is } Y\text{-periodic} \Big\}\,,$$

for $k \geq 0$. If $U \subset Y$ is open, then the completion of $C^k_\sharp(\overline{U};\mathbb{F}^m)$ with respect to the $H^k$-norm is denoted by $H^k_\sharp(U;\mathbb{F}^m)$.

## 2.3 Helmholtz decomposition for periodic vector fields

For a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, the classical Helmholtz decomposition states that every vector field $v \in L^2(\Omega;\mathbb{R}^3)$ can be written as a sum of a gradient vector field and a divergence-free vector field; see, for instance, [Sch18, BPS16]. Such a decomposition does also hold for $L^2$-vector fields on the torus $\mathbb{T}^3$. We remark that the two spaces $H^1_\sharp(Y;\mathbb{F}^3)$ and $H^1(\mathbb{T}^3;\mathbb{F}^3)$ can be identified. Before proving the Helmholtz decomposition , we recall a classical result related to the Sobolev space $H^1(\mathbb{T}^3;\mathbb{F}^3)$.

Lemma 2.4. – *Consider the sesquilinear form $\mathfrak{b}\colon H^1(\mathbb{T}^3;\mathbb{F}^3)\times H^1(\mathbb{T}^3;\mathbb{F}^3) \to \mathbb{F}$,*

$$\mathfrak{b}(v,\varphi) \coloneqq \int_{\mathbb{T}^3} \langle \operatorname{curl} v, \operatorname{curl} \varphi\rangle + \int_{\mathbb{T}^3} \operatorname{div} v \cdot \overline{\operatorname{div} \varphi}\,. \tag{2.3.1}$$

*Then, for all $v \in H^1(\mathbb{T}^3;\mathbb{F}^3)$, there holds*

$$\mathfrak{b}(v,v) = \|\nabla v\|^2_{L^2(\mathbb{T}^3;\mathbb{F}^{3\times 3})}\,. \tag{2.3.2}$$

*In particular, the map $H^1(\mathbb{T}^3;\mathbb{F}^3) \to \mathbb{R}$, $v \mapsto \mathfrak{b}(v,v) + \int_{\mathbb{T}^3}|v|^2$ is the standard $H^1(\mathbb{T}^3;\mathbb{F}^3)$-norm. Moreover, if $v \in L^2(\mathbb{T}^3;\mathbb{F}^3)$ is a vector field for which* $\operatorname{curl} v \in L^2(\mathbb{T}^3;\mathbb{F}^3)$ *and* $\operatorname{div} v \in L^2(\mathbb{T}^3;\mathbb{F})$*, then $v \in H^1(\mathbb{T}^3;\mathbb{F}^3)$.*

*Proof.* We begin by proving equation (2.3.2). For vector fields $v \in C^\infty_\sharp(Y;\mathbb{F}^3)$ the following identity for the vector Laplacian holds:

$$\big\langle(-\Delta v), v\big\rangle_{L^2(\mathbb{T}^3;\mathbb{F}^3)} = \big\langle \text{curl curl } v, v\big\rangle_{L^2(\mathbb{T}^3;\mathbb{F}^3)} - \big\langle \nabla \operatorname{div} v, v\big\rangle_{L^2(\mathbb{T}^3;\mathbb{F}^3)} . \tag{2.3.3}$$

Using integration by parts on both sides of (2.3.3), one readily checks that (2.3.2) holds for all vector fields $v \in C^\infty_\sharp(Y;\mathbb{F}^3)$. By definition of the periodic Sobolev spaces, $C^\infty_\sharp(Y;\mathbb{F}^3)$ is dense in $H^1(\mathbb{T}^3;\mathbb{F}^3)$. Thus, for every $v \in H^1(\mathbb{T}^3;\mathbb{F}^3)$ we find a sequence $(v_k)_{k\in\mathbb{N}}$ in $C^\infty_\sharp(Y;\mathbb{F}^3)$ that converges to $v$ in $H^1(\mathbb{T}^3;\mathbb{F}^3)$. As there are constants $c_1, c_2 > 0$ such that

$$\|\text{curl } v - \text{curl } v_k\|^2_{L^2(\mathbb{T}^3;\mathbb{F}^3)} \leq c_1 \|\nabla v - \nabla v_k\|^2_{L^2(\mathbb{T}^3;\mathbb{F}^{3\times 3})} \tag{2.3.4}$$

and

$$\|\operatorname{div} v - \operatorname{div} v_k\|^2_{L^2(\mathbb{T}^3;\mathbb{F})} \leq c_2 \|\nabla v - \nabla v_k\|^2_{L^2(\mathbb{T}^3;\mathbb{F}^{3\times 3})} ,$$

we deduce that

$$\|\nabla v\|^2_{L^2(\mathbb{T}^3;\mathbb{F}^{3\times 3})} = \lim_{k\to\infty} \|\nabla v_k\|^2_{L^2(\mathbb{T}^3;\mathbb{F}^{3\times 3})} = \lim_{k\to\infty} \mathfrak{b}(v_k, v_k) = \mathfrak{b}(v, v) .$$

This proves the identity (2.3.2) for all vector fields $v \in H^1(\mathbb{T}^3;\mathbb{F}^3)$. Having this identity, it is obvious that $v \mapsto \mathfrak{b}(v,v) + \int_{\mathbb{T}^3} |v|^2$ is equal to the standard norm on $H^1(\mathbb{T}^3;\mathbb{F}^3)$.

In order to prove the second part of the lemma, fix a vector field $v \in L^2(\mathbb{T}^3;\mathbb{F}^3)$ for which its curl and its divergence are $L^2(\mathbb{T}^3)$-maps. Let $(\rho_\varepsilon)_\varepsilon$ be a sequence of mollifiers, and set $v_\varepsilon := v * \rho_\varepsilon$. Then $v_\varepsilon \in C^\infty_\sharp(\mathbb{T}^3;\mathbb{F}^3)$, and for $\varepsilon, \varepsilon' > 0$ there holds

$$\mathfrak{b}(v_\varepsilon - v_{\varepsilon'}) + \|v_\varepsilon - v_{\varepsilon'}\|^2 = \|v_\varepsilon - v_{\varepsilon'}\|^2_{H^1(\mathbb{T}^3;\mathbb{F}^3)} . \tag{2.3.5}$$

As $v_\varepsilon \to v$ in $L^2(\mathbb{T}^3;\mathbb{F}^3)$, curl $v_\varepsilon = (\text{curl } v) * \rho_\varepsilon \to \text{curl } v$ in $L^2(\mathbb{T}^3;\mathbb{F}^3)$ and div $v_\varepsilon = (\operatorname{div} v) * \rho_\varepsilon \to \operatorname{div} v$ in $L^2(\mathbb{T}^3;\mathbb{F})$, we deduce from (2.3.5) that $(v_\varepsilon)_\varepsilon$ is a Cauchy sequence in $H^1(\mathbb{T}^3;\mathbb{F}^3)$; hence, there exists $\tilde{v} \in H^1(\mathbb{T}^3;\mathbb{F}^3)$ such that $v_\varepsilon \to \tilde{v}$ in $H^1(\mathbb{T}^3;\mathbb{F}^3)$. By uniqueness of the limit, $v = \tilde{v} \in H^1(\mathbb{T}^3;\mathbb{F}^3)$, which proves the claim. ☐

Define the space of gradient vector fields as

$$\nabla H^1(\mathbb{T}^3;\mathbb{F}) := \Big\{ u \in L^2(\mathbb{T}^3;\mathbb{F}^3) \,\Big|\, \text{there is a } \Theta \in H^1(\mathbb{T}^3;\mathbb{F}) \text{ with } u = \nabla\Theta \text{ in } \mathbb{T}^3 \Big\}$$

and the space of solenoidal vector fields as

$$L^2_{\text{sol}}(\mathbb{T}^3;\mathbb{F}^3) := \Big\{ u \in L^2(\mathbb{T}^3;\mathbb{F}^3) \,\Big|\, \operatorname{div} u = 0 \text{ in } \mathbb{T}^3 \Big\} .$$

Armed with the characterisation of the $H^1$-norm of vector fields and the above defined spaces, we are now in the position to formulate a first version of the Helmholtz decomposition on the torus, which resembles the classical Helmholtz decomposition for a bounded Lipschitz domain.

Lemma 2.5. – *The following orthogonal decomposition of $L^2(\mathbb{T}^3;\mathbb{F}^3)$ holds,*

$$L^2(\mathbb{T}^3;\mathbb{F}^3) = \nabla H^1(\mathbb{T}^3;\mathbb{F}^3) \oplus L^2_{\text{sol}}(\mathbb{T}^3;\mathbb{F}^3) .$$

*More precisely, for every* $u \in L^2(\mathbb{T}^3;\mathbb{F}^3)$ *there exists a periodic potential* $\Theta \in H^1(\mathbb{T}^3;\mathbb{F})$ *and some vector field* $w \in L^2_{\text{sol}}(\mathbb{T}^3;\mathbb{F}^3)$ *such that* $u = \nabla\Theta + w$ *in* $\mathbb{T}^3$*. Moreover, the potential* $\Theta \in H^1(\mathbb{T}^3;\mathbb{F}^3)$ *is the unique solution to*

$$\int_{\mathbb{T}^3} \langle \nabla\Theta, \nabla\varphi \rangle = \int_{\mathbb{T}^3} \langle u, \nabla\varphi \rangle \quad \textit{for all } \varphi \in H^1(\mathbb{T}^3;\mathbb{F}) .$$

*with* $\int_{\mathbb{T}^3} \Theta = 0$*.*

*Proof.* Fix $u \in L^2(\mathbb{T}^3;\mathbb{F}^3)$. Thanks to the Lax-Milgram lemma, there exists a unique function $\Theta \in H^1(\mathbb{T}^3;\mathbb{F})$ with $\int_{\mathbb{T}^3} \Theta = 0$ such that

$$\int_{\mathbb{T}^3} \langle \nabla\Theta, \nabla\varphi \rangle = \int_{\mathbb{T}^3} \langle u, \nabla\varphi \rangle \quad \text{for all } \varphi \in H^1(\mathbb{T}^3;\mathbb{F}) .$$

Having this potential $\Theta$ at hand, we find that the vector field $v := u - \nabla\Theta$ is an element of $L^2(\mathbb{T}^3;\mathbb{F}^3)$ and satisfies

$$\int_{\mathbb{T}^3} \langle v, \nabla\varphi \rangle = \int_{\mathbb{T}^3} \langle u, \nabla\varphi \rangle - \int_{\mathbb{T}^3} \langle \nabla\Theta, \nabla\varphi \rangle = 0 ,$$

for all $\varphi \in H^1(\mathbb{T}^3;\mathbb{F})$. This shows that $v \in L^2_{\text{sol}}(\mathbb{T}^3;\mathbb{F}^3)$.

In order to show that the decomposition

$$L^2(\mathbb{T}^3;\mathbb{F}^3) = \nabla H^1(\mathbb{T}^3;\mathbb{F}) \oplus L^2_{\text{sol}}(\mathbb{T}^3;\mathbb{F}^3)$$

is direct, we need to prove that only the trivial vector field is an element in both spaces. One readily checks, that $L^2_{\text{sol}}(\mathbb{T}^3;\mathbb{F}^3)$ is a linear subspace of the $L^2$-orthogonal complement of $\nabla H^1(\mathbb{T}^3;\mathbb{F})$, which proves the claim. ☐

The space $L^2_{\text{sol}}(\mathbb{T}^3;\mathbb{F}^3)$ of solenoidal vector fields on the torus can be further decomposed. To this end, we define the space of curl vector fields as

$$\operatorname{curl} H^1(\mathbb{T}^3;\mathbb{F}^3) := \left\{ u \in L^2(\mathbb{T}^3;\mathbb{F}^3) \;\middle|\; \begin{array}{l} \text{there exists } w \in H^1(\mathbb{T}^3;\mathbb{F}^3) \\ \text{such that } u = \operatorname{curl} w \text{ in } \mathbb{T}^3 \end{array} \right\} .$$

We further introduce the space of harmonic vector fields,

$$\mathcal{H}(\mathbb{T}^3;\mathbb{F}^3) := \left\{ u \in L^2(\mathbb{T}^3;\mathbb{F}^3) \;\middle|\; \operatorname{curl} u = 0 \text{ in } \mathbb{T}^3 \text{ and } \operatorname{div} u = 0 \text{ in } \mathbb{T}^3 \right\} .$$

Having these function spaces at hand, we can state an improved Helmholtz decomposition on the torus.

PROPOSITION 2.6. (Helmholtz decomposition on the torus) – *The vector space of harmonic fields* $\mathcal{H}_\sharp(Y;\mathbb{F}^3)$ *is finite dimensional. More precisely,* $\mathcal{H}(\mathbb{T}^3;\mathbb{F}^3) = \mathbb{F}^3$*. Furthermore, the following orthogonal decomposition holds*

$$L^2(\mathbb{T}^3;\mathbb{F}^3) = \nabla H^1(\mathbb{T}^3;\mathbb{F}) \oplus \operatorname{curl} H^1(\mathbb{T}^3;\mathbb{F}^3) \oplus \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3) .$$

*In particular, for every vector field* $u \in L^2(\mathbb{T}^3;\mathbb{F}^3)$ *there exist a scalar potential* $\Theta \in H^1(\mathbb{T}^3;\mathbb{F})$*, a vector potential* $w \in H^1(\mathbb{T}^3;\mathbb{F}^3)$*, and a constant* $c \in \mathbb{F}^3$ *such that* $u = \nabla\Theta + \operatorname{curl} w + c$ *in* $\mathbb{T}^3$*.*

*Proof.* We proceed in two steps.

*Step 1.* We claim that the vector space $\mathcal{H}(\mathbb{T}^3;\mathbb{F}^3) = \mathbb{F}^3$. First, we observe that $\mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)$ is a subspace of $H^1(\mathbb{T}^3;\mathbb{F}^3)$; indeed, if $u$ is an harmonic vector field, both its distributional curl and its distributional divergence vanish in $\mathbb{T}^3$. We can thus deduce from Lemma 2.4 that $u \in H^1(\mathbb{T}^3;\mathbb{F}^3)$. Equation (2.3.2) shows $\nabla u = 0$ in $L^2(\mathbb{T}^3;\mathbb{F}^{3\times 3})$. This implies that $u = c \in \mathbb{F}^3$, since $\mathbb{T}^3$ is connected. Consequently, $\mathcal{H}(\mathbb{T}^3;\mathbb{F}^3) \subset \mathbb{F}^3$. The other inclusion, that is $\mathbb{F}^3 \subset \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)$ is clear.

*Step 2.* We already know, by Lemma 2.5, that $L^2(\mathbb{T}^3;\mathbb{F}^3)$ admits the orthogonal decomposition

$$L^2(\mathbb{T}^3;\mathbb{F}^3) = \nabla H^1(\mathbb{T}^3;\mathbb{F}) \oplus L^2_{\mathrm{sol}}(\mathbb{T}^3;\mathbb{F}^3)\,.$$

One readily checks that $\mathcal{H}(\mathbb{T}^3;\mathbb{F}^3) \subset L^2_{\mathrm{sol}}(\mathbb{T}^3;\mathbb{F}^3)$. As $\mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)$ is finite dimensional, it is a closed subspace of $L^2_{\mathrm{sol}}(\mathbb{T}^3;\mathbb{F}^3)$, and thus

$$L^2_{\mathrm{sol}}(\mathbb{T}^3;\mathbb{F}^3) = \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3) \oplus \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)^{\perp}\,.$$

The claim is proved if we show that $\mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)^{\perp} = \operatorname{curl} H^1_\sharp(Y;\mathbb{F}^3)$ in $L^2_{\mathrm{sol}}(\mathbb{T}^3;\mathbb{F}^3)$.

By Step 1, $\mathcal{H}(\mathbb{T}^3;\mathbb{F}^3) = \mathbb{F}^3$. This implies $\operatorname{curl} H^1(\mathbb{T}^3;\mathbb{F}^3) \subset \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)^{\perp}$. To prove the converse inclusion, fix a vector field $v \in \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)^{\perp}$. Consider the function space

$$H^1_{\mathrm{sol},0}(\mathbb{T}^3;\mathbb{F}^3) \coloneqq \left\{ u \in H^1(\mathbb{T}^3;\mathbb{F}^3) \;\middle|\; \int_{\mathbb{T}^3} u = 0 \text{ and } \operatorname{div}\, u = 0 \text{ in } \mathbb{T}^3 \right\},$$

and let $\mathfrak{b}\colon H^1_{\mathrm{sol},0}(\mathbb{T}^3;\mathbb{F}^3) \times H^1_{\mathrm{sol},0}(\mathbb{T}^3;\mathbb{F}^3) \to \mathbb{F}$ be the sesquilinear form defined in (2.3.1). The Lax-Milgram lemma together with equation (2.3.2) establish the existence of a unique solution $w \in H^1_{\mathrm{sol},0}(\mathbb{T}^3;\mathbb{F}^3)$ to

$$\mathfrak{b}(w,\varphi) = \int_{\mathbb{T}^3} \langle v, \operatorname{curl}\, \varphi\rangle \tag{2.3.6}$$

for all $\varphi \in H^1_{\mathrm{sol},0}(\mathbb{T}^3;\mathbb{F}^3)$. Note that $w$ satisfies (2.3.6) also for all $\varphi \in H^1(\mathbb{T}^3;\mathbb{F}^3)$, since the equation holds for constant vector fields. The claim is proved if the remainder $R \coloneqq v - \operatorname{curl}\, w \in L^2(\mathbb{T}^3;\mathbb{F}^3)$ vanishes.

We claim that $R \in \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)$. Indeed, as $v \in L^2_{\mathrm{sol}}(\mathbb{T}^3;\mathbb{F}^3)$ we find that $\operatorname{div}\, R = \operatorname{div}\, v - \operatorname{div}\operatorname{curl}\, w = 0$. On the other hand, by construction of $w$, there holds

$$\begin{aligned}\int_Y \big\langle R, \operatorname{curl}\, \varphi\big\rangle &= \int_Y \langle v, \operatorname{curl}\, \varphi\rangle - \int_Y \langle \operatorname{curl}\, w, \operatorname{curl}\, \varphi\rangle \\ &= \int_Y \langle v, \operatorname{curl}\, \varphi\rangle - \mathfrak{b}(w,\varphi) = 0\,,\end{aligned}$$

for all $\varphi \in H^1(\mathbb{T}^3;\mathbb{F}^3)$. This shows that $R \in \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)$.

The proof is finished if we can show that $R$ is also an element of $\mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)^{\perp}$. As shown above, $\operatorname{curl}\, H^1(\mathbb{T}^3;\mathbb{F}^3) \subset \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)^{\perp}$ and so $\operatorname{curl}\, w \in \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)^{\perp}$. We chose $v \in \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)^{\perp}$ and hence, $R = v - \operatorname{curl}\, w \in \mathcal{H}(\mathbb{T}^3;\mathbb{F}^3)^{\perp}$. ☐

CHAPTER 3

# Two-scale convergence

In order not to disturb the discussion later on, we present the definition of two-scale convergence, introduced by Nguetseng [Ngu89], and some of its properties in this chapter. The main references are [LNW02] and [All92]; we also mention the book by Pavliotis and Stuart [PS08].

## 3.1 Banach space-valued maps

In this section, we survey the definition and basic properties of $L^2$-spaces for Banach space-valued maps. The appropriate integral is the Bochner integral. We briefly recall some facts on the Bochner integral and refer to the book by Diestel and Uhl [DU77, Chapter 2] for a more detailed discussion on the topic.

Let $\mathfrak{X}$ be a (real or complex) Banach space with norm $\|\cdot\|$, and let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$. A map $f\colon \Omega \to \mathfrak{X}$ is *simple* if there exist $x_1, \ldots, x_n \in \mathfrak{X}$ and measurable subsets $E_1, \ldots, E_n$ of $\Omega$ such that $f = \sum_{j=1}^n x_i \mathbb{1}_{E_i}$. We say $f\colon \Omega \to \mathfrak{X}$ is *strong measurable* if there exists a sequence $(f_k)_k$ of simple functions with $\lim_{k\to\infty} \|f_k(x) - f(x)\| = 0$ for almost all $x \in \Omega$. The following characterisation of Bochner integrable functions is sufficient for our purposes.

Lemma 3.1. – *Let $f\colon \Omega \to \mathfrak{X}$ be a strong measurable map. Then $f$ is Bochner integrable if and only if*

$$\int_\Omega \big\|f(x)\big\| \,\mathrm{d}x < \infty \,.$$

*Proof.* We refer to the proof of Theorem 2 in [DU77, Chapter 2.2]. □

For brevity, we abuse notation and write $\|f\|$ for the map $\Omega \to \mathbb{R}$, $x \mapsto \|f(x)\|$. The previous result motivates the following definition:

$$L^2(\Omega; \mathfrak{X}) \coloneqq \Big\{ f\colon \Omega \to \mathfrak{X} \,\Big|\, f \text{ is strong measurable and } \|f\| \in L^2(\Omega; \mathbb{R}) \Big\} \,.$$

One can show that $L^2(\Omega; \mathfrak{X})$ is a Banach space. In addition, simple functions are dense in $L^2(\Omega; \mathfrak{X})$. If $\mathfrak{X}$ is reflexive, so is $L^2(\Omega; \mathfrak{X})$; see Corollary 2 in [DU77, Chapter IV.1].

## 3.2 The notion of two-scale convergence

Let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$, and denote the closed unit cube in $\mathbb{R}^d$ by $Y := [0,1]^d$. Throughout this section, $m \in \mathbb{N}$ with $m \geq 1$.

DEFINITION 3.2. (Two-scale convergence) – *A sequence $(u^\eta)_\eta$ in $L^2(\Omega;\mathbb{C}^m)$ is said to* two-scale converge *to $u_0 \in L^2(\Omega \times Y;\mathbb{C}^m)$, and we write $u^\eta \xrightarrow{2} u_0$, if*

$$\lim_{\eta\to 0}\int_\Omega \Big\langle u^\eta(x), \psi\Big(x, \frac{x}{\eta}\Big)\Big\rangle \,\mathrm{d}x = \int_\Omega\int_Y \big\langle u_0(x,y), \psi(x,y)\big\rangle \,\mathrm{d}y\,\mathrm{d}x \tag{3.2.1}$$

*for every $\psi \in L^2(\Omega; C^0_\sharp(Y;\mathbb{C}^m))$.*

We remark that different definitions of two-scale convergence are present in the literature; the main difference concerns the space of test functions, $L^2(\Omega; C^0_\sharp(Y;\mathbb{C}^m))$. We refer to [LNW02] for a discussion on different choices of spaces of test functions.

LEMMA 3.3. – *Let $(u^\eta)_\eta$ be a sequence in $L^2(\Omega;\mathbb{C}^m)$ that two-scale converges to $u_0 \in L^2(\Omega\times Y;\mathbb{C}^m)$. Then*

$$u^\eta \to \int_Y u_0(\cdot, y)\,\mathrm{d}y \qquad \textit{weakly in } L^2(\Omega;\mathbb{C}^m) \textit{ as } \eta \to 0\,.$$

*In particular, the sequence $(u^\eta)_\eta$ is bounded in $L^2(\Omega;\mathbb{C}^m)$.*

*Proof.* Fix an arbitrary $\psi \in L^2(\Omega;\mathbb{C}^m)$. We can view the map $x \mapsto \psi(x)$ as an element of $L^2(\Omega; C^0_\sharp(Y;\mathbb{C}^m))$. As $(u^\eta)_\eta$ two-scale converges to $u_0$, we obtain

$$\lim_{\eta\to 0}\int_\Omega \big\langle u^\eta(x), \psi(x)\big\rangle\,\mathrm{d}x = \int_\Omega \Big\langle \int_Y u_0(x,y)\,\mathrm{d}y, \psi(x)\Big\rangle\,\mathrm{d}x\,.$$

As $\psi$ was chosen arbitrarily, the claim follows. □

In [LNW02], [All92], and [PS08], only sequences $(u^\eta)_\eta$ in $L^2(\Omega;\mathbb{R})$ are considered. In order to apply the results obtained in these articles to sequences in $L^2(\Omega;\mathbb{C}^m)$, we need the following two auxiliary lemmas.

LEMMA 3.4. – *Let $\Omega$ be a bounded and open subset of $\mathbb{R}^d$. For a sequence $(u^\eta)_\eta$ in $L^2(\Omega;\mathbb{C})$ the following statements are equivalent:*

*(i)* $(u^\eta)_\eta$ *two-scale converges to $u_0 \in L^2(\Omega\times Y;\mathbb{C})$.*

*(ii) The sequences $(\mathrm{Re}\{u^\eta\})_\eta$ and $(\mathrm{Im}\{u^\eta\})_\eta$ in $L^2(\Omega;\mathbb{R})$ two-scale converge to $\mathrm{Re}\{u_0\} \in L^2(\Omega\times Y;\mathbb{R})$ and $\mathrm{Im}\{u_0\} \in L^2(\Omega\times Y;\mathbb{R})$, respectively.*

*Proof.* Assume $(u^\eta)_\eta$ is a sequence in $L^2(\Omega;\mathbb{C})$ that two-scale converges to $u_0 \in L^2(\Omega\times Y;\mathbb{C})$. We note that $2\,\mathrm{Re}\{u^\eta\} = u^\eta + \overline{u}^\eta$ and that $2\mathrm{i}\,\mathrm{Im}\{u^\eta\} = u^\eta - \overline{u}^\eta$. Thus, using the fact that every $\psi \in L^2(\Omega, C^0_\sharp(Y;\mathbb{R}))$ is an element of $L^2(\Omega; C^2_\sharp(Y;\mathbb{C}))$, the two-scale convergence of $(u^\eta)_\eta$ implies statement *(ii)*.

Conversely, assume that statement *(ii)* holds. Then *(i)* follows from the observation that $\mathrm{Re}\{\psi\}, \mathrm{Im}\{\psi\} \in L^2(\Omega; C^0_\sharp(Y,\mathbb{R}))$ for every $\psi \in L^2(\Omega; C^0_\sharp(Y,\mathbb{C}))$. □

LEMMA 3.5. – *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$, and let $m \geq 2$. For a sequence $(u^\eta)_\eta$ in $L^2(\Omega;\mathbb{C}^m)$ the following statements are equivalent:*

*(i)* $(u^\eta)_\eta$ *two-scale converges to* $u_0 \in L^2(\Omega \times Y;\mathbb{C}^m)$.

*(ii)* *For each* $k \in \{1,\ldots,m\}$*, the sequence* $(u_k^\eta)_\eta$ *two-scale converges to* $(u_0)_k \in L^2(\Omega \times Y;\mathbb{C})$.

*Proof.* Let $(u^\eta)_\eta$ be a sequence in $L^2(\Omega;\mathbb{C}^m)$ that two-scale converges to $u_0 \in L^2(\Omega \times Y;\mathbb{C}^m)$. We observe that for each $k \in \{1,\ldots,m\}$ and for all test functions $\psi \in L^2(\Omega; C^0_\sharp(Y;\mathbb{C}))$, the vector field $(x,y) \mapsto \psi(x,y)\,\mathrm{e}_k$ is an element of $L^2(\Omega; C^0_\sharp(Y;\mathbb{C}^m))$. This together with the two-scale convergence of $(u^\eta)_\eta$ implies statement *(ii)*.

The converse implication, *(ii)* $\Rightarrow$ *(i)*, is straightforward and follows from the linearity of the integral. □

We have the following result.

LEMMA 3.6. – *Let $\Omega \subset \mathbb{R}^d$ be open and bounded. If $\psi \in L^1_\sharp(Y; C^0(\overline{\Omega};\mathbb{R}))$, then for any $\eta > 0$ the map $\Omega \to \mathbb{R}$, $x \mapsto \psi_\eta(x) := \psi(x/\eta, x)$ is measurable and satisfies*

$$\left\|\psi_\eta\right\|_{L^1(\Omega;\mathbb{R})} \leq C(\Omega)\|\psi\|_{L^1_\sharp(Y;C^0(\overline{\Omega};\mathbb{R}))}$$

*Proof.* The statement is proved in [All92, Corollary 5.4]. □

The next result is crucial for the theory of two-scale convergence, and for us in Chapters 5 and 6.

THEOREM 3.7. (Compactness theorem) – *If $(u^\eta)_\eta$ is a bounded sequence in $L^2(\Omega;\mathbb{C}^m)$, then there exists a subsequence $(u^\eta)_\eta$ and a map $u_0 \in L^2(\Omega \times Y;\mathbb{C}^m)$ such that $(u^\eta)_\eta$ two-scale converges to $u_0$.*

*Proof.* The statement for sequences in $L^2(\Omega;\mathbb{R})$ is proved in [LNW02, Theorem 14]; see also [All92, Theorem 1.2].

In order to prove the general statement, let $(u^\eta)_\eta$ be a bounded sequence in $L^2(\Omega;\mathbb{C}^m)$. The sequences $(u_1^\eta)_\eta,\ldots,(u_m^\eta)_\eta$ are then bounded in $L^2(\Omega;\mathbb{C})$. Fix $k \in \{1,\ldots,m\}$. Applying Theorem 14 from [LNW02] to the sequence of the real part and the imaginary part of $u_k^\eta$, respectively, we deduce the existence of a subsequence $(u_k^\eta)_\eta$ that two-scale converges. As $k \in \{1,\ldots,m\}$ was arbitrary, we can successively choose subsequences $(u_1^\eta)_\eta,\ldots,(u_m^\eta)_\eta$ that two-scale converge. This implies the claim, by Lemma 3.5. □

The following theorem shows that if $(u^\eta)_\eta$ two-scale converges, then the identity (3.2.1) holds for test functions in other spaces than $L^2(\Omega; C^0_\sharp(Y;\mathbb{C}^m))$.

THEOREM 3.8. – *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$. If $(u^\eta)_\eta$ is a sequence in $L^2(\Omega;\mathbb{C}^m)$ that two-scale converges to $u_0 \in L^2(\Omega \times Y;\mathbb{C}^m)$, then*

$$\lim_{\eta\to 0}\int_\Omega \left\langle u^\eta(x), \psi\left(x, \frac{x}{\eta}\right)\right\rangle \mathrm{d}x = \int_\Omega\int_Y \left\langle u_0(x,y), \psi(x,y)\right\rangle \mathrm{d}y\,\mathrm{d}x\,, \tag{3.2.2}$$

*for every $\psi \in L^2_\sharp(Y; C^0(\overline{\Omega};\mathbb{C}^m))$.*

*Proof.* Thanks to the linearity of the integral, we may assume that $(u^\eta)_\eta$ is a sequence in $L^2(\Omega;\mathbb{R}^m)$ and that test vector fields are elements of $L^2_\sharp(Y;C^0(\overline{\Omega};\mathbb{R}^m))$. The statement for a sequence in $L^2(\Omega;\mathbb{R})$ is proved in [LNW02, Theorem 15]. In order to prove the general statement, we apply Theorem 15 from [LNW02] to the sequence $(u^\eta_k)_\eta$ for each $k \in \{1,\dots,m\}$, which two-scale converge to $(u_0)_k \in L^2(\Omega\times Y;\mathbb{R}^m)$ by Lemma 3.5. $\square$

The next result is a straightforward variant of a classical rule, which applies to vector fields $u^\eta\colon \Omega\to\mathbb{C}^3$ for which both sequences $(u^\eta)_\eta$ and $(\eta \operatorname{div} u^\eta)_\eta$ are bounded in $L^2(\Omega;\mathbb{C}^3)$; see [All92, Proposition 1.14].

Lemma 3.9. – *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^d$, and let $(u^\eta)_\eta$ be a sequence in $H(\operatorname{curl},\Omega)$. Assume that $(u^\eta)_\eta$ and $(\eta \operatorname{curl} u^\eta)_\eta$ are bounded sequences in $L^2(\Omega;\mathbb{C}^3)$. Then there exists a vector field $u_0 \in L^2(\Omega\times Y;\mathbb{C}^3)$ with $\operatorname{curl}_y u_0(x,\cdot) \in L^2_\sharp(Y;\mathbb{C}^3)$ for almost all $x\in\Omega$ such that*

$$u^\eta \xrightarrow{2} u_0 \quad \text{and} \quad \eta \operatorname{curl} u^\eta \xrightarrow{2} \operatorname{curl}_y u_0\,. \tag{3.2.3}$$

*Proof.* As both sequences $(u^\eta)_\eta$ and $(\eta \operatorname{curl} u^\eta)_\eta$ are bounded, we find subsequences $(u^\eta)_\eta$ and $(\eta \operatorname{curl} u^\eta)_\eta$ as well as a vector field $\xi \in L^2(\Omega\times Y;\mathbb{C}^3)$ such that

$$u^\eta \xrightarrow{2} u_0 \quad \text{and} \quad \eta \operatorname{curl} u^\eta \xrightarrow{2} \xi\,.$$

The claim is proved provided $\xi(x,\cdot) = \operatorname{curl}_y u_0(x,\cdot)$ for almost all $x\in\Omega$. Choose an arbitrary test function $\theta \in C^\infty_c(\Omega;\mathbb{R})$ and a test vector field $\psi \in C^\infty_\sharp(Y;\mathbb{C}^3)$. Define $\varphi(x,y) \coloneqq \theta(x)\psi(y)$ and set $\varphi_\eta(\cdot) \coloneqq \varphi(\cdot,\cdot/\eta)$ for each $\eta>0$. On the one hand, we have

$$\lim_{\eta\to 0}\int_\Omega \big\langle \eta \operatorname{curl} u^\eta(x), \varphi_\eta(x)\big\rangle \,\mathrm{d}x = \int_\Omega \theta(x)\int_Y \big\langle \xi(x,y),\psi(y)\big\rangle \,\mathrm{d}y\,\mathrm{d}x\,. \tag{3.2.4}$$

On the other hand, applying integration by parts and using the boundedness of $(u^\eta)_\eta$, we find that

$$\begin{aligned}\lim_{\eta\to 0}\int_\Omega \big\langle \eta \operatorname{curl} u^\eta(x), \varphi_\eta(x)\big\rangle \,\mathrm{d}x &= \lim_{\eta\to 0}\int_\Omega \theta(x)\big\langle u^\eta(x), \operatorname{curl}\, \psi(x/\eta)\big\rangle \,\mathrm{d}x \\ &\quad + \lim_{\eta\to 0}\eta\int_\Omega \big\langle u^\eta(x), \nabla\theta(x)\wedge\psi(x/\eta)\big\rangle \,\mathrm{d}x \\ &= \int_\Omega \theta(x)\int_Y \big\langle u_0(x,y), \operatorname{curl}\,\psi(y)\big\rangle \,\mathrm{d}y\,\mathrm{d}x\,.\end{aligned}$$

Combining the last equation and (3.2.4) yields the claim. $\square$

# Part II

# Homogenization of time-harmonic Maxwell's equations in general periodic microstructures

*What you say is very fine, Adso, and I thank you. The order that our mind imagines is like a net, or like a ladder, built to attain something. But afterward you must throw the ladder away, because you discover that, even if it was useful, it was meaningless. Er muoz gelîchesame die leiter abewerfen, sô er an ir ufgestigen.*

—Umberto Eco, *The name of the rose*

*He must, so to speak, throw away the ladder after he has climbed up it.*

—Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*

CHAPTER 4

# Introduction

In the mid nineteenth century, James C. Maxwell [Max61, Max65] published twenty equations summarising the state of electromagnetic theory in those days; he thereby added a critical correction to Ampère's circuital law by introducing a displacement current term. Oliver Heaviside later simplified this set of equations via vector notation; the resulting system of four equations involves four time- and space-dependent unknown fields: the electric field $\mathcal{E}$, the displacement field $\mathcal{D}$, the magnetic field $\mathcal{H}$ and the magnetic induction $\mathcal{B}$. The sources of the electromagnetic field are the electric current density $\mathcal{J}$ and the electric charge density $\rho$. Maxwell's equations then read

$$\operatorname{curl} \mathcal{E} = -\partial_t \mathcal{B}\,, \qquad \operatorname{curl} \mathcal{H} = \mathcal{J} + \partial_t \mathcal{D}\,, \tag{4.0.1a}$$
$$\operatorname{div} \mathcal{D} = \rho\,, \qquad \operatorname{div} \mathcal{B} = 0\,. \tag{4.0.1b}$$

The relations between the four fields $\mathcal{D}$, $\mathcal{B}$, $\mathcal{E}$, and $\mathcal{H}$ are determined by the (experimentally derived) material laws; sometimes referred to as the constitutive relations. In this work, we are interested in linear isotropic materials and thus the material laws are of the form

$$\mathcal{D} = \varepsilon \mathcal{E} \quad \text{and} \quad \mathcal{B} = \mu \mathcal{H}\,, \tag{4.0.2}$$

with space-dependent scalars $\varepsilon$ and $\mu$. We refer to $\varepsilon$ as the *permittivity* of the material and to $\mu$ as the *permeability* of the material.

**Simplifications.** We consider Maxwell's equations (4.0.1) in a bounded and simply connected Lipschitz domain $\Omega \subset \mathbb{R}^3$. For simplicity, we assume that there are neither charges nor currents; that is, $\rho = 0$ and $\mathcal{J} = 0$. Let us mention that we have not specified any boundary conditions on $\partial\Omega$, which can be used to encode source terms. The results presented in this part of the thesis are essentially independent of the boundary values of the electromagnetic field on $\partial\Omega$; this is why we will not specify any boundary condition in the sequel.

Instead of the full Maxwell system (4.0.1), one often studies a reduced set of equations—namely, the so called time-harmonic Maxwell equations. In order to derive this reduced system, we make the time-harmonic ansatz

$$\begin{aligned}\mathcal{H}(x,t) &= \mathrm{Re}\big\{H(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\big\}\,, \quad \mathcal{E}(x,t) = \mathrm{Re}\big\{E(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\big\}\\ \mathcal{B}(x,t) &= \mathrm{Re}\big\{B(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\big\}\,, \quad \mathcal{D}(x,t) = \mathrm{Re}\big\{D(x)\,\mathrm{e}^{-\mathrm{i}\omega t}\big\}\,,\end{aligned}$$

with time-independent fields $H, B, E, D\colon \Omega \to \mathbb{C}^3$ and a fixed frequency $\omega > 0$. Substituting the fields $H(x)\,\mathrm{e}^{-\mathrm{i}\omega t}$ and $E(x)\,\mathrm{e}^{-\mathrm{i}\omega t}$ into (4.0.1a) and using the constitutive relations (4.0.2) yield the *time-harmonic Maxwell equations*

$$\begin{cases}\operatorname{curl} E = \phantom{-}\mathrm{i}\omega\mu H & \text{in } \Omega\,, \qquad (4.0.3\mathrm{a})\\ \operatorname{curl} H = -\mathrm{i}\omega\varepsilon E & \text{in } \Omega\,. \qquad (4.0.3\mathrm{b})\end{cases}$$

The two equations in (4.0.1b) are automatically satisfied by $D = \varepsilon E$ and $B = \mu H$, which can be seen by taking the divergence in (4.0.3b) and (4.0.3a), respectively.

**Metamaterials.** For a homogeneous material both the permittivity $\varepsilon$ and the permeability $\mu$ are space-independent; they do depend, however, on the frequency $\omega$ of an electromagnetic wave hitting the material. Whereas the permittivity $\varepsilon = \varepsilon(\omega)$ can be a complex number with non-negative imaginary part for frequencies $\omega$ of visible light, all natural materials show a non-magnetic behaviour for this region of the spectrum; their permeability $\mu$ is hence close to $\mu_0 > 0$, the permeability of free space.

In recent years, there has been a great interest in constructing artificial materials, also called metamaterials, that behave as homogeneous materials with a negative permittivity $\varepsilon^{\mathrm{eff}}$ [FB97] or a non-trivial permeability $\mu^{\mathrm{eff}}$ [BS10, BBF09, BBF17]; we refer to the survey [Sch16] for an overview. Usually such metamaterials are periodically micro-structured. Assuming the scale of the microstructure is small compared to the wavelength of the fields, a rigorous mathematical analysis of the behaviour of the artificial material can be performed using periodic homogenisation theory.

The vital part in the construction of metamaterials with unusual optical properties is the choice of the microstructure. Using highly conductive split-ring resonators, Bouchitté and Schweizer [BS10] designed a metamaterial with negative permeability $\mathrm{Re}\{\mu^{\mathrm{eff}}(\omega)\} < 0$ for special frequencies $\omega$. A similar result holds for perfectly conducting split rings, which was later shown by Lipton and Schweizer [LS18b]. In [LS16], Lamacz and Schweizer construct a metamaterial that exhibits a negative permittivity $\mathrm{Re}\{\varepsilon^{\mathrm{eff}}\} < 0$ as well as a negative permeability $\mathrm{Re}\{\mu^{\mathrm{eff}}(\omega)\} < 0$ for a special frequency $\omega$. Bouchitté, Bourel, and Felbacq [BBF09, BBF17] exploited the so-called Mie resonances to construct a metamaterial with artificial magnetism from dielectric microstructures.

In contrast to the above mentioned articles, our intention in the subsequent chapters is not to propose a new microstructure leading to an artificial material with unusual properties, but to clarify the connection between the topology of the microstructure and transmission properties of the metamaterial. We thereby extend the result obtained in [SU18] and [PSU19].

## 4.1 Geometry and overview of main results

Although we are interested in general microstructures, we stay in the framework of periodic homogenization.

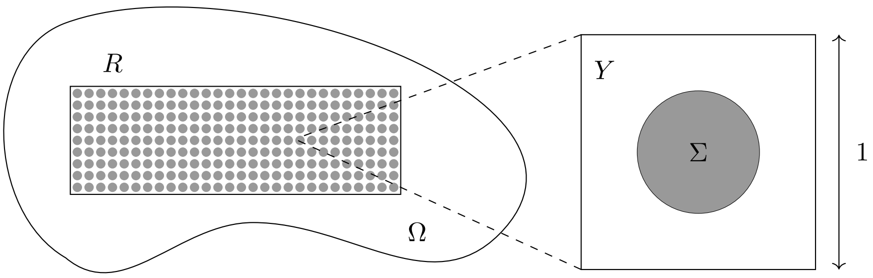


Figure 4.1: Left: The set of obstacles $\Sigma_\eta$, located in a subdomain $R \Subset \Omega$, is obtained by a "periodization" of microstructures $\Sigma \subset Y$. Right: Magnification of the microstructure in $R$.

**Geometric assumptions.** Besides $\Omega \subset \mathbb{R}^3$, we consider a simply connected Lipschitz domain $R \Subset \Omega$ in which the metamaterial is located; see Figure 4.1. The term metamaterial describes a periodic assembly of conductors and free space. More precisely, let $Y$ denote the closed unit cube $[0,1]^3 \subset \mathbb{R}^3$. As discussed in Section 2.2, we sometimes identify $Y$ with the flat three-dimensional torus $\mathbb{T}^3$. We make the following assumptions on the microstructure:

(A1) $\Sigma \subset Y$ is a subdomain of the flat three-dimensional torus $\mathbb{T}^3$ that has a Lipschitz boundary;

(A2) $\Sigma^* := Y \setminus \overline{\Sigma}$ is a connected and non-empty subset of $\mathbb{T}^3$.

We call a microstructure $\Sigma$ *admissible* provided assumptions (A1) and (A2) are satisfied.

Denoting the period of the metamaterial by $\eta > 0$, we set

$$\Sigma_\eta := \bigcup_{m \in \mathcal{M}} \eta(m + \Sigma) \quad \text{with} \quad \mathcal{M} := \left\{ m \in \mathbb{Z}^3 \,\middle|\, \eta(m+Y) \in R \right\}.$$

**Equations.** Let $R \Subset \Omega$ be as described above, and let $\Sigma \subset Y$ be an admissible microstructure. As discussed above, we may assume that the permeability $\mu$ of the microstructure $\Sigma$ coincides with the permeability $\mu_0$ of free space. On the other hand, the permittivity $\varepsilon$ of $\Sigma$ differs from the permittivity of free space $\varepsilon_0$ by a complex scalar, $\varepsilon = \varepsilon_\eta \varepsilon_0$ with $\varepsilon_\eta \colon \Omega \to \mathbb{C}$ being the relative permittivity. In $\Omega \setminus \overline{\Sigma}_\eta$ we set $\varepsilon_\eta = 1$ since we assume free space outside the metamaterial $\Sigma_\eta$. The value of $\varepsilon_\eta$ in $\Sigma_\eta$ is chosen below. The time-harmonic Maxwell equations for a metamaterial with period $\eta > 0$ then read

$$\begin{cases} \operatorname{curl} E^\eta = \phantom{-}\mathrm{i}\omega\mu_0 H^\eta & \text{in } \Omega\,, \quad (4.1.1a) \\ \operatorname{curl} H^\eta = -\mathrm{i}\omega\varepsilon_\eta\varepsilon_0 E^\eta & \text{in } \Omega\,. \quad (4.1.1b) \end{cases}$$

### Main results of Chapter 5.

The microstructure $\Sigma$ is assumed to be perfectly conducting, which formally amounts to $\varepsilon_\eta = +\infty$ in $\Sigma_\eta$. More rigorously, we require $E^\eta$ and $H^\eta$ to vanish identically in $\Sigma_\eta$ and impose equation (4.1.1b) only in $\Omega \setminus \overline{\Sigma}_\eta$.

**Cell problems and their analysis.** In the first part of Chapter 5, we derive the cell problems for the two-scale limits $E_0$ and $H_0$ of $(E^\eta)_\eta$ and $(H^\eta)_\eta$. It is well-known that the solution space $X^E$ of the cell problem for $E_0$ is at most three-dimensional; see, for instance, [SU18, LS18b]. In contrast, the solution space of the cell problem for $H_0$ can have more than three dimensions; indeed, Lipton and Schweizer [LS18b] consider a two-dimensional full torus $\Sigma \Subset Y$ and show that there are four linearly independent solutions to the cell problem of $H_0$. Our main result of this first part states, however, that the two-scale limit $H_0$ lies in an at most three-dimensional subspace $X^H$ of the solution space of the $H_0$-cell problem; see Lemma 5.5 and (5.2.11). Let us emphasize that this is no contradiction to the result by Lipton and Schweizer, because they consider a microstructure $\Sigma = \Sigma(\eta)$ that changes its topology as $\eta \to 0$.

In order to derive the effective equations, we need a characterisation of $X^E$ and $X^H$. It turns out that both spaces depend only on the topology of $\Sigma$. We can thus, for a large class of admissible $\Sigma$, determine bases for $X^E$ and $X^H$ by using special curves in $Y$—so-called *k-loops*—, which were first introduced in [SU18]; see Lemmas 5.2 and 5.9, and Section 5.4.

There is another, more abstract, way to characterise the spaces $X^E$ and $X^H$ using the so-called *geometric average*. This average is a linear map that assigns to each vector field $v \in L^2(\Sigma^*; \mathbb{C}^3)$ with curl $v = 0$ in $\Sigma^*$ a constant vector $\oint_{\Sigma^*} v \in \mathbb{C}^3$, which we call the geometric average of $v$. The geometric average can be applied to every element in $X^H$, allowing us to consider the following subspaces of $\mathbb{C}^3$:

$$A^E := \left\{ \int_Y u \;\middle|\; u \in X^E \right\} \qquad \text{and} \qquad A^H := \left\{ \oint_{\Sigma^*} v \;\middle|\; v \in X^H \right\}. \tag{4.1.2}$$

The maps $\int_Y : X^E \to A^E$ and $\oint_{\Sigma^*} : X^H \to A^H$ are vector space isomorphisms, and hence the characterisation of $X^E$ and $X^H$ is equivalent to the characterisation of $A^E$ and $A^H$. It turns out that $A^H$ is determined by $A^E$. That is, we can focus on the analysis of $A^E$ and can then deduce the characterisation of $X^E$, $A^H$, and $X^H$ from $A^E$. Let us stress that $A^E$ can be determined easily, for a large class of microstructures $\Sigma$, using $k$-loops.

**Effective material parameters and effective equations.** The second part of Chapter 5 is concerned with the definition of effective material parameters and the derivation of effective equations. The relative permittivity $\varepsilon^{\text{eff}}$ as well as the relative permeability $\mu^{\text{eff}}$ of the metamaterial $R$ are linear maps, which are defined via solutions of the cell problems. More precisely, $\varepsilon^{\text{eff}}$ is the *unique* linear map $\varepsilon^{\text{eff}} : A^E \to A^E$ that satisfies

$$\left\langle \varepsilon^{\text{eff}} \left( \int_Y u_1 \right), \int_Y u_2 \right\rangle = \int_Y \langle u_1, u_2 \rangle \quad \text{ for all } u_1, u_2 \in X^E .$$

The relative permeability $\mu^{\text{eff}} : A^H \to \mathbb{C}^3$ is defined by $\mu^{\text{eff}}(\oint_{\Sigma^*} v) := \int_Y v$. We recall that outside the metamaterial, we assume free space; this leads to the following definition of the effective material parameters:

$$\hat{\varepsilon}(x) := \mathrm{id}_{\mathbb{C}^{3\times 3}} \mathbb{1}_{\Omega \setminus \bar{R}}(x) + \varepsilon^{\text{eff}} \mathbb{1}_R(x) \text{ and } \hat{\mu}(x) := \mathrm{id}_{\mathbb{C}^{3\times 3}} \mathbb{1}_{\Omega \setminus \bar{R}}(x) + \mu^{\text{eff}} \mathbb{1}_R(x) .$$

What is noteworthy in the homogenisation of the time-harmonic Maxwell equations is that the effective magnetic field $\hat{H}$ is *not* to be understood in the standard way as a cell-average. Instead we define $\hat{H}\colon \Omega \to \mathbb{C}^3$ using the geometric average,

$$\hat{H}(x) := \oint_{\Sigma^*} H_0(x,\cdot)\,. \tag{4.1.3}$$

The effective electric field $\hat{E}\colon \Omega \to \mathbb{C}^3$ can also be defined as the geometric average of the two-scale limit $E_0$; it turns out, however, that for $E_0$ the volume average and the geometric average coincide. We therefore define $\hat{E}\colon \Omega \to \mathbb{C}^3$ as

$$\hat{E}(x) := \int_Y E_0(x,\cdot)\,. \tag{4.1.4}$$

We remind the reader that $E_0(x,\cdot)$ is an element of $X^E$, and that $H_0(x,\cdot)$ is an element of $X^H$. This fact together with the definitions (4.1.3) and (4.1.4) yield:

$$\hat{E}(x) \in A^E \qquad \text{and} \qquad \hat{H}(x) \in A^H \tag{4.1.5}$$

for almost all $x \in R$. The spaces $A^E$ and $A^H$ depend only on the topology of $\Sigma$ and can be determined easily using $k$-loops, for a large class of microstructures $\Sigma$. Moreover, (4.1.5) allows us to determine whether transmission through the metamaterial is possible, and if transmission is possible, in which directions the fields can point.

Our main contribution in this second part of Chapter 5 is the derivation of the effective system; see Theorem 5.12. Denote by $\pi_{A^E}: \mathbb{C}^3 \to \mathbb{C}^3$ the orthogonal projection onto $A^E$. The effective field $(\hat{E}, \hat{H})\colon \Omega \to \mathbb{C}^3 \times \mathbb{C}^3$ satisfies the following equations:

$$\begin{cases} \operatorname{curl} \hat{E} = \mathrm{i}\omega\mu_0\hat{\mu}\hat{H} & \text{in } \Omega\,, & (4.1.6a)\\ \pi_{A^E}\big(\operatorname{curl} \hat{H}\big) = -\mathrm{i}\omega\varepsilon_0\pi_{A^E}\big(\hat{\varepsilon}\hat{E}\big) & \text{in } \Omega\,, & (4.1.6b)\\ \operatorname{curl} \hat{H} = -\mathrm{i}\omega\varepsilon_0\hat{E} & \text{in } \Omega \setminus \overline{R}\,. & (4.1.6c) \end{cases}$$

The effective system has the form of time-harmonic Maxwell's equations for a linear material. We note, however, that the material parameters $\varepsilon_0\hat{\varepsilon}$ and $\mu_0\hat{\mu}$ are, in general, tensors—although $\varepsilon_0\varepsilon_\eta$ and $\mu_0$ are scalars. Depending on the space $A^E$, equation (4.1.6b) does not determine all components of curl $\hat{H}$. To the best knowledge of the author, the first effective Maxwell system, in which not all components of curl $\hat{H}$ are determined, appeared in [SU18]; see also [PSU19].

Equations (4.1.6a) and (4.1.6b) imply interface conditions across the boundary $\partial R$ of the metamaterial. For a microstructure $\Sigma$ that is compactly contained in $(0,1)^3$, we will see that $A^E = \mathbb{C}^3$ and consequently, the effective system coincides with time-harmonic Maxwell's equations for some linear material. In this situation, the tangential components of $\hat{E}$ and $\hat{H}$ do not jump across $\partial R$. Of particular interest are those microstructures $\Sigma$ for which some components of the effective field $(\hat{E}, \hat{H})$ vanish while other components satisfy certain parts of Maxwell's equations. This occurs, for instance, for cylindrical or plate structures; see Section 5.4.

### Main results of Chapter 6

In this chapter, we consider the more realistic high contrast case; that is, the microstructure $\Sigma$ is highly conductive. The relative permeability $\varepsilon_\eta$ is thus defined as $\varepsilon_\eta = \varepsilon_r \eta^{-2}$ in $\Sigma_\eta$ with $\mathrm{Im}\{\varepsilon_r\} > 0$. This scaling is not new in the theory of homogenisation and leads to metamaterials with astonishing optical properties; see, for instance, [Zhi05, FB05, CC15, LS16, BBF17]. The assumption $\mathrm{Im}\{\varepsilon_r\} > 0$ is vital to ensure existence and uniqueness of solutions to the cell problem.

**Cell problems and their analysis.** Similar to Chapter 5, we first derive the cell problems for the two-scale limits $E_0$ and $H_0$ of $(E^\eta)_\eta$ and $(H^\eta)_\eta$. These cell problems are well-known; see, for instance, [BS10, BBF09, LS13, LS16]. Moreover, the cell problem for $E_0$ coincides with the cell problem of $E_0$ in the case of a perfectly conducting microstructure $\Sigma$. We can therefore use the same techniques to analyse the solution space $X^E$ as in Chapter 5.

The analysis of the cell problem for $H_0$ is much more involved, though. Besides the fields $E^\eta$ and $H^\eta$ one considers a third quantity $J^\eta$, the rescaled displacement current, which was introduced in [BS10]. Denoting the two-scale limit of $(J^\eta)_\eta$ by $J_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$, the cell problem of $H_0$ reads

$$\begin{cases} \mathrm{curl}_y\, H_0(x,\cdot) = -\mathrm{i}\omega\varepsilon_0 J_0(x,\cdot) & \text{in } Y\,, & (4.1.7\mathrm{a}) \\ \mathrm{div}_y\; H_0(x,\cdot) = 0 & \text{in } Y\,, & (4.1.7\mathrm{b}) \\ \mathrm{curl}_y\, J_0(x,\cdot) = \;\; \mathrm{i}\omega\mu_0\varepsilon_r H_0(x,\cdot) & \text{in } \Sigma\,, & (4.1.7\mathrm{c}) \\ \qquad J_0(x,\cdot) = 0 & \text{in } \Sigma^*\,, & (4.1.7\mathrm{d}) \end{cases}$$

for almost all $x \in R$. The solution space to this cell problem can have more than three dimensions. Indeed, Bouchitté and Schweizer [BS10] consider a two-dimensional full torus $\Sigma \Subset Y$ and show that (4.1.7) admits four linearly independent solutions.

Our main result in the first part of this chapter is a new variational equation involving $J_0$ and $H_0$, from which we deduce that $H_0$ lies in an at most three-dimensional subspace of the solution space to (4.1.7). More precisely, we show that for every vector field $v \in L^2_\sharp(Y; \mathbb{C}^3)$ with $\mathrm{curl}\, v = 0$ in $\Sigma^*$ and $\oint_{\Sigma^*} v = 0$ there holds

$$\int_\Sigma \big\langle J_0(x,\cdot), \mathrm{curl}\; v \big\rangle = \mathrm{i}\omega\varepsilon_r\mu_0 \int_Y \big\langle H_0(x,\cdot), v \big\rangle \tag{4.1.8}$$

for almost all $x \in R$. Using properties of the geometric average it is straightforward to show that (4.1.8) implies equations (4.1.7b) and (4.1.7c); the converse, however, is not true, in general. That is, the variational identity (4.1.8) contains more information than the two equations.

Combining (4.1.7a) and (4.1.8) we obtain the following variational characterisation of $H_0$ in $R$,

$$\int_\Sigma \big\langle \mathrm{curl}\; H_0(x,\cdot), \mathrm{curl}\; v \big\rangle = \omega^2 \varepsilon_r \varepsilon_0 \mu_0 \int_Y \big\langle H_0(x,\cdot), v \big\rangle\,, \tag{4.1.9}$$

which holds for all $v \in L^2_\sharp(Y; \mathbb{C}^3)$ with $\mathrm{curl}\; v = 0$ in $\Sigma^*$ and $\oint_{\Sigma^*} v = 0$. Thus, the variational identity (4.1.8) allows us to reduce the cell problem of $H_0$ from a system of coupled partial differential equations to one variational equation, which can be analysed by the Lax-Milgram lemma.

The variational characterisation (4.1.9) was also found by Bouchitté, Bourel, and Felbacq in [BBF17]; they consider, however, only microstructures $\Sigma \Subset (0,1)^3$ which are simply connected and for which $\Sigma^* \subset [0,1]^3$ is simply connected, as well. Moreover, in [BBF17] the variational identity (4.1.9) is directly derived from Maxwell's equations (4.1.1) using more advanced results from the theory of two-scale convergence. Our approach—that is, taking the detour to derive equations (4.1.7a) and (4.1.8) first and then combine them—, yields a more elementary proof of (4.1.9).

Let us comment on the solution space of (4.1.9). The effective system (4.1.7) implies that $H_0(x,\cdot) \in H^1_\sharp(Y;\mathbb{C}^3)$ is curl-free in $\Sigma^*$, for almost all $x \in R$. We therefore define the solution space $X^H$ of (4.1.9) to be the space of vector fields $w \in H^1_\sharp(Y;\mathbb{C}^3)$ with curl $w = 0$ in $\Sigma^*$ such that (4.1.8) holds after substituting $H_0(x,\cdot)$ by $w$. As each $w \in X^H$ satisfies curl $w = 0$ in $\Sigma^*$, we can apply the geometric average and obtain a constant vector $\oint_{\Sigma^*} w \in \mathbb{C}^3$. The linear map $\oint_{\Sigma^*} X^H \to \mathbb{C}^3$ is injective, since $\mathrm{Im}\{\varepsilon_r\} > 0$; consequently, $\dim X^H \leq 3$. Let us define

$$A^H := \left\{ \oint_{\Sigma^*} w \,\middle|\, w \in X^H \right\}.$$

We emphasize that the definition of the space $A^H$ seems to be the same as (4.1.5) from Chapter 5; however, the solution space $X^H$ is defined differently in each chapter. The geometric average $\oint_{\Sigma^*} : X^H \to A^H$ is again a vector space isomorphism, and thus allows us to characterise $A^H$ instead of $X^H$. Similar to Chapter 5, we show that $A^H$ is determined by $A^E$. That is, also in the high contrast case, the analysis of the solution spaces $X^E$ and $X^H$ reduces to a characterisation of $A^E$.

As the cell problem of $E_0$ is the same in Chapters 5 and 6, the spaces $A^E$ coincide as well. Therefore, from an abstract point of view, if we understand the case of perfectly conducting microstructures, then we also understand the high contrast case, and vice versa. Let us emphasis, however, that although the solution spaces of the $H_0$-cell problem in both cases are isomorphic they do *not* coincide, and this influences, for instance, the relative permeability $\mu^{\mathrm{eff}}$ of $R$.

**Effective material parameters and equations.** The effective material parameters $\hat{\varepsilon}$ and $\hat{\mu}$ are defined as in Chapter 5. We also define the effective electromagnetic field $(\hat{E}, \hat{H})$ as in (4.1.3) and (4.1.4), and obtain (4.1.6) as the effective system. This is not surprising but rather an evidence that the effective field $(\hat{E}, \hat{H})$ is defined in the right way, since we obtain an effective system which has the structure of Maxwell's equations for a linear material.

## Concluding comments

Besides our new definition of the geometric average and $k$-loops, we use mainly the tool of two-scale convergence in Chapters 5 and 6.

**Literature.** In [BBF09, BBF17], Bouchitté, Bourel, and Felbacq study the behaviour of solutions $(E^\eta, H^\eta)_\eta$ of (4.1.1) as $\eta \to 0$ based on the following assumptions on the microstructure: $\Sigma$ is connected and compactly contained in $(0,1)^3$ and $\Sigma^* \subset (0,1)^3$ is simply connected. Let us stress that the emphasis in the articles [BBF09, BBF17] is more on constructing metamaterials with a negative effective permeability than on the connection between the topology of $\Sigma$ and transmission properties of the metamaterial. A first step to study this connection is presented in [SU18], in which microstructures $\Sigma$ are allowed that satisfy the following property: every vector field $v \in L^2_\sharp(Y;\mathbb{C}^3)$ with vanishing distributional curl in $\Sigma^*$ can be written as $v = \nabla\Theta + c$ in $\Sigma^*$ for some periodic potential $\Theta \in H^1_\sharp(\Sigma^*;\mathbb{C})$ and some constant vector $c \in \mathbb{C}^3$. Moreover, $\Sigma$ is assumed to be perfectly conducting.

Perfectly conducting microstructures $\Sigma$ are also studied in [PSU19]; there $\Sigma$ is only assumed to satisfy assumptions (A1) and (A2). The main topic of [PSU19], however, is the generalisation of the geometric average to this large class of microstructures. Nevertheless, parts of our results in Chapter 5 coincide with conclusions from [PSU19].

**Technical remark.** As described in the previous sections, we show in Chapters 5 and 6 that the classical cell problems of $H_0$ do not contain all the information about $H_0$. Indeed, in the case of a perfectly conducting microstructure $\Sigma$, our analysis reveals that $H_0$ lies in an at most three-dimensional subspace $X^H$ of the solution space to the cell problem; in the high contrast case, we obtain the new relation (4.1.8) between $J_0$ and $H_0$. We would like to stress that these two new information are obtained in the same way—namely, by testing equation (4.1.1a) with the *same* vector field. More precisely, we choose a cut-off function $\theta \in C_c^\infty(\Omega;\mathbb{R})$, a vector field $v \in L^2_\sharp(Y;\mathbb{C}^3)$ with $\operatorname{curl} v = 0$ in $\Sigma^*$ and $\oint_{\Sigma^*} v = 0$, and define the test vector field $\varphi \in C^0(\overline{\Omega}; L^2_\sharp(Y;\mathbb{C}^3))$ as $\varphi(x,y) := \theta(x)v(y)$. The new information about $H_0$ is then derived by testing equation (4.1.1b) with $\varphi(\cdot,\cdot/\eta)$, using that $(E^\eta)_\eta$ and $(H^\eta)_\eta$ two-scale converge, and from the definition of the geometric average.

**Averaging electromagnetic fields.** The effective magnetic field $\hat{H}$ is not to be understood as a cell-average but as a geometric average; see its definition (4.1.3). In fact, the same holds true for the effective electric field $\hat{E}$. The reason for choosing a different averaging method is the following: the electric field as well as the magnetic field may be viewed as differential one-forms. As such, line integrals are the natural objects to average electromagnetic fields. In the physics literature this is well-known; see, for instance, [PHRS99] or [Ram05, Chapter 2.4]. This fact has also been utilised in the mathematical homogenisation of Maxwell's equations; we mention [KS08] for a two-dimensional homogenisation problem, and [BBF09, BBF17] for an analysis of the full three-dimensional Maxwell system.

Besides this mathematical argument, there is also a physical reason which suggests that taking the volume average as the effective field is not the right choice. Assume $(E^\eta, H^\eta)_\eta$ is a bounded sequence of distributional solutions to time-harmonic Maxwell's equations (4.1.1), and let $H$ denote the weak $L^2$-limit of $(H^\eta)_\eta$. It turns out that the tangential trace of $H$ jumps across the interface $\partial R$, which is not desirable for a physical quantity. Moreover, having the effective system (4.1.6) for $(\hat{E}, \hat{H})$ at hand, and using the fact that $H = \mu^{\text{eff}}\hat{H}$ in $R$, the effective system in terms of $H$ read

$$\begin{cases} \operatorname{curl} \hat{E} = \mathrm{i}\omega\mu_0 H & \text{in } R\,, \\ \pi_{A^E}\big(\operatorname{curl}(\mu^{\text{eff}})^{-1}\hat{H}\big) = -\mathrm{i}\omega\varepsilon_0\pi_{A^E}\big(\hat{\varepsilon}\hat{E}\big) & \text{in } R\,. \end{cases}$$

This set of equations has apparently not the form of Maxwell's equations for some linear material.

Line integrals are well suited for smooth vector fields. However, for less regular fields, which one encounters in the framework of periodic homogenization, a generalisation of line integrals is necessary. Such a generalisation—the so-called geometric average—was first accomplished by Bouchitté, Bourel, and

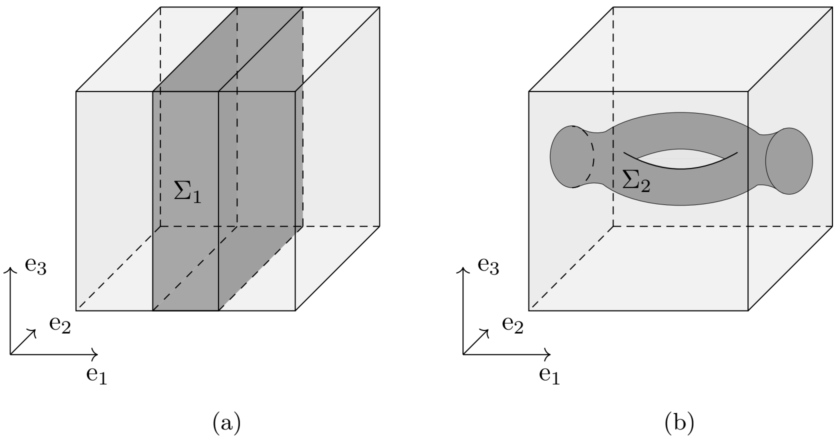


Figure 4.2: The figure shows two admissible microstructures. The unit cell $Y$ is represented by the cube; the dark grey areas represent the microstructures. (a) $\Sigma_1$ represents a metal plate, which is not compactly contained in $(0,1)^3$. (b) $\Sigma_2$ is a torus which connects opposite faces of $Y$. Neither is $\Sigma_2$ compactly contained in $(0,1)^3$ nor is its complement simply connected.

Felbacq [BBF09] and later extended to be applicable to more general microstructures $\Sigma$; see [PSU19] for an overview of the different definitions. Defining the effective fields via the geometric average leads to the effective system (4.1.6), which is in Maxwell form and which expresses, in particular, that the tangential components of $\hat{H}$ do not jump across the interface $\partial R$. Furthermore, the effective magnetic induction $\hat{B} \coloneqq \mu_0 \mu^{\mathrm{eff}} \hat{H}$ agrees with the weak $L^2$-limit of $B^\eta \coloneqq \mu_0 H^\eta$ in the metamaterial $R$.

## 4.2 Main ideas discussed by two examples

The objective of this section is to present the ideas, which are developed in [SU18, PSU19] as well as in this thesis, using two microstructures; in doing so, we postpone involved technical details to later sections. The two microstructures that we consider are sketched in Figures 4.2 (a) and 4.2 (b).

### The metal plate

Choose $\gamma \in (0, 1/2)$ and set

$$\Sigma_1 \coloneqq \left\{ (y_1, y_2, y_3) \in Y \;\middle|\; y_1 \in \left( \frac{1}{2} - \gamma, \frac{1}{2} + \gamma \right) \right\}.$$

A sketch of $\Sigma_1$ is given in Figure 4.2 (a).

Bouchitté and Schweizer analyse in [BS13] the high contrast case, but only in a two-dimensional setting. We note that the results from [BBF09, BBF17] cannot be applied since $\Sigma_1$ is not compactly contained in $(0,1)^3$.

We focus in this section on the case that $\Sigma_1$ is perfectly conducting, which is also discussed in [SU18]; see Section 6.4 for a discussion of the high contrast case.

**Cell problem of $E_0$.** We first analyse the cell problem for the two-scale limit $E_0$. Let us recall from the previous section that the identical cell problem appears in the high contrast case; the following analysis applies thus to both cases. For almost all $x \in R$, the field $E_0 = E_0(x, \cdot) \in L^2_\sharp(Y; \mathbb{C}^3)$ satisfies

$$\begin{cases} \operatorname{curl}_y E_0 = 0 \quad \text{in } Y\,, & (4.2.1\text{a}) \\ \operatorname{div}_y E_0 = 0 \quad \text{in } \Sigma_1^*\,, & (4.2.1\text{b}) \\ E_0 = 0 \quad \text{in } \Sigma_1\,, & (4.2.1\text{c}) \end{cases}$$

in the distributional sense. We ask: how many linearly independent solutions of (4.2.1) exist?

In order to answer this question, we first ask: what are necessary conditions a solution to (4.2.1) has to satisfy? Assume $u \in L^2_\sharp(Y; \mathbb{C}^3)$ is a solution to (4.2.1). Equation (4.2.1a) implies the decomposition $u = \nabla\Theta + c$ in $Y$ for some *periodic* scalar potential $\Theta \in H^1_\sharp(Y; \mathbb{C}^3)$ and a constant vector $c \in \mathbb{C}^3$. Substituting this decomposition into equation (4.2.1c) yields $\nabla\Theta = -c$ in $\Sigma_1$, from which we deduce that

$$\Theta(y_1, y_2, y_3) = -c_1 y_1 - c_2 y_2 - c_3 y_3 \quad \text{for } (y_1, y_2, y_3) \in \Sigma_1\,,$$

where $c_j = \langle c, \mathrm{e}_j\rangle$ for $j \in \{1, 2, 3\}$. As $\Theta$ is periodic, the following identities have to be satisfied

$$\Theta(y_1, 0, y_3) = \Theta(y_1, 1, y_3) \quad \text{and} \quad \Theta(y_1, y_2, 0) = \Theta(y_1, y_2, 1)$$

for all $(y_1, y_2, y_3) \in \Sigma_1$. Hence, $c_2 = c_3 = 0$. On the other hand, there is no restriction on $c_1$.

The above argument can be summarised as follows: if it is possible to connect two opposite points on the faces $\{y_k = 0\}$ and $\{y_k = 1\}$ by a *continuous curve in* $\Sigma_1$, then $c_k = 0$ with $k \in \{1, 2, 3\}$. Indeed, $\gamma_2 \colon [0, 1] \to \Sigma_1$, $\gamma_2(t) := (y_1, t, y_2)$ is a continuous curve in $\Sigma_1$ connecting the opposite points $(y_1, 0, y_3)$ and $(y_1, 1, y_3)$, for all $y_1 \in (-\gamma, \gamma)$ and $y_3 \in [0, 1]$. Similarly, the continuous curve $\gamma_3 \colon [0, 1] \to \Sigma_1$, $\gamma_3(t) := (y_1, y_2, t)$ connects $(y_1, y_2, 0)$ with $(y_1, y_2, 1)$ for all $y_1 \in (-\gamma, \gamma)$ and $y_2 \in [0, 1]$. The curves $\gamma_2$ and $\gamma_3$ are examples of so-called *k-loops in* $\Sigma_1$; more precisely, $\gamma_2$ is an $\mathrm{e}_2$-loop in $\Sigma_1$ and $\gamma_3$ is an $\mathrm{e}_3$-loop in $\Sigma_1$.

We also note that there is *no* continuous curve in $\Sigma_1$ connecting the faces $\{y_1 = 0\}$ and $\{y_1 = 1\}$. This implies the existence of a periodic potential $\Theta \in H^1_\sharp(Y; \mathbb{C})$ with $\nabla\Theta = \mathrm{e}_1$ in $\Sigma_1$. Indeed, a solution $\Theta \in H^1_\sharp(Y; \mathbb{C})$ to

$$\begin{cases} -\Delta\Theta = 0 \quad \text{in } \Sigma_1^*\,, & (4.2.2\text{a}) \\ \Theta = y_1 \quad \text{in } \Sigma_1\,, & (4.2.2\text{b}) \end{cases}$$

satisfies $\nabla\Theta = \mathrm{e}_1$ in $\Sigma_1$. Let us remark, however, that there are admissible microstructures $\Sigma$ for which there is no $\Theta \in H^1_\sharp(Y; \mathbb{C})$ with $\nabla\Theta = \mathrm{e}_1$ in $\Sigma$ although the faces $\{y_1 = 0\}$ and $\{y_1 = 1\}$ cannot be connected by a continuous path in $\Sigma$; see Remarks 3 on page 49.

The above considerations show that every solution $u \in L^2_\sharp(Y; \mathbb{C}^3)$ of (4.2.1) is of the form $u = \nabla\Theta + \lambda\,\mathrm{e}_1$ with $\Theta \in H^1_\sharp(Y; \mathbb{C}^3)$ and $\lambda \in \mathbb{C}$, and thus

$$A^E := \left\{ \int_Y u \;\middle|\; u \in X^E \right\} \subset \mathbb{C}\,\mathrm{e}_1\,.$$

We claim that $A^E = \mathbb{C}\,\mathrm{e}_1$. Indeed, for fixed $\lambda \in \mathbb{C}$, the vector field $E^1\colon Y \to \mathbb{C}^3$, $E^1 := \alpha^{-1}\lambda \mathbb{1}_{\Sigma_1^*}\,\mathrm{e}_1$ with $\alpha := |\Sigma_1^*|$ is a solution to (4.2.1) with $\int_Y E^1 = \lambda\,\mathrm{e}_1$.

In order to answer the question how many linearly independent solutions the system (4.2.1) admits, we show that the linear map $\int_Y\colon X^E \to A^E$ is a vector space isomorphism. By definition of $A^E$, this map is onto. Thus, we are left to prove that every $u \in X^E$ with $\int_Y u = 0$ vanishes identically. Choose such a vector field $u \in X^E$. Then $u = \nabla\Theta + c$ in $Y$ with $c = 0$, since $\int_Y u = 0$. Consequently, due to (4.2.1b) and (4.2.1c), the potential $\Theta \in H^1_\sharp(Y;\mathbb{C}^3)$ solves

$$\begin{cases} -\Delta\Theta = 0 & \text{in } \Sigma_1^*\,, \\ \quad\;\; \Theta = C & \text{in } \Sigma_1\,, \end{cases}$$

for some $C \in \mathbb{C}$. This Dirichlet problem has only the constant solution $\Theta \equiv C$, and hence $u = 0$. We note that this proof of the injectivity of $\int_Y\colon X^E \to A^E$ works for any admissible microstructure $\Sigma$.

We have thus shown that the solution space $X^E$ to (4.2.1) is isomorphic to $A^E = \mathbb{C}\,\mathrm{e}_1$. Moreover, due to its definition in (4.1.4), the effective field $\hat{E}(x) \in A^E = \mathbb{C}\,\mathrm{e}_1$ for almost all $x \in R$.

**Cell problem of $H_0$.** If $\Sigma_1$ is perfectly conducting, then the two-scale limit $H_0(x,\cdot) \in L^2(Y;\mathbb{C}^3)$ satisfies, for almost all $x \in R$,

$$\begin{cases} \operatorname{curl}_y H_0 = 0 & \text{in } \Sigma_1^*\,, & (4.2.3\text{a}) \\ \operatorname{div}_y H_0 = 0 & \text{in } Y\,, & (4.2.3\text{b}) \\ \qquad\; H_0 = 0 & \text{in } \Sigma_1\,, & (4.2.3\text{c}) \end{cases}$$

in the distributional sense. We ask again: how many linearly independent solutions to (4.2.3) exist?

Let $v \in L^2_\sharp(Y;\mathbb{C}^3)$ be a solution to the cell problem (4.2.3). Due to the special structure of $\Sigma_1$, equation (4.2.3a) implies that $v = \nabla\Theta + c$ in $\Sigma_1^*$ for some periodic potential $\Theta \in H^1_\sharp(\Sigma_1^*;\mathbb{C})$ and some constant vector $c \in \mathbb{C}^3$. In contrast to the decomposition of a solution to the $E_0$-cell problem (4.2.1), the constant vector $c$ in the decomposition of $v$ is *not* uniquely defined. Indeed, the function $\Phi\colon \Sigma_1^* \to \mathbb{R}$,

$$\Phi(y_1,y_2,y_3) := \begin{cases} y_1 & \text{for } y_1 \in \left[0, \dfrac{1}{2} - \gamma\right) \\ y_1 - 1 & \text{for } y_1 \in \left(\dfrac{1}{2} + \gamma, 1\right] \end{cases} \tag{4.2.4}$$

is an element of $H^1_\sharp(\Sigma_1^*;\mathbb{C})$ with $\nabla\Phi = \mathrm{e}_1$. Hence, $v$ admits the two decompositions $v = \nabla\Theta + c$ and $v = \nabla(\Theta + \Phi) + (c - \mathrm{e}_1)$. On the other hand, there cannot be a potential $\Psi \in H^1_\sharp(\Sigma_1^*;\mathbb{C})$ with $\nabla\Psi = \mathrm{e}_2$. For there was such a potential $\Psi$, it has to be an affine function of $y_2$ which is periodic as well. But every periodic and affine function is constant and hence $\mathrm{e}_2 = \nabla\Psi = 0$. A similar reasoning shows there cannot exist a potential $\Psi \in H^1_\sharp(\Sigma_1^*;\mathbb{C})$ with $\nabla\Psi = \mathrm{e}_3$.

Given a solution $v \in L^2_\sharp(Y;\mathbb{C}^3)$ of (4.2.3) that can be written as $v = \nabla\Theta + c$ in $\Sigma_1^*$ for some $\Theta \in H^1_\sharp(\Sigma_1^*;\mathbb{C})$ and $c \in \mathbb{C}^3$. The geometric average $\oint_{\Sigma_1^*} v \in \mathbb{C}^3$ of the field $v$ is defined as

$$\oint_{\Sigma_1^*} v := \langle c, \mathrm{e}_2\rangle\,\mathrm{e}_2 + \langle c, \mathrm{e}_3\rangle\,\mathrm{e}_3\ . \tag{4.2.5}$$

Denote the solution space to (4.2.3) by $X^H$. We claim that every $v \in X^H$ with $\oint_{\Sigma_1^*} v = 0$ vanishes identically in $Y$. As the geometric average of $v$ vanishes, $v = \nabla\Theta + \lambda\,\mathrm{e}_1$ in $\Sigma_1^*$ for some $\Theta \in H^1_\sharp(\Sigma_1^*;\mathbb{C})$ and some $\lambda \in \mathbb{C}$. Equations (4.2.3b) and (4.2.3c) imply that $\Theta$ is a weak solution to

$$\begin{cases} -\Delta\Theta = 0 & \text{in } \Sigma_1^*\,, \qquad (4.2.6a)\\ \partial_\nu\Theta = -\lambda\langle \mathrm{e}_1, \nu\rangle & \text{on } \partial\Sigma_1^*\,, \qquad (4.2.6b)\end{cases}$$

where $\nu$ is the outward pointing normal. This Neumann problem is uniquely solvable up to an additive constant. Setting $\Theta := -\lambda\Phi$ provides us with a weak solution to this Neumann problem, where $\Phi$ is defined in (4.2.4). Thus, any weak solution $\Theta \in H^1_\sharp(\Sigma_1^*;\mathbb{C})$ of (4.2.6) satisfies $\nabla\Theta = -\lambda\nabla\Phi = -\lambda\,\mathrm{e}_1$ in $\Sigma_1^*$. From this, we deduce that $v = \nabla\Theta + \lambda\,\mathrm{e}_1 = 0$ in $\Sigma_1^*$. Due to (4.2.3c), the field $v$ vanishes identically in $Y$. We have thus proved that the geometric average $\oint_{\Sigma_1^*} \colon X^H \to \mathbb{C}^3$ is an injective linear map; that is, there are at most three linearly independent solutions to (4.2.3).

By the definition of the geometric average in (4.2.5) there holds

$$A^H := \left\{ \oint_{\Sigma_1^*} v \;\middle|\; v \in X^H \right\} \subset \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3\ .$$

We claim that $A^H = \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$. Indeed, given any $c \in \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$, one readily checks that $v := (c_2\,\mathrm{e}_2 + c_3\,\mathrm{e}_3)\mathbb{1}_{\Sigma_1^*}$ is an element of $X^H$ with $\oint_{\Sigma_1^*} v = c$. Consequently, the geometric average is a vectors space isomorphism between $A^H$ and $\mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$. This together with (4.1.3) implies that $\hat{H}(x) \in A^H = \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$ for almost all $x \in R$.

**The effective material parameters and equations.** The relative permittivity $\varepsilon^{\mathrm{eff}}$ of the metamaterial is given as the diagonal matrix

$$\varepsilon^{\mathrm{eff}} = \operatorname{diag}\Big( \int_Y \big|E^1\big|^2, 0, 0 \Big) = \alpha^{-1} \operatorname{diag}\big(1, 0, 0\big) \in \mathbb{R}^{3\times 3}\,,$$

where the field $E^1$, defined above, is a basis of $X^E$.

The vector fields $H^2, H^3 \colon Y \to \mathbb{R}^3$, $H^2 := \mathbb{1}_{\Sigma^*}\,\mathrm{e}_2$ and $H^3 := \mathbb{1}_{\Sigma^*}\,\mathrm{e}_3$ are solutions of the $H_0$-cell problem with $\oint_{\Sigma_1^*} H^j = \mathrm{e}_j$ for $j \in \{2, 3\}$. Thus, the relative permeability $\mu^{\mathrm{eff}}$ of the metamaterial $R$ is given by

$$\mu^{\mathrm{eff}} = \Big( 0 \,\Big|\, \int_Y H^2 \,\Big|\, \int_Y H^3 \Big) = \alpha \operatorname{diag}\big(0, 1, 1\big) \in \mathbb{R}^{3\times 3}\,.$$

Defining the effective permittivity $\hat{\varepsilon} \colon \Omega \to \mathbb{R}^{3\times 3}$ and the effective permeability $\hat{\mu} \colon \Omega \to \mathbb{R}^{3\times 3}$ as in Section 4.1, the effective system (4.1.6) for the microstructure $\Sigma_1$ reads

$$\begin{cases} \operatorname{curl} \hat{E} = \mathrm{i}\omega\mu_0\hat{\mu}\hat{H} & \text{in } \Omega\,,\\ \partial_2\hat{H}_3 - \partial_3\hat{H}_2 = -\mathrm{i}\omega\varepsilon_0(\hat{\varepsilon}\hat{E})_1 & \text{in } \Omega\,,\\ \operatorname{curl} \hat{H} = -\mathrm{i}\omega\varepsilon_0\hat{E} & \text{in } \Omega \setminus \overline{R}\,. \end{cases}$$

### Torus touching the boundary

In this section, we consider a two-dimensional full torus $\Sigma_2$ which connects the two opposite faces $\{y_1 = 0\}$ and $\{y_1 = 1\}$ of the unit cube $Y$; see Figure 4.2 (b) for a sketch of $\Sigma_2$.

The high contrast case, which we consider in this section, has never been treated before. Indeed, the results from [BBF09, BBF17] are not applicable since $\Sigma_2$ is neither compactly contained in $Y$ nor is $\Sigma_2^*$ simply connected in $Y$; we can also not apply the result from [SU18, PSU19] since $\Sigma_2$ is not perfectly conducting. We discuss the case that $\Sigma_2$ is perfectly conducting in Section 5.4.

**Cell problem of $E_0$.** After replacing $\Sigma_1$ by $\Sigma_2$, the cell problem for the two-scale limit $E_0$ is identical to (4.2.1). We can therefore use the techniques discussed above to determine the solution space $X^E$. Let us first observe that there is a continuous curve $\gamma_1 \colon [0,1] \to \Sigma_2$ connecting the faces $\{y_1 = 0\}$ and $\{y_1 = 1\}$. In contrast, there is no continuous curve in $\Sigma_2$ connecting the opposite faces $\{y_k = 0\}$ and $\{y_k = 1\}$ for $k \in \{2, 3\}$. Thus, every solution $u \in L^2_\sharp(Y;\mathbb{C}^3)$ to (4.2.1) is of the form $u = \nabla\Theta + c_2\,\mathrm{e}_2 + c_3\,\mathrm{e}_3$ in $Y$ for some $\Theta \in H^1_\sharp(Y;\mathbb{C})$ and $c_2, c_3 \in \mathbb{C}$. Consequently,

$$A^E \coloneqq \left\{ \int_Y u \;\middle|\; u \in X^E \right\} \subset \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3 \ .$$

As there are no continuous paths in $\Sigma_2$ connecting the faces $\{y_k = 0\}$ and $\{y_k = 1\}$ for $k \in \{2, 3\}$, we can, similar to (4.2.2), construct potentials $\Theta_2, \Theta_3 \in H^1_\sharp(Y;\mathbb{C})$ that are harmonic in $\Sigma_2^*$ and that satisfy $\nabla\Theta_2 = \mathrm{e}_2$ in $\Sigma_2$ and $\nabla\Theta_3 = \mathrm{e}_3$ in $\Sigma_2$. These two potentials allow us to conclude that $A^E = \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$. Indeed, given any vector $c \in \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$, one readily checks that the vector field $u \colon Y \to \mathbb{C}^3$,

$$u \coloneqq -\langle c, \mathrm{e}_2\rangle \nabla\Theta_2 - \langle c, \mathrm{e}_3\rangle \nabla\Theta_3 + c$$

is an element of $X^E$ with $\int_Y u = c$.

In the previous section, we showed that $\int_Y \colon X^E \to A^E$ is injective; the same proof works for $\Sigma_2$. Consequently, the volume average yields a vector space isomorphism between $X^E$ and $A^E = \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$. Moreover, the effective field satisfies $\hat{E}(x) \in \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$ for almost all $x \in R$.

**Cell problem of $H_0$.** Besides $E_0$ and $H_0$, we introduce a new quantity—namely, the rescaled displacement current $J_0 \in L^2(\Omega \times Y;\mathbb{C}^3)$. For almost all $x \in R$, the fields $H_0 = H_0(x,\cdot) \in L^2_\sharp(Y;\mathbb{C}^3)$ and $J_0 = J_0(x,\cdot) \in L^2_\sharp(Y;\mathbb{C}^3)$ are distributional solutions to (4.1.7).

We saw in the previous section that the geometric average can be used to characterise the solution space to the $H_0$-cell problem. Unfortunately, we cannot simply apply the definition of the geometric average from the previous section, although the field $H_0(x,\cdot)$ is curl-free in $\Sigma_2^*$. Indeed, in definition (4.2.5) we assume that every $v \in L^2_\sharp(Y;\mathbb{C}^3)$ with $\operatorname{curl} v = 0$ in $\Sigma_1^*$ can be written as $v = \nabla\Theta + c$ in $\Sigma_1^*$ for some $\Theta \in H^1_\sharp(\Sigma_1^*;\mathbb{C})$ and for some $c \in \mathbb{C}^3$. This decomposition is not true for curl-free vector fields in $\Sigma_2^*$.

In [PSU19], the definition of the geometric average was extended to vector fields $v \in L^2_\sharp(\Sigma^*;\mathbb{C}^3)$ with $\operatorname{curl} v = 0$ in $\Sigma^*$, where $\Sigma$ is any admissible microstructure. In order to state this definition, we introduce a space of test vector fields,

$$\mathcal{V} := \left\{ \phi \in L^2_\sharp(Y;\mathbb{C}^3) \,\middle|\, \operatorname{curl}\,\phi = 0 \text{ in } Y \text{ and } \phi = 0 \text{ in } \Sigma_2 \right\}.$$

Let $v \in L^2_\sharp(Y;\mathbb{C}^3)$ be a vector field with $\operatorname{curl}\, v = 0$ in $\Sigma_2^*$. One can show that there exists a *unique* vector $\oint_{\Sigma_2^*} v \in \mathbb{C}^3$ such that

$$\int_{\Sigma_2^*} v \wedge \phi = \left( \oint_{\Sigma_2^*} v \right) \wedge \left( \int_{\Sigma_2^*} \phi \right) \quad \text{for all } \phi \in \mathcal{V}. \tag{4.2.7}$$

We remark that the geometric average defined in (4.2.5) satisfies identity (4.2.7); a proof can be found in [SU18, Corollary 2.12].

Let us consider some properties of the geometric average, which will turn out to be useful to characterise the solution space of the $H_0$-cell problem. If $c$ is a constant vector in $\mathbb{C}^3$, then $\oint_{\Sigma_2^*} c = c$. Indeed, as every solution $u \in X^E$ of the $E_0$-cell problem is an element of $\mathcal{V}$ and as $\int_Y \colon X^E \to \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$ is an isomorphism, we find $u^j \in X^E$ with $\int_Y u^j = \mathrm{e}_j$ for $j \in \{2,3\}$. Using identity (4.2.7) with $\phi = u^j$, we find that

$$\left( c - \oint_{\Sigma_2^*} c \right) \wedge \mathrm{e}_j = c \wedge \left( \int_{\Sigma_2^*} u^j \right) - \left( \oint_{\Sigma_2^*} c \right) \wedge \left( \int_{\Sigma_2^*} u^j \right) = 0 \quad \text{for } j \in \{2,3\}.$$

From this equation we deduce that $\oint_{\Sigma_2^*} c = c$ for any $c \in \mathbb{C}^3$. Given a potential $\Theta \in H^1_\sharp(Y;\mathbb{C})$, we compute that

$$\left( \oint_{\Sigma_2^*} \nabla\Theta \right) \wedge \left( \int_{\Sigma_2^*} \phi \right) = \int_{\Sigma_2^*} \nabla\Theta \wedge \phi = \int_Y \nabla\Theta \wedge \phi = \int_Y \operatorname{curl}\big(\Theta\phi\big) = 0,$$

for all $\phi \in \mathcal{V}$. Hence $\oint_{\Sigma_2^*} \nabla\Theta = 0$.

The two-scale limit $H_0(x,\cdot)$ is curl-free in $\Sigma_2^*$ and thus an element of the function space

$$\mathcal{X} := \left\{ v \in H^1_\sharp(Y;\mathbb{C}^3) \,\middle|\, \operatorname{curl}\, v = 0 \text{ in } \Sigma_2^* \right\}.$$

Our main contribution in the analysis of the $H_0$-cell problem in the high contrast case is a new variational equation involving $J_0$ and $H_0$. More precisely, for every $v \in \mathcal{X}$ with $\oint_{\Sigma_2^*} v = 0$ there holds:

$$\int_{\Sigma_2^*} \big\langle J_0, \operatorname{curl}\, v \big\rangle - \mathrm{i}\omega\mu_0\varepsilon_r \int_Y \big\langle H_0, v \big\rangle = 0. \tag{4.2.8}$$

This identity implies equations (4.1.7b) and (4.1.7c). Let us verify this claim. If $\varphi \in C^\infty_\sharp(Y;\mathbb{C})$ is an arbitrary test function, then $\nabla\varphi \in \mathcal{X}$ with $\oint_{\Sigma_2^*} \nabla\varphi = 0$. We can therefore choose $\phi = \nabla\varphi$ in (4.2.8) and find that $H_0$ satisfies (4.1.7b). On the other hand, every $\psi \in C^\infty_c(\Sigma_2;\mathbb{C}^3)$ is an element of $\mathcal{X}$ with $\oint_{\Sigma_2^*} \psi = 0$. Setting $\phi = \psi$ in (4.2.8) yields (4.1.7c).

Combining (4.2.8) with equation (4.1.7a) yields a variational characterisation of the two-scale limit $H_0$: for almost all $x \in R$, the field $H_0 = H_0(x,\cdot) \in \mathcal{X}$ satisfies

$$b\big(H_0, v\big) = 0 \quad \text{for all } v \in \mathcal{X} \text{ with } \oint_{\Sigma_2^*} v = 0, \tag{4.2.9}$$

where the sesquilinear form $b\colon \mathcal{X} \times \mathcal{X} \to \mathbb{C}$ is defined as

$$b(w,v) \coloneqq \int_{\Sigma_2^*} \langle \operatorname{curl} w, \operatorname{curl} v \rangle - \omega^2 \varepsilon_0 \varepsilon_r \mu_0 \int_Y \langle w, v \rangle \,.$$

This variational cell problem (4.2.9) of $H_0$ is equivalent to equations (4.1.7a) and (4.2.8) in the following sense: if $w \in \mathcal{X}$ satisfies $b(w,v) = 0$ for all vector fields $v \in \mathcal{X}$ with $\oint_{\Sigma_2^*} v = 0$, then $(H_0, J_0)$ is a solution to (4.1.7a) and (4.2.8), where $H_0 \coloneqq w$ and $J_0 \coloneqq -(\mathrm{i}\omega\varepsilon_0)^{-1} \operatorname{curl} w$. On the other hand, if $(H_0, J_0)$ solve (4.1.7a) and (4.2.8), then $H_0$ satisfies (4.2.9).

Thanks to the equivalence of the variational cell problem (4.2.9) with the two equations (4.1.7a) and (4.2.8), we can focus on the analysis of the space

$$X^H \coloneqq \left\{ w \in \mathcal{X} \,\middle|\, b(w,v) = 0 \text{ for all } v \in \mathcal{X} \text{ with } \oint_{\Sigma_2^*} v = 0 \right\}.$$

Choose an arbitrary vector field $w \in X^H$. As the map $\oint_{\Sigma_2^*} \colon \mathcal{X} \to \mathbb{C}^3$ is linear, there exists a vector field $w_0 \in \mathcal{X}$ with $\oint_{\Sigma_2^*} w_0 = 0$ such that $w = w_0 + \oint_{\Sigma_2^*} w$ in $Y$. To shorten notation, we set $z \coloneqq \oint_{\Sigma_2^*} w$. Thanks to this decomposition of $w$ there holds

$$0 = b(w,v) = b(w_0, v) - \omega^2 \varepsilon_0 \varepsilon_r \mu_0 \int_Y \langle z, v \rangle \tag{4.2.10}$$

for all $v \in \mathcal{X}$ with $\oint_{\Sigma_2^*} v = 0$. Equation (4.2.10) motivates the question: Given $z \in \mathbb{C}^3$, is it possible to find a *unique* $w_0 \in \mathcal{X}$ with $\oint_{\Sigma_2^*} w_0 = 0$ such that

$$b(w_0, v) = \omega^2 \varepsilon_0 \varepsilon_r \mu_0 \int_Y \langle z, v \rangle \quad \text{for all } v \in \mathcal{X} \text{ with } \oint_{\Sigma_2^*} v = 0\,? \tag{4.2.11}$$

If this question has an affirmative answer, then the map $\oint_{\Sigma_2^*} \colon X^H \to \mathbb{C}^3$ is a vector space isomorphism. Let us check the injectivity of the map first: Assume $w \in X^H$ has a vanishing geometric average. Then, by (4.2.10), the field $w$ satisfies $b(w,v) = 0$ for all $v \in \mathcal{X}$ with $\oint_{\Sigma_2^*} v = 0$. Thus, by uniqueness, $w = 0$. In order to show that the map is onto, we choose $c \in \mathbb{C}^3$. Then there exists $w_0 \in \mathcal{X}$ with $\oint_{\Sigma_2^*} w_0 = 0$ such that

$$b(w_0, v) = \omega^2 \varepsilon_0 \varepsilon_r \mu_0 \int_Y \langle c, v \rangle \quad \text{for all } v \in \mathcal{X} \text{ with } \oint_{\Sigma_2^*} v = 0\,.$$

Setting $w \coloneqq w_0 + c$ yields an element of $X^H$ with $\oint_{\Sigma_2^*} w = \oint_{\Sigma_2^*} w_0 + \oint_{\Sigma_2^*} c = c$.

The question whether for every $z \in \mathbb{C}^3$ there exists a unique $w_0 \in \mathcal{X}$ with $\oint_{\Sigma^*} w_0 = 0$ such that (4.2.11) holds has an affirmative answer; the proof is an application of the Lax-Milgram lemma. We remark that the assumption $\operatorname{Im}\{\varepsilon_r\} > 0$ is vital for the proof.

We have shown above that the geometric average $\oint_{\Sigma_2^*} \colon X^H \to \mathbb{C}^3$ is an isomorphism between the solution space $X^H$ of the variational cell problem (4.2.9) and $\mathbb{C}^3$.

**Effective material parameters and equations.** As the map $\int_Y \colon X^E \to A^E = \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$ is an isomorphism, there exist unique fields $E^2, E^3 \in X^E$ such that $\int_Y E^j = \mathrm{e}_j$ for $j \in \{2,3\}$. The relative permittivity $\varepsilon^{\mathrm{eff}}$ of the metamaterial $R$ is a $3\times 3$-matrix, which is given by

$$\left(\varepsilon^{\mathrm{eff}}\right)_{k,l} = \begin{cases} \left\langle E^k, E^l \right\rangle_{L^2(Y;\mathbb{C}^3)} & \text{if } k,l \in \{2,3\} \\ 0 & \text{if } k = 1 \text{ or } l = 1 \end{cases} .$$

The relative permeability $\mu^{\mathrm{eff}}$ of $R$ is given by

$$\mu^{\mathrm{eff}} = \left( \int_Y H^1 \,\middle|\, \int_Y H^2 \,\middle|\, \int_Y H^3 \right) \in \mathbb{R}^{3\times 3} ,$$

where $H^j$ is the unique element of $X^H$ with $\oint_{\Sigma_2^*} H^j = \mathrm{e}_j$ for $j \in \{1,2,3\}$. Defining the effective permittivity $\hat{\varepsilon}\colon \Omega \to \mathbb{R}^{3\times 3}$ and the effective permeability $\hat{\mu}\colon \Omega \to \mathbb{R}^{3\times 3}$ as in Section 4.1, the effective system (4.1.6) for the microstructure $\Sigma_2$ reads

$$\begin{cases} \operatorname{curl} \hat{E} = \mathrm{i}\omega\mu_0\hat{\mu}\hat{H} & \text{in } \Omega\,, \\ \partial_3\hat{H}_1 - \partial_1\hat{H}_3 = -\mathrm{i}\omega\varepsilon_0\big(\hat{\varepsilon}\hat{E}\big)_2 & \text{in } \Omega\,, \\ \partial_1\hat{H}_2 - \partial_2\hat{H}_1 = -\mathrm{i}\omega\varepsilon_0\big(\hat{\varepsilon}\hat{E}\big)_3 & \text{in } \Omega\,, \\ \operatorname{curl} \hat{H} = -\mathrm{i}\omega\varepsilon_0\hat{E} & \text{in } \Omega \setminus \overline{R}\,. \end{cases}$$

## 4.3 Preliminary geometric results

The derivation of effective Maxwell's equations is based on an analysis of two cell problems, for which we need the notion of a $k$-loop and the geometric average. We introduce both concepts in this section and discuss the relevant properties.

### Geometric averaging

Throughout this section, $\Sigma$ is an admissible microstructure. Following [PSU19], we shall define the geometric average for vector fields $v\colon \Sigma^* \to \mathbb{C}^3$ with vanishing distributional curl in $\Sigma^*$. We extend the notion of a geometric average, which was introduced in [PSU19], to prepare for an analysis of highly conductive materials.

In a first step, the geometric average for real-valued vector fields $\Sigma^* \to \mathbb{R}^3$ is defined. Afterwards, the results are extended to complex-valued fields $\Sigma^* \to \mathbb{C}^3$.

**Function spaces for the geometric average.** The space of vector fields for which we define the geometric average is

$$\mathcal{X}_{\mathbb{F}}(\Sigma^*) := \left\{ v \in L^2_\sharp(\Sigma^*;\mathbb{F}^3) \,\middle|\, \operatorname{curl} v = 0 \text{ in } \Sigma^* \right\} , \tag{4.3.1}$$

where $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. We note that the equation $\operatorname{curl} v = 0$ in $\Sigma^*$ is understood in the distributional sense. In the definition, we also need a space of test vector fields,

$$\mathcal{V}_{\mathbb{F}}(\Sigma^*) := \left\{ \phi \in L^2_\sharp(Y;\mathbb{F}^3) \,\middle|\, \operatorname{curl} \phi = 0 \text{ in } Y \text{ and } \phi = 0 \text{ in } \Sigma \right\} . \tag{4.3.2}$$

The space of attainable volume averages of fields from $\mathcal{V}_{\mathbb{F}}(\Sigma^*)$ is denoted by

$$A^{\mathcal{V}}_{\mathbb{F}}(\Sigma^*) := \left\{ \fint_Y \phi \,\middle|\, \phi \in \mathcal{V}_{\mathbb{F}}(\Sigma^*) \right\}. \tag{4.3.3}$$

Clearly, $A^{\mathcal{V}}_{\mathbb{F}}(\Sigma^*)$ is a linear subspace of $\mathbb{F}^3$. Given any linear subspace $W$ of $\mathbb{F}^3$, we define the wedge-annihilator $W^\wedge$ as the set

$$W^\wedge := \left\{ b \in \mathbb{F}^3 \,\middle|\, b \wedge w = 0 \text{ for all } w \in W \right\} = \begin{cases} \mathbb{F}^3 & \text{if } \dim W = 0 \\ W & \text{if } \dim W = 1 \\ \{0\} & \text{if } \dim W \geq 2 \end{cases}.$$

The set $W^{\wedge\perp} \subset \mathbb{F}^3$ is defined as the orthogonal complement, $W^{\wedge\perp} := (W^\wedge)^\perp$. It turns out that

$$W^{\wedge\perp} = \begin{cases} \{0\} & \text{if } \dim W = 0 \\ W^\perp & \text{if } \dim W = 1 \\ \mathbb{F}^3 & \text{if } \dim W \geq 2 \end{cases}. \tag{4.3.4}$$

*Notation:* In Chapters 5 and 6, we mostly consider complex-valued vector fields $\Sigma \to \mathbb{C}^3$ and hence write $\mathcal{X}(\Sigma^*)$, $\mathcal{V}(\Sigma^*)$, and $A^{\mathcal{V}}(\Sigma^*)$ instead of $\mathcal{X}_{\mathbb{C}}(\Sigma^*)$, $\mathcal{V}_{\mathbb{C}}(\Sigma^*)$, and $A^{\mathcal{V}}_{\mathbb{C}}(\Sigma^*)$.

**Definition of the geometric average—real-valued fields.** In this paragraph, we present the definition of the geometric average for real-valued vector fields $\Sigma \to \mathbb{R}^3$.

Definition 4.1. (Geometric average—real-valued fields) – *The* geometric average *is the unique linear map*

$$\oint_{\Sigma^*} : \mathcal{X}_{\mathbb{R}}(\Sigma^*) \to A^{\mathcal{V}}_{\mathbb{R}}(\Sigma^*)^{\wedge\perp} \tag{4.3.5}$$

*that satisfies the identity*

$$\int_{\Sigma^*} v \wedge \phi = \left( \oint_{\Sigma^*} v \right) \wedge \left( \int_{\Sigma^*} \phi \right), \tag{4.3.6}$$

*for all $v \in \mathcal{X}_{\mathbb{R}}(\Sigma^*)$ and $\phi \in \mathcal{V}_{\mathbb{R}}(\Sigma^*)$. We say that $\oint_{\Sigma^*} v \in \mathbb{R}^3$ is the* geometric average *of $v$.*

Among all vectors $b \in \mathbb{R}^3$ that satisfy the identity

$$\int_{\Sigma^*} v \wedge \phi = b \wedge \left( \int_{\Sigma^*} \phi \right), \tag{4.3.7}$$

for all $\phi \in \mathcal{V}_{\mathbb{R}}(\Sigma^*)$, the geometric average $\oint_{\Sigma^*} v$ is the one with the minimal Euclidean norm. The next result establishes the validity of Definition 4.1.

Theorem 4.2. – *There exists one and only one map that satisfies* (4.3.5) *and* (4.3.6)*. This map is linear, bounded and surjective.*

*Proof.* We refer to [PSU19, Theorem 2.2] for the proof of this theorem. □

**Definition of the geometric average—general case.** In Chapters 5 and 6, we mostly encounter complex-valued vector fields, which forces us to extend the geometric average to vector fields in $\mathcal{X}(\Sigma^*)$. Although this definition is straightforward,

$$\oint_{\Sigma^*} v := \oint_{\Sigma^*} \mathrm{Re}\{v\} + \mathrm{i} \oint_{\Sigma^*} \mathrm{Im}\{v\}\,, \tag{4.3.8}$$

some remarks are in order. Before we prove that the geometric average of a complex-valued vector field is also characterised by the identity (4.3.6), we need the following auxiliary result.

LEMMA 4.3. – *For vector fields $v\colon \Sigma \to \mathbb{C}^3$ and $\phi\colon Y \to \mathbb{C}^3$ the following statements hold.*

*(i) $v \in \mathcal{X}(\Sigma^*)$ if and only if $\mathrm{Re}\{v\}, \mathrm{Im}\{v\} \in \mathcal{X}_{\mathbb{R}}(\Sigma^*)$.*

*(ii) $\phi \in \mathcal{V}(\Sigma^*)$ if and only if $\mathrm{Re}\{\phi\}, \mathrm{Im}\{\phi\} \in \mathcal{V}_{\mathbb{R}}(\Sigma^*)$.*

As the proof is straightforward, we postpone it to Appendix A.

The previous lemma in particular implies that the expressions on the right-hand side of (4.3.8) are well defined.

THEOREM 4.4. – *There is one and only one map*

$$\oint_{\Sigma^*} : \mathcal{X}(\Sigma^*) \to A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}\,, \quad v \mapsto \oint_{\Sigma^*} v := \oint_{\Sigma^*} \mathrm{Re}\{v\} + \mathrm{i} \oint_{\Sigma^*} \mathrm{Im}\{v\} \tag{4.3.9}$$

*that satisfies*

$$\int_{\Sigma^*} v \wedge \phi = \left(\oint_{\Sigma^*} v\right) \wedge \left(\int_{\Sigma^*} \phi\right), \tag{4.3.10}$$

*for all $v \in \mathcal{X}(\Sigma^*)$ and $\phi \in \mathcal{V}(\Sigma^*)$. This map is linear, bounded and surjective.*

*Proof.* We proceed in four steps.

*Step 1. (Existence)* We need to show that the map defined in (4.3.9) satisfies the identity (4.3.10). To this end, fix two fields $v \in \mathcal{X}(\Sigma^*)$ and $\phi \in \mathcal{V}(\Sigma^*)$. A straightforward computation shows that for two complex vectors $\alpha, \beta \in \mathbb{C}^3$ there holds

$$\mathrm{Re}\{\alpha \wedge \beta\} = \mathrm{Re}\{\alpha\} \wedge \mathrm{Re}\{\beta\} - \mathrm{Im}\{\alpha\} \wedge \mathrm{Im}\{\beta\} \tag{4.3.11}$$

and

$$\mathrm{Im}\{\alpha \wedge \beta\} = \mathrm{Re}\{\alpha\} \wedge \mathrm{Im}\{\beta\} + \mathrm{Im}\{\alpha\} \wedge \mathrm{Re}\{\beta\}\,. \tag{4.3.12}$$

Thus, using (4.3.6) for the real-valued vector fields $\mathrm{Re}\{v\}$, $\mathrm{Im}\{v\}$, $\mathrm{Re}\{\phi\}$, and $\mathrm{Im}\{\phi\}$, we find that

$$\begin{aligned}\int_{\Sigma^*} \mathrm{Re}\{v \wedge \phi\} &= \int_{\Sigma^*} \mathrm{Re}\{v\} \wedge \mathrm{Re}\{\phi\} - \int_{\Sigma^*} \mathrm{Im}\{v\} \wedge \mathrm{Im}\{\phi\} \\ &= \Big(\oint_{\Sigma^*} \mathrm{Re}\{v\}\Big) \wedge \Big(\int_{\Sigma^*} \mathrm{Re}\{\phi\}\Big) \\ &\quad - \Big(\oint_{\Sigma^*} \mathrm{Im}\{v\}\Big) \wedge \Big(\int_{\Sigma^*} \mathrm{Im}\{\phi\}\Big) \\ &= \mathrm{Re}\Big\{\oint_{\Sigma^*} v\Big\} \wedge \mathrm{Re}\Big\{\int_{\Sigma^*} \phi\Big\} - \mathrm{Im}\Big\{\oint_{\Sigma^*} v\Big\} \wedge \mathrm{Im}\Big\{\int_{\Sigma^*} \phi\Big\} \\ &= \mathrm{Re}\Big\{\Big(\oint_{\Sigma^*} v\Big) \wedge \Big(\int_{\Sigma^*} \phi\Big)\Big\}, \end{aligned} \tag{4.3.13}$$

where we used (4.3.11) again to obtain the last equality. A similar calculation using (4.3.12) yields

$$\int_{\Sigma^*} \mathrm{Im}\{v \wedge \phi\} = \mathrm{Im}\Big\{\Big(\oint_{\Sigma^*} v\Big) \wedge \Big(\int_{\Sigma^*} \phi\Big)\Big\}. \tag{4.3.14}$$

Combining (4.3.13) and (4.3.14) proves that the map $\oint_{\Sigma^*}$ satisfies (4.3.10).

*Step 2.* Before we prove uniqueness of the map, we need to show that $\operatorname{im} \oint_{\Sigma^*} \subset A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$. If $\dim A^{\mathcal{V}}(\Sigma^*) \in \{0, 2, 3\}$, one readily checks that this inclusion holds.

In the case $\dim A^{\mathcal{V}}(\Sigma^*) = 1$, the space $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ coincides with $A^{\mathcal{V}}(\Sigma^*)^{\perp}$. The claim follows if we show that

$$A^{\mathcal{V}}_{\mathbb{R}}(\Sigma^*)^{\perp} \oplus \mathrm{i}A^{\mathcal{V}}_{\mathbb{R}}(\Sigma^*)^{\perp} \subset A^{\mathcal{V}}(\Sigma^*)^{\perp}. \tag{4.3.15}$$

Fix $c_1, c_2 \in A^{\mathcal{V}}_{\mathbb{R}}(\Sigma^*)^{\perp}$. Every $b \in A^{\mathcal{V}}(\Sigma^*)$ can be written as $b = \mathrm{Re}\{b\} + \mathrm{i}\,\mathrm{Im}\{b\}$ with $\mathrm{Re}\{b\}, \mathrm{Im}\{b\} \in A^{\mathcal{V}}_{\mathbb{R}}(\Sigma^*)$. Thus,

$$\langle c_1 + \mathrm{i}c_2, b\rangle = \big\langle c_1, \mathrm{Re}\{b\}\big\rangle - \big\langle c_2, \mathrm{Im}\{b\}\big\rangle + \mathrm{i}\big(\big\langle c_1, \mathrm{Im}\{b\}\big\rangle + \big\langle c_2, \mathrm{Re}\{b\}\big\rangle\big) = 0.$$

As $b \in A^{\mathcal{V}}(\Sigma^*)$ was chosen arbitrarily, the inclusion (4.3.15) holds, and we infer that $\operatorname{im} \oint_{\Sigma^*} \subset A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$.

*Step 3. (Uniqueness)* Assume $L\colon \mathcal{X}(\Sigma^*) \to A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ satisfies (4.3.10). Then, for any $v \in \mathcal{X}(\Sigma^*)$ and every $\phi \in \mathcal{V}(\Sigma^*)$, there holds

$$\Big(L(v) - \oint_{\Sigma^*} v\Big) \wedge \int_{\Sigma^*} \phi = 0. \tag{4.3.16}$$

Let us set $R(v) \coloneqq L(v) - \oint_{\Sigma^*} v \in A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$. If $\dim A^{\mathcal{V}}(\Sigma^*) = 0$, then $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp} = \{0\}$ and $R(v) = 0$ for all $v \in \mathcal{X}(\Sigma^*)$ by definition.

If $\dim A^{\mathcal{V}}(\Sigma^*) = 1$, we find vectors $c \in A^{\mathcal{V}}(\Sigma^*)$ and $b_2 \in \mathbb{C}^3$ such that $(c, b_2, c \wedge b_2)$ is an orthogonal basis of $\mathbb{C}^3$. Fix $v \in \mathcal{X}(\Sigma^*)$. By definition of $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$, there exists scalars $\alpha_2, \alpha_3 \in \mathbb{C}$ such that $R(v) = \alpha_2 b_2 + \alpha_3(c \wedge b_2)$. Substituting this into (4.3.16) with a vector field $\phi \in \mathcal{V}(\Sigma^*)$ such that $⨏_Y \phi = c$ yields

$$0 = R(v) \wedge c = \alpha_2 b_2 \wedge c + \alpha_3(c \wedge b_2) \wedge c.$$

From this we infer that $\alpha_2 = \alpha_3 = 0$, and hence $R(v) = 0$.

In the case $\dim A^{\mathcal{V}}(\Sigma^*) \geq 2$ we can argue similarly and show that $R = 0$. This establishes the uniqueness of the map $\oint_{\Sigma^*}$.

*Step 4.* The proof of linearity, boundedness, and surjectivity of the map $\oint_{\Sigma^*}$ is similar to the one presented in [PSU19, Proof of Theorem 2.2]. $\square$

The following properties of the geometric average are used later.

LEMMA 4.5. (Properties of the geometric average) – *Let $\Sigma \subset Y$ be an admissible microstructure.*

*(i) In the case $\Sigma = \emptyset$ the geometric average coincides with the volume average. More precisely, for every vector field $v \in \mathcal{X}(Y)$ there holds*

$$\oint_Y v = \fint_Y v\,.$$

*(ii) For every $f \in H^1_\sharp(\Sigma^*;\mathbb{C})$ the vector field $\nabla f$ is an element of $\mathcal{X}(\Sigma^*)$ with*

$$\oint_{\Sigma^*} \nabla f = 0\,.$$

*(iii) Denote by $\pi_{A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}} \colon \mathbb{C}^3 \to \mathbb{C}^3$ the orthogonal projection onto $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$. Any constant vector field $c \in \mathbb{C}^3$ is an element of $\mathcal{X}(\Sigma^*)$ with*

$$\oint_{\Sigma^*} c = \pi_{A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}}(c)\,.$$

*Proof.* *(i)* Let $v \in L^2_\sharp(Y;\mathbb{C}^3)$ be a vector field with $\operatorname{curl} v = 0$ in $Y$. Every vector $c \in \mathbb{C}^3$ is an element of $\mathcal{V}(Y)$. We therefore find that

$$\left(\int_Y v\right) \wedge c = \int_Y v \wedge c = \left(\oint_Y v\right) \wedge c\,,$$

and as $A^{\mathcal{V}}(Y)^{\wedge\perp} = \mathbb{C}^3$, the claim follows.

*(ii)* Let $f \in H^1_\sharp(\Sigma^*;\mathbb{C})$ be arbitrary. Choose an $H^1$-extension $\tilde{f} \in H^1_\sharp(Y;\mathbb{C})$ of $f$. Then, for any $\phi \in \mathcal{V}(\Sigma^*)$ there holds

$$\left(\oint_{\Sigma^*} \nabla f\right) \wedge \left(\int_{\Sigma^*} \phi\right) = \int_{\Sigma^*} \nabla f \wedge \phi = \int_Y \nabla \tilde{f} \wedge \phi = \int_Y \operatorname{curl}\big(\tilde{f}\phi\big) = 0\,.$$

The claim follows as $0 \in A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$.

*(iii)* Fix $c \in \mathbb{C}^3$ and let $\phi \in \mathcal{V}(\Sigma^*)$ be arbitrary. We compute

$$\left(\oint_{\Sigma^*} c\right) \wedge \left(\int_{\Sigma^*} \phi\right) = \int_{\Sigma^*} c \wedge \phi = c \wedge \left(\int_{\Sigma^*} \phi\right) = \pi_{A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}}(c) \wedge \left(\int_{\Sigma^*} \phi\right),$$

where we used the definition of $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ and the fact that $\int_{\Sigma^*} \phi \in A^{\mathcal{V}}(\Sigma^*)$ to obtain the last equation. $\square$

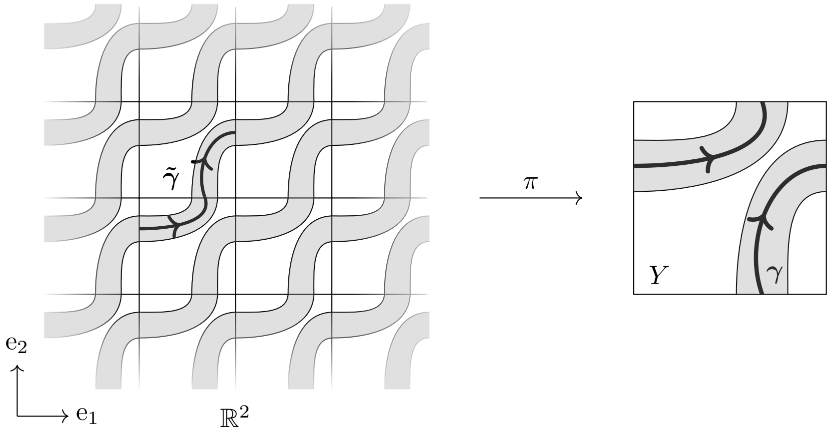


Figure 4.3: On the right, we sketch the cross-section $Y \cap \{y_3 = 1/2\}$ of the unit cell $Y$. We assume that the cross-sections $Y \cap \{y_3 = c\}$ are identical for any $c \in (0,1)$. The gray area represents a subset $U$, which is open in $\mathbb{T}^3$. On the left, a part of the cross-section $(\bigcup_{k\in\mathbb{Z}^3}(k+U)) \cap \{x_3 = 0\}$ is sketched. The universal covering $\mathbb{R}^2 \to Y$, $y \mapsto y \mod \mathbb{Z}^2$ is denoted by $\pi$. The black curve $\gamma$ in $Y$ represents a $k$-loop in $U$. From its lift $\tilde{\gamma}$ on the left, we see that $\gamma$ is a $(1,1,0)$-loop in $U$.

### The notion of $k$-loops

In this section we generalise the concept of a $k$-loop that was introduced in [SU18]. Consider a path $\sigma\colon [0,1] \to \mathbb{T}^3$. We say that $\tilde{\sigma}\colon [0,1] \to \mathbb{R}^3$ is a *lift* of $\sigma$ provided $\sigma = \pi \circ \tilde{\sigma}$, where $\pi$ denotes the universal covering $\mathbb{R}^3 \to \mathbb{T}^3$, $x \mapsto x \mod \mathbb{Z}^3$. Let us recall that besides $\pi$ there is another covering of the torus, $\iota\colon Y \to \mathbb{T}^3$, $x \mapsto x \mod \mathbb{Z}^3$.

Definition 4.6. – *Let $U \subset Y$ be a non-empty subset for which $\iota(U) \subset \mathbb{T}^3$ is open. For a vector $k \in \mathbb{Z}^3$, $k \neq 0$, a continuous and piecewise continuously differentiable closed and simple curve $\gamma\colon [0,1] \to \iota(U)$ is called a* $k$-loop *in $U$ if there is a lift $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^3$ of $\gamma$ satisfying $\tilde{\gamma}(1) - \tilde{\gamma}(0) = k$.*

In Figure 4.3, a $k$-loop and one of its lifts is sketched. We observe that every lift $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^3$ of a continuous and closed path $\gamma\colon [0,1] \to \iota(U)$ satisfies $\tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{Z}^3$. In the definition of a $k$-loop we demand that there is a lift $\tilde{\gamma}$ of $\gamma$ for which $\tilde{\gamma}(1) - \tilde{\gamma}(0) = k$. The next result shows that if there is one such lift, then every lift has this property.

Lemma 4.7. – *Let $U \subset Y$ be a non-empty subset for which $\iota(U) \subset \mathbb{T}^3$ is open, and fix $k \in \mathbb{Z}^3$. For a continuous and piecewise continuously differentiable closed and simple path $\gamma\colon [0,1] \to \iota(U)$ the following statements are equivalent:*

- *(i) There is a lift $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^3$ of $\gamma$ satisfying $\tilde{\gamma}(1) - \tilde{\gamma}(0) = k$.*
- *(ii) Every lift $\tilde{\gamma}\colon [0,1] \to \mathbb{R}^3$ of $\gamma$ satisfies $\tilde{\gamma}(1) - \tilde{\gamma}(0) = k$.*

*Proof.* When *(ii)* holds, then *(i)* is trivially true. In order to show that *(i)* implies *(ii)*, let $\tilde{\gamma}$ be a lift of $\gamma$ with $\tilde{\gamma}(1) - \tilde{\gamma}(0) = k$. Suppose that $\tilde{\sigma}\colon [0,1] \to \mathbb{R}^3$

is another lift of $\gamma$ for which $\tilde{\sigma}(1) - \tilde{\sigma}(0) \neq k$. As $\pi$ is the universal covering of $\mathbb{T}^3$, for every $y \in \iota(U)$ there is a point $p \in U$ such that $\pi(p) = y$. We further have that $\pi^{-1}(y) = \{p\} + \mathbb{Z}^3$. Thus for each $t \in [0,1]$ there is a vector $l \in \mathbb{Z}^3$ such that $\tilde{\gamma}(t) = \tilde{\sigma}(t) + l$. Due to the continuity of $\tilde{\gamma}$ and $\tilde{\sigma}$ it is the same vector $l \in \mathbb{Z}^3$ for all $t \in [0,1]$. Using this relation between $\gamma$ and $\sigma$, we find that

$$k \neq \tilde{\sigma}(1) - \tilde{\sigma}(0) = \tilde{\gamma}(1) - \tilde{\gamma}(0) = k\,.$$

As this is a contradiction, the claim is proved. □

Given a subset $U$ of the unit cell $Y$, we define the two index sets

$$\mathcal{L}_U := \left\{ k \in \mathbb{Z}^3 \;\middle|\; k \neq 0 \text{ and there is a } k\text{-loop in } U \right\} \tag{4.3.17}$$

and

$$\mathcal{N}_U := \left\{ k \in \mathbb{Z}^3 \;\middle|\; k \neq 0 \text{ and there is no } k\text{-loop in } U \right\}. \tag{4.3.18}$$

Clearly, there holds $\mathcal{L}_U \cup \mathcal{N}_U = \mathbb{Z}^3 \setminus \{0\}$.

We will exploit the following connection between $k$-loops and potentials in Chapters 5 and 6.

Proposition 4.8. – *Let $U \subset Y$ be a non-empty subset for which $\iota(U) \subset \mathbb{T}^3$ is connected and has a Lipschitz boundary. If $m \in \mathbb{Z}^3 \setminus \{0\}$ satisfies $\langle m, k\rangle = 0$ for all $k \in \mathcal{L}_U$, then there is a potential $\Theta_m \in H^1_\sharp(U;\mathbb{R})$ with $\nabla\Theta_m = m$.*

*Proof.* We proceed in two steps: at first we construct a potential on the torus, and afterwards we lift this function to $Y$.

*Step 1.* We first construct a potential $\Psi_m \colon \iota(U) \to \mathbb{R}$. To this end, fix a point $y_0 \in \iota(U)$. For every $y \in \iota(U)$ we choose a piecewise smooth path $\gamma_y \colon [0,1] \to \iota(U)$ connecting $\gamma_y(0) = y_0$ and $\gamma_y(1) = y$. Denoting by $\tilde{\gamma}_y \colon [0,1] \to \mathbb{R}^3$ a lift of $\gamma_y$, we define $\Psi_m \colon \iota(U) \to \mathbb{R}$ by $\Psi_m(y) := \langle \tilde{\gamma}_y(1) - \tilde{\gamma}_y(0), m\rangle$.

*Claim: $\Psi_m$ is well defined.* The potential $\Psi_m$ is independent of the chosen path $\gamma_y$. Indeed, let $\sigma_y \colon [0,1] \to \iota(U)$ be another continuous and piecewise continuously differentiable path connecting $\sigma_y(0) = y_0$ and $\sigma_y(1) = y$. Denote by $\tilde{\sigma}_y \colon [0,1] \to \mathbb{R}^3$ the unique lift of $\sigma_y$ with $\tilde{\sigma}_y(0) = \tilde{\gamma}_y(0)$. Setting $\Gamma \colon [0,1] \to \iota(U)$,

$$\Gamma(t) := \begin{cases} \sigma_y(1-2t) & \text{for } t \in [0, 1/2] \\ \gamma_y(2t-1) & \text{for } t \in (1/2, 1] \end{cases},$$

we obtain a continuous and piecewise continuously differentiable closed path. For every lift $\tilde{\Gamma} \colon [0,1] \to \mathbb{R}^3$ of $\Gamma$ there exists $k \in \{0\} \cup \mathcal{L}_U$ such that

$$\tilde{\Gamma}(1) - \tilde{\Gamma}(0) = k\,. \tag{4.3.19}$$

One readily checks that $\tilde{\Gamma} \colon [0,1] \to \mathbb{R}^3$,

$$\tilde{\Gamma}(t) := \begin{cases} \tilde{\sigma}_y(1-2t) & \text{for } t \in [0, 1/2] \\ \tilde{\gamma}_y(2t-1) & \text{for } t \in (1/2, 1] \end{cases},$$

is a lift of $\Gamma$. Thanks to (4.3.19) we find that $\tilde{\sigma}_y(1) = \tilde{\Gamma}(0) = \tilde{\Gamma}(1) - k = \tilde{\gamma}_y(1) - k$ for some $k \in \{0\} \cup \mathcal{L}_U$. As $\tilde{\sigma}_y$ was chosen in such a way that $\tilde{\sigma}_y(0) = \tilde{\gamma}_y(0)$, we infer that

$$\begin{aligned}\Psi_m(y) &= \left\langle \tilde{\gamma}_y(1) - \tilde{\gamma}_y(0), m \right\rangle = \left\langle \tilde{\sigma}_y(1) - \tilde{\sigma}_y(0), m \right\rangle + \langle k, m \rangle \\ &= \left\langle \tilde{\sigma}_y(1) - \tilde{\sigma}_y(0), m \right\rangle .\end{aligned}$$

Thus $\Psi_m$ is independent of the chosen path $\gamma_y$.

The domain $\iota(U)$ is path connected and hence the potential $\Psi_m$ depends on the basepoint $y_0$ in the following sense: if we choose another basepoint $y_1 \in \iota(U)$ and define $\Phi_m$ in the same way as $\Psi_m$ but with respect to the basepoint $y_1$, then $\Psi_m - \Phi_m = c$ for some $c \in \mathbb{R}$.

*Step 2.* Having the potential $\Psi_m$ at hand, we define the $\Theta_m$. Let $\tilde{U} \subset \mathbb{R}^3$ denote the “periodic continuation” of $U$, that is,

$$\tilde{U} := \bigcup_{m \in \mathbb{Z}^3} (m + U) .$$

We recall that $\pi \colon \mathbb{R}^3 \to \mathbb{T}^3$ denotes the universal covering of $\mathbb{T}^3$. Clearly, $\pi(\tilde{U}) = \iota(U)$. Setting $\Theta_m \colon \tilde{U} \to \mathbb{R}$, $\Theta_m := \Psi_m \circ \pi$ provides us with a continuous and $Y$-periodic function. We need to prove that $\nabla \Theta_m = m$. It suffices to show this identity almost everywhere in $\tilde{U}$. Choose $\tilde{y} \in \tilde{U}$ and $y \in U$ such that $\pi(\tilde{y}) = y$. Without loss of generality, we may assume that $\Psi_m$ is constructed with respect to the basepoint $y$ and hence $\Psi_m(y) = 0$. As $\pi$ is a covering, there are open sets $\tilde{V} \subset \tilde{U}$ and $V \subset U$ such that $\pi|_{\tilde{V}} \colon \tilde{V} \to V$ is a homeomorphism. Fix $j \in \{1, 2, 3\}$ and choose $h_0 > 0$ such that $\tilde{y} + h\,\mathrm{e}_j \in \tilde{V}$ for all $h \in (0, h_0)$. The path $\tilde{\gamma} \colon [0,1] \to \mathbb{R}^3$, $\tilde{\gamma}(t) := \tilde{y} + th\,\mathrm{e}_j$ is a lift of $\gamma \colon [0,1] \to \iota(U)$, $\gamma := \pi \circ \tilde{\gamma}$, which satisfies $\gamma(0) = y$ and $\gamma(1) = \pi(\tilde{y} + h\,\mathrm{e}_j)$. Thus, by definition of $\Psi_m$, we find that

$$\Theta_m(\tilde{y} + h\,\mathrm{e}_j) = \left\langle \tilde{\gamma}(1) - \tilde{\gamma}(0), m \right\rangle = h \langle \mathrm{e}_j, m \rangle .$$

As $\Theta_m(\tilde{y}) = \Psi_m(y) = 0$, we compute that

$$\partial_j \Theta_m(\tilde{y}) = \lim_{h \to 0} \frac{\Theta_m(\tilde{y} + h\,\mathrm{e}_j) - \Theta_m(\tilde{y})}{h} = \langle \mathrm{e}_j, m \rangle .$$

The identity $\nabla \Theta_m = m$ is thus proved. We need to show that $\Theta_m \in H^1_\sharp(U; \mathbb{R})$. As $\nabla \Theta_m = m$, we find that $\Theta_m \in C^1(\tilde{U}; \mathbb{R})$. Moreover, $\Theta_m$ as well as its gradient are Lipschitz continuous on $\tilde{U}$. We can thus extend both to continuous maps on the closure of $\tilde{U}$ and find that $\Theta_m \in C^1(\overline{\tilde{U}}; \mathbb{R})$. The claim follows by restricting $\Theta_m$ to $U \subset Y$. □

Remarks 3. – *(a) The assumption on $m \in \mathbb{Z}^3 \setminus \{0\}$ in Proposition 4.8 can, in general, not be replaced by $m \in \mathcal{N}_U$. Indeed, consider the set $U$ in Figure 4.4. Every $k$-loop in $U$ is of the form $k = (n, n, l)$ for $n, l \in \mathbb{Z}$ and thus $\mathrm{e}_1 \in \mathcal{N}_U$. Suppose $\Theta \in H^1_\sharp(U; \mathbb{R})$ satisfies $\nabla \Theta = \mathrm{e}_1$. As the two components $U_1$ and $U_2$ are connected subsets of $[0,1]^3$, there exist constants $c_1, c_2 \in \mathbb{R}$ such that*

$$\Theta(y_1, y_2, y_3) = \begin{cases} y_1 + c_1 & \textit{if } (y_1, y_2, y_3) \in U_1 \\ y_1 + c_2 & \textit{if } (y_1, y_2, y_3) \in U_2 \end{cases} .$$

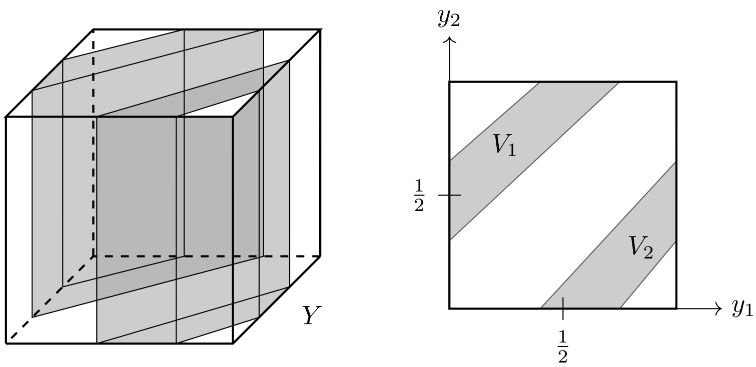


Figure 4.4: On the left: The dark grey area represents a subset $U$ of the unit cube $Y$ such that $\iota(U) \subset \mathbb{T}^3$ is open. Moreover, $U \subset (0,1)^3$ has two connected components $U_1$ and $U_2$. On the right: We sketch a cross-section $Y \cap \{y_3 = c\}$ of the left figure, for some $c \in [0,1]$. The cross section of the two connected components are called $V_1$ and $V_2$, respectively. As can be seen from the left figure, every $k$-loop in $U$ is of the form $k = (n,n,l)$ for $n, l \in \mathbb{Z}$.

*By periodicity,*

$$\Theta\left(0, \frac{1}{2}, y_3\right) = \Theta\left(1, \frac{1}{2}, y_3\right) \quad \textit{and} \quad \Theta\left(\frac{1}{2}, 0, y_3\right) = \Theta\left(\frac{1}{2}, 1, y_3\right)$$

*for all $y_3 \in [0,1]$. The first equality yields the identity $c_1 = 1 + c_2$ and the second equality implies that $c_1 = c_2$. As this is a contradiction, there cannot exist a periodic function $\Theta \in H^1_\sharp(U;\mathbb{R})$ with $\nabla\Theta = \mathrm{e}_1$, although $\mathrm{e}_1 \in \mathcal{N}_U$.*

*(b) In Sections 5.4 and 6.4 we consider microstructures $U$ for which every $m \in \mathcal{N}_U$ satisfies $\langle k, m\rangle = 0$ for all $k \in \mathcal{L}_U$. If $U$ is such a microstructure, we deduce from Proposition 4.8 that, for every $m \in \mathcal{N}_U$, there exists a potential $\Theta_m \in H^1_\sharp(U;\mathbb{R})$ with $\nabla\Theta_m = m$.*

CHAPTER 5

# Effective Maxwell's equations for perfectly conducting microstructures

In this chapter, we focus on a perfectly conducting microstructure $\Sigma$; in view of the time-harmonic Maxwell equations (4.1.1) this formally amounts to $\varepsilon_\eta = +\infty$ in $\Sigma_\eta$. More precisely, using the notation from Section 4.1, we consider a distributional solution $(E^\eta, H^\eta) \in L^2(\Omega; \mathbb{C}^3) \times L^2(\Omega; \mathbb{C}^3)$ to

$$
\begin{cases}
\operatorname{curl} E^\eta = \quad \mathrm{i}\omega\mu_0 H^\eta \text{ in } \Omega\,, & (5.0.1a)\\
\operatorname{curl} H^\eta = -\mathrm{i}\omega\varepsilon_0 E^\eta \text{ in } \Omega \setminus \overline{\Sigma}_\eta\,, & (5.0.1b)\\
\qquad E^\eta = H^\eta = 0 \quad \text{in } \Sigma_\eta\,, & (5.0.1c)
\end{cases}
$$

where $\omega > 0$ is a fixed frequency, and $\varepsilon_0, \mu_0 > 0$ are the permittivity and permeability of free space, respectively.

In this chapter we combine and generalise the results obtained in [SU18, PSU19]. Indeed, $k$-loops were used to characterise the solution spaces of the cell problems and to determine the effective equations in [SU18]. However, the assumptions on the microstructure were rather restrictive. In [PSU19], the above system was homogenised for microstructures $\Sigma$ satisfying assumptions (A1) and (A2), but $k$-loops were not used to characterise the solution spaces of the cell problems.

Besides assumptions (A1) and (A2) discussed in Section 4.1 we make an additional one:

(A3) There exists a sequence $(E^\eta, H^\eta)_\eta$ of distributional solutions to (5.0.1) that satisfies the energy-bound

$$
\sup_{\eta>0} \int_\Omega |E^\eta|^2 + |H^\eta|^2 < \infty\,. \tag{5.0.2}
$$

This assumption together with Theorem 3.7 ensure the existence of a subsequence $(E^\eta, H^\eta)_\eta$ and of fields $E_0, H_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$ such that

$$E^\eta \xrightarrow{2} E_0 \quad \text{and} \quad H^\eta \xrightarrow{2} H_0 \,.$$

Assumption (A3) further implies the existence of a subsequence $(E^\eta, H^\eta)_\eta$ and of fields $E, H \in L^2(\Omega; \mathbb{C}^3)$ such that $(E^\eta)_\eta$ weakly converges to $E$ in $L^2(\Omega; \mathbb{C}^3)$ and $(H^\eta)_\eta$ weakly converges to $H$ in $L^2(\Omega; \mathbb{C}^3)$. The relation between $E$ and $E_0$ as well as between $H$ and $H_0$ is given in Lemma 3.3,

$$E(x) = \fint_Y E_0(x, \cdot) \quad \text{and} \quad H(x) = \fint_Y H_0(x, \cdot)\,, \tag{5.0.3}$$

for almost all $x \in \Omega$.

In the subsequent sections, we define five spaces to which we will often refer. The following list is meant to be an overview and a reference:

- $X^E$ is the space of vector fields $u \in L^2_\sharp(Y; \mathbb{C}^3)$ that are distributional solutions to (5.1.1);
- $A^E$ is the space of attainable volume averages of fields in $X^E$ and it is defined in (5.1.6);
- $X$, defined in (5.2.9), is the space of vector fields $v \in L^2_\sharp(Y; \mathbb{C}^3)$ that are distributional solutions to (5.2.1);
- $X^H$ is the space of fields $w \in X$ that satisfy identity (5.2.2);
- $A^H$ is the space of attainable geometric averages of fields in $X^H$ and it is defined in (5.2.12).

## 5.1 The oscillating electric field

Our aim in this section is to derive the cell problem for the two-scale limit $E_0$ and to analyse its solution space.

LEMMA 5.1. (Cell problem) – *Let $\Omega \subset \mathbb{R}^3$ be as described in Section 4.1 and let $\Sigma \subset Y$ be an admissible microstructure. Let $(E^\eta, H^\eta)_\eta$ be a sequence satisfying assumption (A3), that two-scale converges to $(E_0, H_0)$, and that weakly converges in $L^2(\Omega; \mathbb{C}^3) \times L^2(\Omega; \mathbb{C}^3)$ to $(E, H)$. Then for almost all $x \in R$ the two-scale limit $E_0 = E_0(x, \cdot) \in L^2_\sharp(Y; \mathbb{C}^3)$ satisfies*

$$\begin{cases} \operatorname{curl}_y E_0 = 0 & \text{in } Y\,, & (5.1.1a)\\ \operatorname{div}_y E_0 = 0 & \text{in } \Sigma^*\,, & (5.1.1b)\\ E_0 = 0 & \text{in } \Sigma\,, & (5.1.1c)\end{cases}$$

*in the distributional sense.*

*Outside the meta-material $R$, the two-scale limit $E_0$ is $y$-independent; that is, $E_0(x, y) = E(x)$ for almost all $x \in \Omega \setminus R$ and almost all $y \in Y$.*

*Proof.* We choose $\theta \in C_c^\infty(\Omega;\mathbb{R})$, $\psi \in C_\sharp^\infty(Y;\mathbb{C}^3)$, and define the field $\varphi(x,y) \coloneqq \theta(x)\psi(y)$ for all $x \in \Omega$ and $y \in Y$. For fixed $\eta > 0$ we set $\psi_\eta(x) \coloneqq \psi(x/\eta)$ and $\varphi_\eta(x) \coloneqq \theta(x)\psi_\eta(x)$. Combining assumption (A3) and equation (5.0.1a), we find that $(E^\eta)_\eta$ and $(\eta \operatorname{curl} E^\eta)_\eta$ are bounded in $L^2(\Omega;\mathbb{C}^3)$. Thus, by Lemma 3.9, we know that $\eta \operatorname{curl} E^\eta \xrightarrow{2} \operatorname{curl}_y E_0$; that is,

$$\lim_{\eta\to 0} \int_\Omega \eta \langle \operatorname{curl} E^\eta, \varphi_\eta \rangle = \int_\Omega \theta \int_Y \langle \operatorname{curl}_y E_0, \psi \rangle \,. \tag{5.1.2}$$

On the other hand, for each $\eta > 0$ the field $E^\eta$ is a distributional solution to (5.0.1a), and hence

$$\lim_{\eta\to 0} \int_\Omega \eta \big\langle E^\eta, \operatorname{curl}[\varphi_\eta] \big\rangle = \mathrm{i}\omega\mu_0 \lim_{\eta\to 0} \int_\Omega \eta \big\langle H^\eta, \varphi_\eta \big\rangle = 0 \,, \tag{5.1.3}$$

where we used the boundedness of $(H^\eta)_\eta$ in $L^2(\Omega;\mathbb{C}^3)$ to obtain the last equation. Combining equations (5.1.2) and (5.1.3), we conclude that $E_0(x,\cdot)$ is a distributional solution to (5.1.1a) for almost all $x \in \Omega$.

On account of (5.0.1b), the distributional divergence of $E^\eta$ vanishes in $\Omega \setminus \overline{\Sigma}_\eta$. Fixing $\theta \in C_c^\infty(\Omega;\mathbb{R})$, $\psi \in C_c^\infty(\Sigma^*;\mathbb{C})$, and setting $\psi_\eta(x) \coloneqq \psi(x/\eta)$ for each $\eta > 0$, we calculate

$$0 = \lim_{\eta\to 0} \int_\Omega \eta\psi_\eta \big\langle E^\eta, \nabla\theta \big\rangle + \lim_{\eta\to 0} \int_\Omega \theta \big\langle E^\eta, \nabla\psi_\eta \big\rangle = \int_\Omega \theta \int_{\Sigma^*} \langle E_0, \nabla\psi \rangle \,, \tag{5.1.4}$$

where we used the two-scale convergence of $(E^\eta)_\eta$ to obtain the last equation. From (5.1.4) we deduce that $E_0(x,\cdot)$ satisfies (5.1.1b) for almost all $x \in \Omega$. In order to prove that $E_0(x,\cdot)$ solves (5.1.1c), we choose $\theta \in C_c^\infty(\Omega;\mathbb{R})$ and $\psi \in C_c^\infty(\Sigma;\mathbb{C}^3)$. For each $\eta > 0$ we set $\varphi_\eta(x) \coloneqq \theta(x)\psi(x/\eta)$. As $E^\eta$ satisfies (5.0.1c),

$$0 = \lim_{\eta\to 0} \int_\Omega \big\langle E^\eta, \varphi_\eta \big\rangle = \int_\Omega \theta \int_\Sigma \big\langle E_0, \psi \big\rangle \,. \tag{5.1.5}$$

Thus, the two-scale limit $E_0(x,\cdot)$ satisfies (5.1.1) for almost all $x \in R$.

We claim that $\operatorname{div}_y E_0(x,\cdot) = 0$ in $Y$ for almost all $x \in \Omega \setminus R$. Indeed, thanks to equation (5.0.1b), the distributional divergence of $E^\eta$ vanishes in $\Omega \setminus R$. Fix $\theta \in C_c^\infty(\Omega \setminus R;\mathbb{R})$ and $\psi \in C_\sharp^\infty(Y;\mathbb{C})$. Setting $\psi_\eta(x) \coloneqq \psi(x/\eta)$ for each $\eta > 0$, we deduce that

$$0 = \lim_{\eta\to 0} \int_{\Omega\setminus R} \eta\psi_\eta \big\langle E^\eta, \nabla\theta \big\rangle + \lim_{\eta\to 0} \int_{\Omega\setminus R} \theta \big\langle E^\eta, \nabla\psi_\eta \big\rangle = \int_{\Omega\setminus R} \theta \int_Y \big\langle E_0, \nabla\psi \big\rangle \,.$$

Thus, both $\operatorname{curl}_y E_0(x,\cdot)$ and $\operatorname{div}_y E_0(x,\cdot)$ vanish in $Y$ for almost all $x \in \Omega \setminus R$, from which we infer that $E_0(x,\cdot)$ is a constant vector field, by Lemma 2.4. Using (5.0.3) yields the identity $E_0(\cdot,y) = E(\cdot)$ for almost every $y \in Y$. □

Remark 4. (On the averaging of the electric field) – *As explained in the introduction, the natural object for averaging the electric field is a line integral—or a generalisation thereof. We use the geometric average, introduced in Section 4.3. However, in view of equation* (5.1.1a), *Lemma 4.5(i), and* (5.0.3), *we find that*

$$\oint_Y E_0(x,\cdot) = \fint_Y E_0(x,\cdot) = E(x)$$

*for almost all $x \in \Omega$.*

The next result settles the uniqueness of solutions. We investigate the existence of solutions to (5.1.1) in more detail; to this end, we remind the reader that the index sets $\mathcal{L}_\Sigma$ and $\mathcal{N}_\Sigma$ are defined in (4.3.17) and (4.3.18), respectively.

LEMMA 5.2. (Analysis of the cell problem) – *For an admissible microstructure $\Sigma \subset Y$ the following statements hold:*

*(i) Given any $c \in \mathbb{C}^3$ there is at most one distributional solution $u \in L^2_\sharp(Y;\mathbb{C}^3)$ to* (5.1.1) *with $\fint_Y u = c$.*

*(ii) If $k \in \mathcal{L}_\Sigma$, then there is no distributional solution to* (5.1.1) *with volume average equal to $k$.*

*(iii) If $m \in \mathcal{N}_\Sigma$ satisfies $\langle m, k\rangle = 0$ for every $k \in \mathcal{L}_\Sigma$, then there is a unique distributional solution $u^m \in L^2_\sharp(Y;\mathbb{R}^3)$ to* (5.1.1) *with $\fint_Y u^m = m$.*

*Proof. (i)* It suffices to show that every distributional solution $w \in L^2_\sharp(Y;\mathbb{C}^3)$ to (5.1.1) with $\fint_Y w = 0$ vanishes identically. As $w$ is curl-free in $Y$, we can apply Proposition 2.6 and find a potential $\Theta \in H^1_\sharp(Y;\mathbb{C})$ and a constant $c \in \mathbb{C}^3$ such that $w = \nabla\Theta + c$ in $L^2_\sharp(Y;\mathbb{C}^3)$. The volume average of $w$ vanishes and hence $c = 0$. Equation (5.1.1c) implies $\nabla\Theta = 0$ in $\Sigma$, and since $\Sigma$ is connected, we infer that $\Theta$ is constant in $\Sigma$. Combining this observation with equation (5.1.1b), we deduce that $\Theta \in H^1_\sharp(Y;\mathbb{C})$ is a weak solution to

$$\begin{cases} -\Delta\Theta = 0 \text{ in } \Sigma^*\,, \\ \quad\ \Theta = d \text{ in } \Sigma\,, \end{cases}$$

for some $d \in \mathbb{C}$. The unique solution to this problem is $\Theta \equiv d$. Hence $w = \nabla\Theta = 0$ in $Y$ and the claim is proved.

*(ii)* Suppose $u^k \in L^2_\sharp(Y;\mathbb{C}^3)$ is a distributional solution to (5.1.1) with $\fint_Y u^k = k \in \mathcal{L}_\Sigma$. Due to equation (5.1.1a), there exist a potential $\Theta \in H^1_\sharp(Y;\mathbb{C})$ and a constant $c \in \mathbb{C}^3$ such that $u^k = \nabla\Theta + c$ in $L^2_\sharp(Y;\mathbb{C}^3)$. As $\fint_Y u^k = k$, we infer that $c = k$. Let $\gamma\colon [0,1] \to \iota(\Sigma)$ be a $k$-loop in $\Sigma$ and denote by $\tilde\gamma\colon [0,1] \to \Sigma$ a lift of $\gamma$. By equation (5.1.1c), the field $u$ vanishes identically in $\Sigma$ and hence

$$0 = \int_\gamma u = \int_0^1 \big\langle u(\tilde\gamma(t)), \dot{\tilde\gamma}(t)\big\rangle\,\mathrm{d}t = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big[\Theta(\tilde\gamma(t))\big]\,\mathrm{d}t + \langle k, \tilde\gamma(1) - \tilde\gamma(0)\rangle\,.$$

As $\Theta$ is periodic and $\gamma$ is a $k$-loop, we conclude that

$$0 = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big[\Theta(\tilde\gamma(t))\big]\,\mathrm{d}t + \big\langle k, \tilde\gamma(1) - \tilde\gamma(0)\big\rangle = |k|^2\,.$$

This is a contradiction since $k \neq 0$.

*(iii)* Fix $m \in \mathcal{N}_\Sigma$ which satisfies $\langle m, k\rangle = 0$ for all $k \in \mathcal{N}_\Sigma$. By Proposition 4.8, there exists a potential $\Theta \in H^1_\sharp(\Sigma;\mathbb{R}^3)$ such that $\nabla\Theta = m$ in $L^2_\sharp(\Sigma;\mathbb{R}^3)$. Let $\Phi \in H^1_\sharp(Y;\mathbb{R})$ be a weak solution to

$$\begin{cases} -\Delta\Phi = 0 \ \text{ in } \Sigma^*\,, \\ \quad\ \Phi = \Theta \text{ in } \Sigma\,. \end{cases}$$

Setting $u^m := -\nabla\Phi + m$ provides us with a distributional solution to (5.1.1). As $\Phi$ is a periodic function, $\fint_Y u^m = m$ and the claim is proved. $\square$

We denote the space of distributional solutions to (5.1.1) by $X^E$ and set

$$A^E := \left\{ \fint_Y u \;\middle|\; u \in X^E \right\}. \tag{5.1.6}$$

Clearly, $A^E$ is a subspace of $\mathbb{C}^3$. In the definition of the geometric average, the space $A^{\mathcal{V}}(\Sigma^*)$, given in (4.3.3), plays a crucial role. The next result allows us to simplify the derivation of the effective equations; see Section 5.4.

Lemma 5.3. – *The two spaces $A^E$ and $A^{\mathcal{V}}(\Sigma^*)$ coincide for every admissible microstructure $\Sigma \subset Y$.*

*Proof.* The solution space $X^E$ is a subset of $\mathcal{V}(\Sigma^*)$ and hence $A^E \subset A^{\mathcal{V}}(\Sigma^*)$. In order to show the converse inclusion, we prove that for each $\phi \in \mathcal{V}(\Sigma^*)$ there is a potential $\Theta \in H^1_\sharp(Y;\mathbb{C})$ such that $\phi + \nabla\Theta \in X^E$. Indeed, let $\Theta \in H^1_\sharp(Y;\mathbb{C})$ be the unique distributional solution to

$$\begin{cases} -\Delta\Theta = \operatorname{div} \phi & \text{in } \Sigma^*\,, \\ \quad \Theta = 0 & \text{in } \Sigma\,. \end{cases}$$

One readily checks that $\phi + \nabla\Theta$ satisfies equations (5.1.1a) and (5.1.1c). Moreover, by construction of $\Theta$, the distributional divergence of $\phi + \nabla\Theta$ vanishes in $\Sigma^*$. Consequently, $\phi + \nabla\Theta \in X^E$. The potential $\Theta$ is periodic and hence

$$\fint_Y \phi = \fint_Y \phi + \int_Y \nabla\Theta = \fint_Y \big(\phi + \nabla\Theta\big) \in A^E\,.$$

As $\phi \in \mathcal{V}(\Sigma^*)$ was chosen arbitrarily, $A^{\mathcal{V}}(\Sigma^*) \subset A^E$ and the claim is proved. □

We note that, due to Lemma 5.2 *(i)*, for every vector $c \in A^E$ there exists a unique element $u \in X^E$ such that $\fint_Y u = c$.

Proposition 5.4. (Characterisation of the solution space) – *If $\Sigma \subset Y$ is an admissible microstructure, then there holds*

$$\dim X^E = \dim A^E \le 3\,.$$

*More precisely, let $\{b^j \,|\, j \in I\}$ be a basis of $A^E$ with $I \subset \{1,2,3\}$. For each $j \in I$ we denote by $u^j$ the unique distributional solution to* (5.1.1) *with $\fint_Y u^j = b^j$. Then every $u \in X^E$ can be written as a linear combination,*

$$u = \sum_{j \in I} \alpha_j u^j \quad \text{in } L^2_\sharp(Y;\mathbb{C}^3)\,, \tag{5.1.7}$$

*with coefficients $\alpha_j \in \mathbb{C}$ for $j \in I$.*

*Proof.* Fix $u \in X^E$. As $\fint_Y u \in A^E$, we find $\alpha_j \in \mathbb{C}$ for $j \in I$ such that

$$\fint_Y u = \sum_{j\in I} \alpha_j \fint_Y u^j\,.$$

Define the field $w\colon Y \to \mathbb{C}^3$ by $w := u - \sum_{j\in I} \alpha_j u^j$. Then $w \in X^E$ and

$$\fint_Y w = \fint_Y u - \sum_{j\in I} \alpha_j \fint_Y u^j = 0.$$

Due to Lemma 5.2 *(i)*, distributional solutions to the cell problem (5.1.1) are unique and hence $w = 0$. This proves (5.1.7) and the claim follows. □

This characterisation is somewhat abstract, since it might not be easy to determine a basis of $A^E$. In many cases, however, the solution space can be characterised using $k$-loops; see Proposition 5.13.

## 5.2 The oscillating magnetic field

In this section we derive and analyse the cell problem for the two-scale limit $H_0$. The analysis is more delicate than in the previous section due to the fact that the geometric average can, in general, not be used to show a uniqueness result.

**The cell problem and its solution space.** We recall that the energy-bound (5.0.2) ensures the existence of a subsequence $(H^\eta)_\eta$ and a field $H_0$ such that $H^\eta \xrightarrow{2} H_0$. Moreover, the weak $L^2$-limit $H$ of $(H^\eta)_\eta$ is connected with the two-scale limit $H_0$ by (5.0.3).

In order to state the cell problem of $H_0$ we define the function space

$$\mathcal{X} := \left\{ v \in L^2_\sharp(Y;\mathbb{C}^3) \,\middle|\, \operatorname{curl} v = 0 \text{ in } \Sigma^* \right\}.$$

We recall from Section 4.3 that the geometric average can be applied to every element of $\mathcal{X}$.

Lemma 5.5. (Cell problem) – *Let $\Omega \subset \mathbb{R}^3$ be as described as in Section 4.1 and let $\Sigma \subset Y$ be an admissible microstructure. Let $(E^\eta, H^\eta)_\eta$ be a sequence satisfying (A3), that two-scale converges to $(E_0, H_0)$ and that weakly converges in $L^2(\Omega;\mathbb{C}^3) \times L^2(\Omega;\mathbb{C}^3)$ to $(E, H)$. Then for almost all $x \in R$ the two-scale limit $H_0 = H_0(x,\cdot) \in L^2_\sharp(Y;\mathbb{C}^3)$ satisfies*

$$\begin{cases} \operatorname{curl}_y H_0 = 0 & \textit{in } \Sigma^* , \\ \operatorname{div}_y H_0 = 0 & \textit{in } Y , \\ H_0 = 0 & \textit{in } \Sigma , \end{cases} \qquad \begin{matrix} (5.2.1a) \\ (5.2.1b) \\ (5.2.1c) \end{matrix}$$

*in the distributional sense. Moreover, for almost all $x \in \Omega$ the following identity holds:*

$$\int_Y \big\langle H_0(x,y), v(y) \big\rangle \, \mathrm{d}y = 0 \qquad \textit{for all } v \in \mathcal{X} \textit{ with } \oint_{\Sigma^*} v = 0 . \tag{5.2.2}$$

*Outside the meta-material $R$ the two-scale limit $H_0$ is $y$-independent; that is, $H_0(x,y) = H(x)$ for almost all $x \in \Omega \setminus R$ and almost all $y \in Y$.*

*Proof.* We proceed in three steps.

*Step 1. (Derivation of equations* (5.2.1a)-(5.2.1c)*)* This proof mimics the one for Lemma 5.1. For $\theta \in C_c^\infty(\Omega;\mathbb{R})$ and $\psi \in C_c(\Sigma^*;\mathbb{C}^3)$, we define $\varphi(x,y) := \theta(x)\psi(y)$ for all $x \in \Omega$ and $y \in Y$. We further set $\psi_\eta(\cdot) := \psi(\cdot/\eta)$ and $\varphi_\eta(\cdot) := \theta(\cdot)\psi_\eta(\cdot)$ for each $\eta > 0$. As the sequence $(H^\eta)_\eta$ two-scale converges to $H_0$, which in particular implies that $(H^\eta)_\eta$ is bounded in $L^2(\Omega;\mathbb{C}^3)$, we find that

$$\begin{aligned} \lim_{\eta\to 0} \int_\Omega \eta \langle H^\eta, \operatorname{curl}[\varphi_\eta] \rangle &= \lim_{\eta\to 0} \int_\Omega \eta \big\langle H^\eta, \nabla\theta \wedge \psi_\eta \big\rangle + \lim_{\eta\to 0} \int_\Omega \theta \big\langle H^\eta, \operatorname{curl} \psi_\eta \big\rangle \\ &= \int_\Omega \theta \int_{\Sigma^*} \big\langle H_0, \operatorname{curl} \psi \big\rangle . \end{aligned} \tag{5.2.3}$$

On the other hand, $H^\eta$ is a distributional solution to (5.0.1b) and hence

$$\lim_{\eta\to 0}\int_\Omega \eta\big\langle H^\eta, \operatorname{curl}[\varphi_\eta]\big\rangle = -\mathrm{i}\omega\varepsilon_0 \lim_{\eta\to 0}\int_\Omega \eta\big\langle E^\eta, \varphi_\eta\big\rangle = 0\,. \tag{5.2.4}$$

Combining equations (5.2.3) and (5.2.4), we deduce that $H_0(x,\cdot)$ is a distributional solution to (5.2.1a) for almost all $x \in \Omega$.

Choose $\theta \in C_c^\infty(\Omega;\mathbb{R})$, $\psi \in C_\sharp^\infty(Y;\mathbb{C})$ and define $\psi_\eta(\cdot) \coloneqq \psi(\cdot/\eta)$ for $\eta > 0$. The distributional divergence of $H^\eta$ vanishes in $\Omega$ because of equation (5.0.1a). Thus,

$$0 = \lim_{\eta\to 0}\int_\Omega \eta\psi_\eta\big\langle H^\eta, \nabla\theta\big\rangle + \lim_{\eta\to 0}\int_\Omega \theta\big\langle H^\eta, \nabla\psi_\eta\big\rangle = \int_\Omega \theta \int_Y \big\langle H_0, \nabla\psi\big\rangle\,, \tag{5.2.5}$$

where we used the two-scale convergence of $(H^\eta)_\eta$ to obtain the last equality. Equation (5.2.5) implies that $H_0(x,\cdot)$ satisfies (5.2.1b) for almost all $x \in \Omega$.

In order to prove that $H_0(x,\cdot)$ solves (5.2.1c), we choose $\theta \in C_c^\infty(\Omega;\mathbb{R})$ and $\psi \in C_c^\infty(\Sigma;\mathbb{C}^3)$. For each $\eta > 0$ we set $\varphi_\eta(\cdot) \coloneqq \theta(\cdot)\psi(\cdot/\eta)$. The vector field $H^\eta$ satisfies (5.0.1c) and thus

$$0 = \lim_{\eta\to 0}\int_\Omega \langle H^\eta, \varphi_\eta\rangle = \int_\Omega \theta \int_\Sigma \langle H_0, \psi\rangle\,.$$

Consequently, $H_0(x,\cdot)$ solves (5.2.1) for almost every $x \in R$.

*Step 2. (Derivation of* (5.2.2)*)* Fix a vector field $v \in \mathcal{X}$ with $\oint_{\Sigma^*} v = 0$ and a test function $\theta \in C_c^\infty(\Omega;\mathbb{R})$. For $\eta > 0$ we set $\varphi_\eta(\cdot) \coloneqq \theta(\cdot)v(\cdot/\eta)$. The sequence $(H^\eta)_\eta$ two-scale converges to $H_0$,

$$\lim_{\eta\to 0}\int_\Omega \big\langle H^\eta, \varphi_\eta\big\rangle = \int_\Omega \theta\bigg(\int_Y \big\langle H_0, v\big\rangle\bigg)\,. \tag{5.2.6}$$

On the other hand, $E^\eta$ is a solution to (5.0.1a) and thus

$$\begin{aligned}\int_\Omega \langle H^\eta, \varphi_\eta\rangle &= -\frac{\mathrm{i}}{\omega\mu_0}\int_\Omega \big\langle E^\eta, \operatorname{curl}[\varphi_\eta]\big\rangle \\ &= -\frac{\mathrm{i}}{\omega\mu_0}\int_{\Omega\setminus\Sigma_\eta} \big\langle E^\eta, \nabla\theta \wedge v(\cdot/\eta)\big\rangle\,, \end{aligned}\tag{5.2.7}$$

where we used that $\operatorname{curl} v = 0$ in $\Sigma^*$ and $E^\eta = 0$ in $\Sigma_\eta$ to obtain the last equation. As the two-scale limit $E_0$ satisfies the cell problem (5.1.1), we infer that $E_0(x,\cdot) \in \mathcal{V}(\Sigma^*)$ for almost all $x \in \Omega$; the space $\mathcal{V}(\Sigma^*)$ is defined in (4.3.2). Sending $\eta \to 0$ in equation (5.2.7) and using the definition of the geometric average in Theorem 4.4, we find that

$$\begin{aligned}\lim_{\eta\to 0}\int_\Omega \langle H^\eta, \varphi_\eta\rangle &= -\frac{\mathrm{i}}{\omega\mu_0}\int_\Omega\int_{\Sigma^*} \langle E_0, \nabla\theta\wedge v\rangle = \frac{\mathrm{i}}{\omega\mu_0}\int_\Omega \Big\langle \nabla\theta, \int_{\Sigma^*} E_0 \wedge v\Big\rangle \\ &= \frac{\mathrm{i}}{\omega\mu_0}\int_\Omega \Big\langle \nabla\theta, \Big(\int_{\Sigma^*} E_0\Big) \wedge \Big(\oint_{\Sigma^*} v\Big)\Big\rangle\,. \end{aligned}\tag{5.2.8}$$

Combining equations (5.2.6) and (5.2.8) and using that $\oint_{\Sigma^*} v = 0$, we deduce that

$$\int_\Omega \theta\bigg(\int_Y \big\langle H_0, v\big\rangle\bigg) = 0\,.$$

This proves equation (5.2.2), since $\theta \in C_c^\infty(\Omega;\mathbb{R})$ was chosen arbitrarily.

*Step 3.* For almost every $x \in \Omega \setminus R$ there holds $\operatorname{curl}_y H_0(x\cdot) = 0$ in $Y$. To see this, we choose $\theta \in C_c^\infty(\Omega \setminus R;\mathbb{R})$ and $\psi \in C_\sharp^\infty(Y;\mathbb{C}^3)$, and define the field $\varphi_\eta(\cdot) := \theta(\cdot)\psi(\cdot/\eta)$ for each $\eta > 0$. Due to equation (5.0.1b) we have

$$0 = \lim_{\eta\to 0} \int_{\Omega\setminus R} \eta \langle H^\eta, \operatorname{curl}[\varphi_\eta]\rangle = \int_{\Omega\setminus R} \theta \int_Y \langle H_0, \operatorname{curl}\,\psi\rangle\,,$$

which proves that $H_0(x,\cdot)$ is curl-free in $Y$ for almost all $x \in \Omega \setminus R$. We showed above that $\operatorname{div}_y H_0(x,\cdot) = 0$ in $Y$ for almost all $x \in \Omega \setminus R$. Applying Lemma 2.4 we deduce that $H_0(x,\cdot)$ is a constant vector field in $Y$ and the claim is proved.□

System (5.2.1) is usually referred to as the cell problem for $H_0$. As was shown by Lipton and Schweizer [LS18b, Proposition 2.1], there might be more than three linearly independent solutions to this cell problem. The identity (5.2.2), which will ensure that $H_0$ lies in an at most three-dimensional subspace of the solution space of (5.2.1), was first discussed in [PSU19, Corollary 5.2].

The solution space to the classical cell problem (5.2.1) is defined as

$$X := \left\{ v \in L_\sharp^2(Y;\mathbb{C}^3) \,\middle|\, v \text{ is a distributional solution to (5.2.1)} \right\}. \tag{5.2.9}$$

Equipped with the $L_\sharp^2(Y;\mathbb{C}^3)$-scalar product, $X$ is a Hilbert space. We further introduce the following two subspaces of $X$,

$$X_0 := \left\{ v \in X \,\middle|\, \oint_{\Sigma^*} v = 0 \right\} \tag{5.2.10}$$

and

$$X^H := \left\{ w \in X \,\middle|\, \int_Y \langle w, v\rangle = 0 \text{ for all } v \in X_0 \right\}. \tag{5.2.11}$$

As the geometric average is a bounded operator, $X_0$ is a closed subspace of $X$. Moreover, by definition, $X^H$ is the $L^2$-orthogonal complement of $X_0$; we may therefore write $X = X_0 \oplus X^H$. We already explained that $X$ might have more than three dimensions; it is, however, always a finite dimensional space.

Proposition 5.6. – *For every admissible microstructure $\Sigma \subset Y$, the space $X$ of distributional solutions to* (5.2.1) *is finite dimensional.*

As we do not use this result, we postpone its proof to Appendix B.

**The appropriate function space and its analysis.** The space $X_0$ can be nontrivial and hence, there might be two or more solutions to the cell problem (5.2.1) with the same geometric average. However, if we only consider distributional solutions in $X^H$, then uniqueness of solutions necessarily holds. Thanks to identity (5.2.2) we can indeed focus on $X^H$ as $H_0(x,\cdot) \in X^H$ for almost all $x \in R$.

We observe that the solution space $X$ is a subset of $\mathcal{X}(\Sigma^*)$, the latter space being defined in (4.3.1); the geometric average can thus be applied to elements of $X$. Before analysing the space $X^H$, we introduce the space $A^H$ of attainable geometric averages of fields in $X^H \subset X$,

$$A^H := \left\{ \oint_{\Sigma^*} v \,\middle|\, v \in X^H \right\}. \tag{5.2.12}$$

Let us recall from Section 4.3 that the geometric average is a surjective linear map $\mathcal{X}(\Sigma^*) \to A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$. It is thus remarkable that the two subspaces $A^H$ and $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ of $\mathbb{C}^3$ coincide.

LEMMA 5.7. – *The two spaces $A^H$ and $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ coincide for every admissible microstructure $\Sigma \subset Y$.*

*Proof.* The solution space $X$ of the cell problem (5.2.1), defined in (5.2.9), is a subset of $\mathcal{X}(\Sigma^*)$. As the geometric average maps $\mathcal{X}(\Sigma^*)$ onto $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ and as $X^H \subset X$, we infer that $A^H \subset A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$.

Fix $k \in A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ with $k \neq 0$. Then, by Theorem 4.4, there is an element $v^k \in \mathcal{X}(\Sigma^*)$ with $\oint_{\Sigma^*} v^k = k$. Let $\Theta \in H^1_\sharp(\Sigma^*;\mathbb{C})$ with $⨍_{\Sigma^*} \Theta = 0$ be the unique solution to

$$\int_{\Sigma^*} \langle \nabla\Theta, \nabla\varphi \rangle = -\int_{\Sigma^*} \langle v, \nabla\varphi \rangle \quad \text{for all } \varphi \in H^1_\sharp(\Sigma^*;\mathbb{C})\,. \tag{5.2.13}$$

Defining the vector field

$$w^k := \begin{cases} v^k + \nabla\Theta & \text{in } \Sigma^* \\ 0 & \text{in } \Sigma \end{cases}$$

provides us with an element of $L^2_\sharp(Y;\mathbb{C}^3)$. We claim that $w^k \in X^H$. Clearly, $w^k$ is a distributional solution to equations (5.2.1a) and (5.2.1c). Let us show that the distributional divergence of $w^k$ vanishes in $Y$. To do so, fix $\varphi \in C^\infty_\sharp(Y;\mathbb{C})$. As $\varphi \in H^1_\sharp(\Sigma^*;\mathbb{C})$, we can use (5.2.13) and find that

$$\int_Y \langle w^k, \nabla\varphi \rangle = \int_{\Sigma^*} \langle v^k, \nabla\varphi \rangle + \int_{\Sigma^*} \langle \nabla\Theta, \nabla\varphi \rangle = 0\,.$$

Consequently, $w^k \in X$ and, by Lemma 4.5 *(ii)*, we find that $\oint_{\Sigma^*} w^k = \oint_{\Sigma^*} v^k = k$. As $k \neq 0$, we infer that $w^k \in X^H$. The constant vector field $v^0 := 0$ is an element of $X^H$ with $\oint_{\Sigma^*} v^0 = 0$. This shows that $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp} \subset A^H$. ☐

Having chosen the appropriate functions space $X^H$, in which we seek solutions to the cell problem (5.2.1), the analysis of the cell problem is rather straightforward.

LEMMA 5.8. (Analysis of the cell problem) – *Let $\Sigma \subset Y$ be an admissible microstructure. Then the following statements hold:*

*(i) Given any $c \in \mathbb{C}^3$ there is at most one distributional solution $v \in X^H$ to* (5.2.1) *with $\oint_{\Sigma^*} v = c$.*

*(ii) For every $c \in A^H$ there is a unique element $v^c \in X^H$ with $\oint_{\Sigma^*} v^c = c$.*

*(iii) If $c \notin A^H$, then there is no element $v \in X$ with $\oint_{\Sigma^*} v = c$.*

*Proof. (i)* It suffices to show that every $w \in X^H$ with $\oint_{\Sigma^*} w = 0$ vanishes identically. This, however, follows from the definition of the space $X^H$ as the $L^2$-orthogonal complement of $X_0$.

*(ii)* Given $c \in A^H$, the existence of a field $\tilde{v}^c \in X$ with $\oint_{\Sigma^*} \tilde{v}^c = c$ follows from definition of the set $A^H$. The space $X$ admits the decomposition $X = X_0 \oplus X^H$

and hence we can write $\tilde{v}^c = \tilde{v}^c_1 + \tilde{v}^c_2$, where $\tilde{v}^c_1 \in X_0$ and $\tilde{v}^c_2 \in X^H$. Setting $v^c := \tilde{v}^c_2$ provides us with an element of $X^H$ that satisfies $\oint_{\Sigma^*} v^c = c$. Uniqueness of $v^c$ follows from *(i)*.

*(iii)* The space $X$ admits the orthogonal decomposition $X = X_0 \oplus X^H$. Every element $v \in X_0$ satisfies $\oint_{\Sigma^*} v = 0$. Thus, for any constant vector $c \in \mathbb{C}^3 \setminus \{0\}$ there holds, $c \in A^H$ if and only if there exists $v \in X$ with $\oint_{\Sigma^*} v = c$. This shows statement *(iii)*. ☐

Compared to Lemma 5.2, the previous result seems to be unsatisfactory as it provides only an abstract condition for the existence (and non-existence) of distributional solutions to (5.2.1) with prescribed geometric averages. As it turns out, a statement similar to Lemma 5.2 cannot hold for $X^H$. We do have, however, the following relations.

LEMMA 5.9. – *Let $\Sigma \subset Y$ be an admissible microstructure, and let $\mathcal{L}_{\Sigma^*}$ and $\mathcal{N}_{\Sigma^*}$ be the sets given in* (4.3.17) *and* (4.3.18).

*(i) If $m \in \mathcal{N}_{\Sigma^*}$ satisfies $\langle m, k\rangle = 0$ for all $k \in \mathcal{L}_{\Sigma^*}$, then $m$ is not an element of $A^H$; in particular, there is no $v^m \in X$ with $\oint_{\Sigma^*} v^m = m$.*

*(ii) If $k \in \mathcal{L}_{\Sigma^*}$, then there exists $v^k \in X$ with*

$$\oint_{\Sigma^*} v^k = \pi_{A^H}(k)\,, \tag{5.2.14}$$

*where $\pi_{A^H} : \mathbb{C}^3 \to \mathbb{C}^3$ denotes the orthogonal projection onto $A^H$.*

*Proof. (i)* By Proposition 4.8, there exists $\Theta_m \in H^1_\sharp(\Sigma^*;\mathbb{R})$ with $\nabla\Theta_m = m$. Applying the properties of the geometric average from Lemma 4.5 we obtain that

$$0 = \oint_{\Sigma^*} \nabla\Theta_m = \oint_{\Sigma^*} m = \pi_{A^H}(m)\,,$$

As $m \neq 0$, we deduce that $m \notin A^H$.

*(ii)* Fix $k \in \mathcal{L}_{\Sigma^*}$. Let $\Theta \in H^1_\sharp(\Sigma^*;\mathbb{C})$ be a weak solution to

$$\int_{\Sigma^*} \langle \nabla\Theta, \nabla\varphi\rangle = -\int_{\Sigma^*} \langle k, \nabla\varphi\rangle \quad \text{for all } \varphi \in H^1_\sharp(\Sigma^*;\mathbb{C})\,. \tag{5.2.15}$$

We claim that the vector field

$$v^k := \begin{cases} \nabla\Theta + k & \text{in } \Sigma^* \\ 0 & \text{in } \Sigma \end{cases}$$

is an element of $X^H$. One readily checks that $v^k$ solves the equations (5.2.1a) and (5.2.1c) in the distributional sense. In order to show that the distributional divergence of $v^k$ vanishes in $Y$, we take a test function $\varphi \in C^\infty_\sharp(Y;\mathbb{C})$. As $\varphi \in H^1_\sharp(\Sigma^*;\mathbb{C})$, we can use (5.2.15) and find

$$\int_Y \langle v^k, \nabla\varphi\rangle = \int_{\Sigma^*} \langle \nabla\Theta + k, \nabla\varphi\rangle = \int_{\Sigma^*} \langle \nabla\Theta, \nabla\varphi\rangle + \int_{\Sigma^*} \langle k, \nabla\varphi\rangle = 0\,,$$

which proves that $v^k$ is a distributional solution to the cell problem (5.2.1). Applying the properties of the geometric average from Lemma 4.5 yields

$$\oint_{\Sigma^*} v^k = \oint_{\Sigma^*} \nabla\Theta + \oint_{\Sigma^*} k = \pi_{A^H}(k)\,.$$

This proves the claim. □

Remarks 5. – *(a) Let us discuss equation* (5.2.14) *in more detail. The spaces $A^E$ and $A^{\mathcal{V}}(\Sigma^*)$ coincide, by Lemma 5.3. Moreover, the identity $A^H = A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ holds, due to Lemma 5.7. Thus, using the definition* (4.3.4) *of $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$,*

$$A^H = A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp} = \begin{cases} \{0\} & \textit{if } \dim A^E = 0 \\ \left(A^E\right)^{\perp} & \textit{if } \dim A^E = 1 \\ \mathbb{C}^3 & \textit{if } \dim A^E \geq 2 \end{cases}.$$

*Due to Lemma 5.2(i), the linear map $\fint_Y \colon X^E \to A^E$ is a vector space isomorphism. We consider three cases:*

Case 1. *If the solution space $X^E$ is zero-dimensional, then $A^H = \{0\}$ and hence for every element $v \in X$ there holds $\oint_{\Sigma^*} v = 0$.*

Case 2. *If $\dim X^E = 1$, then $A^H = (A^E)^{\perp}$ and equation* (5.2.14) *reads*

$$\oint_{\Sigma^*} v^k = \pi_{(A^E)^{\perp}}(k)\,.$$

Case 3. *If $\dim X^E \geq 2$, then $A^H = \mathbb{C}^3$ and equation* (5.2.14) *reads*

$$\oint_{\Sigma^*} v^k = k\,.$$

*Consequently, for any $k \in \mathcal{L}_{\Sigma^*}$ there exists a unique $v^k \in X^H$ with $\oint_{\Sigma^*} v^k = k$.*

*(b) The above considerations allow us to discuss the relation between the two subspaces $\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*}$ and $A^H$ of $\mathbb{C}^3$. For a generic microstructure $\Sigma$, neither is $\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*}$ a subspace of $A^H$ nor is $A^H$ a subspace of $\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*}$. We again consider three cases, depending on the dimension of $X^E$.*

Case 1. *If $\dim X^E = 0$, then $A^H = \{0\}$. However, we find admissible microstructures $\Sigma_j$ such that $\dim \operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma_j^*} = j$ for all $j \in \{0,1,2,3\}$. Indeed, fix $r \in (0,1/2)$ and consider for $j \in \{1,2,3\}$ the following sets*

$$\Sigma_0 := \left\{ y \in Y \;\middle|\; \left\| y - \frac{1}{2}(1,1,1) \right\|^2 > r^2 \right\}$$

*and*

$$Z_j := \left\{ (y_1, y_2, y_3) \in Y \;\middle|\; \sum_{\substack{l=1 \\ l \neq j}}^{3} \left( y_l - \frac{1}{2} \right)^2 < r^2 \right\}.$$

*We set*

$$\Sigma_1 \coloneqq Y \setminus \overline{Z}_1\,, \quad \Sigma_2 \coloneqq Y \setminus \big(\overline{Z}_1 \cup \overline{Z}_2\big)\,, \quad \Sigma_3 \coloneqq Y \setminus \big(\overline{Z}_1 \cup \overline{Z}_2 \cup \overline{Z}_3\big)\,.$$

*We note that all four microstructures* $\Sigma_0, \dots, \Sigma_3$ *are admissible. Moreover, for each* $j \in \{0,1,2,3\}$*, there exist* $\mathrm{e}_1$*-,* $\mathrm{e}_2$*-, and* $\mathrm{e}_3$*-loops in* $\Sigma_j$*, which implies, by Proposition 5.13, that the corresponding* $E_0$*-cell problem* (5.1.1) *has only the trivial solution; that is* $\dim X^E = 0$ *for all four microstructures.*

*On the other hand, one readily checks that there is no k-loop in* $\Sigma_0^*$ *for any* $k \in \mathbb{Z}^3 \setminus \{0\}$*, and thus* $\dim \operatorname{span} \mathcal{L}_{\Sigma_0^*} = 0$*. Fix* $j \in \{1,2,3\}$*. If there is a k-loop in* $\Sigma_j^*$*, then k has the form* $k = \sum_{l=1}^{j} \alpha_l \, \mathrm{e}_l$ *for* $\alpha_l \in \mathbb{Z}$*. This shows that* $\dim \operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma_j^*} = j$*.*

*Summarising the above considerations, we find that* $A^H \subset \operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*}$ *if* $\dim X^E = 0$*.*

Case 2. *If* $\dim X^E = 1$*, we do not know which of the two inclusions hold for a generic microstructure* $\Sigma$*. However, the identity* $\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*} = A^H$ *is, in general, wrong. Indeed, consider the two cylinders* $Z_1$*,* $Z_2$ *defined in case 1, and set* $\Sigma \coloneqq Z_1 \cup Z_2$*. As there are* $\mathrm{e}_1$*- and* $\mathrm{e}_2$*-loops in* $\Sigma$ *but no* $\mathrm{e}_3$*-loop, we infer from Proposition 5.13 that* $A^E = \mathbb{C}\,\mathrm{e}_3$*. Consequently,* $A^H = \mathbb{C}\,\mathrm{e}_1 \oplus \mathbb{C}\,\mathrm{e}_2$*. On the other hand, one readily checks that* $\mathcal{L}_{\Sigma^*} = \mathbb{Z}^3 \setminus \{0\}$ *and thus* $\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*} = \mathbb{C}^3$*. That is,* $A^H \subsetneq \operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*}$*.*

Case 3. *If* $\dim X^E \geq 2$*, we deduce from case 3 in (a) that* $\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*} \subset A^H$*. Adding an additional assumption on* $\Sigma$*, we can show that* $\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*} = A^H = \mathbb{C}^3$*; see Proposition 5.14.*

Having Lemma 5.8 at hand, we can turn to the characterisation of the solution space $X^H$. Similar to Proposition 5.4, this result is rather abstract for general microstructures; we stress, however, that this characterisation is not needed to derive the effective equations.

Proposition 5.10. (Characterisation of the solution space) – *If* $\Sigma \subset Y$ *is an admissible microstructure, then*

$$\dim X^H = \dim A^H \leq 3\,.$$

*More precisely, let* $\{b^j \mid j \in I\}$ *be a basis of* $A^H$*, and denote by* $v^j$ *the distributional solution to* (5.2.1) *with geometric average equal to* $b^j$*. Then every* $v \in X^H$ *can be written as a linear combination,*

$$v = \sum_{j \in I} \alpha_j v^j \qquad \text{in } L^2_\sharp(Y;\mathbb{C}^3)\,, \tag{5.2.16}$$

*with coefficients* $\alpha_j \in \mathbb{C}$ *for* $j \in I$*.*

*Proof.* Fix $v \in X^H$. As $\oint_{\Sigma^*} v \in A^H$, we find $\alpha_j \in \mathbb{C}$ for $j \in I$ such that

$$\oint_{\Sigma^*} v = \sum_{j \in J} \alpha_j b^j = \sum_{j \in J} \alpha_j \oint_{\Sigma^*} v^j\,.$$

Define the field $w\colon Y \to \mathbb{C}^3$ by $w := v - \sum_{j\in I} \alpha_j v^j$. Then $w \in X^H$ and

$$\oint_{\Sigma^*} w = \oint_{\Sigma^*} v - \sum_{j\in J} \alpha_j \oint_{\Sigma^*} v^j = 0\,.$$

From Lemma 5.8*(i)*, we deduce that $w = 0$ and the claim is proved. □

## 5.3 Effective material parameters and equations

The objective of this section is the derivation of the effective system as well as the definition of the effective permittivity and effective permeability. Let us recall that the subspaces $A^E$ and $A^H$ of $\mathbb{C}^3$ are defined in (5.1.6) and (5.2.12), respectively. Denote by $\pi_{A^E} : \mathbb{C}^3 \to \mathbb{C}^3$ the orthogonal projection onto $A^E$.

Before we define the effective material parameters, we need the following technical result.

LEMMA 5.11. – *Let $\Sigma \subset Y$ be an admissible microstructure, and let $A^E$ be the space defined in* (5.1.6). *Then there exists one and only one linear map $\varepsilon^{\text{eff}} : A^E \to A^E$ that satisfies the identity*

$$\left\langle \varepsilon^{\text{eff}} \left( \fint_Y u_1 \right), \fint_Y u_2 \right\rangle = \int_Y \langle u_1, u_2 \rangle \tag{5.3.1}$$

*for all $u_1, u_2 \in X^E$.*

*Proof. Existence.* Due to Lemma 5.2, for every $c \in A^E$ there is a unique element $u^c \in X^E$ with $\fint_Y u^c = c$. The uniqueness of the element $u^c$ implies that the map $\varepsilon^{\text{eff}} : A^E \to (A^E)^*$,

$$\left\langle \varepsilon^{\text{eff}} \left( \fint_Y u_1 \right), \fint_Y u_2 \right\rangle_{(A^E)^*, A^E} := \fint_Y \langle u_1, u_2 \rangle \tag{5.3.2}$$

is linear. As $A^E$ is a finite dimensional vector space, we may identify $(A^E)^*$ and $A^E$. In this way, we obtain a linear map $\varepsilon^{\text{eff}} : A^E \to A^E$ that satisfies (5.3.1).

*Uniqueness.* Assume $L\colon A^E \to A^E$ is a linear map such that for all $u_1, u_2 \in X^E$ the identity

$$\left\langle L \left( \fint_Y u_1 \right), \fint_Y u_2 \right\rangle = \int_Y \langle u_1, u_2 \rangle \tag{5.3.3}$$

holds. Identifying the finite dimensional vector space $A^E$ with its anti-dual $(A^E)^*$, we conclude from (5.3.2) and (5.3.3) that $(L - \varepsilon^{\text{eff}})(c) = 0$ in $(A^E)^*$ for all $c \in A^E$. This implies that $L = \varepsilon^{\text{eff}}$ on $A^E$. □

The unique linear map $\varepsilon^{\text{eff}} : A^E \to A^E$ from Lemma 5.11 is called the relative permittivity of the metamaterial located in $R$.

We recall from Lemma 5.8 that for every $k \in A^H$ there is a unique element $v^k \in X^H$ with $\oint_{\Sigma^*} v^k = k$. The linear map $\mu^{\text{eff}} : A^H \to \mathbb{C}^3$ given by

$$\mu^{\text{eff}} \left( \oint_{\Sigma^*} v \right) := \fint_Y v \quad \text{for } v \in X^H$$

is called the relative permeability of the meta-material located in $R$.

For $x \in \Omega$ we define the *effective permittivity* and the *effective permeability* as

$$\hat{\varepsilon}(x) := \begin{cases} \varepsilon^{\mathrm{eff}} & \text{if } x \in R \\ \mathrm{id}_{\mathbb{C}^{3\times 3}} & \text{if } x \in \Omega \setminus R \end{cases} \tag{5.3.4a}$$

and

$$\hat{\mu}(x) := \begin{cases} \mu^{\mathrm{eff}} & \text{if } x \in R \\ \mathrm{id}_{\mathbb{C}^{3\times 3}} & \text{if } x \in \Omega \setminus R \end{cases} . \tag{5.3.4b}$$

We recall that $E_0, H_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$ are the two-scale limits of $(E^\eta)_\eta$ and $(H^\eta)_\eta$, respectively. The *effective electromagnetic field* $(\hat{E}, \hat{H})\colon \Omega \to \mathbb{C}^3 \times \mathbb{C}^3$ is defined by

$$\hat{E}(x) := \fint_Y E_0(x,y)\,\mathrm{d}y \quad \text{and} \quad \hat{H}(x) := \oint_{\Sigma^*} H_0(x,\cdot)\,. \tag{5.3.5}$$

Note that $\hat{H}$ is defined on all of $\Omega$; outside the microstructure $\Omega \setminus R$, we assume free space—in other words, there is no microstructure. Thus, by Lemma 4.5*(i)*, for $x \in \Omega \setminus R$ there holds

$$\hat{H}(x) = \oint_Y H_0(x,\cdot) = \fint_Y H_0(x,y)\,\mathrm{d}y = H(x)\,.$$

Theorem 5.12. (Effective equations) – *Let $\Omega \subset \mathbb{R}^3$ and let $R \Subset \Omega$ be as in Section 4.1. Let $\Sigma \subset Y$ be an admissible microstructure, and let $\hat{\varepsilon}$ and $\hat{\mu}$ be the maps given in* (5.3.4). *Assume $(E^\eta, H^\eta)_\eta$ is a sequence satisfying (A3) on page 51. Then the effective electromagnetic field $(\hat{E}, \hat{H})$ satisfies*

$$\hat{E}(x) \in A^E \quad \textit{and} \quad \hat{H}(x) \in A^H \quad \textit{for almost all } x \in R\,. \tag{5.3.6}$$

*Moreover, $(\hat{E}, \hat{H})$ is a distributional solution to*

$$\begin{cases} \operatorname{curl} \hat{E} = \mathrm{i}\omega\mu_0\hat{\mu}\hat{H} & \textit{in } \Omega\,, & (5.3.7a) \\ \pi_{A^E}\big(\operatorname{curl} \hat{H}\big) = -\mathrm{i}\omega\varepsilon_0\pi_{A^E}\big(\hat{\varepsilon}\hat{E}\big) & \textit{in } \Omega\,, & (5.3.7b) \\ \operatorname{curl} \hat{H} = -\mathrm{i}\omega\varepsilon_0\hat{E} & \textit{in } \Omega \setminus R\,. & (5.3.7c) \end{cases}$$

*Proof.* Thanks to the preparation of the last sections, we can follow standard arguments to derive (5.3.7a)-(5.3.7c). We infer (5.3.6) by combining Lemmas 5.1 and 5.5 with (5.3.5).

*Step 1: Derivation of* (5.3.7a) *and* (5.3.7c). The sequence $(E^\eta, H^\eta)_\eta$ converges weakly in $L^2(\Omega; \mathbb{C}^3) \times L^2(\Omega; \mathbb{C}^3)$ to $(E, H)$. The distributional limit of equation (5.0.1a) thus reads

$$\operatorname{curl} E = \mathrm{i}\omega\mu_0 H \quad \text{in } \Omega\,. \tag{5.3.8}$$

By Lemma 3.3, the two-scale limit $E_0$ and the weak limit $E$ of $(E^\eta)_\eta$ are connected by the identity

$$E(x) = \fint_Y E_0(x,\cdot) \quad \text{for almost all } x \in \Omega\,.$$

Using the definition of the effective electric field $\hat{E}$ in (5.3.5), we find that $\hat{E}(x) = E(x)$ for almost all $x \in \Omega$ and thus curl $\hat{E} =$ curl $E$.

On the other hand, using the definition of the effective permeability $\hat{\mu}$ in $R$, we find that

$$H(x) = \fint_Y H_0(x, \cdot) = \mu^{\text{eff}} \oint_{\Sigma^*} H_0(x, \cdot) = \mu^{\text{eff}} \hat{H}(x) = \hat{\mu}\hat{H}(x)\,, \tag{5.3.9}$$

for almost all $x \in R$. Outside of $R$, the two-scale limit $H_0(x, \cdot)$ coincides with the weak limit $H(x)$ by Lemma 5.5, and thus $\hat{H}(x) = H(x)$ for almost all $x \in \Omega \setminus R$. This together with the definition of $\hat{\mu}$ implies (5.3.9) for almost all $x \in \Omega$. From (5.3.8) and (5.3.9) we conclude that equation (5.3.7a) is satisfied.

In order to prove (5.3.7c), we observe that $\Omega \setminus R \subset \Omega \setminus \Sigma_\eta$. We can therefore take the distributional limit in equation (5.0.1b) and find that

$$\operatorname{curl} H = -\mathrm{i}\omega\varepsilon_0 E \quad \text{in } \Omega \setminus R\,.$$

This shows (5.3.7c), since $\hat{E} = E$ and $\hat{H} = H$ in $\Omega \setminus R$.

*Step 3: Derivation of* (5.3.7b). Choose $\theta \in C_c^\infty(\Omega; \mathbb{R})$, $u \in X^E$, and set $\varphi_\eta(\cdot) \coloneqq \theta(\cdot)u(\cdot/\eta)$ for $\eta > 0$. Due to the two-scale convergence of $(H^\eta)_\eta$,

$$\lim_{\eta \to 0} \int_\Omega \big\langle H^\eta, \operatorname{curl}[\varphi_\eta] \big\rangle = \int_\Omega \int_{\Sigma^*} \langle H_0, \nabla\theta \wedge u \rangle = \int_\Omega \Big\langle \nabla\theta, \int_{\Sigma^*} u \wedge H_0 \Big\rangle\,. \tag{5.3.10}$$

Thanks to Lemma 5.5, for almost all $x \in \Omega$, the field $H_0(x, \cdot)$ is an element of $\mathcal{X}(\Sigma^*)$ such that we can apply the geometric average to $H_0(x, \cdot)$. On the other hand, $X^E$ is a subset of $\mathcal{V}(\Sigma^*)$ and hence $u \in \mathcal{V}(\Sigma^*)$. We can thus apply the defining identity (4.3.10) of the geometric average on the right-hand side of (5.3.10) and obtain

$$\int_{\Sigma^*} u \wedge H_0(x, \cdot) = \Big( \int_{\Sigma^*} u \Big) \wedge \Big( \oint_{\Sigma^*} H_0(x, \cdot) \Big)\,, \tag{5.3.11}$$

for almost all $x \in \Omega$. Combining (5.3.10) and (5.3.11) yields

$$\begin{aligned} \lim_{\eta \to 0} \int_\Omega \big\langle H^\eta, \operatorname{curl}[\varphi_\eta] \big\rangle &= \int_\Omega \Big\langle \oint_{\Sigma^*} H_0, \nabla\theta \wedge \int_{\Sigma^*} u \Big\rangle \\ &= \int_\Omega \Big\langle \oint_{\Sigma^*} H_0, \operatorname{curl}\Big(\theta \int_{\Sigma^*} u\Big) \Big\rangle\,. \end{aligned} \tag{5.3.12}$$

On the other hand, for each $\eta > 0$, the field $H^\eta$ is a distributional solution to (5.0.1b). Thus, by exploiting the two-scale convergence of $(E^\eta)_\eta$,

$$\begin{aligned} \lim_{\eta \to 0} \int_\Omega \big\langle H^\eta, \operatorname{curl}[\varphi_\eta] \big\rangle &= -\mathrm{i}\omega\varepsilon_0 \lim_{\eta \to 0} \int_\Omega \langle E^\eta, \varphi_\eta \rangle \\ &= -\mathrm{i}\omega\varepsilon_0 \int_\Omega \theta \int_Y \langle E_0, u \rangle\,. \end{aligned} \tag{5.3.13}$$

Using the map $\varepsilon^{\text{eff}}$ and the definition of the effective field $\hat{E}$ as well as of the effective permittivity $\hat{\varepsilon}$, we find that

$$\int_Y \Big\langle E_0(x, \cdot), u \Big\rangle = \Big\langle \varepsilon^{\text{eff}}\Big( \int_Y E_0(x, \cdot) \Big), \int_Y u \Big\rangle = \Big\langle \hat{\varepsilon}(x)\hat{E}(x), \int_Y u \Big\rangle\,, \tag{5.3.14}$$

for almost all $x \in R$. For $x \in \Omega \backslash R$ there holds $E_0(x,\cdot) = E(x)$ and $\hat{\varepsilon}(x) = \mathrm{id}_{\mathbb{C}^{3\times 3}}$. Thus, for almost all $x \in \Omega$ we find the identity

$$\int_Y \left\langle E_0(x,\cdot), u \right\rangle = \left\langle \hat{\varepsilon}(x)\hat{E}(x), \int_Y u \right\rangle \tag{5.3.15}$$

Substituting (5.3.15) into equation (5.3.13) yields

$$\lim_{\eta\to 0} \int_\Omega \left\langle H^\eta, \operatorname{curl}[\varphi_\eta] \right\rangle = -\mathrm{i}\omega\varepsilon_0 \int_\Omega \left\langle \hat{\varepsilon}\hat{E}, \theta \int_Y u \right\rangle . \tag{5.3.16}$$

Combining equations (5.3.12) and (5.3.16), and using the definition of $\hat{H}$ from (5.3.5), we obtain

$$\int_\Omega \left\langle \hat{H}, \operatorname{curl}\left(\theta \int_{\Sigma^*} u \right) \right\rangle = -\mathrm{i}\omega\varepsilon_0 \int_\Omega \left\langle \hat{\varepsilon}\hat{E}, \theta \int_Y u \right\rangle .$$

As $u \in X^E$ was chosen arbitrarily, we infer that equation (5.3.7b) holds. $\square$

## 5.4 Discussion of examples

The section is devoted to the discussion of the effective equations by examples.

### Special geometries

All the microstructures we discuss in this section satisfy the following two assumptions:

(A4) There exist an orthogonal basis $(b_1, b_2, b_3)$ of $\mathbb{R}^3$ and two disjoint sets $N, L \subset \{1,2,3\}$ with $N \cup L = \{1,2,3\}$ such that $b_i \in \mathcal{N}_\Sigma$ and $b_j \in \mathcal{L}_\Sigma$ for all $i \in N$ and $j \in L$.

(A4*) There exists an orthogonal basis $(b_1^*, b_2^*, b_3^*)$ of $\mathbb{R}^3$ and two disjoint sets $N^*, L^* \subset \{1,2,3\}$ with $N^* \cup L^* = \{1,2,3\}$ such that $b_i^* \in \mathcal{N}_{\Sigma^*}$ and $b_j^* \in \mathcal{L}_{\Sigma^*}$ for all $i \in N^*$ and $j \in L^*$.

Let us remark that if a microstructure $\Sigma$ satisfies (A4), then for every $m \in \mathcal{N}_\Sigma$ there holds $\langle m, k \rangle = 0$ for all $k \in \mathcal{L}_\Sigma$. Combining this observation with Lemma 5.2*(iii)*, we find that for every $m \in \mathcal{N}_\Sigma$ there exists a unique element $u^m \in X^E$ such that $⨍_Y u^m = m$. Moreover, for such microstructures the solution space $X^E$ of the cell problem (5.1.1) can be characterised in terms of $k$-loops.

Proposition 5.13. – *Let $\Sigma \subset Y$ be an admissible microstructure for which assumption (A4) holds. For each $i \in N$, we denote by $u^i$ the unique distributional solution to* (5.1.1) *with $⨍_Y u^i = b_i$. Then every $u \in X^E$ can be written as a linear combination,*

$$u = \sum_{i\in N} \alpha_i u^i , \tag{5.4.1}$$

*with scalars $\alpha_i \in \mathbb{C}$ for $i \in N$. In particular,* $\dim X^E = |N|$. *Moreover,*

$$A^E = \operatorname{span}_{\mathbb{C}} \mathcal{N}_\Sigma . \tag{5.4.2}$$

*Proof.* We proceed in two steps.

*Step 1. (Proof of* (5.4.1)*)* Let $u \in L^2_\sharp(Y;\mathbb{C}^3)$ be a distributional solution to (5.1.1). As the distributional curl of $u$ vanishes in $Y$, we find a potential $\Theta \in H^1_\sharp(Y;\mathbb{C})$ and a constant $c_0 \in \mathbb{C}^3$ such that $u = \nabla\Theta + c_0$ in $Y$. For each $l \in \{1,2,3\}$, we set $\alpha_l \coloneqq \langle c_0, b_l\rangle$. The vector field

$$v \coloneqq u - \sum_{i\in N} \alpha_i u^i \,,$$

is an element of $X^E$ with

$$\fint_Y v = \sum_{j\in L} \alpha_j b_j \in \operatorname{span}_{\mathbb{C}} \mathcal{L}_\Sigma \,.$$

If $\alpha_j = 0$ for all $j \in L$, we deduce from Lemma 5.2*(i)* that $v = 0$ and the claim is proved.

Suppose there exists $j_0 \in L$ for which $\alpha_{j_0} \neq 0$. Let $\gamma_{j_0}$ be a $b_{j_0}$-loop in $\Sigma$. As $v$ is an element of $X^E$, its distributional curl vanishes in $Y$; we find thus a potential $\Phi \in H^1_\sharp(Y;\mathbb{C})$ such that

$$v = \nabla\Phi + \sum_{j\in L} \alpha_j b_j \quad \text{ in } Y\,. \tag{5.4.3}$$

Using that $v = 0$ in $\Sigma$ and the Helmholtz decomposition (5.4.3), we find that

$$\begin{aligned} 0 = \int_{\gamma_{j_0}} v &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big[\Phi(\tilde{\gamma}_{j_0}(t))\big]\,\mathrm{d}t + \sum_{j\in L}\alpha_j \int_0^1 \big\langle b_j, \dot{\tilde{\gamma}}_{j_0}(t)\big\rangle\,\mathrm{d}t \\ &= \sum_{j\in L} \alpha_j \big\langle b_j, b_{j_0}\big\rangle = \alpha_{j_0}\,. \end{aligned}$$

As this is a contradiction, we infer that $\alpha_j = 0$ for all $j \in L$. This proves the representation (5.4.1).

*Step 2. (Proof of* (5.4.2)*)* By Lemma 5.2*(iii)* and assumption (A4), for every $m \in \mathcal{N}_\Sigma$ there exists $u^m \in X^E$ with $\fint_Y u^m = m$. Thus, $\mathcal{N}_\Sigma \subset A^E$ and hence, since $A^E$ is a vector space, $\operatorname{span}_{\mathbb{C}} \mathcal{N}_\Sigma \subset A^E$.

In order to prove the converse inclusion, choose $c \in A^E$. By definition of $A^E$, there exists $u \in X^E$ with $\fint_Y u = c$. Due to *Step 1*, there exist $\alpha_i \in \mathbb{C}$ for $i \in N$ such that $u = \sum_{i\in N} \alpha_i u^i$. Consequently,

$$c = \fint_Y u = \sum_{i\in N} \alpha_i b_i \in \operatorname{span}_{\mathbb{C}} \mathcal{N}_\Sigma \,.$$

This proves the identity (5.4.2). □

One might think that a similar characterisation holds for the solution space $X^H$ provided the microstructure $\Sigma$ satisfies (A4*). In view of cases 1 and 2 of Remarks 5*(b)*, however, we already know that the solution space $X^H$ cannot be characterised by $k$-loops if $\dim X^E \leq 1$.

PROPOSITION 5.14. – *Let $\Sigma \subset Y$ be an admissible microstructure for which (A4*) holds. If $\dim X^E \geq 2$, then*

$$\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*} = A^H = \mathbb{C}^3 \,. \tag{5.4.4}$$

*Proof.* As $\dim X^E \geq 2$, we deduce from Lemma 5.9 that each $k \in \mathcal{L}_{\Sigma^*}$ induces an element $v^k \in X^H$ with $\oint_{\Sigma^*} v^k = k$. Thus, $\mathcal{L}_{\Sigma^*} \subset A^H$ and hence, since $A^H$ is a vector space, $\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*} \subset A^H$.

In order to prove the converse inclusion, choose $c \in A^H$. By definition of $A^H$ there exist $v \in X^H$ with $\oint_{\Sigma^*} v = c$. Let $(b_1^*, b_2^*, b_3^*)$ be an orthogonal basis of $\mathbb{R}^3$ for which (A4$^*$) is satisfied. Then, for each $j \in L^*$, there exists $v^j \in X^H$ with $\oint_{\Sigma^*} v^j = b_j^*$. For $l \in \{1,2,3\}$, we set $\alpha_l := \langle c, b_l^* \rangle$. The vector field $w := v - \sum_{j \in L^*} \alpha_j v^j$ is an element of $X^H$ with

$$\oint_{\Sigma^*} w = \sum_{i \in N^*} \alpha_i b_i^* \in \operatorname{span}_{\mathbb{C}} \big\{ b_i^* \bigm| i \in N^* \big\} .$$

If $\oint_{\Sigma^*} w = 0$, then

$$c = \oint_{\Sigma^*} v = \oint_{\Sigma^*} w + \sum_{j \in L^*} \alpha_j \oint_{\Sigma^*} v^j = \sum_{j \in L^*} \alpha_j b_j^* \in \operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*} ,$$

which proves the inclusion $A^H \subset \operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*}$. We thus need to show that $w$ has a vanishing geometric average.

From Lemma 5.9 *(i)*, we deduce that $b_i^* \notin A^H$ for every $i \in N^*$. On the other hand, $w$ is an element of $X^H$ and thus

$$\oint_{\Sigma^*} w = \pi_{A^H} \left( \oint_{\Sigma^*} w \right) = \sum_{i \in N^*} \alpha_i \pi_{A^H} \left( b_i^* \right) = 0 .$$

This proves the claim. □

In the examples below, we can always choose $(b_1, b_2, b_3) = (b_1^*, b_2^*, b_3^*) = (\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$. We remark that Proposition 5.14 is *not* needed to derive the effective equations for the examples below; see also Remark 6 *(b)* at the end of this section.

We recall from Lemmas 5.3 and 5.7 that $A^{\mathcal{V}}(\Sigma^*) = A^E$ and $A^H = A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ for any admissible microstructure $\Sigma \subset Y$. Using the definition (4.3.4) of $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ and the fact that the volume average $\fint_Y \colon X^E \to A^E$ is an isomorphism, we find that

$$A^H = \begin{cases} \{0\} & \text{if } \dim X^E = 0 \\ (A^E)^{\perp} & \text{if } \dim X^E = 1 \\ \mathbb{C}^3 & \text{if } \dim X^E \geq 2 \end{cases} . \tag{5.4.5}$$

## Compact inclusions with simply connected complements

Let $\Sigma \subset Y$ be an admissible microstructure that is compactly contained in $(0,1)^3$. Assume further that the complement $(0,1)^3 \setminus \overline{\Sigma}$ is simply connected; we refer to Figure 5.1 (a) for an example of such a microstructure. Such microstructures are, for instance, considered in [BBF09, BBF17]. As $\Sigma$ is compactly contained in $(0,1)^3$, we infer that

$$\mathcal{N}_{\Sigma} = \mathbb{Z}^3 \setminus \{0\} \quad \text{and} \quad \mathcal{L}_{\Sigma^*} = \mathbb{Z}^3 \setminus \{0\} . \tag{5.4.6}$$

A compactly contained microstructure satisfies assumptions (A4) and (A4$^*$) with index sets $N = \{1,2,3\}$, $L = \emptyset$ and $N^* = \emptyset$, $L^* = \{1,2,3\}$. We may therefore

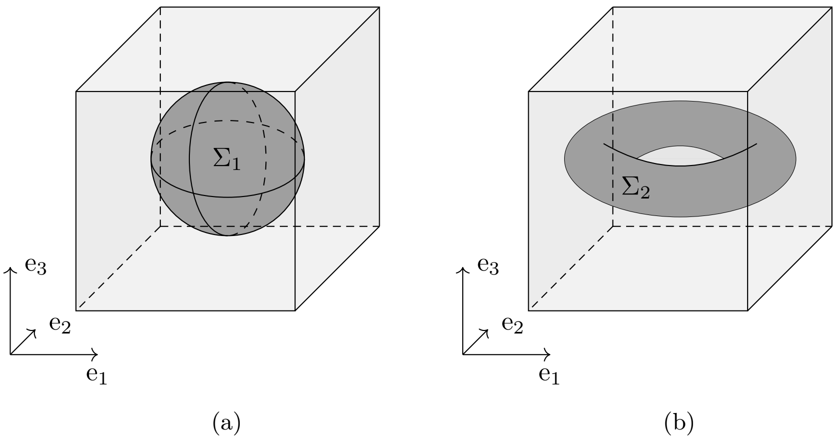


Figure 5.1: The figure shows two admissible microstructures. The unit cell $Y$ is represented by the cube; the dark grey areas represent the microstructures. (a) The microstructure $\Sigma_1$ is compactly contained and its complement $\Sigma_1^* := Y \setminus \overline{\Sigma_1}$ is simply connected in $(0,1)^3$. (b) $\Sigma_2$ is also compactly contained in $(0,1)^3$; its complement $\Sigma_2^*$, however, is not simply connected.

apply Proposition 5.13 and find that the solution space $X^E$ is three-dimensional and that

$$A^E = \operatorname{span}_{\mathbb{C}} \mathcal{N}_\Sigma = \mathbb{C}^3 .$$

As $\dim X^E = 3$, we infer that

$$A^H = \mathbb{C}^3 . \tag{5.4.7}$$

We note that (5.4.7) follows from Proposition 5.14 and (5.4.6), as well.

Having all these at hand, we can apply Theorem 5.12 and find the following effective equations.

Corollary 5.15. (Effective equations) – *Let $\Omega \subset \mathbb{R}^3$ and $R \Subset \Omega$ be as in Section 4.1. Let $\Sigma \subset Y$ be an admissible microstructure that is compactly contained in $(0,1)^3$ and for which $(0,1)^3 \setminus \overline{\Sigma}$ is simply connected. Let $\hat{\varepsilon}, \hat{\mu} \colon \Omega \to \mathbb{C}^{3\times 3}$ be the effective material parameters given in* (5.3.4)*, and let $(E^\eta, H^\eta)_\eta$ be a sequence that satisfies (A3) on page 51. Then the effective field $(\hat{E}, \hat{H}) \colon \Omega \to \mathbb{C}^3 \times \mathbb{C}^3$, defined in* (5.3.5)*, is a distributional solution to*

$$\begin{cases} \operatorname{curl} \hat{E} = \mathrm{i}\omega\mu_0\hat{\mu}\hat{H} & \text{in } \Omega , \\ \operatorname{curl} \hat{H} = -\mathrm{i}\omega\varepsilon_0\hat{\varepsilon}\hat{E} & \text{in } \Omega . \end{cases}$$

## Compact inclusions

Let $\Sigma \subset Y$ be an admissible microstructure that is compactly contained in $(0,1)^3$; see Figure 5.1 (b) for an example. We note that the results from [BBF09, SU18, BBF17] can, in general, not be applied to such microstructures.

As $\Sigma$ is compactly contained in $(0,1)^3$, we can argue as in the previous example, and conclude that

$$A^E = \operatorname{span}_{\mathbb{C}} \mathcal{N}_\Sigma = \mathbb{C}^3 \quad \text{and} \quad A^H = \mathbb{C}^3 . \tag{5.4.8}$$

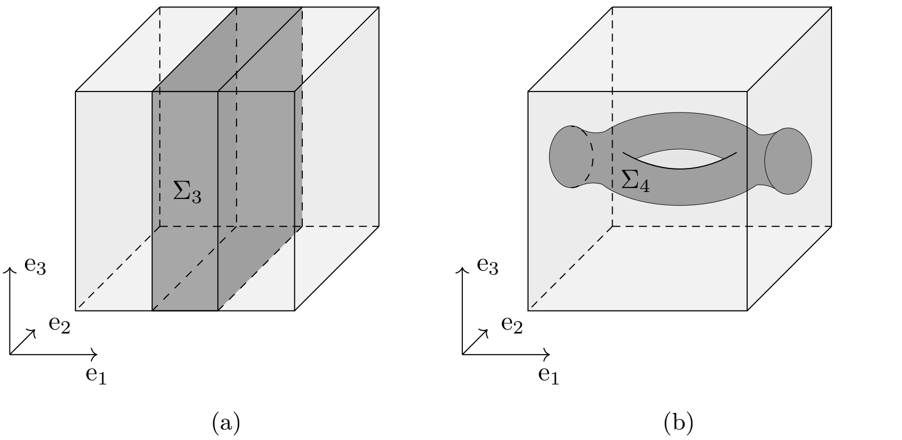


Figure 5.2: The figure shows two admissible microstructures. The unit cell $Y$ is represented by the cube; the dark grey areas represent the microstructures. (a) $\Sigma_3$ represents a metal plate. (b) $\Sigma_4$ is neither compactly contained in $(0,1)^3$ nor is its complement simply connected.

As the geometric average $\oint_{\Sigma^*}: X^H \to A^H$ is an isomorphism, $\dim X^H = 3$. For the sake of completeness, let us mention that the space $X$ of distributional solutions of the cell problem (5.2.1), which is defined in (5.2.9), has, in general, more than three dimensions.

Having (5.4.8) at hand, we can apply Theorem 5.12 and determine the effective equations.

Corollary 5.16. (Effective equations) – *Let $\Omega \subset \mathbb{R}^3$ and $R \Subset \Omega$ be as described as in Section 4.1. Let $\Sigma \subset Y$ be an admissible microstructure that is compactly contained in $(0,1)^3$. Let $\hat{\varepsilon}, \hat{\mu}\colon \Omega \to \mathbb{C}^{3\times 3}$ be the effective material parameters given in* (5.3.4)*, and let $(E^\eta, H^\eta)_\eta$ be a sequence satisfying (A3) on page 51. Then the effective field $(\hat{E}, \hat{H})\colon \Omega \to \mathbb{C}^3 \times \mathbb{C}^3$, defined in* (5.3.5)*, is a distributional solution to*

$$\begin{cases} \operatorname{curl} \hat{E} = \phantom{-}\mathrm{i}\omega\mu_0\hat{\mu}\hat{H} & \text{in } \Omega\,, \\ \operatorname{curl} \hat{H} = -\mathrm{i}\omega\varepsilon_0\hat{\varepsilon}\hat{E} & \text{in } \Omega\,. \end{cases}$$

### The metal plate

We fix $\gamma \in (0, 1/2)$ and consider the set

$$\Sigma \coloneqq \left\{ (y_1, y_2, y_3) \in Y \;\middle|\; y_1 \in \left(\frac{1}{2} - \gamma, \frac{1}{2} + \gamma\right) \right\}. \tag{5.4.9}$$

We refer to Figure 5.2 (a) for a sketch of the microstructure $\Sigma$. Observe that $\Sigma$ is an admissible microstructure. We first note that there is no $k$-loop in $\Sigma$ for $k \in \mathbb{Z}^3 \setminus \{0\}$ with $\langle k, \mathrm{e}_1\rangle \neq 0$. On the other hand, there are $k$-loops in $\Sigma$ for $k \in \mathbb{Z}\,\mathrm{e}_2 \oplus \mathbb{Z}\,\mathrm{e}_3$. Thus,

$$\mathcal{N}_\Sigma = \mathbb{Z}\,\mathrm{e}_1 \setminus \{0\} \quad \text{ and } \quad \mathcal{L}_\Sigma = \left\{ (0, k_2, k_3) \in \mathbb{Z}^3 \;\middle|\; k_2, k_3 \in \mathbb{Z} \setminus \{0\} \right\}.$$

This shows that the metal plate satisfies assumption (A4) with $N = \{1\}$ and $L = \{2, 3\}$. Applying Proposition 5.13 yields

$$A^E = \operatorname{span}_{\mathbb{C}} \mathcal{N}_\Sigma = \mathbb{C}\,\mathrm{e}_1 \ .$$

The permittivity $\varepsilon^{\text{eff}} \colon A^E \to A^E$ of the meta-material located in $R$ can therefore be identified with a complex scalar $\varepsilon^{\text{eff}}$. Moreover, as $\dim X^E = \dim A^E = 1$,

$$A^H = \left(A^E\right)^\perp = \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3 \ .$$

The permeability $\mu^{\text{eff}}$ of the meta-material located in $R$ is thus a linear map $\mu^{\text{eff}} \colon \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3 \to \mathbb{C}^3$.

Having all these information at hand, we can apply Theorem 5.12 and determine the effective equations.

Corollary 5.17. (Effective equations) – *Let $\Omega \subset \mathbb{R}^3$ and $R \Subset \Omega$ be as described as in Section 4.1. Let $\Sigma \subset Y$ be the microstructure defined in* (5.4.9)*. Let $\hat{\varepsilon}$ and $\hat{\mu}$ be the effective material parameters given in* (5.3.4)*, and let $(E^\eta, H^\eta)_\eta$ be a sequence that satisfies (A3) on page 51. Then the effective field $(\hat{E}, \hat{H}) \colon \Omega \to \mathbb{C}^3 \times \mathbb{C}^3$, which is defined in* (5.3.5)*, satisfies*

$$\hat{E}(x) \in \mathbb{C}\,\mathrm{e}_1 \qquad \textit{and} \qquad \hat{H}(x) \in \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3 \qquad \textit{for almost all } x \in R\,.$$

*Moreover, $(\hat{E}, \hat{H})$ is a distributional solution to*

$$\begin{cases} \operatorname{curl} \hat{E} = \mathrm{i}\omega\mu_0\hat{\mu}\hat{H} & \textit{in } \Omega\,, \\ (\operatorname{curl} \hat{H})_1 = -\mathrm{i}\omega\varepsilon_0(\hat{\varepsilon}\hat{E})_1 & \textit{in } \Omega\,, \\ \operatorname{curl} \hat{H} = -\mathrm{i}\omega\varepsilon_0\hat{E} & \textit{in } \Omega \setminus R\,. \end{cases}$$

### The metal torus touching the boundary

Let $\Sigma \subset Y$ be a two-dimensional full torus that connects the two (and only the two) opposite faces $\{y_1 = 0\}$ and $\{y_1 = 1\}$ of the unit cube; see Figure 5.2 (b) for a sketch. As both $\Sigma$ and $\Sigma^*$ are connected, $\Sigma$ is an admissible microstructure. We first note that there is no $k$-loop in $\Sigma$ for $k \in \mathbb{Z}\,\mathrm{e}_2 \oplus \mathbb{Z}\,\mathrm{e}_3$. On the other hand, there is a $k$-loop in $\Sigma$ for every $k \in \mathbb{Z}\,\mathrm{e}_1$ with $k \neq 0$. Combining these two observations, we obtain

$$\mathcal{N}_\Sigma = \left\{ (0, k_2, k_3) \in \mathbb{Z}^3 \,\middle|\, k_2, k_3 \in \mathbb{Z} \setminus \{0\} \right\} \quad \text{and} \quad \mathcal{L}_\Sigma = \mathbb{Z}\,\mathrm{e}_1 \setminus \{0\}\,.$$

Thus, $\Sigma$ satisfies assumption (A4) with index sets $N = \{2, 3\}$ and $L = \{1\}$. From Proposition 5.13 we deduce that

$$A^E = \operatorname{span}_{\mathbb{C}} \mathcal{N}_\Sigma = \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3 \ .$$

The permittivity $\varepsilon^{\text{eff}}$ of the meta-material located in $R$ is thus a linear map $\varepsilon^{\text{eff}} \colon \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3 \to \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3$. Moreover, since $\dim X^E = \dim A^E = 2$,

$$A^H = \mathbb{C}^3\,.$$

Applying Theorem 5.12 we determine the effective equations.

Corollary 5.18. (Effective equations) – *Let* $\Omega \subset \mathbb{R}^3$ *and* $R \Subset \Omega$ *be as described as in Section 4.1. Let* $\Sigma \subset Y$ *be a two-dimensional full-torus that connects the two opposite faces* $\{y_1 = 0\}$ *and* $\{y_1 = 1\}$ *of the unit cube. Let* $\hat{\varepsilon}$ *and* $\hat{\mu}$ *be the effective material parameters given in* (5.3.4)*, and let* $(E^\eta, H^\eta)_\eta$ *be a sequence that satisfies (A3). Then the effective electric field* $\hat{E}\colon \Omega \to \mathbb{C}^3$*, defined in* (5.3.5)*, satisfies*

$$\hat{E}(x) \in \mathbb{C}\,\mathrm{e}_2 \oplus \mathbb{C}\,\mathrm{e}_3 \qquad \textit{for almost all } x \in R\,.$$

*Moreover, the effective electromagnetic field* $(\hat{E}, \hat{H})\colon \Omega \to \mathbb{C}^3 \times \mathbb{C}^3$ *is a distributional solution to*

$$\begin{cases} \operatorname{curl} \hat{E} = \phantom{-}\mathrm{i}\omega\mu_0\hat{\mu}\hat{H} & \textit{in } \Omega\,, \\ (\operatorname{curl} \hat{H})_2 = -\mathrm{i}\omega\varepsilon_0(\hat{\varepsilon}\hat{E})_2 & \textit{in } \Omega\,, \\ (\operatorname{curl} \hat{H})_3 = -\mathrm{i}\omega\varepsilon_0(\hat{\varepsilon}\hat{E})_3 & \textit{in } \Omega\,, \\ \operatorname{curl} \hat{H} = -\mathrm{i}\omega\varepsilon_0\hat{E} & \textit{in } \Omega \setminus R\,. \end{cases}$$

Remarks 6. – *(a) The above examples show that k-loops can be used to characterise the solution space* $X^E$ *of* (5.1.1) *quite easily. Unfortunately, the analysis of the solution space* $X^H$ *of* (5.2.1) *for general microstructures is more involved.*

*(b) We do neither need Proposition 5.14 nor a characterisation of the space* $X^H$ *to derive the effective equations for the four examples. This is no coincidence. The geometric average*

$$\oint_{\Sigma^*} : X^H \to A^H$$

*is an isomorphism between the two vector spaces and thus, instead of characterising the solution space* $X^H$ *of the cell problem* (5.2.1)*, it suffices to analyse the space* $A^H$*. Moreover,* $A^H$ *is determined by* $A^E$*. Thus, the geometric average allows us to bypass the involved analysis of the cell problem* (5.2.1) *of* $H_0$ *and to focus on the analysis of the space* $A^E$ *when deriving the effective equations.*

CHAPTER 6

# Effective Maxwell's equations for highly conductive microstructures

In this chapter, we focus on highly conductive microstructures $\Sigma$. Following the notation of Section 4.1, we denote the period of the metamaterial by $\eta > 0$ and the frequency by $\omega > 0$. In view of the time-harmonic Maxwell equations (4.1.1), we need to specify the relative permittivity $\varepsilon_\eta$ of the medium; we choose a function $\varepsilon_\eta \colon \mathbb{R}^3 \to \mathbb{C}$ of the form

$$\varepsilon_\eta(x) := \frac{\varepsilon_r}{\eta^2} \mathbb{1}_{\Sigma_\eta}(x) + \mathbb{1}_{\Omega \setminus \Sigma_\eta}(x)\,, \tag{6.0.1}$$

where $\varepsilon_r \in \mathbb{C}$ with $\operatorname{Im}\{\varepsilon_r\} > 0$ and $\operatorname{Re}\{\varepsilon_r\} \geq 0$. We study the behaviour of distributional solutions $(E^\eta, H^\eta) \in L^2(\Omega; \mathbb{C}^3) \times L^2(\Omega; \mathbb{C}^3)$ to the system

$$\begin{cases} \operatorname{curl} E^\eta = \phantom{-}\mathrm{i}\omega\mu_0 H^\eta & \text{in } \Omega\,, \qquad (6.0.2a) \\ \operatorname{curl} H^\eta = -\mathrm{i}\omega\varepsilon_0\varepsilon_\eta E^\eta & \text{in } \Omega\,, \qquad (6.0.2b) \end{cases}$$

in the limit $\eta \to 0$.

In Chapter 5 we make an additional assumption (A3) besides assumptions (A1) and (A2) discussed in Section 4.1. Assumption (A3) has to be modified in this chapter, since the fields $E^\eta$ and $H^\eta$ are nontrivial in all of $\Omega$. Throughout this chapter we assume:

(A3′) There exists a sequence $(E^\eta, H^\eta)_\eta$ of distributional solutions to (6.0.2) that satisfies the energy-bound

$$\sup_{\eta > 0} \int_\Omega |E^\eta|^2 + |H^\eta|^2 < \infty\,. \tag{6.0.3}$$

Assumption (A3$'$) ensures the existence of a subsequence $(E^\eta, H^\eta)_\eta$ and fields $E_0, H_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$ such that $(E^\eta, H^\eta) \xrightarrow{2} (E_0, H_0)$. Moreover, there exist fields $E, H \in L^2(\Omega; \mathbb{C}^3)$ such that $(E^\eta)_\eta$ and $(H^\eta)_\eta$ weakly converge in $L^2(\Omega; \mathbb{C}^3)$ to $E$ and $H$, respectively. The two-scale limit fields $E_0$ and $H_0$ and the weak $L^2$-limit fields $E$ and $H$ are related via

$$E(x) = \fint_Y E_0(x, \cdot) \quad \text{and} \quad H(x) = \fint_Y H_0(x, \cdot)\,,$$

for almost all $x \in \Omega$.

The following list of spaces is intended to be an overview and reference of the important spaces, which we define in the subsequent sections.

- $X^E$ is the space of vector fields $u \in L^2_\sharp(Y; \mathbb{C}^3)$ that are distributional solutions to (6.1.1). This space is identical to the corresponding function space defined in Section 5.1;
- $A^E$, defined in (6.1.7), is the space of attainable volume averages of fields $u \in X^E$;
- $\mathcal{X}$ is the space of vector fields $v \in H^1_\sharp(Y; \mathbb{C}^3)$ with curl $v = 0$ in $\Sigma^*$;
- $X^H$, defined in (6.2.26), is the subspace of $\mathcal{X}$ in which we seek $H_0(x, \cdot)$ for almost all $x \in \Omega$;
- $\mathcal{X}_0^{\mathrm{div}}$ is the space of vector fields $v \in \mathcal{X}$ with div $v = 0$ in $Y$ and $\oint_{\Sigma^*} v = 0$;
- $A^H$ is the space of attainable geometric averages of fields in $X^H$ and it is defined in (6.2.33).

## 6.1 The oscillating electric field

In this section we derive and analyse the cell problem for the two-scale limit $E_0$. The analysis is based on results from Section 5.1, since the cell problems are identical.

Lemma 6.1. (Cell problem) – *Let $\Omega \subset \mathbb{R}^3$ be as described in Section 4.1 and let $\Sigma \subset Y$ be an admissible microstructure. Let $(E^\eta, H^\eta)_\eta$ be a sequence satisfying* (A3$'$) *that two-scale converges to $(E_0, H_0)$ and that weakly converges in $L^2(\Omega; \mathbb{C}^3) \times L^2(\Omega; \mathbb{C}^3)$ to $(E, H)$. Then for almost all $x \in R$ the two-scale limit $E_0 = E_0(x, \cdot) \in L^2_\sharp(Y; \mathbb{C}^3)$ satisfies*

$$\begin{cases} \operatorname{curl}_y E_0 = 0 & \text{in } Y\,, & (6.1.1a)\\ \operatorname{div}_y E_0 = 0 & \text{in } \Sigma^*\,, & (6.1.1b)\\ E_0 = 0 & \text{in } \Sigma\,, & (6.1.1c)\end{cases}$$

*in the distributional sense.*

*Outside of the meta-material $R$, the two-scale limit $E_0$ is $y$-independent; that is, $E_0(x, y) = E(x)$ for almost all $x \in \Omega \setminus R$ and almost all $y \in Y$.*

*Proof.* For $\theta \in C_c^\infty(\Omega;\mathbb{R})$ and $\psi \in C_\sharp^\infty(Y;\mathbb{C}^3)$ we define $\varphi(x,y) := \theta(x)\psi(y)$ for all $(x,y) \in \Omega \times Y$. We set $\psi_\eta(\cdot) := \psi(\cdot/\eta)$ and $\varphi_\eta(\cdot) := \theta(\cdot)\psi_\eta(\cdot)$ for each $\eta > 0$. As $(E^\eta)_\eta$ two-scale converges to $E_0 \in L^2(\Omega \times Y;\mathbb{C}^3)$, we find that

$$\begin{aligned}\lim_{\eta\to 0}\int_\Omega \eta\langle E^\eta, \operatorname{curl}[\varphi_\eta]\rangle &= \lim_{\eta\to 0}\int_\Omega \eta\langle E^\eta, \nabla\theta \wedge \psi_\eta\rangle + \lim_{\eta\to 0}\int_\Omega \theta\langle E^\eta, \operatorname{curl}\,\psi_\eta\rangle \\ &= \int_\Omega \theta\bigg(\int_Y \langle E^\eta, \operatorname{curl}\,\psi\rangle\bigg)\,. \end{aligned} \tag{6.1.2}$$

On the other hand, for every $\eta > 0$, the field $E^\eta$ is a distributional solution to equation (6.0.2a) and thus

$$\int_\Omega \eta\big\langle E^\eta, \operatorname{curl}[\varphi_\eta]\big\rangle = \mathrm{i}\omega\mu_0\eta\int_\Omega \big\langle H^\eta, \varphi_\eta\big\rangle\,. \tag{6.1.3}$$

The sequence $(H^\eta)_\eta$ is bounded in $L^2(\Omega;\mathbb{C}^3)$ by assumption. Sending $\eta \to 0$ in (6.1.3) and combining the result with (6.1.2) yields

$$\int_\Omega \theta\bigg(\int_Y \big\langle E_0, \operatorname{curl}\,\psi\big\rangle\bigg) = 0\,.$$

This proves that $E_0(x,\cdot)$ is a distributional solution to (6.1.1a) for almost all $x \in \Omega$.

In order to show that equation (6.1.1b) is valid, we choose $\theta \in C_c^\infty(\Omega;\mathbb{R})$ and $\psi \in C_c^\infty(\Sigma^*;\mathbb{C})$. For $\eta > 0$ we define $\psi_\eta(\cdot) := \psi(\cdot/\eta)$ and $\varphi_\eta(\cdot) := \theta(\cdot)\psi_\eta(\cdot)$. Due to (6.0.2b) the distributional divergence of $E^\eta$ vanishes in $\Omega \setminus \overline{\Sigma}_\eta$ for all $\eta > 0$. Thus,

$$0 = \int_\Omega \eta\big\langle E^\eta, \nabla[\varphi_\eta]\big\rangle = \eta\int_\Omega \psi_\eta\big\langle E^\eta, \nabla\theta\big\rangle + \int_\Omega \theta\big\langle E^\eta, \nabla\psi_\eta\big\rangle\,. \tag{6.1.4}$$

Sending $\eta \to 0$ in (6.1.4) and using the two-scale convergence of $(E^\eta)_\eta$ to $E_0$ shows the validity of equation (6.1.1b).

Choose $\theta \in C_c^\infty(\Omega;\mathbb{R})$, $\psi \in C_c^\infty(\Sigma;\mathbb{C}^3)$, and define $\psi_\eta(\cdot) := \psi(\cdot/\eta)$ and $\varphi_\eta(\cdot) := \theta(\cdot)\psi_\eta(\cdot)$ for each $\eta > 0$. In $\Sigma_\eta$, equation (6.0.2b) reads

$$\eta^2 \operatorname{curl}\, H^\eta = -\mathrm{i}\omega\varepsilon_0\varepsilon_r E^\eta\,,$$

and hence

$$\begin{aligned}-\mathrm{i}\omega\varepsilon_0\varepsilon_r \lim_{\eta\to 0}\int_\Omega \big\langle E^\eta, \varphi_\eta\big\rangle &= \lim_{\eta\to 0}\eta\int_\Omega \eta\big\langle H^\eta, \operatorname{curl}[\varphi_\eta]\big\rangle \\ &= \lim_{\eta\to 0}\eta\Bigg(\int_\Omega \eta\big\langle H^\eta, \nabla\eta \wedge \psi_\eta\big\rangle + \int_\Omega \theta\big\langle H^\eta, \operatorname{curl}\,\psi_\eta\big\rangle\Bigg) \\ &= 0\,. \end{aligned} \tag{6.1.5}$$

On the other hand, $(E^\eta)_\eta$ two-scale converges to $E_0$ and thus,

$$-\mathrm{i}\omega\varepsilon_0\varepsilon_r \lim_{\eta\to 0}\int_\Omega \big\langle E^\eta, \varphi_\eta\big\rangle = -\mathrm{i}\omega\varepsilon_0\varepsilon_r\int_\Omega \theta\bigg(\int_\Sigma \big\langle E_0, \psi\big\rangle\bigg)\,. \tag{6.1.6}$$

Combining (6.1.5) and (6.1.6) yields: $E_0(x,\cdot)$ satisfies (6.1.1c) for almost all $x \in \Omega$.

The proof that $E_0(x,\cdot)$ is independent of $y$ for almost all $x \in \Omega \setminus R$ is identical to the one given for Lemma 5.1. $\square$

As before, let us denote by $X^E$ the set of all distributional solutions to (6.1.1). The space of all attainable volume averages of fields in $X^E$ is defined by

$$A^E := \left\{ \fint_Y u \;\middle|\; u \in X^E \right\}. \tag{6.1.7}$$

Clearly, $A^E$ is a subspace of $\mathbb{C}^3$. The cell problems (6.1.1) and (5.1.1) are identical and thus, by Lemma 5.3, the spaces $A^E$ and $A^{\mathcal{V}}(\Sigma^*)$ coincide.

Remark 7. – *We collect other results of Section 5.1, which follow by combining the proofs of Lemma 5.2 and Proposition 5.4. For every admissible microstructure $\Sigma \subset Y$ the following statements hold:*

*(i) Given any $c \in \mathbb{C}^3$ there is at most on distributional solution $u \in L^2_\sharp(Y;\mathbb{C}^3)$ to (6.0.2) with $\fint_Y u = c$.*

*(ii) If $k \in \mathcal{L}_\Sigma$, then there is no distributional solution to (6.0.2) with volume average equal to $k$.*

*(iii) If $m \in \mathcal{N}_\Sigma$ satisfies $\langle m, k\rangle = 0$ for all $k \in \mathcal{L}_\Sigma$, then there is a unique distributional solution $u^m \in L^2_\sharp(Y;\mathbb{R}^3)$ with $\fint_Y u^m = m$.*

*(iv) The space $X^E$ of distributional solutions to the cell problem (6.1.1) is finite dimensional; more precisely,* $\dim X^E = \dim A^E \leq 3$.

This characterisation of the solution space $X^E$ is somewhat abstract for a general microstructure $\Sigma$. However, in many cases, $X^E$ can be analysed using $k$-loops; see Proposition 5.13 and the examples discussed in Section 5.4.

## 6.2 The oscillating magnetic field

This section is devoted to the derivation and analysis of the cell problem of the magnetic field. In contrast to the previous section, the analysis of the cell problem of $H_0$ for highly conductive microstructures is more involved than for perfectly conducting microstructures due to the appearance of a new quantity, the rescaled displacement current, which was introduced in [BS10].

Following an idea of Bouchitté, Bourel, and Felbacq we show that $H_0$ is not only a solution of the cell problem but satisfies a variational identity. It is thus sufficient to analyse the space that consists of vector fields which satisfy the variational identity and induce solutions to the cell problem.

**The classical cell problem.** Besides the electromagnetic field $(E^\eta, H^\eta)$ we consider a third quantity—namely, the rescaled displacement current

$$J^\eta \colon \Omega \to \mathbb{C}^3, \quad J^\eta := \eta\varepsilon_\eta E^\eta \tag{6.2.1}$$

Assume $(E^\eta, H^\eta)$ is a distributional solution to Maxwell's equations (6.0.2); that is, for every $\varphi \in C_c^\infty(\Omega;\mathbb{C}^3)$ there holds

$$\int_\Omega \langle E^\eta, \operatorname{curl} \varphi\rangle = \; \mathrm{i}\omega\mu_0 \int_\Omega \langle H^\eta, \varphi\rangle \tag{6.2.2}$$

and

$$\int_\Omega \langle H^\eta, \operatorname{curl}\varphi\rangle = -\mathrm{i}\omega\varepsilon_0 \int_\Omega \varepsilon_\eta \langle E^\eta, \varphi\rangle\,. \tag{6.2.3}$$

As $C_c^\infty(\Omega;\mathbb{C}^3)$ is dense in $H_0(\operatorname{curl},\Omega)$, both equations, (6.2.2) and (6.2.3), are still valid for vector fields $\varphi \in H_0(\operatorname{curl},\Omega)$. We observe that for every $\varphi \in C_c^\infty(\Omega;\mathbb{R})$ the vector fields $\varphi E^\eta$ and $\varphi H^\eta$ are elements of $H_0(\operatorname{curl},\Omega)$.

Assumption (A3′) ensures that the sequence $(J^\eta)_\eta$ is bounded in $L^2(\Omega;\mathbb{C}^3)$. This result was proved in [BS10, Section 3.1].

Lemma 6.2. – *Let $\Omega \subset \mathbb{R}^3$ as described in Section 4.1 and let $\Sigma \subset Y$ be an admissible microstructure. Let $(E^\eta, H^\eta)_\eta$ be a sequence that satisfies* (A3′)*. Then the sequence of displacement currents $(J^\eta)_\eta$ is bounded in $L^2(\Omega;\mathbb{C}^3)$.*

*Proof.* We proceed in two steps.

*Step 1.* The claim is a consequence of the estimate

$$\sup_{\eta>0}\int_\Omega \Big(|\varepsilon_\eta||E^\eta|^2 + |H^\eta|^2\Big) < \infty\,. \tag{6.2.4}$$

Indeed, provided the bound (6.2.4) holds, we find

$$\int_\Omega |J^\eta|^2 = \eta^2\int_\Omega |\varepsilon_\eta|^2|E^\eta|^2 \le \sup_\Omega\big(\eta^2|\varepsilon_\eta|\big)\int_\Omega |\varepsilon_\eta||E^\eta|^2 < \infty$$

for all $\eta \in (0,\eta_0)$ and for an arbitrary $\eta_0 > 0$. We are thus left to prove (6.2.4).

*Step 2.* Fix a cut-off function $\chi \in C_c^\infty(\Omega;[0,1])$ with $\chi \equiv 1$ in $R$. Clearly, $\chi E^\eta \in H_0(\operatorname{curl},\Omega)$. Setting $\varphi := \mathrm{i}\omega^{-1}\chi E^\eta$ in (6.2.3), we find that

$$\begin{aligned}\varepsilon_0\int_\Omega \varepsilon_\eta\chi|E^\eta|^2 &= \frac{\mathrm{i}}{\omega}\int_\Omega\langle\chi H^\eta, \operatorname{curl} E^\eta\rangle + \frac{\mathrm{i}}{\omega}\int_\Omega\langle H^\eta, \nabla\chi\wedge E^\eta\rangle\\ &= \mu_0\int_\Omega \chi|H^\eta|^2 + \frac{\mathrm{i}}{\omega}\int_\Omega\langle H^\eta, \nabla\chi\wedge E^\eta\rangle\,,\end{aligned} \tag{6.2.5}$$

where we used integration by parts and the fact that (6.2.2) is valid for $\chi H^\eta \in H_0(\operatorname{curl},\Omega)$. Taking the imaginary part in (6.2.5) and applying Young's inequality yields

$$\varepsilon_0\int_\Omega \operatorname{Im}\{\varepsilon_\eta\}\chi|E^\eta|^2 \le C_1\int_\Omega |H^\eta|^2 + C_2\int_\Omega |E^\eta|^2\,. \tag{6.2.6}$$

Similarly we obtain the estimate

$$\varepsilon_0\int_\Omega \operatorname{Re}\{\varepsilon_\eta\}\chi|E^\eta|^2 \le C_3\int_\Omega |H^\eta|^2 + C_2\int_\Omega |E^\eta|^2\,. \tag{6.2.7}$$

By definition (6.0.1) of $\varepsilon_\eta$, we have that $\operatorname{Re}\{\varepsilon_\eta\} \ge 0$ and $\operatorname{Im}\{\varepsilon_\eta\} \ge 0$ in $\Omega$. Thus, combining (6.2.6) and (6.2.7) we get

$$\begin{aligned}\varepsilon_0\int_{\Sigma_\eta}|\varepsilon_\eta||E^\eta|^2 &\le \varepsilon_0\int_\Omega\big(\operatorname{Re}\{\varepsilon_\eta\}+\operatorname{Im}\{\varepsilon_\eta\}\big)\chi|E^\eta|^2\\ &\le C\sup_{\eta>0}\int_\Omega\Big(|E^\eta|^2+|H^\eta|^2\Big)\,.\end{aligned} \tag{6.2.8}$$

On the other hand,

$$\varepsilon_0 \int_{\Omega\setminus\Sigma_\eta} |\varepsilon_\eta||E^\eta|^2 = \varepsilon_0 \int_{\Omega\setminus\Sigma_\eta} |E^\eta| \le \varepsilon_0 \sup_{\eta>0} \int_\Omega |E^\eta|^2 \,. \tag{6.2.9}$$

Combining (6.2.8) and (6.2.9) yields the estimate (6.2.4) and the claim is proved. □

Lemma 6.2 implies that a subsequence $(J^\eta)_\eta$ of the sequence of displacement currents two-scale converges to some field $J_0 \in L^2(\Omega\times Y;\mathbb{C}^3)$ provided (A3′) is satisfied.

In order to state the cell problem for $H_0$, we introduce the function space

$$\mathcal{X} := \left\{ v \in H^1_\sharp(Y;\mathbb{C}^3) \;\middle|\; \operatorname{curl} v = 0 \text{ in } \Sigma^* \right\}. \tag{6.2.10}$$

We note that the geometric average can be applied to every element of $\mathcal{X}$.

Lemma 6.3. (Cell problem) – *Let $\Omega \subset \mathbb{R}^3$ be as described in Section 4.1 and let $\Sigma \subset Y$ be an admissible microstructure. Let $(E^\eta, H^\eta)_\eta$ be a sequence satisfying* (A3′) *that two-scale converges to $(E_0, H_0, J_0)$ and such that $(E^\eta)_\eta$, $(H^\eta)_\eta$, and $(J^\eta)_\eta$ weakly converge in $L^2(\Omega;\mathbb{C}^3)$ to $E$, $H$, and $J$. Then for almost all $x \in R$ the two-scale limits $H_0 = H_0(x,\cdot) \in L^2_\sharp(Y;\mathbb{C}^3)$ and $J_0 = J_0(x,\cdot) \in L^2_\sharp(Y;\mathbb{C}^3)$ satisfy*

$$\begin{cases} \operatorname{curl}_y H_0 = -\mathrm{i}\omega\varepsilon_0 J_0 & \text{in } Y\,, & (6.2.11a)\\ \operatorname{div}_y H_0 = 0 & \text{in } Y\,, & (6.2.11b)\\ \operatorname{curl}_y J_0 = \ \mathrm{i}\omega\mu_0\varepsilon_r H_0 & \text{in } \Sigma\,, & (6.2.11c)\\ \qquad J_0 = 0 & \text{in } \Sigma^*\,, & (6.2.11d)\end{cases}$$

*in the distributional sense. Moreover, for almost all $x \in \Omega$, the two-scale limits $H_0 = H_0(x,\cdot)$ and $J_0 = J_0(x,\cdot)$ satisfy the identity:*

$$\int_\Sigma \langle J_0, \operatorname{curl} v\rangle = \mathrm{i}\omega\varepsilon_r\mu_0 \int_Y \langle H_0, v\rangle \quad \text{for all } v \in \mathcal{X} \text{ with } \oint_{\Sigma^*} v = 0\,. \tag{6.2.12}$$

*Outside the meta-material R, the two-scale limit $H_0(x,\cdot)$ is y-independent and $J_0(x,\cdot)$ vanishes identically.*

*Proof.* We proceed in three steps.

*Step 1. (Derivation of* (6.2.11)*)* For $\theta \in C^\infty_c(\Omega;\mathbb{R})$ and $\psi \in C^\infty_\sharp(Y;\mathbb{C}^3)$ we define $\varphi(x,y) := \theta(x)\psi(y)$ for almost all $x \in \Omega$ and $y \in Y$. We further set $\varphi_\eta(\cdot) := \varphi(\cdot,\cdot/\eta)$ for each $\eta > 0$. The sequence $(H^\eta)_\eta$ two-scale converges to $H_0 \in L^2(\Omega\times Y;\mathbb{C}^3)$, which implies that

$$\lim_{\eta\to 0} \int_\Omega \eta\langle H^\eta, \operatorname{curl}[\varphi_\eta]\rangle = \int_\Omega \theta\left(\int_Y \langle H_0, \operatorname{curl}\psi\rangle\right). \tag{6.2.13}$$

On the other hand, $H^\eta$ is a distributional solution to (6.0.2b); hence, by definition of the displacement current $J^\eta$ in (6.2.1),

$$\int_\Omega \eta\langle H^\eta, \operatorname{curl}[\varphi_\eta]\rangle = -\mathrm{i}\omega\varepsilon_0 \int_\Omega \eta\varepsilon_\eta\langle E^\eta, \varphi_\eta\rangle = -\mathrm{i}\omega\varepsilon_0 \int_\Omega \langle J^\eta, \varphi_\eta\rangle\,. \tag{6.2.14}$$

The sequence $(J^\eta)_\eta$ two-scale converges and thus sending $\eta \to 0$ in (6.2.14) yields

$$\lim_{\eta\to 0} \int_\Omega \eta \big\langle H^\eta, \operatorname{curl}[\varphi_\eta]\big\rangle = -\mathrm{i}\omega\varepsilon_0 \int_\Omega \theta\bigg(\int_Y \big\langle J^\eta, \psi\big\rangle\bigg)\,. \tag{6.2.15}$$

Combining (6.2.13) and (6.2.15) shows that $H_0(x,\cdot)$ is a distributional solution to (6.2.11a) for almost all $x \in \Omega$.

Thanks to equation (6.0.2a), the distributional divergence of $H^\eta$ vanishes identically in $\Omega$. Thus,

$$0 = \lim_{\eta\to 0} \int_\Omega \eta \big\langle H^\eta, \nabla[\varphi_\eta]\big\rangle = \int_\Omega \theta\bigg(\int_Y \big\langle H_0, \nabla\psi\big\rangle\bigg)\,. \tag{6.2.16}$$

The two-scale limit $H_0(x,\cdot)$ is thus a distributional solution to (6.2.11b) for almost all $x \in \Omega$.

In order to show the validity of equation (6.2.11c), we choose $\theta \in C_c^\infty(\Omega;\mathbb{R})$ and $\psi \in C_c^\infty(\Sigma;\mathbb{C}^3)$. For $\eta > 0$ we define $\varphi_\eta(\cdot) \coloneqq \theta(\cdot)\psi(\cdot/\eta)$. Using the definition of the relative permittivity $\varepsilon_\eta$ in (6.0.1) and Maxwell's equation (6.0.2a), we compute

$$\begin{aligned}\int_\Omega \eta\big\langle J^\eta, \operatorname{curl}[\varphi_\eta]\big\rangle &= \int_{\Sigma_\eta} \eta^2\varepsilon_\eta\big\langle E^\eta, \operatorname{curl}[\varphi_\eta]\big\rangle = \varepsilon_r \int_\Omega \big\langle E^\eta, \operatorname{curl}[\varphi_\eta]\big\rangle \\ &= \mathrm{i}\omega\varepsilon_r\mu_0 \int_\Omega \big\langle H^\eta, \varphi_\eta\big\rangle\,. \end{aligned}\tag{6.2.17}$$

Sending $\eta \to 0$ in (6.2.17) yields

$$\int_\Omega \theta\bigg(\int_Y \big\langle J_0, \operatorname{curl}\,\psi\big\rangle\bigg) = \mathrm{i}\omega\varepsilon_r\mu_0 \int_\Omega \theta\bigg(\int_Y \big\langle H_0, \psi\big\rangle\bigg)\,.$$

This proves the validity of equation (6.2.11c).

Let us fix $\theta \in C_c^\infty(\Omega;\mathbb{R})$ and $\psi \in C_c^\infty(\Sigma^*;\mathbb{C}^3)$. For every $\eta > 0$ we set $\varphi_\eta(\cdot) \coloneqq \theta(\cdot)\psi(\cdot/\eta)$. The two-scale convergence of both sequences $(J^\eta)_\eta$ and $(E^\eta)_\eta$ as well as the definition of $\varepsilon_\eta$ imply that

$$\int_\Omega \theta\bigg(\int_Y \big\langle J_0, \psi\big\rangle\bigg) = \lim_{\eta\to 0}\int_\Omega \big\langle J^\eta, \varphi_\eta\big\rangle = \lim_{\eta\to 0}\int_{\Omega\setminus\Sigma_\eta} \eta\big\langle E^\eta, \varphi_\eta\big\rangle = 0\,,$$

which shows that $J_0(x,\cdot)$ solves equation (6.2.11d) for almost all $x \in \Omega$. Thus, the two-scale limits $H_0(x,\cdot)$ and $J_0(x,\cdot)$ solve (6.2.11) for almost every $x \in R$.

*Step 2. (Derivation of* (6.2.12)*)* Choose a cut-off function $\theta \in C_c^\infty(\Omega;\mathbb{R})$ and a field $v \in \mathcal{X}$ with $\oint_{\Sigma^*} v = 0$. For each $\eta > 0$ we define the fields $v_\eta(\cdot) \coloneqq v(\cdot/\eta)$ and $\varphi_\eta(\cdot) \coloneqq \theta(\cdot)v_\eta(\cdot)$; note that $\varphi_\eta \in H_0^1(\Omega;\mathbb{C}^3)$. As $(H^\eta)_\eta$ two-scale converges to $H_0$,

$$\lim_{\eta\to 0}\int_\Omega \big\langle H^\eta, \varphi_\eta\big\rangle = \int_\Omega \theta\bigg(\int_Y \big\langle H_0, v\big\rangle\bigg)\,. \tag{6.2.18}$$

On the other hand, $H^\eta$ is a solution to (6.0.2a) for each $\eta > 0$. Thus,

$$\mathrm{i}\omega\mu_0 \int_\Omega \big\langle H^\eta, \varphi_\eta\big\rangle = \int_\Omega \big\langle E^\eta, \nabla\theta \wedge v_\eta\big\rangle + \frac{1}{\eta}\int_\Omega \theta\big\langle E^\eta, \operatorname{curl}\, v_\eta\big\rangle$$

As curl $v_\eta = 0$ in $\Omega \setminus \overline{\Sigma}_\eta$ and $\varepsilon_r E^\eta = \eta J^\eta$ in $\Sigma_\eta$, we deduce from the above equation

$$\mathrm{i}\omega\mu_0\varepsilon_r \int_\Omega \big\langle H^\eta, \varphi_\eta \big\rangle = \varepsilon_r \int_\Omega \big\langle E^\eta, \nabla\theta \wedge v_\eta \big\rangle + \int_{\Sigma_\eta} \theta \big\langle J^\eta, \mathrm{curl}\ v_\eta \big\rangle\,. \tag{6.2.19}$$

Due to Lemma 6.1, one readily checks that the two-scale limit $E_0(x,\cdot)$ is an element of the space $\mathcal{V}(\Sigma^*)$, which is defined in (4.3.2). Thus, using the identity (4.3.10) of the geometric average, we obtain

$$\begin{aligned}\lim_{\eta\to 0}\int_\Omega \big\langle E^\eta, \nabla\theta \wedge v_\eta \big\rangle &= \int_\Omega \int_Y \big\langle E_0, \nabla\theta \wedge v \big\rangle = \int_\Omega \Big\langle \nabla\theta, \int_Y v \wedge E_0 \Big\rangle \\ &= \int_\Omega \Big\langle \nabla\theta, \Big(\oint_{\Sigma^*} v\Big) \wedge \Big(\int_Y E_0\Big)\Big\rangle = 0\,, \end{aligned} \tag{6.2.20}$$

where we used the fact $\oint_{\Sigma^*} v = 0$ to obtain the last equation.

Using the two-scale convergence of $(J^\eta)_\eta$, we obtain

$$\lim_{\eta\to 0}\int_{\Sigma_\eta} \theta \big\langle J^\eta, \mathrm{curl}\ v_\eta \big\rangle = \int_\Omega \theta \Big( \int_\Sigma \big\langle J_0, \mathrm{curl}\ v \big\rangle \Big)\,. \tag{6.2.21}$$

Consequently, sending $\eta \to 0$ in (6.2.19) and combining (6.2.20)-(6.2.21), we get

$$\mathrm{i}\omega\mu_0\varepsilon_r \lim_{\eta\to 0}\int_\Omega \big\langle H^\eta, \varphi_\eta \big\rangle = \int_\Omega \theta \Big( \int_\Sigma \big\langle J_0, \mathrm{curl}\ v \big\rangle \Big)\,. \tag{6.2.22}$$

Combining (6.2.22) and (6.2.18) yields the identity (6.2.12).

*Step 3.* For $x \in \Omega \setminus R$ the relative permittivity $\varepsilon_\eta = 1$ is independent of $\eta > 0$. This implies that the two-scale limit $J_0(x,\cdot) = 0$. To see this, fix $\theta \in C_c^\infty(\Omega \setminus R; \mathbb{R})$ and $\psi \in C_\sharp^\infty(Y; \mathbb{C}^3)$. For $\eta > 0$ we define $\varphi_\eta(\cdot) := \theta(\cdot)\psi(\cdot/\eta)$. The two sequences $(J^\eta)_\eta$ and $(E^\eta)_\eta$ two-scale converge and hence

$$\int_{\Omega\setminus R} \theta \Big( \int_Y \langle J_0, \psi \rangle \Big) = \lim_{\eta\to 0} \int_{\Omega\setminus R} \big\langle J^\eta, \varphi_\eta \big\rangle = \lim_{\eta\to 0}\int_{\Omega\setminus R} \eta \big\langle E^\eta, \varphi_\eta \big\rangle = 0\,,$$

which proves the claim. We have proved that equations (6.2.11a) and (6.2.11b) are valid for almost all $x \in \Omega$. Thus, $\mathrm{curl}_y\, H_0(x,\cdot) = 0$ in $Y$ and $\mathrm{div}_y\, H_0(x,\cdot) = 0$ in $Y$ for almost all $x \in \Omega \setminus R$. Applying Lemma 2.4, we deduce that $H_0(x,\cdot)$ is a constant vector field in $Y$. $\square$

The system (6.2.11) is usually called the cell problem of $H_0$; see, for instance, [BS10, BBF09]. The novelty in Lemma 6.2, and our main contribution in this section, is the identity (6.2.12).

We observe that the two-scale limit $H_0(x,\cdot)$ is an element of $H^1_\sharp(Y;\mathbb{C}^3)$, for almost all $x \in \Omega$. Indeed, equations (6.2.11a) and (6.2.11b) imply that curl $H_0(x,\cdot) \in L^2_\sharp(Y;\mathbb{C}^3)$ and div $H_0(x,\cdot) \in L^2_\sharp(Y;\mathbb{C})$ and thus, by Lemma 2.4, the field $H_0(x,\cdot) \in H^1_\sharp(Y;\mathbb{C}^3)$ for almost all $x \in R$. Outside the meta-material, $H_0(x,\cdot)$ is constant in the second argument and hence an $H^1_\sharp(Y;\mathbb{C}^3)$-vector field, as well.

**A variational characterisation of the two-scale limit.** Due to equations (6.2.11a) and (6.2.11d) the two-scale limit $H_0(x,\cdot)$ is an element of the space $\mathcal{X}$, which is defined in (6.2.10). Equipped with the $H^1_\sharp(Y;\mathbb{C}^3)$-scalar product, $\mathcal{X}$ is a Hilbert space.

The equations (6.2.11a) and (6.2.12) allow us to give a variational characterisation of $H_0$. Before we discuss this variational characterisation, we introduce the sesquilinear form $b\colon \mathcal{X}\times\mathcal{X}\to\mathbb{C}$,

$$b(w,v) := \int_\Sigma \langle \operatorname{curl} w, \operatorname{curl} v\rangle - \omega^2\varepsilon_r\varepsilon_0\mu_0 \int_Y \langle w, v\rangle\,. \tag{6.2.23}$$

Lemma 6.4. – *Let $\Sigma\subset Y$ be an admissible microstructure. Then the following statements hold:*

*(i) If $(H_0,J_0)\in H^1_\sharp(Y;\mathbb{C}^3)\times L^2_\sharp(Y;\mathbb{C}^3)$ satisfy* (6.2.11a) *and* (6.2.12)*, then*

$$b\big(H_0,v\big) = 0 \quad \textit{for all } v\in\mathcal{X} \textit{ with } \oint_{\Sigma^*} v = 0\,. \tag{6.2.24}$$

*(ii) Assume $w\in\mathcal{X}$ satisfies $b(w,v)=0$ for all $v\in\mathcal{X}$ with $\oint_{\Sigma^*} v = 0$. Set $H_0 := w$ and $J_0 := -(\mathrm{i}\omega\varepsilon_0)^{-1}\operatorname{curl} w$. Then $(H_0,J_0)$ is a distributional solution to* (6.2.11)*.*

*Proof. (i)* Combining equations (6.2.11a) and (6.2.12) yields (6.2.24).

*(ii)* By definition of the sesquilinear form $b$, the field $w$ satisfies the equation

$$\int_\Sigma \langle \operatorname{curl} w, \operatorname{curl} v\rangle - \omega^2\varepsilon_r\varepsilon_0\mu_0 \int_Y \langle w, v\rangle = 0 \tag{6.2.25}$$

for all $v\in\mathcal{X}$ with $\oint_{\Sigma^*} v = 0$. Defining $J_0 := -(\mathrm{i}\omega\varepsilon_0)^{-1}\operatorname{curl} w$ provides us with an element of $L^2_\sharp(Y;\mathbb{C}^3)$ and the pair $(w,J_0)$ satisfies equation (6.2.11a) trivially. From the definition (6.2.10) of $\mathcal{X}$, we deduce that $J_0$ satisfies equation (6.2.11d).

By Lemma 4.5*(ii)*, the vector field $\nabla\varphi$ is an element of $\mathcal{X}$ with $\oint_{\Sigma^*}\nabla\varphi = 0$, for every test function $\varphi\in C^\infty_\sharp(Y;\mathbb{C})$. Thus, setting $v=\nabla\varphi$ in (6.2.25) yields the identity

$$\int_Y \langle w, \nabla\varphi\rangle = 0 \qquad \text{for all } \varphi\in C^\infty_\sharp(Y;\mathbb{C})\,,$$

which shows that $w$ satisfies equation (6.2.11b).

In order to prove that $(w,J_0)$ solves equation (6.2.11c), we take a test vector field $\varphi\in C^\infty_c(\Sigma;\mathbb{C}^3)$. Clearly, $\operatorname{curl}\varphi = 0$ in $\Sigma^*$. We claim that $\oint_{\Sigma^*}\varphi = 0$. Indeed, as the geometric average $\oint_{\Sigma^*}\varphi$ has the smallest Euclidean norm amongst all vectors $c\in\mathbb{C}^3$ for which

$$c\wedge\left(\int_{\Sigma^*}\phi\right) = \int_{\Sigma^*}\varphi\wedge\phi = 0 \qquad \text{for all } \phi\in\mathcal{V}(\Sigma^*)\,,$$

we deduce that $\oint_{\Sigma^*}\varphi = 0$. We may therefore set $v=\varphi$ in (6.2.25) and find that

$$-\mathrm{i}\omega\varepsilon_0\int_\Sigma \langle J_0, \operatorname{curl}\varphi\rangle = \omega^2\varepsilon_r\varepsilon_0\mu_0\int_\Sigma\langle w,\varphi\rangle \qquad \text{for all } \varphi\in C^\infty_c(\Sigma;\mathbb{C}^3)\,.$$

This proves the validity of equation (6.2.11c). □

The identity (6.2.24) has been formulated for the first time by Bouchitté, Bourel, and Felbacq in [BBF17, Lemma 5.4]. In [BBF17], (6.2.24) is directly derived from Maxwell's equations (6.0.2) using more advanced results from the theory of two-scale convergence.

Statement *(i)* of Lemma 6.4 shows that the two-scale limit $H_0(x,\cdot)$ lies in the space

$$X^H := \left\{ w \in \mathcal{X} \;\middle|\; b(w,v) = 0 \text{ for all } v \in \mathcal{X} \text{ with } \oint_{\Sigma^*} v = 0 \right\}. \tag{6.2.26}$$

Let us stress that, although we denote them identical, the function spaces $X^H$ in Chapters 5 and 6 are *not* identical. Indeed, every element of $X^H$ in Chapter 5 vanishes in $\Sigma$; this is not true for a generic element of the function space defined in (6.2.26).

Before we proceed, let us briefly sketch the idea of the analysis of $X^H$. A generic element $w \in X^H$ has a non-vanishing geometric average, $\oint_{\Sigma^*} w = z \neq 0$. Then $w = (w - z) + z = w_0 + z$ for some $w_0 \in \mathcal{X}$ with $\oint_{\Sigma^*} w_0 = 0$. Substituting this decomposition into the sesquilinear form $b$ and using the definition of $X^H$, we obtain the equation

$$0 = b(w,v) = b\big(w_0, v\big) - \omega^2 \varepsilon_0 \varepsilon_r \mu_0 \int_Y \langle z, v \rangle \,, \tag{6.2.27}$$

which holds for all $v \in \mathcal{X}$ with $\oint_{\Sigma^*} v = 0$. Both vector fields $w_0$ and $v$ are elements of $\mathcal{X}$ with vanishing geometric averages. Moreover, $\ell_z \colon \mathcal{X} \to \mathbb{C}$,

$$\ell_z(v) := \omega^2 \varepsilon_0 \varepsilon_r \mu_0 \int_Y \langle z, v \rangle \tag{6.2.28}$$

is a bounded anti-linear map. Thus, equation (6.2.27) fits in the framework of the Lax-Milgram lemma.

**Analysis of the variational equation** (6.2.27)**.** An immediate consequence of statement *(ii)* in Lemma 6.4 is the following equality:

$$X^H = \left\{ w \in \mathcal{X} \;\middle|\; \operatorname{div} w = 0 \text{ in } Y \text{ and } b(w,v) = 0 \;\; \forall v \in \mathcal{X} \text{ with } \oint_{\Sigma^*} v = 0 \right\},$$

suggesting that instead of thinking of $X^H$ as a subspace of $\mathcal{X}$, we should think of it as a subspace of

$$\mathcal{X}^{\operatorname{div}} := \big\{ v \in \mathcal{X} \;\big|\; \operatorname{div} v = 0 \text{ in } Y \big\}\,. \tag{6.2.29}$$

The geometric average is a bounded linear form and hence

$$\mathcal{X}_0^{\operatorname{div}} := \left\{ v \in \mathcal{X}^{\operatorname{div}} \;\middle|\; \oint_{\Sigma^*} v = 0 \right\} \tag{6.2.30}$$

is a closed subspace of $\mathcal{X}^{\operatorname{div}}$.

Lemma 6.5. (Existence and uniqueness result) – *Let $b \colon \mathcal{X}_0^{\operatorname{div}} \times \mathcal{X}_0^{\operatorname{div}} \to \mathbb{C}$ be the sesquilinear form defined in* (6.2.23)*. Choose $z \in \mathbb{C}^3$ and let $\ell_z \colon \mathcal{X}_0^{\operatorname{div}} \to \mathbb{C}$ be the anti-linear form given in* (6.2.28)*. Assume that* $\operatorname{Im}(\varepsilon_r) > 0$*. Then there is a unique solution $w \in \mathcal{X}_0^{\operatorname{div}}$ to*

$$b(w,\cdot) = \ell_z(\cdot)\,. \tag{6.2.31}$$

*Proof.* Fix $z \in \mathbb{C}^3$. Choose $\alpha > 0$ such that $\alpha \operatorname{Im}\{\varepsilon_r\} \geq 1 + \operatorname{Re}\{\varepsilon_r\}$. For every $w \in \mathcal{X}^{\text{div}}$ we then find that

$$\operatorname{Re}\big\{(1+\mathrm{i}\alpha) b(w,w)\big\} = \|\text{curl } w\|^2_{L^2(\Sigma;\mathbb{C}^3)} + \big(\alpha \operatorname{Im}\{\varepsilon_r\} - \operatorname{Re}\{\varepsilon_r\}\big)\|w\|^2_{L^2(Y;\mathbb{C}^3)} .$$

Due to Lemma 2.4,

$$\|\nabla w\|^2_{L^2(Y;\mathbb{C}^3)} = \|\text{curl } w\|^2_{L^2(Y;\mathbb{C}^3)} + \|\text{div } w\|^2_{L^2(Y;\mathbb{C})} = \|\text{curl } w\|^2_{L^2(\Sigma;\mathbb{C}^3)} ,$$

where we used that $w \in \mathcal{X}_0^{\text{div}}$ to obtain the last equality. These two equations together with the choice of $\alpha$ imply the coercivity of the sesquilinear form $(1+\mathrm{i}\alpha)b$. One readily checks that $(1+\mathrm{i}\alpha)b$ is bounded. Applying the Lax-Milgram lemma, we infer the existence and uniqueness of a solution $w \in \mathcal{X}_0^{\text{div}}$ to (6.2.31), since $\ell_z$ is a continuous anti-linear form on $\mathcal{X}_0^{\text{div}}$. □

**Analysis of the space $X^H$.** The first result is an immediate consequence of Lemma 6.5.

Corollary 6.6. – *If $w \in X^H$ satisfies $\oint_{\Sigma^*} w = 0$, then $w = 0$.*

*Proof.* Thanks to Lemma 6.4 *(ii)*, the field $w$ is an element of $\mathcal{X}^{\text{div}}$. As the geometric average of $w$ vanishes, $w \in \mathcal{X}_0^{\text{div}}$. By definition of $X^H$,

$$b(w,v) = 0 \,, \tag{6.2.32}$$

for all $v \in \mathcal{X}$ with $\oint_{\Sigma^*} v = 0$. Each $v \in \mathcal{X}_0^{\text{div}}$ is an element of $\mathcal{X}$ with $\oint_{\Sigma^*} v = 0$ and hence equation (6.2.32) does hold for all $v \in \mathcal{X}_0^{\text{div}}$. By Lemma 6.5, there is only the trivial solution $w = 0$ in $\mathcal{X}_0^{\text{div}}$ to (6.2.32). This proves the claim. □

From Corollary 6.6 we infer that the geometric average $\oint_{\Sigma^*} : X^H \to A^H$ is an isomorphism, where $A^H$ is the space of attainable geometric averages of fields in $X^H$; that is

$$A^H := \left\{ \oint_{\Sigma^*} v \,\middle|\, v \in X^H \right\}. \tag{6.2.33}$$

We note that $A^H$ is defined in the same was as in (5.2.12); let us stress, however, that the symbol $X^H$ denotes different function spaces in Chapters 5 and 6. The following result, which is an analogue of Lemma 5.7, is thus remarkable.

Proposition 6.7. – *Let $\varepsilon_r \in \mathbb{C}$ with $\operatorname{Im}\{\varepsilon_r\} > 0$. Then the two spaces $A^H$ and $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ coincide for every admissible microstructure $\Sigma \subset Y$.*

*Proof.* We recall from its definition that the geometric average is a linear and surjective map $\mathcal{X}(\Sigma^*) \to A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$. As $X^H$, defined in (6.2.29), is a subset of $\mathcal{X}(\Sigma^*)$, which is defined in (4.3.1), we deduce that $A^H \subset A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$.

Choose an arbitrary $z \in A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$. By Lemma 6.5, there is a unique $w^z \in \mathcal{X}_0^{\text{div}}$ such that $b(w^z, v) = \ell_z(v)$ for all $v \in \mathcal{X}_0^{\text{div}}$. Define the vector field $v^z := w^z + z$. One readily checks that $v^z \in \mathcal{X}^{\text{div}}$ satisfies

$$b\big(v^z, v\big) = b\big(w^z, v\big) - \ell_z(v) = 0 \,,$$

for all $v \in \mathcal{X}_0^{\text{div}}$. Consequently, $v^z \in X^H$. Using the linearity of the geometric average as well as Lemma 4.5*(iii)*, we find that $\oint_{\Sigma^*} v^z = \oint_{\Sigma^*} z = z$. □

Remark 8. – *We recall from its definition in* (4.3.4) *that the space* $A^{\mathcal{V}}(\Sigma^*)^{\wedge\perp}$ *depends on* $A^{\mathcal{V}}(\Sigma^*)$. *As the* $E_0$*-cell problem is identical in Chapters 5 and 6, the spaces* $A^E$ *and* $A^{\mathcal{V}}(\Sigma^*)$ *coincide by Lemma 5.3. Combining this observation with Proposition 6.7 we find that the spaces which we denote by* $A^H$ *are not only defined in the same way in Chapters 5 and 6, but they are identical. This is a nontrivial observation, since the function space* $X^H$ *is defined differently in Chapters 5 and 6.*

The next result clarifies the relation between $k$-loops and $X^H$.

Lemma 6.8. – *Assume* $\varepsilon_r \in \mathbb{C}$ *with* $\operatorname{Im}\{\varepsilon_r\} > 0$. *Let* $\Sigma \subset Y$ *be an admissible microstructure, and let* $\mathcal{L}_{\Sigma^*}$ *and* $\mathcal{N}_{\Sigma^*}$ *be the sets given in* (4.3.17) *and* (4.3.18).

*(i) If* $m \in \mathcal{N}_{\Sigma^*}$ *satisfies* $\langle m, k\rangle = 0$ *for all* $k \in \mathcal{L}_{\Sigma^*}$, *then* $m \notin A^H$; *in particular, there is no element* $v^m \in X^H$ *with* $\oint_{\Sigma^*} v^m = m$.

*(ii) If* $k \in \mathcal{L}_{\Sigma^*}$, *then there exists a unique* $v^k \in X^H$ *with*

$$\oint_{\Sigma^*} v^k = \pi_{A^H}(k)\,, \tag{6.2.34}$$

*where* $\pi_{A^H} : \mathbb{C}^3 \to \mathbb{C}^3$ *denotes the orthogonal projection onto* $A^H$.

*Proof. (i)* By Proposition 4.8, there is a potential $\Theta \in H^1_\sharp(\Sigma^*;\mathbb{R})$ with $\nabla\Theta = m$. Applying the properties of the geometric average from Lemma 4.5, we obtain that

$$0 = \oint_{\Sigma^*} \nabla\Theta = \oint_{\Sigma^*} m = \pi_{A^H}(m)\,.$$

As $m \neq 0$, we deduce that $m \notin A^H$.

*(ii)* Fix $k \in \mathcal{L}_{\Sigma^*}$. Due to Lemma 6.5, there is a unique element $w^k \in \mathcal{X}_0^{\mathrm{div}}$ such that $b(w^k, v) = \ell_k(v)$ for all $v \in \mathcal{X}_0^{\mathrm{div}}$. The vector field $v^k := w^k + k$ is an element of $\mathcal{X}^{\mathrm{div}}$, since both summands are. Furthermore,

$$b\big(v^k, v\big) = b\big(w^k, v\big) - \ell_k(v) = 0$$

for all $v \in \mathcal{X}_0^{\mathrm{div}}$, and hence $v^k \in X^H$. Applying the properties of the geometric average from Lemma 4.5, we obtain that $\oint_{\Sigma^*} v^k = \oint_{\Sigma^*} k = \pi_{A^H}(k)$. This shows the existence of $v^k$; uniqueness follows from Corollary 6.6. □

For a discussion of equation (6.2.34) we refer to Remarks 5 on page 61; see also Proposition 6.11.

Proposition 6.9. (Characterisation of the solution space ) – *If* $\operatorname{Im}\{\varepsilon_r\} > 0$, *then for every admissible microstructure* $\Sigma \subset Y$, *we have that*

$$\dim X^H = \dim A^H \leq 3\,.$$

*More precisely, let* $\{b^j \mid j \in I\}$ *be a basis of* $A^H$ *and denote by* $v^j \in X^H$ *the unique field with* $\oint_{\Sigma^*} v^j = b^j$. *Then every* $v \in X^H$ *can be written as a linear combination,*

$$v = \sum_{j\in I} \alpha_j v^j \qquad \text{in } H^1_\sharp(Y;\mathbb{C}^3)\,, \tag{6.2.35}$$

*with coefficients* $\alpha_j \in \mathbb{C}$ *for* $j \in I$.

*Proof.* Fix $v \in X^H$. As $\oint_{\Sigma^*} v \in A^H$, we find $\alpha_j \in \mathbb{C}$ for $j \in I$ such that

$$\oint_{\Sigma^*} v = \sum_{j\in I} \alpha_j b^j = \sum_{j\in I} \alpha_j \oint_{\Sigma^*} v^j \,.$$

Setting $w\colon Y \to \mathbb{C}^3$ as $w := v - \sum_{j\in I} \alpha_j v^j$ provides us with an element of $X^H$ with $\oint_{\Sigma^*} w = 0$. From Corollary 6.6, we infer that $w = 0$ in $X^H$ and the claim is proved. $\square$

## 6.3 Effective material parameters and equations

This section is devoted to the derivation of the effective system and the definition of the effective material parameters. For convenience of the reader, let us recall that the subspaces $A^E$ and $A^H$ of $\mathbb{C}^3$ are defined in (6.1.7) and (6.2.33), respectively. Denote by $\pi_{A^E} : \mathbb{C}^3 \to \mathbb{C}^3$ the orthogonal projection onto $A^E$.

By Lemma 5.11, there exists a unique linear map $\varepsilon^{\text{eff}}\colon A^E \to A^E$ that satisfies the identity

$$\left\langle \varepsilon^{\text{eff}}\left(\int_Y u_1\right), \int_Y u_2 \right\rangle = \int_Y \langle u_1, u_2\rangle \quad \text{for all } u_1, u_2 \in X^E \,.$$

This map is called the relative permittivity of the metamaterial located in $R$.

Due to Corollary 6.6 the geometric average is an isomorphism between $X^H$ and $A^H$. Thus, for every $k \in A^H$ there is a unique $v^k \in A^H$ with $\oint_{\Sigma^*} v^k = k$. The linear map $\mu^{\text{eff}}\colon A^H \to \mathbb{C}^3$,

$$\mu^{\text{eff}}\left(\oint_{\Sigma^*} v\right) := \fint_Y v \quad \text{for } v \in X^H$$

is called the relative permeability of the meta-material located in $R$.

The *effective permittivity* $\hat{\varepsilon}$ and the *effective permeability* $\hat{\mu}$ are then defined as

$$\hat{\varepsilon}(x) := \begin{cases} \varepsilon^{\text{eff}} & \text{if } x \in R \\ \mathrm{id}_{\mathbb{C}^{3\times 3}} & \text{if } x \in \Omega \setminus R \end{cases} \tag{6.3.1a}$$

and

$$\hat{\mu}(x) := \begin{cases} \mu^{\text{eff}} & \text{if } x \in R \\ \mathrm{id}_{\mathbb{C}^{3\times 3}} & \text{if } x \in \Omega \setminus R \end{cases} . \tag{6.3.1b}$$

We recall that $E_0, H_0 \in L^2(\Omega \times Y; \mathbb{C}^3)$ are the two-scale limits of $(E^\eta)_\eta$ and $(H^\eta)_\eta$, respectively. The *effective electromagnetic field* $(\hat{E}, \hat{H})\colon \Omega \to \mathbb{C}^3 \times \mathbb{C}^3$ is defined by

$$\hat{E}(x) := \fint_Y E_0(x,y)\,\mathrm{d}y \quad \text{and} \quad \hat{H}(x) := \oint_{\Sigma^*} H_0(x,\cdot)\,. \tag{6.3.2}$$

Let us recall from Section 5.3 that outside the microstructure, that is, for $x \in \Omega \setminus R$, the fields $\hat{H}(x)$ and $H(x)$ coincide.

THEOREM 6.10. (Effective equations) – *Let $\Omega \subset \mathbb{R}^3$ and let $R \Subset \Omega$ be as described in Section 4.1. Let $\Sigma \subset Y$ be an admissible microstructure, and let $\hat{\varepsilon}$ and $\hat{\mu}$ be the linear maps given in* (6.3.1). *Assume $(E^\eta, H^\eta)_\eta$ is a sequence of distributional solutions to* (6.0.2) *that satisfies the energy-bound* (6.0.3). *Then the effective electromagnetic field $(\hat{E}, \hat{H})$ satisfies*

$$\hat{E}(x) \in A^E \quad \text{and} \quad \hat{H}(x) \in A^H \quad \text{for almost all } x \in R\,. \tag{6.3.3}$$

*Moreover, $(\hat{E}, \hat{H})$ is a distributional solution to*

$$\begin{cases} \operatorname{curl} \hat{E} = \mathrm{i}\omega\mu_0\hat{\mu}\hat{H} & \text{in } \Omega\,, \quad (6.3.4a) \\ \pi_{A^E}\big(\operatorname{curl} \hat{H}\big) = -\mathrm{i}\omega\varepsilon_0\pi_{A^E}\big(\hat{\varepsilon}\hat{E}\big) & \text{in } \Omega\,, \quad (6.3.4b) \\ \operatorname{curl} \hat{H} = -\mathrm{i}\omega\varepsilon_0\hat{E} & \text{in } \Omega \setminus R\,. \quad (6.3.4c) \end{cases}$$

We note that the relations (6.3.3) and the effective system (6.3.4) coincide with the corresponding results from Section 5.3. Let us stress, however, that the relative permeabilities $\mu^{\text{eff}} \colon A^H \to \mathbb{C}^3$ are different in Sections 5.3 and 6.3, because the function spaces which we denote by $X^H$ are not identical in Chapters 5 and 6. The relative permittivity $\varepsilon^{\text{eff}} \colon A^E \to A^E$, on the other hand, is identical to the one from Section 5.3.

*Proof.* This proof is similar to the proof of the effective system (5.3.5).

*Step 1: Derivation of* (6.3.3). By Lemma 6.1, $E_0(x, \cdot) \in X^E$ for almost all $x \in R$ and thus, $\hat{E}(x) = \fint_Y E_0(x, \cdot) \in A^E$ for almost all $x \in R$, by definition of $A^E$. This shows the first part of (6.3.3).

Combining Lemmas 6.3 and 6.4 yields $H_0(x, \cdot) \in X^H$ for almost all $x \in R$. As $X^H \subset \mathcal{X}^{\text{div}}$, we deduce that (6.3.3) holds by definition of the effective magnetic field $\hat{H}$ as well as the definition of the space $A^H$.

*Step 2: Derivation of* (6.3.4a) *and* (6.3.4c). The verification of (6.3.4a) is analogous to the one for equation (5.3.7a): take the distributional limit of (6.0.2a) as $\eta \to 0$ and use to definition of the effective electromagnetic field $(\hat{E}, \hat{H})$ as well as the definition of $\hat{\mu}$ to obtain (6.3.4a).

In order to show (6.3.4c), we observe that $\Omega \setminus R \subset \Omega \setminus \Sigma_\eta$. We can therefore take the distributional limit in (6.0.2b) as $\eta \to 0$ and find

$$\operatorname{curl} H = -\mathrm{i}\omega\varepsilon_0 E \quad \text{in } \Omega \setminus R\,.$$

From this identity we infer the validity of (6.3.4c), since $E = \hat{E}$ and $H = \hat{H}$ in $\Omega \setminus R$.

*Step 3: Derivation of* (6.3.4b). Choose $\theta \in C_c^\infty(\Omega; \mathbb{R})$ and $u \in X^E$. For $\eta > 0$ we set $\varphi_\eta(\cdot) \coloneqq \eta(\cdot)u(\cdot/\eta)$. Using the two-scale convergence of $(H^\eta)_\eta$ and the fact that $\operatorname{curl} u = 0$ in $Y$, we obtain

$$\lim_{\eta \to 0} \int_\Omega \big\langle H^\eta, \operatorname{curl}[\varphi_\eta]\big\rangle = \int_\Omega \int_Y \big\langle H_0, \nabla\theta \wedge u\big\rangle = \int_\Omega \Big\langle \nabla\theta, \int_Y u \wedge H_0 \Big\rangle\,. \tag{6.3.5}$$

The definition of the geometric average implies that

$$\int_Y u \wedge H_0(x, \cdot) = \left(\int_{\Sigma^*} u\right) \wedge \oint_{\Sigma^*} H_0(x, \cdot) = \left(\int_{\Sigma^*} u\right) \wedge \hat{H}(x)\,, \tag{6.3.6}$$

for almost all $x \in \Omega$. Combining (6.3.5) and (6.3.6) we get

$$\begin{aligned} \lim_{\eta\to 0} \int_\Omega \left\langle H^\eta, \operatorname{curl}[\varphi_\eta] \right\rangle &= \int_\Omega \left\langle \hat{H}, \nabla\theta \wedge \left( \int_{\Sigma^*} u \right) \right\rangle \\ &= \int_\Omega \left\langle \hat{H}, \operatorname{curl}\left( \theta \int_{\Sigma^*} u \right) \right\rangle . \end{aligned} \tag{6.3.7}$$

On the other hand, the field $H^\eta$ is a distributional solution to (6.0.2b). Thus, by exploiting the two-scale convergence of $(E^\eta)_\eta$ and the fact that $u = 0$ in $\Sigma$, we get

$$\begin{aligned} \lim_{\eta\to 0} \int_\Omega \left\langle H^\eta, \operatorname{curl}[\varphi_\eta] \right\rangle &= -\mathrm{i}\omega\varepsilon_0 \lim_{\eta\to 0} \int_{\Omega\setminus\Sigma_\eta} \left\langle E^\eta, \varphi_\eta \right\rangle \\ &= -\mathrm{i}\omega\varepsilon_0 \int_\Omega \theta \int_{\Sigma^*} \left\langle E_0, u \right\rangle . \end{aligned} \tag{6.3.8}$$

We claim that the identity

$$\int_Y \left\langle E_0(x,\cdot), u \right\rangle = \left\langle \hat{\varepsilon}(x)\hat{E}(x), \fint_Y u \right\rangle \tag{6.3.9}$$

holds for almost all $x \in \Omega$. Lemma 6.1 implies that $\hat{E}(x) = E(x)$ for almost all $x \in \Omega \setminus R$. As $\hat{\varepsilon}(x) = \mathrm{id}_{\mathbb{C}^{3\times 3}}$, identity (6.3.9) holds in $\Omega \setminus R$. For $x \in R$, equation (6.3.9) follows from the definitions of $\varepsilon^{\text{eff}}$ and $\hat{E}$. Substituting (6.3.9) into (6.3.8) yields

$$\begin{aligned} \lim_{\eta\to 0} \int_\Omega \left\langle H^\eta, \operatorname{curl}[\varphi_\eta] \right\rangle &= -\mathrm{i}\omega\varepsilon_0 \int_\Omega \left\langle \hat{\varepsilon}\hat{E}, \theta \fint_Y u \right\rangle \\ &= -\mathrm{i}\omega\varepsilon_0 \int_\Omega \left\langle \hat{\varepsilon}\hat{E}, \theta \int_{\Sigma^*} u \right\rangle . \end{aligned} \tag{6.3.10}$$

Combining (6.3.7) and (6.3.10) we infer the validity of equation (6.3.4b), since $u \in X^E$ was chosen arbitrarily. □

## 6.4 Discussion of examples

Let us recall that we only used the spaces $X^E$, $A^E$, and $A^H$ to derive the effective equations in Section 5.4. As explained in Remark 8 on page 84, the spaces $X^E$, $A^E$, and $A^H$ are identical in Chapters 5 and 6. Thus, the effective equations for a perfectly conducting microstructure coincide with the effective equations for a highly conductive microstructure, and we therefore do not state the equations again. We would like to stress, though, that the relative permeability $\mu^{\text{eff}} \colon A^H \to \mathbb{C}^3$ is a different map in Chapters 5 and 6, since the function spaces $X^H$ are not identical.

The next result is an analogue of Proposition 5.14.

PROPOSITION 6.11. – *Let $\varepsilon_r \in \mathbb{C}$ with $\operatorname{Im}\{\varepsilon_r\} > 0$, and let $\Sigma \subset Y$ be an admissible microstructure for which (A4*) holds. If $\dim X^E \geq 2$, then*

$$\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*} = A^H = \mathbb{C}^3 .$$

*Proof.* This proof mimics the one of 5.14. As $\dim X^E \geq 2$, we deduce from equation (5.4.5) and from Lemma 6.8*(ii)* that each $k \in \mathcal{L}_{\Sigma^*}$ induces an element $v^k \in X^H$ with $\oint_{\Sigma^*} v^k = k$. Thus, $\mathcal{L}_{\Sigma^*} \subset A^H$ and hence, since $A^H$ is a vector space, $\operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*} \subset A^H$.

In order to proof the converse inclusion, choose $c \in A^H$. By definition of $A^H$ there exists $v \in X^H$ with $\oint_{\Sigma^*} v = c$. Let $(b_1^*, b_2^*, b_3^*)$ be an orthogonal basis of $\mathbb{R}^3$ for which (A4*) is satisfied. Then, for each $j \in L^*$, there exists $v^j \in X^H$ with $\oint_{\Sigma^*} v^j = b_j^*$. Set $\alpha_l \coloneqq \langle c_l, b_l^* \rangle$ for $l \in \{1, 2, 3\}$. The vector field $w \coloneqq v - \sum_{j \in L^*} \alpha_j v^j$ is an element of $X^H$ with

$$\oint_{\Sigma^*} w = \sum_{i \in N^*} \alpha_i b_i^* \in \operatorname{span}_{\mathbb{C}} \big\{ b_i^* \;\big|\; i \in N^* \big\} . \tag{6.4.1}$$

If $\oint_{\Sigma^*} w = 0$, then

$$c = \oint_{\Sigma^*} v = \oint_{\Sigma^*} w + \sum_{j \in L^*} \alpha_j \oint_{\Sigma^*} v^j = \sum_{j \in L^*} \alpha_j b_j^* \in \operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*} ,$$

which proves the inclusion $A^H \subset \operatorname{span}_{\mathbb{C}} \mathcal{L}_{\Sigma^*}$. We are thus left to prove that $w$ has a vanishing geometric average.

By Lemma 6.8*(i)*, the vector $b_i^* \notin A^H$ for $i \in N^*$. On the other hand, $w$ is an element of $X^H$ and thus

$$\oint_{\Sigma^*} w = \pi_{A^H} \left( \oint_{\Sigma^*} w \right) = \sum_{i \in N^*} \alpha_i \pi_{A^H} \left( b_i^* \right) = 0 . \tag{6.4.2}$$

This proves the claim. $\square$

# Part III

# Limiting absorption principle for a bounded periodic waveguide

*Upwards:- in spite of the spirit that drew it downwards, towards the abyss, the spirit of gravity, my devil and archenemy.*
*Upwards:- although it sat upon me, half-dwarf, half-mole; paralysed, paralysing; dripping lead in my ear, and thoughts like drops of lead into my brain.*
*"Oh Zarathustra," it whispered scornfully, syllable by syllable, "you stone of wisdom! you threw yourself high, but every thrown stone must - fall![...]"*
*"Halt, dwarf!" said I. "Either I- or you! I, however, am the stronger of the two:- you knowest not my abysmal thought! It- could you not endure!"[...]*
*"Look at this gateway! Dwarf!" I continued, "it has two faces. Two roads come together here: these has no one yet gone to the end.[...]"*
*"But should one follow them further- and even further and further on, think you, dwarf, that these roads would be eternally antithetical?"*
*"Everything straight lies," murmured the dwarf, contemptuously. "All truth is crooked; time itself is a circle."*
*"You spirit of gravity!" said I wrathfully, "do not take it too lightly! Or I shall let you squat where you squat, Haltfoot,- and I carried you high!"*

—Friedrich Nietzsche, *Thus Spoke Zarathustra, The Vision and the Riddle*

CHAPTER 7

# Introduction

Consider an electromagnetic wave $(E, H)$ satisfying the time-harmonic Maxwell equations

$$\begin{cases} \operatorname{curl} E = \phantom{-}\mathrm{i}\omega\mu H & \text{in } \Omega\,, \quad (7.0.1a) \\ \operatorname{curl} H = -\mathrm{i}\omega\varepsilon E & \text{in } \Omega\,, \quad (7.0.1b) \end{cases}$$

in a domain $\Omega \subset \mathbb{R}^3$, where $\omega > 0$ is the prescribed frequency of the wave, $\varepsilon$ the permittivity and $\mu$ the permeability of the medium, respectively. The equations are complemented by boundary conditions on $\partial\Omega$. For applications it is interesting to solve the equations in an unbounded domain $\Omega$; in this case, radiation conditions at infinity must additionally be imposed to ensure the well-posedness of the problem. There are many radiation conditions for time-harmonic electromagnetic waves present in the literature, the most prominent being the Silver-Müller radiation condition; see [Sil49], [Mül57], and the survey [Sch92].

In some cases, the system (7.0.1) of two equations for two unknown fields can be reduced to a scalar Helmholtz equation

$$-\Delta u - k^2 u = 0 \quad \text{in } \Omega$$

with $u$ being a component of one of the fields $E$ or $H$. For an unbounded domain $\Omega$, the problem is only well-posed if $u$ also satisfies a radiation condition at infinity such as the classical Sommerfeld condition [Som12]. Which radiation condition needs to be imposed depends on the medium $\Omega$: For a homogeneous medium with a compact boundary, the Sommerfeld radiation condition ensures well-posedness. In the case of an unbounded homogeneous waveguide $\Omega := \mathbb{R} \times (0, 1)$, Svešnikov proved in [Sve50] that the Dirichlet as well as the Neumann boundary value problems are well posed provided the so-called partial radiation conditions are imposed. We mention [Rit09, Section 2.4] for a discussion of different radiation conditions.

The study of wave propagation in a periodic medium $\Omega$ has a long history. Wave propagation in stratified materials has been extensively discussed in the literature; see, for instance, [Wil84, Wed91, BBDT01]. Recently, the interest has turned to radiation conditions for closed periodic waveguides; see, for instance,

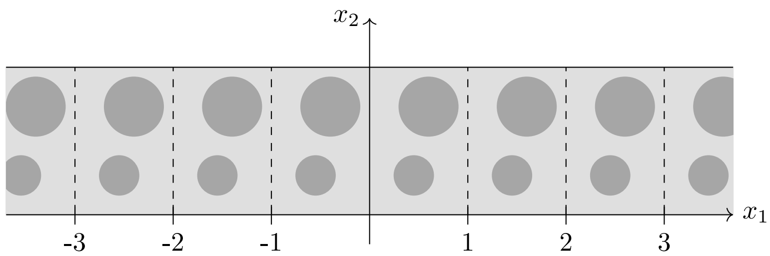


Figure 7.1: Cross section of the unbounded waveguide $\tilde{\Omega} = \mathbb{R} \times \mathbb{T}^2$. The dark gray areas represent a periodic assembly $\tilde{\mathcal{O}}$ of effectively two-dimensional obstacles $\Sigma$.

the articles of Fliss and Joly [FJ15], Lamacz and Schweizer [LS18a], and Kirsch and Lechleiter [KL18].

### From Maxwell to Helmholtz

Instead of studying the full Maxwell system (7.0.1) in a periodic and closed waveguide, we reduce this system to a scalar Helmholtz-like equation. This simplification is only valid if the domain $\Omega$ and the source terms satisfy certain assumptions, which we discuss below.

We denote by $\tilde{\Omega}$ the unbounded waveguide $\mathbb{R} \times \mathbb{T}^2$, which contains a periodic assembly of obstacles $\tilde{\mathcal{O}}$; see Figure 7.1 for a sketch. Let us assume that $\tilde{\mathcal{O}}$ is an open set with Lipschitz boundary. We are interested in the propagation of a time-harmonic electromagnetic wave $(E, H)$ in this waveguide. Assuming the obstacles are perfect conductors, $(E, H) \in L^2_{\text{loc}}(\tilde{\Omega}; \mathbb{C}^3) \times L^2_{\text{loc}}(\tilde{\Omega}; \mathbb{C}^3)$ satisfies the time-harmonic Maxwell equations,

$$\begin{cases} \operatorname{curl} E = \mathrm{i}\omega\mu_0 H + f^H & \text{in } \tilde{\Omega}\,, & (7.0.2\text{a}) \\ \operatorname{curl} H = -\mathrm{i}\omega\varepsilon_0 E + f^E & \text{in } \tilde{\Omega} \setminus \overline{\tilde{\mathcal{O}}}\,, & (7.0.2\text{b}) \\ E = H = 0 & \text{in } \tilde{\mathcal{O}}\,, & (7.0.2\text{c}) \end{cases}$$

in a distributional sense. Here $f^H \in L^2(\tilde{\Omega}; \mathbb{C}^3)$ and $f^E \in H(\operatorname{curl}, \tilde{\Omega} \setminus \overline{\tilde{\mathcal{O}}})$ are given source terms of the form

$$f^H(x_1, x_2, x_3) := f^H_3(x_1, x_2)\, \mathrm{e}_3 \tag{7.0.3a}$$

and

$$f^E(x_1, x_2, x_3) := f^E_1(x_1, x_2)\, \mathrm{e}_1 + f^E_2(x_1, x_2)\, \mathrm{e}_2\ . \tag{7.0.3b}$$

The two functions $f^H$ and $f^E$ also vanish almost everywhere in the obstacles $\tilde{\mathcal{O}}$. We note that the system (7.0.2) admits, in general, no unique solution, since no radiation conditions at infinity are specified.

This system of three equations with two unknown fields can be transformed into a system involving only the magnetic field $H$; indeed, taking the distributional curl in (7.0.2b) and using equation (7.0.2a) yields

$$\operatorname{curl} \operatorname{curl} H = \omega^2 \mu_0 \varepsilon_0 H - \mathrm{i}\omega\varepsilon_0 f^H + \operatorname{curl} f^E \quad \text{in } \tilde{\Omega} \setminus \overline{\tilde{\mathcal{O}}}\,,$$

provided $(E, H)$ is a distributional solution to (7.0.2). Due to (7.0.2a) the distributional divergence of $H$ vanishes in $\tilde{\Omega}$ and hence the normal trace $\langle H, \nu\rangle$ does not jump across the interface $\partial\tilde{\mathcal{O}}$. Consequently, if $(E, H)$ is a distributional solution to (7.0.2), then $H$ solves

$$\begin{cases} -\Delta H = \omega^2\mu_0\varepsilon_0 H - \mathrm{i}\omega\varepsilon_0 f^H + \operatorname{curl} f^E & \text{in } \tilde{\Omega} \setminus \overline{\tilde{\mathcal{O}}}\,, \quad (7.0.4a)\\ \quad H = 0 & \text{in } \tilde{\mathcal{O}}\,, \quad (7.0.4b)\\ \langle H, \nu\rangle = 0 & \text{on } \partial\tilde{\mathcal{O}}\,. \quad (7.0.4c) \end{cases}$$

Equation (7.0.4a) follows from the identity curl curl $H = -\Delta H + \nabla \operatorname{div} H$.

On the other hand, every distributional solution $H$ of (7.0.4) that is divergence free in $\tilde{\Omega} \setminus \overline{\tilde{\mathcal{O}}}$ induces a solution $(E, H)$ to (7.0.2) with the electric field given by $E \coloneqq (\mathrm{i}\omega\varepsilon_0)^{-1}(f^E - \operatorname{curl} H)\mathbb{1}_{\tilde{\Omega}\setminus\tilde{\mathcal{O}}}$.

**Further simplifications.** So far, we have only demanded that the obstacle $\tilde{\mathcal{O}}$ has a Lipschitz boundary. In what follows, we additionally assume that it is independent of $x_3$. We call a set $\Sigma \subset (0,1)^3$ an *effectively two-dimensional obstacle* if

$$\Sigma = \sigma \times (0,1)\,,$$

where $\sigma \Subset (0,1)^2$ is an open, non-empty set with Lipschitz boundary. An example of such an effectively two-dimensional obstacle is given in Figure 7.2.

Given an effectively two-dimensional obstacle $\Sigma \subset (0,1)^3$, we define the set of all obstacles as

$$\tilde{\mathcal{O}} \coloneqq \bigcup_{m\in\mathbb{Z}} \left(m\,\mathrm{e}_1 + \Sigma\right).$$

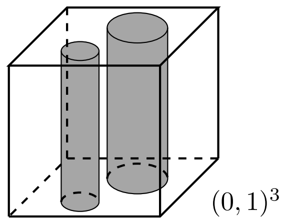


Figure 7.2: The two dark grey cylinders represent an effectively two-dimensional obstacle $\Sigma$.

For an effectively two-dimensional obstacle $\Sigma$, the geometry of $\tilde{\mathcal{O}}$ as well as the geometry of the waveguide $\tilde{\Omega}$ are essentially two-dimensional. We recall from (7.0.3) that the source terms $f^H$ and $f^E$ are also independent of $x_3$. This allows for a special ansatz of the magnetic field transforming the vector-valued problem (7.0.4) into a system of scalar equations. Before presenting this ansatz, we introduce suitable notation:

$$\Omega \coloneqq \mathbb{R} \times \mathbb{S}^1 \quad \text{and} \quad \mathcal{O} \coloneqq \pi_{1,2}\big(\tilde{\mathcal{O}}\big)\,,$$

where $\pi_{1,2}\colon \mathbb{R}^3 \to \mathbb{R}^2$ is given by $\pi_{1,2}(x_1, x_2, x_3) \coloneqq (x_1, x_2)$.

**A special ansatz for the magnetic field.** We look for a distributional solution $H$ to (7.0.4) of the form

$$H(x_1, x_2, x_3) = u(x_1, x_2)\,\mathrm{e}_3\,, \tag{7.0.5}$$

for some function $u\colon \Omega \to \mathbb{C}$. Setting $h \coloneqq -\mathrm{i}\omega\varepsilon_0 f_3^H + \partial_1 f_2^E - \partial_2 f_1^E$ provides us with an element of $L^2(\Omega; \mathbb{C})$ if $f^H \in L^2(\Omega; \mathbb{C}^3)$ and $f^E \in H(\operatorname{curl}, \Omega \setminus \overline{\mathcal{O}})$. Let $u \in H^1_{\mathrm{loc}}(\Omega \setminus \overline{\mathcal{O}}; \mathbb{C})$ be a distributional solution to

$$\begin{cases} -\Delta u = \omega^2\mu_0\varepsilon_0 u + h & \text{in } \Omega \setminus \overline{\mathcal{O}}\,, \quad (7.0.6a)\\ \quad u = 0 & \text{on } \partial\mathcal{O}\,. \quad (7.0.6b) \end{cases}$$

Then the vector field $H$, given in (7.0.5) and extended trivially to $\Omega$, is a distributional solution to (7.0.4). Furthermore, the distributional divergence of $H$ vanishes in $\tilde{\Omega} \setminus \overline{\tilde{\mathcal{O}}}$.

In order to have a well-posed problem, system (7.0.6) has to be complemented by radiation conditions at infinity. Choosing radiation conditions and establishing the existence and uniqueness of a solution to (7.0.6) is out of the scope of this thesis. Instead, we study (7.0.6) in a bounded waveguide. More precisely, we fix $R \in \mathbb{N}$ and $l > 0$, and consider the bounded waveguide $\Omega_R := (-R, R) \times \mathbb{S}^1$ as well as the box $W_{R,l} := (R, R+l) \times \mathbb{S}^1$. Denoting by $\mathcal{O}_R := \mathcal{O} \cap \Omega_R$ set of all obstacles in $\Omega_R$, we prove that up to a countable set of singular frequencies $k^2 := \omega^2 \mu_0 \varepsilon_0 > 0$ there exists a unique distributional solution $u \in H^1(\Omega_{R+l}; \mathbb{C})$ to

$$\begin{cases} -\Delta u = k^2 u + h & \text{in } \Omega_R \setminus \overline{\mathcal{O}}_R \,, & (7.0.7a) \\ \quad u = 0 & \text{on } \partial \mathcal{O}_R \,. & (7.0.7b) \end{cases}$$

The radiation condition at infinity needs to be replaced appropriately. Loosely speaking, we demand that, for some large $l > 0$, the solution $u|_{W_{-R-l,l}}$ transports energy to the left and $u|_{W_{R,l}}$ transports energy to the right; see the next section for a detailed discussion.

Let us mention that this approach can lead to an existence and uniqueness result for the Helmholtz equation in an unbounded waveguide, as has been recently shown by Schweizer in [Sch19].

### Replacement of the radiation condition at infinity

The radiation conditions at infinity have to be suitably replaced by conditions in the boxes $W_{-R-l,l}$ and $W_{R,l}$. An important ingredient in the formulation of those conditions play distributional solutions to

$$\begin{cases} -\Delta u = k^2 u & \text{in } \Omega \setminus \overline{\mathcal{O}} \,, & (7.0.8a) \\ \quad u = 0 & \text{on } \partial \mathcal{O} \,. & (7.0.8b) \end{cases}$$

Let us consider the following vector space, which contains special solutions to the above problem:

$$X := \left\{ u \in H^1_{\mathrm{loc}}(\Omega; \mathbb{C}) \;\middle|\; \begin{array}{l} u \text{ is a distributional solution to (7.0.8),} \\ u = 0 \text{ in } \mathcal{O}, \text{ and } \sup\limits_{r \in \mathbb{Z}} \|u\|_{L^2(W_{r,1}; \mathbb{C})} < \infty \end{array} \right\} . \tag{7.0.9}$$

Elements of $X$ may transport energy in any direction; to indicate in which direction along the $x_1$-axis energy is transported, we introduce the sesquilinear forms $Q_{R,l} \colon H^1(W_{R,l}; \mathbb{C}) \times H^1(W_{R,l}; \mathbb{C}) \to \mathbb{C}$,

$$Q_{R,l}(u, v) := \frac{1}{l} \int_{W_{R,l}} \big\langle \nabla u, v \, \mathrm{e}_1 \big\rangle \,, \tag{7.0.10}$$

for $R, l \in \mathbb{R} \setminus \{0\}$. We say a function $u \colon \Omega \to \mathbb{C}$ is quasiperiodic if there exists $\xi \in [0, 2\pi)$ such that the function $u$ satisfies $u(x + \mathrm{e}_1) = \mathrm{e}^{\mathrm{i}\xi} \, u(x)$ for all $x \in \Omega$; the number $\xi$ is called the *quasimoment of* $u$. The following assumption is vital for our analysis of the Dirichlet problem (7.0.7).

ASSUMPTION 7.1. – *We assume $k^2 > 0$ is non-singular in the following sense:*

*(i) The space $X$ defined in* (7.0.9) *is finite dimensional. More precisely, there exists a basis $(\phi_1^-, \dots, \phi_M^-, \phi_1^+, \dots, \phi_N^+)$ of $X$ and numbers $\xi_k^-, \xi_j^+ \in [0, 2\pi)$ for $k \in \{1, \dots, M\}$ and $j \in \{1, \dots, N\}$ such that*

*(a) $\phi_k^-$ is $\xi_k^-$-quasiperiodic;*

*(b) $\phi_j^+$ is $\xi_j^+$-quasiperiodic, and*

*(c) The quasimoments $\xi_1^-, \dots, \xi_M^-, \xi_1^+, \dots, \xi_N^+$ are pairwise distinct.*

*(ii) There exists $l_0 \in \mathbb{N}$ such that the basis functions $\phi_1^-, \dots, \phi_M^-, \phi_1^+, \dots, \phi_N^+$ satisfy*

$$\operatorname{Im}\Big\{ Q_{0,l_0}\big(\phi_k^-, \phi_k^-\big) \Big\} < 0 \quad \textit{for all } k \in \{1, \dots, M\}\,, \tag{7.0.11a}$$

*and*

$$\operatorname{Im}\Big\{ Q_{0,l_0}\big(\phi_j^+, \phi_j^+\big) \Big\} > 0 \quad \textit{for all } j \in \{1, \dots, N\}\,. \tag{7.0.11b}$$

Let $(\phi_1^-, \dots, \phi_M^-, \phi_1^+, \dots, \phi_N^+)$ be a basis of $X$ for which *(i)* and *(ii)* in Assumption 7.1 are satisfied. We will show that such a basis satisfies *(ii)* for any $l > 0$; see Remark 9. Instead of the space $X$, we consider the following subspaces of $H^1(W_{0,l}; \mathbb{C})$:

$$X_l^- \coloneqq \operatorname{span}_{\mathbb{C}} \Big\{ \phi_k^- |_{W_{0,l}} \;\Big|\; k \in \{1, \dots, M\} \Big\}$$

and

$$X_l^+ \coloneqq \operatorname{span}_{\mathbb{C}} \Big\{ \phi_j^+ |_{W_{0,l}} \;\Big|\; j \in \{1, \dots, N\} \Big\}\,.$$

Loosely speaking, a function in $X_l^-$ transports energy to the left and a function in $X_l^+$ transports energy to the right. Let us recall that we look for a distributional solution $u \in H^1(\Omega_{R+l}; \mathbb{C})$ to (7.0.7) such that $u|_{W_{-R-l,l}}$ transports energy to the left and $u|_{W_{R,l}}$ transports energy to the right. To make this precise, we introduce the operators $\mathcal{R}_{R,l}^{\pm}(u) \colon W_{0,l} \to \mathbb{C}$ by

$$\mathcal{R}_{R,l}^-(u)(x_1, x_2) \coloneqq u\big(x_1 - R - l, x_2\big) \;\text{ and }\; \mathcal{R}_{R,l}^+(u)(x_1, x_2) \coloneqq u\big(x_1 + R, x_2\big)\,.$$

The function space in which we seek a solution $u$ to (7.0.7) is

$$V_{R,l} \coloneqq \left\{ u \in H^1(\Omega_{R+l}; \mathbb{C}) \;\middle|\; \begin{array}{l} \mathcal{R}_{R,l}^-(u) \in X_l^-\,,\; \mathcal{R}_{R,l}^+(u) \in X_l^+ \\ \text{and } u = 0 \text{ in } \mathcal{O}_R \end{array} \right\}. \tag{7.0.12}$$

## 7.1 Main results

A standard tool to establish the existence of a solution to a Helmholtz-like equation,

$$\mathcal{L}u \coloneqq -\nabla\big(a\nabla u\big) - k^2 u = f\,, \tag{7.1.1}$$

in an unbounded domain (with appropriate boundary conditions), is the so-called limiting absorption principle. The idea of this principle is to add a small absorption term leading to the problem

$$\mathcal{L}u_\delta + \mathrm{i}\delta u_\delta = f \tag{7.1.2}$$

for $\delta > 0$. Usually, the existence of a solution $u_\delta$ to (7.1.2) is easy to establish. The challenging part is to show that the sequence $(u_\delta)_\delta$ converges in some sense to a limit function $u$, which is a candidate for a solution to (7.1.1). The first limiting absorption principle was formulated for the Helmholtz equation in $\mathbb{R}^2$ by Ignatowsky [vI05] and since then this program has been applied successfully in several settings [BBDT01, Hoa11, Rad14, FJ15, KL18]. The limiting absorption principle is usually based on operator theoretical results.

As we seek a distributional solution $u \in H^1(\Omega_{R+l};\mathbb{C})$ to (7.0.7) that satisfies the special boundary condition in the boxes $W_{-R-l,l}$ and $W_{R,l}$, it is convenient to formulate the problem using sesquilinear forms instead of operators. We nevertheless follow the approach to use a limiting absorption principle in order to prove the existence (and uniqueness) of a solution $u \in V_{R,l}$ to (7.0.7).

**Abstract limiting absorption principle.** In Chapter 8, we derive an abstract limiting absorption principle for sesquilinear forms on reflexive Banach spaces, generalising the results from [SU19].

Given (real or complex) Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, a bounded sesquilinear form $\mathfrak{b}$ on $\mathfrak{X} \times \mathfrak{Y}$, and $\ell \in \mathfrak{Y}^*$, we seek an element $u \in \mathfrak{X}$ for which

$$\mathfrak{b}(u,\cdot) = \ell(\cdot)\,. \tag{7.1.3}$$

If $\mathfrak{X} = \mathfrak{Y}$ with $\mathfrak{X}$ a Hilbert space, the Lax-Milgram lemma implies the existence of a solution $u$ to (7.1.3) provided $\mathfrak{b}$ is coercive. Even in the general case of two Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, the existence of such an element $u$ is standard provided $\mathfrak{b}$ satisfies the inf-sup condition (also known as Ladyzhenskaya-Babuška-Brezzi condition). But how can we prove the existence of a solution $u \in \mathfrak{X}$ to (7.1.3) if $\mathfrak{b}$ does not satisfy the inf-sup condition? Borrowing the idea of the limiting absorption principle, we propose the following approach: Instead of $\mathfrak{b}$, we consider a family $(\mathfrak{b}_\delta)_\delta$ of sesquilinear forms for which there exists a unique solution $u_\delta \in \mathfrak{X}$ to

$$\mathfrak{b}_\delta\big(u_\delta,\cdot\big) = \ell(\cdot)\,,$$

for each $\delta > 0$. Under suitable assumptions on the sesquilinear forms $(\mathfrak{b}_\delta)_\delta$, the sequence $(u_\delta)_\delta$ converges to a solution $u \in \mathfrak{X}$ of (7.1.3). This statement, a limiting absorption principle for sesquilinear forms, is the main result of Chapter 8 and it has the character of a Fredholm alternative: $\ker(\mathfrak{b}) = \{0\}$ implies the existence (and uniqueness) statement.

One crucial assumption on $(\mathfrak{b}_\delta)_\delta$ is the existence of a compact operator $K\colon \mathfrak{X} \to \mathfrak{Y}^*$ and a constant $c > 0$ such that

$$\sup_{\substack{v\in\mathfrak{Y}\\ \|v\|_{\mathfrak{Y}}=1}} \Big|\mathfrak{b}_\delta(u,v) + \langle Ku, v\rangle_{\mathfrak{Y}^*,\mathfrak{Y}}\Big| \geq c\|u\|_{\mathfrak{X}} \qquad \text{for all } u \in \mathfrak{X}\,,$$

and each $\delta > 0$. Loosely speaking, the operator $B\colon \mathfrak{X} \to \mathfrak{Y}^*$, $u \mapsto \mathfrak{b}(u,\cdot)$ satisfies the inf-sup condition up to a compact perturbation. This assumption essentially restricts the applicability of the abstract limiting absorption principle to equations posed in bounded domains.

A key ingredient for the abstract limiting absorption principle is a Fredholm alternative for sesquilinear forms; see Proposition 8.2. Our result is not new but rather a variant of known results [Stu69, Kre77].

**Application to the Dirichlet problem (7.0.7).** In Chapter 9, we use the abstract limiting absorption principle, derived in Chapter 8, to study the question if there exists a unique distributional solution $u \in V_{R,l}$ to (7.0.7). To this end, we first prove that for each $\delta > 0$ and every $h \in L^2(\Omega_R; \mathbb{C})$ there is a unique distributional solution $u_\delta \in V_{R,l}$ to

$$\begin{cases} -\Delta u_\delta = k^2(1 + \mathrm{i}\delta)u_\delta + h & \text{in } \Omega_R \setminus \overline{\mathcal{O}}_R\,, & (7.1.4a)\\ \quad u_\delta = 0 & \text{on } \partial\mathcal{O}_R\,, & (7.1.4b)\end{cases}$$

provided $l > 0$ is large enough and $k^2 > 0$ is non-singular in the sense of Assumption 7.1. This existence and uniqueness result follows from a Fredholm alternative for sesquilinear forms.

Using an appropriate sesquilinear form $\mathfrak{b} \colon V_{R,l} \times V_{R,l} \to \mathbb{C}$, the limit problem for $\delta = 0$ can be written as

$$\mathfrak{b}(u, \cdot) = \langle h, \cdot \rangle_{V_{R,l}^*, V_{R,l}}\,. \tag{7.1.5}$$

If $k^2 > 0$ is non-singular, $\ker(\mathfrak{b}) = \{0\}$, and $l > 0$ is large enough, then the sequence $(u_\delta)_\delta$ of solutions to (7.1.4) converges weakly in $V_{R,l}$ to a limit function $u$ which satisfies (7.1.5). The assumptions on $k^2$ and $l$ are necessary to ensure the existence of the sequence $(u_\delta)_\delta$. However, whether the kernel of $\mathfrak{b}$ is trivial depends on the value of $k^2 > 0$. We prove, using standard arguments, that there is an at most countable set of non-singular values $k^2 > 0$ such that $\ker(\mathfrak{b}) \neq \{0\}$.

### Similar approaches

Dohnal and Schweizer [DS18] consider, for $H > 0$, the following Helmholtz-like equation

$$\mathcal{L}u := -\nabla\big(a\nabla u\big) - k^2 u = f \tag{7.1.6}$$

in the closed waveguide $\mathbb{R} \times (0, H)_\sharp$; the symbol $\sharp$ indicates periodic boundary conditions on the lateral boundary. The source $f$ has a compact support and the coefficient field $a \colon \mathbb{R}^2 \to \mathbb{R}$ is bounded and satisfies $a(x) \geq c > 0$ for all $x \in \mathbb{R}^2$. Moreover, $a$ is assumed to be periodic in the left half-space and in the right half-space: $a(x) = a_-(x)$ for $x \in (-\infty, 0) \times \mathbb{R}$ and $a(x) = a_+(x)$ for $x \in [0, \infty) \times \mathbb{R}$, where $a_-, a_+ \colon \mathbb{R}^2 \to \mathbb{R}$ are periodic functions with the same period.

In order to treat (7.1.6) numerically, Dohnal and Schweizer choose a number $R > 0$ and truncate the waveguide $\mathbb{R} \times (0, H)_\sharp$ to obtain a bounded domain $(-R, R) \times (0, H)_\sharp$. They further introduce a small absorption term $\delta > 0$ leading to the equation

$$\mathcal{L}u_\delta + \mathrm{i}\delta u_\delta = f \quad \text{ in } (-R, R) \times (0, H)_\sharp\,. \tag{7.1.7}$$

Similar to our approach, discussed above, the radiation condition at infinity is replaced by a "radiation condition at a finite distance". To this end, they define, for $L > 0$, the radiation boxes $(-R - L, -R) \times (0, H)_\sharp$ and $(R, R + L) \times (0, H)_\sharp$.

In [DS18], it is shown that for every $\delta > 0$ there exists a unique distributional solution $u_\delta \in H^1(\Omega_{R+L}; \mathbb{C})$ to (7.1.7) that satisfies the following "radiation condition at a finite distance": $u_\delta$ restricted to $(-R - L, -R) \times (0, H)_\sharp$ can be expanded in finitely many left-going Bloch waves and $u_\delta$ restricted to the box $(R, R + L) \times (0, H)_\sharp$ can be expanded in finitely many right-going Bloch waves.

We stress that this existence result only holds for $\delta > 0$. The limit $\delta \to 0$ was discussed in [SU19]. More precisely, the following is shown in [SU19]: There exists a sequence $(u_\delta)_\delta$ in $H^1(\Omega_{R+L};\mathbb{C})$ of distributional solutions to (7.1.7) and each member $u_\delta$ of this sequence satisfies the "radiation condition at a finite distance". Up to a countable set of singular frequencies $k^2 > 0$, this sequence converges weakly in $H^1(\Omega_{R+L};\mathbb{C})$ to a function $u$ that is a distributional solution to

$$\mathcal{L}u = f \quad \text{in } (-R,R)\times(0,H)_\sharp \tag{7.1.8}$$

and that satisfies the "radiation condition at a finite distance".

Let us stress that, although the approach in Chapters 8 and 9 is inspired by [SU19], there are two main differences: (i) the equations we consider are different. Indeed, in this thesis we focus on the Dirichlet problem (7.0.7) in a perforated, bounded, and closed waveguide whereas in [SU19], the Helmholtz-like equation (7.1.6) is studied in a bounded and closed waveguide. (ii) In contrast to [DS18, SU19], we do not use Bloch waves to formulate the "radiation condition at a finite distance" but solutions to (7.0.8) that belong to the space $V_{R,l}$, which is defined in (7.0.12).

## 7.2 Function spaces and a special boundary condition

In order to replace the radiation conditions at infinity, we introduced the vector space $X$ in (7.0.9). This section is devoted to a more detailed study of elements of $X$. Let us recall that the sesquilinear form $Q_{R,l}\colon H^1(W_{R,l};\mathbb{C})\times H^1(W_{R,l};\mathbb{C})\to\mathbb{C}$ defined in (7.0.10) is used to indicate in which direction along the $x_1$-axis energy is transported by elements of $X$. The following flux identities turn out to be useful.

Lemma 7.1. (Flux identities) – *Let $R_1, R_2 \in \mathbb{R}$ and $l_1, l_2 > 0$ such that $R_1 + l_1 \le R_2$. Then for any $u \in X$ there holds*

$$\operatorname{Im}\{Q_{R_1,l_1}(u,u)\} = \operatorname{Im}\{Q_{R_2,l_2}(u,u)\}\,. \tag{7.2.1}$$

*Moreover, if $u \in X$ is quasiperiodic, then*

$$\operatorname{Im}\{Q_{0,l_1}(u,u)\} = \operatorname{Im}\{Q_{0,l_2}(u,u)\} \tag{7.2.2}$$

*for all $l_1, l_2 > 0$.*

*Proof.* Define the cut-off function $\theta\colon \Omega \to [0,1]$,

$$\theta(x_1,x_2) := \begin{cases} 0 & x_1 < R_1 \text{ and } x_1 > R_2 + l_2 \\ 1 & x_1 \in (R_1 + l_1, R_2) \\ \dfrac{1}{l_1}\big(x_1 - R_1\big) & x_1 \in [R_1, R_1 + l_1] \\ \dfrac{1}{l_2}\big(R_2 + l_2 - x_1\big) & x_1 \in [R_2, R_2 + l_2] \end{cases}. \tag{7.2.3}$$

We refer to Figure 7.3 for a sketch of $\theta$. As $u$ is a distributional solution to (7.0.8), for every $v \in H^1(\Omega;\mathbb{C})$ with compact support that satisfies $v = 0$ in $\mathcal{O}$, there holds

$$0 = \int_\Omega \langle \nabla u, \nabla v\rangle - k^2 \int_\Omega u\bar{v}\,.$$

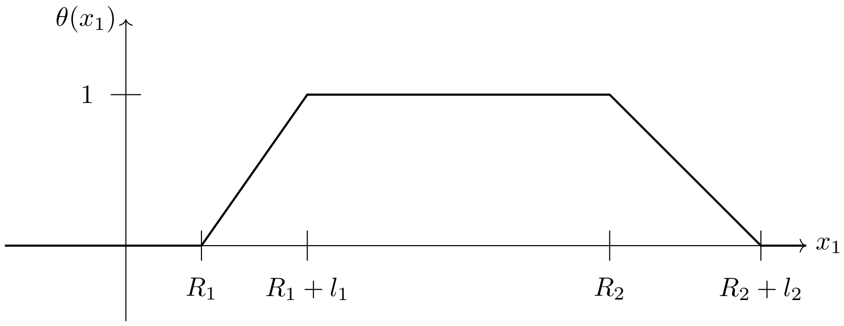


Figure 7.3: Sketch of the cut-off function $\theta$ defined in (7.2.3).

Substituting $v = \theta u$, we find that

$$0 = \int_\Omega \theta \big|\nabla u\big|^2 - k^2 \int_\Omega \theta |u|^2 + \frac{1}{l_1} \int_{W_{R_1,l_1}} \big\langle \nabla u, u\,\mathrm{e}_1 \big\rangle - \frac{1}{l_2} \int_{W_{R_2,l_2}} \big\langle \nabla u, u\,\mathrm{e}_1 \big\rangle\,.$$

Taking the imaginary part of the above equation yields the claim.

Let us assume that $u \in X$ is $\xi$-quasiperiodic. For $R_1 = 0$ and $R_2 = l_1$, we deduce from (7.2.1) that

$$\begin{aligned} \operatorname{Im}\big\{Q_{0,l_1}(u,u)\big\} = \operatorname{Im}\big\{Q_{l_1,l_2}(u,u)\big\} &= \frac{1}{l_2} \operatorname{Im}\bigg\{\int_{W_{l_1,l_2}} \big\langle \nabla u, u\,\mathrm{e}_1 \big\rangle \bigg\} \\ &= \frac{1}{l_2} \operatorname{Im}\bigg\{\int_{W_{0,l_2}} \mathrm{e}^{\mathrm{i}\xi l_1} \big\langle \nabla u, u\,\mathrm{e}_1 \big\rangle\, \mathrm{e}^{-\mathrm{i}\xi l_1} \bigg\} \\ &= \operatorname{Im}\big\{Q_{0,l_2}(u,u)\big\}\,. \end{aligned}$$

This proves the claim. □

We say that a basis $(\phi_1^-, \dots, \phi_M^-, \phi_1^+, \dots, \phi_N^+)$ of $X$ is *admissible* if the conditions *(i)* and *(ii)* of Assumption 7.1 are satisfied.

Remark 9. – *In Assumption 7.1(ii) on page 95 we demand that there exists* $l_0 \in \mathbb{N}$ *such that* (7.0.11) *holds. As the basis functions* $(\phi_1^-, \dots, \phi_M^-, \phi_1^+, \dots, \phi_N^+)$ *are assumed to be quasiperiodic, we infer from the flux identity* (7.2.2) *that* (7.0.11) *is satisfied for all* $l > 0$ *if it is satisfied for some* $l_0 \in \mathbb{N}$.

Instead of $X$, we consider the following subspace of $H^1(W_{0,l};\mathbb{C})$,

$$X_l := \Big\{ u|_{W_{0,l}} \;\Big|\; u \in X \Big\}\,, \tag{7.2.4}$$

for $l \in \mathbb{N}$. We note that if $k^2 > 0$ is non-singular in the sense of Assumption 7.1, then $X_l$ is finite dimensional. More precisely, $X_l$ admits a basis $(\psi_1^-, \dots, \psi_M^-, \psi_1^+, \dots, \psi_N^+)$ of quasiperiodic functions with pairwise distinct quasi-moments such that

$$\operatorname{Im}\big\{Q_{0,l}(\psi_k^-, \psi_k^-)\big\} < 0 \quad \text{and} \quad \operatorname{Im}\big\{Q_{0,l}(\psi_j^+, \psi_j^+)\big\} > 0$$

for all $k \in \{1, \dots, M\}$ and $j \in \{1, \dots, N\}$. Such a basis is *admissible* if there exists an admissible basis $(\phi_1^-, \dots, \phi_M^-, \phi_1^+, \dots, \phi_N^+)$ of $X$ such that

$$\psi_k^- = \phi_k^-|_{W_{0,l}} \quad \text{and} \quad \psi_j^+ = \phi_j^+|_{W_{0,l}} \tag{7.2.5}$$

for all $k \in \{1, \dots, M\}$ and $j \in \{1, \dots, N\}$. We also say that the basis $(\psi_1^-, \dots, \psi_M^-, \psi_1^+, \dots, \psi_N^+)$ is induced by $(\phi_1^-, \dots, \phi_M^-, \phi_1^+, \dots, \phi_N^+)$.

**Some properties of functions in $X_l$.** Throughout this section, $l \in \mathbb{N}$ is arbitrary and fixed unless otherwise stated. The fact that $X_l$ is a finite dimensional subspace of $H^1(W_{0,l}; \mathbb{C})$ yields the following regularity result.

Lemma 7.2. (Regularity estimate) – *Assume $k^2 > 0$ is non-singular. Choose $l \in \mathbb{N}$ and let $X_l$ be the function space defined in* (7.2.4). *Then there exists a constant $C > 0$ such that*

$$\|\phi\|_{H^1(W_{0,l};\mathbb{C})} \leq C \|\phi\|_{L^2(W_{0,l};\mathbb{C})}$$

*for all $\phi \in X_l$.*

*Proof.* Let $(\psi_1, \dots, \psi_K)$ be a basis of $X_l$ that is orthogonal with respect to the $L^2(W_{0,l}; \mathbb{C})$-scalar product. As the basis consists of only a finite number of functions, we find a constant $C_0 > 0$ such that

$$\big\|\nabla \psi_j\big\|^2_{L^2(W_{0,l};\mathbb{C}^2)} \leq C_0 \|\psi_j\|^2_{L^2(W_{0,l};\mathbb{C})} \quad \text{for all } j \in \{1, \dots, K\}\,. \tag{7.2.6}$$

Fix a function $\phi \in X_l$. Then $\phi = \sum_{j=1}^K \alpha_j \psi_j$ in $X_l$ for some constants $\alpha_1, \dots, \alpha_K \in \mathbb{C}$. Using (7.2.6) and the orthogonality of $(\psi_1, \dots, \psi_K)$ with respect to the $L^2(W_{0,l}; \mathbb{C})$-scalar product, we find that

$$\begin{aligned}\big\|\nabla \phi\big\|^2_{L^2(W_{0,l};\mathbb{C}^2)} &= \int_{W_{0,l}} \Big|\sum_{j=1}^K \alpha_j \nabla \psi_j\Big|^2 \leq C_1 \sum_{j=1}^K |\alpha_j|^2 \int_{W_{0,l}} \big|\nabla \psi_j\big|^2 \\ &\leq C_2 \sum_{j=1}^K \int_{W_{0,l}} \big|\alpha_j \psi_j\big|^2 = C_2 \|\phi\|^2_{L^2(W_{0,l};\mathbb{C})}\,.\end{aligned}$$

From this inequality it is straightforward to deduce the claim. □

An admissible basis $(\psi_1^-, \dots, \psi_M^-, \psi_1^+, \dots, \psi_N^+)$ of $X_l$ is not necessarily orthogonal with respect to the sesquilinear form $Q_{0,l}$. However, we have the following "approximate orthogonality" result.

Lemma 7.3. (Approximate orthogonality) – *Let $k^2 > 0$ be non-singular, and let $(\phi_1^-, \dots, \phi_M^-, \phi_1^+, \dots, \phi_N^+)$ be an admissible basis of $X$. For every $\varepsilon > 0$ there exists a number $l_0 \in \mathbb{N}$ such that for all $l \in \mathbb{N}$ with $l \geq l_0$ the following statement holds: If $u, v \in \{\phi_1^-|_{W_{0,l}}, \dots, \phi_M^-|_{W_{0,l}}, \phi_1^+|_{W_{0,l}}, \dots, \phi_N^+|_{W_{0,l}}\}$ with $u \neq v$, then*

$$\big|Q_{0,l}(u, v)\big| \leq \varepsilon\,. \tag{7.2.7}$$

*Proof.* Let $(\phi_1^-, \dots, \phi_M^-, \phi_1^+, \dots, \phi_n^+)$ be an admissible basis of $X$. Fix two distinct elements $\tilde{u}$ and $\tilde{v}$ of $\{\phi_1^-, \dots, \phi_M^-, \phi_1^+, \dots, \phi_n^+\}$ and denote their quasi-moments by $\xi_{\tilde{u}}, \xi_{\tilde{v}} \in [0, 2\pi)$. We recall that $\xi_{\tilde{u}} \neq \xi_{\tilde{v}}$ by Assumption 7.1*(i)(c)*.

Using the quasiperiodicity of $\tilde{u}$ and $\tilde{v}$, we calculate that, for each $l \in \mathbb{N}$,

$$\begin{aligned} Q_{0,l}\Big(\tilde{u}|_{W_{0,l}}, \tilde{v}|_{W_{0,l}}\Big) &= \frac{1}{l}\int_{W_{0,l}} \big\langle \nabla\tilde{u}, \tilde{v}\,\mathrm{e}_1\big\rangle = \frac{1}{l}\sum_{k=0}^{l-1}\int_{W_{k,1}} \big\langle \nabla\tilde{u}, \tilde{v}\,\mathrm{e}_1\big\rangle \\ &= \frac{1}{l}\sum_{k=0}^{l-1}\int_{W_{0,1}} \mathrm{e}^{\mathrm{i}k\xi_{\tilde{u}}}\big\langle \nabla\tilde{v}, \tilde{v}\,\mathrm{e}_1\big\rangle\,\mathrm{e}^{-\mathrm{i}k\xi_{\tilde{v}}} \\ &= \frac{1}{l}\sum_{k=0}^{l-1}\mathrm{e}^{\mathrm{i}k(\xi_{\tilde{u}}-\xi_{\tilde{v}})}\int_{W_{0,1}} \big\langle \nabla\tilde{u}, \tilde{v}\,\mathrm{e}_1\big\rangle\,. \end{aligned}$$

As $\xi_{\tilde{u}} \neq \xi_{\tilde{v}}$, the sum $l^{-1}\sum_{k=0}^{l-1}\mathrm{e}^{\mathrm{i}k(\xi_{\tilde{u}}-\xi_{\tilde{v}})}$ converges to 0 as $l \to \infty$. As the admissible basis $(\phi_1^-,\dots,\phi_M^-,\phi_1^+,\dots,\phi_N^+)$ contains only finitely many elements, we can choose $l_0 \in \mathbb{N}$ large enough such that (7.2.7) holds for every two distinct elements $u$ and $v$ of the admissible basis. Thus the claim is proved. $\square$

Assume $(\phi_1^-,\dots,\phi_M^-,\phi_1^+,\dots,\phi_N^+)$ is an admissible basis of $X$. For each $l > 0$, we define the two subspaces of $X_l$,

$$X_l^- \coloneqq \operatorname{span}_{\mathbb{C}}\Big\{\phi_k^-|_{W_{0,l}} \;\Big|\; k \in \{1,\dots,M\}\Big\} \tag{7.2.8a}$$

and

$$X_l^+ \coloneqq \operatorname{span}_{\mathbb{C}}\Big\{\phi_j^+|_{W_{0,l}} \;\Big|\; j \in \{1,\dots,N\}\Big\}\,. \tag{7.2.8b}$$

The approximate orthogonality of the basis $(\phi_1^-,\dots,\phi_M^-,\phi_1^+,\dots,\phi_N^+)$ allows us to estimate the $L^2(W_{0,l};\mathbb{C})$-norm of elements in $X_l^\pm$ by the bilinear form $\operatorname{Im}\{Q_{W_{0,l}}(\cdot,\cdot)\}$; this observations plays a key role in establishing the existence of a solution to the radiation problem with absorption (7.0.7).

Lemma 7.4. – *Let $k^2 > 0$ be non-singular, and let $(\phi_1^-,\dots,\phi_M^-,\phi_1^+,\dots,\phi_N^+)$ be an admissible basis of $X$. Then there exists $l_0 \in \mathbb{N}$ such that for all $l \in \mathbb{N}$ with $l \geq l_0$ and for some $c = c(l) > 0$ there holds*

$$-\operatorname{Im}\big\{Q_{0,l}(v,v)\big\} \geq c\|v\|^2_{L^2(W_{0,l},\mathbb{C})} \quad \textit{and} \quad \operatorname{Im}\big\{Q_{0,l}(u,u)\big\} \geq c\|u\|^2_{L^2(W_{0,l},\mathbb{C})}\,,$$

*for all $v \in X_l^-$ and $u \in X_l^+$.*

*Proof.* We perform the proof of the statement for $v \in X_l^-$, since the argument for $u \in X_l^+$ is analogous.

For each $l \in \mathbb{N}$ we define two positive constants

$$\gamma_l \coloneqq \min_{k\in\{1,\dots,M\}}\Big(-\operatorname{Im}\big\{Q_{0,l}(\phi_k^-,\phi_k^-)\big\}\Big) \text{ and } \eta_l \coloneqq \max_{\substack{k,n\in\{1,\dots,M\}\\ k\neq n}}\big|Q_{0,l}(\phi_n^-,\phi_k^-)\big|\,.$$

The constant $\gamma_l$ is independent of $l$ because of the flux identity (7.2.2); in other words, $\gamma_l = \gamma_1$ for all $l > 0$. By Lemma 7.3, we find $l_0 \in \mathbb{N}$ such that for all $l \in \mathbb{N}$ with $l \geq l_0$ there holds $\eta_l \leq \gamma_1(2M)^{-1}$.

Fix such an $l \in \mathbb{N}$ and let $(\psi_1^-,\dots,\psi_M^-,\psi_1^+,\dots,\psi_N^+)$ be an admissible basis for $X_l$ that is induced by $(\phi_1^-,\dots,\phi_M^-,\phi_1^+,\dots,\phi_N^+)$ . For every $v \in X_l^-$, there exist constants $\alpha_1, \dots, \alpha_M \in \mathbb{C}$ such that $v = \sum_{k=1}^{M} \alpha_k \psi_k^-$ in $X_l^-$. A straightforward calculation yields that

$$\begin{aligned} -\operatorname{Im}\big\{Q_{0,l}(v,v)\big\} &= -\sum_{k=1}^{M} |\alpha_k|^2 \operatorname{Im}\big\{Q_{0,l}(\psi_k^-, \psi_k^-)\big\} \\ &\quad - \sum_{\substack{k,n=1 \\ k\neq n}}^{M} \operatorname{Im}\big\{\alpha_k \overline{\alpha}_n Q_{0,l}(\psi_k^-, \psi_n^-)\big\} \\ &\geq \gamma_1 \sum_{k=1}^{M} |\alpha_k|^2 - \eta_l \sum_{\substack{k,n=1 \\ k\neq n}}^{M} |\alpha_k|\,|\alpha_n|\,. \end{aligned}$$

As $\eta_l \leq \gamma_1 (2M)^{-1}$, we infer the estimate

$$-\operatorname{Im}\big\{Q_{0,l}(v,v)\big\} \geq \frac{\gamma_1}{2} \sum_{k=1}^{M} |\alpha_k|^2\,. \tag{7.2.9}$$

The following estimate

$$\|v\|^2_{L^2(W_{0,l};\mathbb{C})} = \int_{W_{0,l}} \bigg|\sum_{k=1}^{M} \alpha_k \psi_k^-\bigg|^2 \leq M \sum_{k=1}^{M} |\alpha_k|^2 \bigg(\int_{W_{0,l}} |\psi_k^-|^2\bigg) \leq C \sum_{k=1}^{M} |\alpha_k|^2$$

together with (7.2.9) yields the claim. □

**Variational formulation.** Throughout this section $R \in \mathbb{N}$ and $l > 0$ are fixed. The existence of a solution $u \in V_{R,l}$ to (7.0.7) is established using a limiting absorption principle. Consequently, in a first step, we seek a distributional solution $u_\delta \in V_{R,l}$ to (7.1.4). For a weak formulation of this problem with absorption, we introduce the function $\vartheta \colon \Omega_{R+l} \to [0,1]$,

$$\vartheta(x_1,x_2) \coloneqq \begin{cases} 0 & \text{for } |x_1| \geq R+l\,, \\ 1 & \text{for } |x_1| \leq R\,, \\ \frac{1}{l}\big(R+l-|x_1|\big) & \text{otherwise} \end{cases}\,.$$

With the help of $\vartheta$ we define the sesquilinear form $\mathfrak{b}_\delta \colon V_{R,l} \times V_{R,l} \to \mathbb{C}$,

$$\begin{aligned} \mathfrak{b}_\delta(u,v) \coloneqq \int_{\Omega_{R+l}} \vartheta \big\langle \nabla u, \nabla v \big\rangle - k^2 \big(1 + \mathrm{i}\delta\big) \int_{\Omega_{R+l}} \vartheta u \bar{v} \\ - Q_{0,l}\big(\mathcal{R}^+_{R,l}(u), \mathcal{R}^+_{R,l}(v)\big) + Q_{0,l}\big(\mathcal{R}^-_{R,l}(u), \mathcal{R}^-_{R,l}(v)\big)\,. \end{aligned} \tag{7.2.10}$$

Let us recall that we look for a distributional solution $u$ to (7.0.7) that is an element of $V_{R,l}$ and as such satisfies conditions in the boxes $W_{-R-l,l}$ and $W_{R,l}$. The solution concept we introduce in the next definition contains these boundary conditions.

Definition 7.5. (Solution concept) – *We say that $u \colon \Omega_{R+l} \to \mathbb{C}$ is a* solution to the radiation problem with damping *if $u$ is an element of $V_{R,l}$ and satisfies*

$$\mathfrak{b}_\delta(u,v) = \int_{\Omega_R} h \bar{v} \tag{7.2.11}$$

*for all* $v \in V_{R,l}$*. If* $u$ *satisfies* (7.2.11) *for* $\delta = 0$*, then* $u$ *is a* solution to the radiation problem.

For later purposes, we introduce the sesquilinear form $\mathcal{Q}_{R,l} \colon V_{R,l} \times V_{R,l} \to \mathbb{C}$,

$$\mathcal{Q}_{R,l}(u, v) \coloneqq Q_{0,l}\big(\mathcal{R}^{+}_{R,l}(u), \mathcal{R}^{+}_{R,l}(v)\big) - Q_{0,l}\big(\mathcal{R}^{-}_{R,l}(u), \mathcal{R}^{-}_{R,l}(v)\big)\,. \tag{7.2.12}$$

CHAPTER 8

# Abstract limiting absorption principle for sesquilinear forms

This chapter is devoted to the proof of an abstract limiting absorption principle. One key result in the proof of this principle is a Fredholm alternative for sesquilinear forms, which we discuss in the next section.

## 8.1 Fredholm alternative for sesquilinear forms

We recall from Section 1.1 that a bounded linear operator $T\colon \mathfrak{X} \to \mathfrak{Y}$ between Banach spaces is a Fredholm operator if its kernel $\ker T$ and its cokernel $\operatorname{coker} T$ are finite dimensional vector spaces. The index of $T$ is the integer $\operatorname{ind} T := \dim \ker T - \dim \operatorname{coker} T$. One classical version of Fredholm's alternative is the following: if $T\colon \mathfrak{X} \to \mathfrak{Y}$ is a Fredholm operator with index 0, then either for every $f \in \mathfrak{Y}$ there exists a unique solution $u \in \mathfrak{X}$ to the equation $Tu = f$ or there is a non-trivial solution $u \neq 0$ to $Tu = 0$.

As we mainly focus on sesquilinear forms $\mathfrak{b}$ on normed spaces, we formulate a Fredholm alternative for such maps. Let $E$ and $V$ be normed spaces over the same field $\mathbb{F}$, and let $\mathfrak{b}\colon E \times V \to \mathbb{F}$ be a sesquilinear form. Fredholm alternatives for bilinear forms are well-known in the literature; indeed, Stummel [Stu69, §1 and §2] developed a Fredholm theory for bilinear forms in the case $E = V$ with $V$ a Hilbert space. This theory was later generalised by Kress [Kre77] to the case of bilinear forms $\mathfrak{b}\colon \mathfrak{X} \times \mathfrak{Y} \to \mathbb{R}$ on reflexive Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$. We also mention the more recent article by Arendt and coauthors [AtEKS14, Lemma 4.1] in which a Fredholm-Lax-Milgram lemma is employed to establish the self-adjointness of a Dirichlet-to-Neumann graph.

Let us recall that a sesquilinear form $\mathfrak{b}\colon E \times V \to \mathbb{F}$ is linear in its first argument and anti-linear in its second. Note that $\mathfrak{b}$ is bilinear if $\mathbb{F} = \mathbb{R}$. We define the *kernel of* $\mathfrak{b}$ as the set

$$\ker(\mathfrak{b}) := \left\{ u \in E \,\middle|\, \mathfrak{b}(u,v) = 0 \text{ for all } v \in V \right\}.$$

The *adjoint sesquilinear form* $\mathfrak{b}^* \colon V \times E \to \mathbb{F}$ is given by $\mathfrak{b}^*(v,u) := \overline{\mathfrak{b}(u,v)}$. If $\mathbb{F} = \mathbb{R}$ this reduces to $\mathfrak{b}^*(v,u) = \mathfrak{b}(u,v)$.

We also remind the reader that $V^*$ denotes the set of anti-linear and bounded maps $V \to \mathbb{F}$.

Definition 8.1. (Gårding inequality) – *Let $E$ and $V$ be normed spaces over the field $\mathbb{F}$. A sesquilinear form $\mathfrak{b} \colon E \times V \to \mathbb{F}$ satisfies a* Gårding inequality *if there exist a constant $c > 0$ and a compact linear operator $K \colon E \to V^*$ such that*

$$\sup_{\substack{v \in V \\ \|v\|_V = 1}} \left| \mathfrak{b}(u,v) + \langle Ku, v \rangle_{V^*,V} \right| \geq c \|u\|_E \qquad \textit{for all } u \in E\,.$$

At risk of redundancy, we remind the reader that every reflexive normed space is a Banach space; see Remark 2 on page 14. However, throughout this section we write “reflexive normed space” instead of “reflexive Banach space” to emphasise that completeness is not used in the proofs.

Proposition 8.2. (Fredholm alternative for sesquilinear forms) – *Let $\mathfrak{X}$ be a Banach space over the field $\mathbb{F}$, and let $V$ be a reflexive normed space over the same field. Let $\mathfrak{b} \colon \mathfrak{X} \times V \to \mathbb{F}$ be a bounded sesquilinear form that satisfies a Gårding inequality with the compact operator $K \colon \mathfrak{X} \to V^*$. Assume further that for every $v \in V \setminus \{0\}$ there is an element $u \in \mathfrak{X}$ such that*

$$\mathfrak{b}(u,v) + \langle Ku, v \rangle_{V^*,V} \neq 0\,.$$

*Then the following statements hold:*

*(i) Let $\ell \in V^*$. There exists $u \in \mathfrak{X}$ with*

$$\mathfrak{b}(u,\cdot) = \ell(\cdot) \tag{8.1.1}$$

*if and only if*

$$\ell(v) = 0 \qquad \textit{for all } v \in \ker(\mathfrak{b}^*)\,.$$

*(ii) The spaces $\ker(\mathfrak{b})$ and $\ker(\mathfrak{b}^*)$ are finite dimensional and*

$$\dim \ker(\mathfrak{b}) = \dim \ker(\mathfrak{b}^*)\,.$$

*(iii) If $\ker(\mathfrak{b}) = \{0\}$, then for every $\ell \in V^*$ there is a unique $u \in \mathfrak{X}$ such that (8.1.1) holds.*

The following result will be useful in the proof of Proposition 8.2. We remind the reader that the notion of an $\mathfrak{X}$-elliptic sesquilinear form was defined in (1.2.2).

Lemma 8.3. – *Let $\mathfrak{X}$ be a Banach space over the field $\mathbb{F}$, and let $V$ be a reflexive normed space over the same field. Let $\mathfrak{b}\colon \mathfrak{X} \times V \to \mathbb{F}$ be a bounded sesquilinear form that satisfies a Gårding inequality with compact operator $K\colon \mathfrak{X} \to V^*$. Assume further that for every $v \in V \setminus \{0\}$ there exists an element $u \in \mathfrak{X}$ such that*

$$\mathfrak{b}(u,v) + \langle Ku, v\rangle_{V^*,V} \neq 0\,. \tag{8.1.2}$$

*Then $B\colon \mathfrak{X} \to V^*$, $u \mapsto \mathfrak{b}(u,\cdot)$ is a Fredholm operator with* $\operatorname{ind} B = 0$*.*

*Proof.* The sesquilinear form $\mathfrak{b}$ is linear in its first argument and bounded, which implies linearity and boundedness of $B$. The operator $(B+K)\colon \mathfrak{X} \to V^*$ induces a sesquilinear form $\mathfrak{a}\colon \mathfrak{X} \times V \to \mathbb{F}$,

$$\mathfrak{a}(u,v) := \langle (B+K)u, v\rangle_{V^*,V} = \mathfrak{b}(u,v) + \langle Ku, v\rangle_{V^*,V}\,.$$

Thanks to the boundedness of $\mathfrak{b}$ and $K$, the sesquilinear form $\mathfrak{a}$ is bounded as well. Moreover, $\mathfrak{a}$ is $\mathfrak{X}$-elliptic, since $\mathfrak{b}$ satisfies a Gårding inequality with the compact operator $K$. Due to (8.1.2), the sesquilinear form $\mathfrak{a}$ is non-degenerate. We can therefore apply Proposition 1.13 and find that $(B+K)\colon \mathfrak{X} \to V^*$ is a Banach space isomorphism and as such a Fredholm operator with $\operatorname{ind}(B+K) = 0$. By Lemma 1.12, the Fredholm index is stable under compact perturbations, and hence, $B = (B+K) - K$ is a Fredholm operator with $\operatorname{ind} B = 0$. This proves the claim. ☐

Let $T\colon E \to V$ be a linear operator between two normed vector spaces. We recall from Section 1.1 that the dual operator $T'\colon V' \to E'$ is determined by the identity (1.1.1). Similarly, the anti-dual operator $T^*\colon V^* \to E^*$ is defined by

$$\langle v^*, Tu\rangle_{V^*,V} = \langle T^*v^*, u\rangle_{E^*,E} \qquad \text{for all } u \in E \text{ and } v^* \in V^*\,.$$

The annihilator ${}^\perp(U')$ of a linear subspace $U'$ of $V'$ is defined in (1.1.3); we recall that ${}^\perp(U') \subset V$. If $U^*$ is a linear subspace of $V^*$, we set

$${}^\perp(U^*) := \Big\{ v \in V \;\Big|\; \langle u^*, v\rangle_{V^*,V} = 0 \text{ for all } u^* \in U^* \Big\}\,.$$

If $\mathbb{F} = \mathbb{C}$, the two natural dual spaces $V'$ and $V^*$ of a normed space $V$ are isometrically isomorphic via the map $V' \to V^*$, $\ell \mapsto \overline{\ell}$. Using this isomorphism it is straightforward to show that

$${}^\perp(\ker T') = {}^\perp(\ker T^*) \tag{8.1.3}$$

for every linear and bounded operator $T\colon E \to V$ between normed spaces over $\mathbb{C}$. In the case $\mathbb{F} = \mathbb{R}$, the two spaces $V'$ and $V^*$ coincide and (8.1.3) holds trivially.

Having Lemma 8.3 at hand, we can perform the proof of Proposition 8.2.

*Proof (of Proposition 8.2).* Due to Lemma 8.3, the operator $B\colon \mathfrak{X} \to V^*$, $u \mapsto \mathfrak{b}(u,\cdot)$ is a Fredholm operator with $\operatorname{ind} B = 0$.

*(i)* Problem (8.1.1) is equivalent to $Bu = \ell$ in $V^*$. As $B$ is a Fredholm operator, its image is closed by Lemma 1.9. By Banach's closed range theorem, Proposition 1.1, there holds $\operatorname{im} B = {}^\perp(\ker B')$. This together with (8.1.3) implies that $\ell \in \operatorname{im} B$ if and only if $\ell \in {}^\perp(\ker(B^*))$. As $V$ is reflexive, we shall

identify $V^{**}$ with $V$; in particular, $\ker(B^*)$ is viewed as a subset of $V$. The claim is thus proved if we show that $\ker(B^*) = \ker(\mathfrak{b}^*)$, since $\ell \in {}^\perp(\ker(\mathfrak{b}^*))$ if and only if

$$\ell(v) = 0 \qquad \text{for all } v \in \ker(\mathfrak{b}^*)\,.$$

Using the definition of the adjoint sesquilinear form $\mathfrak{b}^*$, it is straightforward to show the identity $\mathfrak{b}^*(v,u) = \langle B^* v, u\rangle_{\mathfrak{X}^*,\mathfrak{X}}$ for all $u \in \mathfrak{X}$ and $v \in V$. This implies in particular that $\ker(\mathfrak{b}^*) = \ker(B^*)$, and claim *(i)* is proved.

*(ii)* As $B$ is a Fredholm operator, its kernel is finite dimensional and hence $\dim\ker(\mathfrak{b}) < \infty$. Moreover, $B^*\colon V \to \mathfrak{X}^*$ is also a Fredholm operator with $\operatorname{ind} B^* = -\operatorname{ind} B = 0$. As discussed in Section 1.1, the vector spaces $(\operatorname{im} B)^\perp$ and $(V/\operatorname{im} B)'$ are isomorphic, and $(\operatorname{im} B)^\perp = \ker B'$. Consequently, $(V/\operatorname{im} B)^*$ and $\ker B^*$ are isomorphic, finite dimensional vector spaces. As $\operatorname{ind} B = 0$, we therefore find

$$\dim\ker B = \dim\operatorname{coker} B = \dim\bigl(V/\operatorname{im} B\bigr) = \dim\bigl(V/\operatorname{im} B\bigr)^* = \dim\ker(B^*)\,.$$

The claim follows since $\ker B = \ker(\mathfrak{b})$ and $\ker(B^*) = \ker(\mathfrak{b}^*)$.

*(iii)* Existence of an element $u \in \mathfrak{X}$ such that (8.1.1) holds follows from *(i)*. As $\ker(\mathfrak{b}) = \{0\}$, the element $u$ is the only solution to equation (8.1.1). ☐

If we consider a sesquilinear form $\mathfrak{b}\colon \mathfrak{H}\times\mathfrak{H} \to \mathbb{F}$ on a Hilbert space $\mathfrak{H}$, we have the following special case of the Fredholm alternative.

Corollary 8.4. (Fredholm alternative for Hilbert spaces) – *Let $\mathfrak{b}\colon \mathfrak{H}\times\mathfrak{H} \to \mathbb{F}$ be a bounded sesquilinear form on a Hilbert space $\mathfrak{H}$ over the field $\mathbb{F}$. Assume that there exists a compact operator $K\colon \mathfrak{H} \to \mathfrak{H}^*$ and a constant $c > 0$ such that*

$$\bigl|\mathfrak{b}(u,u) + \langle Ku, u\rangle_{\mathfrak{H}^*,\mathfrak{H}}\bigr| \geq c\|u\|_{\mathfrak{H}}^2 \qquad \textit{for all } u \in \mathfrak{H}\,. \tag{8.1.4}$$

*Then the following statements hold:*

*(i) Let $\ell \in \mathfrak{H}^*$. There exists $u \in \mathfrak{H}$ with*

$$\mathfrak{b}(u,\cdot) = \ell(\cdot) \tag{8.1.5}$$

*if and only if*

$$\ell(v) = 0 \qquad \textit{for all } v \in \ker(\mathfrak{b}^*)\,.$$

*(ii) The spaces $\ker(\mathfrak{b})$ and $\ker(\mathfrak{b}^*)$ are finite dimensional and*

$$\dim\ker(\mathfrak{b}) = \dim\ker(\mathfrak{b}^*)\,.$$

*(iii) If $\ker(\mathfrak{b}) = \{0\}$, then for every $\ell \in \mathfrak{H}^*$ there is a unique $u \in \mathfrak{H}$ such that (8.1.5) holds.*

*Proof.* The claim follows from Proposition 8.2 if we show that $\mathfrak{b}$ satisfies a Gårding inequality and that $\mathfrak{b}$ is non-degenerate. Inequality (8.1.4) implies that $\mathfrak{b}$ is non-degenerate. Moreover, for every $u \in \mathfrak{H}$, $u \neq 0$, there holds

$$\sup_{\substack{v\in\mathfrak{H}\\ \|v\|=1}} \bigl|\mathfrak{b}(u,v) + \langle Ku, v\rangle_{\mathfrak{H}^*,\mathfrak{H}}\bigr| \geq \left|\mathfrak{b}\left(u, \frac{u}{\|u\|_{\mathfrak{H}}}\right) + \left\langle Ku, \frac{u}{\|u\|_{\mathfrak{H}}}\right\rangle_{\mathfrak{H}^*,\mathfrak{H}}\right| \geq c\|u\|_{\mathfrak{H}}\,,$$

and hence $\mathfrak{b}$ satisfies a Gårding inequality. ☐

## 8.2 Limiting absorption principle

This section is devoted to a limiting absorption principle for sesquilinear forms.

Assumption 8.1. – *Let $\mathfrak{X}$ be a Banach space over the field $\mathbb{F}$, and let $V$ be a normed space over the same field. We assume that the sesquilinear form $\mathfrak{b}\colon \mathfrak{X} \times V \to \mathbb{F}$ satisfies the following two requirements:*

*(i) (Uniqueness) The kernel of $\mathfrak{b}$ is trivial.*

*(ii) (Gårding) There exist $c > 0$ and a compact linear operator $K\colon \mathfrak{X} \to V^*$ such that*

$$\sup_{\substack{v\in V\\ \|v\|_V=1}} \big|\mathfrak{b}(u,v) + \langle Ku, v\rangle_{V^*,V}\big| \geq c\|u\|_{\mathfrak{X}} \qquad \textit{for all } u \in \mathfrak{X}\,.$$

*(iii) (Nontriviality) For every $v \in V \setminus \{0\}$ there is an element $u \in \mathfrak{X}$ such that*

$$\mathfrak{b}(u,v) + \langle Ku, v\rangle_{V^*,V} \neq 0\,,$$

*where $K\colon \mathfrak{X} \to V^*$ is the compact operator from (ii).*

We observe that every coercive sesquilinear form $\mathfrak{H} \times \mathfrak{H} \to \mathbb{F}$ on a Hilbert space $\mathfrak{H}$ satisfies Assumption 8.1.

Lemma 8.5. – *Let $\mathfrak{X}$ be a Banach space over the field $\mathbb{F}$, and let $V$ be a reflexive normed space over the same field. Let $\mathfrak{b}\colon \mathfrak{X} \times V \to \mathbb{F}$ be a bounded sesquilinear form that satisfies Assumption 8.1. Then, for every $\ell \in V^*$, there is a unique $u \in \mathfrak{X}$ such that*

$$\mathfrak{b}(u,\cdot) = \ell(\cdot)\,.$$

*Proof.* The claim follows from applying Proposition 8.2*(iii)*. ☐

Having this existence result at hand, we can state and proof the main result of this section.

Theorem 8.6. (Abstract limiting absorption principle) – *Let $\mathfrak{X}$ be a reflexive Banach space over the field $\mathbb{F}$, and let $V$ be a reflexive normed space over the same field. Let $\mathfrak{b}\colon \mathfrak{X} \times V \to \mathbb{F}$ be a sesquilinear form, and assume that $(\mathfrak{b}_\delta)_\delta$ is a sequence of bounded sesquilinear forms, $b_\delta\colon \mathfrak{X} \times V \to \mathbb{F}$ for all $\delta > 0$, that satisfies the following requirements:*

*(a) For each $\delta > 0$, the sesquilinear form $\mathfrak{b}_\delta$ satisfies Assumption 8.1. Moreover, the compact operator in Assumption 8.1(ii) can be chosen independently of $\delta > 0$.*

*(b) For fixed $u \in \mathfrak{X}$ and $v \in V$, $\lim_{\delta\to 0} \mathfrak{b}_\delta(u,v) = \mathfrak{b}(u,v)$.*

*(c) For every sequence $(u_\delta)_\delta$ in $\mathfrak{X}$ with $u_\delta \to u$ weakly in $\mathfrak{X}$ there holds:*

$$\lim_{\delta\to 0} \mathfrak{b}_\delta\big(u_\delta - u, v\big) = 0 \qquad \textit{for all } v \in V\,.$$

*Assume further that*

$$\ker(\mathfrak{b}) = \{0\}\,.$$

*Then the following holds: For every $\ell \in V^*$ there exists a sequence $(u_\delta)_\delta$ in $\mathfrak{X}$ such that $\mathfrak{b}_\delta(u_\delta, \cdot) = \ell(\cdot)$, for each $\delta > 0$. The sequence $(u_\delta)_\delta$ weakly converges in $\mathfrak{X}$ to an element $u \in \mathfrak{X}$, which satisfies*

$$\mathfrak{b}(u, \cdot) = \ell(\cdot)\,. \tag{8.2.1}$$

*Moreover, the limit $u$ is the unique solution to* (8.2.1).

*Proof.* Fix $\ell \in V^*$. Assumption *(a)* together with Lemma 8.5 imply that, for each $\delta > 0$, there is a unique $u_\delta \in \mathfrak{X}$ such that $\mathfrak{b}_\delta(u_\delta, \cdot) = \ell(\cdot)$.

*Case 1.* Assume $(u_\delta)_\delta$ is a bounded sequence in $\mathfrak{X}$. We then find a subsequence $(u_\delta)_\delta$ and an element $u \in \mathfrak{X}$ such that $u_\delta \to u$ weakly in $\mathfrak{X}$, since $\mathfrak{X}$ is reflexive. For all $v \in V$ there holds

$$\big|\mathfrak{b}_\delta\big(u_\delta, v\big) - \mathfrak{b}(u, v)\big| \le \big|\mathfrak{b}_\delta\big(u_\delta - u, v\big)\big| + \big|\mathfrak{b}_\delta(u, v) - \mathfrak{b}(u, v)\big|\,. \tag{8.2.2}$$

Using assumptions *(b)* and *(c)*, we deduce from (8.2.2) that

$$\lim_{\delta \to 0} \mathfrak{b}_\delta\big(u_\delta, v\big) = \mathfrak{b}(u, v) \quad \text{ for all } v \in V\,. \tag{8.2.3}$$

On the other hand, we have that $\mathfrak{b}_\delta(u_\delta, \cdot) = \ell(\cdot)$ for each $\delta > 0$. Combining this with (8.2.3) yields $\mathfrak{b}(u, \cdot) = \ell(\cdot)$ and a solution $u \in \mathfrak{X}$ to (8.2.1) is found. Uniqueness of $u$ follows from $\ker(\mathfrak{b}) = \{0\}$. As the limit $u$ is unique, we infer that every subsequence of the bounded sequence $(u_\delta)_\delta$ has a subsequence that weakly converges to $u$ and thus, the whole sequence $(u_\delta)_\delta$ weakly converges to $u$.

*Case 2.* Suppose the sequence $(u_\delta)_\delta$ is unbounded in $\mathfrak{X}$. We shall show that this is impossible. As $(u_\delta)_\delta$ is unbounded, we find a subsequence $(u_\delta)_\delta$ such that $\lim_{\delta\to 0}\|u_\delta\|_{\mathfrak{X}} = +\infty$. Consider the re-scaled sequence $(w_\delta)_\delta$ given by

$$w_\delta := \frac{u_\delta}{\|u_\delta\|_{\mathfrak{X}}}\,.$$

Clearly, $(w_\delta)_\delta$ is a bounded sequence in $\mathfrak{X}$; we therefore find a subsequence $(w_\delta)_\delta$ and an element $w \in \mathfrak{X}$ such that $w_\delta \to w$ weakly in $\mathfrak{X}$. For each $\delta > 0$, the element $w_\delta$ satisfies the equation

$$\mathfrak{b}_\delta\big(w_\delta, v\big) = \frac{\ell(v)}{\|u_\delta\|_{\mathfrak{X}}} \tag{8.2.4}$$

for all $v \in V$. As $(u_\delta)_\delta$ is unbounded, we deduce from *(b)*, *(c)*, and equation (8.2.4) that

$$\mathfrak{b}(w, v) = \lim_{\delta \to 0} \mathfrak{b}_\delta\big(w_\delta, v\big) = \lim_{\delta \to 0} \frac{\ell(v)}{\|u_\delta\|_{\mathfrak{X}}} = 0 \quad \text{ for all } v \in V\,. \tag{8.2.5}$$

Equation (8.2.5) has only the trivial solution $w = 0$, since $\ker(\mathfrak{b}) = \{0\}$.

On the other hand, there exists a compact operator $K\colon \mathfrak{X} \to V^*$ and a constant $c > 0$ such that

$$\sup_{\substack{v \in V \\ \|v\|_V = 1}} \big|\mathfrak{b}_\delta(u, v) + \langle Ku, v\rangle_{V^*,V}\big| \ge c\|u\|_{\mathfrak{X}} \tag{8.2.6}$$

for all $\delta > 0$ and all $u \in \mathfrak{X}$. Substituting $u = w_\delta$ into (8.2.6) yields

$$0 < c = c\|w_\delta\|_{\mathfrak{X}} \leq \sup_{\substack{v \in V \\ \|v\|_V = 1}} \left|\mathfrak{b}_\delta(w_\delta, v) + \langle Kw_\delta, v\rangle_{V^*,V}\right|$$
$$\leq \sup_{\substack{v \in V \\ \|v\|_V \leq 1}} \left|\mathfrak{b}_\delta\big(w_\delta, v\big)\right| + \|Kw_\delta\|_{V^*} \,. \tag{8.2.7}$$

From (8.2.4) we deduce that, for every $v \in V$ with $\|v\|_V \leq 1$ and each $\delta > 0$ there holds

$$\left|b_\delta(w_\delta, v)\right| \leq \frac{\|\ell\|}{\|u_\delta\|_{\mathfrak{X}}} \,.$$

Combining this estimate with (8.2.7) yields the inequality

$$0 < c \leq \frac{\|\ell\|}{\|u_\delta\|_{\mathfrak{X}}} + \left\|Kw_\delta\right\|_{V^*} , \tag{8.2.8}$$

for all $\delta > 0$. The operator $K$ is compact and the sequence $(w_\delta)_\delta$ weakly converges to $w = 0$; we find therefore a subsequence $(Kw_\delta)_\delta$ that converges strongly in $V^*$ to $Kw = 0$. Sending $\delta \to 0$ in (8.2.8) thus yields

$$0 < c \leq \lim_{\delta \to 0} \frac{\|\ell\|}{\|u_\delta\|_{\mathfrak{X}}} + \lim_{\delta \to 0}\left\|Kw_\delta\right\|_{V^*} = 0 \,.$$

This is a contradiction and hence the sequence $(u_\delta)_\delta$ has to be bounded in $\mathfrak{X}$.□

Remark 10. – *In Theorem 8.6, the sesquilinear form $\mathfrak{b}\colon \mathfrak{X} \times V \to \mathbb{F}$ is not assumed to be bounded. In view of (8.2.1) we do, however, implicitly assume that $\mathfrak{b}$ is continuous in its second argument. In fact, $\mathfrak{b}$ is bounded if $(\mathfrak{b}_\delta)_\delta$ is a sequence of bounded sesquilinear forms for which assumption (b) is satisfied. Indeed, for fixed $v \in V$, the map $T\colon \mathfrak{X} \to \mathbb{F}$, $u \mapsto \mathfrak{b}(u, v)$ is the pointwise limit of the family of bounded linear operators $T_\delta\colon \mathfrak{X} \to \mathbb{F}$, $u \mapsto \mathfrak{b}_\delta(u, v)$. Thus, by Banach-Steinhaus, $T$ is a bounded linear operator and hence $\mathfrak{b}$ is continuous in its first argument. As every reflexive normed space is a Banach space, we can repeat the above argument for $S\colon V \to \mathbb{F}$, $v \mapsto \mathfrak{b}(u, v)$ and every fixed $u \in \mathfrak{X}$, and infer that $\mathfrak{b}$ is continuous in its second argument. The uniform boundedness principle then implies the boundedness of $\mathfrak{b}$.*

Let us state a special case of the abstract limiting absorption principle for a sesquilinear form $\mathfrak{b}$ on a Hilbert space $\mathfrak{H}$.

Corollary 8.7. (Limiting absorption principle for Hilbert spaces) – *Let $\mathfrak{H}$ be a Hilbert space over $\mathbb{F}$, and let $\mathfrak{b}\colon \mathfrak{H} \times \mathfrak{H} \to \mathbb{F}$ be a sesquilinear form. Assume that $(b_\delta)_\delta$ is a sequence of bounded sesquilinear forms, $\mathfrak{b}_\delta\colon \mathfrak{H} \times \mathfrak{H} \to \mathbb{F}$ for all $\delta > 0$, that satisfies the following requirements:*

*(a) There exists a compact operator $K\colon \mathfrak{H} \to \mathfrak{H}^*$ and a constant $c > 0$ such that*

$$\left|\mathfrak{b}_\delta(u, u) + \langle Ku, u\rangle_{\mathfrak{H}^*,\mathfrak{H}}\right| \geq c\|u\|^2_{\mathfrak{H}} \tag{8.2.9}$$

*for all $u \in \mathfrak{H}$ and $\delta > 0$. Moreover, $\ker(\mathfrak{b}_\delta) = \{0\}$ for each $\delta > 0$.*

*(b) For fixed $u, v \in \mathfrak{H}$, $\lim_{\delta \to 0} \mathfrak{b}_\delta(u, v) = \mathfrak{b}(u, v)$.*

*(c) For every sequence* $(u_\delta)_\delta$ *in* $\mathfrak{H}$ *with* $u_\delta \to u$ *weakly in* $\mathfrak{H}$ *there holds:*

$$\lim_{\delta \to 0} \mathfrak{b}_\delta\big(u_\delta - u, v\big) = 0 \qquad \textit{for all } v \in \mathfrak{H}\,.$$

*Assume further that* $\ker(\mathfrak{b}) = \{0\}$*. Then the following holds: For every* $\ell \in \mathfrak{H}^*$ *there exists a sequence* $(u_\delta)_\delta$ *in* $\mathfrak{H}$ *such that* $\mathfrak{b}_\delta(u_\delta, \cdot) = \ell(\cdot)$ *for each* $\delta > 0$*. The sequence* $(u_\delta)_\delta$ *weakly converges in* $\mathfrak{H}$ *to an element* $u \in \mathfrak{H}$*, which satisfies*

$$\mathfrak{b}(u, \cdot) = \ell(\cdot)\,. \tag{8.2.10}$$

*Furthermore, the limit* $u$ *is the unique solution to* (8.2.10)*.*

*Proof.* Inequality (8.2.9) implies a Gårding inequality as was shown in the proof of Corollary 8.4. The claim thus follows from Theorem 8.6. □

CHAPTER 9

# Existence and Uniqueness result for a Helmholtz equation in a bounded waveguide with a special boundary condition

In this section we apply the abstract limiting absorption principle as well as the Fredholm alternative for sesquilinear forms from the previous chapter to deduce the existence of a distributional solution $u \in V_{R,l}$ to (7.0.7).

## 9.1 Existence result for the radiation problem with $\delta > 0$

This section is devoted to an existence and uniqueness result for the radiation problem with damping; that is, for given $R \in \mathbb{N}$, $l > 0$ and $h \in L^2(\Omega_R; \mathbb{C})$, we seek a function $u \in V_{R,l}$ such that

$$\mathfrak{b}_\delta(u, v) = \int_{\Omega_R} h\bar{v} \quad \text{for all } v \in V_{R,l} \,.$$

We shall use the Fredholm alternative for sesquilinear forms to establish the existence of a solution. In order to apply Proposition 8.2, we need to show that $\mathfrak{b}_\delta$ satisfies a Gårding inequality. To this end we define the operator $K \colon V_{R,l} \to V_{R,l}^*$, $u \mapsto Ku$ with

$$\langle Ku, v\rangle_{V_{R,l}^*, V_{R,l}} := 2k^2 \int_{\Omega_{R+l}} u\bar{v} + \mathcal{Q}_{R,l}(u, v) \,, \tag{9.1.1}$$

where the sesquilinear form $\mathcal{Q}_{R,l}$ is defined in (7.2.12). We first show that this operator is compact.

LEMMA 9.1. – *Assume $k^2 > 0$ is non-singular. Let $R \in \mathbb{N}$, $l > 0$, and let $V_{R,l}$ be the function space defined in* (7.0.12)*. Then the operator $K$ defined in* (9.1.1) *is compact.*

*Proof.* For brevity we write $L^2$ instead of $L^2(\Omega_{R+l})$. By Lemma 1.5, the compactness of $K$ is equivalent to the following condition: Every bounded sequence $(u_m)_m$ in $V_{R,l}$ admits a subsequence $(u_m)_m$ that weakly converges to some $u \in V_{R,l}$ and that satisfies

$$\lim_{m\to\infty} \langle K u_m, v_m \rangle_{V_{R,l}^*, V_{R,l}} = \langle K u, v \rangle_{V_{R,l}^*, V_{R,l}}$$

for all sequences $(v_m)_m$ in $V_{R,l}$ with $v_m \to v$ weakly in $V_{R,l}$.

Let $(u_m)_m$ and $(v_m)_m$ be sequences in $V_{R,l}$ such that $(u_m)_m$ is bounded in $V_{R,l}$ and $(v_m)_m$ weakly converges in $V_{R,l}$ to some $v \in V_{R,l}$. As $V_{R,l}$ is a Hilbert space, $(u_m)_m$ admits a subsequence $(u_m)_m$ that weakly converges in $V_{R,l}$ to some function $u \in V_{R,l}$. The space $V_{R,l}$ is a subspace of $H^1(\Omega_{R+l};\mathbb{C})$, and thus, by the Rellich-Kondrachov theorem, there exists a subsequence $(u_m)_m$ such that $u_m \to u$ in $L^2$. Similarly, as $(v_m)_m$ weakly converges in $V_{R,l}$ to $v$, we deduce the existence of a subsequence $(v_m)_m$ such that $v_m \to v$ in $L^2$. Consequently,

$$\lim_{m\to\infty} \int_{\Omega_{R+l}} u_m \bar{v}_m = \int_{\Omega_{R+l}} u \bar{v} \,. \tag{9.1.2}$$

Furthermore, the weak convergence of $(\nabla u_m)_m$ in $L^2$ and the strong convergence of $(v_m)_m$ in $L^2$ imply that

$$\lim_{m\to\infty} \mathcal{Q}_{R,l}\big(u_m, v_m\big) = \mathcal{Q}_{R,l}(u, v) \,. \tag{9.1.3}$$

Combining (9.1.2) and (9.1.3) yields

$$\lim_{m\to\infty} \left( 2k^2 \int_{\Omega_{R+l}} u_m \bar{v}_m + \mathcal{Q}_{R,l}\big(u_m, v_m\big) \right) = 2k^2 \int_{\Omega_{R+l}} u \bar{v} + \mathcal{Q}_{R,l}(u, v) \,.$$

and the claim is proved. □

With the compact operator $K$, we obtain a Gårding inequality.

LEMMA 9.2. (Gårding inequality) – *Assume $k^2 > 0$ is non-singular. Let $R \in \mathbb{N}$, $l > 0$, and let $V_{R,l}$ be the function space defined in* (7.0.12)*. Then, for each $\delta \geq 0$, the sesquilinear form $\mathfrak{b}_\delta \colon V_{R,l} \times V_{R,l} \to \mathbb{C}$ defined in* (7.2.10) *satisfies a Gårding inequality. More precisely, there exists a constant $c > 0$ such that for all $u \in V_{R,l}$ there holds*

$$\Big| \mathfrak{b}_\delta(u, u) + \langle K u, u \rangle_{V_{R,l}^*, V_{R,l}} \Big| \geq c \|u\|^2_{H^1(\Omega_{R+l};\mathbb{C})} \,,$$

*where $K \colon V_{R,l} \to V_{R,l}^*$ is the compact operator defined in* (9.1.1)*.*

*Proof.* Fix $u \in V_{R,l}$. Using the definition of $K$, we find that

$$\operatorname{Re}\Big\{ \mathfrak{b}_\delta(u, u) + \langle K u, u \rangle_{V_{R,l}^*, V_{R,l}} \Big\} = \int_{\Omega_{R+l}} \vartheta \big|\nabla u\big|^2 + k^2 \int_{\Omega_{R+l}} (2 - \vartheta) |u|^2 \,. \tag{9.1.4}$$

Equation (9.1.4) and Lemma 7.2 imply

$$
\begin{aligned}
\Big|\mathfrak{b}_\delta(u,u) + \langle Ku,u\rangle_{V_{R,l}^*,V_{R,l}}\Big| &\geq \big\|\nabla u\big\|^2_{L^2(\Omega_{R+l};\mathbb{C})} + k^2\|u\|^2_{L^2(\Omega_{R+l};\mathbb{C})} \\
&\geq c\|u\|^2_{H^1(\Omega_R;\mathbb{C})} + k^2\|u\|^2_{L^2(W_{-R-l,l};\mathbb{C})} + k^2\|u\|^2_{L^2(W_{R,l};\mathbb{C})} \\
&= c\|u\|^2_{H^1(\Omega_R;\mathbb{C})} + k^2\big\|\mathcal{R}^-_{R,l}(u)\big\|^2_{L^2(W_{0,l};\mathbb{C})} + k^2\big\|\mathcal{R}^+_{R,l}(u)\big\|^2_{L^2(W_{0,l};\mathbb{C})} \\
&\geq c\|u\|^2_{H^1(\Omega_{R+l};\mathbb{C})}\,. \qquad (9.1.5)
\end{aligned}
$$

This implies the claim. □

Our next aim is to show that the homogeneous problem $\mathfrak{b}_\delta(u,\cdot) = 0$ has only the trivial solution $u = 0$. Before we do so, let us remind the reader that, due to Lemma 7.4, there exists an $l_0 \in \mathbb{N}$ such that for all $l \in \mathbb{N}$ with $l \geq l_0$ there holds

$$
-\operatorname{Im}\big\{Q_{0,l}(v,v)\big\} \geq c\|v\|^2_{L^2(W_{0,l},\mathbb{C})} \qquad \text{for all } v \in X_l^- \qquad (9.1.6a)
$$

and

$$
\operatorname{Im}\big\{Q_{0,l}(u,u)\big\} \geq c\|u\|^2_{L^2(W_{0,l},\mathbb{C})} \qquad \text{for all } u \in X_l^+\,. \qquad (9.1.6b)
$$

The function spaces $X_l^-$ and $X_l^+$ are defined in (7.2.8).

Lemma 9.3. (Uniqueness of the homogeneous problem) – *Let $k^2 > 0$ be non-singular and let $l \in \mathbb{N}$ be large enough such that* (9.1.6) *holds. Fix $R \in \mathbb{N}$ and let $V_{R,l}$ be function space defined in* (7.0.12)*. For $\delta > 0$ we denote by $\mathfrak{b}_\delta$ the sesquilinear form given in* (7.2.10)*. If $u \in V_{R,l}$ satisfies $\mathfrak{b}_\delta(u,\cdot) = 0$, then $u = 0$.*

*Proof.* Let $u \in \ker(\mathfrak{b}_\delta)$. Taking the imaginary part of $\mathfrak{b}_\delta(u,u)$, we obtain

$$
\delta k^2 \int_{\Omega_{R+l}} \vartheta|u|^2 + \operatorname{Im}\big\{\mathcal{Q}_{R,l}(u,u)\big\} = 0\,. \qquad (9.1.7)
$$

As $u \in V_{R,l}$, the functions $\mathcal{R}^+_{R,l}(u)$ and $\mathcal{R}^-_{R,l}(u)$ are elements of $X_l^+$ and $X_l^-$, respectively. The definition of $\mathcal{Q}_{R,l}$ in (7.2.12) as well as (9.1.6) imply then

$$
\begin{aligned}
\operatorname{Im}\big\{\mathcal{Q}_{R,l}(u,u)\big\} &= \operatorname{Im}\Big\{Q_{0,l}\big(\mathcal{R}^+_{R,l}(u),\mathcal{R}^+_{R,l}(u)\big)\Big\} - \operatorname{Im}\Big\{Q_{0,l}\big(\mathcal{R}^-_{R,l}(u),\mathcal{R}^-_{R,l}(u)\big)\Big\} \\
&\geq c\|u\|^2_{L^2(W_{R,l};\mathbb{C})} + c\|u\|^2_{L^2(W_{-R-l,l};\mathbb{C})}\,. \qquad (9.1.8)
\end{aligned}
$$

Combining (9.1.7) with (9.1.8), we find that

$$
0 \geq \delta k^2 \int_{\Omega_{R+l}} \vartheta|u|^2 + c\|u\|^2_{L^2(W_{R,l};\mathbb{C})} + c\|u\|^2_{L^2(W_{-R-l,l};\mathbb{C})} \geq C\|u\|^2_{L^2(\Omega_{R+l};\mathbb{C})}\,.
$$

Consequently $u = 0$ in $V_{R,l}$ and the claim is proved. □

Theorem 9.4. (Existence result for $\delta > 0$) – *Let $k^2 > 0$ be non-singular and choose $l \in \mathbb{N}$ large enough such that* (9.1.6) *holds. Fix $R \in \mathbb{N}$ and let $V_{R,l}$ be the function space defined in* (7.0.12)*. If $\delta > 0$, then for every $h \in L^2(\Omega_R;\mathbb{C})$ there exists a unique solution $u \in V_{R,l}$ to the radiation problem with damping. More precisely, for every $v \in V_{R,l}$ there holds*

$$
\mathfrak{b}_\delta(u,v) = \int_{\Omega_R} h\bar{v}\,,
$$

*where $\mathfrak{b}_\delta$ is the sesquilinear form given in* (7.2.10)*.*

*Proof.* Due to Assumption 7.1*(i)*, the space $X_l$ is finite dimensional and hence $V_{R,l}$ is a Hilbert space. The sesquilinear form $\mathfrak{b}_\delta$ is bounded and satisfies a Gårding inequality by Lemma 9.2. Due to Lemma 9.3, $\ker(\mathfrak{b}_\delta) = \{0\}$. Thus, the claim follows from Corollary 8.4. □

## 9.2 Existence result for the radiation problem with $\delta = 0$

In this section we apply the abstract limiting absorption principle to establish the existence of a solution to the radiation problem with $\delta = 0$. We recall that, by Definition 7.5, a function $u \in V_{R,l}$ is a solution provided

$$\mathfrak{b}(u, v) = \int_{\Omega_R} h\bar{v} \quad \text{for all } v \in V_{R,l}\,,$$

where the sesquilinear form $\mathfrak{b}\colon V_{R,l} \times V_{R,l} \to \mathbb{C}$ is defined as

$$\mathfrak{b}(u, v) := \int_{\Omega_{R+l}} \vartheta\big\langle \nabla u, \nabla v\big\rangle - k^2 \int_{\Omega_{R+l}} \vartheta u\bar{v} - \mathcal{Q}_{R,l}(u, v)\,. \tag{9.2.1}$$

THEOREM 9.5. (Existence result for $\delta = 0$) – *Let $k^2 > 0$ be non-singular and let $l \in \mathbb{N}$ large enough such that* (9.1.6) *holds. Let $(\delta)_\delta$ be a sequence in* $(0, \infty)$ *with $\delta \to 0$. Fix $R \in \mathbb{N}$, let $V_{R,l}$ be the function space defined in* (7.0.12)*, and let $\mathfrak{b}_\delta, \mathfrak{b}\colon V_{R,l} \times V_{R,l} \to \mathbb{C}$ be the sesquilinear forms from* (7.2.10) *and* (9.2.1)*, respectively. Assume that* $\ker(\mathfrak{b}) = \{0\}$*. Then, for every $h \in L^2(\Omega_R; \mathbb{C})$, there exists a sequence $(u_\delta)_\delta$ of solutions to the radiation problem with damping. That sequence converges weakly in $V_{R,l}$ to a function $u \in V_{R,l}$ which satisfies*

$$\mathfrak{b}(u, v) = \int_{\Omega_R} h\bar{v} \quad \textit{for all } v \in V_{R,l}\,. \tag{9.2.2}$$

*Moreover, the solution $u$ to* (9.2.2) *is unique.*

*Proof.* We use the limiting absorption principle for Hilbert spaces, Corollary 8.7, to deduce the claim. Due to Assumption 7.1*(i)*, the space $X_l$ is finite dimensional and hence $V_{R,l}$ is a Hilbert space. Moreover, $\mathfrak{b}_\delta$ is bounded for each $\delta > 0$. One readily checks that $\lim_{\delta\to 0} \mathfrak{b}_\delta(u, v) = \mathfrak{b}(u, v)$ for all $u, v \in V_{R,l}$. Lemmas 9.2 and 9.3 imply that premise *(a)* from Corollary 8.7 is satisfied. We are thus left to show premise *(c)* of Corollary 8.7.

Let $(u_\delta)_\delta$ be a sequence in $V_{R,l}$ with $u_\delta \to u$ weakly in $V_{R,l}$. Thanks to the Rellich-Kondrachov theorem, we find a subsequence $(u_\delta)_\delta$ that strongly converges to $u$ in $L^2(\Omega_{R+l}; \mathbb{C})$. Consequently, for every $v \in V_{R,l}$,

$$\lim_{\delta\to 0}\left(k^2(1 + \mathrm{i}\delta) \int_{\Omega_{R+l}} \vartheta\big(u_\delta - u\big)\bar{v}\right) = 0\,. \tag{9.2.3}$$

On the other hand, as $\nabla u_\delta \to \nabla u$ weakly in $L^2(\Omega_{R+l}; \mathbb{C})$, we deduce that

$$\lim_{\delta\to 0} \int_{\Omega_{R+l}} \big\langle \nabla u_\delta - \nabla u, v\big\rangle = 0 \;\text{ and }\; \lim_{\delta\to 0} \mathcal{Q}_{R,l}\big(u_\delta - u, v\big) = 0\,, \tag{9.2.4}$$

for all $v \in V_{R,l}$. Combining (9.2.3) and (9.2.4), we get

$$\lim_{\delta\to 0} \mathfrak{b}_\delta(u_\delta - u, v) = 0$$

for every $v \in V_{R,l}$, which shows premise *(c)* of Corollary 8.7

We can therefore apply the abstract limiting absorption principle and the claim is proved. $\square$

### On the number of non-singular values of $k^2$

In Theorem 9.5 we assume the kernel of $\mathfrak{b}$ to be trivial. Whether or not this is the case depends on the value of $k^2$. We shall see that there are at most countable many non-singular values of $k^2$ for which $\ker(\mathfrak{b}) \neq \{0\}$. To be more precise, we define

$$D \coloneqq \Big\{ k^2 \in (0,\infty) \,\Big|\, k^2 \text{ is non-singular in the sense of Assumption 7.1} \Big\}.$$

We have the following result.

PROPOSITION 9.6. – *For every $R \in \mathbb{N}$ there exists an at most countable set $\Sigma_R \subset D$ and a number $l_0 \in \mathbb{N}$ such that for all $l \in \mathbb{N}$ with $l \geq l_0$ there holds* $\ker(\mathfrak{b}) = \{0\}$ *for all $k^2 \in D \setminus \Sigma_R$.*

We need two auxiliary lemmas for the proof of this result, in which we use the following subspace of $H^1(\Omega_R \setminus \overline{\mathcal{O}}_R; \mathbb{C})$:

$$Z \coloneqq \Big\{ u \in H^1(\Omega_R \setminus \overline{\mathcal{O}}_R; \mathbb{C}) \,\Big|\, u = 0 \text{ on } \partial\mathcal{O}_R \text{ and } u = 0 \text{ on } \{\pm R\} \times \mathbb{S}^1 \Big\}, \quad (9.2.5)$$

for $R \in \mathbb{N}$.

LEMMA 9.7. – *Let $k^2 > 0$ be non-singular. Choose the number $l \in \mathbb{N}$ so large that* (9.1.6) *is satisfied. Then, for every $R \in \mathbb{N}$ the following holds: if $u \in V_{R,l}$ is an element of* $\ker(\mathfrak{b})$*, then $u|_{\Omega_R \setminus \overline{\mathcal{O}}_R} \in Z$ and $u$ satisfies*

$$\int_{\Omega_R \setminus \overline{\mathcal{O}}_R} \langle \nabla u, \nabla v \rangle = k^2 \int_{\Omega_R \setminus \overline{\mathcal{O}}_R} u\bar{v} \quad \textit{for all } v \in Z\,. \quad (9.2.6)$$

*Proof.* Let $u \in \ker(\mathfrak{b})$. Then, for any $v \in V_{R,l}$ there holds

$$0 = \int_{\Omega_{R+l}} \langle \nabla u, \nabla v \rangle - k^2 \int_{\Omega_{R+l}} u\bar{v} - \mathcal{Q}_{R,l}(u,v)\,. \quad (9.2.7)$$

Substituting $v = u$ into the above equation and taking the imaginary part yields

$$0 = \operatorname{Im}\big\{Q_{R,l}\big(\mathcal{R}^+_{R,l}(u), \mathcal{R}^+_{R,l}(u)\big)\big\} - \operatorname{Im}\big\{Q_{R,l}\big(\mathcal{R}^-_{R,l}(u), \mathcal{R}^-_{R,l}(u)\big)\big\}\,.$$

As $\mathcal{R}^-_{R,l}(u) \in X^-_l$ and $\mathcal{R}^+_{R,l}(u) \in X^+_l$, we deduce from Lemma 7.4 that

$$0 \geq c\|u\|^2_{L^2(W_{-R-l,l};\mathbb{C})} + c\|u\|^2_{L^2(W_{R,l};\mathbb{C})} \geq 0\,.$$

Consequently, $u = 0$ in $W_{-R-l,l} \cup W_{R,l}$. As $V_{R,l}$ is a subspace of $H^1(\Omega_{R+l};\mathbb{C})$ and $u$ vanishes in the boxes $W_{-R-l,l}$, $W_{R,l}$ as well as in the obstacles $\mathcal{O}_R$, we infer that $u = 0$ on $\{\pm R\} \times \mathbb{S}^1$ and $u = 0$ on $\partial\mathcal{O}_R$. This shows that for every $u \in \ker(\mathfrak{b})$ there holds $u|_{\Omega_R \setminus \overline{\mathcal{O}}_R} \in Z$. Moreover, since $u$ vanishes in the boxes $W_{-R-l,l}$ and $W_{R,l}$, equation (9.2.7) reads

$$0 = \int_{\Omega_R} \langle \nabla u, \nabla v \rangle - k^2 \int_{\Omega_R} u\bar{v}$$

for all $v \in V_{R,l}$. The trivial extension $\tilde{v}$ of any function $v \in Z$ to $\Omega_{R+l}$ is an element of $V_{R,l}$. Thus, for any $v \in Z$ there holds

$$0 = \int_{\Omega_R \setminus \mathcal{O}_R} \langle \nabla u, \nabla v \rangle - k^2 \int_{\Omega_R \setminus \mathcal{O}_R} u \bar{v} \, .$$

This proves the claim. □

LEMMA 9.8. – *For every $R \in \mathbb{N}$ there exists a countable set $\Sigma_R \subset (0, \infty)$ such that* (9.2.6) *admits only the trivial solution $u = 0$ if $k^2 \in (0, \infty) \setminus \Sigma_R$.*

*Proof.* For fixed $R \in \mathbb{N}$, equation (9.2.6) is the weak formulation of the Dirichlet eigenvalue problem

$$\begin{cases} -\Delta u = k^2 u & \text{in } \Omega_R \setminus \overline{\mathcal{O}}_R \, , & (9.2.8a) \\ \quad u = 0 & \text{on } \partial\big(\Omega_R \setminus \overline{\mathcal{O}}_R\big) \, . & (9.2.8b) \end{cases}$$

The Dirichlet Laplacian admits countably many eigenvalues, since $\Omega_R \setminus \overline{\mathcal{O}}_R$ is a bounded Lipschitz domain; that is, there exists a countable set $\Sigma_R \subset (0, \infty)$ such that (9.2.8) has only the trivial solution $u = 0$ if $k^2 \in (0, \infty) \setminus \Sigma_R$. This proves the claim. □

*Proof (of Proposition 9.6).* Fix $R \in \mathbb{N}$, and assume $D \neq \emptyset$. Due to Lemma 9.8, there exists a countable set $\Sigma_R \subset (0, \infty)$ such that (9.2.6) admits only the trivial solution if $k^2 \in (0, \infty) \setminus \Sigma_R$. Let $k^2 \in D \setminus \Sigma_R$ and choose $l \in \mathbb{N}$ so large that (9.1.6) holds. Then, by Lemma 9.7, every $u \in \ker(\mathfrak{b})$ is a distributional solution to (9.2.6). As $k^2 \in D \setminus \Sigma_R \subset (0, \infty) \setminus \Sigma_R$, we infer that $u = 0$. This proves the claim. □

APPENDIX A

# Proof of Lemma 4.3

*(i)* Assume $v \in \mathcal{X}(\Sigma^*)$. Clearly, $\mathrm{Re}\{v\}$ and $\mathrm{Im}\{v\}$ are elements of $L^2_\sharp(\Sigma^*;\mathbb{R}^3)$. As every test vector field $\varphi \in C_c^\infty(\Sigma^*;\mathbb{R}^3)$ lies in $C_c^\infty(\Sigma^*;\mathbb{C}^3)$, we find that

$$0 = \int_{\Sigma^*} \langle v, \mathrm{curl}\ \varphi \rangle = \int_{\Sigma^*} \langle \mathrm{Re}\{v\}, \mathrm{curl}\ \varphi \rangle + \mathrm{i} \int_{\Sigma^*} \langle \mathrm{Im}\{v\}, \mathrm{curl}\ \varphi \rangle \,.$$

This equation implies that, in the sense of distributions, $\mathrm{curl}\ \mathrm{Re}\{v\} = 0$ in $\Sigma^*$ and $\mathrm{curl}\ \mathrm{Im}\{v\} = 0$ in $\Sigma^*$. Consequently, $\mathrm{Re}\{v\}, \mathrm{Im}\{v\} \in \mathcal{X}_{\mathbb{R}}(\Sigma^*)$.

Conversely, assume that $\mathrm{Re}\{v\}$ and $\mathrm{Im}\{v\}$ are elements of $\mathcal{X}_{\mathbb{R}}(\Sigma^*)$. By definition of the complex Lebesgue spaces, we deduce that $v \in L^2_\sharp(\Sigma^*;\mathbb{C}^3)$. We claim that

$$\int_{\Sigma^*} \langle \mathrm{Re}\{v\}, \varphi \rangle = 0 \quad \text{and} \quad \int_{\Sigma^*} \langle \mathrm{Im}\{v\}, \varphi \rangle = 0 \tag{A.0.1}$$

for all $\varphi \in C_c^\infty(\Sigma^*;\mathbb{C}^3)$. Fix an arbitrary $\varphi \in C_c^\infty(\Sigma^*;\mathbb{C}^3)$. As $\mathrm{Re}\{\varphi\}, \mathrm{Im}\{\varphi\} \in \mathbb{C}_c^\infty(\Sigma^*;\mathbb{R}^3)$, we deduce from the fact that $\mathrm{Re}\{v\} \in \mathcal{X}_{\mathbb{R}}(\Sigma^*)$ that

$$\int_{\Sigma^*} \langle \mathrm{Re}\{v\}, \varphi \rangle = \int_{\Sigma^*} \langle \mathrm{Re}\{v\}, \mathrm{Re}\{\varphi\} \rangle + \mathrm{i} \int_{\Sigma^*} \langle \mathrm{Re}\{v\}, \mathrm{Im}\{v\} \rangle = 0 \,.$$

The statement for $\mathrm{Im}\{v\}$ can be shown analogously. Having (A.0.1) at hand, we observe that

$$\int_{\Sigma^*} \langle v, \varphi \rangle = \int_{\Sigma^*} \langle \mathrm{Re}\{v\}, \varphi \rangle + \int_{\Sigma^*} \langle \mathrm{Im}\{v\}, \varphi \rangle = 0$$

for all $\varphi \in C_c^\infty(\Sigma^*;\mathbb{C}^3)$.

*(ii)* Replacing $\Sigma^*$ by $Y$, we can follow the above argument and find that $\mathrm{curl}\ \phi = 0$ in $Y$ if and only if $\mathrm{curl}\ \mathrm{Re}\{\phi\} = 0$ in $Y$ and $\mathrm{curl}\ \mathrm{Im}\{\phi\} = 0$ in $Y$. We are thus left to prove that $\phi = 0$ in $\Sigma^*$ if and only if $\mathrm{Re}\{\phi\} = 0$ in $\Sigma$ and $\mathrm{Im}\{\phi\} = 0$ in $\Sigma$. But this holds trivially.

APPENDIX B

# Proof of Proposition 5.6

We proceed in two steps.

*Step 1.* We claim that the vector space $X$ is isomorphic to the vector space

$$\mathcal{H} := \left\{ v \in L^2_\sharp(\Sigma^*;\mathbb{C}^3) \;\middle|\; \text{curl } v = 0 \text{ in } \Sigma^*,\ \text{div } v = 0 \text{ in } \Sigma^*,\ \langle v, \nu\rangle = 0 \text{ on } \partial\Sigma^* \right\}.$$

Choose $v \in \mathcal{H}$ and extend it trivially to a vector field $\tilde v \in L^2_\sharp(Y;\mathbb{C}^3)$. Clearly, curl $\tilde v = 0$ in $\Sigma^*$ and $\tilde v = 0$ in $\Sigma$. Thus, the map $T\colon \mathcal{H} \to X$, $v \mapsto \tilde v$ is well defined provided $\tilde v$ is divergence free in $Y$. The normal trace of $v$ on $\partial\Sigma^*$ vanishes; thus, the identity

$$\int_{\Sigma^*} \langle v, \nabla\psi\rangle = -\int_{\Sigma^*} (\text{div } v)\, \overline{\psi}$$

holds for all $\psi \in H^1_\sharp(\Sigma^*;\mathbb{R})$. As every $\varphi \in C^\infty_\sharp(Y;\mathbb{R})$ is an element of $H^1_\sharp(\Sigma^*;\mathbb{R})$ and div $v = 0$ in $\Sigma^*$, we infer that

$$\int_Y \big\langle \tilde v, \nabla\varphi \big\rangle = \int_{\Sigma^*} \big\langle v, \nabla\varphi\big\rangle = -\int_{\Sigma^*} (\text{div } v)\, \overline{\varphi} = 0\,.$$

This shows div $\tilde v = 0$ in $Y$. We note that $T$ is a linear map. Its inverse $S$ is given by $S\colon X \to \mathcal{H}$, $v \mapsto v|_{\Sigma^*}$. This map is well defined, since the two equations

$$\begin{cases} \text{div } v = 0 \text{ in } Y\,, \\ \qquad v = 0 \text{ in } \Sigma\,, \end{cases}$$

imply that $\langle v, \nu\rangle = 0$ on $\partial\Sigma^*$; in particular, $v|_{\Sigma^*} \in \mathcal{H}$. The two vector spaces $X$ and $\mathcal{H}$ are thus isomorphic.

*Step 2.* We claim that $\mathcal{H}$ is finite dimensional. Clearly, $\mathcal{H}$ is a closed subspace of

$$M := \left\{ v \in L^2_\sharp(\Sigma^*;\mathbb{C}^3) \;\middle|\; \begin{array}{l} \text{curl } v \in L^2_\sharp(\Sigma^*;\mathbb{C}^3)\,,\ \text{div } v \in L^2_\sharp(\Sigma^*;\mathbb{C}^3) \\ \text{and } \langle v, \nu\rangle = 0 \text{ on } \partial\Sigma^* \end{array} \right\}.$$

Equipped with the norm $\|\cdot\|_M \colon M \to \mathbb{R}$,

$$\|v\|_M := \|v\|_{L^2_\sharp(\Sigma^*;\mathbb{C}^3)} + \|\text{curl } v\|_{L^2_\sharp(\Sigma^*;\mathbb{C}^3)} + \|\text{div } v\|_{L^2_\sharp(\Sigma^*;\mathbb{C})}\,,$$

the space $M$ is a Hilbert space. $\mathcal{H}$ is finite dimensional if every bounded sequence in $\mathcal{H}$ admits a strongly convergent subsequence. Let $(v_k)_k$ be a bounded sequence in $\mathcal{H}$. Then there exists a subsequence $(v_k)_k$ that weakly converges to some field $w \in M$. As $\mathcal{H}$ is a closed subspace of $M$, we deduce that $w \in \mathcal{H}$. The space $M$ embeds compactly into $L^2_\sharp(Y;\mathbb{C}^3)$; see Weber [Web80]. Thus, we find another subsequence $(v_k)_k$ and a field $\tilde{w} \in L^2_\sharp(\Sigma^*;\mathbb{C}^3)$ such that $v_k \to \tilde{w}$ in $L^2_\sharp(\Sigma^*;\mathbb{C}^3)$. By uniqueness of the limit, $\tilde{w} = w \in \mathcal{H}$. Thus,

$$\lim_{k\to\infty} \|w - v_k\|_M = \lim_{k\to\infty} \|w - v_k\|_{L^2_\sharp(\Sigma^*;\mathbb{C}^3)} = 0\,.$$

We have thus proved that each bounded sequence $(v_k)_k$ in $\mathcal{H}$ admits a strongly convergent subsequence and hence the claim is proved.

# Bibliography

[All92] Grégoire Allaire. Homogenization and two-scale convergence. *SIAM J. Math. Anal.*, 23(6):1482–1518, November 1992.

[AtEKS14] Wolfgang Arendt, Antonious F. M. ter Elst, James B. Kennedy, and Manfred Sauter. The Dirichlet-to-Neumann operator via hidden compactness. *J. Funct. Anal.*, 266(3):1757–1786, 2014.

[BBDT01] Anne-Sophie Bonnet-Ben Dhia and Axel Tillequin. A limiting absorption principle for scattering problems with unbounded obstacles. *Math. Methods Appl. Sci.*, 24(14):1089–1111, 2001.

[BBF09] Guy Bouchitté, Christophe Bourel, and Didier Felbacq. Homogenization of the 3D Maxwell system near resonances and artificial magnetism. *C. R. Math. Acad. Sci. Paris*, 347(9-10):571–576, May 2009.

[BBF17] Guy Bouchitté, Christophe Bourel, and Didier Felbacq. Homogenization near resonances and artificial magnetism in three dimensional dielectric metamaterials. *Archive for Rational Mechanics and Analysis*, 225(3):1233–1277, June 2017.

[BK14] Adam Bowers and Nigel J. Kalton. *An introductory course in functional analysis.* New York, NY: Springer, 2014.

[BPS16] Sebastian Bauer, Dirk Pauly, and Michael Schomburg. The Maxwell compactness property in bounded weak Lipschitz domains with mixed boundary conditions. *SIAM J. Math. Anal.*, 48(4):2912–2943, 2016.

[BS10] Guy Bouchitté and Ben Schweizer. Homogenization of Maxwell's equations in a split ring geometry. *Multiscale Model. Simul.*, 8(3):717–750, January 2010.

[BS13] Guy Bouchitté and Ben Schweizer. Plasmonic waves allow perfect transmission through sub-wavelength metallic gratings. *Netw. Heterog. Media*, 8(4):857–878, November 2013.

[CC15] Kirill Cherednichenko and Shane Cooper. Homogenization of the system of high-contrast Maxwell equations. *Mathematika*, 61(2):475–500, 2015.

[Cia13] Philippe G. Ciarlet. *Linear and Nonlinear Functional Analysis with Applications.* SIAM-Society for Industrial and Applied Mathematics, 2013.

[DS18] Tomáš Dohnal and Ben Schweizer. A Bloch wave numerical scheme for scattering problems in periodic wave-guides. *SIAM J. Numer. Anal.*, 56(3):1848–1870, 2018.

[DU77] Joseph Diestel and John J. Uhl, Jr. *Vector measures.* American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[EG15] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions. $2^{nd}$ revised ed.* Boca Raton, FL: CRC Press, $2^{\text{nd}}$ revised ed. edition, 2015.

[FB97] Didier Felbacq and Guy Bouchitté. Homogenization of a set of parallel fibres. *Waves Random Media*, 7(2):245–256, 1997.

[FB05] D. Felbacq and G. Bouchitté. Theory of mesoscopic magnetism in photonic crystals. *Physical Review Letters*, 94(18):183902, 2005.

[FJ15] Sonia Fliss and Patrick Joly. Solutions of the time-harmonic wave equation in periodic waveguides: Asymptotic behaviour and radiation condition. *Archive for Rational Mechanics and Analysis*, 219(1):349–386, June 2015.

[GR86] Vivette Girault and Pierre-Arnaud Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics.* Springer-Verlag, Berlin, 1986. Theory and algorithms.

[Hay68] Thomas L. Hayden. Representation theorems in reflexive Banach spaces. *Math. Z.*, 104:405–406, 1968.

[Heu92] Harro Heuser. *Funktionalanalysis.* Mathematische Leitfäden. [Mathematical Textbooks]. B. G. Teubner, Stuttgart, third edition, 1992. Theorie und Anwendung. [Theory and application].

[Hoa11] Vu Hoang. The limiting absorption principle for a periodic semi-infinite waveguide. *SIAM J. Appl. Math.*, 71(3):791–810, 2011.

[KL18] Andreas Kirsch and Armin Lechleiter. A radiation condition arising from the limiting absorption principle for a closed full- or half-waveguide problem. *Math. Methods Appl. Sci.*, 41(10):3955–3975, 2018.

[Kre77] Rainer Kreß. Fredholm's alternative for compact bilinear forms in reflexive Banach spaces. *J. Diff. Equations*, 25(2):216–226, 1977.

[Kre78] Erwin Kreyszig. *Introductory functional analysis with applications.* John Wiley & Sons, New York-London-Sydney, 1978.

[KS08] Robert V. Kohn and Stephen P. Shipman. Magnetism and homogenization of microresonators. *Multiscale Modeling & Simulation*, 7(1):62–92, January 2008.

[LM54] Peter D. Lax and Arthur N. Milgram. Parabolic equations. In *Contributions to the theory of partial differential equations*, Annals of Mathematics Studies, no. 33, pages 167–190. Princeton University Press, Princeton, N. J., 1954.

[LNW02] Dag Lukkassen, Gabriel Nguetseng, and Peter Wall. Two-scale convergence. *Int. J. Pure Appl. Math.*, 2(1):35–86, 2002.

[LS13] Agnes Lamacz and Ben Schweizer. Effective Maxwell equations in a geometry with flat rings of arbitrary shape. *SIAM J. Math. Anal.*, 45(3):1460–1494, 2013.

[LS16] Agnes Lamacz and Ben Schweizer. A negative index meta-material for Maxwell's equations. *SIAM J. Math. Anal.*, 48(6):4155–4174, 2016.

[LS18a] Agnes Lamacz and Ben Schweizer. Outgoing wave conditions in photonic crystals and transmission properties at interfaces. *ESAIM Math. Model. Numer. Anal.*, 52(5):1913–1945, 2018.

[LS18b] Robert Lipton and Ben Schweizer. Effective maxwell's equations for perfectly conducting split ring resonators. *Archive for Rational Mechanics and Analysis*, 229(3):1197–1221, March 2018.

[Max61] James C. Maxwell. On Physical Lines of Force. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 21(139):161–175, 1861.

[Max65] James C. Maxwell. A dynamical theory of the electromagnetic field. *Philosophical Transactions of the Royal Society of London*, 155:459–513, 1865.

[Mül57] Claus Müller. *Grundprobleme der mathematischen Theorie elektromagnetischer Schwingungen.* Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

[Ngu89] Gabriel Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM J. Math. Anal.*, 20(3):608–623, 1989.

[PHRS99] J.B. Pendry, A.J. Holden, D.J. Robbins, and W.J. Stewart. Magnetism from conductors and enhanced nonlinear phenomena. *IEEE Transactions on Microwave Theory and Techniques*, 47(11):2075–2084, 1999.

[PS08] Grigorios A. Pavliotis and Andrew M. Stuart. *Multiscale Methods: Averaging and Homogenization*, volume 53 of *Texts in Applied Mathematics.* Springer, New York, 2008.

[PSU19] Klaas H. Poelstra, Ben Schweizer, and Maik Urban. The geometric average of curl-free fields in periodic geometries. *Technical Report, TU Dortmund*, 2019.

[Rad14] Maria Radosz. New limiting absorption and limit amplitude principles for periodic operators. *Zeitschrift für angewandte Mathematik und Physik*, 66(2):253–275, February 2014.

[Ram05] S Anantha Ramakrishna. Physics of negative refractive index materials. *Reports on Progress in Physics*, 68(2):449–521, January 2005.

[Rit09] Sebastian Ritterbusch. *Coercivity and the Calderon Operator on an Unbounded Domain*. PhD thesis, Universität Karlsruhe, 2009.

[Sau66] Niko Sauer. Representation theorems in a reflexive $B$-space. *Math. Z.*, 93:202–205, 1966.

[Sch92] Steven H. Schot. Eighty years of Sommerfeld's radiation condition. *Historia Math.*, 19(4):385–401, 1992.

[Sch16] Ben Schweizer. Resonance meets homogenization – construction of meta-materials with astonishing properties. *Jahresberichte der DMV*, 2016. DOI: 10.1365/s13291-016-0153-2.

[Sch18] Ben Schweizer. On Friedrichs inequality, Helmholtz decomposition, vector potentials, and the div-curl lemma. In *Trends in applications of mathematics to mechanics*, volume 27 of *Springer INdAM Ser.*, pages 65–79. Springer, Cham, 2018.

[Sch19] Ben Schweizer. Existence results for the Helmholtz equation in periodic wave-guides with energy methods. *Technical Report, TU Dortmund*, 2019.

[Sil49] Samuel Silver. *Microwave Antenna Theory and Design.* Massachusetts Institute of Technology. Radiation Laboratory Series. no. 12. McGraw-Hill Book Company, 1949.

[Som12] Arnold Sommerfeld. Die Greensche Funktion der Schwingungsgleichung. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 21:309–353, 1912.

[Stu69] Friedrich Stummel. *Rand- und Eigenwertaufgaben in Sobolewschen Räumen.*, volume 102. Berlin-Heidelberg-New York: Springer-Verlag, 1969.

[SU18] Ben Schweizer and Maik Urban. Effective Maxwell's equations in general periodic microstructures. *Appl. Anal.*, 97(13):2210–2230, 2018.

[SU19] Ben Schweizer and Maik Urban. On a limiting absorption principle for sesquilinear forms with an application to the Helmholtz equation in a waveguide. *Accepted for the proceedings volume of the Conference on Mathematics of Wave Phenomena 2018 at Karlsuruhe*, 2019.

[Sve50] A. G. Svešnikov. The principle of radiation. *Doklady Akad. Nauk SSSR (N.S.)*, 73:917–920, 1950.

[Tay11] Michael E. Taylor. *Partial differential equations. I: Basic theory. 2$^{nd}$ ed.*, volume 115. New York, NY: Springer, 2$^{\text{nd}}$ ed. edition, 2011.

[vI05] Vladimir v. Ignatowsky. Reflexion Elektromagnetisches Wellen an einem Draht. *Annalen der Physik*, 323(13):495–522, 1905.

[Viš51] Mark I. Višik. On strongly elliptic systems of differential equations. *Mat. Sbornik N.S.*, 29(71):615–676, 1951.

[Web80] Christian Weber. A local compactness theorem for Maxwell's equations. *Math. Methods Appl. Sci.*, 2(1):12–25, 1980.

[Wed91] Ricardo Weder. *Spectral and scattering theory for wave propagation in perturbed stratified media*, volume 87 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1991.

[Wil84] Calvin H. Wilcox. *Sound propagation in stratified fluids*, volume 50 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1984.

[Zhi05] Vasilii V. Zhikov. On Gaps in the Spectrum of some elliptic operators in divergent form with periodic coefficients. *St. Petersbg. Math. J.*, 16(5):773–790, 2005.