Dissertation

Dimension reduction for elastoplastic rods and homogenization of elastoplastic lattices

Dimension reduction for elastoplastic rods and homogenization of elastoplastic lattices

Dissertation

zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften
(Dr. rer. nat.)

vorgelegt
der Fakultät für Mathematik
der Technischen Universität Dortmund

von

Klaas Hendrik Poelstra

Dortmund, Februar 2021

Dissertation

*Dimension reduction for elastoplastic rods
and homogenization of elastoplastic lattices*

Fakultät für Mathematik
Technische Universität Dortmund

Erstgutachter: Prof. Dr. Ben Schweizer

Zweitgutachter: Prof. Dr. Alexander Mielke

Tag der mündlichen Prüfung: 7. Juni 2021

Acknowledgements

I would like to express my deepest gratitude to my advisor Ben Schweizer for his constant encouragement, his optimism and guidance, and for all the mathematics I would not know without him.

I am also very grateful to my fellow graduate students, notably Maik Urban and Nils Dabrock, for many helpful discussions.

Moreover, I am deeply indebted to my parents for their unwavering support.

Contents

List of Symbols

$*$ denotes entries of a symmetric matrix which can be inferred from the symmetry

$\square$ rescaled periodicity cell $(0,1)^3$

$\square_k^\varepsilon$ periodicity cell $\varepsilon(k + (0,1)^3)$ for $k \in \mathbb{Z}^3$

$\mathbb{C}$ preferred symbol for an elasticity tensor

$\mathcal{B}$ preferred symbol for a stored energy functional

$\mathcal{E}$ preferred symbol for a total energy functional

$\mathcal{H}^2$ two-dimensional Hausdorff measure

$\mathcal{Q}$ preferred symbol for a state space

$\mathcal{R}$ preferred symbol for a dissipation functional

$\mathrm{Diss}_{\mathcal{R}}(q;[s,t])$ total dissipation of q on the interval $[s,t]$ w.r.t. a dissipation functional $\mathcal{R}$

ℓ preferred symbol for mechanical loads

$\lfloor x \rfloor$ largest integer smaller than or equal to $x \in \mathbb{R}$; when x is a vector, it is applied element-wise

$\mathrm{grad}(\alpha, \beta; G)$ limiting graph gradient

$\mathrm{grad}^\varepsilon(\beta; G^\varepsilon)$ graph gradient of a G^ε-node function β

$\mathcal{D}'(\Omega)$ space of distributions on Ω

$\nabla_{2,3}^s$ symmetric gradient of an $\mathbb{R}^2$-valued function containing partial derivatives along the second and third axis

$\nabla^s u$	symmetric gradient of a displacement field u, also denoted ϵ
$\Omega^\varepsilon(G^\varepsilon)$	union of all ε-cells which contain at least one node
$\Omega^\varepsilon_e(G^\varepsilon)$	union of all ε-cells which contain the edge e
$\Omega^\varepsilon_v(G^\varepsilon)$	union of all ε-cells which contain the node v
$\mathbb{1}^\varepsilon(G^\varepsilon)$	characteristic function of $\Omega^\varepsilon(G^\varepsilon)$
$\mathbb{1}^\varepsilon_e(G^\varepsilon)$	characteristic function of $\Omega^\varepsilon_e(G^\varepsilon)$
$\mathbb{1}^\varepsilon_v(G^\varepsilon)$	characteristic function of $\Omega^\varepsilon_v(G^\varepsilon)$
∂	subdifferential or partial derivative
$\Pi_{i\in I} X_i$	Cartesian product of X_i for $i \in I$
$\mathbb{R}^{3\times 3}_{\mathrm{dev}}$	space of symmetric 3×3 matrices with zero trace
$\mathbb{R}_\infty$	$\mathbb{R} \cup \{+\infty\}$
$\mathbb{R}^{3\times 3}_{\mathrm{sym}}$	space of symmetric 3×3 matrices
$\langle \cdot, \cdot \rangle$	scalar product or dual pairing
$\overset{\mathcal{M}}{\to}$	Mosco-convergence
$\overset{\Gamma}{\to}$	Γ-convergence
$\overset{\Gamma}{\rightharpoonup}$	Γ-convergence w.r.t. weak convergence
$\Subset$	compact containment: $A \Subset B$ means that A is bounded and its closure is a subset of the interior of B
θ	asymptotic thickness parameter, limit of $h(\varepsilon)/\varepsilon$ as $\varepsilon \to 0$
$\mathbb{W}$	preferred symbol for a stored energy density
$\rightharpoonup$	weak convergence in Banach spaces
B_e	rescaled cross section of a rod e
$E(G)$	set of edges of a graph G
$H^1_\Gamma(\Omega)$	set of all $f \in H^1(\Omega)$ with $f = 0$ on Γ in the sense of traces
$L(e)$	rescaled length of a rod e

p	preferred symbol for a plastic strain tensor
P^ε	projection onto the space of functions which are constant on each cell $\square_k^\varepsilon$
q	preferred symbol for a state
R	preferred symbol for a dissipation potential
$r(e)$	unit vector indicating the direction of an edge e
S_h	scaling matrix $\mathrm{diag}(1, h, h) \in \mathbb{R}^{3\times 3}$
T_v	translation operator, mapping a function $x \mapsto f(x)$ to $x \mapsto f(x) + v$
$V(G)$	set of nodes (vertices) of a graph G
$v \otimes w$	the matrix $(v_i w_j)_{ij}$
$v_1(e), v_2(e)$	first and second node of an edge e

Chapter 1

Introduction

The object of this work is the derivation of effective equations for periodic frameworks of elastoplastic material in the limit of both infinitesimal periodicity and infinitesimal relative width of the rods of which the framework is composed.

The derivation of effective equations for heterogeneous materials with a fine mixing of different constituents is commonly known as *homogenization*. Such materials may occur in nature or be engineered with the specific aim of obtaining certain effective material properties which may be impossible or difficult to obtain in a homogeneous material. One can describe homogenization as the process of averaging the oscillatory coefficients that describe a heterogeneous material. The key problem is to find the correct notion of averaging.

In engineered materials, but also in nature (e.g. in crystals), the mixing of the constituents often follows a periodic pattern. The study of periodic microstructures is known as *periodic homogenization*. The other direction of study is *stochastic homogenization*, where the coefficients are random variables which satisfy a stationarity and ergodicity assumption. Qualitative results were first obtained by Kozvlov [32], Papanicolaou and Varadhan [50] who studied heat conduction in randomly inhomogeneous media. More recently, quantitative estimates for the approximation error in the homogenization were obtained by Gloria, Otto and Neukamm [24, 25, 23]. We focus on periodic homogenization. Here, an important tool is the notion of *two-scale convergence* which was first proposed by Nguetseng [49] and further developed and popularized by Allaire [5].

We combine homogenization with *dimension reduction*. In dimension reduction one is concerned with the derivation of equations for lower-dimensional objects such as rods, beams, plates and shells from bulk material models. In the realm of finite strains, the seminal work by Friesecke, James and Müller

[22] enabled much progress. The authors provide the famous quantitative rigidity estimate which they use to rigorously derive a plate model for nonlinear elasticity. See also [46] for a model for rods. The rigidity result of [22] gives an estimate for the L^2-distance of deformation gradients to the space of orthogonal matrices in terms of the distance to a single (optimal) orthogonal matrix. We will work in the realm of infinitesimal strains. Therefore a much simpler rigidity estimate is sufficient, the *Korn inequality*, which provides an estimate for the symmetric gradient of displacement fields.

Many of the results in homogenization and dimension reduction are based on the notion of Γ-convergence invented by DeGiorgi [18]. This notion for the convergence of functionals allows rigorous statements about the convergence of material behaviour for materials which are modeled by energy functionals.

For the modeling of plasticity we refer to [3, 26, 19, 54]. We will consider a simple model of elastoplasticity with linear kinematic hardening. Hardening prevents the concentration of displacements. In the absence of hardening (known as *perfect plasticity*), no H^1-estimates are available and one must resort to the space of bounded deformations in which displacement fields u are such that the strain $\epsilon = (\nabla u + \nabla u^T)/2$ is only a measure. This problem was analyzed by Dal Maso, DeSimone and Mora in [17] with groundwork laid by Temam and Kohn in [57, 30]. As we will incorporate hardening into our models, we can work in the classical Hilbert-Sobolev spaces.

Moreover, we work under the assumption of rate-indepence [40]. This means in particular that all processes are quasi-static, inertia terms are neglected. In addition to that, no viscous effects are considered. The system is assumed to be constantly in an equilibrium, an assumption which can be justified when the input of the system evolves slowly compared to the internal restructuring processes. We may view elastoplasticity as a system that maps an input function $\ell : [t_1, t_2] \to \mathcal{L}$ to an output function $q : [t_1, t_2] \to \mathcal{Q}$. Here $\mathcal{Q}$ is the state space of the system, containing for example displacement fields. The input space $\mathcal{L}$ might consist of loads and boundary conditions. Rate-independence manifests itself in the property that q is a solution to the input ℓ if and only if $q \circ \phi$ is a solution to the input $\ell \circ \phi$ for every strictly monotone reparametrization $\phi : [t'_1, t'_2] \to [t_1, t_2]$.

Informal overview

Dimension reduction for elastoplastic rods. When we consider thin rods $\Omega_h = [0, L] \times hB$, with displacement fields $u^h : \Omega_h \to \mathbb{R}^3$, a natural question is how to represent limits of sequences $(u^h)_h$ as $h \to 0$. In particular, one could ask wether sufficiently many features of such limit sequences can be captured in a one-dimensional displacement field $v : [0, L] \to \mathbb{R}^3$.

Leaving plasticity aside for a moment, considering pure elasticity, we note that a rod is fully described by its elastic energy functional, which associates an energy $\mathcal{B}^h(u^h)$ to every displacement field u^h. Certainly, stretching a thin rod costs less elastic energy than stretching a thicker rod by the same amount. The same qualitative statement is true for bending. In other words: When the displacements u^h and the elastic energies $\mathcal{B}^h(u^h)$ converge simulteneously as $h \to 0$, the limit energy will always be zero. We therefore introduce scalings. Specifically, we consider $h^{-6}\mathcal{B}^h$ instead of $\mathcal{B}^h$. But we also rescale the displacements and consider $(h^{-2}u_1^h, h^{-1}u_2^h, h^{-1}u_3^h)$ instead of u^h. The precise choice of the exponents may not be clear at this point. But it can be seen that with this scaling of u^h, we commit ourselves to the study of sequences of displacements where the first component (stretching) is of order h^2, and thus asymptotically smaller than the other two components (bending) which are of order h. It turns out that with this scaling, both stretching and bending contribute to the elastic energy at the same order, namely h^6.

In this scaling, limit displacements can indeed be characterized by one-dimensional displacement fields $v : [0, L] \to \mathbb{R}^3$. Yet some geometric information is lost in the process: The field v is for example not capable of capturing any torsion of the rod, even though such torsion may contribute to the energy. It thus features in the limit energy as a quantity over which the energy is infimized: the rod always relaxes to an energetically optimal state of torsion. Besides torsion, there are two other correcting terms which contribute in this way to the complexity of the limit model (see Figure 4.2 on Page 45 for an illustration).

When we turn to elastoplasticity, the energy also depends on the plastic strain tensor $p^h : \Omega_h \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. This tensor is also rescaled, but it remains genuinely three-dimensional in the limit.

Homogenization of elastoplastic lattices. When we want to study periodic lattices, we first need to describe a single periodicity cell. This cell contains all the joints in the cell as nodes, while rods are represented as edges. There may even exist multiple edges between the same pair of nodes, because the edges may reach for different copies of the target node located in different neighboring cells, see Figure 1.1(a). This periodicity graph can then be "unrolled" to form an infinite periodic graph as depticted in Figure 1.1(b) and (c).

We want to fill a macroscopic domain $\Omega \subset \mathbb{R}^3$ with lattice-material of periodicity ε. Thus we need a reasonable method for taking sections from the infinite periodic graph. In particular, we do not want to end up having loose edges or other subcomponents with floppy modes. The overall structure should possess some form of *rigidity*. This can be achieved by using rigid components, which are usually somewhat larger than a single periodicity cell, as building blocks (see Figure 5.4 on Page 62).

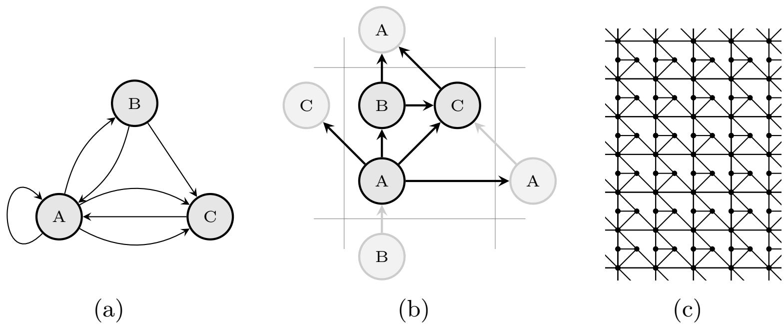


Figure 1.1: A periodicity graph (a) is equipped with node positions and edge cell-offsets and thus "unfolds" to an infinite periodic graph. Figure (b) shows the immediate neighborhood of one periodicity cell, while (c) shows a larger section from the periodic graph. For visual clarity, the example in this figure is in two dimensions.

We then pose the equations of elastoplasticity on each of the edges. In particular, we define an energy functional for the whole system, which is a sum over the energies of all the edges. Each edge has, after a rigid motion, an associated domain of the form $\varepsilon([0, L] \times hB)$. We thus have two infinitesimal parameters: The periodicity $\varepsilon \to 0$, and a relative width $h = h(\varepsilon) \to 0$.

Of course, the equations for the different edges are coupled by compatibility conditions at the nodes. These are encoded by postulating for each node a displacement vector and an infinitesimal rotation matrix.

After proper rescaling, limits can be considered. Let us, for the purpose of this introduction, focus on one particular quantity: the displacement vectors for the nodes. In what sense can they be said to converge? For given ε, let $u^\varepsilon_{k,v} \in \mathbb{R}^3$ denote the displacement of node v from cell $k \in \mathbb{Z}^3$. We then identify $(u^\varepsilon_{k,v})_k$ with the associated piecewise constant interpolation $u^\varepsilon_v : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $u^\varepsilon_v = u^\varepsilon_{k,v}$ on $\varepsilon(k + (0, 1)^3)$. We can require ordinary L^2-convergence for u^ε_v. However, it seems unnatural to consider the sequences $(u^\varepsilon_v)_\varepsilon$ for different node types v separately. There are rods between nodes of different type, and these tend to keep the corresponding displacements close together. We therefore use a two-scale ansatz and write $u^\varepsilon + \varepsilon\xi^\varepsilon_v$ instead of u^ε_v. Here u^ε is the average node displacement in a cell, and ξ^ε_v is the ε-order deviation of node v. It turns out that along sequences of bounded energy, both u^ε and ξ^ε_v are bounded in L^2. Moreover, weak limits u of u^ε are in $H^1(\Omega; \mathbb{R}^3)$. This limit quantity $u : \Omega \to \mathbb{R}^3$ is the primary unknown of the limit model: It is the displacement field of the homogenized material. We refer to the next paragraph for more details on the structure of the limit model.

It remains here to notice the dependency of the limit model on the rate at which the relative width $h(\varepsilon)$ converges to zero as $\varepsilon \to 0$. It turns out that the rate is relevant when volume loads are considered. When the rods are sufficiently thin, $h(\varepsilon)/\varepsilon \to 0$, then the volume loads will only affect the local oscillations. For example, in the presence of gravitation, all non-vertical rods will be sagging. On the other hand, when the rods are sufficiently thick, $h(\varepsilon)/\varepsilon \to \infty$, then the volume loads will only affect the macroscopic displacement field $u : \Omega \to \mathbb{R}^3$. The most interesting case is when $h \sim \varepsilon$. Then both effects are present. In the literature, this case is referred to as *critical thickness* [59, 60].

Outline and main results

The main results of this work are developed in Chapters 4–6. Chapters 2 and 3 are preparatory.

We start in Chapter 2 by introducing the equations for linearized elasto-plasticity with kinematic hardening. We study the rate-independent case and introduce the notion of rate-independent systems and their energetic solutions as developed by Mielke [40]. For the convenience of the reader we give a proof for the by now classical existence and uniqueness result for quadratic rate-independent systems (Theorem 2.1).

An energetic rate-independent system is a triple $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$, where $\mathcal{Q}$ is the state space. In the case of elastoplasticity, the states are of the form $q = (u, p)$, where u is a displacement field and p a plastic strain tensor. The functional $\mathcal{E} = \mathcal{E}(t, q)$ is the total energy. It is time-dependent because it includes the time-dependent loads. The positive one-homogeneous functional $\mathcal{R} = \mathcal{R}(q) = \mathcal{R}(p)$ is the dissipation functional. An evolution $q : [0, T] \to \mathcal{Q}$ is considered to be a solution of that system when

$$0 \in \partial\mathcal{R}(\partial_t q(t)) + \mathrm{D}_q\mathcal{E}(t, q(t))\,.$$

This differential inclusion can also be split into a balance of forces $0 = \mathrm{D}_u\mathcal{E}(t, q(t))$ and the plastic flow rule $0 \in \partial_p\mathcal{R}(\partial_t p(t)) + \mathrm{D}_p\mathcal{E}(t, q(t))$. When $\mathcal{E}(t, \cdot)$ is convex, this is equivalent to the conditions

$$\begin{aligned}\mathcal{E}(t, q(t)) \leq \mathcal{E}(t, q(t) + \overline{q}) + \mathcal{R}(q') \quad \text{for all } q' \in \mathcal{Q}\,,\\ \mathcal{E}(t, q(t)) + \int_0^t \mathcal{R}(\partial_s q(s))\mathrm{d}s = \mathcal{E}(0, q(0)) + \int_0^t \partial_s\mathcal{E}(s, q(s))\mathrm{d}s\,.\end{aligned}$$

These are referred to als *stability* and *energy equality.* Together they consitute the definition of energetic solutions.

In Chapter 3 we cite a result on evolutionary Γ-convergence for rate-indepentent systems with quadratic energies [42]. We give a proof of this theorem (Theorem 3.4) which is somewhat simplified compared to what is found in [42]. The theorem provides conditions under which sequences of solutions q^ε of rate-independent systems $(\mathcal{Q}, \mathcal{E}^\varepsilon, \mathcal{R}^\varepsilon)$ converge to solutions q^0 of a rate-independent system $(\mathcal{Q}, \mathcal{E}^0, \mathcal{R}^0)$. It turns out that the main requirements are the Γ-convergence of $\mathcal{E}^\varepsilon$ to $\mathcal{E}^0$ and of $\mathcal{R}^\varepsilon$ to $\mathcal{R}^0$. More precisely, one needs Γ-convergence with respect to both the weak and the strong convergence in the state space $\mathcal{Q}$, a mode of convergence which is called *Mosco-convergence.* The quadratic nature of the energies $\mathcal{E}^\varepsilon$ and the further assumption that *all* strongly converging sequences are recovery sequences for $\mathcal{R}^\varepsilon$ then allow the construction of so-called *mutual recovery sequences* which enable the proof that solutions of $(\mathcal{Q}, \mathcal{E}^\varepsilon, \mathcal{R}^\varepsilon)$ converge to solutions of $(\mathcal{Q}, \mathcal{E}^0, \mathcal{R}^0)$.

Theorem 3.4 is then applied in Chapter 4 to a single rod and in Chapter 6 to lattices. The challenging part is in both cases the proof of convergence for $\mathcal{E}^\varepsilon$. The energy has always the form $\mathcal{E}^\varepsilon(t, q) = \mathcal{B}^\varepsilon(q) - \langle \ell^\varepsilon(t), q\rangle$ and for the most part we focus on the quadratic form $\mathcal{B}^\varepsilon$.

In Chapter 4 we consider rods, i.e. domains of the form $\Omega_h = I \times hB$. When $\overline{q} = (\overline{u}, \overline{p})$ is the state of the rod expressed in physical variables, we introduce the rescaled state $q^h = (u^h, p^h)$ defined on $\Omega = I \times B$ by the relationship

$$\overline{u}(x) = \left(\begin{smallmatrix} h^2 & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{smallmatrix}\right) u^h \left(x_1, \frac{x_2}{h}, \frac{x_3}{h}\right) , \qquad \overline{p}(x) = h^2 p^h \left(x_1, \frac{x_2}{h}, \frac{x_3}{h}\right) .$$

This leads to an equivalent rate-independent system in which the states $\overline{q}$ are replaced by q^h and the stored energy $\mathcal{B}$ takes the form

$$\mathcal{B}^h(q^h) = \int_\Omega \mathbb{W}\left(S_h \nabla^s u^h S_h, p^h\right) , \qquad S_h = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1/h & 0 \\ 0 & 0 & 1/h \end{smallmatrix}\right).$$

The limit stored energy turns out to be supported on $q = (u, p)$ with $\nabla^s u \in \operatorname{span}(e_1 \otimes e_1)$ a.e., and has the form

$$\mathcal{B}^0(q) = \int_\Omega \mathbb{W}\left(\begin{pmatrix} \partial_1 u_1 & * & * \\ \partial_2 f - g'(x_1)x_3 & & \\ \partial_3 f(x) + g'(x_1)x_2 & \nabla^s_{2,3} w & \end{pmatrix}, \ p\right) \mathrm{d}x \,.$$

Here f, g, w depend on q and are chosen in such a way that the expression given for $\mathcal{B}^0(q)$ is minimized; see (4.11) for the precise definition. The given expression suggests that the upper bound property of $\mathcal{B}^0$ can be proved by considering a recovery sequence of the form

$$u^h(x) = u(x) + 2h \begin{pmatrix} f \\ -g(x_1)x_3 \\ g(x_1)x_2 \end{pmatrix} + h^2 \begin{pmatrix} 0 \\ w_1 \\ w_2 \end{pmatrix} .$$

The lower bound is more difficult to prove as it requires to find appropriate functions f, g, w for any sequence $q^h \rightharpoonup q^0$ with bounded energy $\mathcal{B}^h(q^h)$.

Chapter 5 is preparatory for the homogenization of periodic lattices considered in Chapter 6. We introduce the notion of periodic graphs. We start from a *periodicity graph* G, which is a finite multigraph with edge labels in $\mathbb{Z}^3$. An "unfolding" procedure prescribed by these labels leads to an infinite periodic graph G_{per} (see Figure 5.1). When each node of G is assigned a position in the periodicity cell, this gives rise to an infinite periodic framework in $\mathbb{R}^3$. This framework is scaled by a factor $\varepsilon > 0$ and fitted into a given domain $\Omega \subset \mathbb{R}^3$. In doing so, which requires cropping the infinite framework at the boundary of Ω, we have to be careful not to lose the property of *infinitesimal rigidity*. We develop a procedure according to which such cropped graphs G^ε can be constructed using *rigidity cells*. These are building blocks which are possibly larger than a single periodicity cell but possess rigidity. The notion of rigidity we presuppose can be expressed by the assumption that there is an estimate of the form

$$\sum_{(v_1,v_2)} |u(v_2) - u(v_1)|^2 \leq C \sum_{(v_1,v_2)} \left| \frac{z(v_2) - z(v_1)}{|z(v_2) - z(v_1)|} \cdot (u(v_2) - u(v_1)) \right|^2 .$$

Here, the summation is over all edges (v_1, v_2) of the underlying graph and $z : V \to \mathbb{R}^3$ is a placement of the nodes V of that graph. The estimate must hold uniformly for all node displacement fields $u : V \to \mathbb{R}^3$. When only a single graph with a finite set V of nodes is considered, the estimate is clearly equivalent to the qualitative statement that the left-hand side vanishes whenever the right-hand side vanishes. However, we need the estimate to hold uniformly in $\varepsilon > 0$ for all the graphs G^ε. This can be guaranteed by constructing G^ε with the above-mentioned rigidity cells.

We then go on to introduce notation for dealing with functions defined on the nodes and edges of the graphs G^ε with a view towards appropriate limit notions as $\varepsilon \to 0$. Functions on the set of nodes of G^ε are denoted (e.g.) $\beta_v(x)$ with x corresponding to the periodicity cell $\lfloor x/\varepsilon \rfloor \in \mathbb{Z}^3$ and $v \in V(G)$ selecting a node from that cell. Accordingly, functions on the set of edges of G^ε are denoted (e.g.) $\gamma_e(x)$ with x as before and $e \in E(G)$ selecting an edge from the cell corresponding to x. These functions are assumed to be constant in x on each cell $\varepsilon(k + (0, \varepsilon)^3)$ for $k \in \mathbb{Z}^3$.

Next, we introduce the notion of a graph gradient $\mathrm{grad}^\varepsilon(\beta; G^\varepsilon)$ which turns a node-function β into an edge-function which contains the difference quotients of β along all edges. We prove a corresponding Poincaré inequality $\|\beta\| \lesssim \|\mathrm{grad}^\varepsilon(\beta; G^\varepsilon)\|$ (see Lemma 5.11) and introduce a notion of two-scale convergence which satisfies a corresponding compactness property (see Lemma 5.16).

In the limit, the graph gradient becomes $\mathrm{grad}(\alpha,\beta;G^\varepsilon)$ where $\alpha=\alpha(x)$ contains the macroscopic displacements and $\beta=\beta_v(x)$ captures the microscopic, node-type dependent oscillations of the original sequence.

In Chapter 6 we finally turn to the homogenization of elastoplastic lattices. Here we have two infinitesimal parameters: The periodicity $\varepsilon\to 0$ and the (relative) width of the rods $h=h(\varepsilon)\to 0$. Interestingly, the limit stored energy looks almost the same as in the case of a single rod:

$$\mathcal{B}^0(q)=\sum_{e\in E(G)}\int_{\mathbb{R}^3}\int_{\Omega_e}$$
$$\mathbb{W}_e\left(\begin{pmatrix}\partial_{y_1}v_{e,1} & * & * \\ \partial_{y_2}f_e-\partial_{y_1}g_e(x,y_1)x_3 & \multicolumn{2}{c}{\nabla^s_{y_2,y_3}w_e} \\ \partial_{y_3}f_e(x)+\partial_{y_1}g_e(x,y_1)x_2 & & \end{pmatrix},\ p_e\right)\mathrm{d}y\mathrm{d}x\,.$$

Here the state q is a triple $q=(u,v,p)$ of macroscopic displacements $u:\Omega\to\mathbb{R}^3$, microscopic displacements $v_e(x,\cdot):\Omega_e\to\mathbb{R}^3$ for each macroscopic position $x\in\Omega$ and each edge type e, and plastic strain fields $p_e(x,\cdot):\Omega_e\to\mathbb{R}^{3\times 3}_{\mathrm{dev}}$. For the precise definition see (6.24). The important differences from the rod-energy do not appear in this formula but in the definition of the limit space $\mathcal{Q}^0\subset\mathcal{Q}$ on which alone this formula is valid. The definition of $\mathcal{Q}^0$ includes the condition $\nabla^s_y v\in\mathrm{span}(e_1\otimes e_1)$ (as in the case of rods, this implies that the microscopic displacements are effectively one-dimensional). But it also includes boundary conditions for v and g. The boundary values are defined via node states $A_v(x)$ (infinitesimal rotations) and $(u(x),\xi_v(x))$ (two-scale node displacements).

For a better understanding of the limit energy, we can infimize out v and get an energy of the form

$$(u,p)\mapsto\inf_v\mathcal{B}^0(u,v,p)=\int_{\mathbb{R}^3}F(\nabla^s u(x),p(x))\,,$$
$$F(\epsilon,p)=\sum_{e\in E(G)}\int_{\Omega_e}\mathbb{W}_e\left(\begin{pmatrix}\partial_1 v_{e,1} & * & * \\ \partial_2 f_e-\partial_1 g_e(y_1)x_3 & \multicolumn{2}{c}{\nabla^s_{2,3}w_e} \\ \partial_3 f_e(x)+\partial_1 g_e(y_1)x_2 & & \end{pmatrix},\ p_e\right)\mathrm{d}y\mathrm{d}x\,,$$

where v,f,g,w are minimizers of the expression defining F, and the macroscopic strain ϵ comes in through the boundary conditions imposed on v. For a more transparent description which explicitly features ϵ, see (6.34).

In the case of sufficient thickness, where $h(\varepsilon)/\varepsilon\to\infty$, the microscopic displacement field v can indeed be neglected in this way. In the critical case $h(\varepsilon)\sim\varepsilon$ and in the case of sufficiently thin rods, $h(\varepsilon)/\varepsilon\to 0$, however, the microscopic displacements v appear in the loading term and thus must be accounted for.

Related work

The mathematical theory of rods has rich history. For rigorous results on elastic rods we refer to the works by Mielke [39, 41] and by Mora and Müller [46, 47], which work even in the nonlinear regime.

The general theory of rate-independent systems and evolutionary Γ-convergence is developed in [40, 43] and in book form in [42]. Based on this, Liero and coworkes have developed elastoplastic plate models in [34, 35]. We follow the overall approach and scalings used in these papers (particularly [34]) when we develop a model for elastoplastic rods in Chapter 4.

For the homogenization part (Chapter 6) we refer to previous work on the homogenization of equations for elastoplasticity [4, 45, 58, 20, 55, 27]. There are several lines of research in which lattices, frames or trusses are studied. In most of these, scalar equations or pure elasticity is considered.

One line of research was initiatied by Bouchitté and coauthors [11, 12, 13] with the introduction of the notion of *energies with respect to a measure* [11]. This notion serves the specific aim to study singular (i.e. lower-dimensional) structures. Such structures, for example lattices, are represented by measures. The authors introduce the concept of tangential gradients with respect to a measure which enables them to construct associated Sobolev spaces. In [12] they use this framework for the homogenization of periodic structures. For this, an appropriate notion of two-scale convergence is introduced. This notion is employed to obtain homogenized energy densities for convex integral functionals. However, all of this only applies to scalar problems. In [13] the authors then study vector valued problems, and in particular linear elasticity. For this they introduce an approach which they call *measure fattening*: The low-dimensional structure of periodicity ε is fattened by an amount δ relative so ε. The analysis is carried out under the assumption of a fatness condition. In particular, the authors avoid what is called *critical thickness*, where $\delta(\varepsilon) \sim \varepsilon$.

The work by Zhikov, Pastukhova and coauthors [60, 59, 51, 15, 61] is in many respects parallel to what Bouchitté and coauthors did. In [15] they also first study scalar problems, and in [60] the homogenization of elasticity problems on lower-dimensional structures is considered. In [59] the authors moreover tackle the setting of critical thickness. In [61], various Korn inequalities for thin structures are proved.

A more recent line of work comes from Seppecher and coauthors [1, 2]. They are interested in exotic materials. For this they start with simple elastic networks made up of linear springs which are, however, not rigid in the sense that we will study in our work, but have, as they call it, “a few number of floppy modes”. In the homogenization, the model escapes the classical framework of Cauchy stress theory. The authors get materials with higher order gradients (see [56]). In [1, 2], this idea is carried out for elastic networks. As a first step, the problem posed on a composite domain is reduced to a discrete problem. Subsequently, the discrete problem is tackled.

Lastly, there is a series of publications by Babuška and coauthors [7, 36, 37]. In [7], the authors develop algorithms for the verification of various properties of periodic lattices. Then in [37] they prove existence and uniqueness for elastic equilibrium equations on infinite periodic lattices, and in [36] these equations are homogenized.

For the concept of the rigidity of graphs we refer to [9, 10, 29, 52]. We only use quite elementary notions of this field and do not go into the details of the underlying algebraic theory.

Notation

When we prove various estimates, we often use the notation

$$A \lesssim B\,,$$

which is meant to be equivalent to the statement that $A \leq CB$ for some constant $C > 0$. Here, A and B usually depend on one or more parameters and it is understood that C is independent of these. In particular, we write $A \lesssim 1$ when the quantity A is uniformly bounded. Moreover, we write $A \sim B$ when $A \lesssim B$ and $B \lesssim A$.

Chapter 2

Elastoplasticity

We introduce the classical equations for linearized elastoplasticity in the rate-independent case with linear kinematic hardening (see for example [26, 3]). Subsequently we introduce the concept of energetic solutions as introduced by Mielke [40] and state a by now classical existence result.

2.1 Kinematics

We will study bodies that, from a macroscopic perspective, appear to be continuously distributed. This means that they occupy a region of three-dimensional space. In an undeformed state, a body can be identified with a region $\Omega \subset \mathbb{R}^3$, which we call the *reference domain* of that body.

Any deformed state of the body can be described by specifying for each material point $x \in \Omega$ a *displacement vector* $u(x) \in \mathbb{R}^3$. In its deformed state, the region occupied by the body is $\{x + u(x) \ : \ x \in \Omega\}$. When we study evolutions of the body over a time interval $[0, T]$, the primary unknown variable is therefore the *displacement field*

$$u : [0, T] \times \Omega \to \mathbb{R}^3 \, .$$

We need to distinguish mere rigid body motions, in which the body as a whole is translated and rotated, from deformations that affect the shape of the body or at least locally result in changes of lengths and angles. Such behaviour is fully captured by the (nonlinear) *strain tensor*, which is defined as

$$\eta(u) \coloneqq \frac{1}{2}\Big(\nabla u + (\nabla u)^T + (\nabla u)^T \nabla u\Big) \, .$$

For a rigid body motion, one has $\nabla u = R - I$ for some $R \in SO(3)$ which implies $\eta(u) = 0$. More generally, let us assume $0 \in \Omega$ and $u(0) = 0$, and consider vectors $v, w \in \mathbb{R}^3$ at the origin which are of length $O(\varepsilon)$. The inner product of the "deformed vectors" is

$$\begin{aligned}(v + u(v)) \cdot (w + u(w)) &= (v + (\nabla u)v) \cdot (w + (\nabla u)w) + O(\varepsilon^2) \\ &= v \cdot w + \left(\nabla u + (\nabla u)^T + (\nabla u)^T \nabla u\right) v \cdot w + O(\varepsilon^2) \\ &= v \cdot w + 2\eta(u)v \cdot w + O(\varepsilon^2)\,.\end{aligned}$$

Thus we see that the nonlinear strain tensor $\eta(u)$ indeed fully describes how inner products (and thus lengths and angles) are affected by a displacement u.

We will, however, not work with the full strain tensor $\eta(u)$. By restricting our attention to small displacements, we can assume that ∇u is small enough to justify that we neglect the quadratic term in the definition of $\eta(u)$. This leads to the definition of the *linearized* or *infinitesimal strain tensor*

$$\epsilon(u) \coloneqq \frac{1}{2}\left(\nabla u + (\nabla u)^T\right),$$

which is just the *symmetrized gradient* of u and also denoted by $\nabla^s u$.

2.2 Balance of momentum

From now on we will assume infinitesimal deformations. There are two types of forces that may act in (or on) every part of the body: A *body force* $f : \Omega \times [0, T] \to \mathbb{R}^3$, and surface traction. The surface traction $s_n(x, t)$ depends on a unit vector n and is defined by the following property: When the body is split in two by a regular surface with normal n at point x, then $s_n(x, t)$ is the force per unit area which the part of the body towards which n points exerts on the other part. We assume that there exists a *stress tensor* $\sigma : [0, T] \times \Omega \to \mathbb{R}^{3\times 3}$ such that $s_n = \sigma n$.

For regular subset $U \subset \mathbb{R}^3$, we have the balance of linear momentum,

$$0 = \int_U f \mathrm{d}x + \int_{\partial U} \sigma n \mathrm{d}s = \int_U f + \operatorname{div} \sigma \mathrm{d}x\,.$$

As U was arbitrary, this implies $f = -\operatorname{div} \sigma$. We also have the balance of angular momentum,

$$\begin{aligned}
0 &= \int_U x \wedge f \mathrm{d}x + \int_{\partial U} x \wedge \sigma n \mathrm{d}s \\
&= \int_U x \wedge f \mathrm{d}x + \int_{\partial U} (\varepsilon_{ijk} x_j \sigma_{kl}) n_l \mathrm{d}s \\
&= \int_U x \wedge f \mathrm{d}x + \int_U \partial_l (\varepsilon_{ijk} x_j \sigma_{kl}) \mathrm{d}s \\
&= \int_U x \wedge f \mathrm{d}x + \int_U \varepsilon_{ijk} \sigma_{kj} + x \wedge \operatorname{div} \sigma \mathrm{d}s \\
&= \int_U \begin{pmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} \end{pmatrix} \mathrm{d}x \,.
\end{aligned}$$

Again, as U was arbitrary, this implies $\sigma = \sigma^T$ or $\sigma : [0, T] \times \Omega \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$.

2.3 Constitutive equations

What is still missing is a relation between σ and u. This is where material properties come into play.

We assume an additive decomposition of the strain $\epsilon = \nabla^s u$ into an elastic part e and a plastic part p,

$$\epsilon = e + p \,.$$

We further assume a linear relation between the elastic strain e and the stress σ

$$\sigma = \mathbb{C}\epsilon \,,$$

with an *elasticity tensor* $\mathbb{C} : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. For thermodynamical reasons, it is generally assumed that $\mathbb{C}$ is symmetric and positive. The elasticity tensor may depend on the material point, but we will consider homogeneous materials throughout.

2.4 Plastic flow rule

We complete the equations with an evolution law for p, a *flow rule*. Plastic behaviour only occurs when the stress $\sigma(x)$ reaches a certain limit. We suppose a bounded, closed, convex set $K \subset \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ of attainable stresses with $0 \in K$.

We further assume linear kinematic hardening: There is a positive symmetric tensor $\mathbb{B} : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ which maps the plastic strain $p(x)$ to a *back stress* $\mathbb{B}p(x)$. We assume that $\sigma(x) - \mathbb{B}p(x) \in K$ everywhere and at all times. The plastic strain $p(x)$ only evolves when $\sigma(x) - \mathbb{B}p(x) \in \partial K$, and in that case $\dot{p}(x)$ must be an outer normal vector to K at $\sigma(x) - \mathbb{B}p(x)$.

Moreover, we make the assumption that the plastic behaviour is insensitive to pressure, meaning that all plastic deformations are volume preserving. This is expressed in the condition $p \in \mathbb{R}^{3\times3}_{\text{dev}}$, where $\mathbb{R}^{3\times3}_{\text{dev}}$ denotes the space of *deviatoric* matrices,

$$\mathbb{R}^{3\times3}_{\text{dev}} \coloneqq \{A \in \mathbb{R}^{3\times3}_{\text{sym}} \;:\; \operatorname{tr}(A) = 0\}\,.$$

This is guaranteed by requiring that $K + \mathbb{R}I \subset K$.

2.5 The initial and boundary value problem

To sum up, the material is described by an elasticity modulus $\mathbb{C} : \mathbb{R}^{3\times3}_{\text{sym}} \to \mathbb{R}^{3\times3}_{\text{sym}}$ which is a positive symmetric tensor; a linear kinematic hardening parameter $\mathbb{B} : \mathbb{R}^{3\times3}_{\text{sym}} \to \mathbb{R}^{3\times3}_{\text{sym}}$ which is also a positive symmetric tensor; and a bounded, closed, convex set $K \subset \mathbb{R}^{3\times3}_{\text{sym}}$ with $0 \in K$ and $K + \mathbb{R}I \subset K$. We define $\psi : \mathbb{R}^{3\times3}_{\text{sym}} \to \mathbb{R}_\infty$ to be the indicator function of K, that is,

$$\psi(\sigma) \coloneqq \begin{cases} 0 & \text{if } \sigma \in K \\ +\infty & \text{if } \sigma \notin K\,. \end{cases}$$

Then the subdifferential of ψ is

$$\partial\psi(\sigma) = \begin{cases} N_\sigma K & \text{if } \sigma \in \partial K \\ \{0\} & \text{if } \sigma \in \mathring{K} \\ \emptyset & \text{if } x \notin K\,, \end{cases}$$

where $N_\sigma K \subset \mathbb{R}^3$ is the cone of outer normal vectors to K at σ. Thus the flow rule can be expressed as $\partial_t p(x) \in \partial\psi(\sigma(x) - \mathbb{B}p(x))$.

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain, and Γ_D a nonempty open subset of $\partial\Omega$. We write $\Gamma_N \coloneqq \partial\Omega \setminus \Gamma_D$. For given volume and surface loads

$$f_{\text{vol}} : [0,T] \times \Omega \to \mathbb{R}^3\,, \qquad f_{\text{surf}} : [0,T] \times \Gamma_N \to \mathbb{R}^3\,,$$

the equations of elastoplasticity are

$$-\nabla \cdot \sigma = f_{\text{vol}} \qquad \text{in } [0,T] \times \Omega \tag{2.1a}$$
$$\sigma = \mathbb{C}(\nabla^s u - p) \qquad \text{in } [0,T] \times \Omega \tag{2.1b}$$
$$\partial_t p \in \partial\psi(\sigma - \mathbb{B}p) \qquad \text{in } [0,T] \times \Omega\,. \tag{2.1c}$$

These equations for $u : [0,T] \times \Omega \to \mathbb{R}^3$ and $p : [0,T] \times \Omega \to \mathbb{R}^{3\times 3}_{\text{dev}}$ are completed by initial conditions $u(0,\cdot) = u_0$ and $p(0,\cdot) = 0$, and boundary conditions

$$u = 0 \qquad \text{on } [0,T] \times \Gamma_D \,, \tag{2.2a}$$

$$\sigma \cdot n = f_{\text{surf}} \quad \text{on } [0,T] \times \Gamma_N \,. \tag{2.2b}$$

Here n denotes a field of outer normal vectors to Ω.

2.6 Energetic formulation, Existence and Uniqueness

In this section we want to make the equations (2.1) and (2.2) precise by introducing function spaces and the notion of energetic solutions.

For this we define two scalar quantities that will replace $\mathbb{C}, \mathbb{B}$ and K. The first of these is the stored energy density $\mathbb{W} : \mathbb{R}^{3\times 3}_{\text{sym}} \times \mathbb{R}^{3\times 3}_{\text{dev}} \to \mathbb{R}$,

$$\mathbb{W}(\epsilon, p) := \frac{1}{2}\mathbb{C}(\epsilon - p) : (\epsilon - p) + \frac{1}{2}\mathbb{B}p : p \,. \tag{2.3}$$

The second one, the dissipation potential $R : \mathbb{R}^{3\times 3}_{\text{dev}} \to \mathbb{R}$, is the Fenchel conjugate of ψ,

$$R(p) := \psi^*(p) = \sup_{\sigma \in \mathbb{R}^{3\times 3}_{\text{sym}}} \sigma : p - \psi(\sigma) \,.$$

Since ψ is the indicator function of the elastic region $K \subset \mathbb{R}^{3\times 3}_{\text{sym}}$, the dissipation potential is a positive one-homogeneous function, namely the so-called support function of K, that is,

$$R(p) = \sup_{\sigma \in K} \sigma : p \,.$$

With $\mathbb{W}$ as defined in (2.3), we can reformulate (2.1b) to

$$\sigma = \mathbb{C}(\nabla^s u - p) = \partial_\varepsilon \mathbb{W}(\nabla^s u, p) \tag{2.4}$$

This implies $\partial_p \mathbb{W}(\nabla^s u, p) = -\mathbb{C}(\nabla^s u - p) + \mathbb{B}p = \mathbb{B}p - \sigma$. Thus we can reformulate (2.1c) to

$$\partial_t p \in \partial\psi(\sigma - \mathbb{B}p) = \partial\psi(-\partial_p \mathbb{W}(\nabla^s u, p)) \,.$$

By Fenchel's relations, this is equivalent to

$$-\partial_p \mathbb{W}(\nabla^s u, p) \in \partial\psi^*(\partial_t p) = \partial R(\partial_t p) \,, \tag{2.5}$$

In (2.4) and (2.5) we thus have an equivalent expression for the constitutive equations (2.1b) and (2.1c) in terms of $\mathbb{W}$ and R.

Up to now we have considered only a single material point. We will now introduce integrated quantities and therefore consider fields $u : \Omega \to \mathbb{R}^3$ and $p : \Omega \to \mathbb{R}^{3\times3}_{\text{dev}}$. The corresponding space is

$$\mathcal{Q} := \{(u,p) \in H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R}^{3\times3}_{\text{dev}}) \;:\; u = 0 \text{ on } \Gamma\}\,.$$

We call $\mathcal{Q}$ the *state space* of the system because one element of it can fully describe the state of the system at a particular point in time. This space already encodes the Dirichlet boundary condition (2.2a).

We integrate the pointwise quantities $\mathbb{W}(\nabla^s u, p)$ and $R(p)$ to get the stored energy functional $\mathcal{B} : \mathcal{Q} \to \mathbb{R}$,

$$\mathcal{B}(q) := \int_\Omega \mathbb{W}(\nabla^s u(x), p(x))\mathrm{d}x\,, \qquad q = (u,p) \in \mathcal{Q}\,,$$

and the dissipation functional $\mathcal{R} : \mathcal{Q} \to \mathbb{R}$,

$$\mathcal{R}(q) := \int_\Omega R(p(x))\mathrm{d}x\,, \qquad\qquad q = (u,p) \in \mathcal{Q}\,.$$

Moreover, for $t \in [0,T]$ we define $\ell(t) \in \mathcal{Q}^*$ by

$$\langle \ell(t), q\rangle := \int_\Omega f_{\text{vol}}(t,x)\cdot u(x)\mathrm{d}x + \int_{\Gamma_N} f_{\text{surf}}(t,x)\cdot u(x)\mathrm{d}\mathcal{H}^2(x)$$

for $q = (u,p) \in \mathcal{Q}$. With this we define the total energy $\mathcal{E} : [0,T]\times\mathcal{Q} \to \mathbb{R}$,

$$\mathcal{E}(t,q) := \mathcal{B}(q) - \langle \ell(t), q\rangle\,, \qquad t \in [0,T]\,, \quad q \in \mathcal{Q}\,.$$

We can now combine (2.1a), (2.2b) and (2.4) into

$$-\nabla\cdot\partial_\epsilon\mathbb{W}(\nabla^s u,p) = f_{\text{vol}} \quad \text{in } \Omega\,, \qquad \partial_\epsilon\mathbb{W}(\nabla^s u,p)\cdot n = f_{\text{surf}} \quad \text{on } \Gamma_N\,,$$

which in short is

$$0 = \mathrm{D}_u\mathcal{E}(t,q(t))\,. \tag{2.6}$$

Moreover, (2.5) simply becomes

$$0 \in \partial_p\mathcal{R}(\partial_t q(t)) + D_p\mathcal{E}(t,q(t))\,. \tag{2.7}$$

Now since $\partial_u\mathcal{R} = 0$, we can add (2.6) and (2.7) in order to obtain the simple subdifferential inclusion

$$0 \in \partial\mathcal{R}(\dot q(t)) + \mathrm{D}_q\mathcal{E}(t,q(t))\,, \tag{2.8}$$

which is sufficient to replace (2.1) and (2.2).

Given any evolution $q : [0,T] \to \mathcal{Q}$ and any time inverval $[s,t] \subset [0,T]$, the *total dissipation* is defined as

$$\mathrm{Diss}_{\mathcal{R}}(q;[s,t]) := \sup \left\{ \sum_{k=1}^{N} \mathcal{R}(q(t_k) - q(t_{k-1})) \ : \ N \in \mathbb{N}\,,\ s = t_0 \leq \cdots \leq t_N = t \right\}.$$

Due to the one-homogeneity of $\mathcal{R}$, when q is absolutely continuous, we have

$$\mathrm{Diss}_{\mathcal{R}}(q;[s,t]) = \int_s^t \mathcal{R}(\partial_t q(s))\mathrm{d}s\,.$$

The differential inclusion (2.8) is equivalent to the so-called energetic formulation for rate-independent systems: An evolution $q \in L^1(0,T;\mathcal{Q})$ is said to be a solution to the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{R})$ if and only if for every $t \in [0,T]$ the following two conditions are satisfied.

(S) *Stability.* For all $t \in [0,T]$ and $q' \in \mathcal{Q}$

$$\mathcal{E}(t,q(t)) \leq \mathcal{E}(t,q(t)+q') + \mathcal{R}(q')\,.$$

(E) *Energy equality.* For all $t \in [0,T]$:

$$\mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{R}}(q;[0,t]) = \mathcal{E}(0,q(0)) - \int_0^t \langle \partial_s \ell(s), q(s)\rangle \mathrm{d}s\,.$$

Notice that this formulation is free of time-derivatives of $q : [0,T] \to \mathcal{Q}$. It was developed by Mielke and coauthors in [44].

Quadratic rate-independent systems

We now give the classical existence and uniqueness result for quadratic rate-independent systems. For this, we abstract away from the equations of elasto-plasticity. Let $\mathcal{Q}$ denote a separable Hilbert space. We suppose to have the following ingredients:

(i) A *stored energy functional* $\mathcal{B} : \mathcal{Q} \to [0,\infty]$, which is a lower semi-continuous and coercive quadratic form;

(ii) a *dissipation functional* $\mathcal{R} : \mathcal{Q} \to [0,\infty]$, which is positive one-homogeneous, convex and lower semi-continuous;

(iii) *loads* $\ell \in W^{1,\infty}(0,T;\mathcal{Q}^*)$.

By saying that $\mathcal{B}$ is a quadratic form we mean that $\mathcal{V} := \{q \in \mathcal{Q} \,:\, \mathcal{B}(q) < \infty\}$ is a linear subspace of $\mathcal{Q}$, and

$$\mathcal{V} \times \mathcal{V} \to \mathbb{R}\,, \qquad (q, q') \mapsto \langle q, q' \rangle_{\mathcal{V}} := \frac{1}{4}\Big(\mathcal{B}(q + q') - \mathcal{B}(q - q')\Big) \tag{2.9}$$

defines a bilinear form. Note that (2.9) implies $\mathcal{B}(q) = \langle q, q \rangle$ for $q \in \mathcal{V}$. By saying that $\mathcal{B}$ is *coervice*, we mean that there is a constant $\beta > 0$ with

$$\beta \|q\|^2 \leq \mathcal{B}(q) \quad \forall\, q \in \mathcal{Q}\,.$$

As a derived quantity we have the *total energy*

$$\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}\,, \qquad \mathcal{E}(t, q) := \mathcal{B}(q) - \langle \ell(t), q \rangle\,.$$

We say that $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ is a quadratic coercive rate-independent system.

Let us now state the classical existence and uniqueness result. It is basically the same as Theorem 3.5.2 of [42]. However, we give it in a slightly weaker form by assuming the loads to be Lipschitz continuous as opposed to only being absolutely continuous.

Theorem 2.1 (Existence and uniqueness). *Let $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ be a quadratic coercive rate-independent system. Consider any initial state $q_0 \in \mathcal{Q}$ such that*

$$\mathcal{E}(0, q_0) \leq \mathcal{E}(0, q_0 + q') + \mathcal{R}(q') \qquad \textit{for all } q' \in \mathcal{Q}\,.$$

Then there exists one and only one solution $q \in L^1(0, T; \mathcal{Q})$ for $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ with $q(0) = q_0$. Moreover, $q \in W^{1,\infty}(0, T; \mathcal{Q})$ and

$$\|\partial_t q(t)\|_{\mathcal{Q}} \leq \frac{1}{2\beta} \|\partial_t \ell(t)\|_{\mathcal{Q}^*} \qquad \textit{for a.e. } t \in [0, T]\,.$$

In particular, $\|q\|_{W^{1,\infty}(0,T;\mathcal{Q})} \leq \|q_0\|_{\mathcal{Q}} + \frac{T}{2\beta} \|\partial_t \ell\|_{L^\infty(0,T;\mathcal{Q}^*)}$.

Before we come to the proof of the above theorem, let us begin with a basic statement about quadratic forms.

Lemma 2.2 (On quadratic forms). *Suppose that $\mathcal{B} : \mathcal{Q} \to [0, \infty]$ is a lower semi-continuous and coercive quadratic form on a separable Hilbert space $\mathcal{Q}$. Then*

$$\mathcal{V} := \{q \in \mathcal{Q} \,:\, \mathcal{B}(q) < \infty\}$$

is a Hilbert space when it is equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{V}}$ defined in (2.9)*.*

Proof. The inner product is positive definite since

$$\langle q, q\rangle_{\mathcal{V}} = \mathcal{B}(q) \geq \beta\|q\|^2 .$$

It remains only to show that $\mathcal{V}$ is complete. Let us suppose that $(q_n)_n$ is a Cauchy sequence in $\mathcal{V}$. Then, by the coercivity of $\mathcal{B}$, it is also a Cauchy sequence in $\mathcal{Q}$. By completeness of $\mathcal{Q}$, there exists $q \in \mathcal{Q}$ such that $q_n \to q$ in $\mathcal{Q}$. Then, by the lower semi-continuity of $\mathcal{B}$,

$$\mathcal{B}(q) \leq \liminf_{n\to\infty} \mathcal{B}(q_n) = \liminf_{n\to\infty} \|q_n\|_{\mathcal{V}}^2 < \infty .$$

Thus $q \in \mathcal{V}$. For any $\delta > 0$ we have

$$\|q_n - q\|_{\mathcal{V}}^2 = \mathcal{B}(q_n - q) \leq \liminf_{k\to\infty} \mathcal{B}(q_n - q_k) = \liminf_{k\to\infty} \|q_n - q_k\|_{\mathcal{V}}^2 \leq \delta \qquad (2.10)$$

for large values of n. The first inequality again follows from the lower semi-continuity of $\mathcal{B}$. The second inequality is a consequence of $(q_n)_n$ being a Cauchy sequence in $\mathcal{V}$. As $\delta > 0$ was arbitrary, (2.10) implies $q_n \to q$ in $\mathcal{V}$. □

We also need the following statement of lower semi-continuity for the total dissipation.

Lemma 2.3. *Suppose that $\mathcal{R} : \mathcal{Q} \to [0,\infty]$ is weakly lower-semicontinuous. Let $q_n, q : [0,T] \to \mathcal{Q}$ with $q_n(t) \rightharpoonup q(t)$ in $\mathcal{Q}$ for all $t \in [0,T]$. Then*

$$\mathrm{Diss}_{\mathcal{R}}(q; [s,t]) \leq \liminf_{n\to\infty} \mathrm{Diss}_{\mathcal{R}}(q_n; [s,t]) .$$

Proof. Let $\varepsilon > 0$. Then by the definition of $\mathrm{Diss}_{\mathcal{R}}$ there exist $N \in \mathbb{N}$ and $s = t_0 \leq \cdots \leq t_N = t$ such that

$$\mathrm{Diss}_{\mathcal{R}}(q; [s,t]) - \varepsilon \leq \sum_{k=1}^{N} \mathcal{R}(q(t_k) - q(t_{k-1})) .$$

By the lower-semicontinuity of $\mathcal{R}$ and again the definition of $\mathrm{Diss}_{\mathcal{R}}$, this implies

$$\begin{aligned}\mathrm{Diss}_{\mathcal{R}}(q; [s,t]) - \varepsilon &\leq \sum_{k=1}^{N} \liminf_{n\to\infty} \mathcal{R}(q_n(t_k) - q_n(t_{k-1})) \\ &\leq \liminf_{n\to\infty} \sum_{k=1}^{N} \mathcal{R}(q_n(t_k) - q_n(t_{k-1})) \leq \liminf_{n\to\infty} \mathrm{Diss}_{\mathcal{R}}(q_n; [s,t]) .\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, this finishes the proof. □

We now come to the proof of Theorem 2.1.

Proof of Theorem 2.1. Step 1: Time discretization and a priori estimates. For $N \in \mathbb{N}$, we consider a partition

$$0 = t_0^N < t_1^N < \cdots < t_N^N = T$$

of the interval $[0, T]$. We set $q_0^N \coloneqq q_0$, and inductively define

$$q_k^N = \arg\min \left\{ \mathcal{E}(t_k^N, q) + \mathcal{R}(q - q_{k-1}^N) \ : \ q \in \mathcal{Q} \right\} , \qquad 1 \le k \le N \, .$$

Existence and uniqueness of these minimizers follows from the strong convexity, lower semi-continuity, and coercivity of the functional which is minimized,

$$\mathcal{I}_k^N(q) \coloneqq \|q\|_{\mathcal{V}}^2 - \langle \ell(t_k^N), q \rangle + \mathcal{R}(q - q_{k-1}^N) \, , \qquad q \in \mathcal{V} \, .$$

By Lemma A.1, we have the estimate

$$\|q_k^N - q\|_{\mathcal{V}}^2 \le \mathcal{I}_k^N(q) - \mathcal{I}_k^N(q_k^N) \qquad \text{for all } q \in \mathcal{V} \, . \tag{2.11}$$

Using (2.11) with $q = q_{k+1}^N$, we derive the following estimate for the time-discrete solution $(q_k^N)_{k=0}^N$:

$$\begin{aligned} &\|q_{k+1}^N - q_k^N\|_{\mathcal{V}}^2 \le \mathcal{I}_k^N(q_{k+1}^N) - \mathcal{I}_k^N(q_k^N) \\ = &\, \mathcal{E}(t_k^N, q_{k+1}^N) + \mathcal{R}(q_{k+1}^N - q_{k-1}^N) - \mathcal{E}(t_k^N, q_k^N) - \mathcal{R}(q_k^N - q_{k-1}^N) \, . \end{aligned}$$

Here we can make use of the fact that $\mathcal{R}$, being positive one-homogeneous and convex, satisfies a triangle inequality of the form $\mathcal{R}(c-a) = 2\mathcal{R}(\frac{1}{2}(c-b) + \frac{1}{2}(b-a)) \le \mathcal{R}(c-b) + \mathcal{R}(b-a)$, and continue,

$$\begin{aligned} &\le \mathcal{E}(t_k^N, q_{k+1}^N) - \mathcal{E}(t_k^N, q_k^N) + \mathcal{R}(q_{k+1}^N - q_k^N) \\ &\le \mathcal{E}(t_{k+1}^N, q_{k+1}^N) - \mathcal{E}(t_k^N, q_k^N) - \int_{t_k^N}^{t_{k+1}^N} \partial_s \mathcal{E}(s, q_{k+1}^N) \mathrm{d}s + \mathcal{R}(q_{k+1}^N - q_k^N) \, . \end{aligned}$$

We now use $\mathcal{I}_{k+1}^N(q_{k+1}^N) \le \mathcal{I}_{k+1}^N(q_k^N)$ in order to continue,

$$\begin{aligned} &\le \mathcal{E}(t_{k+1}^N, q_k^N) - \mathcal{E}(t_k^N, q_k^N) - \int_{t_k^N}^{t_{k+1}^N} \partial_s \mathcal{E}(s, q_{k+1}^N) \mathrm{d}s \\ &= \int_{t_k^N}^{t_{k+1}^N} \partial_s \mathcal{E}(s, q_k^N) - \partial_s \mathcal{E}(s, q_{k+1}^N) \mathrm{d}s = \int_{t_k^N}^{t_{k+1}^N} \langle \partial_s \ell(s), q_{k+1}^N - q_k^N \rangle \mathrm{d}s \\ &\le \|\partial_t \ell\|_{L^\infty(0,T;\mathcal{V}^*)} \cdot (t_{k+1}^N - t_k^N) \cdot \|q_{k+1}^N - q_k^N\|_{\mathcal{V}} \, . \end{aligned}$$

We conclude that

$$\|q_{k+1}^N - q_k^N\|_{\mathcal{V}} \le (t_{k+1}^N - t_k^N)\sqrt{\beta}\|\partial_t \ell\|_{L^\infty(0,T;\mathcal{Q}^*)}\,, \tag{2.12}$$

where we used that $\|q\| \le \beta^{-1/2}\|q\|_{\mathcal{V}}$ for $q \in \mathcal{V}$ and therefore $\|\alpha\|_{\mathcal{V}^*} \le \sqrt{\beta}\|\alpha\|$ for $\alpha \in \mathcal{Q}^*$. We define $\overline{q}^N : [0,T] \to \mathcal{Q}$ and $\hat{q}^N : [0,T] \to \mathcal{Q}$ to be the piecewise constant and piecewise affine interpolations:

$$\overline{q}^N(t) := q_{k-1}^N\,, \qquad \overline{q}^N(T) := q_N^N\,,$$
$$\hat{q}^N(t) := \frac{t_k^N - t}{t_k^N - t_{k-1}^N} q_{k-1}^N + \frac{t - t_{k-1}^N}{t_k^N - t_{k-1}^N} q_k^N\,, \qquad \hat{q}^N(T) := q_N^N\,,$$

for $t \in [t_{k-1}^N, t_k^N)$ and $1 \le k \le N$. From (2.12) we derive the estimates

$$\|\partial_t \hat{q}^N\|_{L^\infty(0,T;\mathcal{V})} \le \sqrt{\beta}\|\partial_t \ell\|_{L^\infty(0,T;\mathcal{Q}^*)}\,, \tag{2.13}$$
$$\|\overline{q}^N - \hat{q}^N\|_{L^\infty(0,T;\mathcal{V})} \le \sqrt{\beta}\Delta t^N \|\partial_t \ell\|_{L^\infty(0,T;\mathcal{Q}^*)}\,, \tag{2.14}$$

where $\Delta t^N := \max_{1\le k\le N}(t_k^N - t_{k-1}^N)$.

Step 2: Selection of a subsequence. We now choose a sequence of partitions such that $\Delta t^N \to 0$ as $N \to \infty$. As $\hat{q}^N$ is uniformly bounded in $W^{1,\infty}(0,T;\mathcal{V})$ by (2.13), we can apply the Arzelà-Ascoli Theorem (Lemma A.2) in order to find a subsequence and a limit function $q \in W^{1,\infty}(0,T;\mathcal{V})$ such that

$$\hat{q}^N(t) \overset{\mathcal{V}}{\rightharpoonup} q(t) \qquad \text{(and thus by (2.14) also } \overline{q}^N(t) \overset{\mathcal{V}}{\rightharpoonup} q(t))$$

for all $t \in [0,T]$. In particular $q(0) = q_0$.

Step 3: Stability of the limit function. For any $N \in \mathbb{N},\ 1 \le k \le N$ and $q \in \mathcal{Q}$, one has, by the definition of q_k^N and the triangle inequality for $\mathcal{R}$,

$$\begin{aligned} 0 &\le \mathcal{E}(t_k^N, q) + \mathcal{R}(q - q_{k-1}^N) - \big(\mathcal{E}(t_k^N, q_k^N) + \mathcal{R}(q_k^N - q_{k-1}^N)\big) \\ &\le \mathcal{E}(t_k^N, q) - \mathcal{E}(t_k^N, q_k^N) + \mathcal{R}(q - q_k^N) \end{aligned} \tag{2.15}$$

which is just the stability of q_k^N at $t = t_k^N$. Given any $t \in [0,T]$, we choose $1 \le k_N \le N$ such that $t^N := t_{k_N}^N \to t$ as $N \to \infty$. Then also $q^N := q_{k_N}^N \rightharpoonup q(t)$ in $\mathcal{Q}$ as $N \to \infty$. Indeed, $q^N = \hat{q}^N(t^N) - \hat{q}^N(t) + \hat{q}^N(t)$ with $\|\hat{q}^N(t^N) - \hat{q}^N(t)\| \le |t^N - t|\sqrt{\beta}\|\partial_t \ell\|_{L^\infty(0,T;\mathcal{Q})} \to 0$ and $\hat{q}^N(t) \rightharpoonup q(t)$ in $\mathcal{Q}$. Inserting $q + (q^N - q(t))$ for q in (2.15), we have, as $N \to \infty$,

$$\begin{aligned} 0 &\le \|q + q^N - q(t)\|_{\mathcal{V}}^2 - \|q^N\|_{\mathcal{V}}^2 - \langle \ell(t^N), q - q(t)\rangle + \mathcal{R}(q - q(t)) \\ &= \|q - q(t)\|_{\mathcal{V}}^2 + 2\langle q - q(t), q^N\rangle_{\mathcal{V}} - \langle \ell(t^N), q - q(t)\rangle + \mathcal{R}(q - q(t)) \\ &\to \|q - q(t)\|_{\mathcal{V}}^2 + 2\langle q - q(t), q(t)\rangle_{\mathcal{V}} - \langle \ell(t), q - q(t)\rangle + \mathcal{R}(q - q(t)) \\ &= \|q\|_{\mathcal{V}}^2 - \|q(t)\|_{\mathcal{V}}^2 - \langle \ell(t), q - q(t)\rangle + \mathcal{R}(q - q(t)) \\ &= \mathcal{E}(t, q) - \mathcal{E}(t, q(t)) + \mathcal{R}(q - q(t))\,, \end{aligned}$$

which is the desired stability.

Step 4: Upper energy estimate. By the definition of q_k^N,

$$\begin{aligned}\mathcal{E}(t_k^N, q_k^N) + \mathcal{R}(q_k^N - q_{k-1}^N) &\le \mathcal{E}(t_k^N, q_{k-1}^N) \\ &= \mathcal{E}(t_{k-1}^N, q_{k-1}^N) - \int_{t_{k-1}^N}^{t_k^N} \langle \partial_s \ell(s), q_{k-1}^N \rangle \mathrm{d}s\,.\end{aligned}$$

Summing this inequality from $k = 1$ to $k = k_N \le N$ and using the definition of $\mathrm{Diss}_{\mathcal{R}}$ yields

$$\mathcal{E}(t^N, q^N) + \mathrm{Diss}_{\mathcal{R}}(\overline{q}^N; [0, t^N]) \le \mathcal{E}(0, q_0) - \int_0^{t^N} \langle \partial_s \ell(s), \overline{q}^N(s) \rangle \mathrm{d}s\,.$$

Using lower semi-continuity on the left-hand side, and dominated convergence on the right-hand side, we get in the limit $N \to \infty$:

$$\mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{R}}(q; [0, t]) \le \mathcal{E}(0, q(0)) - \int_0^t \langle \partial_s \ell(s), q(s) \rangle \mathrm{d}s\,.$$

Step 5: Lower energy estimate. The lower energy estimate follows from stability of the limit function q. Let $t \in [0, T]$. Given any $N \in \mathbb{N}$, we let $t_k := \frac{tk}{N}$ for $0 \le k \le N$. For $1 \le k \le N$ we have

$$\begin{aligned}&\mathcal{E}(t_k, q(t_k)) + \mathcal{R}(q(t_k) - q(t_{k-1})) \\ &= -\int_{t_{k-1}}^{t_k} \langle \partial_s \ell(s), q(t_k) \rangle \mathrm{d}s + \mathcal{E}(t_{k-1}, q(t_k)) + \mathcal{R}(q(t_k), q(t_{k-1})) \\ &\ge -\int_{t_{k-1}}^{t_k} \langle \partial_s \ell(s), q(t_k) \rangle \mathrm{d}s + \mathcal{E}(t_{k-1}, q(t_{k-1}))\,.\end{aligned}$$

Summation over $1 \le k \le N$ gives us

$$\mathcal{E}(t, q(t)) + \sum_{k=1}^N \mathcal{R}(q(t_k) - q(t_{k-1})) \ge \mathcal{E}(0, q(0)) - \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \langle \partial_s \ell(s), q(t_k) \rangle \mathrm{d}s\,.$$

By the definition of $\mathrm{Diss}_{\mathcal{R}}$,

$$\mathrm{Diss}_{\mathcal{R}}(q; [0, t]) \ge \sum_{k=1}^N \mathcal{R}(q(t_k) - q(t_{k-1}))\,.$$

On the other hand,

$$\sum_{k=1}^N \int_{t_{k-1}}^{t_k} \langle \partial_s \ell(s), q(t_k) \rangle \mathrm{d}s \to \int_0^t \langle \partial_s \ell(s), q(s) \rangle \mathrm{d}s$$

as $N \to \infty$ which follows with the continuity of q from the dominated convergence theorem. In combination, we arrive at

$$\mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{R}}(q; [0, t]) \geq \mathcal{E}(0, q(t)) - \int_0^t \langle \partial_s \ell(s), q(s) \rangle \mathrm{d}s \,.$$

Step 6: Regularity and uniqueness. We first show that all solutions are Lipschitz continuous. Then we prove that Lipschitz continuous solutions are unique.

Similar to what we did in *Step 1*, we consider for $t \in [0, T]$ the functional

$$\mathcal{I}(q) \coloneqq \mathcal{E}(t, q) + \mathcal{R}(q - q(t)) \,, \qquad q \in \mathcal{Q} \,.$$

By the stability of $q(t)$ at time t, $q(t)$ minimizes $\mathcal{I}$, and by Lemma A.1 we have the estimate

$$\|q - q(t)\|_{\mathcal{V}}^2 \leq \mathcal{I}(q) - \mathcal{I}(q(t)) \,, \qquad q \in \mathcal{Q} \,.$$

With $q = q(t')$ for $t' > t$ this implies

$$\begin{aligned}
\|q(t') - q(t)\|_{\mathcal{V}}^2 &\leq \mathcal{E}(t, q(t')) + \mathcal{R}(q(t') - q(t)) - \mathcal{E}(t, q(t)) \\
&\leq \mathcal{E}(t', q(t')) - \int_t^{t'} \partial_s \mathcal{E}(s, q(t'))\mathrm{d}s + \mathrm{Diss}_{\mathcal{R}}(q; [t, t']) - \mathcal{E}(t, q(t)) \\
&= \int_t^{t'} \langle \partial_s \ell(s), q(t') - q(s) \rangle \mathrm{d}s \qquad \text{(by the energy equality)} \\
&\leq \int_t^{t'} \|\partial_s \ell(s)\|_{\mathcal{V}'} \|q(t') - q(s)\|_{\mathcal{V}} \mathrm{d}s \,.
\end{aligned}$$

By Lemma B.1 this implies $q \in W^{1,\infty}(0, T; \mathcal{V})$ and $\|\partial_t q(t)\|_{\mathcal{V}} \leq \frac{1}{2} \|\partial_t \ell(t)\|_{\mathcal{V}^*}$. In particular $q \in W^{1,\infty}(0, T; \mathcal{Q})$ and $\|\partial_t q(t)\| \leq \frac{1}{2\beta} \|\partial_t \ell(t)\|$.

Suppose $q_1, q_2 \in W^{1,\infty}(0, T; \mathcal{Q})$ are solutions for the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$ with $q_1(0) = q_2(0)$. We claim that $q_1 = q_2$.

The stability relation for q_j gives for $q \in \mathcal{V}$:

$$\begin{aligned}
0 &\leq \frac{1}{\varepsilon} \Big(\|q_j(t) + \varepsilon q\|_{\mathcal{V}}^2 + \mathcal{R}(\varepsilon q) - \|q_j(t)\|_{\mathcal{V}}^2 - \langle \ell(t), \varepsilon q \rangle \Big) \\
&= \langle 2 q_j(t) - \ell(t), q \rangle_{\mathcal{V}} + \mathcal{R}(q) + \varepsilon \|q\|_{\mathcal{V}}
\end{aligned}$$

for all $\varepsilon > 0$ and therefore

$$0 \leq \langle 2 q_j(t) - \ell(t), q \rangle_{\mathcal{V}} + \mathcal{R}(q) \qquad \text{for all } q \in \mathcal{V} \,. \tag{2.16}$$

On the other hand, the energy equality gives us

$$\mathcal{E}(0, q_j(0)) = \|q_j(t)\|_{\mathcal{V}}^2 - \langle \ell(t), q_j(t)\rangle + \int_0^t R(\partial_s q_j(s))\mathrm{d}s + \int_0^t \langle \partial_s \ell(s), q_j(s)\rangle \mathrm{d}s$$

Taking the time-derivative in the sense of distributions, this yields

$$0 = \langle 2q_j(t) - \ell(t), \partial_t q_j(t)\rangle_{\mathcal{V}} + R(\partial_t q_j(t))\,. \tag{2.17}$$

Applying (2.16) and (2.17), we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|q_1(t) - q_2(t)\|_{\mathcal{V}}^2 &= 2\langle q_1(t) - q_2(t), \partial_t q_1(t) - \partial_t q_2(t)\rangle_{\mathcal{V}} \\ &= \langle 2q_1(t) - \ell(t), \partial_t q_1(t)\rangle_{\mathcal{V}} + \mathcal{R}(\partial_t q_1(t)) \\ &\quad + \langle 2q_2(t) - \ell(t), \partial_t q_2(t)\rangle_{\mathcal{V}} + \mathcal{R}(\partial_t q_2(t)) \\ &\quad - \langle 2q_1(t) - \ell(t), \partial_t q_2(t)\rangle_{\mathcal{V}} - \mathcal{R}(\partial_t q_1(t)) \\ &\quad - \langle 2q_2(t) - \ell(t), \partial_t q_1(t)\rangle_{\mathcal{V}} - \mathcal{R}(\partial_t q_2(t)) \\ &\leq 0\,.\end{aligned}$$

Hence $q_1(t) = q_2(t)$ for all $t \in [0, T]$. □

Chapter 3

Evolutionary Γ-convergence

In this chapter we provide a survey of a method that enables our proof of Theorem 4.2. The notion of Γ-convergence, developed by DeGiorgi [18], is primarily designed to deal with static problems of energy minimization. However, it can also be employed to show that solutions of rate-independent systems $(\mathcal{Q}, \mathcal{E}^\varepsilon, \mathcal{R}^\varepsilon)$ converge to solutions of a rate-indepentent system $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$. This was first explored in [43]. We recall the theory from [42, Section 3.5.4] in the special case of quadratic energies.

3.1 Γ-convergence and Mosco-convergence

Definition 3.1 (Γ-convergence). *Let X denote a topological space, and $f_\varepsilon : X \to \mathbb{R}_\infty$ a sequence of functionals. We say that f_ε converges in the sense of Γ-convergence to a limit functional $f : X \to \mathbb{R}_\infty$ if the following two conditions are satisfied:*

(i) Lower bound: For every sequence $(x_\varepsilon)_\varepsilon \subset X$ with $x_\varepsilon \to x$ in X there holds

$$f(x) \leq \liminf_{\varepsilon \to 0} f_\varepsilon(x_\varepsilon) .$$

(ii) Upper bound: For every $x \in X$ there exists a sequence $(x_\varepsilon)_\varepsilon \subset X$ such that $x_\varepsilon \to x$ in X and

$$f(x) \geq \limsup_{\varepsilon \to 0} f_\varepsilon(x_\varepsilon) .$$

Any such sequence $(x_\varepsilon)_\varepsilon$ is called a recovery sequence *for x.*

In this case we write $f_\varepsilon \xrightarrow{\Gamma} f$. When X is a Banach space equipped with its weak topology, we write $f_\varepsilon \overset{\Gamma}{\rightharpoonup} f$.

Remark. The notion defined above is generally known as *sequential* Γ-convergence. By naming it simply Γ-convergence, we deviate from the terminology commonly used in the literature. In the literature, Γ-convergence is defined in terms of the open sets of the underlying topology: The functional f is the Γ-limit of f_ε if

$$f(x) = \sup_{U \in \mathcal{N}(x)} \liminf_{\varepsilon \to 0} \inf_{x_\varepsilon \in U} f_\varepsilon(x_\varepsilon) = \sup_{U \in \mathcal{N}(x)} \limsup_{\varepsilon \to 0} \inf_{x_\varepsilon \in U} f_\varepsilon(x_\varepsilon)$$

for all $x \in X$, where $\mathcal{N}(x)$ denotes the family of all open sets in X that contain x. The two notions coincide when the underlying toplogy is first-countable [38, Proposition 8.1]. In particular this is the case when the topology is metrizable. Although we will use Γ-convergence with respect to the weak topology of Banach spaces, and the weak topology is not metrizable, it *is* metrizable on bounded sets when the Banach space is reflexive and separable. When the functionals are equicoercive, as will be the case in our application, this can be shown to imply that the notions coincide again [38, Propopsition 8.10]. We will, however, avoid these questions altogether by directly employing the above given definition in terms of sequences. This is possible as we make no use of results from the literature (which would be stated in terms of the commonly used definition of Γ-convergence).

We gather a few well-known facts about Γ-convergence which will be used later on.

Lemma 3.2 (Elementary properties of Γ-convergence). *Suppose* $f_\varepsilon \xrightarrow{\Gamma} f$ *on a topological space* X*.*

(i) Suppose $g : X \to \mathbb{R}$ *is continuous. Then* $f_\varepsilon + g \xrightarrow{\Gamma} f + g$*.*

(ii) Suppose $(x_\varepsilon)_\varepsilon$ *is a sequence of almost-minimizers of* f_ε*, that is,*

$$f_\varepsilon(x_\varepsilon) - \inf f_\varepsilon \to 0 \quad \textit{for } \varepsilon \to 0\,.$$

Suppose further that $x_\varepsilon \to x$ *for some* $x \in X$*. Then* x *is a minimizer of* f *and* $f_\varepsilon(x_\varepsilon) \to f(x)$*.*

(iii) Suppose that f *is not identical* $+\infty$*. Suppose further that* $(f_\varepsilon)_\varepsilon$ *is equi-compact in the sense that any sequence* $(x_\varepsilon)_\varepsilon$ *for which* $f_\varepsilon(x_\varepsilon)$ *is uniformly bounded contains a convergent subsequence.*

Then every sequence of almost-minimizers of f_ε *contains a convergent subsequence.*

(iv) If X *is metrizable,* f *is lower-semicontinuous.*

Proof. *(i)* This easily follows from the definition of Γ-convergence since $g(x_\varepsilon) \to g(x)$ whenever $x_\varepsilon \to x$.

(ii) Let $y \in X$ and consider a recovery sequence $(y_\varepsilon)_\varepsilon$ for y. Then

$$f(x) \le \liminf_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon) \le \limsup_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon) \le \limsup_{\varepsilon\to 0} f_\varepsilon(y_\varepsilon) \le f(y)\,.$$

As this can be done for every $y \in X$, we conclude that x is a minimizer of f. Moreover, choosing $y = x$, we see that the chain of inequalities becomes an equality and therefore $f_\varepsilon(x_\varepsilon) \to f(x)$.

(iii) Let $(x_\varepsilon)_\varepsilon$ denote a sequence of almost-minimizers of f_ε. Consider any $y \in X$ with $f(y) < \infty$ and let $(y_\varepsilon)_\varepsilon$ denote a recovery sequence for y. Then

$$\limsup_{\varepsilon\to 0} f_\varepsilon(x_\varepsilon) \le \limsup_{\varepsilon\to 0} f_\varepsilon(y_\varepsilon) \le f(y) < \infty\,.$$

Thus $f_\varepsilon(x_\varepsilon)$ is bounded along all sequences $\varepsilon \to 0$. Therefore, by the equi-covercivity of $(f_\varepsilon)_\varepsilon$ there is some convergent subsequence of $(x_\varepsilon)_\varepsilon$.

(iv) Let $x_n \to x$ as $n \to \infty$ and $\delta > 0$. We work with an arbitrary subsequence $(f_m)_m$ of $(f_\varepsilon)_\varepsilon$. For all $n \in \mathbb{N}$ there exists a recovery sequence $(x_n^m)_m$ with

$$x_n^m \to x_n\,, \qquad f(x_n) \ge \limsup_{m\to\infty} f_m(x_n^m)\,.$$

Since X is metrizable, we can choose $(m_n)_n$ such that $m_n \to \infty$ and $x_n^{m_n} \to x$ as $n \to \infty$ as well as

$$f(x_n) + \delta \ge f_{m_n}(x_n^{m_n}) \qquad \text{for all } n \in \mathbb{N}\,.$$

This implies

$$\liminf_{n\to\infty} f(x_n) + \delta \ge \liminf_{n\to\infty} f_{m_n}(x_n^{m_n}) \ge f(x)\,,$$

where the last inequality follows from the lower bound of $f_{m_n} \xrightarrow{\Gamma} f$. As $\delta > 0$ was arbitrary, this finishes the proof. □

We also need the notion of *Mosco-convergence.* Mosco-convergence is Γ-convergence for functionals on a Banach space with respect to *both* weak and strong convergence.

Definition 3.3 (Mosco-convergence). *Let X denote a Banach space, and $f_\varepsilon : X \to \mathbb{R}_\infty$ a sequence of functionals. We say that f_ε converges in the sense of Mosco-convergence to a limit functional $f : X \to \mathbb{R}_\infty$ if the following two conditions are satisfied:*

(i) Lower bound: For every sequence $(x_\varepsilon)_\varepsilon \subset X$ *with* $x_\varepsilon \rightharpoonup x$ *in* X *there holds*

$$f(x) \leq \liminf_{\varepsilon \to 0} f_\varepsilon(x_\varepsilon)\,.$$

(ii) Upper bound: For every $x \in X$ *there exists a sequence* $(x_\varepsilon)_\varepsilon \subset X$ *such that* $x_\varepsilon \to x$ *in* X *and*

$$f(x) \geq \limsup_{\varepsilon \to 0} f_\varepsilon(x_\varepsilon)\,.$$

Any such sequence $(x_\varepsilon)_\varepsilon$ *is called a* recovery sequence *for* x*.*

In this case we write $f_\varepsilon \overset{\mathcal{M}}{\to} f$*.*

3.2 Abstract convergence result

We work with a separable Hilbert space $\mathcal{Q}$ as our state space. For each $\varepsilon \in [0,1]$, where $\varepsilon = 0$ corresponds to the limit, we have three ingredients:

(a) a *stored energy functional* $\mathcal{B}^\varepsilon : \mathcal{Q} \to [0,\infty]$,

(b) a *dissipation functional* $\mathcal{R}^\varepsilon : \mathcal{Q} \to [0,\infty]$,

(c) *loads* $\ell^\varepsilon \in W^{1,\infty}(0,T;\mathcal{Q}^*)$.

As a derived quantity we have the *total energy*

$$\mathcal{E}^\varepsilon : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty\,, \qquad \mathcal{E}^\varepsilon(t,q) := \mathcal{B}^\varepsilon(q) - \langle \ell^\varepsilon(t), q\rangle\,.$$

For $\varepsilon > 0$ we make the following assumptions:

(A) The stored energy functionals $\mathcal{B}^\varepsilon$ are quadratic forms and lower semicontinuous. Moreover, they satisfy an equicoercivity estimate

$$\beta \|q\|^2 \leq \mathcal{B}^\varepsilon(q) \quad \forall\, q \in \mathcal{Q}$$

for some $\beta > 0$.

(B) The dissipation functionals $\mathcal{R}^\varepsilon$ are positive one-homogeneous, convex and lower-semicontinuous.

(C) The loads satisfy a uniform Lipschitz bound

$$\|\ell^\varepsilon\|_{W^{1,\infty}(0,T;\mathcal{Q}^*)} \leq C\,.$$

(D) The following convergences hold:

$$\mathcal{B}^\varepsilon \overset{\mathcal{M}}{\to} \mathcal{B}^0 \qquad\qquad \mathcal{R}^\varepsilon \overset{\mathcal{M}}{\to} \mathcal{R}^0$$
$$\mathcal{R}^\varepsilon \overset{c}{\to} \mathcal{R}^0 \qquad\qquad \ell^\varepsilon(t) \to \ell^0(t) \quad \forall\, t \in [0,T]\,.$$

Here, $\mathcal{R}^\varepsilon \overset{c}{\to} \mathcal{R}^0$ denotes *continuous convergence*, which means that $\mathcal{R}^\varepsilon(q^\varepsilon) \to \mathcal{R}^0(q)$ whenever $q^\varepsilon \to q$. In association with Mosco-convergence this implies that *every* strongly convergent sequence is a recovery sequence.

Remark. (i) The assumed convergences in (D) imply that the assumptions (A)–(C), which were made only for $\varepsilon > 0$, still hold true for $\varepsilon = 0$.

(ii) The continuous convergence $\mathcal{R}^\varepsilon \overset{c}{\to} \mathcal{R}^0$ implies that $\mathcal{R}^0$ is continuous. Indeed, the continuous convergence implies both $\mathcal{R}^\varepsilon \overset{\Gamma}{\to} \mathcal{R}^0$ and $-\mathcal{R}^\varepsilon \overset{\Gamma}{\to} -\mathcal{R}^0$. Thus by Lemma 3.2(iv), both $\mathcal{R}^0$ and $-\mathcal{R}^0$ are lower-semicontinuous. Hence $\mathcal{R}^0$ is continuous.

Theorem 3.4 (Convergence, see [42, Theorem 3.5.14]). *Let* $(\mathcal{Q}, \mathcal{E}^\varepsilon, \mathcal{R}^\varepsilon)$ *for* $\varepsilon \in [0,1]$ *be a family of rate-independent systems that satisfies the assumptions* (A)–(D) stated above. Consider a corresponding family of energetic solutions $q^\varepsilon : [0,T] \to \mathcal{Q}$ with

$$q^\varepsilon(0) \rightharpoonup q^0(0)\,, \qquad \mathcal{B}^\varepsilon(q^\varepsilon(0)) \to \mathcal{B}^0(q^0(0))$$

as $\varepsilon \to 0$. Then also

$$q^\varepsilon(t) \to q^0(t)\,, \qquad \mathcal{B}^\varepsilon(q^\varepsilon(t)) \to \mathcal{B}^0(q^0(t))$$

for all $t \in [0,T]$ as $\varepsilon \to 0$. Moreover,

$$\mathrm{Diss}_{\mathcal{R}^\varepsilon}(q^\varepsilon; [0,t]) \to \mathrm{Diss}_{\mathcal{R}^0}(q^0; [0,t])\,, \qquad \langle \partial_t \ell^\varepsilon(t), q^\varepsilon(t)\rangle \to \langle \partial_t \ell^0(t), q^0(t)\rangle\,.$$

The proof of this theorem is given in Section 3.4.

Lemma 3.5 (Lower bound for the total dissipation). *Let* $(\mathcal{R}^\varepsilon)_\varepsilon$ *satisfy the assumptions outlined in this section. We assume further that* $q^\varepsilon : [0,T] \to \mathcal{Q}$ *with* $q^\varepsilon(t) \rightharpoonup q^0(t)$ *for all* $t \in [0,T]$. *Then*

$$\mathrm{Diss}_{\mathcal{R}^0}(q^0; [s,t]) \le \liminf_{\varepsilon \to 0} \mathrm{Diss}_{\mathcal{R}^\varepsilon}(q^\varepsilon; [s,t])$$

for all $[s,t] \subset [0,T]$.

Proof. Let $\varepsilon > 0$. Then there exist $n \in \mathbb{N}$ and $s = t_0 \leq \cdots \leq t_n = t$ such that

$$\begin{aligned}
\mathrm{Diss}_{\mathcal{R}^0}(q^0;[s,t]) - \varepsilon &\leq \sum_{k=1}^{n} \mathcal{R}^0(q^0(t_k) - q^0(t_{k-1})) \\
&\leq \sum_{k=1}^{n} \liminf_{\varepsilon\to 0} \mathcal{R}^\varepsilon(q^\varepsilon(t_k) - q^\varepsilon(t_{k-1})) \\
&\leq \liminf_{\varepsilon\to 0} \sum_{k=1}^{n} \mathcal{R}^\varepsilon(q^\varepsilon(t_k) - q^\varepsilon(t_{k-1})) \\
&\leq \liminf_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{R}^\varepsilon}(q^\varepsilon;[s,t]) \,,
\end{aligned}$$

where we used the lower bound property of $\mathcal{R}^\varepsilon \xrightharpoonup{\Gamma} \mathcal{R}^0$. □

3.3 Quadratic forms

In this section, $\mathcal{Q}$ denotes any separable Hilbert space. We consider a family of lower-semicontinuous quadratic forms

$$\mathcal{A}_\varepsilon : \mathcal{Q} \to \mathbb{R}_\infty \,, \qquad \varepsilon \in [0,1] \,,$$

which satisfies an equi-coercivity estimate: $\beta\|q\|^2 \leq \mathcal{A}_\varepsilon(q)$ for some $\beta > 0$ and all $q \in \mathcal{Q}$. As above, for $\mathcal{A}_\varepsilon$ to be a quadratic form means that

$$\mathrm{dom}\,\mathcal{A}_\varepsilon = \{q \in \mathcal{Q} \;:\; \mathcal{A}_\varepsilon(q) < \infty\}$$

is a linear subspace of $\mathcal{Q}$ and that the map

$$(\mathrm{dom}\,\mathcal{A}_\varepsilon)^2 \to \mathbb{R} \,, \qquad (q,q') \mapsto \frac{1}{4}\Big(\mathcal{B}(q+q') - \mathcal{B}(q-q')\Big)$$

is a bilinear form.

We begin with a lemma that is similar to Proposition 3.5.16 in [42]. Unter the assumption of Mosco-convergence, it first shows that all weakly converging recovery sequences for $\mathcal{A}^\varepsilon$ indeed converge strongly. (We know that strongly converging recovery sequences do exist, but a priori there could be strictly more weakly converging recovery sequences.) It then shows the existence of so-called *mutual recovery sequences* $(q_\varepsilon)_\varepsilon$. The Lemma crucially relies on the quadratic nature of the functionals. It employs a "quadratic trick" which was first introduced in [45] for homogenization in elastoplasticity.

Lemma 3.6. *Suppose $(\mathcal{A}_\varepsilon)_{\varepsilon\geq 0}$ is a family of lower-semicontinuous quadratic forms on a separable Hilbert space $\mathcal{Q}$ with $\mathcal{A}_\varepsilon \xrightarrow{\mathcal{M}} \mathcal{A}_0$. Then there holds:*

(i) If $q_\varepsilon \rightharpoonup q_0$ *and* $\mathcal{A}_\varepsilon(q_\varepsilon) \to \mathcal{A}_0(q_0) < \infty$*, then* $q_\varepsilon \to q_0$*.*
That is: all recovery sequences converge strongly.

(ii) We endow $\mathcal{V}_0 := \{q \in \mathcal{Q} \,:\, \mathcal{A}_0(q) < \infty\}$ *with the norm* $\|\cdot\|_0 := \mathcal{A}_0(\cdot)^{1/2}$*.*
Then there exists a dense subset $\mathcal{D} \subset \mathcal{V}_0$ *with the following property:*
For every $q_0 \in \mathcal{D}$ *there exists a sequence* $(q_\varepsilon)_\varepsilon$ *in* $\mathcal{Q}$ *such that*

(A) $q_\varepsilon \to q_0$ *and* $\mathcal{A}_\varepsilon(q_\varepsilon) \to \mathcal{A}_0(q_0)$

(B) if $\tilde{q}_\varepsilon \rightharpoonup \tilde{q}_0$ *for some sequence* $(\tilde{q}_\varepsilon)_{\varepsilon \geq 0}$ *in* $\mathcal{Q}$ *with* $\sup_{\varepsilon>0} \mathcal{A}_\varepsilon(\tilde{q}_\varepsilon) < \infty$*, then*

$$\mathcal{A}_\varepsilon(q_\varepsilon + \tilde{q}_\varepsilon) - \mathcal{A}_\varepsilon(\tilde{q}_\varepsilon) \to \mathcal{A}_0(q_0 + \tilde{q}_0) - \mathcal{A}_0(\tilde{q}_0)\,. \tag{3.1}$$

Remark. (i) The sequence (q_ε) in part (ii) of the lemma is a *mutual recovery sequence* in the following sense: (a) it is a recovery sequence for $\mathcal{R}^\varepsilon$ because it converges strongly and all strongly converging sequences are recovery sequences for $\mathcal{R}^\varepsilon$; and (b) it is by (3.1) also a recovery sequence for $\mathcal{A}_\varepsilon(\cdot + \tilde{q}_\varepsilon) - \mathcal{A}_\varepsilon(\tilde{q}_\varepsilon)$. This property will be helpful for the limit passage in the stability relation in Theorem 3.4 (*Step 3* of the proof).

(ii) Our proof is somewhat simplified as compared to the one in [42]. In particular our proof of part (i), which is part (ii) in [42], bypasses the intricate constructions of several linear maps related to $\mathcal{A}_\varepsilon$, subspaces of $\mathcal{Q}$ and projections onto these subspaces, which we were not able to figure out in full detail. Instead, it is a simple application of the parallelogram identity. In part (ii) of our proof (corresponding to part (i) in [42]) a few of these constructions resurface, although in a quite transparent manner.

Proof. Part (i). Let $q_\varepsilon \rightharpoonup q_0$ be a weakly convergent sequence in $\mathcal{Q}$ with $\mathcal{A}_\varepsilon(q_\varepsilon) \to \mathcal{A}_0(q_0)$ and $q_0 \in \operatorname{dom} \mathcal{A}_0$. We know from the definition of Mosco-convergence that there exists a strongly converging recovery sequence $(\tilde{q}_\varepsilon)_\varepsilon$ for q_0,

$$\tilde{q}_\varepsilon \to q_0\,, \qquad \mathcal{A}_\varepsilon(\tilde{q}_\varepsilon) \to \mathcal{A}_0(q_0)\,.$$

Using the equi-covercivity and the parallelogram identity for $\mathcal{A}_\varepsilon$, which holds since $\mathcal{A}_\varepsilon$ is the square of a norm induced by an inner product, we conclude that

$$\beta\|\tilde{q}_\varepsilon - q_\varepsilon\|^2 \leq \mathcal{A}_\varepsilon(\tilde{q}_\varepsilon - q_\varepsilon) = 2\mathcal{A}_\varepsilon(\tilde{q}_\varepsilon) + 2\mathcal{A}_\varepsilon(q_\varepsilon) - \mathcal{A}_\varepsilon(\tilde{q}_\varepsilon + q_\varepsilon)\,.$$

Taking the limes superior on both sides, we get

$$\beta \limsup_{\varepsilon\to 0}\|\tilde{q}_\varepsilon - q_\varepsilon\|^2 \leq 2\mathcal{A}_0(q_0) + 2\mathcal{A}_0(q_0) - \liminf_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\tilde{q}_\varepsilon + q_\varepsilon) \leq 0\,,$$

since $\tilde{q}_\varepsilon + q_\varepsilon \rightharpoonup 2q_0$ and thus $\liminf_{\varepsilon\to 0} \mathcal{A}_\varepsilon(\tilde{q}_\varepsilon + q_\varepsilon) \geq \mathcal{A}_0(2q) = 4\mathcal{A}_0(q_0)$. We conclude that $q_\varepsilon = (q_\varepsilon - \tilde{q}_\varepsilon) + \tilde{q}_\varepsilon \to q_0$.

Part (ii). We denote by

$$\mathcal{V}_\varepsilon \coloneqq \{q \in \mathcal{Q} \ : \ \mathcal{A}_\varepsilon(q) < \infty\}, \qquad \varepsilon \geq 0\,,$$

the domains of the quadratic forms $\mathcal{A}_\varepsilon$, which (by Lemma 2.2) are Hilbert spaces when we equip them with the norms $\|q\|_\varepsilon \coloneqq \mathcal{A}_\varepsilon(q)^{1/2}$. The corresponding inner products are denoted $\langle \cdot\,, \cdot \rangle_\varepsilon$.

We denote by $\mathcal{Q}_0$ the closure of $\mathcal{V}_0$ in $\mathcal{Q}$. By the coercicity of $\mathcal{A}_0$, the inclusion map $\jmath : \mathcal{V}_0 \hookrightarrow \mathcal{Q}_0$ is a bounded linear operator. As $\jmath$ injective, its Hilbert adjoint $\jmath' : \mathcal{Q}_0 \to \mathcal{V}_0$ has dense image $\mathcal{D} \coloneqq \jmath'(\mathcal{Q}_0) \subset \mathcal{V}_0$ (see part (i) of Lemma A.5). Moreover, as $\mathcal{V}_0$ is dense in $\mathcal{Q}_0$, $\jmath'$ is injective (see part (ii) of Lemma A.5). We denote by $A : \mathcal{D} \to \mathcal{Q}_0$ the inverse of $\jmath'$.

Let $q_0 \in \mathcal{D}$. Observe that

$$\langle Aq_0, q\rangle = \langle Aq_0, \jmath(q)\rangle = \langle \jmath'(Aq_0), q\rangle_0 = \langle q_0, q\rangle_0\,, \qquad q \in \mathcal{V}_0 \subset \mathcal{Q}_0\,. \tag{3.2}$$

We define

$$q_\varepsilon \coloneqq \arg\min \left\{ \frac{1}{2}\mathcal{A}_\varepsilon(q) - \langle Aq_0, q\rangle \ : \ q \in \mathcal{Q} \right\}.$$

See (3.3) below for an explicit description of q_ε. By Lemma 3.2(i) we know that

$$\frac{1}{2}\mathcal{A}_\varepsilon - Aq_0 \xrightharpoonup{\Gamma} \frac{1}{2}\mathcal{A}_0 - Aq_0\,.$$

Moreover, the functionals $\frac{1}{2}\mathcal{A}_\varepsilon - Aq_0$ are equi-covercive with respect to the weak convergence in $\mathcal{Q}$. Thus by Lemma 3.2(ii)-(iii) we find that

$$\begin{aligned} q_\varepsilon &\rightharpoonup \arg\min \left\{ \frac{1}{2}\mathcal{A}_0(q) - \langle Aq_0, q\rangle \ : \ q \in \mathcal{Q} \right\} \\ &\overset{(3.2)}{=} \arg\min \left\{ \frac{1}{2}\|q\|_0^2 - \langle q_0, q\rangle_0 \ : \ q \in \mathcal{V}_0 \right\} = q_0 \end{aligned}$$

and $\frac{1}{2}\mathcal{A}_\varepsilon(q_\varepsilon) - \langle Aq_0, q_\varepsilon\rangle \to \frac{1}{2}\mathcal{A}_0(q_0) - \langle Aq_0, q_0\rangle$, which implies $\mathcal{A}_\varepsilon(q_\varepsilon) \to \mathcal{A}_0(q_0)$. Thus $(q_\varepsilon)_\varepsilon$ is a weakly converging recovery sequence for q_0. The strong convergence $q_\varepsilon \to q_0$ then follows from part (i).

We denote by $\imath_\varepsilon : \mathcal{V}_\varepsilon \hookrightarrow \mathcal{Q}$ the inclusion map which is bounded because of the coercivity of $\mathcal{A}_\varepsilon$. Observe that

$$\begin{aligned} q_\varepsilon &= \arg\min \left\{ \frac{1}{2}\|q\|_\varepsilon^2 - \langle Aq_0, \imath_\varepsilon q\rangle \ : \ q \in \mathcal{V}_\varepsilon \right\} \\ &= \arg\min \left\{ \frac{1}{2}\|q\|_\varepsilon^2 - \langle \imath_\varepsilon' Aq_0, q\rangle_\varepsilon \ : \ q \in \mathcal{V}_\varepsilon \right\} = \imath_\varepsilon' Aq_0\,. \end{aligned} \tag{3.3}$$

Given a weakly converging sequence $\tilde{q}_\varepsilon \rightharpoonup \tilde{q}_0$ in $\mathcal{Q}$ with $\sup_{\varepsilon>0} \mathcal{A}_\varepsilon(\tilde{q}_\varepsilon) < \infty$ (and hence also $\mathcal{A}_0(\tilde{q}_0) < \infty$ by the lower bound of $\mathcal{A}_\varepsilon \overset{\Gamma}{\rightharpoonup} \mathcal{A}_0$), we therefore have,

$$\begin{aligned}\mathcal{A}_\varepsilon(q_\varepsilon + \tilde{q}_\varepsilon) - \mathcal{A}_\varepsilon(\tilde{q}_\varepsilon) &= \mathcal{A}_\varepsilon(q_\varepsilon) + 2\langle q_\varepsilon, \tilde{q}_\varepsilon\rangle_\varepsilon \overset{(3.3)}{=} \mathcal{A}_\varepsilon(q_\varepsilon) + 2\langle A q_0, \tilde{q}_\varepsilon\rangle \\ &\to \mathcal{A}_0(q_0) + 2\langle A q_0, \tilde{q}_0\rangle \overset{(3.2)}{=} \mathcal{A}_0(q_0) + 2\langle q_0, \tilde{q}_0\rangle_0 \\ &= \mathcal{A}_0(q_0 + \tilde{q}_0) - \mathcal{A}_0(\tilde{q}_0)\,. \end{aligned}$$ □

3.4 Proof of the abstract convergence result

Proof of Theorem 3.4. Step 1: A priori estimates. By Theorem 2.1, we have a uniform bound

$$\|q^\varepsilon\|_{W^{1,\infty}(0,T;\mathcal{Q})} \le \|q^\varepsilon(0)\|_{\mathcal{Q}} + \frac{T}{2\sqrt{\beta}}\|\partial_t \ell^\varepsilon\|_{L^\infty(0,T;\mathcal{Q})} \le C\,.$$

Step 2: Selection of subsequences. The Arzelá-Ascoli theorem (Lemma A.2) guarantees the existence of a subsequene and a limit function $q \in W^{1,\infty}(0,T;\mathcal{Q})$ such that $q^\varepsilon(t) \rightharpoonup q(t)$ for all $t \in [0,T]$. In particular $q(0) = q^0(0)$. We now show that q is a solution. By the uniqueness of solutions this implies $q = q^0$.

Step 3: Stability of the limit. Fix $t \in [0,T]$. For $\varepsilon > 0$ we have the stability of q^ε at time t. This means that

$$\mathcal{J}^\varepsilon(\overline{q}) := \mathcal{B}^\varepsilon(q^\varepsilon(t) + \overline{q}) - \mathcal{B}^\varepsilon(q^\varepsilon(t)) + \mathcal{R}^\varepsilon(\overline{q}) - \langle \ell^\varepsilon(t), \overline{q}\rangle \ge 0 \quad \text{for all } \overline{q} \in \mathcal{Q}\,.$$

We want to conclude that

$$\mathcal{J}^0(\overline{q}) := \mathcal{B}^0(q(t) + \overline{q}) - \mathcal{B}^0(q^0(t)) + \mathcal{R}^0(\overline{q}) - \langle \ell^0(t), \overline{q}\rangle \ge 0 \quad \text{for all } \overline{q} \in \mathcal{Q}\,.$$

We start by showing $\mathcal{J}^0(\overline{q}) \ge 0$ for $\overline{q} \in \mathcal{D}$ with $\mathcal{D}$ from Lemma 3.6(ii). The Mosco-convergence $\mathcal{B}^\varepsilon \overset{\mathcal{M}}{\to} \mathcal{B}^0$ and Lemma 3.6(ii) imply that we find a sequence $\overline{q}^\varepsilon \to \overline{q}$ in $\mathcal{Q}$ such that

$$\mathcal{B}^\varepsilon(q^\varepsilon(t) + \overline{q}^\varepsilon) - \mathcal{B}^\varepsilon(q^\varepsilon(t)) \to \mathcal{B}^0(q(t) + \overline{q}) - \mathcal{B}^0(q(t))\,,$$

and therefore $\mathcal{J}^0(\overline{q}) = \lim_{\varepsilon\to 0} \mathcal{J}^\varepsilon(\overline{q}^\varepsilon) \ge 0$ by the continuous convergence $\mathcal{R}^\varepsilon \overset{c}{\to} \mathcal{R}^0$ and the strong convergence $\ell^\varepsilon(t) \to \ell^0(t)$ in $\mathcal{Q}^*$. We have thus shown that $\mathcal{J}^0 \ge 0$ on $\mathcal{D}$.

As $\mathcal{D}$ is a dense subset of $\mathcal{V}_0$, we also have $\mathcal{J}^0 \ge 0$ on $\mathcal{V}_0$. Indeed, this immediately follows from the fact that $\mathcal{J}^0|_{\mathcal{V}_0} : \mathcal{V}_0 \to \mathbb{R}$ is continuous w.r.t. the norm $\|\cdot\|_0 = \mathcal{B}^0(\cdot)^{1/2}$ of $\mathcal{V}_0$. This continuity property is easily seen by inspecting each term of the formula

$$\mathcal{J}^0(\overline{q}) = \|q(t)+\overline{q}\|_0^2 - \|q(t)\|_0^2 + \mathcal{R}^0(\jmath\overline{q}) - \langle \ell^0(t), \jmath\overline{q}\rangle\,, \qquad \overline{q}\in\mathcal{V}_0\,,$$

where $\jmath : \mathcal{V}_0 \hookrightarrow \mathcal{Q}$ denotes the continuous inclusion map. (For the continuity of $\mathcal{R}^0$ see Remark (ii) on Page 29.)

It remains to verify $\mathcal{J}^0 \geq 0$ on $\mathcal{Q}\backslash\mathcal{V}_0$, but here we trivially have $\mathcal{J}^0 = \infty \geq 0$.

Step 4: Upper energy estimate. By the lower bound of $\mathcal{E}^\varepsilon(t,\cdot) \xrightharpoonup{\Gamma} \mathcal{E}^0(t,\cdot)$ and Lemma 3.5, the pointwise weak convergence $q^\varepsilon(t) \rightharpoonup q^0(t)$ implies together with the energy balance for q^ε that, for arbitrary t,

$$\begin{aligned}
&\mathcal{E}^0(t,q(t)) + \mathrm{Diss}_{\mathcal{R}^0}(q;[0,t])\\
&\quad\leq \liminf_{\varepsilon\to 0}\big(\mathcal{E}^\varepsilon(t,q^\varepsilon(t)) + \mathrm{Diss}_{\mathcal{R}^\varepsilon}(q^\varepsilon;[0,t])\big)\\
&\quad= \liminf_{\varepsilon\to 0}\left(\mathcal{E}^\varepsilon(0,q^\varepsilon(0)) - \int_0^t \langle\partial_t\ell^\varepsilon(s), q^\varepsilon(s)\rangle \mathrm{d}s\right)\\
&\quad= \mathcal{E}^0(0,q(0)) - \int_0^t \langle \partial_t \ell^0(s), q(s)\rangle\mathrm{d}s\,.
\end{aligned}$$

Regarding the last equality, the reasoning is as follows. With an integration by parts we have

$$\begin{aligned}
-\int_0^t \langle\partial_t\ell^\varepsilon(s), q^\varepsilon(s)\rangle\mathrm{d}s &= \int_0^t \langle \ell^\varepsilon(s), \partial_t q^\varepsilon(s)\rangle \mathrm{d}s - \langle \ell^\varepsilon(t), q^\varepsilon(t)\rangle + \langle\ell^\varepsilon(0), q^\varepsilon(0)\rangle\\
&\to \int_0^t \langle \ell^0(s), \partial_t q(s)\rangle\mathrm{d}s - \langle\ell^0(t), q(t)\rangle + \langle \ell^0(0), q(0)\rangle\\
&= -\int_0^t \langle\partial_t \ell^0(s), q(s)\rangle\mathrm{d}s\,.
\end{aligned}$$

Here the convergence of the boundary parts is obvious since $\ell^\varepsilon(s)\to\ell^0(s)$ in $\mathcal{Q}^*$ and $q^\varepsilon(s)\rightharpoonup q(s)$ in $\mathcal{Q}$ for every $s\in[0,T]$. The integral term however also converges as $\ell^\varepsilon\to\ell^0$ in $L^2(0,T;\mathcal{Q}^*)$ (by dominated convergence) and $q^\varepsilon \rightharpoonup q$ in $L^2(0,T;\mathcal{Q})$.

Step 5: Lower energy estimate. The lower energy estimate can be derived from the stability proved in *Step 3.* Given $t\in[0,T]$, consider a partition

$0 = t_0^N < t_1^N < \ldots < t_N^N = t$. For $1 \leq k \leq N$ we have

$$\begin{aligned}
&\mathcal{E}^0(t_{k+1}^N, q(t_{k+1}^N)) + \mathcal{R}(q(t_{k+1}^N) - q(t_k^N)) \\
&\quad = \int_{t_k^N}^{t_{k+1}^N} \partial_t \mathcal{E}^0(s, q(s))\mathrm{d}s + \mathcal{E}^0(t_k^N, q(t_{k+1}^N)) + \mathcal{R}(q(t_{k+1}^N) - q(t_k^N)) \\
&\qquad \geq - \int_{t_k^N}^{t_{k+1}^N} \langle \partial_t \ell^0(s), q(s)\rangle \mathrm{d}s + \mathcal{E}^0(t_k^N, q(t_k^N))\,,
\end{aligned}$$

where we used the stability of q at t_k^N. Summing this over $1 \leq k \leq N$, we obtain

$$\begin{aligned}
&\mathcal{E}^0(t, q(t)) + \mathrm{Diss}_{\mathcal{R}^0}(q; [0,t]) \\
&\quad \geq \mathcal{E}^0(0, q(0)) - \sum_{k=1}^{N} \int_{t_k^N}^{t_{k+1}^N} \langle \partial_t \ell^0(s), q(t_k^N)\rangle \mathrm{d}s \\
&\quad = \mathcal{E}^0(0, q(0)) - \int_0^t \langle \partial_t \ell^0(s), \overline{q}^N(s)\rangle \mathrm{d}s\,,
\end{aligned}$$

where $\overline{q}^N$ is the piecewise constant approximation of q defined by $\overline{q}^N = q(t_k^N)$ on (t_k^N, t_{k+1}^N). As the fineness of the partition converges to zero, the right-hand side converges to

$$\mathcal{E}^0(0, q(0)) - \int_0^t \langle \partial_t \ell^0(s), q(s)\rangle \mathrm{d}s$$

by the dominated convergence theorem.

Step 6: Improved convergence. We know from *Step 5*, that we have equality in the calculation of *Step 4*. Therefore

$$\mathcal{E}^\varepsilon(t, q^\varepsilon(t)) + \mathrm{Diss}_{\mathcal{R}^\varepsilon}(q^\varepsilon; [0,t]) \to \mathcal{E}^0(t, q(t)) + \mathrm{Diss}_{\mathcal{R}^0}(q; [0,t])\,.$$

But for the individual terms we have the lower bounds

$$\begin{aligned}
\liminf_{\varepsilon\to 0} \mathcal{E}^\varepsilon(t, q^\varepsilon(t)) &\geq \mathcal{E}^0(t, q(t))\,, \\
\liminf_{\varepsilon\to 0} \mathrm{Diss}_{\mathcal{R}^\varepsilon}(q^\varepsilon; [0,t]) &\geq \mathrm{Diss}_{\mathcal{R}^0}(q; [0,t])\,.
\end{aligned}$$

Thus both terms must converge individually,

$$\mathcal{E}^\varepsilon(t, q^\varepsilon(t)) \to \mathcal{E}^0(t, q(t))\,, \qquad \mathrm{Diss}_{\mathcal{R}^\varepsilon}(q^\varepsilon; [0,t]) \to \mathrm{Diss}_{\mathcal{R}^0}(q; [0,t])\,.$$

With the help of Lemma 3.6(i) we conclude that $q^\varepsilon(t) \to q^0(t)$ strongly. □

Chapter 4

Dimension reduction for elastoplastic rods

In this chapter, we study the elastoplastic behaviour of a single rod with a thickness parameter $h > 0$ in the limit $h \to 0$. We use Theorem 3.4 to perform a 3D-1D dimension reduction, i. e. we rigorously derive a material model with which the original model can be replaced when the thickness h is small. Our approach is inspired by [34, 35], where a plate model is derived via 3D-2D dimension reduction. The models derived in [34, 35] are obtained by pointwise minimization of the original energy density in some of its components. With rods, the situation is more complicated and certain global features of the displacements have to be injected into the limit model.

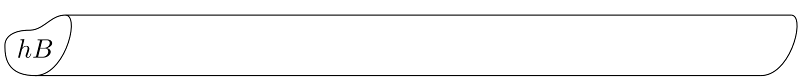
A rod is described by a reference domain $\Omega_h \subset \mathbb{R}^3$,

$$\Omega_h := I \times hB\,, \qquad I := (0, L)\,, \qquad B \subset \mathbb{R}^2\,, \qquad L\,,\ h > 0\,.$$

We assume that B is a bounded Lipschitz domain which is centered in the sense that $\int_B (x_2, x_3)^\top \mathrm{d}x_2 \mathrm{d}x_3 = 0$. For simplicity, we will in this chapter prescribe zero displacements on $\Gamma_h := \partial I \times hB \subset \partial\Omega_h$. When in Chapter 6 we consider lattices of many rods, we will have linear but nonzero displacements at both ends of each rod.

As outlined in Chapter 2, the elastoplastic behaviour of a solid body as

Figure 4.1: Geometry of a rod $\Omega_h = I \times hB$.



above can be described by evolutions $\overline{q} : [0,T] \to \overline{\mathcal{Q}}^h$ in a state space

$$\overline{\mathcal{Q}}^h := H^1_{\Gamma_h}(\Omega_h;\mathbb{R}^3) \times L^2(\Omega_h;\mathbb{R}^{3\times3}_{\mathrm{dev}}) \,,$$

where $H^1_{\Gamma_h}(\Omega;\mathbb{R}^3) := \{u \in H^1(\Omega;\mathbb{R}^3) \ : \ u = 0 \text{ on } \Gamma_h\}$. The driving force of the evolution is a load function $\overline{\ell}^h \in W^{1,\infty}(1,T;(\overline{\mathcal{Q}}^h)^*)$. We use overscored symbols for all variables (and spaces of variables) in physical dimensions. Later on, we will work with rescaled quantities, and there the overscores will disappear.

The rate-independent system that describes the rod is $(\overline{\mathcal{Q}}^h, \overline{\mathcal{E}}^h, \overline{\mathcal{R}}^h)$, where

$$\begin{aligned}
\overline{\mathcal{E}}^h(\overline{q},t) &:= \overline{\mathcal{B}}^h(\overline{q}) - \langle \overline{\ell}^h(t), \overline{q}\rangle \,, \\
\overline{\mathcal{B}}^h(\overline{q}) &:= \int_{\Omega_h} \mathbb{W}(\nabla^s \overline{u}(x),\, \overline{p}(x))\mathrm{d}x \,, \\
\overline{\mathcal{R}}^h(\overline{q}) &:= \int_{\Omega_h} \overline{R}(\overline{p}(x))\mathrm{d}x \,,
\end{aligned}$$

for $\overline{q} = (\overline{u},\overline{p}) \in \overline{\mathcal{Q}}^h$ and $t \in [0,T]$. We recall from Chapter 2 that the stored energy density $\mathbb{W} : \mathbb{R}^{3\times3}_{\mathrm{asym}} \times \mathbb{R}^{3\times3}_{\mathrm{dev}} \to \mathbb{R}$ is a positive quadratic form and that the dissipation potential $\overline{R} : \mathbb{R}^{3\times3}_{\mathrm{dev}} \to \mathbb{R}$ is positive one-homogeneous and convex.

By way of example, we can consider the case that the loads are composed of volume loads

$$\overline{f}^h_{\mathrm{vol}} \in W^{1,\infty}(0,T;L^2(\Omega_h;\mathbb{R}^3))$$

and surface loads

$$\overline{f}^h_{\mathrm{surf}} \in W^{1,\infty}(0,T;L^2(I \times h\partial B;\mathbb{R}^3)) \,.$$

In that case $\overline{\ell}^h$ is defined by

$$\langle \overline{\ell}^h(t), \overline{q}\rangle = \int_{\Omega_h} \overline{f}^h_{\mathrm{vol}}(t,x)\cdot\overline{u}(x)\mathrm{d}x + \int_{I\times h\partial B} \overline{f}^h_{\mathrm{surf}}(t,x)\cdot\overline{u}(x)\mathrm{d}\mathcal{H}^2(x) \qquad (4.1)$$

for $t \in [0,T]$ and $\overline{q} = (\overline{u},\overline{p}) \in \overline{\mathcal{Q}}^h$.

4.1 Scalings

In order to compare displacement fields

$$\overline{u} : \Omega_h = I \times hB \to \mathbb{R}^3$$

across different values of the thickness parameter $h > 0$, we have to pull them back to a common reference domain. The obvious choice for this reference domain is

$$\Omega := I \times B\,.$$

It would, however, be overly simplistic to study the limit behaviour of

$$\Omega \to \mathbb{R}^3\,, \qquad x \mapsto \overline{q}(x_1, hx_2, hx_3)\,.$$

Physical intuition and experience suggest that bending a thin object needs considerably less energy than stretching it. More precisely, the elastic energy of a fixed amount of bending tends to zero at a faster rate than that of a fixed amount of stretching as the thickness h of the object approaches 0. This indicates that $\overline{u}_1$ (stretching) should be scaled differently from $\overline{u}_2$ and $\overline{u}_3$ (bending).

We therefore propose to look at the scaled quantities

$$u^h(x) := \begin{pmatrix} h^{-\alpha} & & \\ & h^{-\beta} & \\ & & h^{-\beta} \end{pmatrix} \overline{u}(x_1, hx_2, hx_3)\,, \qquad x \in \Omega\,.$$

This means that our limit theory (yielding u^h) will be able to predict stretching of order h^α and bending of order h^β. In terms of u^h, the linearized strain tensor is

$$\nabla^s \overline{u} = \begin{pmatrix} h^\alpha \partial_1 u_1^h & * & * \\ \left(h^{\alpha-1}\partial_2 u_1^h + h^\beta \partial_1 u_2^h\right)/2 & h^{\beta-1}\partial_2 u_2^h & * \\ \left(h^{\alpha-1}\partial_3 u_1^h + h^\beta \partial_1 u_3^h\right)/2 & h^{\beta-1}\left(\partial_3 u_2^h + \partial_2 u_3^h\right)/2 & h^{\beta-1}\partial_3 u_3^h \end{pmatrix}.$$

A $*$-symbol in a symmetric matrix denotes entries which can be inferred from the explicitly given ones. The fact that $\nabla^s \overline{u}$ has entries which contain both terms of oder $h^{\alpha-1}$ and of order h^β, suggests we should take $\beta := \alpha - 1$ so that in the limit process both terms survive. The choice of α is not completely arbitrary since the equations of elastoplasticiy are nonlinear. More precisely, when the material of which the rods are made is fixed (independent of h), the fixed yield surface defines the typical magnitude of stresses, and hence of strains. When the actual strains do not match this, we will either have purely elastic or purely plastic behaviour in the limit. We will, however, not assume the material to be h-independent: we will scale the dissipation potential (and hence the yield surface). Therefore we can make an arbitrary choice for α, and we choose $\alpha = 2$. Accordingly, $\beta = 1$. In particular, stretching will be of order h^2 and bending of order h.

Using the scaling matrix

$$S_h := \begin{pmatrix} 1 & & \\ & 1/h & \\ & & 1/h \end{pmatrix} \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}\,, \tag{4.2}$$

we thus have
$$\overline{u}(x) = h^2 S_h u^h(S_h x)\,. \tag{4.3}$$
With (4.3), we can now write the strain tensor as
$$\begin{aligned}\nabla^s\overline{u} &= \begin{pmatrix} h^2\partial_1 u_1^h & * & * \\ h\left(\partial_2 u_1^h + \partial_1 u_2^h\right)/2 & \partial_2 u_2^h & * \\ h\left(\partial_3 u_1^h + \partial_1 u_3^h\right)/2 & \left(\partial_3 u_2^h + \partial_2 u_3^h\right)/2 & \partial_3 u_3^h \end{pmatrix} \\ &= h^2 S_h \nabla^s u^h(S_h x) S_h\,,\end{aligned}$$
Since $\nabla^s\overline{u}$ is of order h^2, we scale $\overline{p}$ so that it will be of the same order:
$$\overline{p}(x) = h^2 p^h(S_h x)\,. \tag{4.4}$$
We could have also scaled the individual components of $\overline{p}$ differently so that the scaling matches the scaling of the individual components of $\nabla^s\overline{u}$ (and not just the overall scaling). In the case of plates, this has been done in [35], whereas the uniform scaling approach was carried out in [34]. It turns out that the resulting models are similar and differ only in the flow rule [35, Section 4.1]. We therefore follow the simpler approach of [34].

We now express the stored energy $\overline{\mathcal{B}}^h(\overline{q})$ in terms of the rescaled quantity $q^h = (u^h, p^h)$ which is an element of the rescaled state space $\mathcal{Q} := H^1_\Gamma(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{dev}})$:
$$\begin{aligned}\overline{\mathcal{B}}^h(\overline{q}) &= \int_{\Omega_h} \mathbb{W}(\nabla^s\overline{u}(x),\, \overline{p}(x))\mathrm{d}x \\ &= \int_\Omega \mathbb{W}(h^2 S_h\nabla^s u^h(x)S_h,\, h^2 p^h(x))h^2\mathrm{d}x \\ &= h^6\int_\Omega \mathbb{W}(S_h\nabla^s u^h(x)S_h, p^h(x))\mathrm{d}x\,.\end{aligned}$$
We thus have $\overline{\mathcal{B}}^h(\overline{q}) = h^6\mathcal{B}^h(q^h)$ when we define
$$\mathcal{B}^h(q^h) := \int_\Omega \mathbb{W}(S_h\nabla^s u^h(x)S_h, p^h(x))\mathrm{d}x\,, \qquad q^h = (u^h,p^h)\in\mathcal{Q}\,.$$
This determines the overall scaling of the rate-independent system. We now must scale the loads and the dissipation accordingly in order to arrive at a rate-independent system which is equivalent to the original one.

As for the loads, we define $\ell^h \in W^{1,\infty}(0,T;\mathcal{Q}^*)$ by
$$\langle \ell^h(t), q^h\rangle := h^{-6}\langle \overline{\ell}^h(t), \overline{q}\rangle\,, \qquad t\in[0,T]\,.$$
With $\mathcal{E}^h(t,q^h) := \mathcal{B}^h(q^h) - \langle \ell^h(t), q^h\rangle$ we then have $\overline{\mathcal{E}}^h(t,\overline{q}) = h^6\mathcal{E}^h(t,q^h)$.

Remark. In the case where the loads are given by a volume force and surface traction as defined in (4.1), we have

$$\langle \ell^h(t), q^h \rangle = \int_\Omega f^h_{\mathrm{vol}}(t,x) \cdot u^h(x) \mathrm{d}x + \int_{I \times \partial B} f^h_{\mathrm{surf}}(t,x) \cdot u^h(x) \mathrm{d}x$$

with

$$f^h_{\mathrm{vol}}(t,x) := h^{-2} S_h \overline{f}^h_v(t, S_h x)\,, \qquad f^h_{\mathrm{surf}}(t,x) := h^{-3} S_h \overline{f}^h_s(t, S_h x)\,.$$

As noted above, we also rescale the dissipation potential. We let $R := h^{-2}\overline{R}$. This amounts to the assumption that the radius of the yield surface in physical variables is of the order h^2. We now express the total dissipation $\overline{\mathcal{R}}^h(\overline{q})$ in terms of the rescaled variables:

$$\begin{aligned} \overline{\mathcal{R}}^h(\overline{q}) &= \int_{\Omega_h} \overline{R}(\overline{p}(x)) \mathrm{d}x = \int_\Omega h^2 R(h^2 p^h(x)) h^2 \mathrm{d}x \\ &= h^6 \int_\Omega R(p^h(x)) \mathrm{d}x\,, \end{aligned}$$

where we made use of the positive one-homogeneity of R. We thus have $\overline{\mathcal{R}}^h(\overline{q}) = h^6 \mathcal{R}^h(q^h)$ when we define

$$\mathcal{R}^h(q^h) := \int_\Omega R(p^h(x)) \mathrm{d}x\,, \qquad q^h = (u^h, p^h) \in \mathcal{Q}\,.$$

Since $\mathcal{E}^h$ and $\mathcal{R}^h$ have the same scaling, we now have the equivalence:

$$q^h \text{ is a solution of } (\mathcal{Q}, \mathcal{E}^h, \mathcal{R}^h) \quad \Longleftrightarrow \quad \overline{q} \text{ is a solution of } (\overline{\mathcal{Q}}^h, \overline{\mathcal{E}}^h, \overline{\mathcal{R}}^h)\,.$$

We will therefore study the asymptotic behaviour of the rate-independent system $(\mathcal{Q}, \mathcal{E}^h, \mathcal{R}^h)$.

4.2 Summary of the setting

We will now exclusively work with the rescaled rate-independent systems $(\mathcal{Q}, \mathcal{E}^h, \mathcal{R}^h)$. Before we state the convergence result, let us give a concise summary of the setting.

The material of the rod is described by a stored energy density $\mathbb{W} : \mathbb{R}^{3\times 3}_{\mathrm{asym}} \times \mathbb{R}^{3\times 3}_{\mathrm{asym}} \to \mathbb{R}$ which is a positive quadratic form, and a (rescaled) dissipation potential $R : \mathbb{R}^{3\times 3}_{\mathrm{dev}} \to \mathbb{R}$ which is positive one-homogeneous and convex. We work on the reference domain

$$\Omega := I \times B\,, \qquad I := (0, L)\,, \qquad B \subset \mathbb{R}^2\,, \qquad L > 0\,.$$

We assume that B is a bounded Lipschitz domain with $\int_B (x_2, x_3)^\top \mathrm{d}x_2\mathrm{d}x_x = 0$. We denote by

$$\Gamma \coloneqq \partial I \times B \subset \partial\Omega$$

the union of the two opposite faces of the rod. On these, boundary values are prescribed. Hence the overall state space is $\mathcal{Q} \coloneqq \mathcal{U} \times \mathcal{P}$ with

$$\mathcal{U} \coloneqq H^1_\Gamma(\Omega;\mathbb{R}^3)\,, \qquad \mathcal{P} \coloneqq L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{dev}})\,.$$

The stored energy and the dissipation function are given by

$$\mathcal{B}^h(q) \coloneqq \int_\Omega \mathbb{W}(S_h \nabla^s u(x) S_h,\, p(x))\mathrm{d}x\,, \tag{4.5}$$

$$\mathcal{R}^h(q) \coloneqq \int_\Omega R(p(x))\mathrm{d}x \tag{4.6}$$

for $q = (u, p) \in \mathcal{Q}$. We note that the dissipation functional is h-independent, we therefore write $\mathcal{R} \coloneqq \mathcal{R}^h$. Given loads $\ell^h \in W^{1,\infty}(0, T; \mathcal{Q}^*)$, we also define the total energy

$$\mathcal{E}^h(t, q) \coloneqq \mathcal{B}^h(q) - \langle \ell^h(t), q\rangle\,, \qquad t \in [0, T]\,, \qquad q = (u, p) \in \mathcal{Q}\,.$$

4.3 Description of the limit system

We now come to a description of the limiting rate-independent system. For this we first define the subspace $\mathcal{U}^0$ of admissible limit displacements:

$$\mathcal{U}^0 \coloneqq \{u \in \mathcal{U} \;:\; \nabla^s u \in \mathrm{span}(e_1 \otimes e_1) \text{ a.e.}\}\,, \qquad e_1 \otimes e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.7}$$

This is motivated by the fact that sequences $q^h = (u^h, p^h)$ of bounded stored energy $\mathcal{B}^h(q^h)$ have the property that

$$S_h \nabla^s u^h S_h = \begin{pmatrix} \partial_1 u_1 & \frac{1}{2h}(\partial_2 u_1 + \partial_1 u_2) & \frac{1}{2h}(\partial_3 u_1 + \partial_1 u_3) \\ \frac{1}{2h}(\partial_1 u_2 + \partial_2 u_1) & \frac{1}{h^2}\partial_2 u_2 & \frac{1}{2h^2}(\partial_2 u_3 + \partial_3 u_2) \\ \frac{1}{2h}(\partial_1 u_3 + \partial_3 u_1) & \frac{1}{2h^2}(\partial_2 u_3 + \partial_3 u_2) & \frac{1}{h^2}\partial_3 u_3 \end{pmatrix}$$

bounded in $L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$. Hence any weak H^1-limit u of such u^h must lie in $\mathcal{U}^0$. The overall space of admissible limit states is

$$\mathcal{Q}^0 \coloneqq \mathcal{U}^0 \times \mathcal{P}\,. \tag{4.8}$$

The displacements contained in $\mathcal{U}^0$ are effectively one-dimensional in the sense that they are uniquely determined by the midline displacement $x_1 \mapsto u(x_1, 0, 0)$ (see Lemma 4.1 below).

The limit dissipation functional is just

$$\mathcal{R}^0 := \mathcal{R} = \mathcal{R}^h \tag{4.9}$$

as defined in (4.6). We now define the limit stored energy

$$\mathcal{B}^0 : \mathcal{Q} \to \mathbb{R}_\infty\,. \tag{4.10}$$

For $q \in \mathcal{Q} \setminus \mathcal{Q}^0$, the limiting stored energy ist set to $\mathcal{B}^0(q) := \infty$. For $q \in \mathcal{Q}^0$ we let

$$\mathcal{B}^0(q) := \inf_g \int_I \inf_{f,w} \int_B \mathbb{W}\left(\begin{pmatrix} \partial_1 u_1(x) & * \quad * \\ \partial_2 f(x') - g'(x_1)x_3 & \\ \partial_3 f(x') + g'(x_1)x_2 & \nabla^s_{2,3} w(x') \end{pmatrix},\ p(x)\right) \mathrm{d}x'\mathrm{d}x_1\,, \tag{4.11}$$

where $x' = (x_2, x_3)$ such that $x = (x_1, x')$ and the infima are taken over all

$$f \in H^1(B)\,, \qquad g \in H^1_0(I)\,, \qquad w \in H^1(B;\mathbb{R}^2)\,.$$

As above, by $*$ we denote matrix entries which are determined by the condition that the first argument of $\mathbb{W}$ must be a symmetric matrix. By $\nabla^s_{2,3} w(x)$ we denote for $w = (w_2, w_3)$ the matrix

$$\nabla^s_{2,3} w(x') := \begin{pmatrix} \partial_2 w_2(x') & \frac{1}{2}(\partial_3 w_2(x') + \partial_2 w_3(x')) \\ \frac{1}{2}(\partial_2 w_3(x') + \partial_3 w_2(x')) & \partial_3 w_3(x') \end{pmatrix}.$$

The definition of $\mathcal{B}^0$ given above will be justified by the Mosco-convergence

$$\mathcal{B}^h \overset{\mathcal{M}}{\to} \mathcal{B}^0$$

stated in Proposition 4.6. Under the assumptions of Theorem 4.2, we will also have a load function $\ell^0 \in W^{1,\infty}(0,T;\mathcal{Q})$. As usual, we then define the total energy $\mathcal{E}^0 : [0,T] \times \mathcal{Q} \to \mathbb{R}_\infty$ by

$$\mathcal{E}^0(t,q) := \mathcal{B}^0(q) - \langle \ell^0(t), q\rangle\,, \qquad t \in [0,T]\,, \qquad q \in \mathcal{Q}\,.$$

Discussion of the limit stored energy

At a first glance it is not obvious that the limit stored energy $\mathcal{B}^0$ must have the form given above. As a naive guess one might have proposed to define the limit energy as

$$(u,p) \mapsto \int_\Omega \mathbb{W}_{\mathrm{relax}}(\partial_1 u_1(x), p(x))\mathrm{d}x \tag{4.12}$$

with a relaxed energy density

$$\mathbb{W}_{\mathrm{relax}}(a,P)=\inf\{\mathbb{W}(A,P)\ :\ A\in\mathbb{R}^{3\times3}_{\mathrm{sym}}\,,\ A_{11}=a\} \tag{4.13}$$

for $a\in\mathbb{R}$ and $P\in\mathbb{R}^{3\times3}_{\mathrm{dev}}$. Indeed, this is the form the limit energy takes in the case of the plate models considered in [34] and [35]. Contrary to what one might expect, the situation is more complicated with rods. This is the case because integrability conditions prevent the pointwise minimization implied in (4.13) to be realized by actual displacement fields. These integrability conditions are more restrictive in higher codimension. Hence the difficulty lies in the number of dimensions reduced, not in the number of dimensions left.

It is clear that the energy defined by (4.12) and (4.13) is a lower bound for $\mathcal{B}^h$ in the sense of Γ-convergence (with respect to the weak topology of $\mathcal{Q}$). Indeed, whenever $(u^h,p^h)\rightharpoonup(u,p)$ in $\mathcal{Q}$, we have

$$\begin{aligned}\liminf_{h\to0}\mathcal{B}^h(u^h,p^h)&=\liminf_{h\to0}\int_\Omega\mathbb{W}(S_h\nabla^s u^h S_h,p^h)\mathrm{d}x\\&\geq\liminf_{h\to0}\int_\Omega\overline{\mathbb{W}}(\partial_1u_1^h,p^h)\mathrm{d}x\geq\int_\Omega\overline{\mathbb{W}}(\partial_1u_1,p)\mathrm{d}x\,.\end{aligned}$$

However, the bound is too low. It cannot in general be attained by a recovery sequence. The pointwise relaxation of $\mathbb{W}$ to $\overline{\mathbb{W}}$ is inadequate. That is why we have a milder degree of relaxation in our definition of $\mathcal{B}^0$. It is not accomplished by pointwise minimization but instead by global adjustments of u. These adjustments are parametrized by the functions f,g,w in (4.11). The idea behind the matrix argument of $\mathbb{W}$ in (4.11) is to write

$$u^h(x)\coloneqq u(x)+2h\begin{pmatrix}f(x)\\-g(x_1)x_3\\g(x_1)x_2\end{pmatrix}+h^2\begin{pmatrix}0\\w_1(x)\\w_2(x)\end{pmatrix}\,,\qquad x\in\Omega\,. \tag{4.14}$$

This leads to

$$S_h\nabla^s u^h(x)S_h=\begin{pmatrix}\partial_1u_1 & * \quad * \\ \partial_2 f(x)-g'(x_1)x_3 & \\ \partial_3 f(x)+g'(x_1)x_2 & \nabla^s_{2,3}w(x)\end{pmatrix}+o(1)\,,$$

which is up to the $o(1)$ error in h exactly what we have in (4.11). This indicates that recovery sequences can be constructed as in (4.14). The harder task will be to show that $\mathcal{B}^0$ is a lower bound.

To provide more intuition for (4.11) and (4.14), let us indicate the geometric meaning of the displacement-corrections g, f and w:

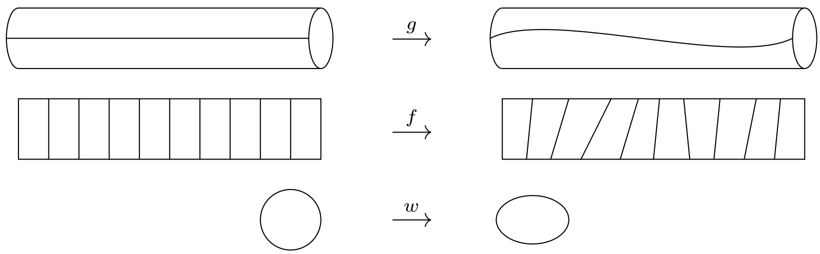


Figure 4.2: Illustration of the effect of g, f and w in the definition of $\mathcal{B}^0$: g is longitudinal torsion; f is an out-of-plane deformation of cross-sections; w is an in-plance deformation of cross-sections.

- $g(x_1)$ measures the torsion at each longitudinal position $x_1 \in I$. The torsion is not captured by the one-dimensional limit displacement u but has a nonvanishing contribution to the stored energy.
- For fixed longitudinal position $x_1 \in I$, $f(x')$ is an out-of-plane deformation of the corresponding cross sections. Again, variations inside a cross-section are not captured by the one-dimensional limit displacement.
- For fixed longitudinal position $x_1 \in I$, $w(x')$ is an in-plane deformation of the corrresponding cross sections. The same comment applies as for f. However, the in-plane deformations affected by w are smaller than the out-of-plane deformations affected by w, as can be seen in (4.14).

The above described effects of f, g and w are illustrated in Figure 4.2.

We now prove that any limit displacement field $u \in \mathcal{U}^0$, and more generally any displacement field $u \in H^1(\Omega;\mathbb{R}^3)$ with $\nabla^s u \in \operatorname{span}(e_1 \otimes e_1)$ a.e., is effectively one-dimensional.

Lemma 4.1 (On limit displacement fields). *Given any $u \in H^1(\Omega;\mathbb{R}^3)$ with $\nabla^s u \in \operatorname{span}(e_1 \otimes e_1)$ almost everywhere, there exist $v \in H^1(I;\mathbb{R}^3)$ and $\alpha \in \mathbb{R}$ such that*

$$u(x) \coloneqq v(x_1) + \begin{pmatrix} -\partial_1 v_2(x_1)x_2 - \partial_1 v_3(x_1)x_3 \\ -\alpha x_3 \\ \alpha x_2 \end{pmatrix}. \tag{4.15}$$

Moreover, $v_2, v_3 \in H^2(I)$.

Remarks. (i) This is analogous to the so-called Kirchhoff-Love displacements for plates (discussed in [34]) which are characterized by the condition $(\nabla^s u)_{13} = (\nabla^s u)_{23} = (\nabla^s u)_{33} = 0$ and can be reconstructed from midplane displacements $(x_1, x_2) \mapsto u(x_1, x_2, 0)$.

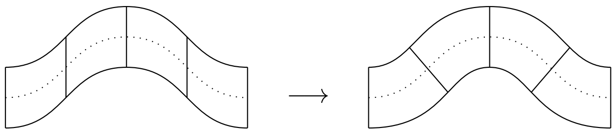


Figure 4.3: Effect of the term $\partial_1 v_2(x_1)x_2 + \partial_1 v_3(x_1)x_3$ in (4.15). The uniform application of the midline-displacement across all of the rod's fibers is replaced by an energetically more favourable layout.

(ii) For $u \in \mathcal{U}^0$, the boundary values $u = 0$ on Γ imply that $\alpha = 0$, $v_1 \in H_0^1(I)$ and $v_2, v_3 \in H_0^2(I)$.

(iii) The constant $\alpha > 0$ specifies the fixed rotational state of the rod.

(iv) Figure 4.3 shows the effect of the first component of the second summand in (4.15).

Proof. We write $\epsilon := \nabla^s u$. Then $\epsilon_{ij} = 0$ for $(i,j) \neq (1,1)$. Thus

$$\begin{aligned} 0 &= \partial_1 \epsilon_{23} + \partial_2 \epsilon_{13} - \partial_3 \epsilon_{12} \\ &= \frac{1}{2} \left(\partial_1(\partial_2 u_3 + \partial_3 u_2) + \partial_2(\partial_1 u_3 + \partial_3 u_1) - \partial_3(\partial_1 u_2 + \partial_2 u_1) \right) \\ &= \partial_1 \partial_2 u_3 \, . \end{aligned} \tag{4.16}$$

Similarly, we get $\partial_1 \partial_3 u_2 = 0$. Together with $\partial_2 u_2 = 0$ and $\partial_3 u_3 = 0$ this implies

$$\begin{aligned} \partial_2(u_1 + \partial_1 u_2 x_2 + \partial_1 u_3 x_3) &= \partial_2 u_1 + \partial_1 u_2 = 0 \\ \partial_3(u_1 + \partial_1 u_2 x_2 + \partial_1 u_3 x_3) &= \partial_3 u_1 + \partial_1 u_3 = 0 \, . \end{aligned}$$

Thus the expression inside the brackets on the left-hand side depends only on $x_1 \in I$. We therefore have

$$u_1(x) + \partial_1 u_2(x) x_2 + \partial_1 u_3(x) x_3 = v_1(x_1) \tag{4.17}$$

for a function $v_1 \in L^2(I)$. Next, we show that $\partial_2 u_3$ is constant by evaluating its partial derivatives:

$$\begin{aligned} \partial_1 \partial_2 u_3 &= 0 \text{ by } (4.16) \, , \\ \partial_2 \partial_2 u_3 &= \partial_2(2\varepsilon_{23} - \partial_3 u_2) = 0 - \partial_3 \partial_2 u_2 = -\partial_3 \epsilon_{22} = 0 \, , \\ \partial_3 \partial_2 u_3 &= \partial_2 \partial_3 u_3 = \partial_2 \epsilon_{33} = 0 \, . \end{aligned}$$

Thus there exists $\alpha \in \mathbb{R}$ with $\partial_2 u_3 = \alpha$ and therefore $\partial_3 u_2 = 2\varepsilon_{32} - \partial_2 u_3 = -\alpha$. As $\partial_2 u_2 = \partial_3 u_3 = 0$, this implies that there exist $v_2, v_3 \in L^2(I)$ such that

$$u_2(x) = -\alpha x_3 + v_2(x_1)\,, \qquad u_3(x) = \alpha x_2 + v_3(x_1)\,. \tag{4.18}$$

It follows that $v_2, v_3 \in H^1(I)$. Moreover, starting from (4.17) and then using (4.18), we have

$$\begin{aligned} u_1(x) &= v_1(x_1) - \partial_1 u_2(x) x_2 - \partial_1 u_3(x) x_3 \\ &= v_1(x_1) - \partial_1 v_2(x_1) x_2 - \partial_1 v_3(x_1) x_3\,. \end{aligned}$$

Now we have shown that u has the form (4.15). From

$$\partial_1 u_1(x) = \partial_1 v_1(x_1) - \partial_1^2 v_2(x_1) x_2 - \partial_1^2 v_3(x_1) x_3\,,$$

and $\partial_1 u_1 \in L^2(\Omega)$ we conclude that $v_1 \in H^1(I)$ and $v_2, v_3 \in H^2(I)$. □

Isotropic elasticity

By way of example, let us consider pure isotropic elasticity,

$$\mathbb{W}(A, P) = \frac{\lambda}{2}(\operatorname{tr} A)^2 + \mu |A|^2\,, \qquad \lambda, \mu > 0\,.$$

The stored energy density $\mathbb{W}$ does not depend on the plastic variable P. This choice of $\mathbb{W}$ implies that the infimum in (4.11) is attained with $f = g = 0$, since populating the off-diagonal entries of A only increases the value of $\mathbb{W}(A, P)$.

In order to find the optimal function w in (4.11), we exploit that $u \in \mathcal{U}^0$ has the form (4.15) for some $v \in H_0^1(I) \times H_0^2(I; \mathbb{R}^2)$ and $\alpha = 0$. Let us consider $w \in L^2(I; H^1(B; \mathbb{R}^2))$ defined by

$$\begin{aligned} w_1 &:= -\nu \left(\partial_1 v_1 x_2 - \partial_1^2 v_3 x_2 x_3 + \frac{1}{2} \partial_1^2 v_2 (x_3^2 - x_2^2) \right) \\ w_2 &:= -\nu \left(\partial_1 v_1 x_3 - \partial_1^2 v_2 x_2 x_3 + \frac{1}{2} \partial_1^2 v_2 (x_2^2 - x_3^2) \right) \end{aligned} \tag{4.19}$$

with some constant $\nu > 0$. We then have $\nabla_{2,3}^s w = -\operatorname{diag}(\nu, \nu) \partial_1 u_1$. Our choice of w may not be optimal, but when we use this ansatz combined with $f = g = 0$ in (4.11), we get

$$\mathcal{B}^0(q) \leq \int_\Omega \mathbb{W}(\operatorname{diag}(1, -\nu, -\nu) \partial_1 u_1, 0) \mathrm{d}x\,. \tag{4.20}$$

Now it turns out that the relaxed energy density $\mathbb{W}_{\text{relax}}$ defined in (4.13) in our case takes the form

$$\mathbb{W}_{\text{relax}}(a, P) = \mathbb{W}(\text{diag}(1, -\nu, -\nu)a, 0)\,.$$

Here, of course, ν is no longer arbitrary. It is *Poisson's ratio* $\nu := \frac{\lambda}{2(\lambda+\mu)}$. This constant measures, when the material is expanded in one direction, how much it is contracted in directions perpendicular to that direction. Using the same constant in (4.19), we can now continue (4.20) to

$$\mathcal{B}^0(q) \leq \int_\Omega \mathbb{W}(\text{diag}(1, -\nu, -\nu)\partial_1 u_1, 0)\mathrm{d}x = \int_\Omega \mathbb{W}_{\text{relax}}(\partial_1 u_1, 0)\mathrm{d}x \leq \mathcal{B}^0(q)\,.$$

In this particular case we see that $\mathcal{B}^0$ agrees with the trivial lower bound defined in (4.12) in terms of $\mathbb{W}_{\text{relax}}$.

4.4 Statement of the convergence result

In this section, we formulate the main convergence result. We also give a proof, but in doing so we refer to the results of the following sections.

Let us suppose that $\ell^h \in W^{1,\infty}(0, T; \mathcal{Q}^*)$ satisfies $\ell^h(t) \to \ell^0(t)$ for all $t \in [0, T]$, and moreover $\|\ell^h\|_{W^{1,\infty}(0,T;\mathcal{Q}^*)} \leq C$ for all $h \in [0, 1]$.

We claim that the rate-independent system $(\mathcal{Q}, \mathcal{E}^0, \mathcal{R}^0)$ is the limit of the systems $(\mathcal{Q}, \mathcal{E}^h, \mathcal{R}^h)$ in the following sense.

Theorem 4.2. *Consider a family of energetic solutions $q^h \in L^1(0, T; \mathcal{Q})$ for the rate-independent system $(\mathcal{Q}, \mathcal{E}^h, \mathcal{R}^h)$ for $h \geq 0$ such that*

$$q^h(0) \rightharpoonup q^0(0)\,, \qquad \mathcal{B}^h(q^h(0)) \to \mathcal{B}^0(q^0(0))$$

as $h \to 0$. Then also

$$q^h(t) \to q^0(t)\,, \qquad \mathcal{B}^h(q^h(t)) \to \mathcal{B}^0(q^0(t))$$

for all $t \in [0, T]$ als $h \to 0$. Moreover,

$$\text{Diss}_{\mathcal{R}^h}(q^h; [0, t]) \to \text{Diss}_{\mathcal{R}^0}(q^0; [0, t])\,, \quad \langle \partial_t \ell^h(t), q^h(t)\rangle \to \langle \partial_t \ell^0(t), q^0(t)\rangle\,.$$

Proof. The statement of the theorem follows from Theorem 3.4. We only need to check that the assumptions (A)–(D) on Pages 28 and 29 are satisfied:

(A) The stored energy functionals $\mathcal{B}^h$ are quadratic forms since $\mathbb{W}$ is a quadratic form. Moreover, $\mathcal{B}^h$ is continuous, hence lower-semicontinuous. What remains to be proved is the equicoercivity. This is done in Proposition 4.3 below.

(B) The dissipation functionals $\mathcal{R}^h$ are all equal to $\mathcal{R}$. The function $\mathcal{R}$ is positive one-homogeneous and convex because R is positive one-homogeneous and convex. Moreover, $\mathcal{R}$ is continuous, hence lower-semicontinuous.

(C) The assumption on the Lipschitz bound of the loads ℓ^h was just repeated in Theorem 4.2.

(D) The Mosco-convergence of $\mathcal{B}^h$ is proved in Proposition 4.6 below. The Mosco-convergence and continuous convergence of $\mathcal{R}^h$ immediately follows from the continuity and weak lower-semicontinuity of $\mathcal{R}^h = \mathcal{R}$. The assumption on the convergence of the loads ℓ^h was just repeated in Theorem 4.2.

Thus the theorem is proved once Propositions 4.3 and 4.6 are established. □

In the following sections, we provide the missing parts referred to in the above proof: equi-coercivity and Mosco-convergence of $\mathcal{B}^h$.

4.5 Proof of the equi-coercivity

Proposition 4.3 (Equi-coercivity). *We consider $\mathcal{B}^h$ of* (4.5)*, describing the stored energy of thin rods. There is a constant $\beta > 0$ such that*

$$\mathcal{B}^h(q) \geq \beta \|q\|^2$$

for all $q \in \mathcal{Q}$ and $h \in (0,1)$.

Proof. For $q = (u,p) \in \mathcal{Q}$, we have

$$\mathcal{B}^h(q) = \int_\Omega \mathbb{W}(S_h \nabla^s u(x) S_h, p(x)) \mathrm{d}x \,.$$

By the positivity of the quadratic form $\mathbb{W}$ this implies

$$\begin{aligned} \mathcal{B}^h(q) &\gtrsim \int_\Omega |S_h \nabla^s u(x) S_h|^2 + |p(x)|^2 \mathrm{d}x \\ &\geq \|\nabla^s u\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})} + \|p\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{dev}})} \,. \end{aligned}$$

By Korn's inequality (see Lemma A.4(i); we recall that boundary conditions for u are imposed in the space $\mathcal{Q}$) this implies

$$\mathcal{B}^h(q) \gtrsim \|u\|^2_{H^1(\Omega;\mathbb{R}^3)} + \|p\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{dev}})} = \|q\|^2 \,.$$

Tracking the constants in the $\gtrsim$-steps, we see that the constant β claimed in the lemma depends only on the quadratic form $\mathbb{W}$ and the Poincaré-Korn-constant of Ω and Γ. □

4.6 Proof of the Mosco-convergence

In order to prove the Mosco-convergence of $\mathcal{B}^h$, we need a Korn-type inequality for thin domains. In general, the constant in a Korn inequality depends on the domain under consideration. In particular, when a domain gets thinner, its Korn constant increases. This is because a bending deformation u takes progressively less energy per volume (measured in terms of $\nabla^s u$) when the thickness decreases. In a two-dimensional setting this can be seen by considering a displacement field u^h,

$$u^h : (0,1) \times (0,h) \to \mathbb{R}^2\,, \qquad u^h(x) = \begin{pmatrix} -\phi'(x_1)x_2 \\ \phi(x_1) \end{pmatrix}\,, \qquad \phi \in C_c^\infty((0,1))\,.$$

Here the values of u^h are of order 1, but the values of $\nabla^s u(x) = -\phi''(x_1)x_2 e_1 \otimes e_1$ are only of order h.

However, when we disallow large deformations in the thin directions, this effect no longer occurs. We then get a Korn inequality which is independent of the thickness parameter. This is basically what the next lemma expresses, where such a restriction is achieved by subtracting from an arbitrary displacement at every point its mean value over the whole cross section to which that point belongs. The lemma is stated in rescaled variables (as this is the form in which we will use it). The physical intuition however is better grasped when the provided estimate is looked at as stated in physical variables in (4.22).

Lemma 4.4 (Korn's inequality for thin domains). *Let* $\Omega := (0,L) \times B$ *for a constant* $L > 0$ *and a bounded Lipschitz domain* $B \subset \mathbb{R}^2$*. There is a constant* $C = C(B) > 0$ *such that*

$$\left\| S_h \left(\nabla u - \fint_B \nabla u \right) S_h \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C \| S_h \nabla^s u S_h \|_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \tag{4.21}$$

for all $u \in H^1(\Omega;\mathbb{R}^3)$ *and* $h \in (0,L)$*, where*

$$\left(\fint_B \nabla u \right)(x) := \fint_B \nabla u(x_1, x')\mathrm{d}x'\,.$$

Proof. We pull-back u to the thin domain $\Omega_h := (0,L) \times hB$ by considering $u_h(x) := S_h u(S_h x)$ for $x \in \Omega_h$. Since

$$\nabla u_h(x) = S_h \nabla u(S_h x) S_h\,, \qquad x \in \Omega_h\,,$$

the claimed inequality (4.21) is equivalent to

$$\left\| \nabla u_h - \fint_{hB} \nabla u_h \right\|^2_{L^2(\Omega_h;\mathbb{R}^{3\times 3})} \leq C \|\nabla^s u_h\|^2_{L^2\left(\Omega_h,\mathbb{R}^{3\times 3}_{\mathrm{sym}}\right)}\,. \tag{4.22}$$

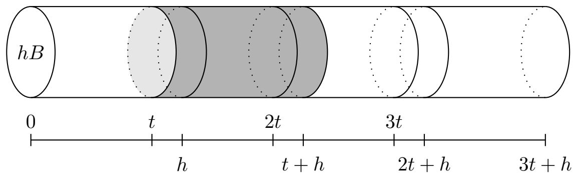


Figure 4.4: Illustration of the decomposition $\Omega^h = \bigcup_{k=0}^K \Omega_h^{kt}$ in the case $K = 3$. The shaded region denotes Ω_h^t. When h gets smaller, the number of patches increases, but their shape remains the same.

We therefore proceed to prove (4.22).

For this we consider subdomains of the form $\Omega_h^t := (t, t+h) \times hB$. On each of these we have

$$\left\| \nabla u_h - \fint_{hB} \nabla u_h \right\|^2_{L^2\left(\Omega_h^t;\mathbb{R}^{3\times 3}\right)} \leq \left\| \nabla u_h - \fint_{\Omega_h^t} \nabla u_h \right\|^2_{L^2\left(\Omega_h^t;\mathbb{R}^{3\times 3}\right)} . \tag{4.23}$$

Here we used Fubini and a pointwise estimate for each longitudinal position $s \in (t, t+h)$, namely that the algebraic mean of a function over $\{s\} \times hB$ is the optimal constant to subtract from that function when the objective is to minimize the $L^2(\{s\} \times hB)$-norm; subtracting the mean over Ω_h^t can only yield a larger norm.

We observe that all domains Ω_h^t are homothetic to $(0,1) \times B$ and thus have the same Korn constant which only depends on B (see Lemma A.4(ii)). Therefore

$$\left\| \nabla u_h - \fint_{\Omega_h^t} \nabla u_h \right\|^2_{L^2\left(\Omega_h^t;\mathbb{R}^{3\times 3}\right)} \lesssim \|\nabla^s u_h\|^2_{L^2\left(\Omega_h^t;\mathbb{R}^{3\times 3}_{\mathrm{sym}}\right)} . \tag{4.24}$$

We now use the (non-disjoint) decomposition $\Omega^h = \bigcup_{k=0}^K \Omega_h^{kt}$, where $K := \lceil (L-h)/h \rceil$ and $t := (L-h)/K$. Here we denote by $\lceil x \rceil$ the smallest integer greater than or equal to x. See Figure 4.4 for an illustration. Then

$$\begin{aligned}
\left\| \nabla u_h - \fint_{hB} \nabla u_h \right\|^2_{L^2(\Omega_h;\mathbb{R}^{3\times 3})} &\leq \sum_{k=0}^K \left\| \nabla u_h - \fint_{hB} \nabla u_h \right\|^2_{L^2\left(\Omega_h^{kt};\mathbb{R}^{3\times 3}\right)} \\
&\overset{(4.23)}{\leq} \sum_{k=0}^K \left\| \nabla u_h - \fint_{\Omega_h^{kt}} \nabla u_h \right\|^2_{L^2\left(\Omega_h^{kt},\mathbb{R}^{3\times 3}_{\mathrm{sym}}\right)} \overset{(4.24)}{\lesssim} \sum_{k=0}^K \|\nabla^s u_h\|^2_{L^2\left(\Omega_h^{kt},\mathbb{R}^{3\times 3}_{\mathrm{sym}}\right)} \\
&\leq 2 \|\nabla^s u_h\|^2_{L^2\left(\Omega_h,\mathbb{R}^{3\times 3}_{\mathrm{sym}}\right)} ,
\end{aligned}$$

where the factor 2 accounts for the possible overlap of the patches Ω_h^{kt}. With this we have proved (4.22) and thus the lemma. □

We now give an alternative description of the limit stored energy $\mathcal{B}^0$.

Lemma 4.5. *We use* $\mathcal{Q}^0$ *from* (4.8) *and* $\mathcal{B}^0$ *from* (4.10) *and* (4.11). *For* $q = (u, p) \in \mathcal{Q}^0$ *there holds*

$$\mathcal{B}^0(q) = \inf_{f,g,w} \int_\Omega \mathbb{W}\left(\begin{pmatrix} \partial_1 u_1(x) & * \quad * \\ \partial_2 f(x) - g'(x_1)x_3 & \\ \partial_3 f(x) + g'(x_1)x_2 & \nabla^s_{2,3} w(x) \end{pmatrix}, \; p(x) \right) \mathrm{d}x \,, \quad (4.25)$$

where the infimum is taken over all

$$f \in H^1_\Gamma(\Omega) \,, \qquad g \in H^1_0(I) \,, \qquad w \in H^1_\Gamma(\Omega; \mathbb{R}^2) \,.$$

Proof. We only have to prove "$\geq$", the opposite inequality is clear. For brevity, we denote the integrand on the right-hand side of (4.25) with ellipses ("…"). The statement now follows from from Lemma B.2:

$$\inf_{\substack{f \in H^1_\Gamma(\Omega) \\ g \in H^1_0(I) \\ w \in H^1_\Gamma(\Omega;\mathbb{R}^2)}} \int_\Omega \mathbb{W}(\dots)\mathrm{d}x \leq \inf_{g \in H^1_0(I)} \inf_{\substack{f \in H^1_0(I;H^1(B)) \\ w \in H^1_0(I;H^1(B;\mathbb{R}^2))}} \int_I \int_B \mathbb{W}(\dots)\mathrm{d}x'\mathrm{d}x_1$$

$$\overset{\text{Lemma B.2}}{\leq} \inf_{g \in H^1_0(I)} \int_I \inf_{\substack{f \in H^1(B) \\ w \in H^1(B;\mathbb{R}^2)}} \int_B \mathbb{W}(\dots)\mathrm{d}x'\mathrm{d}x_1 = \mathcal{B}^0(q) \,.$$

This shows the claim. □

The lemma is important for the construction of recovery sequences: It provides functions f, g, w of sufficient regularity to define q^h in terms of these functions as indicated in (4.14).

We now proceed to the main proposition in this chapter: the Mosco-convergence of the stored energy. It is the last missing piece used in the proof of Theorem 4.2.

Proposition 4.6. *Consider* $\mathcal{B}^h$ *as defined in* (4.5) *and* $\mathcal{B}^0$ *as defined in* (4.11). *Then there holds the Mosco-convergence* $\mathcal{B}^h \overset{\mathcal{M}}{\to} \mathcal{B}^0$.

Proof. **Part I: Lower bound.** Consider any weakly converging sequence $q^h = (u^h, p^h) \rightharpoonup q = (u, p)$ in $\mathcal{Q}$. We claim that

$$\liminf_{h \to 0} \mathcal{B}^h(q^h) \geq \mathcal{B}^0(q) \,. \quad (4.26)$$

Step 1. Without loss of generality, we may assume that $\mathcal{B}^h(q^h)$ is uniformly bounded along a subsequence. We consider a subsequence with $\mathcal{B}^h(q^h) \to \liminf_{h\to 0} \mathcal{B}^h(q^h)$.

The bound on $\mathcal{B}^h(q^h)$ implies that $S_h \nabla^s u^h S_h$ is uniformly bounded in $L^2(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$. Therefore $(\nabla^s u^h)_{ij} \to 0$ in $L^2(\Omega)$ for $(i,j) \neq (1,1)$. Because of $\nabla^s u^h \rightharpoonup \nabla^s u$ in $L^2(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, this implies $\nabla^s u \in \operatorname{span}(e_1 \otimes e_1)$ a.e. This proves $q \in \mathcal{Q}^0$, compare (4.7) and (4.8).

Step 2. Since $S_h \nabla^s u^h S_h$ is uniformly bounded in $L^2(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$, there exists a subsequence and some $E \in L^2(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$ with

$$S_h \nabla^s u^h S_h \rightharpoonup E\,.$$

Our aim is to find

$$f \in L^2(I; H^1(B))\,, \qquad g \in H^1_0(I)\,, \qquad w \in L^2(I; H^1(B; \mathbb{R}^2))$$

such that

$$E(x) = \begin{pmatrix} \partial_1 u_1(x) & * \quad * \\ \partial_2 f(x) - g'(x_1)x_3 & \\ \partial_3 f(x) + g'(x_1)x_2 & \nabla^s_{2,3} w(x) \end{pmatrix}, \qquad x \in \Omega\,. \tag{4.27}$$

Once (4.27) ist shown, the lower bound (4.26) follows since

$$\begin{aligned} \liminf_{h\to 0} \mathcal{B}^h(q^h) &= \liminf_{h\to 0} \int_\Omega \mathbb{W}(S_h \nabla^s u^h(x) S_h, p^h(x))\mathrm{d}x \\ &\geq \int_\Omega \mathbb{W}(E,p) \geq \mathcal{B}^0(q)\,. \end{aligned}$$

Step 3. In order to define (f,g,w), we first consider

$$\tilde{u}^h_j(x) := u^h_j(x) - \fint_B u^h_j(x_1, x')\mathrm{d}x'\,, \qquad j \in \{2,3\}\,.$$

By Korn's inequality of Lemma 4.4 and the boundedness of $\mathcal{B}^h(q^h)$,

$$\left\| \frac{1}{2h} \begin{pmatrix} \partial_1 \tilde{u}^h_2 \\ \partial_1 \tilde{u}^h_3 \end{pmatrix} \right\|_{L^2(\Omega;\mathbb{R}^2)} \leq \left\| S_h \left(\nabla u - \fint_B \nabla u \right) S_h \right\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \lesssim 1\,. \tag{4.28}$$

Since $(\tilde{u}^h_2, \tilde{u}^h_3) = 0$ on $\{0\} \times B$, this implies that

$$\left\| \frac{1}{2h} \begin{pmatrix} \tilde{u}^h_2 \\ \tilde{u}^h_3 \end{pmatrix} \right\|_{L^2(\Omega;\mathbb{R}^2)} \lesssim 1\,. \tag{4.29}$$

We define $g^h \in L^2(I)$ as the unique minimizer of

$$\left\| \frac{1}{2h} \begin{pmatrix} \tilde{u}_2^h(x) \\ \tilde{u}_3^h(x) \end{pmatrix} - g^h(x_1) \begin{pmatrix} -x_3 \\ x_2 \end{pmatrix} \right\|_{L^2(\Omega;\mathbb{R}^2)} .$$

By (4.29), the sequence $(g^h)_h$ is uniformly bounded. Hence there exists a subsequence and a limit function $g \in L^2(I)$ such that

$$g^h \rightharpoonup g \quad \text{in } L^2(I) . \tag{4.30}$$

By Korn's inequality on $\{x_1\} \times B$ (see Lemma A.4(ii)) and the fact that $\frac{1}{2h}(\tilde{u}_2^h, \tilde{u}_3^h)^\top - g^h(x_1)(-x_3, x_2)^\top$ vanishes in the mean on each $\{x_1\} \times B$, we have

$$\begin{aligned} \left\| \frac{1}{2h} \begin{pmatrix} \tilde{u}_2^h \\ \tilde{u}_3^h \end{pmatrix} - g^h(x_1) \begin{pmatrix} -x_3 \\ x_2 \end{pmatrix} \right\|_{L^2(\Omega;\mathbb{R}^2)} &\lesssim \left\| \frac{1}{h} \nabla^s_{2,3} \tilde{u}^h_{2,3} \right\|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} \\ = \left\| \frac{1}{h} \nabla^s_{2,3} u^h_{2,3} \right\|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} &\leq h \left\| S_h \nabla^s u^h S_h \right\|_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \to 0 . \end{aligned}$$

In particular,

$$\frac{1}{2h} \begin{pmatrix} \partial_1 \tilde{u}_2^h \\ \partial_1 \tilde{u}_3^h \end{pmatrix} \to g'(x_1) \begin{pmatrix} -x_3 \\ x_2 \end{pmatrix}$$

in the sense of distributions on Ω, and by the bound (4.28) this implies

$$\frac{1}{2h} \begin{pmatrix} \partial_1 \tilde{u}_2^h \\ \partial_1 \tilde{u}_3^h \end{pmatrix} \rightharpoonup g'(x_1) \begin{pmatrix} -x_3 \\ x_2 \end{pmatrix} \tag{4.31}$$

in $L^2(\Omega;\mathbb{R}^2)$. In particular, $g \in H^1(I)$. Integrating (4.31) over $I \times B'$ for $B' \subset B$, we get

$$0 = (g(L) - g(0)) \int_{B'} (-x_3, x_2)^\top .$$

When we choose B' such that B' is not centered, i.e. $\int_{B'} (-x_3, x_2)^\perp \neq 0$, this implies $g(L) = g(0)$. If $g \notin H^1_0(I)$, we can replace g with $g - g(0)$. Then $g \in H^1_0(I)$ and (4.31) remains true in the process.

Step 4. We define $\tilde{u}_1^h \in L^2(I; H^1(B))$ by

$$\tilde{u}_1^h(x) := u_1^h(x) + \left(x_2 \partial_1 \fint_B u_2^h(x_1, x')\mathrm{d}x' + x_3 \partial_1 \fint_B u_3^h(x_1, x')\mathrm{d}x' \right) .$$

We know from (4.28) and the boundedness of $S_h \nabla^s u^h S_h$ in $L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ that

$$\frac{1}{2h} \begin{pmatrix} \partial_1 \tilde{u}_2^h \\ \partial_1 \tilde{u}_3^h \end{pmatrix} \quad \text{and} \quad \frac{1}{2h} \begin{pmatrix} \partial_1 \tilde{u}_2^h + \partial_2 \tilde{u}_1^h \\ \partial_1 \tilde{u}_3^h + \partial_3 \tilde{u}_1^h \end{pmatrix} = \frac{1}{2h} \begin{pmatrix} \partial_1 u_2^h + \partial_2 u_1^h \\ \partial_1 u_3^h + \partial_3 u_1^h \end{pmatrix}$$

are bounded in $L^2(\Omega;\mathbb{R}^2)$. But then

$$\frac{1}{2h}\begin{pmatrix}\partial_2\tilde{u}_1^h\\ \partial_3\tilde{u}_1^h\end{pmatrix}$$

is also bounded in $L^2(\Omega;\mathbb{R}^2)$. Thus there exists (by Poincaré's inequality from Lemma A.3(ii) and a compactness argument) a subsequence and a function

$$f\in L^2(I;H^1(B))$$

such that

$$\frac{1}{2h}\begin{pmatrix}\partial_2\tilde{u}_1^h\\ \partial_3\tilde{u}_1^h\end{pmatrix}\rightharpoonup\begin{pmatrix}\partial_2 f\\ \partial_3 f\end{pmatrix}$$

in $L^2(\Omega;\mathbb{R}^2)$. Combining this with (4.31), we find that

$$\frac{1}{2h}\begin{pmatrix}\partial_1 u_2^h+\partial_2 u_1^h\\ \partial_1 u_3^h+\partial_3 u_1^h\end{pmatrix}=\frac{1}{2h}\begin{pmatrix}\partial_1 \tilde{u}_2^h+\partial_2 \tilde{u}_1^h\\ \partial_1 \tilde{u}_3^h+\partial_3 \tilde{u}_1^h\end{pmatrix}\rightharpoonup\begin{pmatrix}\partial_2 f-g'(x_1)x_3\\ \partial_3 f+g'(x_1)x_2\end{pmatrix}\tag{4.32}$$

in $L^2(\Omega;\mathbb{R}^2)$.

Step 5. It remains to construct w. As

$$\|h^{-2}\nabla^s_{2,3}u^h_{2,3}\|_{L^2(\Omega;\mathbb{R}^{2\times2}_{\mathrm{sym}})}\le\|S_h\nabla^s u^h S_h\|_{L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})}\lesssim 1\,,$$

by Korn's inequality (see Lemma A.4(ii)) and a compactness argument, there exists a subsequence and a function

$$w\in L^2(I;H^1(B;\mathbb{R}^2))$$

such that

$$\frac{1}{h^2}\nabla^s_{2,3}u^h_{2,3}\rightharpoonup\nabla^s_{2,3}w\tag{4.33}$$

in $L^2(\Omega;\mathbb{R}^{2\times2}_{\mathrm{sym}})$.

Step 6. We conclude, using the weak convergence $u^h\rightharpoonup u$ in $H^1(\Omega;\mathbb{R}^3)$ as well as (4.32) and (4.33) that

$$S_h\nabla^s u^h(x)S_h\rightharpoonup\begin{pmatrix}\partial_1u_1(x) & * \quad * \\ \partial_2 f(x)-g'(x_1)x_3 & \\ \partial_3 f(x)+g'(x_1)x_2 & \nabla^s_{2,3}w(x)\end{pmatrix}=E(x)$$

in $L^2(\Omega;\mathbb{R}^{3\times3}_{\mathrm{sym}})$. As noted at the end of *Step 2*, this concludes the proof of the lower bound.

Part II: Upper bound. Let $q = (u, p) \in \mathcal{Q}$ and $\delta > 0$. We need to find a sequence $(q^h)_h \subset \mathcal{Q}$ with $q^h \to q$ and

$$\limsup_{h\to 0} \mathcal{B}^h(q^h) \le \mathcal{B}^0(q) + \delta\,.$$

We can assume that $q \in \mathcal{Q}^0$ as otherwise $\mathcal{B}^0(q) = \infty$. By Lemma 4.5, there exist

$$f \in H^1_\Gamma(\Omega)\,, \qquad g \in H^1_0(I)\,, \qquad w \in H^1_\Gamma(\Omega;\mathbb{R}^2)$$

such that

$$\int_\Omega \mathbb{W}\left(\begin{pmatrix} \partial_1 u_1(x) & * \;\; * \\ \partial_2 f(x) - g'(x_1)x_3 & \multirow{2}{*}{$\nabla^s_{2,3} w(x)$} \\ \partial_3 f(x) + g'(x_1)x_2 & \end{pmatrix},\ p(x)\right) \mathrm{d}x \;\le\; \mathcal{B}^0(q) + \delta\,.$$

We now define $u^h \in H^1_\Gamma(\Omega;\mathbb{R}^3)$ by

$$u^h(x) := u(x) + 2h \begin{pmatrix} f(x) \\ -g(x_1)x_3 \\ g(x_1)x_2 \end{pmatrix} + h^2 \begin{pmatrix} 0 \\ w_1(x) \\ w_2(x) \end{pmatrix}, \qquad x \in \Omega\,.$$

Defining $q^h := (u^h, p)$, we have $q^h \to q$ in $\mathcal{Q}$. Moreover,

$$\begin{aligned} S_h \nabla^s u^h(x) S_h &= \begin{pmatrix} \partial_1 u_1(x) & * \;\; * \\ \partial_2 f(x) - g'(x_1)x_3 & \multirow{2}{*}{$\nabla^s_{2,3} w(x)$} \\ \partial_3 f(x) + g'(x_1)x_2 & \end{pmatrix} + \begin{pmatrix} 2h\partial_1 f(x) & * & * \\ \frac{h}{2}\partial_1 w_2(x) & 0 & 0 \\ \frac{h}{2}\partial_1 w_3(x) & 0 & 0 \end{pmatrix} \\ &\to \begin{pmatrix} \partial_1 u_1(x) & * \;\; * \\ \partial_2 f(x) - g'(x_1)x_3 & \multirow{2}{*}{$\nabla^s_{2,3} w(x)$} \\ \partial_3 f(x) + g'(x_1)x_2 & \end{pmatrix} \end{aligned}$$

in $L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$. This implies

$$\begin{aligned} \lim_{h\to 0} \mathcal{B}^h(q^h) &= \int_\Omega \mathbb{W}\left(\begin{pmatrix} \partial_1 u_1(x) & * \;\; * \\ \partial_2 f(x) - g'(x_1)x_3 & \multirow{2}{*}{$\nabla^s_{2,3} w(x)$} \\ \partial_3 f(x) + g'(x_1)x_2 & \end{pmatrix},\ p(x)\right) \mathrm{d}x \\ &\le \mathcal{B}^0(q) + \delta\,, \end{aligned}$$

and thus the claim. $\square$

Chapter 5

Periodic graphs

For the homogenization of periodic lattices we need to describe the underlying periodic structure in the language of graph theory. For this purpose, we introduce here the concept of periodic graphs in $\mathbb{R}^3$. We show how, based on this notion, one can construct lattices with a periodicity parameter $\varepsilon > 0$ that approximate a macrosopic domain $\Omega \subset \mathbb{R}^3$. We also discuss the crucial property of (infinitesimal) rigidity. Moreover, we introduce notation for dealing with functions defined on the nodes or edges of the ε-lattices. This will lead to a notion of convergence for such functions, including a notion of two-scale convergence together with an appropriate compactness result.

This chapter prepares for Chapter 6 where we will state the equations of elastoplasticity on the edges of the periodic graph, coupled by boundary values encoded in the state of the nodes.

5.1 The infinite periodic graph

In this section we describe how to construct an infinite periodic graph G_{per} by an unfolding procedure from a finite *periodicity graph* G. The graph G describes a single periodicity cell. Each of its edges has a label (a vector in $\mathbb{Z}^3$) which informs the unfolding procedure about the cell-offset of edges of that type.

Definition 5.1 (Periodicity graph). *Let G be a finite directed multigraph with edges $E(G)$ and vertices $V(G)$. An edge $e \in E(G)$ connects $v_1 = v_1(e) \in V(G)$ with $v_2 = v_2(e) \in V(G)$. Let*

$$z : V(G) \to \square \coloneqq (0,1)^3$$

be a placement function *which assigns to each node $v \in V(G)$ a position $z(v)$ in the periodicity cell $\Box$. Moreover, let*

$$d : E(G) \to \mathbb{Z}^3$$

be a function which assigns a label $d(e) \in \mathbb{Z}^3$ to each edge $e \in E(G)$. We assume that $(v_1, v_2; d)$ uniquely identifies the edge e, i.e., any two edges between the same pair of vertices must have differing labels. We also require for any $e = (v_1, v_2; d) \in E(G)$ that $-e := (v_2, v_1; -d) \notin E(G)$.

The triple (G, z, d) is called a periodicity graph*. When z and d are clear from the context, we simply call G a* periodicity graph*.*

Remark. (i) A *multigraph* is a graph that is allowed to have loops (edges beginning and ending at the same vertex) and also multiple edges between the same pair of vertices. For brevity, once a particular multigraph is introduced, we will subsequently refer to it simply as a *graph*.

(ii) A periodicity graph may contain loops $e = (v, v; d) \in E(G)$, but the requirement $-e \notin E(G)$ implies $d \neq 0$ in this case.

Below we will add two further requirements for a periodicity graph G which, however, will be expressed in terms of the derived periodic graph G_{per} which we introduce now.

Given any periodicity graph (G, z, p), we construct an infinite directed graph G_{per} by defining

$$V(G_{\text{per}}) := V(G) \times \mathbb{Z}^3 \,, \tag{5.1a}$$

$$E(G_{\text{per}}) := \{((v_1, k), (v_2, k + d)) \;:\; (v_1, v_2; d) \in E(G)\,,\; k \in \mathbb{Z}^3\} \,. \tag{5.1b}$$

We will identify $E(G_{\text{per}})$ with $E(G) \times \mathbb{Z}^3$ and thus, by abuse of notation, write $(e, k) \in E(G_{\text{per}})$ when $e \in E(G)$ and $k \in \mathbb{Z}^3$. Observe that G_{per} cannot have any loops, since $(v_1, k) = (v_2, k + d)$ implies $v_1 = v_2$ and $d = 0$, and therefore $(v_1, v_2; d) \notin E(G)$.

The node placement $z : V(G) \to \Box$ now induces a node placement

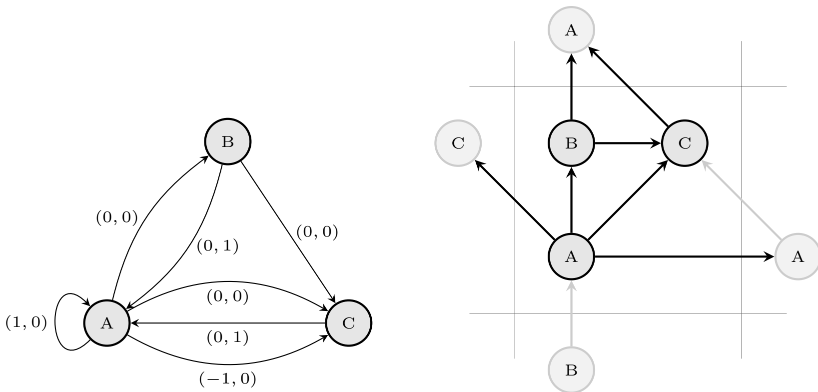
$$z : V(G_{\text{per}}) \to \mathbb{R}^3 \,, \qquad z((v, k)) := z(v) + k \,. \tag{5.2}$$

By this, (G_{per}, z) is a *frame* in $\mathbb{R}^3$. The length of an edge of type $e \in E(G)$ is

$$L(e) := |d(e) + z(v_2(e)) - z(v_1(e))| \,.$$

Furtheremore, we denote by

$$r(e) := \frac{d(e) + z(v_2(e)) - z(v_1(e))}{L(e)} \tag{5.3}$$



(a) A graph G with $V(G) = \{A, B, C\}$ and edges indicated by arrows. The labels are elements of $\mathbb{Z}^2$. A vector (n, m) means: "go n cells to the right and m cells to the top".

(b) A section from the corresponding infinite periodic graph G_{per}, realized by a placement function $z : V(G) \to (0, 1)^2$ with, e.g., $z(A) = (0.25, 0.25)$.

Figure 5.1: Example of a graph G labeled with integer vectors which gives rise to a periodic graph G_{per}. For visual clarity, we provide this example in two space dimensions. In the main text everything is stated in three dimensions.

the unit vector that indicates the direction of an edge of type e. For convenience, we also use

$$\begin{aligned} k &: V(G_{\text{per}}) \to \mathbb{Z}^3\,, & k((v,k)) &:= k\,,\\ v_1 &: E(G_{\text{per}}) \to V(G_{\text{per}})\,, & v_1((e,k)) &:= (v_1(e), k)\,,\\ v_2 &: E(G_{\text{per}}) \to V(G_{\text{per}})\,, & v_2((e,k)) &:= (v_2(e), k + d(e))\,. \end{aligned}$$

As mentioned above, we will make two further assumptions on G which are expressed in terms of (G_{per}, z):

Connectivity. The graph G_{per} must be connected. On the level of G this can be expressed in the following way: For every $v_0, v \in V(G)$ and $k \in \mathbb{Z}^3$, there exists a sequence of edges

$$(v_0, v_1; d_1)\,,\ (v_1, v_2; d_2)\,,\ \cdots\,,\ (v_{n-1}, v_n; d_n)$$

in $\pm E(G) = \{e, -e \ :\ e \in E(G)\}$ with $v = v_n$ and $k = d_1 + \cdots + d_n$. Recall that for $e = (v_1, v_2; d) \in E(G)$ we defined $-e := (v_2, v_1; -d)$.

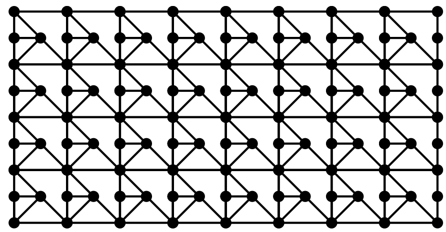


Figure 5.2: A two-dimensional example.

Infinitesimal rigidity. The frame (G_{per}, z) must be infinitesimally rigid. By this we mean that any displacement field on the nodes of G_{per},

$$u : V(G_{\text{per}}) \to \mathbb{R}^3 ,$$

which at first order (i.e. from a geometrically linearized viewpoint) preserves the lengths of all the edges, i. e. which satisfies,

$$r(e) \cdot (u(v_2(e)) - u(v_1(e))) = 0 \qquad \text{for all } e \in E(G_{\text{per}}) , \tag{5.4}$$

also satisfies $u(v_2(e)) - u(v_1(e)) = 0$ for all $e \in E(G_{\text{per}})$. In conjunction with the connectivity of G_{per} this implies that u is in fact constant.

Remark. There are various notions of rigidity for frames. Let us for the purpose of this remark assume that we are given any graph $G = (V, E)$ with a node placement $z : V \to \mathbb{R}^n$.

(i) Generally, the frame (G, z) is said to be rigid if any continuous motion of the nodes which preserves the length of edges, also preserves the distance between *any* pair of two nodes (see for example [28]). More precisely, (G, z) is called rigid if, given any continuous function $Z : [0, 1] \times V \to \mathbb{R}$ which satisfies $Z(0, \cdot) = z$ and

$$|z(v) - z(v')| = |Z(t, v) - Z(t, v')| \quad \text{for all } t \in [0, t] ,\ (v, v') \in E , \tag{5.5}$$

also satisfies

$$|z(v) - z(v')| = |Z(t, v) - Z(t, v')| \quad \text{for all } t \in [0, t] ,\ v, v' \in V . \tag{5.6}$$

(ii) The notion of infinitesimal rigidity, which we introduced above, and which is suitable in the geometrically linearized setting, arises from continuous rigidity by linearization. It is *stronger* than rigidity.

(a) The deformation of the vertical rods is perpendicular to the direction of the rods. This means that lengths are preserved at first order. However, the deformation is not constant.

(b) We see a deformation which (at first order) changes the length of the diagonal edge. In fact, such a change of length occurs for *every* non-constant deformation of this triangle.

Figure 5.3: Example of a non-rigid graph (a) and a rigid graph (b). Dashed lines show the undeformed, solid lines the deformed state.

Indeed, a degenerate triangle defined by three collinear nodes is rigid (which is trivially true for every complete graph), but not infinitesimally rigid (all displacements perpendicular to the straight line containing the triangle preserve edge lengths at first order).

On the other hand, let us assume that (G, z) is flexible (not rigid). Then There exists a continuous motion $Z : [0, 1] \times V \to \mathbb{R}$ with $Z(0, \cdot) = z$ which satisfies (5.5) but not (5.6). It can be shown that it is even possible to assume that Z is smooth and $\partial_t |Z(0, v_0) - Z(0, v_0')| \neq 0$ for some $v_0, v_0' \in V$.

We let $u(v) := \partial_t Z(0, v)$ for $v \in V(G)$. Then (5.5) implies

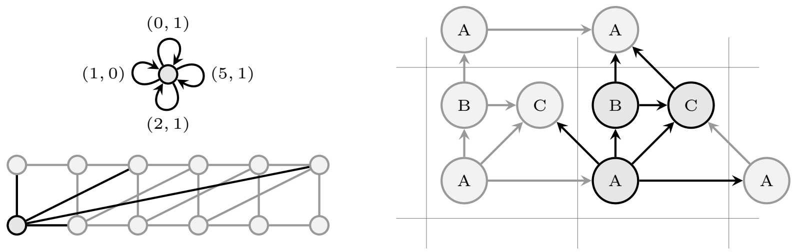
$$0 = \partial_t |Z(0, v) - Z(0, v')| = (u(v) - u(v')) \cdot \frac{z(v) - z(v')}{|z(v) - z(v')|}$$

for all $(v, v') \in E$. If (G, z) were infinitesimally rigid, this would imply that u is constant. But from $\partial_t |Z(0, v_0) - Z(0, v_0')| \neq 0$ we conclude that

$$0 \neq \partial_t Z(0, v_0) - \partial_t Z(0, v_0') = u(v_0) - u(v_0'),$$

so that u is not constant and hence (G, z) not infinitesimally rigid.

(iii) We also mention that it is an interesting question to ask in how far rigidity can be viewed not only as a property of the frame (G, z), but as an inherent property of the graph G, independent of the node placement z. Indeed, there is the notion of n-rigidity: The graph G is said to be *n-rigid* if (G, z) is rigid for almost all node placements z. Sometimes this is also called *generic rigidity* [53].



(a) This example of a graph G with only one node but four edges shows that rigidity cells sometimes need to span many perdiodicity cells. Observe that without the $(5, 1)$-edge, the graph would not be rigid.

(b) A rigidity cell corresponding to the example of Figure 5.1.

Figure 5.4: Example of rigidity cells.

Existence of a rigidity cell. We make an even slightly stronger assumption than infinitesimal rigidity which also subsumes the condition of connectivity. We require the existence of a *rigidity cell* G_r with the following properties:

(R1) G_r is a finite connected subgraph of G_{per}.

(R2) G_r contains the cell with index $k = 0 \in \mathbb{Z}^3$ in the sense that

$$V(G) \times \{0\} \subset V(G_r)\,, \qquad E(G) \times \{0\} \subset E(G_r)\,. \tag{5.7}$$

(R3) G_r is infinitesimally rigid: Every $u : V(G_r) \to \mathbb{R}^3$ with $r(e) \cdot (u(v_2(e)) - u(v_1(e))) = 0$ for all $e \in E(G_r)$ is constant.

(R4) For $1 \leq i \leq 3$ the graph $G_r \cup (G_r + e_i)$ is connected (notation introduced below).

The rigidity cell G_r will serve as a building block for larger subgraphs of G_{per}. In order to give assumption (R4) a precise meaning, we introduce the following bits of notation:

- Having two subgraphs G_1 and G_2 of a graph G, we denote by $G_1 \cup G_2$ the subgraph of G defined by $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

- We write $G_r + d$ with $d \in \mathbb{Z}^3$ for the subgraph of G_{per} defined by

$$V(G_r + d) = V(G_r) + d\,, \qquad E(G_r + d) = E(G_r) + d\,,$$

where $d \in \mathbb{Z}^3$ acts on vertices $(v, k) \in V(G_{\mathrm{per}})$ and edges $(v_1, v_2) \in E(G_{\mathrm{per}})$ by $(v, k) + d \coloneqq (v, k + d)$ and $(v_1, v_2) + d \coloneqq (v_1 + d, v_2 + d)$.

With this notation we can say that (5.7) implies that G_r spans G_{per} in the sense that

$$G_{\mathrm{per}} = \bigcup_{d \in \mathbb{Z}^3} G_r + d\,. \tag{5.8}$$

The following lemma shows that the infinitesimal rigidity of G_r implies the infinitesimal rigidity of G_{per} and even yields a quantitative rigidiy estimate (which serves a purpose similar to that of a Korn inequality).

Lemma 5.2 (Rigidity estimate). *Let (G, z, d) be a periodicity graph. Suppose that G_r is an infinitesimally rigid finite subgraph of G_{per} that spans G_{per} in the sense of* (5.8)*. Then there exists a constant $C > 0$ such that for all $u : V(G^{\mathrm{per}}) \to \mathbb{R}^3$,*

$$\sum_{e \in E(G_{\mathrm{per}})} |u(v_2(e)) - u(v_1(e))|^2 \\ \leq C \sum_{e \in E(G_{\mathrm{per}})} |r(e) \cdot (u(v_2(e)) - u(v_1(e)))|^2 \tag{5.9}$$

as an inequality in $\mathbb{R}_\infty \coloneqq \mathbb{R} \cup \{+\infty\}$.

Proof. The rigidity of G_r can be expressed by saying that the kernel of the linear map $A : (\mathbb{R}^3)^{V(G_r)} \to (\mathbb{R}^3)^{E(G_r)}$, defined by

$$Au \coloneqq (u(v_2(e)) - u(v_1(e)))_{e \in E(G_r)}\,, \quad u : V(G_r) \to \mathbb{R}^3\,,$$

is contained in the kernel of the linear map $B : (\mathbb{R}^3)^{V(G_r)} \to \mathbb{R}^{E(G_r)}$, defined by

$$Bu \coloneqq (r(e) \cdot (u(v_2(e)) - u(v_1(e))))_{e \in E(G_r)}\,, \quad u : V(G_r) \to \mathbb{R}^3\,.$$

As $V(G_r)$ is finite, this implies the quantitative estimate $\|Au\|^2 \leq C\|Bu\|^2$ for some $C > 0$ independent of u, i. e.

$$\sum_{e \in E(G_r)} |u(v_2(e)) - u(v_1(e))|^2 \leq C \sum_{e \in E(G_r)} |r(e) \cdot (u(v_2(e)) - u(v_1(e)))|^2\,.$$

When we now consider $u : V(G_{\text{per}}) \to \mathbb{R}^3$, we have

$$\begin{aligned}\sum_{e\in E(G_{\text{per}})} |u(v_2(e)) - u(v_1(e))|^2 &\leq \sum_{d\in\mathbb{Z}^3} \sum_{e\in E(G_r)+d} |u(v_2(e)) - u(v_1(e))|^2 \\ &\lesssim \sum_{d\in\mathbb{Z}^3} \sum_{e\in E(G_r)+d} |r(e)\cdot(u(v_2(e)) - u(v_1(e)))|^2 \\ &\lesssim \sum_{e\in E(G_{\text{per}})} |r(e)\cdot(u(v_2(e)) - u(v_1(e)))|^2 \,.\end{aligned}$$

For the last inequality we used that by the finiteness of G_r there is a bound on the number of ways that any edge $e \in E(G_{\text{per}})$ can be written as $e = e_r + d$ with $e_r \in E(G_r)$ and $d \in \mathbb{Z}^3$. □

Remark. There is some literature on the (infinitesimal) rigidity of periodic graphs. In the work by E. Ross [52] it is considered under the assumption of forced rigidity, meaning that only displacement fields are considered which have the same periodicity as the underlying graph. In a series of works by Borcea and Streinu [8, 9, 10] this assumption is droppped.

5.2 Finite graphs adapted to a domain

Our object of study will not be an *infinite* lattice corresponding to the full periodic graph G_{per}, but rather a sequence of finite (and scaled) sublattices that occupy some bounded domain Ω. In this section we will therefore undertake the construction of appropriate subgraphs G^ε of G_{per}. For the scaling we introduce the periodicity cells

$$\square_k^\varepsilon := \varepsilon(\square + k)\,, \qquad k \in \mathbb{Z}^3\,, \qquad \square := (0,1)^3\,.$$

We start with a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ and a nonempty open subset Γ of $\partial\Omega$. On Γ we will prescribe Dirichlet boundary values, whereas on $\partial\Omega \setminus \Gamma$ we will have Neumann boundary values. We write

$$H^1_\Gamma(\Omega) := \{u \in H^1(\Omega) \ : \ u = 0 \text{ on } \Gamma\}\,.$$

Since Γ is an open subset of $\partial\Omega$, we can find a set $\Omega_\Gamma \subset \mathbb{R}^3$ such that $\Omega_\Gamma \cup \Omega$ is open, $\Omega_\Gamma \cap \Omega = \emptyset$ and $\Omega_\Gamma \cap \partial\Omega = \Gamma$. We assume that Ω_Γ can be chosen in a way that $\Omega \cup \Omega_\Gamma$ is a Lipschitz domain. In the case of pure Dirichlet data ($\Gamma = \partial\Omega$) we can simply choose $\Omega_\Gamma = \mathbb{R}^3 \setminus \Omega$. For a simple example see Figure 5.5.

We will not only construct subgraphs G^ε, but also subsets $V^\varepsilon_\Gamma \subset V(G^\varepsilon)$ of their respective sets of vertices which correspond to Γ in the sense that they contain the nodes on which values (of displacements) are prescribed.

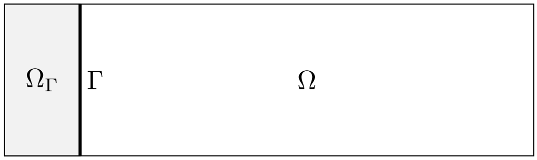


Figure 5.5: Example showing a rectangular domain Ω with Γ being the left face of Ω. An appropriate domain Ω_Γ is denoted by the shaded region.

The main challenge in the construction of G^ε is that the uniform ridigity of G^ε must be ensured. This can be done by using translated copies of the rigidty cell G_r as building blocks. Our approach can be summarized as follows:

(1) Define $\overline{G}^\varepsilon$ as the union of translated copies of G_r which "fit" into $\Omega \cup \Omega_\Gamma$.

(2) Define G^ε as the subgraph of $\overline{G}^\varepsilon$ that consists of all nodes that fit into Ω or are at most one edge away from Ω.

(3) Those nodes of $\overline{G}^\varepsilon$ that are one edge away from Ω make up V^ε_Γ.

In order to formalize these steps, we proceed as follows: We first choose appropriate index sets $D^\varepsilon \subset \mathbb{Z}^3$ and use these to define $\overline{G}^\varepsilon$:

$$\overline{G}^\varepsilon := \bigcup_{d \in D^\varepsilon} G_r + d\,, \qquad \varepsilon > 0\,.$$

For the choice of the sets D^ε we impose the following requirements:

(D1) *Approximation from within.* There holds $\square^\varepsilon_{k(v)} \subset \Omega \cup \Omega_\Gamma$ for all $v \in V(G_r) + D^\varepsilon$.
Moreover, there exists a constant $C > 0$ such that

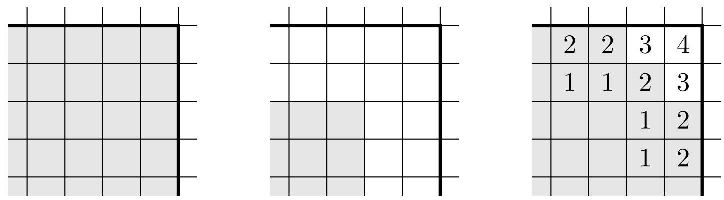
$$\operatorname{dist}(\varepsilon(D^\varepsilon + [0,1]^3)\,,\, \mathbb{R}^3 \setminus (\Omega \cup \Omega_\Gamma)) < C\varepsilon$$

for all $\varepsilon > 0$.

(D2) *Uniform connectivity.* There exist constants $C \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and $d, d' \in D^\varepsilon$ there exist $d_0, \ldots, d_N \in D^\varepsilon$ with $N \le C\|d - d'\|_1$ and

$$d = d_0\,, \qquad d' = d_N\,, \qquad \|d_i - d_{i-1}\|_1 \le 1 \text{ for } i = 1 \le i \le N\,.$$

In other words: The minimum number of unit-steps in $\mathbb{Z}^3$ needed to connect any two given points of D^ε gets at most C times larger when the restriction is added that only points of D^ε may be visited.



(a) The shaded region shows the original D^ε.

(b) After two steps of shrinking we have the shaded region D^ε_2.

(c) Two growing-steps yield $D^\varepsilon_{2,2}$, but four are needed to arrive back at $D^\varepsilon = D^\varepsilon_{2,4}$.

Figure 5.6: The shrink-and-grow property illustrated at a right angle. With sharper angles, the number of growing steps needed would further increase.

(D3) *Shrink-and-grow property.* For $r, j \in \mathbb{N}$ we define

$$D^\varepsilon_r := D^\varepsilon_{r,0} := \{d \in D^\varepsilon \ : \ \operatorname{dist}_{(\mathbb{Z}^3, \|\cdot\|_1)}(d, \mathbb{Z}^3 \setminus D^\varepsilon) > r\}\,, \tag{5.10}$$

$$D^\varepsilon_{r,j} := \{d \in D^\varepsilon \ : \ \|d - d'\|_1 \leq 1 \text{ for some } d' \in D^\varepsilon_{r,j-1}\}\,. \tag{5.11}$$

Given any $r \in \mathbb{N}$ there exist constants $R \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that $D^\varepsilon_{r,R} = D^\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0)$.

Here r is the number of shrinking steps, and R is the number of growing steps needed to compensate for the shrinking. For an illustration see Figure 5.6.

Remark. A canonical way to define D^ε is the following:

$$D^\varepsilon := \{d \in \mathbb{Z}^3 \ : \ \square^\varepsilon_{k(v)} \subset \Omega \cup \Omega_\Gamma \text{ for all } v \in V(G_r) + d\}\,. \tag{5.12}$$

With D^ε defined this way, condition (D1) is obviously satisfied. We conjecture that for Lipschitz Domains Ω with D^ε defined as in (5.12), (D2) and (D3) are satisfied too. However, we will not set out to prove this statement, because in practical situations one would in general have a domain where it is obvious how to define D^ε such that (D1)–(D3) are satisfied.

Following to item (1) on Page 65, we now construct $\overline{G}^\varepsilon$. For this we use D^ε and define $\overline{G}^\varepsilon := G_r + D^\varepsilon$, which means that

$$V(\overline{G}^\varepsilon) = V(G_r) + D^\varepsilon\,, \qquad E(\overline{G}^\varepsilon) = E(G_r) + D^\varepsilon\,. \tag{5.13}$$

As outlined in items (2)–(3) on Page 65, we then define subgraphs G_0^ε and G^ε of $\overline{G}^\varepsilon$ as well as the set of nodes V_Γ^ε in the following way:

$$V(G_0^\varepsilon) := \{(v,k) \in V(\overline{G}^\varepsilon) \ : \ \Box_k^\varepsilon \cap \Omega \neq \emptyset\}, \tag{5.14}$$

$$E(G_0^\varepsilon) := \{e \in E(\overline{G}^\varepsilon) \ : \ v_1(e), v_2(e) \in V(G_0^\varepsilon)\}, \tag{5.15}$$

$$V(G^\varepsilon) := \{v_2(e) \ : \ e \in \pm E(\overline{G}^\varepsilon) \text{ with } v_1(e) \in V(G_0^\varepsilon)\}, \tag{5.16}$$

$$E(G^\varepsilon) := \{e \in E(\overline{G}^\varepsilon) \ : \ v_1(e) \in V(G_0^\varepsilon) \text{ or } v_2(e) \in V(G_0^\varepsilon)\} \tag{5.17}$$

$$V_\Gamma^\varepsilon := V(G^\varepsilon) \setminus V(G_0^\varepsilon). \tag{5.18}$$

For later reference, we summarize these constructions in the following definition.

Definition 5.3 (Graph realizations). *Let (G, z, d) be a periodicity graph according to Definition 5.1 and G_{per} the associated periodic graph as defined in* (5.1)*.*

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and Γ a nonempty open subset of $\partial\Omega$. We assume that there exists a set $\Omega_\Gamma \subset \mathbb{R}^3$ with $\Omega_\Gamma \cap \Omega = \emptyset$ such that $\Omega \cup \Omega_\Gamma$ is a bounded Lipschitz domain and $\Omega_\Gamma \cap \partial\Omega = \Gamma$.

Let $(D^\varepsilon)_\varepsilon$ be a family of subsets of $\mathbb{Z}^3$ satisfying (D1)–(D3). We then say that $(G^\varepsilon)_\varepsilon$ as defined in (5.13) *to* (5.18) *constitutes a family of* graph realizations*.*

We draw two important consequences from the above constructions. The first one is an ε-uniform rigidity estimate which quantifies the infinitesimal rigidity and is similar in character to a Korn inequality. The second one will be a Poincaré inequality.

Lemma 5.4 (Uniform rigidity estimate). *Let $(G^\varepsilon)_\varepsilon$ be a family of graph realizations with $(V_\Gamma^\varepsilon)_\varepsilon$ defined by* (5.18) *in the setting of Definition 5.3. Then there exist constants $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $u : V(G^\varepsilon) \to \mathbb{R}^3$ with $u = 0$ on V_Γ^ε,*

$$\sum_{e \in E(G^\varepsilon)} |u(v_2(e)) - u(v_1(e))|^2 \leq C \sum_{e \in E(G^\varepsilon)} |r(e) \cdot (u(v_2(e)) - u(v_1(e)))|^2 . \tag{5.19}$$

Proof. We extend u by zero to $V(\overline{G}^\varepsilon)$. Then we use $E(G^\varepsilon) \subset E(\overline{G}^\varepsilon) = E(G_r) + D^\varepsilon$ in order to see that

$$\sum_{e \in E(G^\varepsilon)} |u(v_2(e)) - u(v_1(e))|^2 \leq \sum_{d \in D^\varepsilon} \sum_{e \in E(G_r)+d} |u(v_2(e)) - u(v_1(e))|^2 .$$

Using the rigidity of G_r (see Lemma 5.2), this implies

$$\sum_{e\in E(G^\varepsilon)} |u(v_2(e)) - u(v_1(e))|^2 \lesssim \sum_{d\in D^\varepsilon} \sum_{e\in E(G_r)+d} |r(e)\cdot(u(v_2(e)) - u(v_1(e)))|^2 .$$

As every element of $E(\overline{G}^\varepsilon)$ is only contained in $E(G_r)+d$ for a finite and uniformly bounded number of vectors $d\in D^\varepsilon$, we then have

$$\sum_{e\in E(G^\varepsilon)} |u(v_2(e)) - u(v_1(e))|^2 \lesssim \sum_{e\in E(\overline{G}^\varepsilon)} |r(e)\cdot(u(v_2(e)) - u(v_1(e)))|^2 .$$

Finally we use that by (5.17) for $e\in E(\overline{G}^\varepsilon)\setminus E(G^\varepsilon)$ there holds $v_1(e)\notin V(G_0^\varepsilon)$ and $v_2(e)\notin V(G_0^\varepsilon)$ and therefore $u(v_1(e)) = u(v_2(e)) = 0$ by (5.18). Thus we have

$$\sum_{e\in E(G^\varepsilon)} |u(v_2(e)) - u(v_1(e))|^2 \lesssim \sum_{e\in E(G^\varepsilon)} |r(e)\cdot(u(v_2(e)) - u(v_1(e)))|^2 . \qquad \square$$

As already mentioned, a second important consequence which we can draw from our constructions is an ε-uniform Poincaré inequality.

Lemma 5.5 (Uniform Poincaré inequality). *Let $(G^\varepsilon)_\varepsilon$ be a family of graph realizations with $(V_\Gamma^\varepsilon)_\varepsilon$ defined by* (5.18) *in the setting of Definition 5.3.*

Then there exist constants $C>0$ and $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and $u: V(G^\varepsilon)\to\mathbb{R}^3$ with $u=0$ on V_Γ^ε,

$$\sum_{v\in V(G^\varepsilon)} |u(v)|^2 \le C \sum_{e\in E(G^\varepsilon)} \left|\frac{u(v_2(e)) - u(v_1(e))}{\varepsilon}\right|^2 .$$

The proof of this lemma is most conveniently executed in the framework developed in the next section. We thus postpone it to Page 71.

Here is another important lemma which leverages the uniform connectivity assumption (D2) for D^ε:

Lemma 5.6 (Uniform connectivity). *Let $(\overline{G}^\varepsilon)_\varepsilon$ be defined by* (5.13) *in the setting of Definition 5.3. Then there exist constants $C>0$ and $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and any two nodes $(v,k),(v',k')\in V(\overline{G}^\varepsilon)$ there holds*

$$\operatorname{dist}_{\overline{G}^\varepsilon}((v,k),(v',k')) \le C(1+\|k-k'\|_1) .$$

Here $\operatorname{dist}_{\overline{G}^\varepsilon}$ *measures the length of the shortest undirected path in $\overline{G}^\varepsilon$ between two nodes (in terms of numbers of edges).*

Proof. Consider any $(v,k),(v',k') \in V(\overline{G}^\varepsilon)$. Since $V(\overline{G}^\varepsilon) = V(G_r) + D^\varepsilon$, there exist $d, d' \in D^\varepsilon$ such that $(v,k) \in V(G_r) + d$ and $(v',k') \in V(G_r) + d'$. Since G_r is finite and connected (see (R1) on Page 62),

$$\operatorname{dist}_{\overline{G}^\varepsilon}((v,k),(v,d)) \lesssim 1\,, \qquad \operatorname{dist}_{\overline{G}^\varepsilon}((v,d'),(v',k')) \lesssim 1\,.$$

It remains to prove an estimate for $\operatorname{dist}_{\overline{G}^\varepsilon}((v,d),(v,d'))$.

By the uniform connectivity assumption (D2) for D^ε, there exist $d_0,\dots,d_N \in D^\varepsilon$ with $d = d_0, d_N = d'$ and $\|d_i - d_{i-1}\|_1 = 1$ for $1 \le i \le N$ and $N \lesssim \|d-d'\|_1$. Since $G_r \cup (G_r + e)$ is connected for all unit vectors $e \in \mathbb{Z}^3$, see (R4) on Page 62, and in particular for $e = d_i - d_{i-1}$, there exists in $\overline{G}^\varepsilon$ a corresponding path from (v,d) to (v,d') via $(v,d_1),(v,d_2),\dots(v,d_{N-1})$ with a total length $\lesssim \|d-d'\|_1$. Thus we have

$$\begin{aligned}\operatorname{dist}_{\overline{G}^\varepsilon}((v,d),(v,d')) &\lesssim \|d-d'\|_1 \le \|d-k\|_1 + \|k-k'\|_1 + \|k'-d'\|_1 \\ &\lesssim 1 + \|k-k'\|_1\,. \qquad \square\end{aligned}$$

5.3 Calculus on periodic graphs

When we consider elastoplastic lattices, the state of such a lattice will be the union of the states of the individual nodes and edges. We will thus naturally deal with functions of the form

$$\beta : V(G^\varepsilon) \to X \quad \text{and} \quad \gamma : E(G^\varepsilon) \to X$$

for some separable Banach spaces X, that is, with functions defined on the nodes or edges of the graph G^ε.

We implicitly extend such functions β and γ by zero to all of $V(G_{\text{per}})$ or $E(G_{\text{per}})$, respectively. Then we can identify β and γ with functions defined on $\mathbb{R}^3$, piecewise constant on each cell $\square_k^\varepsilon$, and having *multiple* values in X – one for each node type $v \in V(G)$ or edge type $e \in E(G)$, respectively:

$$\begin{aligned}\beta &: \mathbb{R}^3 \to X^{V(G)}\,, & \beta_v|_{\square_k^\varepsilon} &\equiv \beta((v,k)) & &\text{for } (v,k) \in V(G_{\text{per}})\,,\\ \gamma &: \mathbb{R}^3 \to X^{E(G)}\,, & \gamma_e|_{\square_k^\varepsilon} &\equiv \gamma((e,k)) & &\text{for } (e,k) \in E(G_{\text{per}})\,.\end{aligned}$$

Since edges may represent rods of, e.g., different cross sections, the target space for an edge-function γ will usually depend on the class $e \in E(G)$ of an edge $(e,k) \in E(G^\varepsilon)$, so that we will replace X with a separate Banach space Y_e for each $e \in E(G)$. This leads to Definition 5.8. But first we introduce the cell projectors P^ε.

Definition 5.7 (ε-cell projections). *With P^ε we denote the operator that takes a locally integrable function f defined on $\mathbb{R}^3$ with values in some separable Banach space and averages it on each cell $\square_k^\varepsilon$,*

$$P^\varepsilon f(x) := \fint_{\square_k^\varepsilon} f(x)\mathrm{d}x \quad \text{for } x \in \square_k^\varepsilon\,, \quad k \in \mathbb{Z}^3\,. \tag{5.20}$$

Definition 5.8 (Functions on periodic graphs). *Let (G, z, d) be a periodicity graph and G_{per} the associated periodic graph as defined in (5.1). Now let G^ε be a subgraph of G_{per}, and let X and $(Y_e)_{e\in E(G)}$ be separable Banach spaces. For $v \in V(G)$ and $e \in E(G)$ we consider the domains*

$$\Omega_v^\varepsilon(G^\varepsilon) := \bigcup_{k\in D_v^\varepsilon(G^\varepsilon)} \square_k^\varepsilon\,, \quad D_v^\varepsilon(G^\varepsilon) := \{k \in \mathbb{Z}^3 \text{ with } (v,k) \in V(G^\varepsilon)\}\,, \tag{5.21}$$

$$\Omega_e^\varepsilon(G^\varepsilon) := \bigcup_{k\in D_e^\varepsilon(G^\varepsilon)} \square_k^\varepsilon\,, \quad D_e^\varepsilon(G^\varepsilon) := \{k \in \mathbb{Z}^3 \text{ with } (e,k) \in E(G^\varepsilon)\}\,, \tag{5.22}$$

$$\Omega^\varepsilon(G^\varepsilon) := \bigcup_{v\in V(G)} \Omega_v^\varepsilon(G^\varepsilon) \tag{5.23}$$

and the corresponding characteristic functions

$$\mathbb{1}_v^\varepsilon(G^\varepsilon) = \mathbb{1}_{\Omega_v^\varepsilon(G^\varepsilon)}\,,\ \mathbb{1}_e^\varepsilon(G^\varepsilon) = \mathbb{1}_{\Omega_e^\varepsilon(G^\varepsilon)}\,,\ \mathbb{1}^\varepsilon(G^\varepsilon) = \mathbb{1}_{\Omega^\varepsilon(G^\varepsilon)}\ :\ \mathbb{R}^3 \to \{0,1\}\,.$$

We then introduce the following terminology:

(i) A function $\alpha : \mathbb{R}^3 \to \mathbb{R}$ is a G^ε-cell function if $\alpha = P^\varepsilon\alpha$ and $\alpha = \mathbb{1}^\varepsilon(G^\varepsilon)\alpha$.

(ii) A function $\beta : \mathbb{R}^3 \to \Pi_{v\in V(G)}X$ is a G^ε-node function if $\beta = P^\varepsilon\beta$ and $\beta_v = \mathbb{1}_v^\varepsilon(G^\varepsilon)\beta_v$ for all $v \in V(G)$.

(iii) A function $\gamma : \mathbb{R}^3 \to \Pi_{e\in E(G)}Y_e$ is a G^ε-edge function if $\gamma = P^\varepsilon\gamma$ and $\gamma_e = \mathbb{1}_e^\varepsilon(G^\varepsilon)\gamma_e$ for all $e \in E(G)$.

Moreover, given any G^ε-cell function $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$, we also interpret it as a G^ε-node function by the convention

$$\alpha_v := \mathbb{1}_v^\varepsilon(G^\varepsilon)\alpha\,, \qquad v \in V(G)\,. \tag{5.24}$$

This just means that we assign the cell-value to all existing nodes in the cell.

Remark. It would be more precise to speak of $(G^\varepsilon, \varepsilon)$-cell functions etc. instead of G^ε-cell functions etc.: The graph G^ε is in itself ignorant of the ε-scale, it is just a subgraph of G_{per}. We will, however, always use graphs where the intended value of ε is indicated with a superscript. Thus no ambiguity should arise.

Definition 5.9 (Periodic graph gradient). *Let $u : \mathbb{R}^3 \to \Pi_{v\in V(G)}X$ be a G^ε-node function. We define the G^ε-edge function $\mathrm{grad}^\varepsilon(u;G^\varepsilon) : \mathbb{R}^3 \to \Pi_{e\in E(G)}X$ by*

$$\mathrm{grad}^\varepsilon_e(u;G^\varepsilon) \coloneqq \mathbb{1}^\varepsilon_e(G^\varepsilon)\frac{T_{\varepsilon d(e)}u_{v_2(e)} - u_{v_1(e)}}{\varepsilon}\,, \quad e \in E(G)\,, \tag{5.25}$$

where T_v for $v \in \mathbb{R}^3$ is the translation operator $(T_v f)(x) \coloneqq f(x+v)$.

Remark. The graph-gradient $\mathrm{grad}^\varepsilon(u;G^\varepsilon)$ contains difference quotients of u along the edges of the graph G^ε. Observe, however, that in the denominator we do not have the actual length $\varepsilon L(e)$ of an edge of type $e \in E(G)$, but instead just ε.

We can now cast Lemma 5.4 (the already proved uniform rigidity estimate) and Lemma 5.5 (the uniform Poincaré inequality which we still need to prove) into the new language of functions on periodic graphs. In this framework, boundary values will no longer be prescribed on V^ε_Γ as defined in (5.18), but for each $v \in V(G)$ on Γ^ε_v,

$$\Gamma^\varepsilon_v \coloneqq \{x \in \mathbb{R}^3 \ :\ x \in \Box^\varepsilon_k \text{ for some } k \in \mathbb{Z}^3 \text{ with } (v,k) \in V^\varepsilon_\Gamma\}\,. \tag{5.26}$$

Lemma 5.10 (Uniform Rigidity estimate II). *Let $(G^\varepsilon)_\varepsilon$ be a family of graph realizations and Γ^ε_v as defined by (5.26) in the setting of Definition 5.3. Then there exists a constant $C > 0$ such that for all $\varepsilon \in (0,1)$ and all G^ε-node functions $u : \mathbb{R}^3 \to \Pi_{v\in V(G)}\mathbb{R}^3$ with $u_v = 0$ on Γ^ε_v,*

$$\sum_{e\in E(G)} \|\mathrm{grad}^\varepsilon_e(u;G^\varepsilon)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \le C \sum_{e\in E(G)} \|r(e)\cdot \mathrm{grad}^\varepsilon_e(u;G^\varepsilon)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,.$$

Proof. This lemma is equivalent to Lemma 5.4. Indeed, the summation over all the cells is implicit in the squared $L^2(\mathbb{R}^3;\mathbb{R}^3)$-norm. ☐

Lemma 5.11 (Uniform Poincaré inequality II). *Let $(G^\varepsilon)_\varepsilon$ be a family of graph realizations and Γ^ε_v as defined by (5.26) in the setting of Definition 5.3. Then there exist constants $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and all G^ε-node functions $u : \mathbb{R}^3 \to \Pi_{v\in V(G)}\mathbb{R}^3$ with $u_v = 0$ on Γ^ε_v,*

$$\sum_{v\in V(G)} \|u_v\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \le C \sum_{e\in E(G)} \|\mathrm{grad}^\varepsilon_e(u;G^\varepsilon)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}\,.$$

In order to prove Lemma 5.11, and thus Lemma 5.5, we first give the following lemma.

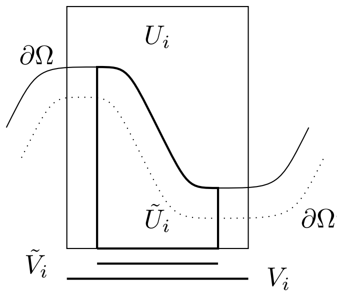


Figure 5.7: Illustration of the shrinking and the Lipschitz covering.

Lemma 5.12 (Shrinking-uniform Poincaré inequality). *Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain and $U \subset \Omega$ a nonempty open subset. For $s \geq 0$ we consider the shrinked domain*

$$\Omega^s := \{x \in \mathbb{R}^n \;:\; \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) > s\} \subset \Omega\,.$$

Then there exist $s_0 > 0$ and $C > 0$ such that

$$\|u\|_{L^2(\Omega^s)} \leq C\|\nabla u\|_{L^2(\Omega^s;\mathbb{R}^n)}$$

for all $s \in [0, s_0]$ and $u \in H^1(\Omega^s)$ with $u = 0$ in $U \cap \Omega^s$.

Proof. We may assume that $U \Subset \Omega$, since shrinking U makes the statement only stronger. Since Ω is a bounded Lipschitz domain, we can cover $\overline{\Omega}$ with open sets $U_0, \ldots, U_N \subset \mathbb{R}^n$ such that $U \subset U_0 \Subset \Omega$ and for $1 \leq i \leq N$ there exist rigid transformations $\phi_i : \mathbb{R}^n \to \mathbb{R}^n$, bounded open sets $V_i \subset \mathbb{R}^{n-1}$ and Lipschitz continuous functions $f_i : \overline{V}_i \to (0, h_i)$ with

$$\begin{aligned}
\phi_i(U_i) &= V_i \times (0, h_i)\,, \\
\phi_i(\Omega \cap U_i) &= \{(x,t) \;:\; x \in V_i\,,\; t \in (0, f_i(x))\}\,, \\
\phi_i(\partial\Omega \cap \overline{U}_i) &= \{(x,t) \;:\; x \in \overline{V}_i\,,\; t = f_i(x)\}\,.
\end{aligned}$$

Moreover, we can choose $\tilde{V}_i \Subset V_i$ such that Ω is still covered by $U_0, \tilde{U}_1, \ldots, \tilde{U}_N$, where

$$\tilde{U}_i := \phi_i^{-1}(\{(x,t) \;:\; x \in \tilde{V}_i\,,\; t \in (0, f_i(x))\}\,.$$

See Figure 5.7 for an illustration.

We now fix some $1 \leq i \leq N$ and assume without loss of generality that the rigid transformation ϕ_i is the identity. We claim: For small $s > 0$ the shrinked patch $\Omega^s \cap \tilde{U}_i$ is a subgraph above $\tilde{V}_i$. More precisely, there exists a function $f_i^s : \tilde{V}_i \to (0, h_i)$ (not necessarily Lipschitz continuous) such that

$$\Omega^s \cap \tilde{U}_i = \{(x,t) \ : \ x \in \tilde{V}_i \, , \ t \in (0, f_i^s(x))\} \, . \tag{5.27}$$

Proof of the claim. Let us choose $s > 0$ small enough such that $V_i \times (-s, 0] \subset \Omega$ and $\operatorname{dist}(\tilde{V}_i, \partial V_i) > s$. In order to prove the subgraph property for $\Omega^s \cap \tilde{U}_i$, we consider any $(x,t) \in \Omega^s \cap \tilde{U}_i$ and show that $(x,r) \in \Omega^s \cap \tilde{U}_i$ for every $0 < r < t$. It is clear that $(x,r) \in \tilde{U}_i$ since $\tilde{U}_i$ has the subgraph property by definition. It remains to show that $(x,r) \in \Omega^s$, i. e. $\operatorname{dist}((x,r), \partial\Omega) > s$. Since $\partial\Omega$ is compact, it suffices to show that $\operatorname{dist}((x,r),(x',r')) > s$ for all $(x',r') \in \partial\Omega$.

Case 1: $(x',r') \notin U_i = V_i \times (0,h_i)$. Since $V_i \times (-s,0] \subset \Omega$ this implies $(x',r') \notin V_i \times (-s,h_i)$. On the other hand, $(x,t) \in \Omega^s$ implies $r < t < f_i(x) - s \leq h_i - s$. Thus $(x,r) \in \tilde{V}_i \times (0, h_i - s)$. This implies

$$\operatorname{dist}((x,r),(x',r')) \geq \operatorname{dist}(\tilde{V}_i \times (0,h_i - s), \mathbb{R}^n \setminus (V_i \times (-s,h_i))) > s \, .$$

Case 2: $(x',r') \in U_i$. Then $r' = f_i(x')$. Now let $t' := \max(r',t)$. Then

$$|t' - t| = \max(r' - t, 0) \leq \max(r' - r, 0) \leq |r' - r| \, .$$

Moreover, $(x',t') \in \mathbb{R}^n \setminus \Omega$, and therefore

$$\operatorname{dist}((x,r),(x',r')) \geq \operatorname{dist}((x,t),(x',t')) \geq \operatorname{dist}((x,t), \mathbb{R}^n \setminus \Omega) > s \, .$$

We have thus proved (5.27).

We now drop the assumption that ϕ_i is the identity so that instead of (5.27) we have

$$\phi_i(\Omega^s \cap \tilde{U}_i) = \{(x,t) \ : \ x \in \tilde{V}_i \, , \ t \in (0, f_i^s(x))\} \, .$$

Writing $v := u \circ \phi_i^{-1}$ we have

$$\begin{aligned} \|u\|^2_{L^2(\Omega^s \cap \tilde{U}_i)} &= \int_{\tilde{V}_i} \int_0^{f_i^s(x)} |v(x,t)|^2 \mathrm{d}t \mathrm{d}x \\ &\lesssim \int_{\tilde{V}_i} |v(x,0)|^2 + \int_0^{f_i^s(x)} |\nabla v(x,t)|^2 \mathrm{d}t \mathrm{d}x \\ &= \|u\|^2_{L^2(\phi_i^{-1}(\tilde{V}_i \times \{0\}))} + \|\nabla u\|^2_{L^2(\Omega^s \cap \tilde{U}_i)} \, , \end{aligned} \tag{5.28}$$

with a constant independent of s for small values $s > 0$. Thus we have, using that $U_0 \subset \Omega^s$ for small $s > 0$,

$$\begin{aligned}\|u\|^2_{L^2(\Omega^s)} &\leq \|u\|^2_{L^2(U_0)} + \sum_{i=1}^N \|u\|^2_{L^2(\Omega^s\cap\tilde{U}_i)} \\ &\overset{(5.28)}{\lesssim} \|u\|^2_{L^2(U_0)} + \sum_{i=1}^N \|u\|^2_{L^2(\phi^{-1}(\tilde{V}_i\times\{0\}))} + \|\nabla u\|^2_{L^2(\Omega^s\cap\tilde{U}_i)} \\ &\lesssim \|u\|^2_{H^1(U_0)} + \sum_{i=1}^N \|\nabla u\|^2_{L^2(\Omega^s\cap\tilde{U}_i)} \quad \text{(trace estimate)} \\ &\lesssim \|\nabla u\|^2_{L^2(U_0)} + \sum_{i=1}^N \|\nabla u\|^2_{L^2(\Omega^s\cap\tilde{U}_i)} \quad \text{(Poincaré on } U_0) \\ &\lesssim \|\nabla u\|^2_{L^2(\Omega^s)}\,.\end{aligned}$$

This shows Lemma 5.12. □

Proof of Lemma 5.11. We fix $v \in V(G)$ and give an estimate for $\|u_v\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)}$. This is sufficient, since $V(G)$ is finite. The problem can be reduced to the ordinary Poincaré inequality, and we do so by considering appropriate approximations to which the Poincaré inequality applies. We denote by $\hat{u}_v : \mathbb{R}^3 \to \mathbb{R}^3$ the trilinear interpolation of u_v with respect to the grid $\varepsilon\mathbb{Z}^3$ for which $\hat{u}_v(\varepsilon k)$ is the value of u_v on $\square^\varepsilon_k$ for $k \in \mathbb{Z}^3$. Since u_v is constant on cells, the interpolation is explicitly given by the formula

$$\hat{u}_v(x) := \fint_{(0,\varepsilon)^3} u_v(x+y)\mathrm{d}y = \int_{(0,1)^3} u_v(x+\varepsilon y)\mathrm{d}y\,, \qquad x \in \mathbb{R}^3\,. \tag{5.29}$$

The main challenge is to deal with the behaviour of u_v near the Neumann-boundary $\partial\Omega\setminus\Gamma$ of Ω. We therefore consider shrinked versions of $\Omega\cup\Omega_\Gamma$. Given any $r \in \mathbb{N}_0$ and $s \geq 0$, we define

$$\Omega^\varepsilon_r := \bigcup_{k\in D^\varepsilon_r} \square^\varepsilon_k\,, \qquad \hat{\Omega}^\varepsilon_s := \{x \in \mathbb{R}^3 \ :\ \mathrm{dist}(x, \mathbb{R}^3\setminus(\Omega\cup\Omega_\Gamma)) > \varepsilon s\}\,,$$

where $D^\varepsilon_r \subset D^\varepsilon$ is the set defined in (5.10). Observe that by (D1), there exists a constant $r_0 \in \mathbb{N}$ such that $\Omega^\varepsilon_0 \subset \hat{\Omega}^\varepsilon_0$ and $\hat{\Omega}^\varepsilon_0 \subset \Omega^\varepsilon_{r_0}$ for all $\varepsilon > 0$. By the equivalence of the euclidean and the ℓ^1 norm in $\mathbb{R}^3$, there thus exists a constant $C \in \mathbb{N}$ such that

$$\Omega^\varepsilon_r \subset \hat{\Omega}^\varepsilon_{Cr} \quad \text{and} \quad \hat{\Omega}^\varepsilon_r \subset \Omega^\varepsilon_{C(r+1)} \quad \text{for all } \varepsilon > 0\,,\ r \in \mathbb{N}\,. \tag{5.30}$$

Note that $u_v = 0$ on Γ_v^ε implies $\operatorname{grad}^\varepsilon(u; G^\varepsilon) = \operatorname{grad}^\varepsilon(u; \overline{G}^\varepsilon)$. Indeed, for any edge $e \in E(\overline{G}^\varepsilon) \setminus E(G^\varepsilon)$ we have $v_1(e), v_2(e) \notin V(G_0^\varepsilon) = V(G^\varepsilon) \setminus V_\Gamma^\varepsilon$ by (5.17) and (5.18). Hence both ends of e are outside the support of u and e does not contribute to the graph gradient.

By (5.13) and (R2) we have $E(\overline{G}^\varepsilon) \supset E(G) \times D^\varepsilon$ and thus

$$\begin{aligned}\sum_{e\in E(G)} \|\operatorname{grad}_e^\varepsilon(u; G^\varepsilon)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} &= \sum_{e\in E(G)} \|\operatorname{grad}_e^\varepsilon(u; \overline{G}^\varepsilon)\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\ &\geq \sum_{e\in E(G)} \left\| \frac{u_{v_2(e)}(\cdot + \varepsilon d(e)) - u_{v_1(e)}}{\varepsilon} \right\|^2_{L^2(\Omega_0^\varepsilon;\mathbb{R}^3)} .\end{aligned}$$

By (R4), there exists for $1 \leq i \leq 3$ a path in G_{per} joining $(v, 0)$ with (v, e_i). Moreover, these paths only visit nodes (v', k') with $\|k'\|_1 \leq r_1$, where r_1 is a constant only depending in G. Using the triangle inequality along (translated copies of) these paths, we get the estimate

$$\sum_{e\in E(G)} \left\| \frac{u_{v_2(e)}(\cdot + \varepsilon d(e)) - u_{v_1(e)}}{\varepsilon} \right\|^2_{L^2(\Omega_0^\varepsilon;\mathbb{R}^3)} \gtrsim \sum_{i=1}^3 \left\| \frac{u_v(\cdot + \varepsilon e_i) - u_v}{\varepsilon} \right\|^2_{L^2(\Omega_{r_1}^\varepsilon;\mathbb{R}^3)} .$$

The derivatives of $\hat{u}_v$ can be explicitly computed from (5.29) to be, e.g.,

$$\begin{aligned}\partial_1 \hat{u}_v(x) &= \fint_{\{0\}\times(0,\varepsilon)^2} \frac{u_v(x + y + \varepsilon e_1) - u_v(x + y)}{\varepsilon} \mathrm{d}y \\ &= \fint_{x+\{0\}\times(0,\varepsilon)^2} \frac{u_v(\cdot + \varepsilon e_1) - u_v}{\varepsilon} \mathrm{d}y \,,\end{aligned}$$

with similar formulas for $\partial_2 \hat{u}_v$ and $\partial_3 \hat{u}_v$. This implies that

$$\sum_{i=1}^3 \left\| \frac{u_v(\cdot + \varepsilon e_i) - u_v}{\varepsilon} \right\|^2_{L^2(\Omega_{r_1}^\varepsilon;\mathbb{R}^3)} \gtrsim \|\nabla \hat{u}_v\|^2_{L^2(\Omega_{r_1+1}^\varepsilon;\mathbb{R}^3)} .$$

Now by (5.30) there exists $r_2 > 0$ such that $\Omega_{r_1+1}^\varepsilon \supset \hat{\Omega}_{r_2}^\varepsilon$ for all $\varepsilon > 0$ and thus

$$\|\nabla \hat{u}_v\|^2_{L^2(\Omega_{r_1+1}^\varepsilon;\mathbb{R}^3)} \geq \|\nabla \hat{u}_v\|^2_{L^2(\hat{\Omega}_{r_2}^\varepsilon;\mathbb{R}^3)} .$$

Consider some nonempty open subset $U \Subset \Omega_\Gamma$. For small enough $\varepsilon > 0$, we have $\hat{u}_v = 0$ in U (since $u_v = 0$ in Ω_Γ) and hence, by Lemma 5.12, we have

$$\|\nabla \hat{u}_v\|^2_{L^2(\hat{\Omega}_{r_2}^\varepsilon;\mathbb{R}^3)} \gtrsim \|\hat{u}_v\|^2_{L^2(\hat{\Omega}_{r_2}^\varepsilon;\mathbb{R}^3)} .$$

Again by (5.30) there exists $r_3 > 0$ such that $\hat{\Omega}^\varepsilon_{r_2} \supset \Omega^\varepsilon_{r_3-3}$ for all $\varepsilon > 0$ and thus, by the definition of $\hat{u}_v$ (observe that $\Omega^\varepsilon_{r_3} + [0,\varepsilon]^3 \subset \Omega^\varepsilon_{r_3-3}$),

$$\|\hat{u}_v\|^2_{L^2(\hat{\Omega}^\varepsilon_{r_2};\mathbb{R}^3)} \geq \|\hat{u}_v\|^2_{L^2(\Omega^\varepsilon_{r_3-3};\mathbb{R}^3)} \gtrsim \|u_v\|^2_{L^2(\Omega^\varepsilon_{r_3};\mathbb{R}^3)} \,.$$

In the following we use the notation $G_D \coloneqq G_r + D$ for any $D \subset \mathbb{Z}^3$. Now with a constant $r_4 > 0$ depending only on r_3 and G_r we have the implication

$$(v,k) \in G_{D^\varepsilon_{r_4}} \implies k \in D^\varepsilon_{r_3} \,,$$

which means that $\Omega^\varepsilon_{r_3} \supset \Omega^\varepsilon_v(G_{D^\varepsilon_{r_4}})$, where $\Omega^\varepsilon_v(G_{D^\varepsilon_{r_3}})$ is defined as in (5.21), and therefore

$$\|u_v\|^2_{L^2(\Omega^\varepsilon_{r_3};\mathbb{R}^3)} \geq \|u_v\|^2_{L^2(\Omega^\varepsilon_v(G_{D^\varepsilon_{r_4}});\mathbb{R}^3)} \,.$$

Joining all the above inequalities and writing $r \coloneqq r_4$, we have

$$\sum_{e\in E(G)} \|\mathrm{grad}^\varepsilon_e(u;G^\varepsilon)\|^2_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^3)} \gtrsim \|u_v\|^2_{L^2(\Omega^\varepsilon_v(G_{D^\varepsilon_r});\mathbb{R}^3)} \,. \tag{5.31}$$

The proof is not finished yet since we have an estimate u_v only on a shrinked domain.

We now make use of the *shrink-and-grow property* (D3): There exists some $R \in \mathbb{N}$, depending only r, such that $D^\varepsilon_{r,R} = D^\varepsilon$ for small enough $\varepsilon > 0$.

For sets $D \subset \mathbb{Z}^3$ of the form $D \coloneqq \{d, d \pm e_i\}$ with $d \in \mathbb{Z}^3$ und $i = 1,2,3$ we have the uniform Poincaré estimate

$$\|u\|_{L^2(\Omega^\varepsilon(G_D);\mathbb{R}^3)} \lesssim \varepsilon\|\mathrm{grad}^\varepsilon(u;G_D)\| + \|u\|_{L^2(\Omega^\varepsilon(G_{\{d\}});\mathbb{R}^3)} \,. \tag{5.32}$$

For the meaning of $\Omega^\varepsilon(\cdot)$ we refer to (5.23). Indeed, when the right-hand side of (5.32) vanishes, u is constant on $\Omega^\varepsilon(G_D)$ and vanishes on $\Omega^\varepsilon(G_{\{d\}})$. Thus it vanishes on $\Omega^\varepsilon(G_D)$ so that the left-hand side also vanishes. As the space of G_D-node functions is finite-dimensional, this implies the quantitative estimate (5.32) for fixed D and $\varepsilon > 0$. But (5.32) is clearly translation-invariant (independent of d) and scaling-invariant (independent of ε).

Summing (5.32) over all $D = \{d, d \pm e_i\}$ with $D \subset D^\varepsilon$ and $d \in D^\varepsilon_{r,k}$, we get

$$\|u\|_{L^2(\Omega^\varepsilon(G_{D^\varepsilon_{r,k}});\mathbb{R}^3)} \lesssim \varepsilon\|\mathrm{grad}^\varepsilon(u;G^\varepsilon)\| + \|u\|_{L^2(\Omega^\varepsilon(G_{D^\varepsilon_{r,k-1}});\mathbb{R}^3)} \,. \tag{5.33}$$

Summing (5.33) over $1 \leq k \leq R$ and using (5.31) finally yields

$$\begin{aligned}
\|u\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} &= \|u\|_{L^2(\Omega^\varepsilon(G_{D^\varepsilon_{r,R}});\mathbb{R}^3)} \\
&\lesssim \varepsilon\|\mathrm{grad}^\varepsilon(u;G^\varepsilon)\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^3)} + \|u\|_{L^2(\Omega^\varepsilon(G_{D^\varepsilon_r});\mathbb{R}^3)} \\
&\lesssim \|\mathrm{grad}^\varepsilon(u;G^\varepsilon)\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^3)} \,.
\end{aligned}$$

This finishes the proof of Lemma 5.11. □

5.4 Two-scale convergence

Until now we have never used the notion of G^ε-cell functions introduced in Definition 5.8. In Chapter 6, G^ε-cell functions $\alpha^\varepsilon : \mathbb{R}^3 \to \mathbb{R}^3$ will serve to describe cell-averages of node-displacements. The displacements of the individual nodes relative to the cell-average will then be denoted as $\varepsilon\beta_v^\varepsilon$ with a G^ε-node function $\beta^\varepsilon : \mathbb{R}^3 \to \Pi_{v\in V(G)}\mathbb{R}^3$ that satisfies $\sum_{v\in V(G)} \beta_v^\varepsilon = 0$. We thus make the following definition.

Definition 5.13 (G^ε-function pairs). *Suppose $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ is a G^ε-cell function and $\beta : \mathbb{R}^3 \to \Pi_{v\in V(G)}\mathbb{R}^3$ is a G^ε-node function in the sense of Definition 5.8. When $\sum_{v\in V(G)} \beta_v = 0$ as a function on $\mathbb{R}^3$, we call (α, β) a G^ε*-function pair*.*

According to (5.24), every G^ε-cell function can also be interpreted as a G^ε-cell function by using each cell-value for all available nodes of the cell. Thus, when (α, β) is a G^ε-function pair, $\alpha + \varepsilon\beta$ is an ε-node function and its graph-gradient as defined in (5.25) is

$$\operatorname{grad}_e^\varepsilon(\alpha + \varepsilon\beta; G^\varepsilon)(x) = \frac{\alpha(x + \varepsilon d(e)) - \alpha(x)}{\varepsilon} + \beta_{v_2(e)}(x + \varepsilon d(e)) - \beta_{v_1(e)}(x)$$

for $e \in E(G)$ and $x \in \Omega_e^\varepsilon(G^\varepsilon)$. This motivates the following definition of a two-scale limit graph gradient.

Definition 5.14 (Limiting periodic graph gradient). *For locally integrable functions $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$ and $\beta : \mathbb{R}^3 \to \Pi_{v\in V(G)}\mathbb{R}^3$ with $\sum_{v\in V(G)} \beta_v = 0$, we let*

$$\operatorname{grad}_e(\alpha, \beta; G) := d(e) \cdot \nabla\alpha + \beta_{v_2(e)} - \beta_{v_1(e)}\,, \qquad e \in E(G)\,,$$

in the sense of distributions on $\mathbb{R}^3$.

As a preliminary for a two-scale compactness result, we have the following lemma which improves upon the Poincaré inequality of Lemma 5.11 by providing an additional estimate for local oscillations.

Lemma 5.15. *Let $(G^\varepsilon)_\varepsilon$ be a family of graph realizations and Γ_v^ε as defined by* (5.26) *in the setting of Definition 5.3. Then there exist constants $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all G^ε-function pairs*

$$(\alpha, \beta) : \mathbb{R}^3 \to \mathbb{R}^3 \times \Pi_{v\in V(G)}\mathbb{R}^3$$

with $\alpha_v + \varepsilon\beta_v = 0$ on Γ_v^ε,

$$\|\alpha\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} + \sum_{v\in V(G)} \|\beta_v\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \le C \sum_{e\in E(G)} \|\operatorname{grad}_e^\varepsilon(\alpha+\varepsilon\beta; G^\varepsilon)\|^2_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^3)}\,.$$

Proof. By Lemma 5.11, we already have an estimate for $\alpha + \varepsilon\beta$. It is therefore sufficient to provide an estimate for β.

By Lemma 5.6, any two vertices $(v, k), (v', k) \in V(G^\varepsilon)$ from the same cell $k \in \mathbb{Z}^3$ can be joined with a path in $\overline{G}^\varepsilon$ which is uniformly bounded in length. Suppose this path is

$$(e^1, k^1), \ldots, (e^J, k^J) \in \pm E(G^\varepsilon)\,.$$

We can assume that no edge occurs more than once in this list. Now we can write

$$\beta_v - \beta_{v'} = \sum_{j=1}^{J} T_{\varepsilon(k^j - k)} \operatorname{grad}^\varepsilon_{e^j}(\alpha + \varepsilon\beta; G^\varepsilon) \quad \text{on } \square^\varepsilon_k\,,$$

where T_v is the translation operator $T_v f(x) = f(x + v)$. Thus by the triangle inequality,

$$\begin{aligned}\|\beta_v - \beta_{v'}\|^2_{L^2(\square^\varepsilon_k;\mathbb{R}^3)} &\le \left(\sum_{j=1}^{J} \|\operatorname{grad}^\varepsilon_{e^j}(\alpha + \varepsilon\beta; G^\varepsilon)\|_{L^2(\square^\varepsilon_{k^j};\mathbb{R}^3)}\right)^2 \\ &\le J \sum_{j=1}^{J} \|\operatorname{grad}^\varepsilon_{e^j}(\alpha + \varepsilon\beta; G^\varepsilon)\|^2_{L^2(\square^\varepsilon_{k^j};\mathbb{R}^3)}\,.\end{aligned}$$

The uniform bound of the path length J implies that the relevant cells $\square^\varepsilon_{k^j}$ are all contained in a ball $\varepsilon B_R(k)$ with a uniform radius $R > 0$. We thus have

$$\|\beta_v - \beta_{v'}\|^2_{L^2(\square^\varepsilon_k;\mathbb{R}^3)} \lesssim \|\operatorname{grad}^\varepsilon(\alpha + \varepsilon\beta; G^\varepsilon)\|^2_{L^2(\varepsilon B_R(k);\Pi_{e\in E(G)}\mathbb{R}^3)}\,.$$

As this is true for every $(v, k), (v', k) \in V(G^\varepsilon)$ and we also have $\sum_{v\in V(G)} \beta_v = 0$, this implies

$$\sum_{v\in V(G)} \|\beta_v\|^2_{L^2(\square^\varepsilon_k;\mathbb{R}^3)} \lesssim \|\operatorname{grad}^\varepsilon(\alpha + \varepsilon\beta; G^\varepsilon)\|^2_{L^2(B_{\varepsilon R}(k);\Pi_{e\in E(G)}\mathbb{R}^3)}\,.$$

A further summation over $k \in \mathbb{Z}^3$ yields

$$\sum_{v\in V(G)} \|\beta_v\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \lesssim \|\operatorname{grad}^\varepsilon(\alpha + \varepsilon\beta; G^\varepsilon)\|^2_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^3)}\,.$$

This proves Lemma 5.15. □

We now state a lemma about two-scale convergence in the spirit of [5]. Note, however, that our setting is very simple: the microsopic variable ranges only over the finite set $V(G)$ as opposed to a continuous periodicity cell.

Lemma 5.16 (Two-scale compactness). *Let $(G^\varepsilon)_\varepsilon$ be a family of graph realizations and Γ_v^ε as defined by (5.26) in the setting of Definition 5.3. Let*

$$(\alpha^\varepsilon, \beta^\varepsilon)_\varepsilon \subset L^2\left(\mathbb{R}^3; \mathbb{R}^3 \times \Pi_{v\in V(G)}\mathbb{R}^3\right)$$

be a sequence of G^ε-function pairs with $\alpha_v^\varepsilon + \varepsilon\beta_v^\varepsilon = 0$ on Γ_v^ε. Suppose that there exists a constant $C > 0$ such that

$$\|\mathrm{grad}^\varepsilon(\alpha^\varepsilon + \varepsilon\beta^\varepsilon; G^\varepsilon)\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^3\right)} \le C \qquad \text{for all } \varepsilon > 0\,.$$

Then there exists a subsequence and $(\alpha, \beta) \in L^2\left(\mathbb{R}^3; \mathbb{R}^3 \times \Pi_{v\in V(G)}\mathbb{R}^3\right)$ such that $\sum_{v\in V(G)} \beta_v = 0$, $\alpha|_\Omega \in H^1_\Gamma(\Omega; \mathbb{R}^3)$, $(\alpha, \beta) = 0$ in $\mathbb{R}^3 \setminus \Omega$, and

$$\alpha^\varepsilon \rightharpoonup \alpha \qquad \text{in } L^2(\mathbb{R}^3; \mathbb{R}^3)\,, \tag{5.34}$$

$$\beta^\varepsilon \rightharpoonup \beta \qquad \text{in } L^2(\mathbb{R}^3; \Pi_{v\in V(G)}\mathbb{R}^3)\,, \tag{5.35}$$

$$\mathrm{grad}^\varepsilon(\alpha^\varepsilon + \varepsilon\beta^\varepsilon; G^\varepsilon) \rightharpoonup \mathrm{grad}(\alpha, \beta; G) \quad \text{in } L^2\left(\Omega; \Pi_{e\in E(G)}\mathbb{R}^3\right)\,. \tag{5.36}$$

Proof. By Lemma 5.15, the bound on $\mathrm{grad}^\varepsilon(\alpha^\varepsilon + \varepsilon\beta^\varepsilon; G^\varepsilon)$ implies bounds on α^ε in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ and on β^ε in $L^2(\mathbb{R}^3; \Pi_{v\in V(G)}\mathbb{R}^3)$. Thus there exist a subsequence and limit functions $(\alpha, \beta) \in L^2(\mathbb{R}^3; \mathbb{R}^3 \times \Pi_{v\in V(E)}\mathbb{R}^3)$ with $\sum_{v\in V(G)} \beta_v = 0$ such that

$$\alpha^\varepsilon \rightharpoonup \alpha \ \text{ in } L^2(\mathbb{R}^3; \mathbb{R}^3)\,, \qquad \beta^\varepsilon \rightharpoonup \beta \ \text{ in } L^2(\mathbb{R}^3; \Pi_{v\in V(E)}\mathbb{R}^3)\,.$$

Moreover, $(\alpha, \beta) = 0$ in $\mathbb{R}^3 \setminus \Omega$ since α^ε and β^ε vanish on every $U \Subset \mathbb{R}^3 \setminus \Omega$ for sufficiently small $\varepsilon > 0$. This is a consequence of the approximation property of D^ε and the construction of G^ε as $G^\varepsilon = G_r + D^\varepsilon$.

Consider any $U \Subset \Omega \cup \Omega_\Gamma$. For sufficiently small $\varepsilon > 0$, we have $U \subset \Omega_e^\varepsilon(G^\varepsilon)$ for all $e \in E(G)$ and therefore

$$\begin{aligned}
&\mathrm{grad}_e^\varepsilon(\alpha^\varepsilon + \varepsilon\beta^\varepsilon; G^\varepsilon)(x) \\
&\quad = \frac{\alpha^\varepsilon(x + \varepsilon d(e)) - \alpha^\varepsilon(x)}{\varepsilon} + \beta^\varepsilon_{v_2(e)}(x + \varepsilon d(e)) - \beta^\varepsilon_{v_1(e)}(x) \\
&\quad \to d(e) \cdot \nabla\alpha(x) + \beta_{v_2(e)}(x) - \beta_{v_1(e)}(x) \\
&\quad = \mathrm{grad}_e(\alpha, \beta; G)(x)
\end{aligned}$$

in the sense of distributions on U. Therefore

$$\mathrm{grad}^\varepsilon(\alpha^\varepsilon + \varepsilon\beta^\varepsilon; G^\varepsilon) \to \mathrm{grad}(\alpha, \beta; G) \qquad \text{in } \mathcal{D}'(\Omega \cup \Omega_\Gamma)\,.$$

By the uniform bound on $\mathrm{grad}^\varepsilon(\alpha^\varepsilon + \varepsilon\beta^\varepsilon; G^\varepsilon)$ we even have

$$\mathrm{grad}^\varepsilon(\alpha^\varepsilon + \varepsilon\beta^\varepsilon; G^\varepsilon) \rightharpoonup \mathrm{grad}(\alpha, \beta; G) \qquad \text{in } L^2(\Omega \cup \Omega_\Gamma; \Pi_{e\in E(G)}\mathbb{R}^3)\,.$$

In particular, $\mathrm{grad}(\alpha, \beta; G) \in L^2(\Omega \cup \Omega_\Gamma; \Pi_{v\in V(G)}\mathbb{R}^3)$. As $\{d(e) : e \in E(G)\}$ spans $\mathbb{R}^3$ (otherwise G_{per} would not be connected) this implies $\alpha \in H^1(\Omega \cup \Omega_\Gamma; \mathbb{R}^3)$. Since $\alpha = 0$ in Ω_Γ, we have $\alpha|_\Omega \in H^1_\Gamma(\Omega; \mathbb{R}^3)$. □

Remark. In addition to the weak convergence of α^ε expressed in (5.34), we even have strong convergence $\alpha^\varepsilon \to \alpha$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$. We do not give a proof of this fact since we will not make use of it.

5.5 Recovery Lemma

In this section, we show how continuous quantities can be approximated by functions on the graphs G^ε. We start with a lemma which states a simple and well-known fact.

Lemma 5.17 (Discretization). *Let $\alpha \in L^2(\mathbb{R}^3; X)$, where X is a reflexive Banach space. Then $P^\varepsilon\alpha \to \alpha$ in $L^2(\mathbb{R}^3; X)$ as $\varepsilon \to 0$ with P^ε from Definition 5.7.*

Proof. We first give a proof for $\alpha \in H^1(\mathbb{R}^3; X)$. By Poincaré's inequality, we have

$$\left\| \alpha - \fint_{\square_k^\varepsilon} \alpha \right\|_{L^2(\square_k^\varepsilon)} \lesssim \varepsilon \|\nabla\alpha\|_{L^2(\square_k^\varepsilon)}$$

for all $\varepsilon > 0$ and $k \in \mathbb{Z}^3$. This implies

$$\begin{aligned}\|\alpha - P^\varepsilon\alpha\|^2_{L^2(\mathbb{R}^3)} &= \sum_{k\in\mathbb{Z}^3} \left\| \alpha - \fint_{\square_k^\varepsilon} \alpha \right\|^2_{L^2(\square_k^\varepsilon)} \lesssim \varepsilon^2 \sum_{k\in\mathbb{Z}^3} \|\nabla\alpha\|^2_{L^2(\square_k^\varepsilon)} \\ &= \varepsilon^2 \|\nabla\alpha\|^2_{L^2(\mathbb{R}^3)} \to 0\,.\end{aligned}$$

Now for a general $\alpha \in L^2(\mathbb{R}^3; X)$ and $\delta > 0$ we can find a function $\phi \in H^1(\mathbb{R}^3; X)$ with $\|\alpha - \phi\|_{L^2(\mathbb{R}^3;X)} \le \delta/2$. Then

$$\begin{aligned}\|\alpha - P^\varepsilon\alpha\| &\le \|\alpha - \phi\| + \|\phi - P^\varepsilon\phi\| + \|P^\varepsilon(\phi - \alpha)\| \\ &\le \|\alpha - \phi\| + \|\phi - P^\varepsilon\phi\| + \|\phi - \alpha\| \\ &\le \delta + \|\phi - P^\varepsilon\phi\| \to \delta\,,\end{aligned}$$

where we have used that $\|P^\varepsilon\| = 1$ in the L^2-operator norm. As $\delta > 0$ was arbitrary, this finishes the proof. □

In the following lemma, we show how, starting from a limit function on cells/nodes/edges, we can define discretizations that approximate this function. This will be important in the construction of recovery sequences.

Lemma 5.18 (Recovery). *Let $(G^\varepsilon)_\varepsilon$ be a family of graph realizations and Γ_v^ε as defined by (5.26) in the setting of Definition 5.3. Let X and $(Y_e)_{e\in E(G)}$ be reflexive Banach spaces. In this lemma, all functions defined on Ω are implicitly extended by zero to all of $\mathbb{R}^3$ (in particular $f|_\Omega = f\mathbb{1}_\Omega$ for any function f on $\mathbb{R}^3$).*

(i) Let $\beta \in L^2(\Omega; \Pi_{v\in V(G)} X)$. Then the G^ε-node function β^ε, defined by

$$\beta_v^\varepsilon := \mathbb{1}_v^\varepsilon(G^\varepsilon) P^\varepsilon \beta_v \qquad \textit{for } v \in V(G)\,,$$

satisfies $\beta_v^\varepsilon = 0$ on Γ_v^ε and $\beta^\varepsilon \to \beta$ in $L^2(\mathbb{R}^3; \Pi_{v\in V(G)} X)$.

(ii) Let $\gamma \in L^2(\Omega; \Pi_{e\in E(G)} Y_e)$. Then the G^ε-edge function γ^ε, defined by

$$\gamma_e^\varepsilon := \mathbb{1}_e^\varepsilon(G^\varepsilon) P^\varepsilon \gamma_e \qquad \textit{for } e \in E(G)\,,$$

satisfies $\gamma^\varepsilon \to \gamma$ in $L^2(\mathbb{R}^3; \Pi_{e\in E(G)} Y_e)$.

(iii) Let $\alpha \in H^1_\Gamma(\Omega; \mathbb{R}^3)$ and $\beta \in L^2(\Omega; \Pi_{v\in V(G)} \mathbb{R}^3)$ with $\sum_{v\in V(G)} \beta_v = 0$. Then the ε-node function η^ε, defined by

$$\eta_v^\varepsilon := \mathbb{1}_v^\varepsilon(G^\varepsilon) P^\varepsilon (\alpha + \varepsilon\beta_v) \qquad \textit{for } v \in V(G)\,,$$

satisfies $\eta_v^\varepsilon = 0$ on Γ_v^ε and

$$\operatorname{grad}^\varepsilon(\eta^\varepsilon; G^\varepsilon) \to \operatorname{grad}(\alpha, \beta; G)|_\Omega \qquad \textit{in } L^2(\mathbb{R}^3; \Pi_{e\in E(G)} \mathbb{R}^3)\,.$$

Moreover, for the unique G^ε-function pairs $(\alpha^\varepsilon, \beta^\varepsilon)$ with $\eta^\varepsilon = \alpha^\varepsilon + \varepsilon\beta^\varepsilon$, there holds

$$(\alpha^\varepsilon, \beta^\varepsilon) \to (\alpha, \beta) \qquad \textit{in } L^2(\mathbb{R}^3; \mathbb{R}^3 \times \Pi_{v\in V(G)} \mathbb{R}^3)\,.$$

Proof. (i) Using that $\mathbb{1}_v^\varepsilon(G^\varepsilon)$ and P^ε commute, and that $\|P^\varepsilon\| \le 1$, we find that

$$\begin{aligned}\|\beta_v - \beta_v^\varepsilon\| &\le \|\beta_v - P^\varepsilon\beta_v\| + \|P^\varepsilon\beta_v - P^\varepsilon \mathbb{1}_v^\varepsilon(G^\varepsilon)\beta_v\| \\ &\le \|\beta_v - P^\varepsilon\beta_v\| + \|\beta_v - \mathbb{1}_v^\varepsilon(G^\varepsilon)\beta_v\| \\ &= \|\beta_v - P^\varepsilon\beta_v\| + \|\beta_v|_{\Omega\setminus\Omega_v^\varepsilon(G^\varepsilon)}\|\,.\end{aligned}$$

From Lemma 5.17 and the fact that $|\Omega \setminus \Omega_v^\varepsilon(G^\varepsilon)| \to 0$ as $\varepsilon \to 0$ we conclude that $\|\beta_v - \beta_v^\varepsilon\| \to 0$.

Observe that β is supported in Ω. Now whenever $\square_k^\varepsilon$ intersects Ω and $(v,k) \in V(G^\varepsilon)$, we also have $(v,k) \in V(G_0^\varepsilon)$, see (5.14). This implies $\mathbb{1}_v^\varepsilon(G^\varepsilon)P^\varepsilon\beta_v = \mathbb{1}_v^\varepsilon(G_0^\varepsilon)P^\varepsilon\beta_v$, and therefore $\beta_v^\varepsilon = 0$ on Γ_v^ε.

(ii) The same reasoning as in *(i)* applies.

(iii) We first prove $\operatorname{grad}^\varepsilon(\eta^\varepsilon; G^\varepsilon) \to \operatorname{grad}(\alpha, \beta; G)|_\Omega$ without making use of the decomposition of η^ε. Let us fix $e \in E(G)$. We can then compute,

$$
\begin{aligned}
&\|\operatorname{grad}^\varepsilon_e(\eta^\varepsilon; G^\varepsilon) - \operatorname{grad}_e(\alpha, \beta; G)|_{\Omega^\varepsilon_e(G^\varepsilon)}\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \\
&= \sum_{k \in D^\varepsilon_e(G^\varepsilon)} \int_{\square^\varepsilon_k} \Bigg| ⨍_{\square^\varepsilon_k} \frac{\alpha(y + \varepsilon d(e)) - \alpha(y)}{\varepsilon} \\
&\quad + \beta_{v_2(e)}(y + \varepsilon d(e)) - \beta_{v_1(e)}(y) \mathrm{d}y - \operatorname{grad}_e(\alpha, \beta; G)(x) \Bigg|^2 \mathrm{d}x \\
&= \sum_{k \in D^\varepsilon_e(G^\varepsilon)} \int_{\square^\varepsilon_k} \Bigg| ⨍_{\square^\varepsilon_k} ⨍_0^\varepsilon \Big(d(e) \cdot \nabla\alpha(y + sd(e)) - d(e) \cdot \nabla\alpha(x) \\
&\quad + \beta_{v_2(e)}(y + \varepsilon d(e)) - \beta_{v_1(e)}(y) + \beta_{v_2(e)}(x) - \beta_{v_1(e)}(x) \Big) \mathrm{d}s\mathrm{d}y \Bigg|^2 \mathrm{d}x \\
&\leq \sum_{k \in D^\varepsilon_e(G^\varepsilon)} \int_{\square^\varepsilon_k} ⨍_{\square^\varepsilon_k} ⨍_0^\varepsilon \Big| d(e) \cdot \nabla\alpha(y + sd(e)) - d(e) \cdot \nabla\alpha(x) \\
&\quad + \beta_{v_2(e)}(y + \varepsilon d(e)) - \beta_{v_1(e)}(y) + \beta_{v_2(e)}(x) - \beta_{v_1(e)}(x) \Big|^2 \mathrm{d}s\mathrm{d}y\mathrm{d}x \\
&\overset{(*)}{\leq} \int_{(-1,1)^3} \int_0^1 \int_{\mathbb{R}^3} \Big| d(e) \cdot \nabla\alpha|_\Omega(x + \varepsilon z + \varepsilon sd(e)) - d(e) \cdot \nabla\alpha|_\Omega(x) \\
&\quad + \beta_{v_2(e)}(x + \varepsilon z + \varepsilon d(e)) - \beta_{v_1(e)}(x + \varepsilon z) + \beta_{v_2(e)}(x) - \beta_{v_1(e)}(x) \Big|^2 \mathrm{d}x\mathrm{d}s\mathrm{d}z \\
&= \int_{(-1,1)^3} \int_0^1 \Big\| d(e) \cdot \nabla\alpha|_\Omega(\cdot + \varepsilon z + \varepsilon sd(e)) - d(e) \cdot \nabla\alpha|_\Omega \\
&\quad + \beta_{v_2(e)}(\cdot + \varepsilon z + \varepsilon d(e)) - \beta_{v_1(e)}(\cdot + \varepsilon z) + \beta_{v_2(e)} - \beta_{v_1(e)} \Big\|^2_{L^2(\mathbb{R}^3;\mathbb{R}^3)} \mathrm{d}s\mathrm{d}z \\
&\to 0\,.
\end{aligned}
$$

For $(*)$ we have used that for $x, y \in \square^\varepsilon_k$ there holds $y = x + \varepsilon z$ for some $z \in (-1,1)^3$. The final convergence follows from the dominated convegence theorem: The integrand converges by the Fréchet-Kolmogorov theorem and it is dominated by

$$
\Big(2\|d(e) \cdot \nabla\alpha|_\Omega\|_{L^2(\mathbb{R}^3)} + 2\|\beta_{v_2(e)}\|_{L^2(\mathbb{R}^3)} + 2\|\beta_{v_1(e)}\|_{L^2(\Omega)} \Big)^2 \in \mathbb{R}\,.
$$

As $|\Omega \setminus \Omega_e^\varepsilon(G^\varepsilon)| \to 0$ for $\varepsilon \to 0$, the above convergence implies

$$\mathrm{grad}^\varepsilon(\eta^\varepsilon; G^\varepsilon) \to \mathrm{grad}(\alpha, \beta; G)|_\Omega$$

in $L^2(\mathbb{R}^3; \Pi_{e\in E(G)}\mathbb{R}^3)$.

It remains to show the convergence of α^ε and β^ε. We claim that the deomposition $\eta^\varepsilon = \alpha^\varepsilon + \varepsilon\beta^\varepsilon$ is given by

$$\alpha^\varepsilon := \mathbb{1}^\varepsilon(G^\varepsilon)P^\varepsilon\alpha + \varepsilon M^\varepsilon\beta\,, \qquad \beta_v^\varepsilon := \mathbb{1}_v^\varepsilon(G^\varepsilon)(P^\varepsilon\beta_v - M^\varepsilon\beta)\,, \tag{5.37}$$

where

$$M^\varepsilon\beta = \frac{\sum_{v\in V(G)} \mathbb{1}_v^\varepsilon(G^\varepsilon)P^\varepsilon\beta_v}{\sum_{v\in V(G)} \mathbb{1}_v^\varepsilon(G^\varepsilon)}\,.$$

Indeed, with α^ε and β^ε defined by (5.37), we have

$$\begin{aligned}\alpha_v^\varepsilon + \varepsilon\beta_v^\varepsilon &= \mathbb{1}_v^\varepsilon(G^\varepsilon)(P^\varepsilon\alpha + \varepsilon M^\varepsilon\beta) + \varepsilon\mathbb{1}_v^\varepsilon(G^\varepsilon)(P^\varepsilon\beta_v - M^\varepsilon\beta)\\ &= \mathbb{1}_v^\varepsilon(G^\varepsilon)P^\varepsilon(\alpha + \varepsilon\beta_v) = \eta_v^\varepsilon\,.\end{aligned}$$

Moreover,

$$\begin{aligned}\sum_{v\in V(G)} \beta_v^\varepsilon &= \sum_{v\in V(G)} \mathbb{1}_v^\varepsilon(G^\varepsilon)\left(P^\varepsilon\beta_v - \frac{\sum_{v'\in V(G)} \mathbb{1}_{v'}^\varepsilon(G^\varepsilon)P^\varepsilon\beta_{v'}}{\sum_{v'\in V(G)} \mathbb{1}_{v'}^\varepsilon(G^\varepsilon)}\right)\\ &= \sum_{v\in V(G)} \mathbb{1}_v^\varepsilon(G^\varepsilon)P^\varepsilon\beta_v - \frac{\sum_{v\in V(G)} \mathbb{1}_v^\varepsilon(G^\varepsilon)}{\sum_{v'\in V(G)} \mathbb{1}_{v'}^\varepsilon(G^\varepsilon)} \sum_{v'\in V(G)} \mathbb{1}_{v'}^\varepsilon(G^\varepsilon)P^\varepsilon\beta_{v'}\\ &= 0\,.\end{aligned}$$

We now observe that $M^\varepsilon\beta \to 0$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$. Indeed, $|M^\varepsilon\beta| \le \sum_{v\in V(G)} |P^\varepsilon\beta_v|$ and $M^\varepsilon\beta = |V(G)|^{-1}P^\varepsilon \sum_{v\in V(G)} \beta_v = 0$ in $U^\varepsilon := \bigcap_{v\in V(G)} \Omega_v^\varepsilon(G^\varepsilon)$. Therefore

$$\begin{aligned}\|M^\varepsilon\beta\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)} &= \|M^\varepsilon\beta\|_{L^2(\mathbb{R}^3\setminus U^\varepsilon;\mathbb{R}^3)}\\ &\le \sum_{v\in V(G)} \|P^\varepsilon\beta_v\|_{L^2(\mathbb{R}^3\setminus U^\varepsilon;\mathbb{R}^3)}\\ &\le \sum_{v\in V(G)} \|\beta_v\|_{L^2(\Omega\setminus U^\varepsilon;\mathbb{R}^3)} \to 0\,,\end{aligned}$$

since $|\Omega \setminus U^\varepsilon| \to 0$. The convergence of $M^\varepsilon\beta$ and the same reasoning as in (i) now imply that $\alpha^\varepsilon \to \alpha$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ and $\beta^\varepsilon \to \beta$ in $L^2(\mathbb{R}^3; \Pi_{v\in V(G)}\mathbb{R}^3)$. □

Chapter 6

Homogenization of elastoplastic lattices

In this chapter, we consider the equations of elastoplasticity on the graphs G^ε introduced in the previous chapter. For this we introduce a parameter $h = h(\varepsilon) > 0$ which describes the relative thickness of the rods in the lattice corresponding to G^ε.

As in Chapter 4, we start by modeling the physical situation. We describe a lattice made of elastoplastic material for fixed ε and h, and then we introduce appropriate scalings. Our main result includes simultaneous homogenization ($\varepsilon \to 0$) and dimension-reduction ($h = h(\varepsilon) \to 0$).

In all of this chapter, we work in the setting of Definition 5.3. In particular, (G, z, d) is a *periodicity graph* which is "unfolded" to the infinite periodic graph G_{per} defined by (5.1). Moreover, we have a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ and a nonempty open subset Γ of $\partial\Omega$. Along a sequence $\varepsilon \to 0$, we have subgraphs G^ε of G_{per} which approximate the domain Ω on an ε-scale, and V_Γ^ε is for each ε a set of nodes corresponding to Γ.

We also make free use of the language introduced for functions on periodic graphs: See Definition 5.8, Definition 5.9, Definition 5.13 and Definition 5.14.

6.1 Elastoplasticity on periodic graphs

We can picture G^ε as a one-dimensional structure in $\mathbb{R}^3$. But in order to impose the laws of elastoplasticity, we need a three-dimensional domain. This is where the thickness-parameter $h = h(\varepsilon)$ comes into play. We blow up all the edges of G^ε so that they have a thickness of order εh (they have a length of order ε). We then consider the displacement fields on each rod separately. This allows for scalings which depend on the different orientations of the rods. All the nodes (joints) are assumed to be rigid. Each node is therefore fully described by a displacement vector and an infinitesimal rotation (i. e. an antisymmetric matrix). The equations on different rods are only coupled via the state of adjacent nodes.

We assume that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. This means that the relative width of the individual rods tends to zero. The rate of convergence is only relevant for the loading terms. As in [60], we distinguish between three cases:

(i) *Sufficiently thick rods:* $h(\varepsilon)/\varepsilon \to \infty$ as $\varepsilon \to 0$,

(ii) *Sufficiently thin rods:* $h(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$,

(iii) *Critical case:* $h(\varepsilon)/\varepsilon \to \theta \in (0, \infty)$ as $\varepsilon \to 0$. The number θ is an asymptotical thickness parameter.

In the presence of volume loads, we will observe a qualitatively different behaviour in the three cases. The critical case is the most complex, as it combines the behaviour of the other two cases.

The state space

For the displacements of the nodes we assume from the outset a decomposition $\overline{u} + \varepsilon\overline{\xi}$, where $\overline{u} \in \mathbb{R}^3$ is the overall displacement of a cell $\square_k^\varepsilon$, and $\overline{\xi} \in \mathbb{R}^3$ is the ε-order relative displacement of a particular node from that cell. In the language of Definition 5.13 we can say that

$$(\overline{u}, \overline{\xi}) \in L^2(\mathbb{R}^3; \mathbb{R}^3 \times \Pi_{v \in V(G)} \mathbb{R}^3)$$

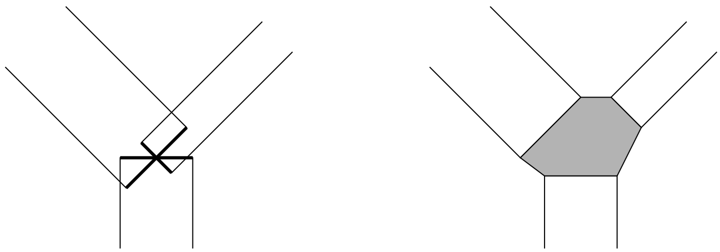
is a G^ε-function pair. The displacement of a node $(v, k) \in V(G^\varepsilon)$ is given by the value of $\overline{u} + \varepsilon\overline{\xi}_v$ on $\square_k^\varepsilon$. The rotational state of the nodes is given by a G^ε-node function

$$\overline{A} \in L^2(\mathbb{R}^3; \Pi_{v \in V(G)} \mathbb{R}^{3\times 3}_{\mathrm{asym}}) .$$

Let us now turn to the edges. We first have to specify the exact domain that each edge occupies. For this purpose we fix for each class $e \in E(G)$ of edges a rescaled cross section $B_e \subset \mathbb{R}^2$ which is assumed to be a bounded and centered Lipschitz domain. We further fix an orthogonal matrix $R(e) \in SO(3)$. The first column of $R(e)$ must be the edge-orientation vector $r(e)$ as defined in (5.3), $r(e) = R(e)e_1$. The remaining degree of freedom in the matrix $R(e)$ specifies the rotational alignment of B_e along the edge.

The domain occupied by an edge $(e, k) \in E(G^\varepsilon)$ at rest is $\varepsilon k + \Omega_e^\varepsilon$ with

$$\Omega_e^\varepsilon := \varepsilon(z(v_1(e)) + R(e)(I_e \times hB_e)) , \qquad I_e := (0, L(e)) .$$



(a) Three rods with overlap at a joint. The bold segments indicate where boundary values are prescribed via the state of the rigid joint.

(b) The more physical situation without overlap. The grey area indicates the rigid joint.

Figure 6.1: Illustration of the overlap of rods at joints.

Here $z : V(G_{\text{per}}) \to \mathbb{R}^3$ is the node placement from (5.2) on Page 58.

Observe that, in this description, the rods will have unphysical overlaps at the joints (see Figure 6.1). This could be prevented by shortening the rods at both ends by a length of order εh and attaching the rods at a distance from the centers of the adjacent joints. As this offset approaches zero as $\varepsilon \to 0$, it would not appear in the limit equations. For notational ease we will not incorporate these offsets. For rigorous approaches to the modeling of junctions in the case of elasticity we refer to [14, 33, 16].

For each edge $(e,k) \in E(G^\varepsilon)$ we have a displacement field and a plastic strain tensor. These are conveniently described as G^ε-edge functions

$$\overline{v} \in L^2(\mathbb{R}^3; \Pi_{e\in E(G)} H^1(\Omega_e^\varepsilon; \mathbb{R}^3)), \qquad \overline{p} \in L^2(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(\Omega_e^\varepsilon; \mathbb{R}^{3\times 3}_{\text{dev}}))\,.$$

Clearly, the set of possible displacements of a rod must be restricted by state of its neighboring nodes. The state of the nodes provides the boundary data for the displacement fields of the edge. The precise conditions are

$$\overline{v}_e(x, \varepsilon z(v_1(e)) + y) = \overline{u}(x) + \varepsilon \overline{\xi}_{v_1(e)}(x) + \overline{A}_{v_1(e)}(x) y \tag{6.1a}$$

for the first node and

$$\begin{aligned}\overline{v}_e(x, \varepsilon z(v_2(e)) + y) = \overline{u}(x + \varepsilon d(e)) &+ \varepsilon \overline{\xi}_{v_2(e)}(x + \varepsilon d(e)) \\ &+ \overline{A}_{v_2(e)}(x + \varepsilon d(e)) y\end{aligned} \tag{6.1b}$$

for the second node, for all $y \in \varepsilon R(e)(\{0\} \times hB_e)$.

We see that the state $(\overline{u}, \overline{\xi}, \overline{A})$ of the nodes is implicitly contained in the state of the edges. In particular, when we define the overall state space, we can leave out $(\overline{\xi}, \overline{A})$ and just require the existence of some $(\overline{\xi}, \overline{A})$ which satisfy (6.1):

$$\overline{\mathcal{Q}}^{\varepsilon} \coloneqq \{(\overline{u}, \overline{v}, \overline{p}) \ : \ (6.1) \text{ holds for some } (\overline{\xi}, \overline{A})\}$$

We decided to keep $\overline{u}$ in the state space since $\overline{u}$ is the macroscopic quantity we are most interested in.

The rate-independent system

As nodes are assumed to be rigid and mass-free, there is neither energy nor dissipation associated with them. The overall stored energy and dissipation of the system is thus simply the sum of the stored energy and dissipation of all the rods.

As in Chapter 4, the material of the rods is described by a stored energy density $\mathbb{W} : \mathbb{R}^{3\times 3}_{\mathrm{asym}} \times \mathbb{R}^{3\times 3}_{\mathrm{asym}} \to \mathbb{R}$ which is a positive quadratic form, and a dissipation potential $\overline{R} : \mathbb{R}^{3\times 3}_{\mathrm{dev}} \to \mathbb{R}$ which is positive one-homogeneous and convex. We could easily assume different material properties for different classes $e \in E(G)$ of rods. Besides bloating notation, this would have no effect on the analysis. For notational ease, we do not pursue this generalization. Note however, that in the case of non-isotropic $\mathbb{W}$ one might want to account for the spatial orientation of the rods and replace the occurrences of $\mathbb{W}_e$ (defined in (6.8)) with $\mathbb{W}$.

Similar to (4.5) and (4.6), we define

$$\overline{\mathcal{E}}^{\varepsilon}(\overline{q}) \coloneqq \varepsilon^{-3} \sum_{e\in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e^{\varepsilon}} \mathbb{W}(\nabla_y^s \overline{v}_e(x,y), \overline{p}_e(x,y)) \mathrm{d}y \mathrm{d}x \,, \tag{6.2}$$

$$\overline{\mathcal{R}}^{\varepsilon}(\overline{q}) \coloneqq \varepsilon^{-3} \sum_{e\in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e^{\varepsilon}} \overline{R}(\overline{p}_e(x,y)) \mathrm{d}y \mathrm{d}x \tag{6.3}$$

for all $\overline{q} = (\overline{u}, \overline{v}, \overline{p}) \in \overline{\mathcal{Q}}^{\varepsilon}$. The factor ε^{-3} compensates for the x-integration in which every cell $\square_k^{\varepsilon}$, and thus every rod, is discounted with a weight factor $|\square_k^{\varepsilon}| = \varepsilon^3$.

Regarding the loads, we restrict our attention to macroscopic volume loads

$$\overline{f}^{\varepsilon} \in W^{1,\infty}(0,T; L^2(\mathbb{R}^3; \mathbb{R}^3)) \,.$$

With these, we define $\overline{\ell}^{\varepsilon} \in W^{1,\infty}(0,T; \mathcal{Q})$ by

$$\langle \overline{\ell}^{\varepsilon}(t), \overline{q} \rangle \coloneqq \varepsilon^{-3} \sum_{e\in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e^{\varepsilon}} \overline{f}^{\varepsilon}(t, \varepsilon\lfloor x/\varepsilon \rfloor + y) \cdot \overline{v}_e(y) \mathrm{d}y \mathrm{d}x$$

for $t \in [0,T]$ and $\overline{q} = (\overline{u}, \overline{v}, \overline{p}) \in \overline{\mathcal{Q}}^{\varepsilon}$. Here, $\lfloor z \rfloor$ denotes for any $z \in \mathbb{R}^3$ the unique integer vector in $\lfloor z \rfloor \in \mathbb{Z}^3$ with $z - \lfloor z \rfloor \in [0,1)^3$. As usual, the total energy $\overline{\mathcal{E}}^{\varepsilon}$ is

$$\overline{\mathcal{E}}^{\varepsilon}(t, \overline{q}) \coloneqq \overline{\mathcal{B}}^{\varepsilon}(\overline{q}) - \langle \overline{\ell}^{\varepsilon}(t), \overline{q} \rangle\,, \qquad t \in [0,T]\,, \qquad \overline{q} \in \overline{\mathcal{Q}}^{\varepsilon}\,.$$

6.2 Scalings

We now perform a rescaling that resembles what we have done in Section 4.1. Starting from $\overline{v}_e(x, \cdot)$ and $\overline{p}_e(x, \cdot)$, we construct functions $v_e(x, \cdot)$ and $p_e(x, \cdot)$ defined on $\Omega_e \coloneqq I_e \times B_e$ in the following manner. As in (4.3) and (4.4), we use $S_h \coloneqq \mathrm{diag}(1\,, h^{-1},\, h^{-1})$ to define

$$v_e(x,y) \coloneqq \varepsilon^{-1} h^{-2} S_h^{-1} R(e)^{-1} \Big(\overline{v}_e \left(x, \varepsilon \left(z(v_1(e)) + R(e) S_h^{-1} y \right) \right) - \overline{u}(x) \Big)\,, \tag{6.4a}$$

$$p_e(x,y) \coloneqq h^{-2} \overline{p}_e(\varepsilon(z(v_1(e)) + R(e) S_h^{-1} y))) \tag{6.4b}$$

for $x \in \mathbb{R}^3$ and $y \in \Omega_e$. We also rescale $\overline{A}$, $\overline{u}$ and $\overline{\xi}$ by introducing

$$A \coloneqq h^{-1} \overline{A}\,, \qquad u \coloneqq h^{-2} \overline{u}\,, \qquad \xi \coloneqq h^{-2} \overline{\xi}\,. \tag{6.5}$$

Using (6.4), we can now express the compatibility conditions (6.1) in rescaled variables: For $x \in \mathbb{R}^3$ and $y \in \{0\} \times B_e$ we have, using $S_h^{-1} y = hy$,

$$\begin{aligned} v_e(x,y) &= \varepsilon^{-1} h^{-2} S_h^{-1} R(e)^{-1} \Big(\varepsilon \overline{\xi}_{v_1(e)}(x) \\ &\qquad + \overline{A}_{v_1(e)}(x) \varepsilon R(e) S_h^{-1} y \Big) \\ &= S_h^{-1} R(e)^{-1} \Big(\xi_{v_1(e)}(x) + A_{v_1(e)}(x) R(e) y \Big) \end{aligned} \tag{6.6a}$$

$$\begin{aligned} v_e(x, y + L(e) e_1) &= \varepsilon^{-1} h^{-2} S_h^{-1} R(e)^{-1} \Big(\overline{u}(x + \varepsilon d(e)) - \overline{u}(x) \\ &\qquad + \varepsilon \overline{\xi}_{v_2(e)}(x + \varepsilon d(e)) + \overline{A}_{v_2(e)} \varepsilon R(e) S_h^{-1} y \Big) \\ &= S_h^{-1} R(e)^{-1} \Big(\mathrm{grad}_e^{\varepsilon}(u + \varepsilon \xi; G^{\varepsilon})(x) + \xi_{v_1(e)}(x) \\ &\qquad + A_{v_2(e)}(x + \varepsilon d(e)) R(e) y \Big)\,. \end{aligned} \tag{6.6b}$$

We now express $\overline{\mathcal{B}}^\varepsilon$ and $\overline{\mathcal{R}}^\varepsilon$, defined in (6.2) and (6.3), in terms of the rescaled variables (v, p):

$$\overline{\mathcal{B}}^\varepsilon(\overline{q}) = h^6 \sum_{e \in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e} \mathbb{W}_e(S_h \nabla_y^s v_e(x,y) S_h \,,\ p_e(x,y)) \mathrm{d}y \mathrm{d}x \,, \tag{6.7a}$$

$$\overline{\mathcal{R}}^\varepsilon(\overline{q}) = h^6 \sum_{e \in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e} R(p_e(x,y)) \mathrm{d}y \mathrm{d}x \,, \quad R := h^{-2} \overline{R} \,, \tag{6.7b}$$

for all $\overline{q} = (\overline{v}, \overline{p}) \in \overline{\mathcal{Q}}^\varepsilon$, where

$$\mathbb{W}_e(A, P) := \mathbb{W}(R(e) A R(e)^{-1}, P) \,, \qquad A \in \mathbb{R}^{3\times 3}_{\mathrm{sym}} \,,\ P \in \mathbb{R}^{3\times 3}_{\mathrm{dev}} \,. \tag{6.8}$$

Note that in the transformation leading to (6.7), a factor $\varepsilon^3 h^2$ comes from the change of variables from Ω_e^ε to Ω_e, the ε^3 being immediately cancelled by the prefactor in (6.2) and (6.3), while another factor h^4 is contributed by the values of the integrands. We also need to rescale the volume loads $\overline{f}^\varepsilon : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$. In the case of *sufficiently thick rods*, we define

$$f^\varepsilon(t,x) := h^{-2} \overline{f}^\varepsilon(t,x) \,.$$

This gives us

$$\begin{aligned} \langle \overline{\ell}^\varepsilon(t), \overline{q} \rangle &= \varepsilon^{-3} \sum_{e \in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e^\varepsilon} \overline{f}^\varepsilon\big(t, \varepsilon \lfloor x/\varepsilon \rfloor + y\big) \cdot \overline{v}_e(y) \mathrm{d}y \mathrm{d}x \\ &= h^6 \sum_{e \in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e} f^\varepsilon\Big(t, \varepsilon(\lfloor x/\varepsilon \rfloor + z(v_1(e)) + R(e) S_h^{-1} y\Big) \\ &\qquad \cdot \Big(u(x) + \varepsilon R(e) S_h v_e(y)\Big) \mathrm{d}y \mathrm{d}x \,. \end{aligned} \tag{6.9}$$

In the *critical case* and in the case of *sufficiently thin rods*, however, we define

$$f^\varepsilon(t,x) := \varepsilon h^{-3} \overline{f}^\varepsilon(t,x) \,.$$

This gives us

$$\begin{aligned} \langle \overline{\ell}^\varepsilon(t), \overline{q} \rangle &= h^6 \sum_{e \in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e} f^\varepsilon\Big(t, \varepsilon(\lfloor x/\varepsilon \rfloor + z(v_1(e)) + R(e) S_h^{-1} y\Big) \\ &\qquad \cdot \Big(\frac{h}{\varepsilon} u(x) + h R(e) S_h v_e(y)\Big) \mathrm{d}y \mathrm{d}x \,. \end{aligned} \tag{6.10}$$

Looking at (6.7), (6.9) and (6.10), we see that we have a factor h^6 in front of $\overline{\mathcal{E}}^\varepsilon(t,\overline{q}) = \overline{\mathcal{B}}^\varepsilon(\overline{q}) - \langle \overline{\ell}^\varepsilon(t), \overline{q}\rangle$ and $\overline{\mathcal{R}}^\varepsilon(\overline{q})$ when expressed in terms of $q = (u, v, p)$. This suggests that as in Chapter 4 the right way to proceed is to define $\mathcal{E}^\varepsilon$ and $\mathcal{R}^\varepsilon$ by

$$\mathcal{E}^\varepsilon(t,q) \coloneqq h^{-6}\overline{\mathcal{E}}^\varepsilon(t,\overline{q}), \qquad \mathcal{R}^\varepsilon(q) \coloneqq h^{-6}\overline{\mathcal{R}}^\varepsilon(\overline{q}).$$

This is done in the next section.

6.3 Summary of the setting

In the preceding section, we performed a rescaling, starting from physical variables $\overline{q} = (\overline{u}, \overline{v}, \overline{p})$ and yielding rescaled variables $q = (u, v, p)$ defined according to (6.4) and (6.5). We have seen that the necessary compatibility condition in rescaled variables is that u, ξ and A exist such that (6.6) holds.

We define the overall state space

$$\mathcal{Q} \coloneqq L^2\left(\mathbb{R}^3; \mathbb{R}^3 \times \prod_{e\in E(G)} H^1(\Omega_e;\mathbb{R}^3) \times \prod_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^{3\times 3}_{\mathrm{dev}})\right). \tag{6.11}$$

For each $\varepsilon > 0$, the subspace $\mathcal{Q}^\varepsilon \subset \mathcal{Q}$ of compatible states consists of all $q = (u, v, p) \in \mathcal{Q}$ such that

(i) u is a G^ε-cell function; v and p are G^ε-edge functions;

(ii) there exist G^ε-node functions

$$(A, \xi) \in L^2\left(\mathbb{R}^3; \Pi_{v\in V(G)}\mathbb{R}^{3\times 3}_{\mathrm{asym}} \times \Pi_{v\in V(G)}\mathbb{R}^3\right)$$

such that (u, ξ) is a G^ε-function pair with $u_v + \varepsilon\xi_v = 0$ on Γ^ε_v and

$$v_e(x,y) = \begin{pmatrix} 1 & & \\ & h & \\ & & h\end{pmatrix} R(e)^{-1}\Big(\xi_{v_1(e)}(x) + A_{v_1(e)}(x)R(e)y\Big) \tag{6.12a}$$

$$\begin{aligned} v_e(x, L(e)e_1 + y) = \begin{pmatrix} 1 & & \\ & h & \\ & & h\end{pmatrix} R(e)^{-1}\Big(&\mathrm{grad}^\varepsilon_e(u + \varepsilon\xi; G^\varepsilon)(x) \\ &+ \xi_{v_1(e)}(x) + A_{v_2(e)}(x + \varepsilon d(e))R(e)y\Big) \end{aligned} \tag{6.12b}$$

for all $x \in \Omega^\varepsilon_e(G^\varepsilon)$ and $y \in \{0\} \times B_e$.

As suggested by (6.7), (6.9) and (6.10), the right scaling for energy and dissipation is

$$\mathcal{B}^\varepsilon(q) = h^{-6}\overline{\mathcal{B}}^\varepsilon(\overline{q}), \qquad \mathcal{R}^\varepsilon(q) = h^{-6}\overline{\mathcal{R}}^\varepsilon(\overline{q}).$$

Accordingly we define the stored energy $\mathcal{B}^\varepsilon : \mathcal{Q} \to \mathbb{R}_\infty$ and the dissipation $\mathcal{R}^\varepsilon : \mathcal{Q} \to \mathbb{R}_\infty$ for $q = (u, v, p) \in \mathcal{Q}$ by

$$\mathcal{B}^\varepsilon(q) := \begin{cases} \sum_{e \in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e} \mathbb{W}_e(S_h \nabla_y^s v_e(x,y) S_h \,,\; p_e(x,y)) \mathrm{d}y \mathrm{d}x, & \text{if } q \in \mathcal{Q}^\varepsilon \,, \\ +\infty & \text{otherwise,} \end{cases} \tag{6.13}$$

$$\mathcal{R}^\varepsilon(q) := \sum_{e \in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e} R(p_e(x,y)) \mathrm{d}y \mathrm{d}x \,. \tag{6.14}$$

We could have set $\mathcal{R}^\varepsilon(q) = +\infty$ for $q \notin \mathcal{Q}^\varepsilon$. However, this makes no difference as the definition of the stored energy $\mathcal{B}^\varepsilon$ already enforces $q(t) \in \mathcal{Q}^\varepsilon$ along energetic solutions $q : [0, T] \to \mathcal{Q}$. The total energy is

$$\mathcal{E}^\varepsilon(t, q) := \mathcal{B}^\varepsilon(q) - \langle \ell^\varepsilon(t), q \rangle \tag{6.15}$$

with $\ell^\varepsilon \in W^{1,\infty}(0, T; \mathcal{Q}^*)$. In the case of rescaled volume loads

$$f^\varepsilon \in W^{1,\infty}(0, T, L^2(\mathbb{R}^3; \mathbb{R}^3))$$

we define in accordance with (6.9), (6.10) and the scaling,

$$\langle \ell^\varepsilon(t), q \rangle := \sum_{e \in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e} f^\varepsilon \Big(t, \varepsilon([x/\varepsilon] + z(v_1(e)) + R(e) S_h^{-1} y) \Big) \cdot \left(\frac{h}{\varepsilon} \right)^\gamma \Big(u(x) + \varepsilon R(e) S_h v_e(x,y) \Big) \mathrm{d}y \mathrm{d}x \tag{6.16}$$

for $t \in [0, T]$ and $q = (u, v, p) \in \mathcal{Q}$, where $\gamma = 0$ in the case of *sufficient thickness* and $\gamma = 1$ otherwise. We then have the equivalence

$$q^\varepsilon \text{ is a solution of } (\mathcal{Q}, \mathcal{E}^\varepsilon, \mathcal{R}^\varepsilon) \quad \Longleftrightarrow \quad \overline{q} \text{ is a solution of } (\overline{\mathcal{Q}}^\varepsilon, \overline{\mathcal{E}}^\varepsilon, \overline{\mathcal{R}}^\varepsilon) \,,$$

We will therefore study the asymptotic behaviour of the rate-independent system $(\mathcal{Q}, \mathcal{E}^\varepsilon, \mathcal{R}^\varepsilon)$.

Lemma 6.1 (Convergence of volume loads). *Consider a bounded sequence* $(f^\varepsilon)_\varepsilon \subset W^{1,\infty}(0, T; L^2(\mathbb{R}^3; \mathbb{R}^3))$ *and let* $\ell^\varepsilon \in W^{1,\infty}(0, T; \mathcal{Q}^*)$ *be defined by* (6.16)*. Suppose that there exists* $f^0 \in W^{1,\infty}(0, T; L^2(\mathbb{R}^3; \mathbb{R}^3))$ *with* $f^\varepsilon(t) \to f^0(t)$

in $L^2(\mathbb{R}^3;\mathbb{R}^3)$ *for all* $t \in [0,T]$. *Define*

$$\langle \ell^0_{\text{thick}}(t), q\rangle := \int_{\mathbb{R}^3} f^0(t,x)\cdot \rho u(x)\mathrm{d}x\,, \tag{6.17}$$

$$\langle \ell^0_{\text{thin}}(t), q\rangle := \int_{\mathbb{R}^3} f^0(t,x)\cdot \sum_{e\in E(G)} R(e)\begin{pmatrix}0 & & \\ & 1 & \\ & & 1\end{pmatrix}\int_{\Omega_e} v_e(x,y)\mathrm{d}y\mathrm{d}x\,, \tag{6.18}$$

$$\ell^0(t) := \begin{cases} \ell^0_{\text{thick}} & \textit{for sufficiently thick rods,}\\ \ell^0_{\text{thin}} & \textit{for sufficiently thin rods,}\\ \theta\ell^0_{\text{thick}} + \ell^0_{\text{thin}} & \textit{in the case of critical thickness,}\end{cases} \tag{6.19}$$

for $t \in [0,T]$ *and* $q = (u,v,p) \in \mathcal{Q}$, *where* $\theta = \lim_{\varepsilon\to 0} h(\varepsilon)/\varepsilon$ *is the asymptotical thickness parameter and* $\rho := \sum_{e\in E(G)} |\Omega_e|$. *Then* $\ell^\varepsilon(t) \to \ell^0(t)$ *in* $\mathcal{Q}^*$ *for all* $t \in [0,T]$.

Proof. Let us consider any weakly converging sequence $q^\varepsilon = (u^\varepsilon, v^\varepsilon, p^\varepsilon) \rightharpoonup (u,v,p) = q$ in $\mathcal{Q}$. Then

$$\left(\frac{h}{\varepsilon}\right)^\gamma (u^\varepsilon(x) + \varepsilon R(e)S_h v^\varepsilon_e(x,y)) \rightharpoonup \begin{cases} u(x) & \text{thick}\\ R(e)\begin{pmatrix}0 & & \\ & 1 & \\ & & 1\end{pmatrix} v_e(x,y) & \text{thin}\\ \theta u(x) + R(e)\begin{pmatrix}0 & & \\ & 1 & \\ & & 1\end{pmatrix} v_e(x,y) & \text{critical}\end{cases} \tag{6.20}$$

in $L^2(\mathbb{R}^3\times\Omega_e;\mathbb{R}^3)$.

We now observe that, for fixed $t\in[0,T]$,

$$f^\varepsilon(t, \varepsilon(\lfloor x/\varepsilon\rfloor + z(v_1(e)) + R(e)S_h^{-1}y)) \to f^0(t,x) \tag{6.21}$$

in $L^2(\mathbb{R}^3\times\Omega_e;\mathbb{R}^3)$. This is by the Fréchet-Kolmogorov theorem a consequence of the $L^2(\mathbb{R}^3;\mathbb{R}^3)$-convergence $f^\varepsilon(t,\cdot)\to f^0(t,\cdot)$ and the uniform convergence

$$\varepsilon(\lfloor x/\varepsilon\rfloor + z(v_1(e)) + R(e)S_h^{-1}y) \to x\,, \qquad x\in\mathbb{R}^3\,,\ y\in\Omega_e\,.$$

Using (6.20) and (6.21), we can now pass to the limit in the term $\langle \ell^\varepsilon(t), q^\varepsilon\rangle$, see (6.16). With ℓ^0 as defined by (6.17)–(6.19), we indeed find that

$$\langle \ell^\varepsilon(t), q^\varepsilon\rangle \to \langle \ell^0(t), q^0\rangle\,, \qquad t\in[0,T]\,.$$

This implies $q^\varepsilon(t) \to q^0(t)$ in $\mathcal{Q}^*$ for all $t\in[0,T]$. □

Remark. Equations (6.17) to (6.19) show that in the thick case, the volume loads only affect the macroscopic displacements u. In the thin case, however, the volume loads also affect the local oscillations v_e (when the volume loads describe gravitation, e.g., the non-vertical rods will be sagging). In the critical case, both effects coexist.

6.4 Description of the limit system

The limit state space is the linear subspace $\mathcal{Q}^0 \subset \mathcal{Q}$ which consists of all $q = (u, v, p) \in \mathcal{Q}$ such that $q = 0$ in the complement of Ω and $u|_\Omega \in H^1_\Gamma(\Omega; \mathbb{R}^3)$ as well as $\nabla^s_y v \in \operatorname{span}(e_1 \otimes e_1)$ a. e., and such that there holds the following compatibility condition: There exist

$$(A, \xi) \in L^2\left(\mathbb{R}^3; \Pi_{v \in V(G)} \mathbb{R}^{3\times 3}_{\text{asym}} \times \Pi_{v \in V(G)} \mathbb{R}^3\right)$$

such that $(A, \xi) = 0$ in the complement of Ω with $\sum_{v \in V(G)} \xi_v = 0$ and

$$v_e(x, y) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} R(e)^{-1} \Big(\xi_{v_1(e)}(x) + A_{v_1(e)}(x) R(e) y \Big) , \tag{6.22a}$$

$$\begin{aligned} v_e(x, y + L(e) e_1) = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} R(e)^{-1} \Big(& \operatorname{grad}_e(u, \xi; G)(x) \\ & + \xi_{v_1(e)}(x) + A_{v_2(e)}(x) R(e) y \Big) \end{aligned} \tag{6.22b}$$

for all $e \in E(G)$, $x \in \Omega$ and $y \in \{0\} \times B_e$.

The limit dissipation functional is just

$$\mathcal{R}^0 \coloneqq \mathcal{R}^\varepsilon \tag{6.23}$$

with $\mathcal{R}^\varepsilon$ as defined in (6.14). We now define the limit stored energy

$$\mathcal{B}^0 : \mathcal{Q} \to \mathbb{R}_\infty .$$

For $q \in \mathcal{Q} \setminus \mathcal{Q}^0$ we set $\mathcal{B}^0(q) \coloneqq \infty$. For $q = (u, v, p) \in \mathcal{Q}^0$ we set

$$\begin{aligned} \mathcal{B}^0(q) \coloneqq \sum_{e \in E(G)} \int_{\mathbb{R}^3} \inf_g \int_{I_e} \inf_{f,w} \int_{B_e} & \\ \mathbb{W}_e \left(\begin{pmatrix} \partial_{y_1} v_{e,1}(x, y) & * \quad * \\ \partial_2 f(y') - g'(y_1) y_3 & \multirow{2}{*}{$\nabla^s w(y')$} \\ \partial_3 f(y') + g'(y_1) y_2 & \end{pmatrix} , \; p_e(x, y) \right) & \mathrm{d}y \mathrm{d}x , \end{aligned} \tag{6.24}$$

where $y = (y_1, y')$. The infima are taken over all

$$g \in H^1(I_e) , \qquad f \in H^1(B_e) , \qquad w \in H^1(B_e; \mathbb{R}^2)$$

such that

$$\begin{aligned} g(0) &= \frac{1}{2} \left(R(e)^{-1} A_{v_1(e)}(x) R(e) \right)_{23} , \\ g(L(e)) &= \frac{1}{2} \left(R(e)^{-1} A_{v_2(e)}(x) R(e) \right)_{23} . \end{aligned}$$

For the volume loads, we consider $f^0 \in W^{1,\infty}(0,T;L^2(\mathbb{R}^3;\mathbb{R}^3))$ and define $\ell^0 \in W^{1,\infty}(0,T;\mathcal{Q}^*)$ by (6.19). The limiting total energy is set to

$$\mathcal{E}^0(t,q) := \mathcal{B}^0(q) - \langle \ell^0(t), q\rangle \,.$$

The following lemma gives an alternative description of $\mathcal{B}^0$, where the infima are outside the sum and integral signs. Moreover, the infimized quantities possess some additional regularity and additional boundary conditions. This will be beneficial in the construction of recovery sequences, where these quantities are used.

Lemma 6.2. *For $q = (u,v,p) \in \mathcal{Q}^0$ there holds*

$$\mathcal{B}^0(q) = \inf_{f,\,g,\,w} \sum_{e\in E(G)} \int_{\mathbb{R}^3}\int_{\Omega_e} \mathbb{W}_e\left(\begin{pmatrix} \partial_{y_1} v_{e,1}(x,y) & * \quad * \\ \partial_{y_2} f_e(x,y) - \partial_{y_1} g_e(x,y_1) y_3 & \\ \partial_{y_3} f_e(x,y) + \partial_{y_1} g_e(x,y_1) y_2 & \nabla^s_{y_2,y_3} w_e(x,y) \end{pmatrix},\ p_e(x,y)\right) \mathrm{d}y\mathrm{d}x\,, \tag{6.25}$$

where the infimum is taken over all

$$\begin{aligned} f &\in L^2(\mathbb{R}^3; \Pi_{e\in E(G)} H^1(\Omega_e))\,,\\ g &\in L^2(\mathbb{R}^3; \Pi_{e\in E(G)} H^1(I_e))\,,\\ w &\in L^2(\mathbb{R}^3; \Pi_{e\in E(G)} H^1(\Omega_e;\mathbb{R}^2)) \end{aligned}$$

such that $f_e = w_e = 0$ on $\mathbb{R}^3 \times \partial I_e \times B_e$ and

$$\begin{aligned} g_e(x,0) &= \frac{1}{2}\left(R(e)^{-1} A_{v_1(e)}(x) R(e)\right)_{23}\,,\\ g_e(x,L(e)) &= \frac{1}{2}\left(R(e)^{-1} A_{v_2(e)}(x) R(e)\right)_{23} \end{aligned}$$

for $e \in E(G)$ and $x \in \mathbb{R}^3$.

Proof. We only have to prove "$\geq$" in (6.25), the opposite inequality is clear. For brevity, we denote the integrand on the right-hand side of (6.25) with ellipses ("…"). The statement now follows from Lemma B.2. Applying Lemma B.2

with $J = \Omega$, we get

$$\begin{aligned}
&\inf_{\substack{f\in L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(\Omega_e))\\ g\in L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(I_e))\\ w\in L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(\Omega_e)\\ +\text{boundary conditions}}} \sum_{e\in E(G)} \int_{\mathbb{R}^3}\int_{\Omega_e} \dots \mathrm{d}y\mathrm{d}x \\
&= \sum_{e\in E(G)} \inf_{\substack{f\in L^2(\mathbb{R}^3;H^1(\Omega_e))\\ g\in L^2(\mathbb{R}^3;H^1(I_e))\\ w\in L^2(\mathbb{R}^3;H^1(\Omega_e))\\ +\text{boundary conditions}}} \int_{\mathbb{R}^3}\int_{\Omega_e} \dots \mathrm{d}y\mathrm{d}x \\
&\overset{\text{Lemma B.2}}{\leq} \sum_{e\in E(G)} \int_{\mathbb{R}^3} \inf_{\substack{f\in H^1(\Omega_e)\\ g\in H^1(I_e)\\ w\in H^1(\Omega_e;\mathbb{R}^2)\\ +\text{boundary conditions}}} \int_{\Omega_e} \dots \mathrm{d}y\mathrm{d}x\,. \qquad (6.26)
\end{aligned}$$

Applying Lemma B.2 once more, now with $J = I_e$, we also get

$$\begin{aligned}
\inf_{\substack{f\in H^1(\Omega_e)\\ w\in H^1(\Omega_e;\mathbb{R}^2)\\ +\text{boundary cond.}}} \int_{\Omega_e} \dots \mathrm{d}y &\leq \inf_{\substack{f\in H^1_0(I_e;H^1(B_e))\\ w\in H^1_0(I_e;H^1(B_e;\mathbb{R}^2))}} \int_{I_e}\int_{B_e} \dots \mathrm{d}y'\mathrm{d}y_1 \\
&\overset{\text{Lemma B.2}}{\leq} \int_{I_e} \inf_{\substack{f\in H^1(B_e)\\ w\in H^1(B_e;\mathbb{R}^2)}} \int_{B_e} \dots \mathrm{d}y'\mathrm{d}y_1\,. \qquad (6.27)
\end{aligned}$$

Continuing (6.26) with the help of (6.27), we get

$$\begin{aligned}
&\inf_{\substack{f\in L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(\Omega_e))\\ g\in L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(I_e))\\ w\in L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(\Omega_e)\\ +\text{boundary conditions}}} \sum_{e\in E(G)} \int_{\mathbb{R}^3}\int_{\Omega_e} \dots \mathrm{d}y\mathrm{d}x \\
&\leq \sum_{e\in E(G)} \int_{\mathbb{R}^3} \inf_{\substack{g\in H^1(I_e)\\ +\text{boundary cond.}}} \int_{I_e} \inf_{\substack{f\in H^1(B_e)\\ w\in H^1(B_e;\mathbb{R}^2)}} \int_{B_e} \dots \mathrm{d}y'\mathrm{d}y_1\mathrm{d}x = \mathcal{B}^0(q)\,,
\end{aligned}$$

where we used the definition of $\mathcal{B}^0$ from (6.24). □

Discussion of the limit stored energy

Let us derive an alternative description of the stored limit energy $\mathcal{B}^0$ defined in (6.24). The results of the following considerations are stated in Proposition 6.3

below. We consider the reduced stored limit energy

$$\mathcal{B}^0_{\mathrm{red}}(u,p) := \inf_v \mathcal{B}^0(u,v,p)\,.$$

Observe that by (6.24) we can write

$$\mathcal{B}^0_{\mathrm{red}}(u,p) = \int_{\mathbb{R}^3} F(\nabla^s u(x), p(x))\mathrm{d}x$$

with a limit energy density $F : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \times \Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^{3\times 3}_{\mathrm{dev}}) \to \mathbb{R}$ which is defined as

$$F(\epsilon,p) := \inf_{A,\xi,v} \sum_{e\in E(G)} \inf_g \int_{I_e} \inf_{f,w} \int_{B_e} \mathbb{W}_e\left(\begin{pmatrix} \partial_1 v_{e,1}(y) & * & * \\ \partial_2 f(y') - g'(y_1)y_3 & \multicolumn{2}{c}{\nabla^s w(y')} \\ \partial_3 f(y') + g'(y_1)y_2 & & \end{pmatrix}, p_e(y)\right) \mathrm{d}y'\mathrm{d}y_1\,. \quad (6.28)$$

Here, the first infimum is taken over all $\xi \in \Pi_{v\in V(G)}\mathbb{R}^3$, $A \in \Pi_{v\in V(G)}\mathbb{R}^{3\times 3}_{\mathrm{asym}}$, and $v \in \Pi_{e\in E(G)} H^1(\Omega_e;\mathbb{R}^3)$ such that $\nabla^s v_e \in \mathrm{span}(e_1\otimes e_1)$ a. e. and

$$v_e(y) = \big(r(e)\cdot\big(\xi_{v_1(e)} + A_{v_1(e)}R_e y\big)\big)\, e_1\,, \quad (6.29\text{a})$$

$$v_e(y + L(e)e_1) = \big(r(e)\cdot\big(\xi_{v_2(e)} + A_{v_2(e)}R_e y + \epsilon d(e)\big)\big)\, e_1 \quad (6.29\text{b})$$

Observe that the argument ϵ in (6.28) only enters in (6.29b). The second infimum in (6.28) is taken over all $g \in H^1(I_e)$ with

$$g(0) = \frac{1}{2}\left(R(e)^{-1}A_{v_1(e)}R(e)\right)_{23}\,, \qquad g(L(e)) = \frac{1}{2}\left(R(e)^{-1}A_{v_2(e)}R(e)\right)_{23}\,. \quad (6.30)$$

The third infimum in (6.28) is taken over all $f \in H^1(B_e)$ and $w \in H^1(B_e;\mathbb{R}^2)$ without any further constraints.

Applying Lemma 4.1 to v_e and using the boundary conditions of (6.29) to conclude that the α of Lemma 4.1 must be $\alpha = 0$, we see that we can write

$$v_e(y) = \overline{v}_e(y_1) - \begin{pmatrix} \partial_1\overline{v}_{e,2}(y_1)y_2 + \partial_1\overline{v}_{e,3}(y_1)y_3 \\ 0 \\ 0 \end{pmatrix} \quad (6.31)$$

with $\overline{v}_{e,1} \in H^1(I_e)$ and $\overline{v}_{e,2}, \overline{v}_{e,3} \in H^2(I_e)$. The boundary values that follow from (6.29) are

$$\overline{v}_{e,1}(0) = r(e)\cdot\xi_{v_1(e)} \qquad \overline{v}_{e,1}(L(e)) = r(e)\cdot\big(\xi_{v_2(e)} + \epsilon d(e)\big) \quad (6.32\text{a})$$

$$\overline{v}_{e,j}(0) = 0 \qquad \overline{v}_{e,j}(L(e)) = 0 \quad (6.32\text{b})$$

$$\partial_1\overline{v}_{e,j}(0) = A_{v_1(e)}r(e)\cdot R(e)e_j \quad \partial_1\overline{v}_{e,j}(L(e)) = A_{v_2(e)}r(e)\cdot R(e)e_j\,, \quad (6.32\text{c})$$

where $j \in \{2, 3\}$.

In order to make the formula for the energy more transparent (in particular its dependence on ϵ), we want to get rid of the dependendencies between the arguments and the infimized quantities expressed in (6.29) and (6.30). We achieve this decoupling by using the unique decomposition $\overline{v}_{e,j} = \overline{v}^0_{e,j} + \overline{v}^1_{e,j}$ in which

(i) $\overline{v}^0_{e,1} \in H^1_0(I_e)$ and $\overline{v}^0_{e,2}, \overline{v}^0_{e,3} \in H^2_0(I_e)$,

(ii) $\overline{v}^1_{e,1}$ is a polynomial of degree one (i.e. affine) and $\overline{v}^1_{e,2}, \overline{v}^1_{e,3}$ are polynomials of degree three.

We now can give an explicit formula for $\overline{v}^1_{e,j}$ and independently infimize over $\overline{v}^0_{e,j}$. By (6.32a) we clearly must have

$$\overline{v}^1_{e,1}(y_1) = r(e) \cdot \xi_{v_1(e)} + \frac{y_1}{L(e)} r(e) \cdot \big(\epsilon d(e) + \xi_{v_2(e)} - \xi_{v_1(e)}\big) \, .$$

As for $\overline{v}^1_{e,2}$ and $\overline{v}^1_{e,2}$, we use the following general fact which is easy to verify: For a third-order polynomial $f : [0, L] \to \mathbb{R}$ the boundary conditions

$$f(0) = f(L) = 0 \, , \qquad f'(0) = a \, , \qquad f'(L) = b$$

imply

$$f(x) = ax - \frac{2a+b}{L} x^2 + \frac{a+b}{L^2} x^3 \, , \quad f''(x) = -2\frac{2a+b}{L} + 6\frac{a+b}{L^2} x \, . \qquad (6.33)$$

In the end, we are interested in an expression for $\partial_1 v_1(y)$, because this term enters into the formular for $F(\eta, p)$ in (6.28). We have:

$$
\begin{aligned}
\partial_1 v_1(y) &= \sum_{i=0}^{1} \partial_1 \overline{v}_1^i(y_1) - \partial_1^2 \overline{v}_2^i(y_1) y_2 - \partial_1^2 \overline{v}_3^i(y_1) y_3 \\
&= \partial_1 \overline{v}_1^0(y_1) - \partial_1^2 \overline{v}_2^0(y_1) y_2 - \partial_1^2 \overline{v}_3^0(y_1) y_3 \\
&\quad + \frac{\eta d(e) + \xi_{v_2(e)} - \xi_{v_1(e)}}{L(e)} \cdot r(e) \\
&\quad + 2\frac{2A_{v_1(e)} + A_{v_2(e)}}{L(e)} r(e) \cdot R(e) e_2 y_2 - 6\frac{A_{v_1(e)} + A_{v_2(e)}}{L(e)^2} r(e) \cdot R(e) e_2 y_1 y_2 \\
&\quad + 2\frac{2A_{v_1(e)} + A_{v_2(e)}}{L(e)} r(e) \cdot R(e) e_3 y_3 - 6\frac{A_{v_1(e)} + A_{v_2(e)}}{L(e)^2} r(e) \cdot R(e) e_3 y_1 y_3 \\
&= \partial_1 \overline{v}_1^0(y_1) - \partial_1^2 \overline{v}_2^0(y_1) y_2 - \partial_1^2 \overline{v}_3^0(y_1) y_3 \\
&\quad + \frac{\eta d(e) + \xi_{v_2(e)} - \xi_{v_1(e)}}{L(e)} \cdot r(e) \\
&\quad + \frac{1}{L(e)} \left(4A_{v_1(e)} + 2A_{v_2(e)} - 6\left(A_{v_1(e)} + A_{v_2(e)}\right) \frac{y_1}{L(e)} \right) r(e) \cdot R(e) y \\
&= \partial_1 \overline{v}_1^0(y_1) - \partial_1^2 \overline{v}_2^0(y_1) y_2 - \partial_1^2 \overline{v}_3^0(y_1) y_3 + \frac{r(e)}{L(e)} \cdot \Bigg(\\
&\quad \eta d(e) + \xi_{v_2(e)} - \xi_{v_1(e)} \\
&\quad + \left(\left(A_{v_2(e)} - A_{v_1(e)}\right) - 3\left(1 - \tfrac{2y_1}{L(e)}\right) \left(A_{v_2(e)} + A_{v_1(e)}\right) \right) R(e) y \Bigg).
\end{aligned}
$$

Inserting this back into (6.28), we arrive at the following description of the limit stored energy.

Proposition 6.3. *Let $\mathcal{B}^0 : \mathcal{Q}^0 \to \mathbb{R}$ denote the stored limit energy defined by (6.24) and let $\mathcal{B}^0_{\mathrm{red}}(u,p) := \inf_v \mathcal{B}^0(u,v,p)$ Then*

$$
\mathcal{B}^0_{\mathrm{red}}(u,p) = \int_\Omega F(\nabla^s u(x), p(x)) \mathrm{d}x,
$$

where the limit energy density

$$
F : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \times \Pi_{e \in E(G)} L^2(\Omega_e; \mathbb{R}^{3\times 3}_{\mathrm{dev}}) \to \mathbb{R}
$$

is defined by

$$
\begin{aligned}
F(\epsilon,p) := \inf_{A,\xi} \sum_{e\in E(G)} \inf_{v,g} \int_{I_e} \inf_{f,w} \int_{B_e} \mathbb{W}_e \Bigg(\\
\frac{1}{L(e)} \bigg(\Big(\epsilon d(e) + \xi_{v_2(e)} - \xi_{v_1(e)} \Big) \cdot r(e) - 6\Big(1 - \tfrac{2y_1}{L(e)}\Big) \Big(A_e^+ y\Big)_1 \bigg) e_1 \otimes e_1 \\
+ \frac{1}{L(e)} \left(2A_e^- \begin{pmatrix} 0 \\ y' \end{pmatrix}, 0, 0 \right)_{\mathrm{sym}} + \begin{pmatrix} v_1'(y_1) - v_2''(y_1)y_2 - v_3''(y_1)y_3 & * & * \\ \partial_2 f(y') - g'(y_1)y_3 & & \\ \partial_3 f(y') + g'(y_1)y_2 & & \nabla^s w \end{pmatrix}, \\
p_e(y) \Bigg) \mathrm{d}y' \mathrm{d}y_1 \,. \qquad (6.34)
\end{aligned}
$$

Here, the infimization takes place over all

$$
A \in \Pi_{v\in V(G)} \mathbb{R}^{3\times 3}_{\mathrm{asym}}\,, \quad \xi \in \Pi_{v\in V(G)} \mathbb{R}^3\,, \quad v \in H_0^1(I_e) \times H_0^2(I_e) \times H_0^2(I_e)\,,
$$
$$
g \in H_0^1(I_e)\,, \quad f \in H^1(B_e)\,, \quad w \in H^1(B_e;\mathbb{R}^2)\,,
$$

and we use the abbreviations $A_e^\pm := \frac{1}{2} R(e)^{-1} \left(A_{v_2(e)} \pm A_{v_1(e)} \right) R(e)$.

Remark. In order to completely reduce the rate-independent system $(\mathcal{Q}, \mathcal{E}^0, \mathcal{R}^0)$ from the state space containing $q = (u,v,p)$ to the space containing only (u,p), we also need to express the load term $\langle \ell^0(t), q\rangle$ and the dissipation $\mathcal{R}^0$ in terms of (u,p). The dissipation depends by definition only on p, see (6.23). But for the load-term, this reduction is not always possible. For example, when ℓ^0 is defined by volume-loads as in (6.17)–(6.19), the reduction is only possible in the case of sufficient thickness. In the critical and thin case, one has to deal with the expression $\int_{\Omega_e} v_{e,j}(y)\mathrm{d}y$ for $j = 2,3$ which translates in our setting into

$$
|B_e| \left(\int_{I_e} v_{e,j}(y_1) \mathrm{d}y_1 + \frac{1}{6} (A_e^-)_{j1} L(e)^2 \right) .
$$

Here we made use of the fact that $\int_0^L f(x)\mathrm{d}x = \frac{1}{12}(a-b)L^2$ when f is defined as in (6.33). This reveals a dependence of the load-term on v_2, v_3 and A. These variables therefore cannot be independently reduced in the stored energy. This issue can be resolved by incorporating v_2, v_3 and A into the state space and defining $\mathcal{B}^0_{\mathrm{red}}(u,p,v_2,v_3,A)$ via an energy density of the form $F(\epsilon,p,v_2,v_3,A)$ which differs from the definition of $F(\epsilon,p)$ in (6.34) only in that no infimization takes place over v_2, v_3 and A.

6.5 Statement of the convergence result

In this section, we formulate the main convergence result. We also give a proof, but in doing so we refer to the results of the following sections.

Let us suppose that $\ell^\varepsilon \in W^{1,\infty}(0,T;\mathcal{Q}^*)$ satisfies $\ell^\varepsilon(t) \to \ell^0(t)$ for all $t \in [0,T]$, and moreover $\|\ell^\varepsilon\|_{W^{1,\infty}(0,T;\mathcal{Q}^*)} \le C$ for all $\varepsilon \in [0,1]$. Such sequences of loads can be realized, e.g., by volume loads as in Lemma 6.1.

We claim that the rate-independent system $(\mathcal{Q}, \mathcal{E}^0, \mathcal{R}^0)$ is the limit of the systems $(\mathcal{Q}, \mathcal{E}^\varepsilon, \mathcal{R}^\varepsilon)$ in the following sense.

Theorem 6.4. *Consider a family of energetic solutions $q^\varepsilon \in L^1(0,T;\mathcal{Q})$ for the rate-independent system $(\mathcal{Q}, \mathcal{E}^\varepsilon, \mathcal{R}^\varepsilon)$ for $\varepsilon \ge 0$ such that*

$$q^\varepsilon(0) \rightharpoonup q^0(0)\,, \qquad \mathcal{B}^\varepsilon(q^\varepsilon(0)) \to \mathcal{B}^0(q^0(0))$$

as $\varepsilon \to 0$. Then also

$$q^\varepsilon(t) \to q^0(t)\,, \qquad \mathcal{B}^\varepsilon(q^\varepsilon(t)) \to \mathcal{B}^0(q^0(t))$$

for all $t \in [0,T]$ as $\varepsilon \to 0$. Moreover,

$$\mathrm{Diss}_{\mathcal{R}^\varepsilon}(q^\varepsilon;[0,t]) \to \mathrm{Diss}_{\mathcal{R}^0}(q^0;[0,t])\,, \quad \langle \partial_t \ell^\varepsilon(t), q^\varepsilon(t)\rangle \to \langle \partial_t \ell^0(t), q^0(t)\rangle\,.$$

Proof. The statement of the theorem follows from Theorem 3.4. We only need to check that the assumptions (A)–(D) on Pages 28 and 29 are satisfied:

(A) The stored energy functionals $\mathcal{B}^\varepsilon$ are quadratic forms since $\mathbb{W}$ is a quadratic form. Moreover, $\mathcal{B}^\varepsilon$ is continuous, hence lower-semicontinuous. What remains to be proved is the equicoercivity. This is done in Proposition 6.6 below.

(B) The dissipation functionals $\mathcal{R}^\varepsilon$ are all equal to $\mathcal{R}$. The function $\mathcal{R}$ is positive one-homogeneous and convex because R is positive one-homogeneous and convex. Moreover, $\mathcal{R}$ is continuous, hence lower-semicontinuous.

(C) The assumption on the Lipschitz bound of the loads ℓ^ε was just repeated in Theorem 6.4.

(D) The Mosco-convergence of $\mathcal{B}^\varepsilon$ is proved in Propositions 6.7 and 6.8 below. The Mosco-convergence and continuous convergence of $\mathcal{R}^\varepsilon$ immediately follows from the continuity and weak lower-semicontinuity of $\mathcal{R}^\varepsilon = \mathcal{R}$. The assumption on the convergence of the loads ℓ^ε was just repeated in Theorem 6.4.

Thus the theorem is proved once Propositions 6.6 to 6.8 are established. □

In the following sections, we provide the missing parts referred to in the above proof: equi-coercivity and Mosco-convergence of $\mathcal{B}^\varepsilon$.

6.6 Proof of the equicoercivity

In this section, we derive energy estimates for lattices of rods. We start by considering a single rod. Its elastic energy is estimated in terms of prescribed Dirichlet boundary values, that is, in terms of the state of its neighboring nodes.

Lemma 6.5 (Estimates for a single rod). *Let $\Omega = (0, L) \times B$ with $L > 0$ and $B \subset \mathbb{R}^2$ a bounded and centered Lipschitz domain, i.e. $\int_B (y_2, y_3) \mathrm{d}y_2 \mathrm{d}y_3 = 0$. For $v \in H^1(\Omega; \mathbb{R}^3)$ and $h \in (0, 1)$ we consider the affine boundary conditions*

$$v(y) = S_h^{-1} \left(A^0 y + d^0 \right) , \qquad v(y + Le_1) = S_h^{-1} \left(A^1 y + d^1 \right) . \tag{6.35}$$

for $y \in \{0\} \times B$ with given $d^0, d^1 \in \mathbb{R}^3$ and A^0, $A^1 \in \mathbb{R}^{3\times 3}_{\text{asym}}$.

We use the shorthands $A^\pm := \frac{1}{2}(A^1 \pm A^0)$ and $d := d^1 - d^0$. There exist constants $C_1, C_2 > 0$ such that:

(i) For all $h \in (0, 1)$, $d^0, d^1 \in \mathbb{R}^3$, A^0, $A^1 \in \mathbb{R}^{3\times 3}_{\text{asym}}$ and $v \in H^1(\Omega; \mathbb{R}^3)$ such that (6.35) is satisfied, there holds

$$|A^-|^2 + |S_h^{-1} d - A^+ Le_1|^2 \leq C_1 \| S_h \nabla^s v S_h \|^2_{L^2(\Omega; \mathbb{R}^{3\times 3}_{\text{sym}})} . \tag{6.36}$$

(ii) For all $h \in (0, 1)$, $d^0, d^1 \in \mathbb{R}^3$, A^0, $A^1 \in \mathbb{R}^{3\times 3}_{\text{asym}}$, there exists $v \in H^1(\Omega; \mathbb{R}^3)$ such that (6.35) is satisfied and

$$\| S_h \nabla^s v S_h \|^2_{L^2(\Omega; \mathbb{R}^{3\times 3}_{\text{sym}})} \leq C_2 \left(|A^-|^2 + |S_h^{-1} d - A^+ Le_1|^2 \right) . \tag{6.37}$$

Remark. Observe that

$$\begin{aligned} &|A^-|^2 + |S_h^{-1} d - A^+ Le_1|^2 \\ &\qquad = |A^-|^2 + |d_1|^2 + |hd_2 - LA^+_{21}|^2 + |hd_3 - LA^+_{31}|^2 . \end{aligned}$$

In particular, (6.36) implies that the elastic energy $\| S_h \nabla^s v S_h \|^2_{L^2(\Omega; \mathbb{R}^{3\times 3}_{\text{sym}})}$ controls the longitudinal displacement d_1. When considering not one isolated rod, but a rigid system of rods, the rigidity of that system will enable us to control the full set of displacement vectors d.

Proof. Throughout this lemma, we can assume without loss of generality that $A^0 = 0$ and $d^0 = 0$. Indeed, consider any $v \in H^1(\Omega; \mathbb{R}^3)$ such that (6.35) is satisfied. We then transform it to

$$\tilde{v}(y) := v(y) - h^{-1} S_h^{-1} A^0 S_h^{-1} y - S_h^{-1} d^0 .$$

This transformation leaves the symmetric gradient unchanged: We have $\nabla^s v = \nabla^s \tilde{v}$ since $S_h^{-1} A^0 S_h^{-1}$ is antisymmetric. Moreover, with

$$\tilde{d}^0 = 0\,, \qquad \tilde{d}^1 = d - A^0 L e_1\,, \qquad \tilde{A}^0 = 0\,, \qquad \tilde{A}^1 = A^1 - A^0\,.$$

the transformed field $\tilde{v}$ satisfies

$$\begin{aligned} \tilde{v}(y) &= 0 = S_h^{-1}\left(\tilde{A}^0 y + \tilde{d}^0\right)\,, \\ \tilde{v}(y + Le_1) &= S_h^{-1}(A^1 y + d^1) - h^{-1} S_h^{-1} A^0 S_h^{-1}(y + Le_1) - S_h^{-1} d^0 \\ &= S_h^{-1}\left((A^1 - A^0)y + d - A^0 L e_1\right) \\ &= S_h^{-1}(\tilde{A}^1 y + \tilde{d}^1) \end{aligned}$$

for $y \in \{0\} \times B$. We also write $\tilde{A}^{\pm} \coloneqq \frac{1}{2}(\tilde{A}^1 \pm \tilde{A}^0)$ and $\tilde{d} \coloneqq \tilde{d}^1 - \tilde{d}^0 = \tilde{d}^1$. Then $\tilde{A}^- = A^-$ and

$$S_h^{-1}\tilde{d} - \tilde{A}^+ L e_1 = S_h^{-1}(d - A^0 L e_1) - A^- L e_1 = S_h^{-1}(d - A^+ L e_1)\,.$$

We have thus seen that both sides of (6.36) and (6.37) are invariant under the transformation from v to $\tilde{v}$. We therefore can from now on assume that $A^0 = 0$ and $d^0 = 0$.

Once we have $A^0 = 0$, we furthermore note that

$$|A^-|^2 + |S_h^{-1} d - A^+ L e_1|^2 \sim |A^1|^2 + |S_h^{-1} d|^2\,, \tag{6.38}$$

since $A^+ = A^- = \frac{1}{2} A^1$.

(i) Using the Poincaré-Korn inequality (Lemma A.4(i)) on Ω with zero boundary values on $\{0\} \times B$, and a trace theorem, we find that

$$\begin{aligned} \|S_h \nabla^s v S_h\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \geq \|\nabla^s v\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})} &\gtrsim \|v\|^2_{H^1(\Omega;\mathbb{R}^3)} \\ &\gtrsim \|v|_{\{L\}\times B}\|^2_{L^2(\{L\}\times B;\mathbb{R}^3)} \\ &= \|S_h^{-1}(A^1 y + d)\|^2_{L^2(\{0\}\times B;\mathbb{R}^3)} \\ &\gtrsim |A^1_{12}|^2 + |A^1_{13}|^2 + |S_h^{-1} d|^2\,. \end{aligned}$$

Because of (6.38) and $|A^1|^2 \sim |A^1_{12}|^2 + |A^1_{13}|^2 + |A^1_{23}|^2$, this almost proves (6.36). It remains only to provide an estimate for $|A^1_{23}|^2$.

Observe that

$$v_2(y) = 0\,, \qquad v_2(y + Le_1) = hA^1_{23} y_3 + hd_2$$

for $y \in \{0\} \times B$. It follows from the fundamental theorem of calculus that

$$h^{-1}\int_\Omega \partial_1 v_2(y) y_3 \mathrm{d}y = \int_B A^1_{23} y_3^2 + d_2 y_3 \mathrm{d}y' = A^1_{23}\int_B y_3^2 \mathrm{d}y'$$

with $y' = (y_2, y_3)$. This implies

$$A^1_{23} = h^{-1}\left(\int_B y_3^2 \mathrm{d}y'\right)^{-1} \int_\Omega \left(\partial_1 v_2(y) - \fint_B \partial_1 v_2(y_1, \overline{y}')\mathrm{d}\overline{y}'\right) y_3 \mathrm{d}y\,,$$

since the term with the averaged integral vanhishes after the Ω-integration by $\int_\Omega y_3 = 0$. Therefore

$$\begin{aligned}|A^1_{23}| &\lesssim \left\| h^{-1}\left(\partial_1 v_2 - \fint_B \partial_1 v_2(\cdot, \overline{y}')\mathrm{d}\overline{y}'\right)\right\|_{L^2(\Omega)} \\ &\lesssim \|S_h \nabla^s v S_h\|_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})}\,,\end{aligned}$$

where the last inequality follows by Korn's inequality for thin domains, Lemma 4.4.

(ii) Given $h > 0$, $A^1 \in \mathbb{R}^{3\times 3}_{\mathrm{asym}}$ and $d \in \mathbb{R}^3$, let us define

$$g : [0, L] \to \mathbb{R}\,, \qquad w : [0, L] \to \mathbb{R}^3$$

by the following conditions: Both g and w_1 are affine, whereas w_2 and w_3 are polynomials of order 3, and there holds

$$\begin{aligned} g(0) = 0\,, \quad g(L) = A^1_{23}\,, \quad w(0) = 0\,, \quad w(L) = S_h^{-1} d\,, \\ w_2'(0) = 0\,, \quad w_2'(L) = A^1_{12}\,, \quad w_3'(0) = 0\,, \quad w_3'(L) = A^1_{13}\,. \end{aligned} \tag{6.39}$$

With these functions, we define

$$v(y) := w(y_1) + \begin{pmatrix} w_2'(y_1)y_2 + w_3'(y_1)y_3 \\ hg(y_1)y_3 \\ -hg(y_1)y_2 \end{pmatrix}.$$

One can easily check that (6.35) holds (recall that we assumed $A^0 = 0$). Moreover,

$$S_h \nabla^s v(y) S_h = \begin{pmatrix} w_1'(y_1) + w_2''(y_1)y_2 + w_3''(y_1)y_3 & * & * \\ \frac{1}{2}g'(y_1)y_3 & 0 & 0 \\ -\frac{1}{2}g'(y_1)y_2 & 0 & 0 \end{pmatrix}.$$

As w_1, w_2, w_3 and g are polynomials which are solely defined by the boundary conditions (6.39), we have

$$\|w_1'\|^2_{L^2(0,L)} + \|w_2''\|^2_{L^2(0,L)} + \|w_3''\|^2_{L^2(0,L)} + \|g'\|_{L^2(0,L)} \lesssim |A^1|^2 + |S_h^{-1} d|^2$$

and thus

$$\|S_h \nabla^s v S_h\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \lesssim |A^1|^2 + |S_h^{-1} d|^2\,.$$

By (6.38), this finishes the proof of (6.37). □

Proposition 6.6 (Equi-coercivity). *We consider $\mathcal{B}^\varepsilon$ of (6.13), describing the stored energy of a lattice of thin rods. There exists a constant $\beta > 0$ such that*

$$\beta \|q^\varepsilon\|^2 \le \mathcal{B}^\varepsilon(q^\varepsilon) \tag{6.40}$$

for all $q^\varepsilon \in \mathcal{Q}$ and $\varepsilon \in (0,1)$. Moreoever, there exists a constant $C > 0$ such that for all $q^\varepsilon = (u^\varepsilon, v^\varepsilon, p^\varepsilon) \in \mathcal{Q}^\varepsilon$ there holds

$$\begin{aligned} &\|v^\varepsilon\|^2_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} H^1(\Omega_e;\mathbb{R}^3)\right)} + \|\operatorname{grad}^\varepsilon(u^\varepsilon + \varepsilon\xi^\varepsilon; G^\varepsilon)\|^2_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^3\right)} + \\ &\quad \|A^\varepsilon\|^2_{L^2\left(\mathbb{R}^3;\Pi_{v\in V(G)}\mathbb{R}^{3\times 3}_{\mathrm{asym}}\right)} + \|p^\varepsilon\|^2_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^{3\times3}_{\mathrm{dev}})\right)} \le C\mathcal{B}^\varepsilon(q) \end{aligned} \tag{6.41}$$

where

$$(A^\varepsilon, \xi^\varepsilon) \in L^2\left(\mathbb{R}^3; \Pi_{v\in V(G)} \mathbb{R}^{3\times3}_{\mathrm{asym}} \times \mathbb{R}^3\right)$$

are G^ε-node functions such that $(u^\varepsilon, \xi^\varepsilon)$ is a G^ε-function pair with $u^\varepsilon_v + \varepsilon\xi^\varepsilon_v = 0$ on Γ^ε_v and the compatibility condition (6.12) holds.

Proof. The estimate (6.40) immediately follows from (6.41) with the help of Lemma 5.15. We therefore directly give a proof for (6.41).

In the proof we will drop the ε-superscripts for better readability. We consider the terms on the left-hand side of (6.41). The p-term is trivially estimated from

$$\begin{aligned} \mathcal{B}^\varepsilon(q) &= \sum_{e\in E(G)} \int_{\mathbb{R}^3}\int_{\Omega_e} \mathbb{W}_e(S_h \nabla^s_y v_e(x,y) S_h\,,\ p_e(x,y))\mathrm{d}y\mathrm{d}x \\ &\gtrsim \sum_{e\in E(G)} \int_{\mathbb{R}^3} \|S_h\nabla^s_y v_e(x,\cdot)S_h\|^2_{L^2(\Omega_e;\mathbb{R}^{3\times3}_{\mathrm{sym}})} + \|p_e(x,\cdot)\|^2_{L^2(\Omega_e;\mathbb{R}^{3\times3}_{\mathrm{dev}})}\mathrm{d}x\,. \end{aligned}$$

For the other terms we need to invoke Lemma 6.5(i) with

$$\begin{aligned} d &:= R(e)^{-1}\operatorname{grad}^\varepsilon_e(u + \varepsilon\xi; G^\varepsilon)(x)\,, \\ A^0 &:= R(e)^{-1} A_{v_1(e)}(x) R(e)\,, \\ A^1 &:= R(e)^{-1} A_{v_2(e)}(x + \varepsilon d(e)) R(e)\,, \end{aligned}$$

according to compatibility condition (6.12) for v. In a first step, we use the estimate for d_1 in (6.36) in order to find that

$$\begin{aligned} \mathcal{B}^\varepsilon(q) &\gtrsim \sum_{e\in E(G)} \int_{\mathbb{R}^3} \|S_h\nabla^s_y v_e(x,\cdot)S_h\|^2_{L^2(\Omega_e;\mathbb{R}^{3\times3}_{\mathrm{sym}})}\mathrm{d}x \\ &\gtrsim \sum_{e\in E(G)} \int_{\mathbb{R}^3} \left|\left(R(e)^{-1}\operatorname{grad}^\varepsilon_e(u+\varepsilon\xi;G^\varepsilon)(x)\right)_1\right|^2\mathrm{d}x\,. \end{aligned}$$

Noting that $(R(e)^{-1}a)_1 = a \cdot R(e)e_1 = a \cdot r(e)$ for any $a \in \mathbb{R}^3$, and using the uniform rigidity estimate from Lemma 5.10, we then have

$$\begin{aligned}\mathcal{B}^\varepsilon(q) &\gtrsim \sum_{e \in E(G)} \int_{\mathbb{R}^3} \left|\operatorname{grad}_e^\varepsilon(u + \varepsilon\xi; G^\varepsilon)(x) \cdot r(e)\right|^2 \mathrm{d}x \\ &\gtrsim \sum_{e \in E(G)} \int_{\mathbb{R}^3} \left|\operatorname{grad}_e^\varepsilon(u + \varepsilon\xi; G^\varepsilon)(x)\right|^2 \mathrm{d}x \\ &= \|\operatorname{grad}^\varepsilon(u + \varepsilon\xi; G^\varepsilon)\|^2_{L^2\left(\mathbb{R}^3; \Pi_{e \in E(G)}\mathbb{R}^3\right)} . \end{aligned} \tag{6.42}$$

Having thus obtained a bound on *all* components of d, we can use (6.36) again to get estimates for A^0 and A^1. Indeed, in the setting of Lemma 6.5, it follows from (6.36) that

$$|A^0 e_1|^2 + |A^1 e_1|^2 \lesssim \|S_h \nabla^s v S_h\|^2_{L^2(\Omega; \mathbb{R}^{3\times 3}_{\mathrm{sym}})} + |d|^2 .$$

Therefore, recalling that $r(e) = R(e)e_1$, we have

$$\begin{aligned}\mathcal{B}^\varepsilon(q) &\gtrsim \sum_{e \in E(G)} \int_{\mathbb{R}^3} \left|A_{v_1(e)}(x) r(e)\right|^2 + \left|A_{v_2(e)}(x + \varepsilon d(e)) r(e)\right|^2 \mathrm{d}x \\ &= \sum_{e \in E(G)} \int_{\mathbb{R}^3} \left|A_{v_1(e)}(x) r(e)\right|^2 + \left|A_{v_2(e)}(x) r(e)\right|^2 \mathrm{d}x . \end{aligned} \tag{6.43}$$

Now we observe that for all $v \in V(G)$,

$$\operatorname{span}\{r(e) \ : \ e \in E(G) \text{ with } v_1(e) = v \text{ or } v_2(e) = v\} = \mathbb{R}^3 . \tag{6.44}$$

This is a consequence of the infinitesimal rigidity of G_{per} (see Lemma 5.2). Indeed, let w denote a vector from the orthogonal complement of the left-hand side of (6.44) and define $u : V(G_{\mathrm{per}}) \to \mathbb{R}^3$ by $u(v, 0) = w$ and $u = 0$ everywhere else. Then (5.9) yields that u must be constant and it follows that $w = 0$. Now a direct consequence of (6.44) is that

$$\int_{\mathbb{R}^3} |A_v(x)|^2 \mathrm{d}x \lesssim \sum_{e \in E(G)} \int_{\mathbb{R}^3} |A_{v_1(e)}(x) r(e)|^2 \mathrm{d}x + \int_{\mathbb{R}^3} |A_{v_2(e)}(x) r(e)|^2 \mathrm{d}x$$

for all $v \in V(G)$. Hence we can continue (6.43) and get

$$\mathcal{B}^\varepsilon(q) \gtrsim \sum_{v \in V(G)} \int_{\mathbb{R}^3} |A_v(x)|^2 \, \mathrm{d}x = \|A\|_{L^2(\mathbb{R}^3; \Pi_{v \in V(G)}\mathbb{R}^3)} . \tag{6.45}$$

In order to get an estimate for v, we first define

$$\tilde{v}_e(x,y) := v_e(x,y) - h^{-1} S_h^{-1} R(e)^{-1} A_{v_1(e)}(x) R(e) S_h^{-1} y - S_h^{-1} R(e)^{-1} \xi_{v_1(e)}(x)\,.$$

Then $\nabla_y^s v = \nabla_y^s \tilde{v}$ and $\tilde{v}_e(x,y) = 0$ for $y \in \{0\} \times B_e$. We can thus apply Korn's inequality from Lemma A.4(i) to find

$$\|\tilde{v}\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} H^1(\Omega_e;\mathbb{R}^3)\right)} \leq C \|\nabla_y^s v\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^{3\times 3}_{\mathrm{sym}})\right)}\,.$$

But then

$$\begin{aligned}
&\|v\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} H^1(\Omega_e;\mathbb{R}^3)\right)} \\
&\quad\lesssim \|\tilde{v}\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} H^1(\Omega_e;\mathbb{R}^3)\right)} + \|\xi\|_{L^2\left(\mathbb{R}^3;\Pi_{v\in V(G)} \mathbb{R}^3\right)} \\
&\qquad + \|h^{-1} S_h^{-1} R(e)^{-1} A R(e) S_h^{-1}\|_{L^2\left(\mathbb{R}^3;\Pi_{v\in V(G)} \mathbb{R}^{3\times 3}_{\mathrm{asym}}\right)} \\
&\quad\lesssim \|\nabla_y^s v\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^{3\times 3}_{\mathrm{sym}})\right)} + \|A\|_{L^2\left(\mathbb{R}^3;\Pi_{v\in V(G)} \mathbb{R}^{3\times 3}_{\mathrm{asym}}\right)} \\
&\qquad + \|\mathrm{grad}^\varepsilon(u + \varepsilon\xi; G^\varepsilon)\|^2_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} \mathbb{R}^3\right)} \\
&\quad\lesssim \mathcal{B}^\varepsilon(q)\,,
\end{aligned}$$

where we have used Lemma 5.15 for the estimate of ξ, and (6.42) and (6.45) in the last step. ☐

6.7 Proof of the Mosco-convergence

Proposition 6.7 (Lower bound). *Consider $\mathcal{B}^\varepsilon$ as defined in* (6.13) *and $\mathcal{B}^0$ as defined in* (6.24)*. Given any weakly convergent sequence $q^\varepsilon \rightharpoonup q$ in $\mathcal{Q}$ there holds*

$$\liminf_{\varepsilon\to 0} \mathcal{B}^\varepsilon(q^\varepsilon) \geq \mathcal{B}^0(q)\,. \tag{6.46}$$

Proof. Step 1. We write $(u^\varepsilon, v^\varepsilon, p^\varepsilon) := q^\varepsilon$ and $(u, v, p) := q$. Without loss of generality, we may assume that $\mathcal{B}^\varepsilon(q^\varepsilon)$ is uniformly bounded along a subsequence. We consider a subsequence with $\mathcal{B}^\varepsilon(q^\varepsilon) \to \liminf_{\varepsilon\to 0} \mathcal{B}^\varepsilon(q^\varepsilon)$.

Recall that $q^\varepsilon \in \mathcal{Q}^\varepsilon$ consists of G^ε-node and G^ε-edge functions. It thus follows immediately from $q^\varepsilon \rightharpoonup q$ and our construction of G^ε (going back to (D1) on Page 65) that $q = 0$ in $\mathbb{R}^3 \setminus \Omega$. Moreover, the bound on $\mathcal{B}^\varepsilon(q^\varepsilon)$ implies by the definition of $\mathcal{B}^\varepsilon$ a bound on $S_h \nabla_y^s v^\varepsilon S_h$ in $L^2(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(\Omega_e; \mathbb{R}^{3\times 3}_{\mathrm{sym}}))$. Therefore $(\nabla_y^s v^\varepsilon)_{ij} \to 0$ in $L^2(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(\Omega_e))$ and $(\nabla_y^s v)_{ij} = 0$ for $(i,j) \neq (1,1)$. This shows that $\nabla_y^s v \in \mathrm{span}(e_1 \otimes e_1)$ a. e.

As $q^\varepsilon \in \mathcal{Q}^\varepsilon$, there exist G^ε-node functions

$$(A^\varepsilon, \xi^\varepsilon) \in L^2\left(\mathbb{R}^3; \Pi_{v\in V(G)}\mathbb{R}^{3\times 3}_{\text{asym}} \times \Pi_{v\in V(G)}\mathbb{R}^3\right)$$

such that (u, ξ) is a G^ε-function pair with $u_v + \varepsilon\xi_v = 0$ on Γ^ε_v and the compatibility conditions (6.12) are satisfied. We know from Proposition 6.6 that

$$\begin{aligned} \|A^\varepsilon\|_{L^2(\mathbb{R}^3;\Pi_{v\in V(G)}\mathbb{R}^{3\times 3}_{\text{asym}})} &\lesssim 1\,, \\ \|\text{grad}^\varepsilon(u^\varepsilon + \varepsilon\xi^\varepsilon; G^\varepsilon)\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^3)} &\lesssim 1\,. \end{aligned}$$

First, this implies that there exists a subsequence and $A \in L^2(\mathbb{R}^3; \Pi_{v\in V(G)}\mathbb{R}^{3\times 3}_{\text{asym}})$ such that

$$A^\varepsilon \rightharpoonup A \qquad \text{in } L^2(\mathbb{R}^3; \Pi_{v\in V(G)}\mathbb{R}^{3\times 3}_{\text{asym}})\,.$$

Second, we can use the two-scale compactness of Lemma 5.16. It provides a subsequence and $\xi \in L^2(\mathbb{R}^3; \Pi_{v\in V(G)}\mathbb{R}^3))$ such that

$$\begin{aligned} \xi^\varepsilon &\rightharpoonup \xi && \text{in } L^2(\mathbb{R}^3; \Pi_{v\in V(G)}\mathbb{R}^3)\,, \\ \text{grad}^\varepsilon(u^\varepsilon + \varepsilon\xi^\varepsilon; G^\varepsilon) &\rightharpoonup \text{grad}(u, \xi; G) && \text{in } L^2\left(\Omega; \Pi_{e\in E(G)}\mathbb{R}^3\right)\,. \end{aligned}$$

Moreoever, $u|_\Omega \in H^1_\Gamma(\Omega; \mathbb{R}^3)$ and $u^\varepsilon \rightharpoonup u$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$.

We now prove that (A, ξ) is admissible for q in the sense that the compatibility conditions (6.22) hold, and thus $q \in \mathcal{Q}^0$. For this we observe that

$$\begin{aligned} v^\varepsilon_e(\cdot, \cdot) &\rightharpoonup v_e(\cdot, \cdot)\,, \\ v^\varepsilon_e(\cdot, \cdot + L(e)e_1) &\rightharpoonup v_e(\cdot, \cdot + L(e)e_1) \end{aligned}$$

in $L^2(\mathbb{R}^3; L^2(\{0\} \times B_e; \mathbb{R}^3))$. But on the other hand, we have the compatibility condition (6.12) and therefore

$$\begin{aligned} v^\varepsilon_e(x, y) &= S_h^{-1} R(e)^{-1}\Big(\xi^\varepsilon_{v_1(e)}(x) + A^\varepsilon_{v_1(e)}(x) R(e) y\Big) \\ &\rightharpoonup \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} R(e)^{-1}\Big(\xi_{v_1(e)}(x) + A_{v_1(e)}(x) R(e) y\Big) \end{aligned}$$

and

$$\begin{aligned} v^\varepsilon_e(x, y + L(e)e_1) = S_h^{-1} R(e)^{-1}\Big(&\text{grad}^\varepsilon_e(u^\varepsilon + \varepsilon\xi^\varepsilon; G^\varepsilon)(x) \\ &+ \xi^\varepsilon_{v_1(e)}(x) + A^\varepsilon_{v_2(e)}(x + \varepsilon d(e)) R(e) y\Big) \\ \rightharpoonup \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} R(e)^{-1}\Big(&\text{grad}_e(u, \xi; G)(x) \\ &+ \xi_{v_1(e)}(x) + A_{v_2(e)}(x) R(e) y\Big)\,. \end{aligned}$$

This establishes (6.22).

Step 2. Since $S_h \nabla_y^s v^\varepsilon S_h$ is uniformly bounded in $L^2(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(\Omega_e; \mathbb{R}^{3\times3}_{\mathrm{sym}}))$, there exists a subsequence and some

$$E \in L^2\left(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(\Omega_e; \mathbb{R}^{3\times3}_{\mathrm{sym}})\right)$$

such that $S_h \nabla_y^s v^\varepsilon S_h \rightharpoonup E$. Our aim is to find

$$\begin{aligned} f &\in L^2\left(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(I_e; H^1(B_e))\right) , \\ g &\in L^2\left(\mathbb{R}^3; \Pi_{e\in E(G)} H^1(I_e)\right) , \\ w &\in L^2\left(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(I_e; H^1(B_e; \mathbb{R}^2))\right) , \end{aligned}$$

such that

$$g_e(x,0) = \frac{1}{2}\left(R(e)^{-1} A_{v_1(e)}(x) R(e)\right)_{23} , \tag{6.47a}$$

$$g_e(x,L(e)) = \frac{1}{2}\left(R(e)^{-1} A_{v_2(e)}(x) R(e)\right)_{23} \tag{6.47b}$$

for almost every $x \in \mathbb{R}^3$, and

$$E_e(x,y) = \begin{pmatrix} \partial_{y_1} v_{e,1}(x,y) & * \quad * \\ \partial_{y_2} f_e(x,y) + \partial_{y_1} g_e(x,y_1) y_3 & \\ \partial_{y_3} f_e(x,y) - \partial_{y_1} g_e(x,y_1) y_2 & \nabla^s_{y_2,y_3} w_e(x,y) \end{pmatrix} . \tag{6.48}$$

Once (6.48) is shown, the lower bound (6.46) follows since

$$\begin{aligned} \liminf_{\varepsilon\to0} \mathcal{B}^\varepsilon(q^\varepsilon) &= \liminf_{\varepsilon\to0} \sum_{e\in E(G)} \int_{\mathbb{R}^3}\int_{\Omega_e} \mathbb{W}_e(S_h \nabla_y^s v_e^\varepsilon(x,y) S_h,\, p_e^\varepsilon(x,y)) \mathrm{d}y\mathrm{d}x \\ &\geq \sum_{e\in E(G)} \int_{\mathbb{R}^3}\int_{\Omega_e} \mathbb{W}_e\left(E_e(x,y),\, p_e(x,y)\right) \mathrm{d}y\mathrm{d}x \\ &\geq \mathcal{B}^0(q) \end{aligned}$$

by weak lower semi-continuity and the definition of $\mathcal{B}^0$ in (6.24).

Step 3. In order to define (f,g,w), we first consider

$$\begin{aligned} \tilde{v}^\varepsilon_{e,2}(x,y) &\coloneqq v^\varepsilon_{e,2}(x,y) - \fint_{B_e} v^\varepsilon_{e,2}(x,y_1,y')\mathrm{d}y' , \\ \tilde{v}^\varepsilon_{e,3}(x,y) &\coloneqq v^\varepsilon_{e,3}(x,y) - \fint_{B_e} v^\varepsilon_{e,3}(x,y_1,y')\mathrm{d}y' . \end{aligned}$$

By Korn's inequality on thin domains (Lemma 4.4), there holds

$$\left\| \frac{1}{2h} \begin{pmatrix} \partial_{y_1} \tilde{v}_2^\varepsilon \\ \partial_{y_1} \tilde{v}_3^\varepsilon \end{pmatrix} \right\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^2)\right)} \lesssim \|S_h \nabla_y^s v^\varepsilon S_h\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^{3\times 3}_{\mathrm{sym}})\right)} \lesssim 1 \,. \tag{6.49}$$

But by (6.12a), we also have the boundary estimate

$$\left\| \frac{1}{2h} \begin{pmatrix} \tilde{v}_2^\varepsilon \\ \tilde{v}_3^\varepsilon \end{pmatrix} \right\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\{0\}\times B_e;\mathbb{R}^2))} \lesssim \|A^\varepsilon\|_{L^2(\mathbb{R}^3;\Pi_{v\in V(G)} \mathbb{R}^{3\times 3}_{\mathrm{asym}})} \lesssim 1 \,.$$

In combination this yields by the fundamental theorem of calculus (applied to the interval I_e) the estimate

$$\left\| \frac{1}{2h} \begin{pmatrix} \tilde{v}_2^\varepsilon \\ \tilde{v}_3^\varepsilon \end{pmatrix} \right\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^2))} \lesssim 1 \,. \tag{6.50}$$

We define $g^\varepsilon \in L^2\left(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(I_e)\right)$ as the unique minimizer of

$$\left\| \frac{1}{2h} \begin{pmatrix} \tilde{v}_2^\varepsilon(x,y) \\ \tilde{v}_3^\varepsilon(x,y) \end{pmatrix} - g^\varepsilon(x,y_1) \begin{pmatrix} -y_3 \\ y_2 \end{pmatrix} \right\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^2)\right)} .$$

By (6.50), the sequence $(g^\varepsilon)_\varepsilon$ is uniformly bounded. Hence there exists a subsequence and a limit function $g \in L^2(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(I_e))$ such that

$$g^\varepsilon \rightharpoonup g \quad \text{in } L^2(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(I_e)) \,.$$

By Korn's inquality on B_e (see Lemma A.4(ii)),

$$\begin{aligned} &\left\| \frac{1}{2h} \begin{pmatrix} \tilde{v}_2^\varepsilon(x,y) \\ \tilde{v}_3^\varepsilon(x,y) \end{pmatrix} - g^\varepsilon(x,y_1) \begin{pmatrix} -y_3 \\ y_2 \end{pmatrix} \right\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^2)\right)} \\ &\lesssim \left\| \frac{1}{h} \nabla^s_{y_2,y_3} \tilde{v}^\varepsilon_{2,3} \right\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^{2\times 2}_{\mathrm{sym}})\right)} \\ &= \left\| \frac{1}{h} \nabla^s_{y_2,y_3} v^\varepsilon_{2,3} \right\|_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^{2\times 2}_{\mathrm{sym}})\right)} \to 0 \,. \end{aligned}$$

In particular,

$$\frac{1}{2h} \begin{pmatrix} \partial_{y_1} \tilde{v}_2^\varepsilon \\ \partial_{y_1} \tilde{v}_3^\varepsilon \end{pmatrix} \to \partial_{y_1} g \begin{pmatrix} -y_3 \\ y_2 \end{pmatrix}$$

in the sense of distributions, and by the bound (6.49) this implies

$$\frac{1}{2h}\begin{pmatrix}\partial_{y_1}\tilde{v}_2^\varepsilon\\ \partial_{y_1}\tilde{v}_3^\varepsilon\end{pmatrix} \rightharpoonup \partial_{y_1} g\begin{pmatrix}-y_3\\ y_2\end{pmatrix} \tag{6.51}$$

in $L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)}L^2(\Omega_e;\mathbb{R}^2)\right)$. In particular, $g\in L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(I_e))$. Integrating (6.51) over $I_e\times B_e'$ for $B_e'\subset B_e$ and taking the limit $\varepsilon\to 0$ yields

$$\frac{1}{2}\left(R(e)^{-1}(A_{v_2(e)}-A_{v_1(e)})(x)R(e)\right)_{23} = g_e(x,L(e))-g_e(x,0)$$

by (6.12). If g does not yet satisfy (6.47), we simply replace $g_e(x,y)$ with

$$g_e(x,y)-g_e(x,0)+\frac{1}{2}\left(R(e)^{-1}A_{v_1(e)}R(e)\right)_{23}\,.$$

Then (6.47) is satisfied, and (6.51) remains true in the process.

Step 4. We define $\tilde{v}_1^\varepsilon\in L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(\Omega_e))$ by

$$\tilde{v}_{e,1}^\varepsilon(x,y) \coloneqq v_{e,1}^\varepsilon(x,y)+\left(y_2\partial_{y_1}\fint_{B_e} v_{e,2}^\varepsilon(x,y_1,y')\mathrm{d}y' + y_3\partial_{y_1}\fint_{B_e} v_{e,3}^\varepsilon(x,y_1,y')\mathrm{d}y'\right).$$

We know from (6.49) and $\|S_h\nabla_y^s v S_h\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}L^2(\Omega_e;\mathbb{R}^{3\times 3}_{\mathrm{sym}}))}\lesssim 1$ that

$$\frac{1}{2h}\begin{pmatrix}\partial_{y_1}\tilde{v}_2^\varepsilon\\ \partial_{y_1}\tilde{v}_3^\varepsilon\end{pmatrix} \quad\text{and}\quad \frac{1}{2h}\begin{pmatrix}\partial_{y_1}\tilde{v}_2^\varepsilon+\partial_{y_2}\tilde{v}_1^\varepsilon\\ \partial_{y_1}\tilde{v}_3^\varepsilon+\partial_{y_3}\tilde{v}_1^\varepsilon\end{pmatrix} = \frac{1}{2h}\begin{pmatrix}\partial_{y_1}v_2^\varepsilon+\partial_{y_2}v_1^\varepsilon\\ \partial_{y_1}v_3^\varepsilon+\partial_{y_3}v_1^\varepsilon\end{pmatrix}$$

are bounded in $L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)}L^2(\Omega_e;\mathbb{R}^2)\right)$. But then

$$\frac{1}{2h}\begin{pmatrix}\partial_{y_2}\tilde{v}_1^\varepsilon\\ \partial_{y_3}\tilde{v}_1^\varepsilon\end{pmatrix}$$

is also bounded in $L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)}L^2(\Omega_e;\mathbb{R}^2)\right)$. Thus there exists (by Poincaré's inequality and a compactness argument) a subsequence and function

$$f\in L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)}L^2(I_e;H^1(B_e))\right)$$

such that

$$\frac{1}{2h}\begin{pmatrix}\partial_{y_2}\tilde{v}_1^\varepsilon\\ \partial_{y_3}\tilde{v}_1^\varepsilon\end{pmatrix} \rightharpoonup \begin{pmatrix}\partial_{y_2}f\\ \partial_{y_3}f\end{pmatrix} \tag{6.52}$$

in $L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)}L^2(\Omega_e;\mathbb{R}^2)\right)$.

Step 5. It remains to construct w. As

$$\begin{aligned}\|h^{-2}\nabla^s_{y_2,y_3}v^\varepsilon_{2,3}\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^{2\times 2}_{\mathrm{sym}})}\\ \le \|S_h\nabla^w_{y_2,y_3}v^\varepsilon_{2,3}S_h\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^{2\times 2}_{\mathrm{sym}})}\lesssim \mathcal{B}^\varepsilon(q^\varepsilon)\lesssim 1\,,\end{aligned}$$

by Korn's inequality (see Lemma A.4(ii)) and a compactness argument, there exsists a subsequence and a function

$$w \in L^2\left(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(I_e; H^1(B_e;\mathbb{R}^2))\right)$$

such that

$$\frac{1}{h^2}\nabla^s_{y_2,y_3} v^\varepsilon_{2,3} \rightharpoonup \nabla^s_{y_2,y_3} w \tag{6.53}$$

in $L^2\left(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^{2\times 2}_{\mathrm{sym}})\right)$.

Step 6. We conclude, using the weak convergence

$$v^\varepsilon \rightharpoonup v \quad \text{in } L^2(\mathbb{R}^3; \Pi_{e\in E(G)} H^1(\Omega_e;\mathbb{R}^3))$$

as well as (6.51), (6.52), and (6.53) that

$$S_h \nabla^s_y v^\varepsilon_e(x,y) S_h \rightharpoonup \begin{pmatrix} \partial_{y_1} v_{e,1}(x,y) & * \quad * \\ \partial_{y_2} f_e(x,y) - \partial_{y_1} g_e(x,y_1) y_3 & \\ \partial_{y_3} f_e(x,y) + \partial_{y_1} g_e(x,y_1) y_2 & \nabla^s_{y_2,y_3} w_e(x,y) \end{pmatrix}$$

in $L^2\left(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(\Omega_e;\mathbb{R}^{3\times 3}_{\mathrm{sym}})\right)$. This implies (6.48). As noted at the end of *Step 2*, this concludes the proof of the lower bound. □

Proposition 6.8 (Upper bound). *Consider* $\mathcal{B}^\varepsilon$ *as defined in* (6.13) *and* $\mathcal{B}^0$ *as defined in* (6.24). *For every* $q \in \mathcal{Q}$ *there exists a sequence* $(q^\varepsilon)_\varepsilon \subset \mathcal{Q}$ *such that* $q^\varepsilon \to q$ *in* $\mathcal{Q}$ *and*

$$\limsup_{\varepsilon\to 0} \mathcal{B}^\varepsilon(q^\varepsilon) \le \mathcal{B}^0(q)\,.$$

Proof. Step 1. It is sufficient for every $\delta > 0$ and $q \in \mathcal{Q}$ to find a sequence $q^\varepsilon \to q$ in $\mathcal{Q}$ such that

$$\limsup_{\varepsilon\to 0} \mathcal{B}^\varepsilon(q^\varepsilon) \le \mathcal{B}^0(q) + \delta\,. \tag{6.54}$$

We can assume that $q = (u, v, p) \in \mathcal{Q}^0$, as otherwise $\mathcal{B}^0(q) = \infty$. Then there exists

$$(A,\xi) \in L^2\left(\mathbb{R}^3; \Pi_{v\in V(G)} \mathbb{R}^{3\times 3}_{\mathrm{asym}} \times \Pi_{v\in V(G)} \mathbb{R}^3\right)$$

vanishing outside Ω with $\sum_{v\in V(G)} \xi_v = 0$ such that (6.22) holds. According to Lemma 6.2, there exist

$$\begin{aligned} &f \in L^2(\mathbb{R}^3; \Pi_{e\in E(G)} H^1(\Omega_e))\,, && f_e = 0 \text{ on } \mathbb{R}^3 \times \partial I_e \times B_e\,,\\ &g \in L^2(\mathbb{R}^3; \Pi_{e\in E(G)} H^1(I_e))\,, && \\ &w \in L^2(\mathbb{R}^3; \Pi_{e\in E(G)} H^1(\Omega_e;\mathbb{R}^2))\,, && w_e = 0 \text{ on } \mathbb{R}^3 \times \partial I_e \times B_e\,, \end{aligned}$$

such that

$$g_e(x,0) = \frac{1}{2}\left(R(e)^{-1}A_{v_1(e)}(x)R(e)\right)_{23}\,,$$
$$g_e(x,L(e)) = \frac{1}{2}\left(R(e)^{-1}A_{v_2(e)}(x)R(e)\right)_{23}$$

for $e \in E(G)$ and $x \in \mathbb{R}^3$, and

$$\sum_{e\in E(G)} \int_{\mathbb{R}^3}\int_{\Omega_e} \mathbb{W}_e\left(\begin{pmatrix} \partial_{y_1} v_{e,1} & * \quad * \\ \partial_{y_2} f_e - \partial_{y_1} g_e(x,y_1)y_3 & \\ \partial_{y_3} f_e + \partial_{y_1} g_e(x,y_1)y_2 & \nabla^s_{y_2,y_3} w_e \end{pmatrix}, \, p_e\right) \le \mathcal{B}^0(q) + \delta\,.$$

Step 2. We define discretizations as introduced in Lemma 5.18,

$$\begin{aligned}(\eta_v^\varepsilon, A_v^\varepsilon) &\coloneqq \mathbb{1}_v^\varepsilon(G^\varepsilon)P^\varepsilon(u+\varepsilon\xi_v, A_v) && \text{for } v \in V(G)\,,\\ (\overline{v}_e^\varepsilon, p_e^\varepsilon, f_e^\varepsilon, g_e^\varepsilon, w_e^\varepsilon) &\coloneqq \mathbb{1}_e^\varepsilon(G^\varepsilon)P^\varepsilon(v_e, p_e, f_e, g_e, w_e) && \text{for } e \in E(G)\,.\end{aligned}$$

Furthermore, we we denote by $(u^\varepsilon, \xi^\varepsilon) \in L^2(\mathbb{R}^3; \mathbb{R}^3 \times \Pi_{v\in V(G)}\mathbb{R}^3)$ the unique G^ε-function pair such that $\eta^\varepsilon = u + \varepsilon\xi$.

With these discretized functions, we define

$$v_e^\varepsilon(x,y) \coloneqq \overline{v}_e^\varepsilon(x,y) + 2h\begin{pmatrix} f_e^\varepsilon(x,y) \\ -g_e^\varepsilon(x,y_1)y_3 \\ g_e^\varepsilon(x,y_1)y_2\end{pmatrix} + h^2\begin{pmatrix} 0 \\ w_{e,1}^\varepsilon(x,y) \\ w_{e,2}^\varepsilon(x,y)\end{pmatrix} + \phi_e^\varepsilon(x,y) \tag{6.55}$$

for $e \in E(G)\,,\ x \in \mathbb{R}^3$ and $y \in \Omega_e$. Here ϕ_e^ε is a small correction term which is necessary because without it, $(u^\varepsilon, v^\varepsilon, p^\varepsilon)$ would in general not satisfy the boundary conditions (6.12) required for elements of $\mathcal{Q}^\varepsilon$. This is because the boundary conditions that $\overline{v}^\varepsilon$ has to satisfy are spanning across neighboring cells (see the term $x + \varepsilon d(e)$ in (6.12) which is also implicit in the definition of $\operatorname{grad}^\varepsilon$), whereas in the limit $\varepsilon = 0$ the boundary conditions fully decompose over $x \in \mathbb{R}^3$ (see (6.22)).

The correction term ϕ^ε is defined to be the unique minimizer of the elastic energy

$$\|S_h \nabla^s \phi^\varepsilon S_h\|^2_{L^2\left(\mathbb{R}^3;\Pi_{e\in E(G)}L^2(\Omega_e;\mathbb{R}^{3\times 3}_{\mathrm{sym}})\right)} \tag{6.56}$$

among all G^ε-edge functions $\phi^\varepsilon \in L^2\left(\mathbb{R}^3; \Pi_{e\in E(G)} H^1(\Omega_e;\mathbb{R}^3)\right)$ which satisfy the boundary conditions

$$\phi_e^\varepsilon(x,y) = \begin{pmatrix} 0 & & \\ & h & \\ & & h \end{pmatrix} R(e)^{-1}\xi^\varepsilon_{v_1(e)}(x) \tag{6.57a}$$

$$\begin{aligned}\phi_e^\varepsilon(x,y+L(e)e_1) = {} & \begin{pmatrix} 1 & & \\ & h & \\ & & h \end{pmatrix} R(e)^{-1}\Big(\operatorname{grad}_e^\varepsilon(\eta^\varepsilon;G^\varepsilon)(x) + \xi^\varepsilon_{v_1(e)}(x) \\ & + A^\varepsilon_{v_2(e)}(x+\varepsilon d(e))R(e)y - A^\varepsilon_{v_2(e)}(x)R(e)y\Big) \\ & - \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} R(e)^{-1}\Big(P^\varepsilon \operatorname{grad}_e(u,\xi;G)(x) + \xi_{v_1(e)}(x)\Big)\end{aligned} \tag{6.57b}$$

for all $e \in E(G)$, $y \in \{0\}\times B_e$ and $x \in \Omega_e^\varepsilon(G^\varepsilon)$.

We check that v^ε then satisfies the compatibility condition (6.12). By (6.22), the boundary values of g, and (6.57) we have:

$$\begin{aligned} v_e^\varepsilon(x,y) &= \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} R(e)^{-1}\Big(\xi^\varepsilon_{v_1(e)}(x) + A^\varepsilon_{v_1(e)}(x)R(e)y\Big) \\ &\quad + \begin{pmatrix} 0 & & \\ & h & \\ & & h \end{pmatrix} R(e)^{-1}\Big(\xi^\varepsilon_{v_1(e)}(x) + A^\varepsilon_{v_1(e)}(x)R(e)y\Big) \\ &= \begin{pmatrix} 1 & & \\ & h & \\ & & h \end{pmatrix} R(e)^{-1}\Big(\xi^\varepsilon_{v_1(e)}(x) + A^\varepsilon_{v_1(e)}(x)R(e)y\Big) \\ v_e^\varepsilon(x,y+L(e)e_1) &= \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} R(e)^{-1}\Big(P^\varepsilon \operatorname{grad}_e(u,\xi;G)(x) + \xi^\varepsilon_{v_1(e)}(x) \\ &\qquad + A^\varepsilon_{v_2(e)}(x)R(e)y\Big) \\ &\quad + \begin{pmatrix} 0 & & \\ & h & \\ & & h \end{pmatrix} R(e)^{-1}A^\varepsilon_{v_2(e)}(x)R(e)y \\ &\quad + \begin{pmatrix} 1 & & \\ & h & \\ & & h \end{pmatrix} R(e)^{-1}\Big(\operatorname{grad}_e^\varepsilon(\eta^\varepsilon;G^\varepsilon)(x) + \xi^\varepsilon_{v_1(e)}(x) \\ &\qquad + A^\varepsilon_{v_2(e)}(x+\varepsilon d(e))R(e)y - A^\varepsilon_{v_2(e)}(x)R(e)y\Big) \\ &\quad - \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} R(e)^{-1}P^\varepsilon\Big(\operatorname{grad}_e(u,\xi;G) + \xi_{v_1(e)}\Big)(x) \\ &= \begin{pmatrix} 1 & & \\ & h & \\ & & h \end{pmatrix} R(e)^{-1}\Big(\operatorname{grad}_e^\varepsilon(\eta^\varepsilon;G^\varepsilon)(x) + \xi^\varepsilon_{v_1(e)}(x) \\ &\qquad + A^\varepsilon_{v_2(e)}(x+\varepsilon d(e))R(e)y\Big) \end{aligned}$$

for all $e \in E(G)$, $y \in \{0\}\times B_e$ and $x \in \Omega_e^\varepsilon(G^\varepsilon)$, which is in accordance with (6.12). We have thus shown that

$$q^\varepsilon := (u^\varepsilon, v^\varepsilon, p^\varepsilon) \in \mathcal{Q}^\varepsilon\,.$$

We still have to show that $q^\varepsilon \to q$ in $\mathcal{Q}$ and $\limsup_{\varepsilon\to 0}\mathcal{B}^\varepsilon(q^\varepsilon) \le \mathcal{B}^0(q) + \delta$.

Step 3. We claim that $q^\varepsilon = (u^\varepsilon, v^\varepsilon, p^\varepsilon) \to q = (u,v,p)$ as $\varepsilon \to 0$. The convergences $u^\varepsilon \to u$ and $p^\varepsilon \to p$ follow from Lemma 5.18. We therefore turn our attention to v^ε. First, we will show that ϕ^ε is small. From the fact that the elastic energy (6.56) is minimized subject to the boundary values (6.57), we can use part (*ii*) of Lemma 6.5 with

$$\begin{aligned} A^0 &= 0 \\ A^1 &= R(e)^{-1}\Big(A^\varepsilon_{v_2(e)}(x+\varepsilon d(e)) - A^\varepsilon_{v_2(e)}(x)\Big)R(e) \\ d = d^1 - d^0 &= R(e)^{-1}\operatorname{grad}^\varepsilon_e(\eta^\varepsilon;G^\varepsilon)(x) - \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} R(e)^{-1}P^\varepsilon \operatorname{grad}_e(u,\xi;G)(x) \end{aligned}$$

for $e \in E(G)$ and $x \in \Omega^\varepsilon_e(G^\varepsilon)$ on Ω_e in order to infer that

$$\|S_h \nabla^s_y \phi^\varepsilon_e(x) S_h\|_{L^2(\Omega_e;\mathbb{R}^{3\times 3}_{\mathrm{sym}})} \lesssim |A^1| + |d_1| + |hd_2| + |hd_3| \,.$$

Integrating x over $\mathbb{R}^3$ and summing e over $E(G)$ yields

$$\begin{aligned} &\|S_h \nabla^s_y \phi^\varepsilon S_h\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}L^2(\Omega_e;\mathbb{R}^{3\times3}_{\mathrm{sym}}))} \\ &\quad\lesssim \Big\|A^\varepsilon_{v_2(e)}(\cdot+\varepsilon d(e)) - A^\varepsilon_{v_2(e)}\Big\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^{3\times3}_{\mathrm{asym}})} \\ &\qquad + \|\operatorname{grad}^\varepsilon(\eta^\varepsilon;G^\varepsilon) - \operatorname{grad}(u,\xi;G)\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^3)} \\ &\qquad + h\|\operatorname{grad}^\varepsilon(\eta^\varepsilon;G^\varepsilon)\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}\mathbb{R}^3)} \to 0\,. \end{aligned}$$

Here we use for the final convergence the Kolmogorov-Riesz theorem and the strong convergence $A^\varepsilon \to A$ in $L^2(\mathbb{R}^3;\Pi_{v\in V(G)}\mathbb{R}^{3\times3}_{\mathrm{asym}})$ for the first term, and part (*iii*) of Lemma 5.18 for the second term.

As $\phi^\varepsilon_e(x,\cdot) - \begin{pmatrix} 0 & & \\ & h & \\ & & h \end{pmatrix} R(e)^{-1}\xi^\varepsilon_{v_1(e)}(x)$ vanishes on $\{0\}\times B_e$, see (6.57a), we can use Korn's inequality (see Lemma A.4(i)) to conclude that

$$\begin{aligned} &\|\phi^\varepsilon\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(\Omega_e;\mathbb{R}^3))} \\ &\qquad\lesssim h\|\xi\|_{L^2(\mathbb{R}^3;\Pi_{v\in V(G)}\mathbb{R}^3)} + \|\nabla^s_y\phi^\varepsilon\|_{L^2(\mathbb{R}^3;\Pi_{e\in E(G)}L^2(\Omega_e;\mathbb{R}^{3\times3}_{\mathrm{asym}}))} \to 0\,. \end{aligned}$$

We can now conclude the convergence of v^ε (as defined in (6.55)): The convergence of $\overline{v}^\varepsilon$ to v in $L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(\Omega_e;\mathbb{R}^3))$ follows from Lemma 5.18, the boundedness of $(f^\varepsilon, g^\varepsilon, w^\varepsilon)$ also follows from Lemma 5.18, the convergence $\phi^\varepsilon \to 0$ in $L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(\Omega_e;\mathbb{R}^3))$ was just shown, and thus

$$v^\varepsilon \to v \quad \text{in } L^2(\mathbb{R}^3;\Pi_{e\in E(G)}H^1(\Omega_e;\mathbb{R}^3))\,.$$

Furthermore, we have the convergences $u^\varepsilon \to u$ and $p^\varepsilon \to p$ in their respective spaces according to Lemma 5.18. This implies $q^\varepsilon \to q$ in $\mathcal{Q}$.

Step 4. It remains to show that $\limsup_{\varepsilon\to 0} \mathcal{B}^\varepsilon(q^\varepsilon) \le \mathcal{B}^0(q) + \delta$. For this we observe that

$$\begin{aligned} S_h \nabla_y^s v_e^\varepsilon(x,y) S_h = &\begin{pmatrix} \partial_{y_1} \overline{v}_{e,1}^\varepsilon(x,y) & * \quad * \\ \partial_{y_2} f_e^\varepsilon(x,y) - \partial_{y_1} g_e^\varepsilon(x,y_1) y_3 & \\ \partial_{y_3} f_e^\varepsilon(x,y) + \partial_{y_1} g_e^\varepsilon(x,y_1) y_2 & \nabla_{y_2,y_3}^s w_e^\varepsilon(x,y) \end{pmatrix} \\ &+ \begin{pmatrix} 2h \partial_{y_1} f_e^\varepsilon(x,y) & * & * \\ \frac{h}{2} \partial_{y_1} w_{e,2}^\varepsilon(x,y) & 0 & 0 \\ \frac{h}{2} \partial_{y_1} w_{e,3}^\varepsilon(x,y) & 0 & 0 \end{pmatrix} + S_h \nabla_y^s \phi_e^\varepsilon(x,y) S_h \\ &\to \begin{pmatrix} \partial_{y_1} v_{e,1}(x,y) & * \quad * \\ \partial_{y_2} f_e(x,y) - \partial_{y_1} g_e(x,y_1) y_3 & \\ \partial_{y_3} f_e(x,y) + \partial_{y_1} g_e(x,y_1) y_y & \nabla_{y_2,y_3}^s w_e(x,y) \end{pmatrix} \end{aligned}$$

in $L^2(\mathbb{R}^3; \Pi_{e\in E(G)} L^2(\Omega_e; \mathbb{R}^{3\times 3}_{\mathrm{sym}}))$. From this we can finally conclude that

$$\begin{aligned} &\mathcal{B}^\varepsilon(q^\varepsilon) = \sum_{e\in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e} \mathbb{W}_e(S_h \nabla_y^s v_e^\varepsilon(x,y) S_h \,,\ p_e^\varepsilon(x,y)) \mathrm{d}y \mathrm{d}x \\ &\to \sum_{e\in E(G)} \int_{\mathbb{R}^3} \int_{\Omega_e} \mathbb{W}_e \left(\begin{pmatrix} \partial_{y_1} v_{e,1} & * \quad * \\ \partial_{y_2} f_e - \partial_{y_1} g_e(x,y_1) y_3 & \\ \partial_{y_3} f_e + \partial_{y_1} g_e(x,y_1) y_y & \nabla_{y_2,y_3}^s w_e \end{pmatrix} ,\ p_e \right) \mathrm{d}y \mathrm{d}x \\ &\le \mathcal{B}^0(q) + \delta \,. \end{aligned}$$

This shows (6.54) and thus finishes the proof. □

Appendix A

Tools from Analysis

A.1 Strong convexity

Lemma A.1. *Let X be a Hilbert space, $f : X \to \mathbb{R}_\infty$ a convex function, and define $I(x) \coloneqq \|x\|^2 + f(x)$. Suppose that $\hat{x}$ is a minimizer of I. Then*

$$\|x - \hat{x}\|^2 \leq I(x) - I(\hat{x})\,, \qquad x \in X\,.$$

Proof. Given any $x \in X$ and $\varepsilon > 0$, the minimizer property of $\hat{x}$ implies

$$\begin{aligned} 0 &\leq \frac{I((1-\varepsilon)\hat{x} + \varepsilon x) - I(\hat{x})}{\varepsilon} \\ &= \frac{\|\hat{x} + \varepsilon(x - \hat{x})\|^2 - \|\hat{x}\|^2}{\varepsilon} + \frac{f((1-\varepsilon)\hat{x} + \varepsilon x) - f(\hat{x})}{\varepsilon}\,. \end{aligned}$$

By the convexity of f, this implies

$$\begin{aligned} 0 &\leq \frac{\|\hat{x} + \varepsilon(x - \hat{x})\|^2 - \|\hat{x}\|^2}{\varepsilon} - f(\hat{x}) + f(x) \\ &= 2\langle \hat{x}, x - \hat{x}\rangle + \varepsilon\|x - \hat{x}\|^2 - f(\hat{x}) + f(x)\,. \end{aligned}$$

Taking the limit $\varepsilon \to 0$, this implies

$$\begin{aligned} 0 &\leq 2\langle \hat{x}, x - \hat{x}\rangle - f(\hat{x}) + f(x) \\ &= -\|\hat{x} - x\|^2 - \|\hat{x}\|^2 + \|x\|^2 - f(\hat{x}) + f(x) \\ &= -\|\hat{x} - x\|^2 - I(\hat{x}) + I(x)\,, \end{aligned}$$

and therefore $\|\hat{x} - x\|^2 \leq I(x) - I(\hat{x})$. ∎

A.2 Arzelà-Ascoli

Lemma A.2. *Let X be a reflexive Banach space, $T > 0$ and $(u_n)_n$ a bounded sequence in $W^{1,\infty}(0,T;X)$. Then there exists a subsequence $(u_{n_k})_k$ and a limit function $u \in W^{1,\infty}(0,T;X)$ such that $u_{n_k}(t) \rightharpoonup u(t)$ as $k \to \infty$ for almost every $t \in (0,T)$.*

Proof. By altering u_n on a null set, we can assume that u_n is Lipschitz continuous. By the boundednes of $(u_n)_n$ in $W^{1,\infty}(0,T;X)$, there exists a uniform bound $C > 0$ on the Lipschitz constants. Now for every $t \in (0,T)$ the sequence $(u_n(t))_n$ is bounded in X. Thus there exists a subsequence and a limit element $u(t) \in X$ such that $u_n(t) \rightharpoonup u(t)$. By doing this for the countably many $t \in (0,T) \cap \mathbb{Q}$, iteratively choosing subsequences and in the end taking the diagonal sequence, we end up with a subsequence $(u_{n_k})_k$ along which $u_{n_k}(t)$ weakly converges to some $u(t) \in X$ for every $t \in (0,T) \cap \mathbb{Q}$. This defines a function $u : (0,T) \cap \mathbb{Q} \to X$. But then we have

$$\|u(s) - u(t)\| \leq \liminf_{k\to\infty} \|u_{n_k}(s) - u_{n_k}(t)\| \leq C|s-t| \qquad \forall\, s,\, t \in (0,T) \cap \mathbb{Q}\,.$$

Therefore we can uniquely extend u to a Lipschitz continuous function $u \in W^{1,\infty}(0,T;X)$.

Now consider any (possibly irrational) $t \in (0,T)$. We want to show that $u_{n_k}(t) \rightharpoonup u(t)$ in X. For this we consider any $f \in X' \setminus \{0\}$ and $\varepsilon > 0$. We can find $t^* \in (0,T) \cap \mathbb{Q}$ such that

$$|t - t^*| \leq \frac{\varepsilon}{3C\|f\|}\,.$$

As $u_{n_k}(t^*) \rightharpoonup u(t^*)$ in X, for large $n \in \mathbb{N}$ we have $|f(u_{n_k}(t^*) - u(t^*))| \leq \frac{\varepsilon}{3}$ and consequently

$$\begin{aligned} &|f(u_{n_k}(t) - u(t))| \\ &\qquad \leq |f(u_{n_k}(t) - u_{n_k}(t^*))| + |f(u_{n_k}(t^*) - u(t^*))| + |f(u(t^*) - u(t))| \\ &\qquad \leq C\|f\||t - t^*| + \frac{\varepsilon}{3} + C\|f\||t - t^*| \leq \varepsilon\,. \end{aligned}$$

Thus $f(u_{n_k}(t)) \to f(u(t))$, and hence $u_{n_k}(t) \rightharpoonup u(n)$ in X. □

A.3 Poincaré and Korn inequalities

In the main text we make use of the following well-known Poincaré and Korn inequalities. For Poincaré inequalities see for example [6, 54]. For Korn inequalities see [21, 48, 31, 54].

Lemma A.3 (Poincaré inequalities). *Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain.*

(i) Let Γ be a subset of $\partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma) > 0$. Then there exists a constant $C > 0$ such that

$$\|u\|_{H^1(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$$

for all $u \in H^1(\Omega)$ with $u = 0$ on Γ in the sense of traces.

(ii) Let U be a nonempty open subset of Ω. Then there exists a constant $C > 0$ such that

$$\|u - \overline{u}\|_{H^1(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)},$$

for all $u \in H^1(\Omega)$ and $\overline{u} := \fint_U u(x)\mathrm{d}x$.

Remark. We often encounter the special cases $U = \Omega$, $\overline{u} = 0$ or $\Gamma = \partial\Omega$.

Lemma A.4 (Poincaré-Korn inequalities). *Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain.*

(i) Let Γ be a subset of $\partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma) > 0$. Then there exists a constant $C > 0$ such that

$$\|u\|_{H^1(\Omega;\mathbb{R}^n)} \leq C\|\nabla^s u\|_{L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}$$

for all $u \in H^1(\Omega;\mathbb{R}^n)$ with $u = 0$ on Γ in the sense of traces.

(ii) There exists a constant $C > 0$ such that

$$\inf_{A\in\mathbb{R}^{n\times n}_{\mathrm{asym}},\, b\in\mathbb{R}^n} \|u(x) - Ax - b\|_{H^1(\Omega;\mathbb{R}^n)} \leq C\|\nabla^s u\|_{L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}$$

for all $u \in H^1(\Omega;\mathbb{R}^n)$. Moreover,

$$\|\nabla u - A\|_{L^2(\Omega;\mathbb{R}^{n\times n})} \leq C\|\nabla^s u\|_{L^2(\Omega;\mathbb{R}^{n\times n}_{\mathrm{sym}})}$$

for $A = \fint_\Omega \nabla^a u$ or $A = \fint_\Omega \nabla u$.

A.4 Hilbert Adjoints

The following basic facts from functional analysis are gathered here for the convenience of the reader.

Lemma A.5. *Let $T : X \to Y$ be a bounded linear operator between Hilbert spaces, and $T^* : Y \to X$ its adjoint operator.*

(i) If T is injective, then the image $R(T^*)$ of T^* is dense in X.

(ii) If the image $R(T)$ of T is dense in Y, then T^* is injective.

Proof. Ad (i). Consider any $x \in R(T^*)^\perp$. Then

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = 0$$

for all $y \in Y$, and consequently $Tx = 0$. As T is injective, this implies $x = 0$. Hence we have $R(T^*)^\perp = 0$ and thus $\overline{R(T^*)} = X$.

Ad (ii). Consider any $y \in Y$ with $T^*y = 0$. Then

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = 0$$

for all $x \in X$. Hence $y \in R(T)^\perp$. As $R(T)$ is dense in Y, this implies $y = 0$. □

Appendix B

Technical proofs

B.1 An integral inequality

Lemma B.1. *For a reflexive Banach space X and $I = (0, T)$ let $u \in L^1(I; X)$ and $g \in L^\infty(I)$ satisfy*

$$\|u(t_2) - u(t_1)\|^2 \leq \int_{t_1}^{t_2} g(s) \cdot \|u(t_2) - u(s)\| \, \mathrm{d}s$$

for almost every $0 < t_1 < t_2 < T$. Then $u \in W^{1,\infty}(I)$ and $\|\partial_t u(t)\| \leq \frac{1}{2} g(t)$ for almost every $t \in I$.

Proof. Step 1: Regularity of u. Fix $t_2 \in I$ such that the inequality holds for almost every $t_1 \in (0, t_2)$ and let

$$f(t) := \int_{t_2 - t}^{t_2} \|u(t_2) - u(s)\| \mathrm{d}s \, .$$

We assume without loss of generality that $\|g\|_{L^\infty(I)} \leq 1$. Then we have the estimate

$$f'(t) = \|u(t_2) - u(t_2 - t)\| \leq \left(\int_{t_2 - t}^{t_2} \|u(t_2) - u(s)\| \mathrm{d}s \right)^{1/2} = \sqrt{f(t)} \, .$$

This implies

$$\partial_t \sqrt{f(t)} = \frac{f'(t)}{2\sqrt{f(t)}} \leq \frac{1}{2} \, .$$

Since $f(0) = 0$, we then have $\sqrt{f(t)} \leq \frac{1}{2} t$. Thus, for almost every $t_1 < t_2$,

$$\|u(t_2) - u(t_1)\| = f'(t_2 - t_1) \leq \sqrt{f(t_2 - t_1)} \leq \frac{1}{2}(t_2 - t_1) \, .$$

Thus u is Lipschitz continuous with Lipschitz constant $\frac{1}{2}$.

Step 2: Estimate for $\partial_t u$. Consider any $t \in I$ where u is differentiable and which is a Lebesgue point of g. By Rademacher's theorem this is true for almost every $t \in I$. We claim that

$$g(t) = \lim_{\varepsilon \to 0} \fint_t^{t+\varepsilon} g(s) \cdot \frac{2(s-t)}{\varepsilon} \mathrm{d}s \,. \tag{B.1}$$

Indeed, writing $g_\varepsilon(t) := \fint_t^{t+\varepsilon} g(s) \frac{2(s-t)}{\varepsilon} \mathrm{d}s$, we have

$$\begin{aligned} |g(t) - g_\varepsilon(t)| &\leq \fint_t^{t+\varepsilon} |g(t) - g(s)| \cdot \frac{2(s-t)}{\varepsilon} \mathrm{d}s \\ &\leq 4 \fint_{t-\varepsilon}^{t+\varepsilon} |g(t) - g(s)| \mathrm{d}s \to 0 \,, \end{aligned}$$

where the convergence ist just the Lebesgue point property. By the assumption of the lemma we have

$$\frac{\|u(t+\varepsilon) - u(t)\|^2}{\varepsilon^2} \leq \fint_t^{t+\varepsilon} |g(s)| \cdot \frac{s-t}{\varepsilon} \cdot \frac{\|u(s) - u(t)\|}{|s-t|} \mathrm{d}s \,.$$

The difference quotients converge by choice of t, and because of (B.1) we get in the limit $\varepsilon \to 0$,

$$\|u'(t)\|^2 \leq \frac{1}{2} |g(t)| \cdot \|u'(t)\| \,.$$

Thus $\|u'(t)\| \leq \frac{1}{2} |g(t)|$. □

B.2 Infimization

Lemma B.2. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $\mathcal{U}, \mathcal{V}$ denote separable Hilbert spaces. Suppose that $\mathcal{F} : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ is a positive semidefinite continuous quadratic form. Then for any $u \in L^2(\Omega; \mathcal{U})$,*

$$\begin{aligned} \inf_{v \in H_0^1(\Omega;\mathcal{V})} \int_\Omega \mathcal{F}(u(x), v(x)) \mathrm{d}x &= \inf_{v \in L^2(\Omega;\mathcal{V})} \int_\Omega \mathcal{F}(u(x), v(x)) \mathrm{d}x \\ &= \int_\Omega \inf_{v \in \mathcal{V}} \mathcal{F}(u(x), v) \mathrm{d}x \,. \end{aligned}$$

Proof. We only have to prove "$\leq$" in both instances, the opposite inequality is clear.

Step 1. Let $v \in L^2(\Omega; \mathcal{V})$. As $H_0^1(\Omega; \mathcal{V})$ is dense in $L^2(\Omega; \mathcal{V})$, we find a sequence $v_n \in H_0^1(\Omega; \mathcal{V})$ such that $v_n \to v$ in $L^2(\Omega; \mathcal{V})$ as $n \to \infty$. Along a subsequence, we have $v_n(x) \to v(x)$ and thus $\mathcal{F}(u(x), v_n(x)) \to \mathcal{F}(u(x), v(x))$ for almost every $x \in \Omega$. With the quadratic bound of $\mathcal{F}$ it follows then by the dominated convergence theorem that $\int_\Omega \mathcal{F}(u(x), v_n(x))\mathrm{d}x \to \int_\Omega \mathcal{F}(u(x), v(x))\mathrm{d}x$, which establishes the first (in)equality. Indeed, we have a sequence of majorants $g_n(x) \coloneqq C(\|u(x)\|^2 + \|v_n(x)\|^2)$ which converge almost everywhere to $g(x) \coloneqq C(\|u(x)\|^2 + \|v(x)\|^2)$.

Step 2. Let $\varepsilon > 0$. There is a continuous linear function $A^\varepsilon : \mathcal{U} \to \mathcal{V}$ such that

$$A^\varepsilon u = \arg\min\{\mathcal{F}(u, v) + \varepsilon\|v\|^2 \ : \ v \in \mathcal{V}\}\,.$$

Indeed, we can write $\mathcal{F}(u, v) = \frac{1}{2}\langle Au, u\rangle + \frac{1}{2}\langle Bv, v\rangle + \langle Cu, v\rangle$ with linear operators $A : \mathcal{U} \to \mathcal{U}^*$, $B : \mathcal{V} \to \mathcal{V}^*$, $C : \mathcal{U} \to \mathcal{V}^*$, and thus $A^\varepsilon = -(B + \varepsilon)^{-1}C$. We therefore let $v^\varepsilon(x) \coloneqq A^\varepsilon u(x)$ and find that $v^\varepsilon \in L^2(\Omega; \mathcal{V})$ and

$$\begin{aligned}\int_\Omega \mathcal{F}(u(x), v^\varepsilon(x)) &\leq \int_\Omega \mathcal{F}(u(x), v^\varepsilon(x)) + \varepsilon\|v^\varepsilon(x)\|^2\mathrm{d}x \\ &= \int_\Omega \inf_{v\in\mathcal{V}} \mathcal{F}(u(x), v) + \varepsilon\|v\|^2\mathrm{d}x \to \int_\Omega \inf_{v\in\mathcal{V}} \mathcal{F}(u(x), v)\mathrm{d}x\end{aligned}$$

as $\varepsilon \to 0$, where we used monotone convergence of the integrand on the right-hand side. The monotone convergence theorem applies for this decreasing sequence since the integrals that are involved exist and are finite. □

Bibliography

[1] H. Abdoul-Anziz and P. Seppecher. Homogenization of periodic graph-based elastic structures. *Journal de l'Ecole Polytechnique - Mathematiques*, 5, 01 2018.

[2] H. Abdoul-Anziz and P. Seppecher. Strain gradient and generalized continua obtained by homogenizing frame lattices. *Mathematics and Mechanics of Complex Systems*, 6:213–250, 07 2018.

[3] H.-D. Alber. *Materials with memory*, volume 1682 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1998. Initial-boundary value problems for constitutive equations with internal variables.

[4] H.-D. Alber. Evolving microstructure and homogenization. *Continuum Mechanics and Thermodynamics*, 12(4):235–286, 2000.

[5] G. Allaire. Homogenization and two-scale convergence. *SIAM Journal on Mathematical Analysis*, 23(6):1482–1518, 1992.

[6] H. W. Alt. *Linear functional analysis*. Universitext. Springer-Verlag London, Ltd., London, 2016. An application-oriented introduction, Translated from the German edition by Robert Nürnberg.

[7] I. Babuška and S. Sauter. Algebraic algorithms for the analysis of mechanical trusses. *Mathematics of Computation*, 73(248):1601–1622, 2004.

[8] C. Borcea, I. Streinu, and S.-i. Tanigawa. Periodic body-and-bar frameworks. In *Computational geometry (SCG'12)*, pages 347–356. ACM, New York, 2012.

[9] C. S. Borcea and I. Streinu. Periodic frameworks and flexibility. *Proceedings of The Royal Society of London. Series A. Mathematical, Physical and Engineering Sciences*, 466(2121):2633–2649, 2010.

[10] C. S. Borcea and I. Streinu. Minimally rigid periodic graphs. *Bulletin of the London Mathematical Society*, 43(6):1093–1103, 2011.

[11] G. Bouchitte, G. Buttazzo, and P. Seppecher. Energies with respect to a measure and applications to low-dimensional structures. *Calculus of Variations and Partial Differential Equations*, 5(1):37–54, 1997.

[12] G. Bouchitté and I. Fragalà. Homogenization of thin structures by two-scale method with respect to measures. *SIAM Journal on Mathematical Analysis*, 32(6):1198–1226, 2001.

[13] G. Bouchitté and I. Fragalà. Homogenization of elastic thin structures: A measure-fattening approach. *Journal of Convex Analysis*, 9(2):339–362, 2002. Special issue on optimization (Montpellier, 2000).

[14] F. Bourquin and P. Ciarlet. Modeling and justification of eigenvalue problems for junctions between elastic structures. *Journal of functional analysis*, 87(2):392–427, 1989.

[15] G. A. Chechkin, V. V. Jikov, D. Lukkassen, and A. L. Piatnitski. On homogenization of networks and junctions. *Asymptotic Analysis*, 30(1):61–80, 2002.

[16] P. G. Ciarlet. *Mathematical elasticity. Vol. II*, volume 27 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1997. Theory of plates.

[17] G. Dal Maso, A. DeSimone, and M. G. Mora. Quasistatic evolution problems for linearly elastic-perfectly plastic materials. *Archive for Rational Mechanics and Analysis*, 180(2):237–291, 2006.

[18] E. De Giorgi and T. Franzoni. Su un tipo di convergenza variazionale. *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali. Serie VIII*, 58(6):842–850, 1975.

[19] C. Eck, H. Garcke, and P. Knabner. *Mathematical modeling*. Springer Undergraduate Mathematics Series. Springer, Cham, 2017.

[20] G. Francfort and A. Giacomini. On periodic homogenization in perfect elasto-plasticity. *Journal of the European Mathematical Society (JEMS)*, 16(3):409–461, 2014.

[21] K. O. Friedrichs. On the boundary-value problems of the theory of elasticity and Korn's inequality. *Annals of Mathematics. Second Series*, 48:441–471, 1947.

[22] G. Friesecke, R. D. James, and S. Müller. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Communications on Pure and Applied Mathematics*, 55(11):1461–1506, 2002.

[23] A. Gloria, S. Neukamm, and F. Otto. Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics. *Inventiones Mathematicae*, 199(2):455–515, 2015.

[24] A. Gloria and F. Otto. An optimal variance estimate in stochastic homogenization of discrete elliptic equations. *The Annals of Probability*, 39(3):779–856, 2011.

[25] A. Gloria and F. Otto. An optimal error estimate in stochastic homogenization of discrete elliptic equations. *The Annals of Applied Probability*, 22(1):1–28, 2012.

[26] W. Han and B. D. Reddy. *Plasticity.* Springer New York, 2013.

[27] M. Heida and B. Schweizer. Stochastic homogenization of plasticity equations. *ESAIM. Control, Optimisation and Calculus of Variations*, 24(1):153–176, 2018.

[28] I. Izmestiev. Projective background of the infinitesimal rigidity of frameworks. *Geometriae Dedicata*, 140(1):183–203, 2008.

[29] D. Kitson and S. C. Power. The rigidity of infinite graphs. *Discrete & Computational Geometry*, 60(3):531–557, 2018.

[30] R. Kohn and R. Temam. Dual spaces of stresses and strains, with applications to Hencky plasticity. *Applied Mathematics and Optimization*, 10(1):1–35, 1983.

[31] V. A. Kondrat'ev and O. A. Oleinik. Boundary value problems for a system in elasticity theory in unbounded domains. Korn inequalities. *Akademiya Nauk SSSR i Moskovskoe Matematicheskoe Obshchestvo. Uspekhi Matematicheskikh Nauk*, 43(5(263)):55–98, 239, 1988.

[32] S. M. Kozlov. The averaging of random operators. *Matematicheskiĭ Sbornik. Novaya Seriya*, 109(151)(2):188–202, 327, 1979.

[33] H. Le Dret. Modeling of the junction between two rods. *Journal de mathématiques pures et appliquées*, 68(3):365–397, 1989.

[34] M. Liero and A. Mielke. An evolutionary elastoplastic plate model derived via Γ-convergence. *Mathematical Models and Methods in Applied Sciences*, 21(9):1961–1986, 2011.

[35] M. Liero and T. Roche. Rigorous derivation of a plate theory in linear elastoplasticity via Γ-convergence. *NoDEA. Nonlinear Differential Equations and Applications*, 19(4):437–457, 2012.

[36] P. Martinsson and I. Babuška. Homogenization of materials with periodic truss or frame micro-structures. *M^3AS. Mathematical Models & Methods in Applied Sciences*, 17(5):805–832, 2007.

[37] P.-G. Martinsson and I. Babuška. Mechanics of materials with periodic truss or frame micro-structures. *Archive for Rational Mechanics and Analysis*, 185(2):201–234, 2007.

[38] G. Maso. *An Introduction to* Γ*-convergence*. Birkhäuser, 1993.

[39] A. Mielke. Saint-venant's problem and semi-inverse solutions in nonlinear elasticity. *Archive for Rational Mechanics and Analysis*, 102(3):205–229, 1988.

[40] A. Mielke. Evolution of rate-independent systems. In *Evolutionary equations. Vol. II*, Handb. Differ. Equ., pages 461–559. Elsevier/North-Holland, Amsterdam, 2005.

[41] A. Mielke. *Hamiltonian and Lagrangian flows on center manifolds: with applications to elliptic variational problems*. Springer, 2006.

[42] A. Mielke and T. Roubíček. *Rate-Independent systems. Theory and application*, volume 193 of *Applied Mathematical Sciences*. Springer, New York, 2015.

[43] A. Mielke, T. Roubíček, and U. Stefanelli. Γ-limits and relaxations for rate-independent evolutionary problems. *Calculus of Variations and Partial Differential Equations*, 31(3):387–416, 2008.

[44] A. Mielke, F. Theil, and V. I. Levitas. A variational formulation of rate-independent phase transformations using an extremum principle. *Archive for Rational Mechanics and Analysis*, 162(2):137–177, 2002.

[45] A. Mielke and A. M. Timofte. Two-scale homogenization for evolutionary variational inequalities via the energetic formulation. *SIAM Journal on Mathematical Analysis*, 39(2):642–668, 2007.

[46] M. G. Mora and S. Müller. Derivation of the nonlinear bending-torsion theory for inextensible rods by Γ-convergence. *Calculus of Variations and Partial Differential Equations*, 18(3):287–305, 2003.

[47] M. G. Mora and S. Müller. A nonlinear model for inextensible rods as a low energy γ-limit of three-dimensional nonlinear elasticity. In *Annales de l'Institut Henri Poincaré, Anal. Non Linéaire*, volume 21, pages 271–293. Elsevier, 2004.

[48] J. Nečas and I. Hlaváček. *Mathematical theory of elastic and elasto-plastic bodies: an introduction*, volume 3 of *Studies in Applied Mechanics*. Elsevier Scientific Publishing Co., Amsterdam-New York, 1980.

[49] G. Nguetseng. A general convergence result for a functional related to the theory of homogenization. *SIAM Journal on Mathematical Analysis*, 20(3):608–623, 1989.

[50] G. C. Papanicolaou and S. R. S. Varadhan. Boundary value problems with rapidly oscillating random coefficients. In *Random fields, Vol. I, II (Esztergom, 1979)*, volume 27 of *Colloq. Math. Soc. János Bolyai*, pages 835–873. North-Holland, Amsterdam-New York, 1981.

[51] S. Pastukhova. Homogenization of problems of elasticity theory on periodic box and rod frames of critical thickness. *Journal of Mathematical Sciences (New York)*, 130(5):4954–5004, 2004.

[52] E. Ross. The rigidity of periodic frameworks as graphs on a fixed torus. *Contributions to Discrete Mathematics*, 9, 02 2012.

[53] B. Schulze. Symmetric versions of Laman's theorem. *Discrete & Computational Geometry. An International Journal of Mathematics and Computer Science*, 44(4):946–972, 2010.

[54] B. Schweizer. *Partielle Differentialgleichungen*. Springer-Verlag, Berlin, 2013. Eine anwendungsorientierte Einführung. [An application-oriented introduction].

[55] B. Schweizer and M. Veneroni. Homogenization of plasticity equations with two-scale convergence methods. *Applicable Analysis. An International Journal*, 94(2):376–399, 2015.

[56] P. Seppecher, J.-J. Alibert, and F. Dell Isola. Linear elastic trusses leading to continua with exotic mechanical interactions. *Journal of Physics: Conference Series*, 319, 09 2011.

[57] R. Temam. *Problèmes mathématiques en plasticité*, volume 12 of *Méthodes Mathématiques de l'Informatique [Mathematical Methods of Information Science]*. Gauthier-Villars, Montrouge, 1983.

[58] A. Visintin. On homogenization of elasto-plasticity. *Journal of Physics: Conference Series*, 22:222–234, 2005.

[59] V. Zhikov and S. Pastukhova. Homogenization for elasticity problems on periodic networks of critical thickness. *Sbornik: Mathematics*, 194(5):697–732, 2003.

[60] V. V. Zhikov. Homogenization of elasticity problems on singular structures. *Izvestiya: Mathematics*, 66(2):299–365, 2002.

[61] V. V. Zhikov and S. E. Pastukhova. On the korn inequalities on thin periodic frames. *Journal of Mathematical Sciences*, 123(5):4499–4521, oct 2004.