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On the correspondence between abstract dialectical frameworks and nonmonotonic conditional logics

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Abstract

The exact relationship between formal argumentation and nonmonotonic logics is a research topic that keeps on eluding researchers despite recent intensified efforts. We contribute to a deeper understanding of this relation by investigating characterizations of abstract dialectical frameworks in conditional logics for nonmonotonic reasoning. We first show that in general, there is a gap between argumentation and conditional semantics when applying several intuitive translations, but then prove that this gap can be closed when focusing on specific classes of translations.

Keywords Abstract argumentation · Abstract dialectical frameworks · Conditional logics · Non-monotonic logics · Non-monotonic conditionals · Non-monotonic reasoning · Defeasible reasoning

Mathematics Subject Classification (2010) 68T27 · 68T30 · 68T37

1 Introduction

It is well-known that argumentation and nonmonotonic resp. default logics are closely connected: In [10] it is shown that Reiter's default logic can be implemented by abstract argumentation frameworks, a most basic form of computational model of argumentation

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to which many existing approaches to formal argumentation refer. On the other hand, it is clear that argumentation allows for nonmonotonic, defeasible reasoning, and in [30] computational models of argumentation are assessed by formal properties that have been adapted from nonmonotonic logics. Nevertheless, argumentation and nonmonotonic reasoning are perceived as two different fields which do not subsume each other, and indeed, often attempts to transform reasoning systems from one side into systems of the other side have been revealing gaps that could not be closed (cf., e.g., [15, 17, 21, 34]). While one might argue that this is due to the seemingly richer, dialectical structure of argumentation, in the end the evaluation of arguments often boils down to comparing arguments with their attackers, and comparing degrees of belief is a basic operation in qualitative nonmonotonic reasoning. Therefore, in spite of the abundance of existing work studying connections between the two fields, the true nature of the relationship between argumentation and nonmonotonic reasoning has not been fully understood.

We aim at deepening the understanding of the relationships between argumentation and nonmonotonic logics and establishing a theoretical basis for integrative approaches by focusing on most fundamental approaches on either side: *Abstract Dialectical Frameworks* (ADFs) [7] for argumentation, and *Conditional Logics*[1] (CL) [28, 31] for nonmonotonic logics. ADFs are an approach to formal argumentation, which subsumes many other argumentative formalisms in a generic, logic-based way. On the side of nonmonotonic logics, conditionals have been shown (and often used) to implement nonmonotonic inferences and provide expressive formalisms to represent knowledge bases; some of the most popular nonmonotonic inference systems (e. g., system Z [13]) make use of conditionals. Both ADFs and CL can be considered as high-level formalisms implementing properly the basic nature of the respective field without being restricted too much by subtleties of specific approaches, and both are based on 3-valued logics.

In this paper we investigate the correspondence between abstract dialectical frameworks and conditional logic. Syntactically, both frameworks focus on pairs of objects such as (ϕ, ψ). In conditional logic, these pairs are interpreted as conditionals with the informal meaning "if ϕ is true then, usually, ψ is true as well" and written as $(\psi|\phi)$. In abstract dialectical frameworks, these pairs are interpreted as acceptance conditions, and interpreted as "if ϕ is accepted then ψ is accepted as well". The resemblance of these informal interpretations is striking, but both approaches use fundamentally different semantics to formalise these interpretations. In this paper, we use this syntactical similarities as the basis of a comparison between abstract dialectical frameworks and conditional logics. In more detail, here we ask the question of whether, and how we can interpret abstract dialectical frameworks in terms of conditional logic so that acceptance in the argumentative system is defined by a nonmonotonic inference relation based on conditionals. We continue work from [22] by considering several translations of ADFs into conditional knowledge bases and applying conditional inference relations based on *ordinal conditional frameworks* [31], including the Z-inference relation [13], to these knowledge bases. We first show that there is a gap between argumentation and conditional semantics when applying several intuitive translations, but then define a class of translations that are *OCF-adequate* or *Z-adequate*—i. e.

[1] In this paper, we set out to compare argumentation with conditional logics for non-monotonic reasoning. There are many other logics for conditionals around, which formalise different kinds of monotonic conditional reasoning, see e.g. [2] for an overview. The conditionals we employ are a well-accepted formalism, which is, however, based on a rather restricted syntax when compared to monotonic conditional logics. For example, these conditionals for non-monotonic reasoning are not allowed to be nested or occur in complex formulas.

they preserve the semantics, see Section 3 for the formal definition—for the 2-valued model semantics of ADFs, and for other semantics under certain conditions on the ADFs. Furthermore, we show that none of the translations studied in this paper are Z-adequate for the grounded semantics and for the preferred and stable semantics in general. We furthermore show that our translations satisfy several desirable properties for translations between formalisms for knowledge representation.

The results in this paper are a substantially extended and revised version of the paper [15]. The main contributions beyond [15] are: (1) a new standard of evaluation, dubbed *OCF-adequacy* (Definition 6); (2) two additional translations Θ_6 and Θ_7 (cf. Section 3); (3) an in-depth study of the consistency of the translations (Section 4.2); (4) additional results on the Z- and OCF-adequacy of our translations w.r.t. all of the well-known ADF-semantics (Section 5 and 6).

Outline of this paper After stating all the necessary preliminaries in Section 2 on propositional logic (Section 2.1), nonmonotonic conditionals (Section 2.2) and abstract dialectical argumentation (Section 2.3). Then we investigate a family of translations from ADFs into conditional knowledge bases by first introducing the translations and defining OCF- and Z-adequacy for such translations (Section 3), after which we investigate the translations in-depth. In more detail, in Section 4 we study the OCF- and Z-adequacy of the translations under the two-valued model semantics by investigating Z-adequacy (Section 4.1), consistency of the translations (Section 4.2) and making some remarks on OCF-adequacy w.r.t. two-valued semantics (Section 4.3). Thereafter, we investigate the Z- and OCF-adequacy w.r.t. the stable and preferred semantics (Section 5) and w.r.t. the grounded semantics (Section 6). Several properties of the translations presented in this paper are discussed in Section 7. We finally discuss related work in Section 8 before concluding (Section 9).

2 Preliminaries

In the following, we we briefly recall some general preliminaries on propositional logic, as well as technical details on conditional logic and ADFs [7].

2.1 Propositional logic

For a set At of atoms let $\mathcal{L}(\mathsf{At})$ be the corresponding propositional language constructed using the usual connectives $\wedge$ (*and*), $\vee$ (*or*), $\neg$ (*negation*) and $\rightarrow$ (*material implication*). A (classical) *interpretation* (also called *possible world*) ω for a propositional language $\mathcal{L}(\mathsf{At})$ is a function $\omega : \mathsf{At} \rightarrow \{\mathsf{T}, \mathsf{F}\}$. Let $\Omega(\mathsf{At})$ denote the set of all interpretations for At. We simply write Ω if the set of atoms is implicitly given. An interpretation ω *satisfies* (or is a *model* of) an atom $a \in \mathsf{At}$, denoted by $\omega \models a$, if and only if $\omega(a) = \mathsf{T}$. The satisfaction relation $\models$ is extended to formulas as usual. As an abbreviation we sometimes identify an interpretation ω with its *complete conjunction*, i. e., if $a_1, \ldots, a_n \in \mathsf{At}$ are those atoms that are assigned T by ω and $a_{n+1}, \ldots, a_m \in \mathsf{At}$ are those propositions that are assigned F by ω we identify ω by $a_1 \ldots a_n \overline{a_{n+1}} \ldots \overline{a_m}$ (or any permutation of this). For example, the interpretation ω_1 on $\{a, b, c\}$ with $\omega_1(a) = \omega_1(c) = \mathsf{T}$ and $\omega_1(b) = \mathsf{F}$ is abbreviated by $a\overline{b}c$. For $\Phi \subseteq \mathcal{L}(\mathsf{At})$ we also define $\omega \models \Phi$ if and only if $\omega \models \phi$ for every $\phi \in \Phi$. Define the set of models $\mathsf{Mod}(X) = \{\omega \in \Omega(\mathsf{At}) \mid \omega \models X\}$ for every formula or set of formulas X. A formula or set of formulas X_1 *entails* another formula or set of formulas X_2, denoted by $X_1 \vdash X_2$, if $\mathsf{Mod}(X_1) \subseteq \mathsf{Mod}(X_2)$.

2.2 Reasoning with nonmonotonic conditionals

There are many different conditional logics (cf., e. g., [23, 28]), we will just use basic properties of conditionals that are common to many conditional logics and are especially important for nonmonotonic reasoning: Basically, we follow the approach of de Finetti ([11]) who considered conditionals as *generalized indicator functions* for possible worlds resp. propositional interpretations ω:

$$((\psi|\phi))(\omega) = \begin{cases} 1 & : \quad \omega \models \phi \wedge \psi \\ 0 & : \quad \omega \models \phi \wedge \neg\psi \\ u & : \quad \omega \models \neg\phi \end{cases} \tag{1}$$

where u stands for *unknown* or *indeterminate*. In other words, a possible world ω *verifies* a conditional $(\psi|\phi)$ iff it satisfies both antecedent ϕ and conclusion ψ ($(\psi|\phi)(\omega) = 1$); it *falsifies, or violates* it iff it satisfies the antecedence but not the conclusion ($(\psi|\phi)(\omega) = 0$); otherwise the conditional is *not applicable*, i. e., the interpretation does not satisfy the antecedence ($(\psi|\phi)(\omega) = u$). We say that ω *satisfies* a conditional $(\psi|\phi)$ iff it does not falsify it, i. e., iff ω satisfies its *material counterpart* $\phi \to \psi$. Hence, conditionals are three-valued logical entities and thus extend the binary setting of classical logics substantially in a way that is compatible with the probabilistic interpretation of conditionals as conditional probabilities. Such a conditional $(\psi|\phi)$ can be accepted as plausible if its verification $\phi \wedge \psi$ is more plausible than its falsification $\phi \wedge \neg\psi$, where plausibility is often modelled by a total preorder on possible worlds. This is in full compliance with nonmonotonic inference relations $\phi \mathrel{|\!\sim} \psi$ [27] expressing that from ϕ, ψ may be plausibly/defeasibly derived. An obvious implementation of total preorders are *ordinal conditional functions (OCFs)*, (also called *ranking functions*) $\kappa : \Omega \to \mathbb{N} \cup \{\infty\}$ [31]. They express degrees of (im)plausibility of possible worlds and propositional formulas ϕ by setting $\kappa(\phi) := \min\{\kappa(\omega) \mid \omega \models \phi\}$. OCFs κ provide a particularly convenient formal environment for nonmonotonic and conditional reasoning, allowing for simply expressing the acceptance of conditionals and nonmonotonic inferences via stating that $(\psi|\phi)$ is accepted by κ iff $\phi \mathrel{|\!\sim}_\kappa \psi$ iff $\kappa(\phi \wedge \psi) < \kappa(\phi \wedge \neg\psi)$, implementing formally the intuition of conditional acceptance based on plausibility mentioned above. For an OCF κ, $Bel\,(\kappa)$ denotes the propositional beliefs that are implied by all most plausible worlds, i. e. $Bel\,(\kappa) = \{\phi \mid \forall\omega \in \kappa^{-1}(0) : \omega \models \phi\}$. We denote with CL the framework of reasoning from conditional knowledge bases based on OCFs.

Specific examples of ranking models are system Z yielding the inference relation $\mathrel{|\!\sim}_Z$ [13] and c-representations [20]. We focus on system Z defined as follows. A conditional $(\psi|\phi)$ is tolerated by a finite set of conditionals Δ if there is a possible world ω with $(\psi|\phi)(\omega) = 1$ and $(\psi'|\phi')(\omega) \neq 0$ for all $(\psi'|\phi') \in \Delta$, i. e. ω verifies $(\psi|\phi)$ and does not falsify any (other) conditional in Δ. The Z-partitioning $(\Delta_0, \ldots, \Delta_n)$ of Δ is defined as:

- $\Delta_0 = \{\delta \in \Delta \mid \Delta \text{ tolerates } \delta\}$;
- $\Delta_1, \ldots, \Delta_n$ is the Z-partitioning of $\Delta \setminus \Delta_0$.

For $\delta \in \Delta$ we define: $Z_\Delta(\delta) = i$ iff $\delta \in \Delta_i$ and $(\Delta_0, \ldots, \Delta_n)$ is the Z-partioning of Δ. Finally, the ranking function κ^Z_Δ is defined via: $\kappa^Z_\Delta(\omega) = \max\{Z(\delta) \mid \delta(\omega) = 0, \delta \in \Delta\} + 1$, with $\max \emptyset = -1$. We can now define $\Delta \mathrel{|\!\sim}_Z \phi$ iff $\phi \in Bel\,(\kappa^Z_\Delta)$. Below the following results about system Z will prove useful:

Lemma 1 *$\omega \notin (\kappa^Z_\Delta)^{-1}(0)$ iff $\delta(\omega) = 0$ for some $\delta \in \Delta$.*

Proof This follows immediately in view of the fact that $\omega \in (\kappa^Z_\Delta)^{-1}(0)$ iff $\delta(\omega) \neq 0$ for every $\delta \in \Delta$. □

Theorem 1 ([13], Theorem 4) *Δ is consistent iff in every non-empty subset $\Delta' \subseteq \Delta$ there exists a rule tolerated by Δ'.*

Example 1 Let $\Delta = \{(b|\neg a), (a|\neg b), (c|\neg a \vee \neg b)\}$. For this set of conditionals, $\Delta = \Delta_0$ and therefore we have:

ω	κ^z_Δ	ω	κ^z_Δ	ω	κ^z_Δ	ω	κ^z_Δ
abc	0	$ab\overline{c}$	0	$a\overline{b}c$	0	$a\overline{b}\overline{c}$	1
$\overline{a}bc$	0	$\overline{a}b\overline{c}$	1	$\overline{a}\overline{b}c$	1	$\overline{a}\overline{b}\overline{c}$	1

Thus, $(\kappa^Z_\Delta)^{-1}(0) = \{abc, ab\overline{c}, a\overline{b}c, \overline{a}bc\}$. This means that, for example, $\Delta \mathrel{|\!\sim}_Z a \vee b$ and $\Delta \not\mathrel{|\!\sim}_Z c$.

2.3 Abstract dialectical frameworks

We now recall some technical details on ADFs following loosely the notation from [7]. We can depict an ADF D as a directed graph whose nodes represent statements or arguments which can be accepted or not. With links we represent dependencies between nodes. A node s is dependant on the status of the nodes with a direct link to s, denoted parent nodes $par_D(s)$. With an acceptance function C_s we define the cases when the statement s can be accepted (truth value $\top$), depending on the acceptance status of its parents in D.

An ADF D is a tuple $D = (S, L, C)$ where S is a set of statements, $L \subseteq S \times S$ is a set of links, and $C = \{C_s\}_{s \in S}$ is a set of total functions $C_s : 2^{par_D(s)} \to \{\top, \bot\}$ for each $s \in S$ with $par_D(s) = \{s' \in S \mid (s', s) \in L\}$. By abuse of notation, we will often identify an acceptance function C_s by its equivalent *acceptance condition* which models the acceptable cases as a propositional formula.

Example 2 We consider the following ADF $D_1 = (\{a, b, c\}, L, C)$ with
$L = \{(a, b), (b, a), (a, c), (b, c)\}$; $C_a = \neg b$; $C_b = \neg a$; $C_c = \neg a \vee \neg b$. The corresponding graph for D_1 can be found in Fig. 1.

Informally, the acceptance conditions can be read as "a is accepted if b is not accepted", "b is accepted if a is not accepted" and "c is accepted if a is not accepted or b is not accepted".

An ADF $D = (S, L, C)$ is interpreted through 3-valued interpretations $v : S \to \{\top, \bot, u\}$, which assign to each statement in S either the value $\top$ (true, accepted), $\bot$

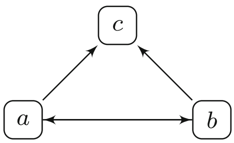

Fig. 1 Graph representing links between nodes for D_1 in Example 2

(false, rejected), or u (unknown). A 3-valued interpretation v can be extended to arbitrary propositional formulas over S via strong Kleene semantics:

1. $v(\neg\phi) = \bot$ iff $v(\phi) = \top$, $v(\neg\phi) = \top$ iff $v(\phi) = \bot$, and $v(\neg\phi) = u$ iff $v(\phi) = u$;
2. $v(\phi \wedge \psi) = \top$ iff $v(\phi) = c(\psi) = \top$, $v(\phi \wedge \psi) = \bot$ iff $v(\phi) = \bot$ or $v(\psi) = \bot$, and $v(\phi \wedge \psi) = u$ otherwise;
3. $v(\phi \vee \psi) = \top$ iff $v(\phi) = \top$ or $v(\psi) = \top$, $v(\phi \vee \psi) = \bot$ iff $v(\phi) = c(\psi) = \bot$, and $v(\phi \vee \psi) = u$ otherwise.

$\mathcal{V}$ consists of all three-valued interpretations whereas $\mathcal{V}^2$ consists of all the two-valued interpretations (i.e. interpretations such that for every $s \in S$, $v(s) \in \{\top, \bot\}$). Then v is a *model* of D if for all $s \in S$, if $v(s) \neq u$ then $v(s) = v(C_s)$. We define the information order $\leq_i$ over $\{\top, \bot, u\}$ by making u the minimal element: $u <_i \top$ and $u <_i \bot$ and this order is lifted pointwise as follows (given two valuations v, w over S): $v \leq_i w$ iff $v(s) \leq_i w(s)$ for every $s \in S$. The truth ordering $\leq_t$ over $\{\top, \bot, u\}$ is defined as $\bot \leq_t u \leq_t \top$ and is lifted to interpretations similarly. The set of two-valued interpretations extending a valuation v is defined as $[v]^2 = \{w \in \mathcal{V}^2 \mid v \leq_i w\}$. Given a set of valuations V, $\sqcap_i V(s) = v(s)$ if for every $v' \in V$, $v(s) = v'(s)$ and $\sqcap_i V(s) = u$ otherwise. $\Gamma_D(v) : S \to \{\top, \bot, u\}$ where $s \mapsto \sqcap_i\{w(C_s) \mid w \in [v]^2\}$.

Definition 1 Let $D = (S, L, C)$ be an ADF with $v : S \to \{\top, \bot, u\}$ an interpretation:

- v is a 2-valued model iff $v \in \mathcal{V}^2$ and v is a model.
- v is complete for D iff $v = \Gamma_D(v)$.
- v is preferred for D iff v is $\leq_i$-maximally complete.
- v is grounded for D iff v is $\leq_i$-minimally complete.[2]

We denote by $\mathsf{2mod}(D)$, $\mathsf{complete}(D)$, $\mathsf{preferred}(D)$, $\mathsf{grounded}(D)$ respectively $\mathsf{stable}(D)$ the sets of 2-valued models and complete, preferred, grounded respectively stable interpretations of D. We will sometimes denote the grounded interpretation by v_G.

We finally define consequence relations for ADFs:

Definition 2 Given $\mathsf{sem} \in \{\mathsf{2mod}, \mathsf{preferred}, \mathsf{grounded}, \mathsf{stable}\}$, an ADF $D = (S, L, C)$ and $s \in S$ we define: $D \mid\!\sim^{\cap}_{\mathsf{sem}} s[\neg s]$ iff $v(s) = \top[\bot]$ for all $v \in \mathsf{sem}(D)$.[3]

Example 3 (Example 2 continued) The ADF of Example 2 has three complete models v_1, v_2, v_3 with:
$v_1(a) = \top$ $v_1(b) = \bot$ $v_1(c) = \top$
$v_2(a) = \bot$ $v_2(b) = \top$ $v_2(c) = \top$
$v_3(a) = u$ $v_3(b) = u$ $v_3(c) = u$
v_3 is the grounded interpretation whereas v_1 and v_2 are both preferred, stable and 2-valued.

We recall the following relationships between the semantics defined above:

[2] We notice that [7] showed the grounded extension to be unique for any ADF.
[3] Since [7] showed the grounded extension to be unique for any ADF, we will omit $\cap$ from $\mid\!\sim_{\mathsf{grounded}}$.

Theorem 2 ([7]) *Given any ADF D, the following relationships hold:*

- *stable*$(D) \subseteq$ *2mod*(D);
- *2mod*$(D) \subseteq$ *preferred*(D);
- *preferred*$(D) \subseteq$ *complete*(D);
- *grounded*$(D) \subseteq$ *complete*(D).

Below we will make use of the following *semantic ADF-subclasses*:

Definition 3 ([9]) An ADF D is called:

- *weakly coherent* if 2mod$(D) \subseteq$ stable(D);
- *coherent* if preferred$(D) \subseteq$ stable(D);
- *semi-coherent* if preferred$(D) \subseteq$ 2mod(D).

Notice that any coherent ADF is also semi-coherent in view of Theorem 2 and transitivity of $\subseteq$.

We furthermore recall some *syntactic ADF-subclasses*. We first have to distinguish between different kinds of links:

Definition 4 ([9]) Given an ADF $D = (S, L, C)$:

- the update of an interpretation v with a truth value $x \in \{\top, \bot\}$ for a node $b \in S$ $v|^b_x$ is defined as:

$$\begin{cases} v|^b_x(a) = v(a) & \text{if } a \neq b \\ v|^b_x(b) = x & \text{otherwise} \end{cases}$$

- a link $(b, a) \in L$ is called:
  1. *supporting* (in D) if for every $v \in \mathcal{V}^2$, $v(C_a) = \top$ implies $v|^b_\top(C_a) = \top$.
  2. *attacking* (in D) if for every $v \in \mathcal{V}^2$, $v(C_a) = \bot$ implies $v|^b_\top(C_a) = \bot$.
  3. *redundant* (in D) if it is both attacking and supporting.
  4. *dependent* (in D) it if is neither attacking nor supporting.

The set of supporting, respectively attacking links of an ADF $D = (S, L, C)$ will be denoted by L^+, respectively L^-.

Example 4 Let $D = (\{a, b, c, d\}, L, C)$ with:

$$C_a = a \wedge \neg a; \ C_b = a; \ C_c = \neg a; \ C_d = (\neg a \wedge \neg b) \vee (a \wedge b)$$

The corresponding graph can be found in Fig. 2.

(a, b) is a supporting link, (a, c) is an attacking link, (a, a) is a redundant and (a, d) is dependent.

We can now define the following syntactic subclasses of ADFs:

Definition 5 ([9]) An ADF $D = (S, L, C)$ is called:

- *acyclic* (in short, AADF) if its corresponding directed graph (S, L) is acyclic.
- *symmetric* if its corresponding directed graph (S, L) is irreflexive and symmetric and L does not contain any redundant links.

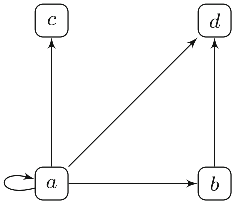

Fig. 2 Graph representing links between nodes for D in Example 4

- *bipolar* if every link in L is either attacking, supporting or redundant.
- *support free* (in short, SFADF) if it is bipolar and does not have any supporting links.
- *support free symmetric* (in short, SFSADF) if it is symmetric and does not have any supporting links.
- *acyclic support symmetric* (in short, ASSADF) if it is symmetric, bipolar, and (S, L^+) is acyclic.

We will furthermore denote the class of SFADFs that do not contain any odd-length cycles as $\text{ASSADF}^{\overline{\text{OLC}}}$s.

The following results on syntactic subclasses of ADFs will prove useful below:

Theorem 3 ([9]) – *For any acyclic ADF D, 2mod*$(D) =$ *grounded*(D) *(*[9, Theorem 2]*).*
- *The class of SFADFs that do not contain any odd-length cycle is coherent (*[9, Corollary 20]*).*
- *The class of SFSADFs is weakly coherent (*[9, Theorem 8]*).*
- *The class of ASSADFs is weakly coherent (*[9, Theorem 5]*).*

3 Translations from ADFs to conditional logics

The general aim of this paper is to study translations of ADFs in CL. In more detail, where S is a set of atoms and $\mathfrak{D}_S$ is the set of all ADFs defined on the basis of S (i.e. all ADFs $D = (S, L, C)$), and $(\mathcal{L}(S)|\mathcal{L}(S))$ is the set of all condtionals over the propositional language generated by S, we investigate mappings $\mathfrak{T} : \mathfrak{D}_S \rightarrow \wp((\mathcal{L}(S)|\mathcal{L}(S)))$ (for arbitrary S).

Definition 6 Let S be a set of atoms and $\mathfrak{T} : \mathfrak{D}_S \rightarrow \wp((\mathcal{L}(S)|\mathcal{L}(S)))$ be a translation from ADFs to conditional knowledge bases. $\mathfrak{T}$ is:

- *OCF-adequate with respect to semantics sem* if: for every $D = (S, L, C)$ there is some κ s.t. (1) $\kappa \models \mathfrak{T}(D)$ and (2) for every $s \in S$, $D \mid\!\sim^{\cap}_{\text{sem}} s$ iff $s \in Bel(\kappa)$.
- *Z-adequate with respect to semantics sem* if: for every $D = (S, L, C)$ and every $s \in S$ it holds that: $D \mid\!\sim^{\cap}_{\text{sem}} s$ iff $\mathfrak{T}(D) \mid\!\sim_Z s$.

We notice that, given some semantics sem, any translation $\mathfrak{T}$ that is Z-adequate (w.r.t. sem) is also OCF-adequate (w.r.t. sem). The other direction, however, does not necessarily hold.

There is a whole family of translations from ADFs to conditional logics which are *prima facie* apt to express the links between nodes s and their acceptance conditions C_s:

- $\Theta_1(D) = \{(s|C_s) \mid s \in S\}$
- $\Theta_2(D) = \{(C_s|s) \mid s \in S\}$
- $\Theta_3(D) = \Theta_1(D) \cup \Theta_2(D)$
- $\Theta_4(D) = \Theta_1(D) \cup \{(\neg s|\neg C_s) \mid s \in S\}$
- $\Theta_5(D) = \{((C_s \equiv s)|\top) \mid s \in S\}$
- $\Theta_6(D) = \Theta_2(D) \cup \{(\neg C_s|\neg s) \mid s \in S\}$.
- $\Theta_7(D) = \{(\neg s|\neg C_s) \mid s \in S\} \cup \{(\neg C_s|\neg s) \mid s \in S\}$.

Notice that all of these translations are based on the idea that there is a strong connection between the acceptance of an acceptance condition C_s and the acceptance of the corresponding node s. Indeed, as [7] puts it: "each node s has an associated acceptance condition C_s specifying the exact conditions under which s is accepted". However, in this formulation, it is not specified (1) when a formula is true according to a three-valued interpretation (i.e. is $a \vee \neg a$ true according to an interpretation v with $v(a) = u$? Different three-valued logics give different answers to this question), (2) what to accept when there are conflicts between different acceptance conditions (e.g. if $C_a = \neg b$ and $C_b = \neg a$) and (3) under which conditions we are justified in rejecting a node. Therefore, we systematically investigate different forms of conditionals based on the common idea that "the influence a node may have on another node is entirely specified through the acceptance condition" [7].

We now explain in more detail every translation. Θ_1 formalizes the intuition that whenever the condition of a node s is believed, normally, s should be believed as well. Likewise, Θ_2 formalizes the idea that if a node is believed, its condition should be believed as well. Θ_3 combines the two aforementioned intuitions. Θ_4 is a slight variation on this idea, combining Θ_1 with the constraint that whenever the negation of a condition of a node is believed, the negation of the node itself should be believed as well. Θ_5 postulates that a node should be equivalent to its condition. Θ_6, formalizes the following intuition: if s is believed, C_s has to be believed, and if $\neg s$ is believed, $\neg C_s$ has to be believed as well. Finally, Θ_7 is a formalization of the idea that whenever the negation of a node, respectively the negation of the condition of a node is believed, the negation of the condition of the node, respectively the negation of the node should be believed. Note that Θ_1 has already been investigated to some small extent in [22]. In the following sections, we will study the Z-adequacy and OCF-adequacy of these translations in depth. In Table 1 these results are summarized.

There are, of course, many more translations possible, for example one could suggest instead of $\Theta_1(D)$ the following $\Theta'_1(D) = \{(C_s \to s|\top) \mid s \in S\}$. However, "shifting" the conditional to the right hand side does not impact the consequences of a translation under system Z:

Proposition 1 *Given a set of conditionals* Δ, $\Delta \cup \{(\psi|\phi)\} \mathrel{|\!\!\sim}_Z \theta$ *iff* $\Delta \cup \{(\phi \to \psi|\top)\} \mathrel{|\!\!\sim}_Z \theta$.

Proof Suppose Δ is a set of conditionals. In what follows, we will denote $\kappa^Z_{\Delta\cup\{(\psi|\phi)\}}$ by κ and $\kappa^Z_{\Delta\cup\{(\phi\to\psi|\top)\}}$ by κ'. We show that $\kappa^{-1}(0) = \kappa'^{-1}(0)$, which implies the proposition. For this, suppose towards a contradiction that $\omega \in \kappa'^{-1}(0)$ yet $\omega \notin \kappa^{-1}(0)$. By Lemma 1 this means that there is some $(\lambda|\delta) \in \Delta \cup \{(\psi|\phi)\}$ s.t. $(\lambda|\delta)(\omega) = 0$.Since κ' accepts Δ, $(\lambda|\delta) =$

Table 1 Schematic summary of the results on Z-adequacy and OCF-adequacy of the translations in this paper

Z-adequacy				
i	2mod	Preferred	Stable	Grounded
1	×	×	×	×
2	×	×	×	×
3	✓	× (✓[ASSADFs ∪ SFSADFs])	× (✓[ASSADF$^{\text{OLC}}$s])	× (✓[AADFs])
4	✓	× (✓[ASSADFs ∪ SFSADFs])	× (✓[ASSADF$^{\text{OLC}}$s])	× (✓[AADFs])
5	✓	× (✓[ASSADFs ∪ SFSADFs])	× (✓[ASSADF$^{\text{OLC}}$s])	× (✓[AADFs])
6	✓	× (✓[ASSADFs ∪ SFSADFs])	× (✓[ASSADF$^{\text{OLC}}$s])	× (✓[AADFs])
7	✓	× (✓[ASSADFs ∪ SFSADFs])	× (✓[ASSADF$^{\text{OLC}}$s])	× (✓[AADFs])
OCF-adequacy				
i	2mod	Preferred	Stable	Grounded
1	✓	×	×	×
2	✓	×	×	×
3	✓	× (✓[ASSADFs ∪ SFSADFs])	× (✓[ASSADF$^{\text{OLC}}$s])	× (✓[AADFs])
4	✓	× (✓[ASSADFs ∪ SFSADFs])	× (✓[ASSADF$^{\text{OLC}}$s])	× (✓[AADFs])
5	✓	× (✓[ASSADFs ∪ SFSADFs])	× (✓[ASSADF$^{\text{OLC}}$s])	× (✓[AADFs])
6	✓	× (✓[ASSADFs ∪ SFSADFs])	× (✓[ASSADF$^{\text{OLC}}$s])	× (✓[AADFs])
7	✓	× (✓[ASSADFs ∪ SFSADFs])	× (✓[ASSADF$^{\text{OLC}}$s])	× (✓[AADFs])

✓ means that the selected form of adequacy w.r.t. the semantics of the respective column is satisfied for the translation in the respective row. × means that the translation in the respective row is Z- resp. OCF-inadequate w.r.t. the semantics in the respective column. ×(✓[xADF]), finally, means that the translation in the respective row is Z- resp. OCF-inadequate w.r.t. the semantics in the respective column in general, but is Z- resp. OCF-adequate w.r.t. the semantics for the class of xADFs in square brackets

$(\psi|\phi)$. Thus, $\omega \models \phi \wedge \neg\psi$. This means that $\omega \models \top \wedge \neg(\phi \rightarrow \psi)$, i.e. $(\phi \rightarrow \psi|\top)(\omega) = 0$. This contradicts $\kappa'(\omega) = 0$ and the assumption that κ' accepts $\Delta \cup \{(\phi \rightarrow \psi|\top)\}$ and thus we have shown that $\kappa'^{-1}(0) \subseteq \kappa^{-1}(0)$. Analogously, we can show that $\kappa'^{-1}(0) \supseteq \kappa^{-1}(0)$ and thus $\kappa'^{-1}(0) = \kappa^{-1}(0)$. This implies $Bel(\kappa) = Bel(\kappa')$ □

The above proposition thus establishes that within our perspective, it does not matter if we consider the conditional "ψ is plausible if ϕ is the case" or the conditional "$\phi \rightarrow \psi$ is plausible". Notice that this does not imply that we can equivalently consider $\phi \rightarrow \psi$ to be *true*. However, the above proposition does not generalize for arbitrary κ, i.e. there might be an OCF κ that accept $(\psi \rightarrow \phi|\top)$ but not $(\phi|\psi)$:

Example 5 Consider an OCF over the signature $\{p, q\}$ with:

ω	pq	$\overline{p}q$	$q\overline{p}$	$\overline{p}\,\overline{q}$
κ	1	1	1	0

Notice that $\kappa(\top \wedge (q \rightarrow p)) = \kappa(\neg q \vee p) = 0 < \kappa(\top \wedge \neg(q \rightarrow p)) = \kappa(q \wedge \neg p) = 1$, whereas $\kappa(q \wedge p) = 1 = \kappa(q \wedge \neg p)$. Therefore, κ accepts $(q \rightarrow p|\top)$ but not $(p|q)$.

Remark 1 We have implemented a reasoner in java by use of the TweetyProject[4] library which calculates the translations $\Theta_1, \ldots, \Theta_7$ and compares $\mid\!\sim_Z$-inference of these translations with inferences of the translated ADF under the grounded, preferred, stable and two-valued model-semantics.

4 Two-valued semantics

In this section we discuss the adequacy of our translations w.r.t. the 2mod-semantics. We will show the Z-inadequacy of the translations Θ_1 and Θ_2 w.r.t. the 2mod-semantics and the Z-adequacy of $\Theta_3,\ldots,\Theta_7$ w.r.t. the 2mod-semantics in Section 4.1. The results in Section 4.2 establish conditions for the consistency of these translations. In Section 4.3, we finally make some observations on the OCF-adequacy of the translations $\Theta_3,\ldots,\Theta_7$.

4.1 Z-adequacy w.r.t. two-valued semantics

In this section we study Z-adequacy with respect to the 2mod-semantics for the translations suggested in the previous section. In particular, we will show that Θ_1 and Θ_2 are not Z-adequate whereas Θ_3, Θ_4, Θ_5, Θ_6 and Θ_7 are in fact Z-adequate for the 2mod-semantics.

We first observe that Θ_1 and Θ_2 are *not* Z-adequate w.r.t. two-valued semantics.

Example 6 (Z-Inadequacy of Θ_1 w.r.t. 2mod) We consider the following ADF D_1 from Example 2. Notice that $\Theta_1(D_1) = \{(b|\neg a), (a|\neg b), (c|\neg a \vee \neg b)\}$, which is the conditional knowledge base considered in Example 1. We therefore see that $\Theta_1(D_1) \not\mid\!\sim_Z c$ even though $D \mid\!\sim^{\cap}_{\mathsf{2mod}} c$ and thus Θ_1 is not Z-adequate with respect to the 2mod-semantics.

Example 7 (Z-Inadequacy of Θ_2 w.r.t. 2mod) We consider the following ADF $D_2 = (\{a, b, c\}, L, C)$ where: $C_a = \neg b \quad C_b = \neg a \quad C_c = a \vee b$ D_2 has three complete models v_1, v_2, v_3 with: $v_1(a) = v_2(b) = v_1(c) = v_2(c) = \top$, $v_1(b) = v_2(a) = \bot$ and $v_3(a) = v_3(b) = v_3(c) = u$. Only v_1 and v_2 are 2-valued.

Moving to $\Theta_2(D) = \{(\neg a|b), (\neg b|a), (a \vee b|c)\}$, we see that $(\kappa^Z_{\Theta_2(D)})^{-1}(0) = \{a\overline{b}c, a\overline{b}\overline{c}, \overline{a}bc, \overline{a}b\overline{c}, \overline{a}\overline{b}\overline{c}\}$. This means that $\Theta_2(D_2) \not\mid\!\sim_Z c$ even though $D \mid\!\sim^{\cap}_{\mathsf{2mod}} c$, i.e. Θ_2 is not Z-adequate with respect to the 2mod-semantics.

We will now show that the translations Θ_3, Θ_4, Θ_5, Θ_6 and Θ_7 are Z-adequate for 2-valued models. For these results, the following conditions on translations will prove useful:

- **C1**: $\kappa^Z_{\Theta(D)}(C_s \wedge \neg s) > 0$ and $\kappa^Z_{\Theta(D)}(\neg C_s \wedge s) > 0$ for every $s \in S$.
- **C2**: $\{\bigwedge_{s \in S} C_s \equiv s\} \vdash \bigwedge_{(\psi|\phi) \in \Theta(D)} (\phi \to \psi)$

Θ_3, Θ_4, Θ_5, Θ_6 and Θ_7 satisfy both of the above conditions:

Proposition 2 *For any $i \in \{3, 4, 5, 6, 7\}$ and any ADF D, if $\Theta_i(D)$ is consistent then $\Theta_i(D)$ satisfies* ***C1*** *and* ***C2****.*

[4] http://tweetyproject.org/index.html

Proof We show the claim for $i = 3$ and **C1**, the proofs for $i \in \{4, 5, 6, 7\}$ and **C2** are analogous. Suppose towards a contradiction that there is some ADF $D = (S, L, D)$ and some $s \in S$ s.t. $\kappa^Z_{\Theta_3(D)}(C_s \wedge \neg s) = 0$ or $\kappa^Z_{\Theta_3(D)}(\neg C_s \wedge s) = 0$. Suppose the former. Then $\kappa^Z_{\Theta_3(D)}(C_s \wedge \neg s) \geq \kappa^Z_{\Theta_3(D)}(C_s \wedge s)$, which contradicts $(s|C_s) \in \Theta_3(D)$. Likewise, $\kappa^Z_{\Theta_3(D)}(\neg C_s \wedge s) = 0$ contradicts $(C_s|s) \in \Theta_3(D)$. □

Proposition 3 *For any Θ that satisfies **C1** for the ADF D, $\omega \in (\kappa^Z_{\Theta(D)})^{-1}(0)$ implies $\omega \in$ 2mod(D).*

Proof Suppose that $\Theta(D)$ satisfies **C1** for the ADF $D = (S, L, C)$ and that $\omega \in (\kappa^Z_{\Theta(D)})^{-1}(0)$. We show that ω is a two-valued model of D. Indeed suppose towards a contradiction that $\omega(s) \neq \omega(C_s)$ for some $s \in S$. This means that $\omega \models s \wedge \neg C_s$ or $\omega \models \neg s \wedge C_s$. Since $\omega \in (\kappa^Z_{\Theta(D)})^{-1}(0)$, this contradicts $\Theta(D)$ satisfying **C1** for D. Thus, it has to be the case that ω is a model of D. That $\omega \in \mathcal{V}^2$ is clear from the fact that $\omega \models s \vee \neg s$ for every $s \in S$. □

Proposition 4 *For any Θ that satisfies **C2** for the ADF D, $\omega \in (\kappa^Z_{\Theta(D)})^{-1}(0)$ if $\omega \in$ 2mod(D).*

Proof Suppose that $\Theta(D)$ satisfies **C2** for the ADF D, and suppose that ω is a 2-valued model of D. Suppose towards a contradiction that $\omega \notin (\kappa^Z_{\Theta(D)})^{-1}(0)$. By Lemma 1 this means that $\omega \models \phi' \wedge \neg\psi'$ for some $(\psi'|\phi') \in \Theta(D)$. But then since $\{\bigwedge_{s\in S} C_s \equiv s\} \vdash \bigwedge_{(\psi|\phi)\in\Theta(D)}(\phi \rightarrow \psi)$, by contraposition, and since $\{\phi' \wedge \neg\psi'\} \vdash \neg(\bigwedge_{(\psi|\phi)\in\Theta(D)}(\phi \rightarrow \psi))$, $\omega \models \neg \bigwedge_{s\in S} C_s \equiv s$. But then there is some $s \in S$ s.t. $\omega \models s \wedge \neg C_s$ or $\omega \models \neg s \wedge C_s$. But then $\omega(s) \neq \omega(C_s)$, contradiction to ω being a 2-valued model of D. □

We can now derive the Z-adequacy with respect to the 2-valued model semantics for the translations Θ_3, Θ_4, Θ_5, Θ_6 and Θ_7, under the condition that $\Theta_i(D)$ is consistent:

Theorem 4 *For any ADF D, and $i \in \{3, 4, 5, 6, 7\}$: if $\Theta_i(D)$ is consistent then $D \mathrel{|\!\!\sim}^{\cap}_{\mathsf{2mod}} s[\neg s]$ iff $\Theta_i(D) \mathrel{|\!\!\sim}_Z s[\neg s]$ for any $s \in S$.*

Proof Let $i \in \{3, 4, 5, 6, 7\}$ and D be an ADF. By definition, $D \mathrel{|\!\!\sim}^{\cap}_{\mathsf{2mod}} s[\neg s]$ iff for every model $v \in \mathcal{V}^2$, $v(s) = \top[\bot]$. By Propositions 2, 3 and 4, $(\kappa^Z_{\Theta_i(D)})^{-1}(0) = \{\omega_v \mid v \in \mathcal{V}^2, v \in \mathsf{2mod}(D)\}$. Thus, $D \mathrel{|\!\!\sim}^{\cap}_{\mathsf{2mod}} s[\neg s]$ iff for every $\omega \in (\kappa^Z_{\Theta_i(D)})^{-1}(0)$, $\omega \models s[\neg s]$, which implies: $D \mathrel{|\!\!\sim}^{\cap}_{\mathsf{2mod}} s[\neg s]$ iff $\Theta_i(D) \mathrel{|\!\!\sim}_Z s[\neg s]$. □

4.2 Consistency of translations

We now discuss the requirement in Theorem 4 of $\Theta_i(D)$ being consistent.[5] We first show that there might be ADFs D for which $\Theta_i(D)$ is inconsistent (for $i \in \{3, 4, 6\}$):

[5] In [15] it was claimed that if Θ_i is inconsistent, there will be no two-valued models. However, the claim that $(C_s \wedge s)$ being unsatisfiable leads to there being no two-valued model of D is false, since as Example 8 shows, we can still have a two-valued model for which $\omega \models \neg C_s \wedge \neg s$.

Example 8 Let $D = (\{a, b\}, L, C)$ with $C_a = \neg a \wedge \neg b$ and $C_b = b$. Notice that $\mathsf{2mod}(D) = \{\overline{a}b\}$. Then we have $\Theta_3(D) = \{(a|\neg a \wedge \neg b), (\neg a \wedge \neg b|a), (b|b)\}$. For the first two conditionals, there is no κ that accepts these conditionals, since this would mean that $\kappa(a \wedge \neg a \wedge \neg b) < \kappa(\neg a \wedge \neg b)$ respectively $\kappa(a \wedge \neg a \wedge \neg b) < \kappa(a \wedge (a \vee b))$. It can easily be seen that also for $i \in \{4, 6\}$, there is no κ that accepts $\Theta_i(D)$.

We now show that also for Θ_7, there might be ADFs D for which the translation is inconsistent:

Example 9 Let $D = (\{a\}, L, C)$ with $C_a = \top$. Notice that $\mathsf{2mod}(D) = \{a\}$. We have $\Theta_7(D) = \{(\neg a|\bot), (\bot|\neg a)\}$. There is no κ that accepts $(\bot|\neg a)$ since this would mean that $\kappa(\bot \wedge \neg a) < \kappa(\top \wedge \neg a)$.

Observe that $\Theta_5(D)$ *is* consistent for D as in Example 9. In fact, we can show the following proposition, which not only establishes consistency of $\Theta_5(D)$ whenever $\mathsf{2mod}(D) \neq \emptyset$, but also ascertains that consistency of $\Theta_5(D)$ guarantees $\mathsf{2mod}(D) \neq \emptyset$:

Theorem 5 *Given an ADF $D = (S, L, C)$, $\mathsf{2mod}(D) \neq \emptyset$ iff $\Theta_5(D)$ is consistent.*

Proof We first show that if $\mathsf{2mod}(D) \neq \emptyset$ then $\Theta_5(D)$ is consistent. We show this by constructing a κ that accepts $\Theta_5(D)$. Take some $\omega \in \mathsf{2mod}(D)$. Since ω is a two-valued model of D, for every $s \in S$, either $\omega \models s \wedge C_s$ or $\omega \models \neg s \wedge \neg C_s$, which implies †: $\omega \models s \equiv c_s$ for every $s \in S$. We construct κ by setting $\kappa(\omega) = 0$ and $\kappa(\omega') = 1$ for any $\omega' \in \Omega(S) \setminus \omega$. Since $\omega \models s \equiv C_s$ for any $s \in S$, $\kappa(\top \wedge (s \equiv C_s)) = 0$. Since $\kappa(\omega') = 1$ for any $\omega' \in \Omega(S) \setminus \{\omega\}$, we know that for any $\omega' \in \Omega(S)$ s.t. $\omega' \models \neg(s \equiv C_s)$, $\kappa(\omega') = 1$. Thus, $\kappa(\top \wedge \neg(s \equiv C_s)) = 1$, which implies $\kappa(\top \wedge (s \equiv C_s)) < \kappa(\top \wedge \neg(s \equiv C_s))$. Thus, κ accepts $(s \equiv C_s|\top)$ for any $s \in S$ which implies that κ accepts $\Theta_5(D)$.

We now show that $\mathsf{2mod}(D) \neq \emptyset$ if $\Theta_5(D)$ is consistent. Indeed, if $\Theta_5(D)$ is consistent, there is an OCF κ s.t. κ accepts $\Theta_5(D)$. By definition of an OCF, there is an $\omega \in \Omega(S)$ s.t. $\kappa(\omega) = 0$, i.e. $(s \equiv C_s|\top)(\omega) \neq 0$ for any $s \in S$. Thus, $\omega \models s \equiv C_s$ for any $s \in S$. But then $\omega \in \mathsf{2mod}(D)$. □

We can now show the Z-adequacy of Θ_5 w.r.t. two-valued model semantics (without having to assume the consistency of Θ_5):

Corollary 1 *For any ADF D, $D \mid\!\sim^{\cap}_{\mathsf{2mod}} s[\neg s]$ iff $\Theta_i(D) \mid\!\sim_Z s[\neg s]$ for any $s \in S$.*

Proof Follows from Theorem 4 and Theorem 5. □

Remark 2 It is perhaps interesting to notice that we can obtain a translation $\Theta'(D)$ closer to the translations Θ_i for $i \in \{3, 4, 6, 7\}$ which is consistent. Indeed, we can do this by using our "shifting" procedure (see e.g. Proposition 1). For the conditional $(a|\neg a \wedge \neg b)$ in example above, we do this as follows:

$$\begin{array}{rl} (a|\neg a \wedge \neg b) & \\ ((\neg a \wedge \neg b) \to a|\top) & \text{with "shifting down"} \\ (\neg a \to (a \vee b)|\top) & \text{with contraposition} \\ (a \vee b|\neg a) & \text{with "shifting up"} \end{array}$$

Likewise, we can transform $(\neg a \wedge \neg b|a)$ into $(\neg a|a \vee b)$, and we obtain $\Theta'(D) = \{(a \vee b|\neg a), (\neg a|a \vee b), (b|b)\}$. We easily observe that $(\kappa^Z_{\Theta'(D)})^{-1}(0) = \{\overline{a}b\}$, which gives use Z-adequacy w.r.t. the two-valued semantics for Δ.

One could ask now, for the translations Θ_3, Θ_4, Θ_6 and Θ_7, whether there are conditions under which they are consistent. One conjecture could be that ADFs without self-attacking nodes make Θ_3 consistent.

Definition 7 An ADF $D = (S, L, C)$ *contains no self-attacking nodes* iff for no $s \in S$, $C_s \vdash \neg s$.

We now give an example of an ADF without self-attacking nodes for which Θ_3 is inconsistent. For this it is convenient to define an *exclusive disjunction* $\phi \overline{\vee} \psi := (\phi \vee \psi) \wedge \neg(\psi \wedge \phi)$.

Example 10 Let $D = (\{a, b, c\}, L, C)$ with $C_a = C_b = a\overline{\vee}b$ and $C_c = \top$. Then the unique two-valued model of D is v with $v(a) = \bot$, $v(b) = \bot$ and $v(c) = \top$. We have $\Theta_3(D) = \big\{(a|a\overline{\vee}b),\ (b|a\overline{\vee}b),\ (c|\top),\ (a\overline{\vee}b|a),\ (a\overline{\vee}b|b),\ (\top|c)\big\}$ We now show that there is no κ s.t. $\kappa \models \Theta_3(D)$. For this, with Theorem 4, it suffices to there is a $\Delta \subseteq \Theta_3(D)$ s.t. no $\delta \in \Delta$ is tolerated in Δ. Setting $\Delta = \{(a|a\overline{\vee}b), (b|a\overline{\vee}b)\}$, we see that there is no ω s.t. $\omega \models a\overline{\vee}b \wedge a$ and $\omega \models (a\overline{\vee}b) \rightarrow b$. For reasons of symmetry, it is also clear that Δ does not tolerate $(b|a\overline{\vee}b)$ either.

The node c in the above example is meant to take away the presumption that the inconsistency of $\Theta_3(D)$ is caused by there being no two-valued model for which some node is validated (i.e. there existing no $\omega \in \mathsf{2mod}(D)$ s.t. for some $s \in S$, $\omega(s) = \top$). Observe that for $D = (\{a, b\}, L, C)$ with $C_a = C_b = a\overline{\vee}b$, this would be the case.

Definition 8 An ADF $D = (S, L, C)$ is:

- *non-refuting* if there is no $s \in S$ s.t. $\sqcap_i \mathsf{2mod}(s) = \bot$.
- *non-validating* if there is no $s \in S$ s.t. $\sqcap_i \mathsf{2mod}(s) = \top$.

Theorem 6 *Given an ADF D, $\Theta_3(D)$ is consistent if D is non-refuting.*

Proof Suppose that D is non-refuting, i.e. for every $s \in S$, $\sqcap_i \mathsf{2mod}(D)(s) \in \{u, \top\}$. This means that (†): for every $s \in S$, there is an $\omega \in \mathsf{2mod}(D)$ s.t. $\omega(s) = \top$. We show now that for every non-empty $\Delta \subseteq \Theta_3(D)$, there is some $\delta \in \Delta$ s.t. δ is tolerated by Δ, which with Theorem 1 suffices to show consistency. Indeed, consider some arbitrary but fixed non-empty $\Delta \subseteq \Theta_3(D)$. Suppose first there is some $s \in S$ s.t. $(s|C_s) \in \Delta$. With †, there is some $\omega \in \mathsf{2mod}(D)$ s.t. $\omega(s) = \top$. Since $\omega \in \mathsf{2mod}(D)$, $\omega(C_s) = \top$ and thus $\omega \models s \wedge C_s$. Furthermore, for every $s' \in S$, $\omega \models s' \equiv C_{s'}$, i.e. $\omega \models C_{s'} \rightarrow s'$ and $\omega \models s' \rightarrow C_{s'}$. But then $\omega((C_{s'}|s')) \neq 0$ and $\omega((s'|C_{s'})) \neq 0$. A fortiori, $\omega(\delta') \neq 0$ for any $\delta \in \Delta$. Suppose now that $(s|C_s) \notin \Delta$ for any $s \in S$. Since $\Delta \neq \emptyset$, there is some $s \in S$ for which $(C_s|s) \in \Delta$. With †, there is some $\omega \in \mathsf{2mod}(D)$ s.t. $\omega(s) = \top$. Since $\omega \in \mathsf{2mod}(D)$, $\omega(C_s) = \top$ and thus $\omega \models s \wedge C_s$. Furthermore, for every $s' \in S$, $\omega \models s' \equiv C_{s'}$, i.e. $\omega \models C_{s'} \rightarrow s'$ and $\omega \models s' \rightarrow C_{s'}$. But then $\omega((C_{s'}|s')) \neq 0$ and $\omega((s'|C_{s'})) \neq 0$. A fortiori, $\omega(\delta') \neq 0$ for any $\delta \in \Delta$. □

Unfortunately, non-refutingness of D is not a necessary condition for the consistency of $\Theta_3(D)$:

Example 11 Consider the ADF $D = (\{a, b\}, L, C)$ with $C_a = \top$ and $C_b = \neg a$. Notice that D is refuting since $\mathsf{2mod}(D) = \{a\overline{b}\}$ and thus $\sqcap_i \mathsf{2mod}(D)(b) = \bot$. However, $\Theta_3(D)$ is not inconsistent, since e.g. κ with $\kappa(a\overline{b}) = 0$, $\kappa(\overline{a}b) = 1$ and $\kappa(ab) = \kappa(\overline{a}\overline{b}) = 2$ accepts $\Theta_3(D)$.

The following example shows that non-refutingness of D is not a sufficient condition for consistency of $\Theta_4(D)$, $\Theta_6(D)$ or $\Theta_7(D)$:

Example 12 Let $D = (\{a\}, L, C)$ with $C_a = \top$. Then $\Theta_4(D) = \{(b|\top), (\neg b|\bot)\}$. Notice that $\{(\neg b|\bot)\}$ does not tolerate $(\neg b|\bot)$ since there is no world $\omega \in \Omega(\{b\})$ s.t. $\omega(\bot \wedge \neg b) = \top$. Thus, with Theorem 1, $\Theta_4(D)$ is inconsistent (and likewise for $\Theta_6(D)$ and $\Theta_7(D)$).

Theorem 7 *Given an ADF D, $\Theta_4(D)$ is consistent if D is non-validating and non-refuting.*

Proof Suppose that D is non-validating, i.e. for every $s \in S$, $\sqcap_i \mathsf{2mod}(D)(s) \in \{u, \bot\}$. This means that (†): for every $s \in S$, there is an $\omega \in \mathsf{2mod}(D)$ s.t. $\omega(s) = \bot$. We show for every $\Delta \subseteq \Theta_4(D)$ tolerates some $\delta \in \Delta$, by Theorem 1 suffices to show consistency. Indeed, consider some arbitrary but fixed $\Delta \subseteq \Theta_4(D)$. Suppose first that for some $s \in S$, $(\neg s|\neg C_s) \in \Delta$. Since D is non-validating, there is some $\omega \in \mathsf{2mod}(D)$ s.t. $\omega(s) = \bot$. Since $\omega(s) = \omega(C_s)$, $\omega \models \neg C_s$. Thus, $\omega \models \neg s \wedge \neg C_s$. Since $\omega \in \mathsf{2mod}(D)$, for every $s' \in S$, $\omega(s') = \omega(C_{s'})$ and thus $\omega \not\models C_{s'} \wedge \neg s'$ and $\omega \not\models \neg C_{s'} \wedge s'$, i.e. $\omega(\delta) \neq 0$ for any $\delta \in \Theta_4(D)$. A fortiori, $\omega(\delta) \neq 0$ for any $\delta \in \Delta$. The proof for $(s|C_s) \in \Delta$ is similar. □

Theorem 8 *Given an ADF D, $\Theta_6(D)$ is consistent if D is non-validating and non-refuting.*

Proof The proof is similar to that of Theorem 7. □

Theorem 9 *Given an ADF D, $\Theta_7(D)$ is consistent if D is non-validating.*

Proof The proof is similar to that of Theorem 7. □

These theorems allow us to make the following statements about the Z-adequacy of the translations Θ_3, Θ_4, Θ_6 and Θ_7:

Corollary 2 *For any $D = (S, L, C)$:*

- *If D is non-refuting, $D \mathrel{|\!\sim}^{\cap}_{\mathsf{2mod}} s[\neg s]$ iff $\Theta_3(D) \mathrel{|\!\sim}_Z s[\neg s]$ for any $s \in S$.*
- *For any $i \in \{4, 6\}$, if D is non-refuting and non-validating, $D \mathrel{|\!\sim}^{\cap}_{\mathsf{2mod}} s[\neg s]$ iff $\Theta_i(D) \mathrel{|\!\sim}_Z s[\neg s]$ for any $s \in S$.*
- *If D is non-validating, $D \mathrel{|\!\sim}^{\cap}_{\mathsf{2mod}} s[\neg s]$ iff $\Theta_7(D) \mathrel{|\!\sim}_Z s[\neg s]$ for any $s \in S$.*

Proof Follows from Theorems 6, 7, 8 and 9. □

Table 2 Summary of the conditions for consistency of translations $\Theta_3, \ldots, \Theta_7$ established in this section

i	Condition on D for Consistency of $\Theta_i(D)$	Result
3	non-refuting	Theorem 6
4	non-refuting and non-validating	Theorem 7
5	$\mathsf{2mod}(D) \neq \emptyset$	Theorem 5
6	non-refuting and non-validating	Theorem 8
7	non-validating	Theorem 9

We close this sub-section by making some obeservations on both the consequence relations resulting from non-refuting and non-validating ADFs and their corresponding translations. We first observe that from a conditional logic perspective, non-refuting and non-validating ADFs are rather simple

Proposition 5 *For any ADF D:*

- *If D is non-refuting* $\Theta_3(D) = (\Theta_3(D))_0$.[6]
- *If D is non-refuting and non-validating,* $\Theta_i(D) = (\Theta_i(D))_0$ *for* $i \in \{4, 6\}$.
- *If D is non-validating,* $\Theta_7(D) = (\Theta_7(D))_0$.

Proof Notice that in the proof of Theorem 7, we have actually established that for every $s \in S$, $(s|C_s)$ and $(C_s|s)$ are tolerated by $\Theta_3(D)$. Similarly for Θ_4, Θ_5 and Θ_7, using Theorems 7, 8 respectively 9 instead of Theorem 7. □

We furthermore observe that ADFs that are both non-validating and non-refuting are inconclusive, in the sense that they do not allow to infer a conclusive judgement about any node:

Proposition 6 *If D is non-refuting and non-validating,* $D \not\mathrel{|\!\sim}_{\mathsf{2mod}} \dot{s}$ *for any* $s \in S$.

Proof Suppose D is non-refuting and non-validating. Then $\sqcap_i \mathsf{2mod}(s) = u$ for any $s \in S$ and thus $D \not\mathrel{|\!\sim}^{\cap}_{\mathsf{2mod}} \dot{s}$ for any $s \in S$. □

Table 2 summarizes the conditions for consistency established in this section.

4.3 OCF-adequacy w.r.t. two-valued semantics

In this section, we generalize some of the results from the previous section in order to show OCF-adequacy of translations based on $\Theta_3, \ldots, \Theta_7$ w.r.t. the two-valued model semantics.

We first show the OCF-adequacy of Θ_1 and Θ_2:

Proposition 7 Θ_1 *(respectively* Θ_2*) is OCF-adequate w.r.t.* 2mod *for the class of ADFs for which* $\Theta_1(D)$ *(respectively* $\Theta_2(D)$*) is consistent.*

[6]Recall, $(\Theta_3(D))_0 = \{\delta \in \Theta_3(D) \mid \Delta \text{ tolerates } \delta\}$ is the set of conditionals tolerated by $\Theta_3(D)$ (i.e. the set of conditionals which have $Z_{\Theta_3(D)}$-rank 0).

Proof We first show that for any $1 \leq i \leq 2$ and any ADF D, $\mathsf{2mod}(D) \subseteq (\kappa^Z_{\Theta_i(D)})^{-1}(0)$. Indeed, consider $\omega \in \mathsf{2mod}(D)$. For any $s \in S$, $\omega(s) = \top$ iff $\omega(C_s) = \top$, i.e. $(C_s|s)(\omega) \in \{1, u\}$ and $(s|C_s)(\omega) \in \{1, u\}$. Thus, $\kappa^Z_{\Theta_i(D)}(\omega) = 0$.

We can now obtain an OCF κ that accepts $\Theta_i(D)$ by setting $\kappa(\omega) = 0$ iff $\omega \in \mathsf{2mod}(D)$ and $\kappa(\omega) = \kappa^Z_{\Theta_i(D)}(\omega) + 1$ otherwise. We show that κ accepts $\Theta_i(D)$ for $i = 1$, the case for $i = 2$ is similar. Indeed consider $(s|C_s) \in \Theta_1(D)$. Suppose first that for some $\omega \in \mathsf{2mod}(D)$, $\omega \models C_s$. Then $\kappa(s \wedge C_s) = 0 < \kappa(\neg s \wedge C_s) = \kappa^Z_{\Theta_1(D)} + 1$. Suppose now that for no $\omega \in \mathsf{2mod}(D)$, $\omega \models C_s$. Since $\Theta_1(D)$ is consistent, $\kappa^Z_{\Theta_1(D)}(C_s \wedge s) < \kappa^Z_{\Theta_1(D)}(C_s \wedge \neg s)$, thus, $\kappa(C_s \wedge s) = \kappa^Z_{\Theta_1(D)}(C_s \wedge s) + 1 < \kappa^Z_{\Theta_1(D)}(C_s \wedge \neg s) + 1 = \kappa(C_s \wedge s \neg s)$.

Finally, notice that since $\kappa^{-1}(0) = \mathsf{2mod}(D)$, $D \mid\!\sim^{\cap}_{\mathsf{2mod}} s$ iff $s \in Bel(\kappa)$. Thus, Θ_i is OCF-adequate w.r.t. 2mod for $1 \leq i \leq 2$. □

We now show the following result on the relationship between OCFs induced by a Z-partitioning and other OCFs.

Proposition 8 *Given a set of conditionals Δ, for any κ s.t. $\kappa \models \Delta$, $\kappa^{-1}(0) \subseteq (\kappa^Z_\Delta)^{-1}(0)$.*

Proof Suppose that for some $\omega \in \Omega$, $\kappa(\omega) = 0$ yet $\kappa^Z_\Delta(\omega) \neq 0$. The latter means (with Lemma 1) that $\omega \models \phi \wedge \neg\psi$ for some $(\psi|\phi) \in \Delta$. But then $\kappa(\phi \wedge \psi) \not< \kappa(\phi \wedge \neg\psi)$, contradiction to $\kappa \models \Delta$. □

For any OCF κ that accepts Θ_3, Θ_4, Θ_5 or Θ_6, the most plausible worlds according to κ will be a subset of the two-valued models of the translated ADF:

Proposition 9 *For any $3 \leq i \leq 7$, if $\Theta_i(D)$ is consistent and $\kappa \models \Theta_i(D)$ then $\kappa^{-1}(0) \subseteq$ 2mod(D).*

Proof By Propositions 2, 3 and 4, for any $3 \leq i \leq 7$, $(\kappa^Z_{\Theta_i(D)})^{-1}(0) = \mathsf{2mod}(D)$. By Proposition 8, if $\kappa \models \theta_i(D)$ then $\kappa^{-1}(0) \subseteq (\kappa^Z_{\Theta_i(D)})^{-1}(0)$. □

Notice that this result is, in a sense, stronger than just establishing OCF-adequacy. In fact, OCF-adequacy of $\Theta_3, \ldots, \Theta_7$ follows from the Z-adequacy of these translations. What Proposition 9 establishes is that any κ that accepts Θ_i (for $3 \leq i \leq 7$) will give rise to a set of beliefs that is determined by $\mathsf{2mod}(D)$, in the sense that the beliefs also follow from a subset of the two-valued models. Furthermore, $\mid\!\sim^{\cap}_{\mathsf{2mod}}$ forms a lower bound on $Bel(\kappa)$ in the sense that everything that is derivable using $\mid\!\sim^{\cap}_{\mathsf{2mod}}$ from D will be in $Bel(\kappa)$. These two insights are shown in the following proposition:

Proposition 10 *For any $3 \leq i \leq 7$, if $\Theta_i(D)$ is consistent and $\kappa \models \Theta_i(D)$:*

- *if $D \mid\!\sim^{\cap}_{\mathsf{2mod}} \phi$ then $\phi \in Bel(\kappa)$.*
- *if $\phi \in Bel(\kappa)$ then there is some $\Delta \subseteq$ 2mod(D) s.t. $Bel(\kappa) = \{\phi \mid \forall\omega \in \Delta : \omega \models \phi\}$.*

Proof Both statements follow immediately from Proposition 9. □

5 Z-adequacy and OCF-adequacy w.r.t. stable and preferred semantics

In this section, we study the Z- and OCF-adequacy of the translations $\Theta_3,\ldots,\Theta_7$ w.r.t. the stable and preferred semantics.

We can strengthen Theorem 4 to obtain Z-adequacy with respect to the stable and preferred semantics for specific subclasses of ADFs:

Theorem 10 *For any $i \in \{3, 4, 5, 6, 7\}$ the following results hold:*

1. *Θ_i is Z-adequate w.r.t. the stable semantics for the class of weakly coherent ADFs for which $\Theta_i(D)$ is consistent.*
2. *Θ_i is Z-adequate w.r.t. the preferred semantics for the class of semi-coherent ADFs for which $\Theta_i(D)$ is consistent.*

Proof Ad 1. By Theorem 2, $\mathsf{2mod}(D) \supseteq \mathsf{stable}(D)$ for any ADF D. If D is weakly coherent, this means $\mathsf{2mod}(D) = \mathsf{stable}(D)$. Thus for any $s \in S$, $D \mid\!\sim^{\cap}_{\mathsf{stable}} s[\neg s]$ iff $D \mid\!\sim^{\cap}_{\mathsf{2mod}} s[\neg s]$. By Theorem 4 $D \mid\!\sim^{\cap}_{\mathsf{stable}} s[\neg s]$ iff $\Theta_i(D) \mid\!\sim_{Z} s[\neg s]$.

Ad 2. By Theorem 2, $\mathsf{preferred}(D) \supseteq \mathsf{2mod}(D)$. If D is semi-coherent, this means $\mathsf{preferred}(D) = \mathsf{2mod}(D)$. Thus for any $s \in S$, $D \mid\!\sim^{\cap}_{\mathsf{preferred}} s[\neg s]$ iff $D \mid\!\sim^{\cap}_{\mathsf{2mod}} s[\neg s]$. By Theorem 4, $D \mid\!\sim^{\cap}_{\mathsf{preferred}} s[\neg s]$ iff $\Theta_i(D) \mid\!\sim_{Z} s[\neg s]$. □

These results can now be rephrased for syntactic subclasses of ADFs as follows:

Corollary 3 *For any $i \in \{3, 4, 5, 7\}$ the following results hold:*

1. *Θ_i is Z-adequate w.r.t. the stable semantics for the class of ASSADFs and the class of SFSADFs, whenever $\Theta_i(D)$ is consistent.*
2. *Θ_i is Z-adequate w.r.t. the preferred semantics for the class of SFADFs that do not contain any odd-length cycles, whenever $\Theta_i(D)$ is consistent.*

Proof This follows from Theorem 3 and Theorem 10. □

In general, however, any translation based on $\Theta_3,\ldots,\Theta_7$ will be OCF-inadequate w.r.t. preferred semantics as well:

Proposition 11 *There is an ADF D s.t. for no $3 \leq i \leq 7$, $\Theta_i(D)$ is OCF-adequate w.r.t. preferred semantics.*

Proof We consider the following ADF $D = (\{a, b, c\}, L, C)$ where:

$C_a = \neg b$; $C_b = \neg a$; $C_c = \neg b \wedge \neg c$; This ADF has the following unique 2-valued models: $v(a) = v(c) = \bot$ and $v(b) = \top$. Thus, by Proposition 9, if $\Theta_i(D)$ is consistent, this implies that if $\kappa \models \Theta_i(D)$ for some $3 \leq i \leq 7$ then $\kappa \models b$. However, there is a second preferred interpretation v' with $v'(a) = \top$, $v'(b) = \bot$ and $v'(c) = u$. Thus, $D \mid\!\not\sim^{\cap}_{\mathsf{preferred}} b$. □

A critical reader might remark that the above example is pathological since it depends on the node c being "self-attacking" in the sense that $C_c \vdash \neg c$. The following alternative yet more involving example shows that a similar behaviour can be created with an odd cycle (notice that an example without an odd-length cycle cannot be found, since any SFADF without odd-length cycle is coherent and therefore any preferred interpretation will also be a two-valued interpretation):

Example 13 We consider the following ADF $D = (\{a, b, c, d, e\}, L, C)$ where:
$C_a = \neg b$; $C_b = \neg a$; $C_c = \neg b \wedge \neg e$; $C_d = \neg c$; $C_e = \neg d$;
This ADF has one 2-valued model: $v(a) = v(c) = v(e) = \bot$ and $v(b) = v(d) = \top$. Thus, by Proposition 9, this implies that if $\kappa \models \Theta_i(D)$ for some $3 \leq i \leq 7$ then $\kappa \models d$. However, there is a second preferred interpretation v' with $v'(a) = \top$, $v'(b) = \bot$ and $v'(c) = v'(d) = v'(e) = u$. Thus, $D \not\mid\!\sim^{\cap}_{\mathsf{preferred}} d$.

Remark 3 Notice that these propositions also imply the Z-inadequacy of Θ_3, Θ_4 and Θ_5 w.r.t. preferred semantics.

One proposal to avoid the impossibility result of Proposition 11 above would be to add some conditionals to Θ_i (for $3 \leq i \leq 7$). Unfortunately, such an escape route does not hold much promise:

Proposition 12 *Where $\Theta \supseteq \Theta_i$ and $\kappa \models \Theta$, $\kappa^{-1}(0) \supseteq \{\omega_v \mid v \in \mathsf{2mod}(D)\}$ (for any $3 \leq i \leq 7$).*

Proof To show this proposition we first show the following Lemma:

Lemma 2 *For two sets of conditionals Δ and Δ', $(\kappa^Z_\Delta)^{-1}(0) \supseteq (\kappa^Z_{\Delta\cup\Delta'})^{-1}(0)$.*

Proof Suppose $\omega \in (\kappa^Z_{\Delta\cup\Delta'})^{-1}(0)$, i.e. $\omega \not\models \phi \wedge \neg\psi$ for any $(\psi|\phi) \in \Delta \cup \Delta'$. Then clearly $\omega \not\models \phi \wedge \neg\psi$ for any $(\psi|\phi) \in \Delta$ and thus $\omega \in (\kappa^Z_\Delta)^{-1}(0)$. □

Consider now some $\Theta \supseteq \Theta_i$ (for some $3 \leq i \leq 7$). By Proposition 8, any κ s.t. $\kappa \models \Theta$, $\kappa^{-1}(0) \subseteq (\kappa^Z_\Theta)^{-1}(0)$. By Lemma 2, $(\kappa^Z_\Theta)^{-1}(0) \subseteq (\kappa^Z_{\Theta_i})^{-1}(0)$. By Propositions 2, 3 and 4, this means that $\kappa^{-1}(0) \subseteq \mathsf{2mod}(D)$. □

We can now reproduce the inadequacy results we had before (Proposition 11) for any $\Theta \supseteq \Theta_i$ (for $3 \leq i \leq 7$):

Proposition 13 *There is an ADF D s.t. for every $3 \leq i \leq 7$, there is no $\Theta \supseteq \Theta(D)$ is OCF-adequate w.r.t. preferred semantics.*

Proof The proof of claim 1 respectively claim 2 is identical to the proofs of Proposition 15 respectively 11 except that instead of Proposition 9 we use Proposition 12. □

Remark 4 Again these propositions imply the Z-inadequacy of any $\Theta \supseteq \Theta_i$ (for $3 \leq i \leq 7$).

6 OCF- and *Z*-inadequacy w.r.t. the grounded semantics

In this section we show the OCF- and Z-inadequacy of all translations $\Theta_1, \ldots, \Theta_7$ w.r.t. the grounded semantics.

We start by showing the OCF- and Z-inadequacy of Θ_1 w.r.t. the grounded semantics:

Example 14 Consider the ADF $D = (\{p, q, r\}, L, C)$ with $C_p = \top$, $C_q = \neg p$, $C_r = \neg q$. The grounded interpretation v_G of D assigns $v_G(p) = v_G(r) = \top$ and $v_G(q) = \bot$. However, $\Theta_1(D) = \{(p|\top), (q|\neg p), (\overline{p}|q)\}$. Then $(\kappa^Z_{\Theta_1(D)})^{-1} = \{p\overline{q}\,\overline{r}, p\overline{q}r\}$ and thus $\Theta_1(D) \not\mid\!\sim_Z r$ whereas $D \mid\!\sim_{\text{grounded}} r$.

Example 15 Consider the ADF $D = (\{p, q\}, L, C)$ with $C_p = \top$ and $C_q = \neg p$. The grounded interpretation of D assigns $v(p) = \top$ and $v(q) = \bot$. $\Theta_2(D) = \{(\top|p), (\neg p|q)\}$. Then $(\kappa^Z_{\Theta_2(D)})^{-1} = \{p\overline{q}, \overline{p}q, pq\}$ and thus $\Theta_2(D) \not\mid\!\sim_Z p$ whereas $D \mid\!\sim_{\text{grounded}} p$.

Theorem 4 can also be used to derive the Z-inadequacy of Θ_3, Θ_4, Θ_5, Θ_6 and Θ_7 with respect to the grounded semantics:

Proposition 14 *For any $\Theta(D)$ that satisfies **C1** and **C2**, Θ is not Z-adequate with respect to grounded.*

Proof We consider the following ADF $D = (\{a, b, c, d\}, L, C)$ where:
$C_a = \neg b$; $C_b = \neg a$; $C_c = \neg a \wedge \neg b$; $C_d = \neg c$ This ADF has the following 2-valued models: v_1 which assigns $v_1(a) = v_1(d) = \top$ and $v_1(b) = v_1(c) = \bot$ and v_2 with $v_2(b) = v_2(d) = \top$ and $v_2(a) = v_2(c) = \bot$. Since $v_1(d) = v_2(d) = \top$, by Proposition 3 and Proposition 4, $\Theta(D) \mid\!\sim_Z d$. However, the grounded assignment v_G sets $v_G(a) = v_G(b) = v_G(c) = v_G(d) = u$. □

We can generalize this result for OCF-inadequacy (w.r.t. grounded semantics):

Proposition 15 *There is an ADF D s.t. for no $3 \leq i \leq 7$, $\Theta_i(D)$ is OCF-adequate w.r.t. grounded semantics.*

Proof We consider the following ADF $D = (\{a, b, c, d\}, L, C)$ where:
$C_a = \neg b$; $C_b = \neg a$; $C_c = \neg a \wedge \neg b$; $C_d = \neg c$. This ADF has the following 2-valued interpretation: v_1 which assigns $v_1(a) = v_1(d) = \top$ and $v_1(b) = v_1(c) = \bot$ and v_2 with $v_2(b) = v_2(d) = \top$ and $v_2(a) = v_2(c) = \bot$. Since $v_1(d) = v_2(d) = \top$, by Proposition 9, this implies that if $\kappa \models \Theta_i(D)$ for some $3 \leq i \leq 5$ then $\kappa \models d$. However, the grounded assignment v_G sets $v_G(a) = v_G(b) = v_G(c) = v_G(d) = u$. □

We can again reproduce the inadequacy results we had before (Proposition 15) for any $\Theta \supseteq \Theta_i$ $(3 \leq i \leq 7)$:

Proposition 16 *There is an ADF D s.t. for any $3 \leq i \leq 7$, for no $\Theta \supseteq \Theta_i(D)$ is Θ OCF-adequate w.r.t. grounded semantics.*

Proof The proof of claim 1 respectively claim 2 is identical to the proofs of Proposition 15 respectively 11 except that instead of Proposition 9 we use Proposition 12. □

Nevertheless, we can report on some classes of ADFs for which $\Theta_3,\ldots,\Theta_7$ are Z-adequate. In particular, for acyclic ADFs, Z-adequacy w.r.t. the grounded semantics is guaranteed:

Theorem 11 *Given some* $3 \leq i \leq 7$, Θ_i *is Z-adequate w.r.t. the two-valued model semantics w.r.t. the class of acyclic ADFs for which* Θ_i *is consistent.*

Proof Follows from Theorem 3 and Propositions 2, 3 and 4. □

7 Properties of the translations

In this section, we study several general properties of our translations. In Section 7.1, we study several desirable properties for translations between non-monotonic formalisms, originally proposed by [14]. In Section 7.2, we make some remarks on the properties of the translations from a conditional perspective.

7.1 Properties for translations between non-monotonic formalisms

In this section we want to look at several desirable properties, proposed by [14] for translations between non-monotonic formalisms like *adequacy*, *polynomiality* and *modularity*. In Section 3 we already discussed *adequacy* in-depth and we have shown, that translations Θ_1 and Θ_2 are never OCF- or Z-adequate where as Θ_3, Θ_4, Θ_5, Θ_6 and Θ_7 are OCF- and Z-adequate for 2mod-semantics (Section 4.1). However these translations are not inadequate for preferred and grounded semantics as shown in Proposition 11 and Proposition 15.

A translation satisfies *polynomiality* if the translation is computable with reasonable bounds, i.e. within time bounded by a polynomial of the input. It is easy to see, that our translations are *polynomial* in the number of statements.

For *modularity* we follow the formulation of [32] for a translation from *ADFs* to a target formalism, even though *modularity* was originally defined for translations between circumscription and default logic [19]. In more detail, a translation being *modular* means that "local" changes in the translated ADF results in "local" changes in the translation. A minimal notion of modularity of a translation Θ would be that given two syntactically disjoint ADFs D_1 and D_2, i.e. two ADFs $D_1 = (S_1, L_1, C_1)$ and $D_2 = (S_2, L_2, C_2)$ s.t. $S_1 \cap S_2 = \emptyset$, $\Theta(D_1 \cup D_2) = \Theta(D_1) \cup \Theta(D_2)$. Clearly the translations presented in this paper are modular in this sense.

None of our translations needs a language extensions therefore they are *language-preserving*.

Finally, it is clear, that the translations are *syntax-based*, in the sense that the translations $\Theta_i(D)$ (for any $1 \leq i \leq 7$) can be derived purely on the basis of the syntactic form of the ADF D.

7.2 Properties from a conditional perspective

In this section we make some observations on the *conditional structure* of the translations suggested in this paper. In particular, we observe that for any consistent translation, the conditional structure is *flat* in the sense that all conditionals get assigned the same Z-rank 0. We first show this for Θ_5:

Proposition 17 $\Theta_5(D) = (\Theta_5(D))_0$ *for any* ADF *D for which* 2mod$(D) \neq \emptyset$.

Proof We show that there is an $\omega \in \Omega(S)$ s.t. for every $\delta \in \Theta_5(D)$, $\omega(\delta) = 1$. Indeed, consider some $\omega \in$ 2mod(D). Then since $\omega \models C_s \equiv s$ for any $s \in S$, we immediately see that for any $(s \equiv C_s|\top) \in \Theta_5(D)$, $\omega \models \top \wedge C_s \equiv s$. Since $\Theta_5(D) = \{(s \equiv C_s|\top) \mid s \in S\}$, this concludes the proof. □

A similar result has already been shown in Theorem 5 for Θ_i for any $3 \leq i \leq 7$ for which $\Theta_i(D)$ is consistent. We summarize these results in the following corollary:

Corollary 4 *For any* $3 \leq i \leq 7$, *if* $\Theta_i(D)$ *is consistent* $\Theta_i(D) = (\Theta_i(D))_0$.

8 Related works

Our aim in this paper is to lay foundations of integrative techniques for argumentative and conditional reasoning. There are previous works, which have similar aims or are otherwise related to this endeavour. We will discuss those in the following.

First, there is huge body of work on *structured argumentation* (see e. g. [3]). In these approaches, arguments are constructed on the basis of a knowledge base possibly consisting of conditionals. An attack relation between these arguments is constructed based on some syntactic criteria. Acceptable arguments are then identified by applying argumentation semantics to the resulting argumentation frameworks. Thus, even though structured argumentation syntactically uses conditional knowledge bases, it relies semantically on formal argumentation.

There have been some attempts to bridge the gap between specific structured argumentation formalisms and conditional reasoning. For example, in [21] conditional reasoning based on system Z [13] and DeLP [12] are combined in a novel way. Roughly, the paper provides a novel semantics for DeLP by borrowing concepts from system Z that allows using *plausibility* as a criterion for comparing the strength of arguments and counterarguments. Our approach differs both in goal (we investigate the correspondence between argumentation and conditional logics instead of integrating insights from the latter into the former) and generality (DeLP is specific and arguably rather peculiar argumentation formalism whereas ADFs are the most general formalism around).

Several works investigate postulates for nonmonotonic reasoning known from conditional logics [23] for specific structured argumentation formalisms, such as assumption-based argumentation [1, 8, 16, 18] and ASPIC$^+$ [25]. These works revealed gaps between nonmonotonic reasoning and argumentation which we try to bridge in this paper.

Besnard et al. [4] develop a structured argumentation approach where general conditional logic is used as the base knowledge representation formalism. Their framework is constructed in a similar fashion as the deductive argumentation approach [5] but they also provide with *conditional contrariety* a new conflict relation for arguments, based on conditional logical terms. Even though insights from conditional logics are used in that paper, this approach stays well within the paradigm of structured argumentation. In [35] a new semantics for abstract argumentation is presented, which is also rooted in conditional logical terms. In more detail, a ranking interpretation is provided for extensions of arguments instantiated by strict and defeasible rules by using conditional ranking semantics. Thus, Weydert presupposes a conditional knowledge base that is used to contruct an argumentation framework whereas we investigate what are sensible translations of ADFs into conditional knowledge bases. In [33] Strass presents a translation from an ASPIC-style defeasible logic theory to ADFs. While actually Strass embeds one argumentative formalism (the ASPIC-style theory) into another argumentative formalism (ADFs) and shows how the latter can simulate the former, the process of embedding is similar to our approach.

9 Conclusion

In this paper we investigated the correspondence between abstract dialectical frameworks and conditional logics based on the syntactic similarities between the two frameworks. We have investigated seven different translations from ADFs into conditional logics and were able to show the OCF- and Z-adequacy of five of these translations under the two-valued semantics. Furthermore, we have shown that for certain classes of ADFs, these results carry on to the preferred and stable semantics, whereas for the grounded semantics there is a signifcant difference between the semantics of ADFs and conditional logics. Furthermore we have shown several desirable properties of these translations.

Since this paper investigates connections between two highl-level formalisms implementing the basic nature of two fields which have co-existed peacefully but largely independent of each-other for at least 25 years, it sheds important light on the exact relationship between these two fields by showing precisely where these two formalisms behave similarly and where these approaches are actually different. As such, this paper provides a foundation for cross-fertilization between the two fields as well as a justification for the adaption of ideas from conditional reasoning into ADFs. For example, in view of the OCF- and Z-adequacy of five of the presented translations with respect to the two-valued semantics, we can look at other inference relations for CL (e.g. c-representations [20], lexicographic closure [24] or disjunctive rational entailment [6]) and compare these with $\mid\!\sim^{\cap}_{\mathsf{2mod}}$. On the other hand, our results showed that for preferred, stable and grounded semantics in general, the translations are neither OCF- nor Z-adequate. This can be used as a justification for incorporating ideas from conditional reasoning into argumentative formalisms. For example, one might extend ADFs to allow for conditional acceptance conditions (e.g. “ if ϕ then normally s is accepted if and only if C_s is accepted”). Furthermore, in view of the findings in Section 7.2, we plan to take the investigation of the correspondences between ADFs and conditional logics beyond the level of beliefs (i.e. beyond $\kappa(0)$). We plan to do this by defining conditional derivations in ADFs based on the Ramsey-test [29], which says that a conditional $(\phi|\psi)$ is valid in a context if ϕ is believed after revision of the knowledge context by ψ. To model such conditionals, we will make use of work on revision of ADFs [26].

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