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A finite element formulation for a simplified, relaxed micromorphic continuum model

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We discuss a simplified problem derived from the relaxed micromorphic continuum model in two dimensions. The model captures important aspects of the micromorphic approach even as a degeneration of the bulk model. Typically, the employed mechanical strain combines the gradient of displacements with the microdistortion field. The interaction between both fields is ruled by the minimization of the overall free energy, where we employ the Curl of the microdistortion. The Curl significantly influences the resulting equations for the balance of linear and angular momentum. Further, we explain the necessity of an extended finite element method. Finite elements based on solely the H^1-Hilbert space are not sufficient for the efficient approximation of the Curl based microdistortion. Therefore, we suggest using a hybrid scheme employing both, H^1 and $H(\mathrm{Curl})$ based functions. The resulting hybrid element formulation is successfully tested for a problem with a predefined Dirichlet boundary condition.

1 The simplified relaxed micromorphic continuum

In classical continuum models each material point can undergo translation in three directions, i.e. has three degrees of freedom. The relaxed micromorphic continuum model extends this kinematic, such that each material point can also rotate and deform with a total of 12 degrees of freedom. This characteristic translates to more complex mechanical behaviours, commonly present in metamaterials. In the relaxed micromorphic continuum [1, 2] the free energy functional takes the form

$$I(\mathbf{u},\boldsymbol{P}) = \frac{1}{2}\int_\Omega \langle \mathbb{C}_\mathrm{e}\,\mathrm{sym}(\nabla\mathbf{u}-\boldsymbol{P}),\,\mathrm{sym}(\nabla\mathbf{u}-\boldsymbol{P})\rangle + \langle \mathbb{C}_\mathrm{micro}\,\mathrm{sym}\,\boldsymbol{P},\,\mathrm{sym}\,\boldsymbol{P}\rangle$$
$$+ \langle \mathbb{C}_\mathrm{c}\,\mathrm{skew}(\nabla\mathbf{u}-\boldsymbol{P}),\,\mathrm{skew}(\nabla\mathbf{u}-\boldsymbol{P})\rangle + \mu_\mathrm{macro}\,L_\mathrm{c}^2\,\|\mathrm{Curl}\boldsymbol{P}\|^2 - \langle \mathbf{f},\mathbf{u}\rangle - \langle \boldsymbol{M},\boldsymbol{P}\rangle\,\mathrm{d}X\,,\quad \Omega\subset\mathbb{R}^3\,,$$

$$\nabla\mathbf{u} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3}\\ u_{2,1} & u_{2,2} & u_{2,3}\\ u_{3,1} & u_{3,2} & u_{3,3}\end{bmatrix}\,,\quad \mathrm{Curl}\boldsymbol{P} = \begin{bmatrix} \mathrm{curl}\begin{bmatrix}P_{1,1} & P_{1,2} & P_{1,3}\end{bmatrix}^T\\ \mathrm{curl}\begin{bmatrix}P_{2,1} & P_{2,2} & P_{2,3}\end{bmatrix}^T\\ \mathrm{curl}\begin{bmatrix}P_{3,1} & P_{3,2} & P_{3,3}\end{bmatrix}^T\end{bmatrix}\,,\quad \mathrm{curl}\,\mathbf{v} = \nabla\times\mathbf{v}\,, \tag{1}$$

with $\mathbf{u}:\Omega\subset\mathbb{R}^3\to\mathbb{R}^3$ and $\boldsymbol{P}:\Omega\subset\mathbb{R}^3\to\mathbb{R}^{3\times 3}$ being the displacement and microdistortion, respectively. External loading is denoted by $\mathbf{f}$ and $\boldsymbol{M}$. Further, $\mathbb{C}_\mathrm{e}$ and $\mathbb{C}_\mathrm{micro}$ are standard elasticity tensors and $\mathbb{C}_\mathrm{c}$ is a positive semi-definite rotational coupling tensor. μ_macro is a typical shear modulus and $L_\mathrm{c}>0$ is the characteristic length scale.

For our formulations we consider a simplified 2D version of the relaxed micromorphic continuum. In this model the elastic free energy functional reads

$$I(u,\boldsymbol{\zeta}) = \int_\Omega \mu_\mathrm{e}||\nabla u-\boldsymbol{\zeta}||^2 + \mu_\mathrm{micro}||\boldsymbol{\zeta}||^2 + \mu_\mathrm{macro}\frac{L_\mathrm{c}^2}{2}(\mathrm{curl}\,\boldsymbol{\zeta})^2 - \langle f,u\rangle - \langle \mathbf{m},\boldsymbol{\zeta}\rangle\,\mathrm{d}X\,,\qquad \begin{array}{c} u:\Omega\subset\mathbb{R}^2\to\mathbb{R}\,,\\ \boldsymbol{\zeta}:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2\,,\\ \mu_\mathrm{e},\,\mu_\mathrm{micro},\,\mu_\mathrm{macro} > 0\,,\end{array} \tag{2}$$

where $f:\Omega\subset\mathbb{R}^2\to\mathbb{R}$ and $\mathbf{m}:\Omega\subset\mathbb{R}^2\to\mathbb{R}^2$ are the body forces and couple moments, respectively. We define $u=u(x,y)$ as a displacement field perpendicular to the x,y plane, and $\boldsymbol{\zeta}=\boldsymbol{\zeta}(x,y)$ as the microdistortion. The curl operator in 2D is given by $\mathrm{curl}\,\boldsymbol{\zeta} := \zeta_{2,1}-\zeta_{1,2}\,,\ \boldsymbol{\zeta}\in\mathbb{R}^2$. From Eq. (2) the variational principle yields

$$\forall\,\delta u:\quad \int_\Omega 2\mu_\mathrm{e}\langle\nabla u-\boldsymbol{\zeta},\nabla\delta u\rangle - \langle f,\delta u\rangle\,\mathrm{d}X = 0\,, \tag{3}$$

$$\forall\,\delta\boldsymbol{\zeta}:\quad \int_\Omega 2\mu_\mathrm{e}\langle\nabla u-\boldsymbol{\zeta},-\delta\boldsymbol{\zeta}\rangle + 2\mu_\mathrm{micro}\langle\boldsymbol{\zeta},\delta\boldsymbol{\zeta}\rangle + \mu_\mathrm{macro}\,L_\mathrm{c}^2\langle\mathrm{curl}\,\boldsymbol{\zeta},\mathrm{curl}\,\delta\boldsymbol{\zeta}\rangle - \langle\mathbf{m},\delta\boldsymbol{\zeta}\rangle\,\mathrm{d}X = 0\,. \tag{4}$$

From Eq. (3) and Eq. (4) we obtain the weak form for further numerical investigation

$$\int_\Omega 2\mu_\mathrm{e}\langle\nabla u-\boldsymbol{\zeta},\nabla\delta u-\delta\boldsymbol{\zeta}\rangle + 2\mu_\mathrm{micro}\langle\boldsymbol{\zeta},\delta\boldsymbol{\zeta}\rangle + \mu_\mathrm{macro}\,L_\mathrm{c}^2\langle\mathrm{curl}\,\boldsymbol{\zeta},\mathrm{curl}\,\delta\boldsymbol{\zeta}\rangle - \langle f,\delta u\rangle - \langle\mathbf{m},\delta\boldsymbol{\zeta}\rangle\,\mathrm{d}X = 0\,. \tag{5}$$

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Existence and uniqueness follows from the Lax-Milgram theorem in the combined space $H^1(\Omega) \times H(\mathrm{curl}, \Omega)$.

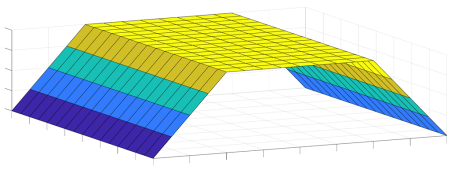
2 Numerical examples and conclusions

For the finite element formulation we employ $H^1(\Omega)$ nodal base functions for u and Nédélec $H(\mathrm{curl}, \Omega)$ edge base functions [3, 4] for $\boldsymbol{\zeta}$. For comparison, we also derive a purely $H^1(\Omega) \times H^1(\Omega)$ element. We set $\Omega = [-4, 4] \times [-4, 4] \subset \mathbb{R}^2$ and impose the lift and material constants

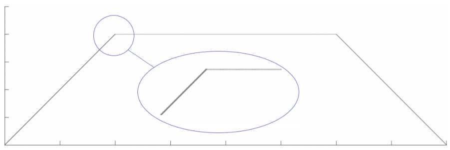
$$u(-4, y) = u(4, y) = 0\,, u(-2, y) = u(2, y) = 2\,, \quad \begin{matrix} \mu_{\mathrm{e}},\, \mu_{\mathrm{micro}}, \\ \mu_{\mathrm{macro}}\,, L_{\mathrm{c}} \end{matrix} = 1\,,$$

for which the analytical solution reads

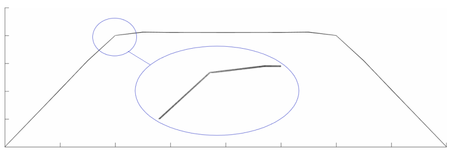
$$\widetilde{u} = \begin{cases} 4 + x & \text{for} \quad -4 \leq x < -2 \\ 2 & \text{for} \quad -2 \leq x < 2 \\ 4 - x & \text{for} \quad 2 \leq x \leq 4 \end{cases}, \widetilde{\boldsymbol{\zeta}} = \begin{cases} \begin{bmatrix} 0.5 & 0 \end{bmatrix}^T & \text{for} \quad -4 \leq x < -2 \\ \begin{bmatrix} 0 & 0 \end{bmatrix}^T & \text{for} \quad -2 \leq x < 2 \\ \begin{bmatrix} -0.5 & 0 \end{bmatrix}^T & \text{for} \quad 2 \leq x \leq 4 \end{cases}.$$



(a) Analytical solution



(b) Front view with 144 $H^1 \times H(\mathrm{curl})$ elements



(c) Front view with 144 $H^1 \times H^1$ elements

Fig. 1: Solutions of the displacement $u(x, y)$.

The microdistortion field takes in fact the form $\widetilde{\boldsymbol{\zeta}} = \nabla \widetilde{u}/2$. The solution is $C^0(\Omega)$ continuous. Thus, its gradient is discontinuous. The microdistortion $\boldsymbol{\zeta}$ formulated using $H^1(\Omega) \times H^1(\Omega)$ requires continuity across both its tangential and normal components and is therefore incapable of correctly capturing the discontinuity of the analytical solution, see Fig. 1c. However, it is capable of approximating the correct solution and further localize the error via mesh refinement, see Fig. 2. The element formulation using $H^1(\Omega) \times H(\mathrm{curl}, \Omega)$ requires only tangential continuity across element boundaries and finds the exact solution directly as seen in Fig. 1b. Its errors in both u and $\boldsymbol{\zeta}$ are not listed in Fig. 2 as they are always at a factor of 10^{-14}.

In conclusion, we recognize that the $H^1 \times H(\mathrm{curl})$ element reproduces discontinuous gradients of the displacement field without difficulty. Thus, the finite element formulation combining both H^1 and $H(\mathrm{curl})$ Hilbert spaces can represent folds in the displacement field of the relaxed micromorphic theory, as shown in our numerical example. As the relaxed micromorphic theory aims to capture more complex mechanical behaviour from e.g. micro effects, the approach using Nédélec finite elements is promising.

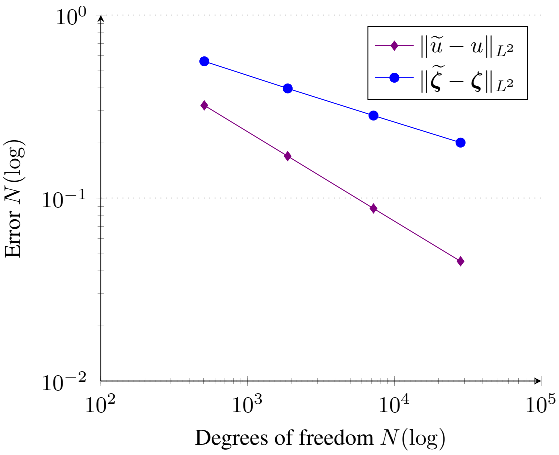


Fig. 2: Convergence of the $H^1 \times H^1$ element in the Lebesgue norm over the domain.

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