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Malmquist-Type Theorems for Cubic Hamiltonians

Norbert Steinmetz[1]

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Abstract

The aim of this paper is to classify the cubic polynomials

$$H(z,x,y)=\sum_{j+k\le 3} a_{jk}(z)x^j y^k$$

over the field of algebraic functions such that the corresponding Hamiltonian system $x'=H_y$, $y'=-H_x$ has at least one transcendental algebroid solution. Ignoring trivial subcases, the investigations essentially lead to several non-trivial Hamiltonians which are closely related to Painlevé's equations $\mathrm{P_I}$, $\mathrm{P_{II}}$, $\mathrm{P_{34}}$, and $\mathrm{P_{IV}}$. Up to normalisation of the leading coefficients, common Hamiltonians are

$$\begin{array}{rl}
\mathrm{H_I}: & -2y^3+\frac{1}{2}x^2-zy\\
\mathrm{H_{II/34}}: & x^2y-\frac{1}{2}y^2+\frac{1}{2}zy+\kappa x\\
\mathrm{H_{IV}}: & \begin{array}{l} x^2y+xy^2+2zxy+2\kappa x+2\lambda y\\ \frac{1}{3}(x^3+y^3)+zxy+\kappa x+\lambda y,\end{array}
\end{array}$$

but the zoo of non-equivalent Hamiltonians turns out to be much larger.

Keywords Hamiltonian system · Painlevé differential equation · Painlevé property · Malmquist property · Algebroid function

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Dedicated to the memory of Stephan Ruscheweyh.

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✉ Norbert Steinmetz
stein@math.tu-dortmund.de
http://www.mathematik.tu-dortmund.de/steinmetz/

[1] Fakultät für Mathematik, Technische Universität Dortmund, Beethovenstrasse 17, 67360 Lingenfeld, Germany

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1 Introduction

Malmquist's so-called *First Theorem* [6] singles out (linear and) Riccati differential equations

$$w' = a_0(z) + a_1(z)w + a_2(z)w^2 \tag{1}$$

among the variety of differential equations

$$w' = R(z, w) \quad (R \text{ rational}) \tag{2}$$

by postulating the *existence of some transcendental meromorphic solution* (always in the whole plane). For a long time, Malmquist's theorem was viewed as a singular and isolated result in the field of complex differential equations. With Nevanlinna theory as a tool it became the template for various theorems of this kind. Instead of citing the legion of original papers the reader is referred to Laine's monograph [5] and the more recent book [8].

If R is merely rational in w with coefficients analytic on some planar domain, the very same result is obtained by postulating the *absence of movable critical and essential singularities* of the solutions. This is abbreviated by saying that among the Eqs. (2) only (1) has the *Painlevé property*. For the first- and second-order case (Fuchs and Painlevé, respectively) the reader is referred to the books of Ince [3] and Golubew [1]. It is quite reasonable to believe that (algebraic) differential equations having the Painlevé property may also be characterised by the aforementioned *Malmquist property*, although the situation is quite different: arbitrary *analytic* versus *rational* coefficients on one hand, and the *totality* of solutions versus *one* transcendental solution on the other.

The nature of the problem makes the appearance of 'many-valued' algebroid instead of 'single-valued' meromorphic functions inevitable. One of our main tools will therefore be the Selberg–Valiron theory of algebroid functions in place of Nevanlinna theory. The interested reader will find a rudimentary description in the appendix at the end of this paper.

2 Six Theorems of Malmquist-Type

The aim of this paper is to support the aforementioned duality principle by proving Malmquist-type theorems for two-dimensional Hamiltonian systems

$$x' = H_y(z, x, y), \;\; y' = -H_x(z, x, y) \tag{3}$$

with cubic Hamiltonians

$$H(z, x, y) = \sum_{j+k\le 3} a_{jk}(z)x^j y^k \tag{4}$$

over the field of algebraic functions. It is nothing more than an exercise to show that our results may be generalised insofar as the terms ‘algebraic coefficients’ and ‘transcendental algebroid solutions’ may be replaced with ‘algebroid coefficients’ and ‘admissible algebroid solutions’, that is, solutions that grow much faster than the coefficients measured in terms of the Selberg–Valiron characteristic. Any Hamiltonian (4) such that the corresponding Hamiltonian system possesses some *transcendental algebroid* solution is said to have the *Malmquist property*.

We will start with polynomials

$$H(z,x,y)=\tfrac{1}{3}(x^3+y^3)+\sum_{j+k\le 2}a_{jk}(z)x^jy^k.$$

Replacing x and y with $x-a_{20}$ and $y-a_{02}$, respectively, the corresponding Hamiltonian system may easily be transformed into

$$\begin{aligned} x' &= y^2+cx+a\\ y' &= -x^2-cy-b \end{aligned} \tag{5}$$

with Hamiltonian

$$H(z,x,y)=\tfrac{1}{3}(x^3+y^3)+c(z)xy+b(z)x+a(z)y. \tag{6}$$

This system has the Painlevé property if and only if

$$c''=a'=b'\equiv 0 \tag{7}$$

holds, see Kecker [4]. Of course, the trivial case $a=b=c=0$ will be excluded.

Theorem 1 *Suppose the Hamiltonian* (6) *has the Malmquist property. Then the resonance condition*

$$c''(z)+\omega b'(z)-\bar{\omega}a'(z)\equiv 0 \tag{8}$$

holds for either one or two or all third roots of unity ($\omega^3=1$). *In the third case,* (6) *certainly has the Malmquist and Painlevé property* (7).

The question whether or not just one or two of the necessary conditions (8) are also sufficient for the polynomial (6) to have the Malmquist resp. Painlevé property will be answered in the following theorem.

Theorem 2 *Suppose the necessary condition* (8) *holds*

(i) *for* $\omega=1$, *say, but not for* $\omega=e^{\pm 2\pi i/3}$*;*
(ii) *for* $\omega=e^{\pm 2\pi i/3}$, *say, but not for* $\omega=1$.

Then (6) *has the Malmquist property in both cases. In case* (ii)*, the necessary condition $b = -a$ holds in addition to the Painlevé property, while in case* (i) *even*

$$x + y - c = c' + b - a \equiv 0$$

is true, but the Painlevé property fails.

Remark In case (i), x satisfies some Riccati differential equation

$$x' = x^2 - cx + a - c^2.$$

Then $(x, c - x)$ solves (5), but is 'too weak' to enforce the Painlevé property. In any other case with $c(z) = z$, say, and a and b constant, $x + y - c$ satisfies some second-order differential equation which is closely related to Painlevé's fourth differential equation

$$2ww'' = w'^2 + 3w^4 + 8zw^3 + 4(z^2 - \alpha)w^2 + 2\beta, \qquad (\mathrm{P_{IV}})$$

see [7]. Moreover, in case (ii) the functions $\bar{\omega}x + \omega y - c$ ($\omega = e^{\pm 2\pi i/3}$) satisfy simple Riccati equations.

Next we will consider the (already simplified) polynomials

$$H(z, x, y) = a(z)y + b(z)xy + \tfrac{1}{2}x^2 - 2y^3 \qquad (9)$$

containing only one third power.

Theorem 3 *Suppose the Hamiltonian* (9) *has the Malmquist property. Then the coefficients a and b are coupled,*

$$a = -\alpha + \tfrac{1}{24}(-2b''' + 4bb'' + 3b'^2 + 2b^2b' - b^4), \qquad (10)$$

where α is linear; in that case the corresponding Hamiltonian system has the Painlevé property without any restriction on b.

Remark If α is non-constant we may assume $\alpha(z) = z$ by a linear change of the independent variable. Then $w = y + \beta/12$ with $\beta = b^2 - b'$ satisfies Painlevé's first equation

$$w'' = z + 6w^2, \qquad (\mathrm{P_I})$$

hence y is transcendental algebroid and so is $x = -y' - by$, and the Malmquist and Painlevé property hold. Nevertheless the occurrence of the Hamiltonians with $b \not\equiv 0$ is really surprising.

By

$$x = u + \tfrac{1}{12} b\beta, \;\; y = v - \tfrac{1}{12}\beta \quad (\beta = b^2 - b')$$

the Hamiltonian (9) is transformed into

$$K(z, u, v) = \left(a - \tfrac{1}{24}\beta^2 + \tfrac{1}{12} b^2 \beta\right) v + \tfrac{1}{2}u^2 + buv + \tfrac{1}{2}\beta v^2 - 2v^3. \tag{11}$$

Our next Theorem shows that the Hamiltonian systems corresponding to (9) and (11) simultaneously can have the Malmquist (and Painlevé) property in very special cases only. In particular, this shows that in general neither the Malmquist nor the Painlevé property is invariant under a linear change of variables of the Hamiltonian.

Theorem 4 *Suppose the Hamiltonians* (9) *and* (11) *have the Malmquist property. Then* b *is either constant or has the form*

$$b(z) = \frac{R}{z - z_0} \quad (R = -1 \text{ or } R = 6).$$

Under these circumstances both Hamiltonian systems have the Painlevé property.

We note the different cases explicitly:

$$\begin{aligned}
&-\left(\alpha(z) + \tfrac{b^4}{24}\right) y + \tfrac{1}{2}x^2 + bxy - 2y^3;\\
&-\alpha(z) y + \tfrac{1}{2}x^2 - \tfrac{xy}{z - z_0} - 2y^3;\\
&-\left(\alpha(z) + \tfrac{105}{2(z - z_0)^4}\right) y + \tfrac{1}{2}x^2 + \tfrac{6xy}{z - z_0} - 2y^3;
\end{aligned}$$

b and z_0 are arbitrary constants, and α is constant or linear.

Finally we will consider cubic polynomials with dominating terms $x^2 y$, xy^2, again in simplified form:

$$H(z, x, y) = x^2 y - \tfrac{1}{2}y^2 + a(z)x^2 + b(z)xy + c(z)x + d(z)y \tag{12}$$

and

$$H(z, x, y) = x^2 y + xy^2 + c(z)xy + b(z)y + a(z)x \tag{13}$$

(a, b, c, d algebraic functions). The reader will not have any difficulty to adapt the proofs to more sophisticated cases.

Theorem 5 *Suppose the Hamiltonian* (12) *has the Malmquist property. Then* $u = x + b/2$ *and* $v = y + a$ *separately solve second-order differential equations*

$$u'' = \alpha + \beta u + 2u^3 \tag{14}$$

and

$$2vv'' - v'^2 = 4v^3 - 2\beta v^2 + (\beta'' - 2\alpha')v - \tfrac{1}{4}(\beta' - 2\alpha)^2, \tag{15}$$

respectively. The coefficients

$$\beta = 2a + b' - \tfrac{1}{2}b^2 + 2d \quad \text{and} \quad \alpha = -ab + c - a' + \tfrac{1}{2}\beta' \tag{16}$$

satisfy either (i) $\alpha = \pm\beta'/2$ *for one sign or else* (ii) $\alpha' = \beta'' \equiv 0$.

Remark In the second case of Theorem 5, Eq. (15) takes the form

$$2vv'' - v'^2 = 4v^3 - 2\beta v^2 - \lambda;$$

$\lambda = (2\alpha - \beta')^2/4$ is constant. In particular, each Hamiltonian with $\alpha' = \beta'' \equiv 0$ has the Malmquist and Painlevé property. In the most important case $\beta(z) = z$ (remember $\beta'' = 0$) we obtain Painlevé's equation

$$2vv'' = v'^2 + 4v^3 - 2zv^2 + \left(\alpha - \tfrac{1}{2}\right)^2, \tag{P_{34}}$$

which is closely related to equation XXXIV in Ince's book [3, p. 340], and, of course, Painlevé's second equation

$$u'' = \alpha + zu + 2u^3. \tag{P_{II}}$$

Theorem 6 *Suppose the Hamiltonian* (13) *has the Malmquist property. Then either the Painlevé property*

$$c'' = a' = b' \equiv 0$$

or else $x + y - c \equiv 0$ *holds.*

Remark In the most important case $c(z) = z$ and a and b constant, x and y separately satisfy Painlevé equations P_{IV}. In the exceptional case, x and y satisfy Riccati equations

$$x' = -x^2 + 3c(z)x + b(z) \quad \text{and} \quad y' = -y^2 - 3c(z)y - a(z).$$

3 Proof of Theorem 1

From

$$\begin{aligned} x' &= \ \ y^2 + cx + a \\ y' &= -x^2 - cy - b \end{aligned} \tag{17}$$

it follows that our algebroid solutions satisfy $2m(r, y) \leq m(r, x) + O(\log(rT(r, x)))$ and $2m(r, x) \leq m(r, y) + O(\log(rT(r, y)))$, hence

$$m(r, x) + m(r, y) = O(\log(rT(r, x)) + \log(rT(r, y)))$$

(for notations and results in Nevanlinna–Selberg–Valiron theory see the appendix). In particular, x and y have infinitely many poles. It is easily seen that the poles of (x, y) are simple with residues $(-\bar{\omega}, \omega)$ restricted to $\omega^3 = 1$. Assuming

$$\begin{aligned} x &= -\bar{\omega}\frac{1}{t} + \xi_0 + \xi_1 t + \xi_2 t^2 + \cdots \\ y &= \omega\frac{1}{t} + \eta_0 + \eta_1 t + \eta_2 t^2 + \cdots \end{aligned} \qquad (t = z - p)$$

it turns out that ξ_0, η_0, ξ_1, and η_1, but not ξ_2 and η_2 may be computed (and one of these numbers may be prescribed), but the *resonance condition*

$$c''(p) + \bar{\omega}b'(p) - \omega a'(p) = 0[^1]$$

[1] is obtained instead. Thus (8) holds if infinitely many poles with residues $(-\bar{\omega}, \omega)$ exist, and $a' = b' = c'' \equiv 0$ if this is true for each third root of unity. It is, however, not at all clear that the poles are regular (not branched)! To exclude this case let p be any pole of (x, y) with residues $(-\bar{\omega}, \omega)$ and assume that a, b, and c are regular at $z = p$, but x and/or y have a branched pole there:

$$x = \frac{-\bar{\omega}}{t} + \sum_{j=-q+1}^{\infty} \xi_j t^{j/q} \quad \text{and} \quad y = \frac{\omega}{t} + \sum_{j=-q+1}^{\infty} \eta_j t^{j/q} \quad (t = z - p).$$

Let n and m denote the first index, if any, such that $n \not\equiv 0 \bmod q$, $\xi_n \neq 0$ and $m \not\equiv 0 \bmod q$, $\eta_m \neq 0$. Then the first branched terms

$$\tfrac{n}{q}\xi_n t^{-1+n/q} \quad \text{and} \quad \tfrac{m}{q}\eta_m t^{-1+m/q}$$

on the left hand sides of the Hamiltonian system (17) are equal to the first branched terms

$$2\omega\eta_m t^{-1+m/q} \quad \text{and} \quad 2\bar{\omega}\xi_n t^{-1+n/q}$$

on the right hand sides corresponding to y^2 and $-x^2$, respectively. This implies

$$n = m, \quad n\xi_n = 2q\omega\eta_n, \quad \text{and} \quad n\eta_n = 2q\bar{\omega}\xi_n,$$

hence also $n^2 = 4q^2$, which contradicts $n \not\equiv 0 \bmod q$ and proves Theorem 1.

[1] Assuming the Painlevé property this holds for *every* p in the domain of the coefficients and *every* third root of unity, see [4]. This illuminates the difference between both concepts.

4 Proof of Theorem 2

In the first case, (x, y) has simple poles with residues $(-1, 1)$ and almost no others, hence

$$\begin{aligned} x &= -(z-p)^{-1} + \tfrac{1}{2}c(p) + \left(\tfrac{1}{4}c(p)^2 + c'(p) - \tfrac{1}{3}a(p) + \tfrac{2}{3}b(p)\right)(z-p) + \cdots \\ y &= \ (z-p)^{-1} + \tfrac{1}{2}c(p) - \left(\tfrac{1}{4}c(p)^2 - c'(p) + \tfrac{2}{3}a(p) - \tfrac{1}{3}b(p)\right)(z-p) + \cdots \end{aligned}$$

holds, and we obtain

$$x + y - c(z) = (c'(p) + b(p) - a(p))(z-p) + \cdots$$

at almost every pole. Then $x + y - c$ has at most finitely many poles, and from $m(r, x+y-c) = O(\log r + \log T(r, x) + \log T(r, y))$ and $m(r, c'+b-a) = O(\log r)$ it follows that

$$x + y - c \quad \text{and} \quad c' + b - a$$

vanish identically. Also x and y satisfy Riccati differential equations

$$x' = a + c^2 - cx + x^2 \quad \text{and} \quad y' = -b - c^2 + cy - y^2, \tag{18}$$

respectively. Conversely, starting with any solution x to the first equation (18), the pair $(x, c-x)$ solves the Hamiltonian system provided $c' + b - a \equiv 0$ holds. This yields the Malmquist property.

In the second case the substitution $u = \bar{\omega}x + \omega y - c$, $v = \omega x + \bar{\omega} y - c$, $\zeta = iz/\sqrt{3}$ transforms the given Hamiltonian system (17) into

$$\begin{aligned} u' &= -c' + A - 3cu - 2uv - u^2 \\ v' &= -c' + B + 3cv + 2uv + v^2 \end{aligned} \tag{19}$$

with $A = (\omega - 1)a + (\bar{\omega} - 1)b$, $B = (1 - \bar{\omega})a + (1 - \omega)b$, and Hamiltonian

$$(c' - B')u + (A - c')v - 3cuv - u^2v - uv^2.$$

The functions u and v can have only finitely many poles in common. The second equation (19) shows that v even vanishes at almost every pole of u (since uv has to be regular), and from $\operatorname{res}_p u = 1$ it follows that $-v'(p) = B(p) - c'(p)$ and so

$$uv \equiv c' - B$$

since $m(r, uv) = O(\log r + \log T(r, x) + \log T(r, y))$. Similarly $u'(q) = A(q) - c'(q)$ is obtained at poles of v ($\operatorname{res}_q v = -1$), hence

$$uv \equiv A - c',$$

and

$$2c' = A + B = i\sqrt{3}(a - b) \quad \text{and} \quad 2c'' = i\sqrt{3}(a' - b')$$

holds. On the other hand the hypothesis $c'' + \omega b' - \bar{\omega} a' \equiv 0$ for $\omega = e^{\pm 2\pi i/3}$ implies $c'' = b' = -a'$, hence $c'' = b' = a' \equiv 0$, this proving the Malmquist and Painlevé property. We note that (19) together with $uv = A - c' = c' - B$ imply the Riccati differential equations

$$u' = c' - A - 3cu - u^2 \quad \text{and} \quad v' = c' - B + 3cv + v^2.$$

5 Proof of Theorem 3

From the Hamiltonian system

$$\begin{aligned} x' &= a + bx - 6y^2 \\ y' &= -by - x \end{aligned} \tag{20}$$

the second-order equation

$$y'' = -a + (b^2 - b')y + 6y^2$$

is easily obtained: differentiate the second equation in (20) and then eliminate x and x'. The substitution $w = y + \frac{1}{12}\beta$ with $\beta = b^2 - b'$ leads to

$$w'' = \alpha(z) + 6w^2 \tag{21}$$

with

$$\alpha = -a + \tfrac{1}{12}\beta'' - \tfrac{1}{24}\beta^2. \tag{22}$$

To proceed we will derive the resonance condition $\alpha'' = 0$, which holds for algebraic α and algebroid solutions to (21) as well as for rational α and transcendental meromorphic solutions; here the argument is due to Wittich [9]. From

$$6|w|^2 \le |\alpha(z)| + |w| \frac{|w''|}{|w'|} \frac{|w'|}{|w|}$$

it follows that $m(r, w) = O(\log(rT(r, w)))$ as $r \to \infty$ outside possibly some exceptional set, hence w has infinitely many poles. Assuming

$$w(z) = \frac{1}{(z-p)^2} + \sum_{j=-1}^{\infty} c_j (z-p)^j,$$

an elementary computation gives $c_{-1} = c_0 = c_1 = 0$, $c_2 = -\alpha(p)/10$, and $c_3 = -\alpha'(p)/6$, while c_4 remains undetermined (and free), but

$$w'' - \alpha(z) - 6w^2 = -\tfrac{1}{2}\alpha''(p)(z-p)^2 + O((z-p)^3) \quad (z \to p)$$

holds instead. This requires $\alpha''(p) = 0$, and since w has infinitely many poles and α is algebraic, the assertion $\alpha'' \equiv 0$ follows. Again we have to assure that the poles p of w are not branched, at least when α is regular at $z = p$. To this end write

$$w(z) = \frac{1}{(z-p)^2} + \sum_{j=-2q+1}^{\infty} c_j (z-p)^{j/q}$$

and let n denote the smallest index, if any, such that $c_n \neq 0$ and $n \not\equiv 0 \bmod q$. Then the first branched terms on the left and right hand side of the differential equation (21) are

$$\tfrac{n}{q}\left(\tfrac{n}{q} - 1\right) c_n (z-p)^{-2+n/q} \quad \text{and} \quad 12 c_n (z-p)^{-2+n/q}.$$

Thus $\xi = n/q$ satisfies $\xi^2 - \xi = 12$, which is absurd since the roots $\xi = 4$ and $\xi = -3$ are integers. This finishes the proof of Theorem 3 since

$$2\beta'' - \beta^2 = -2b''' + 4bb'' + 3b'^2 + 2b^2b' - b^4.$$

6 Proof of Theorem 4

From

$$\begin{aligned} u' &= \left(a - \tfrac{1}{24}\beta^2 + \tfrac{1}{12}b^2\beta\right) + bu + \beta v - 6v^2 \\ v' &= -u - bv \end{aligned}$$

the second-order differential equation

$$v'' = \left(-a + \tfrac{1}{24}\beta^2 - \tfrac{1}{12}b^2\beta\right) + 6v^2$$

easily follows, hence

$$\alpha_1 = -a + \tfrac{1}{24}\beta^2 - \tfrac{1}{12}b^2\beta \tag{23}$$

is also linear. In combination with (22) we obtain

$$\beta^2 - b^2\beta - \beta'' = 12(\alpha_1 - \alpha), \tag{24}$$

hence b satisfies the differential equation

$$\Omega = b''' - 2bb'' - b'^2 - b^2b' = \kappa z + \lambda. \tag{25}$$

The proof of Theorem 4 then follows from the subsequent

Proposition *Algebraic solutions to* (25) *do not exist if* $\kappa z + \lambda \not\equiv 0$. *If* $\kappa = \lambda = 0$, *the non-constant algebraic solutions have the form*

$$b(z) = \frac{R}{z - z_0} \quad (R = -1 \text{ or } R = 6,\ z_0 \text{ arbitrary}). \tag{26}$$

Proof Let b be any algebraic solution with p-fold pole at $z = z_0$. Then the single terms in (25) have poles of order $p+3, 2p+2, 2p+2$, and $3p+1$, respectively. For $p > 1, 3p+1$ dominates the other orders, while for $p < 1$ this role is taken by $p+3$. This means $p = 1$ and $b(z) \sim R/(z - z_0)$ as $z \to 0$ with

$$-6R - 4R^2 - R^2 + R^3 = 0, \tag{27}$$

hence $R = -1$ or $R = 6$. Now suppose

$$b(z) = \frac{R}{z - z_0} + c(z - z_0)^q + o(|z - z_0|^q) \quad \text{as } z \to z_0 \quad (c \neq 0)$$

is any *local* solution, where $q > -1$ is some rational number. Then (25) yields

$$\Omega(z) = (R^3 - 5R^2 - 6R)z^{-4} + J(R,q)\,c(z - z_0)^{q-3} + o(|z - z_0|^{q-3})$$

as $z \to z_0$ with

$$J(R,q) = 4R - 2R^2 - (2 + 4R - R^2)q + (3 + 2R)q^2 - q^3.$$

This requires (27) and

$$\begin{aligned} J(6,q) &= \ -48 + 10q + 15q^2 - q^3 = 0 \quad \text{resp.} \\ J(-1,q) &= \ -6 + 3q + q^2 - q^3 = 0 \end{aligned}$$

if $\kappa = \lambda = 0$.[2] Apart from $q = -2 < -1$ in both cases, the roots are irrational ($q = (17 \pm \sqrt{193})/2$) and non-real ($q = (3 \pm i\sqrt{3})/2$), respectively, which is absurd. Non-constant algebraic solutions have either finite poles or a pole at infinity of order $p > 0$, say. Here the term $-b^2 b' \sim cz^{3p-1}$ dominates the other terms which are $O(|z|^{2p-2}) = o(|z|^{3p-1})$ as $z \to \infty$ and inhibits $\Omega \equiv 0$. This proves (26) for non-constant solutions in case of $\kappa = \lambda = 0$. We have to exclude the case $|\kappa| + |\lambda| > 0$. Here

$$\Omega(z) = J(R,q)\,c(z - z_0)^{q-3} + o(|z - z_0|^{q-3}) = \kappa z_0 + \lambda + \kappa(z - z_0)$$

[2] Like many other computations also these were performed by my favourite tool maple.

requires $q = 3$ and $J(R, 3)\, c = \kappa z_0 + \lambda$ if $\kappa z_0 + \lambda \neq 0$, and $q = 4$ and $J(R, 4)\, c = \kappa$ if $\kappa z_0 + \lambda = 0$. This way we get a unique formal solution

$$\frac{R}{z - z_0} + \sum_{q=3}^{\infty} c_q (z - z_0)^q$$

by successively solving rather elaborate equations

$$J(R, q)\, c_q = \Phi_q(c_3, \ldots, c_{q-1}) \quad (q = 4, 5, \ldots)$$

for c_q; this is possible since $J(R, q) \neq 0$ for $q \in \mathbb{N}$ and $R = -1, 6$. Thus our algebraic function b has no finite algebraic poles. To exclude other algebraic singularities, that is, to prove that b is a rational function, it does not suffice to indicate that every initial value problem $\Omega = \kappa z + \lambda$, $b(z_0) = b_0$, $b'(z_0) = b_1$, $b''(z_0) = b_2$ has a unique local solution. Assume

$$b(z) = \sum_{\nu=0}^{\infty} c_\nu (z - z_0)^{\nu/q}$$

and let n denote the smallest integer, if any, such that $c_n \neq 0$ and $n \not\equiv 0 \bmod q$; call this n *index* of b. Then b''' has index $n - 3q$, while the indices of bb'', b'^2, and $b^2 b'$ are $\geq n - 2q$, which is not possible. Also at $z = \infty$,

$$-b^2 b' + o(|b^2 b'|) = \kappa z + \lambda \quad (z \to \infty)$$

holds, and again we obtain a contradiction since $|\kappa| + |\lambda| > 0$ was assumed. This finishes the proof of the proposition. □

To finish the proof of Theorem 4 we just note that for any (local) solution to the Hamiltonian systems $x' = H_y$, $y' = -H_x$ resp. $u' = K_v$, $v' = -K_u$,

$$w = y + \tfrac{1}{12} b^2, \;\; w = y, \;\; w = y + \tfrac{7}{2} z^{-2} \quad \text{resp.} \quad v$$

solve Painlevé's first equation $w'' = \alpha(z) + 6w^2$ resp. $v'' = \alpha(z) + 6v^2$ (with the very same α). Thus y and v are meromorphic in $\mathbb{C}$, and so are $u = -v' - bv$ and $x = -y' - by$, hence H has the Malmquist (and Painlevé) property.

7 Proof of Theorem 5

Starting with the Hamiltonian system

$$\begin{aligned} x' &= x^2 - y + bx + d \\ y' &= -2xy - 2ax - by - c \end{aligned}$$

we obtain

$$\begin{aligned} u' &= \ u^2 - v + a + d + \tfrac{1}{2}b' - \tfrac{1}{4}b^2 \\ v' &= -2uv + ab + a' - c \end{aligned} \tag{28}$$

by the transformation

$$u = x + \tfrac{1}{2}b, \ \ v = y + a.$$

Then (14) and (15) with coefficients (16) are obtained in the usual way from (28): differentiate the first and second equation and replace the variable v and u with the help of the second and first equation, respectively. To proceed we note that any algebroid solution to (14) has infinitely many poles, almost all of them simple with residues ± 1. The latter follows by inspection, while the former stems from the dominating term $2u^3$, which immediately implies

$$m(r, u) = O(\log(rT(r, u))).$$

Proposition *Suppose the differential equation (14) has some transcendental algebroid solution having infinitely many poles with residue $\epsilon \in \{-1, 1\}$. Then*

$$\alpha' = \tfrac{1}{2}\epsilon\beta'' \tag{29}$$

holds, and

$$\alpha' = \beta'' \equiv 0 \tag{30}$$

if u has infinitely many poles also with residue $-\epsilon$. Otherwise u satisfies

$$u' = -\tfrac{1}{2}\epsilon\beta - \epsilon u^2 \quad \text{and} \quad \alpha = \tfrac{1}{2}\epsilon\beta'. \tag{31}$$

Proof Let p be a pole of u with residue ϵ and assume that α and β are regular at $z = p$. Then

$$u(z) = \epsilon(z-p)^{-1} - \tfrac{1}{6}\epsilon\beta(p)(z-p) - \tfrac{1}{2}(\alpha(p) + \epsilon\beta'(p))(z-p)^2 + c_3(z-p)^3 + \cdots$$

and

$$u'' - \alpha(z) - \beta(z)u - 2u^3 = -\tfrac{1}{2}(2\alpha'(p) - \epsilon\beta''(p))(z-p) + O(|z-p|^2)$$

hold. Thus

$$u'' - \alpha(z) - \beta(z)u - 2u^3, \quad 2\alpha' - \epsilon\beta'' \quad \text{and} \quad u' + \epsilon u^2 + \tfrac{1}{2}\epsilon\beta$$

vanish at $z = p$. If u has infinitely many poles with residue ϵ, (29) follows at once, and also (30) if u in addition has infinitely many poles with residue $-\epsilon$. If, however, only finitely many poles with residue $-\epsilon$ exist,

$$\Omega = u' + \epsilon u^2 + \tfrac{1}{2}\epsilon\beta$$

vanishes identically since Ω has at most finitely many poles, vanishes at almost each pole of u with residue ϵ, and has characteristic

$$T(r, \Omega) \leq 2m(r, u) + O(\log r) = O(\log(rT(r, u)))$$

as $r \to \infty$, possibly outside some set of finite measure. □

8 Proof of Theorem 6

Hamiltonians $H(z, x, y)$ and

$$K(z, u, v) = kH(z, au + bv, cu + dv)$$

with $a, b, c, d, k \in \mathbb{C}$ and $k(ad - bc) \neq 0$ simultaneously have or fail to have the Malmquist resp. Painlevé property. Choosing $a = d = 1$, $b = c = \lambda = e^{2\pi i/3}$, and $k = -1/3$ we obtain

$$K(z, u, v) = -\tfrac{1}{3}(\lambda a + b)u - \tfrac{1}{3}(a + \lambda b)v - \tfrac{1}{3}\lambda c(u^2 - uv + v^2) + \tfrac{1}{3}u^3 + \tfrac{1}{3}v^3$$

and

$$\begin{aligned} u' &= -\tfrac{1}{3}(a + \lambda b) - \tfrac{2}{3}\lambda cv + \tfrac{1}{3}cu + v^2 \\ v' &= \ \tfrac{1}{3}(\lambda a + b) + \tfrac{2}{3}\lambda cu - \tfrac{1}{3}cv - u^2. \end{aligned} \tag{32}$$

By $u = U + \lambda c/3$ and $v = V + \lambda c/3$, system (32) is transformed into

$$\begin{aligned} U' &= \ A + CU + V^2 \\ V' &= -B - CV - U^2 \end{aligned} \tag{33}$$

with

$$A = -\tfrac{1}{3}(\lambda a + b + \lambda c'), \quad B = -\tfrac{1}{3}(a + \lambda b - \lambda c'), \quad \text{and} \quad C = \tfrac{1}{3}\lambda c.$$

Again (32) and (33) simultaneously have or fail to have the Malmquist resp. Painlevé property. By Theorem 1, the latter holds for (33) if and only if

$$C'' = B' = A' \equiv 0,$$

that is, if and only if $c'' = b' = a' \equiv 0$. And so the circle is complete, since by Theorems 1 and 2 this is true except in one particular case, namely when

$$U + V - C = C' + B - A \equiv 0.$$

It is easily seen that this is equivalent with $x + y - c = c' + b - a \equiv 0$.

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Appendix: Algebroid Functions and the Selberg–Valiron Theory

For the convenience of the reader we will give a short overview of the Selberg–Valiron theory. Let

$$P(z, w) = \sum_{\kappa=0}^{k} A_\kappa(z) w^\kappa \quad (A_k(z) \not\equiv 0)$$

be any irreducible polynomial in w over the ring of entire functions. Then the solutions $w = f_\kappa(z)$ $(1 \le \kappa \le k)$ to the equation $P(z, w) = 0$ admit unrestricted analytic continuation into $\mathbb{C} \setminus S_P$, where S_P denotes the set of singularities; it consists of the zeros of A_k and the discriminant of P w.r.t. w. The singularities (including poles) are algebraic; ordinary poles will not be viewed as singularities. The branches f_κ form the *algebroid function*

$$\mathfrak{f} = \{f_1, \ldots, f_k\}.$$

For algebroid functions, Selberg and Valiron independently developed an analogous Nevanlinna theory as follows (see, for example [8]):

$$m(r, \mathfrak{f}) = \frac{1}{2k\pi} \sum_{\kappa=1}^{k} \int_0^{2\pi} \log^+ |f_\kappa(re^{i\theta})|\, d\theta,\ \ N(r, \mathfrak{f}) = \frac{1}{k} N(r, 1/A_k)$$

and

$$T_S(r, \mathfrak{f}) = m(r, \mathfrak{f}) + N(r, \mathfrak{f})$$

denote the *proximity function*, the *counting function* of poles, and the *Selberg characteristic* of $\mathfrak{f}$, respectively. Up to a bounded term the latter coincides with the *Valiron characteristic*

$$T_V(r, \mathfrak{f}) = \frac{1}{2k\pi} \int_0^{2\pi} U(re^{i\theta})\, d\theta,$$

where

$$\begin{aligned} U(z) = & \frac{1}{2\pi} \int_0^{2\pi} \log |P(z, e^{i\phi})|\, d\phi \\ = & \log |A_k(z)| + \sum_{\kappa=1}^{k} \log^+ |f_\kappa(z)| \\ = & \max_{1\le\kappa\le k} \log |A_\kappa(z)| + O(1); \end{aligned}$$

the definition of T_V is similar to the *Ahlfors–Shimizu formula* in ordinary Nevanlinna theory (see, for example, Hayman's monograph [2]). We will not distinguish between both characteristics and just write $T(r, \mathfrak{f})$. The *First Main Theorem*

$$T\Big(r, \frac{1}{\mathfrak{f} - c}\Big) = T(r, \mathfrak{f}) + O(1)$$

follows from $T_V(r, 1/\mathfrak{f}) = T_V(r, \mathfrak{f})$ (based on $w^k P(z, 1/w) = \sum_{\kappa=0}^{k} A_{k-\kappa}(z) w^\kappa$) combined with $T_S(r, \mathfrak{f} - c) = T_S(r, \mathfrak{f}) + O(1)$ (based on $||f_\kappa| - |f_\kappa - c|| \le |c|$). Algebraic functions have characteristic $T(r, \mathfrak{f}) = O(\log r)$ as $r \to \infty$, while for transcendental (non-algebraic) algebroid functions $\log r = o(T(r, \mathfrak{f}))$ holds. The fundamental result in Nevanlinna theory, the *Lemma on the proximity function of the logarithmic derivative* remains valid ($\mathfrak{f}'/\mathfrak{f}$ has branches f_κ'/f_κ):

$$m(r, \mathfrak{f}'/\mathfrak{f}) = O(\log(rT(r, \mathfrak{f}))) \quad (r \to \infty)$$

holds (possibly) outside some set $E \subset (0, \infty)$ of finite measure.

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