The symplectic Dirac and Dolbeault operators and the Lichnerowicz Laplacian

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Dissertation

The symplectic Dirac and Dolbeault operators and the Lichnerowicz Laplacian

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Abstract

The Laplace type operator arising as a commutator of two symplectic Dirac operators introduced in [9] in the context of Schrödinger picture and rediscovered in [4] as a commutator of two to each other formal adjoint differential operators in the Fock picture admits a natural geometric interpretation, which is described in this thesis.

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Contents

Introduction

The symplectic Dirac operator is constructed in complete analogy to the Dirac operator on Riemannian manifolds. That means, one has to think first of all about the structure group an almost symplectic manifold (M, ω) admits as well as the *symplectic spinor space* to construct a symplectic spinor bundle $\mathcal{S}$. While it is always possible to provide a symplectic manifold with an Mp^{c}–structure, one has only to bother about the symplectic spinor space, which in contrast to the Riemannian setting can not be chosen as finite dimensional. The Bargmann transform enables to choose a symplectic spinor space between the so called *Schrödinger* and *Fock pictures.*

In the Schrödinger picture one uses the space of square integrable functions $L^2(W)$ with W a Lagrangian subspace of a symplectic vector space (V, Ω). With the assumption (M, ω) being metaplectic this approach was used in [9] to construct two symplectic Dirac operators D and $\tilde{D}$, which in their definitions only differ by the way of identifying the tangent bundle of M with the cotangent bundle. On the one hand, it can be achieved by means of ω and, on the other hand with the aid of a Riemannian metric g, which is defined by means of an $\omega-$*compatible almost complex structure* J on M. The two different Dirac operators arise then as the compositions of the maps occurring in the diagram below.

$$\begin{tikzcd}
\Gamma(\mathcal{S}) \arrow[r, "\nabla"] \arrow[rrr, bend left, "D"] \arrow[rrr, bend right, "\tilde{D}"'] & \Gamma(T^*M \otimes \mathcal{S}) \arrow[r, phantom, "\overset{\omega}{\underset{g}{\cong}}"] & \Gamma(TM \otimes \mathcal{S}) \arrow[r, "Cl"] & \Gamma(\mathcal{S})
\end{tikzcd}$$

It was then observed, that the commutator $\mathcal{P} := \mathrm{i}[\tilde{D}, D]$ yields a second-order differential operator, which was shown to be a Laplacian and to admit a decomposition similar to that of *Weitzenböck decomposition*, which however in the Kähler case becomes the initial Weitzenböck decomposition.

In the Fock picture one uses, after choosing an Ω–compatible complex structure j on V, the Segal-Bargmann spaces. These are spaces of entire functions on the complex vector space V_j, which are square integrable with respect to a parameter-scaled Gaussian weight function. The set of all monomials on V_j constitutes an orthogonal basis for all Segal-Bargmann spaces, which justifies the name of this approach. That is, a Segal-Bargmann space is isomorphic to a Hilbert space completion of the direct sum of the symmetric powers of V_j. By a particular choice of a Segal-Bargmann space it was shown in [4], that by using a unitary connection on M with vanishing torsion vector field the operator $\mathcal{P}$ arises up to a scalar multiple as a commutator of two to each other formal adjoint operators $D^{1,0}$ and $D^{0,1}$. These operators, referred to as *the symplectic Dirac-Dolbeault operators*, describe the additive splitting of D after decomposing the covariant derivative according to the decomposition of the tangent bundle into the eigenbundles of the chosen ω–compatible almost complex structure J on M.

$$
\begin{tikzcd}
 & T^{1,0^*}M \otimes \mathcal{S} \arrow[dr, "Cl"] & \\
\mathcal{S} \arrow[ur, "\nabla^{1,0}"] \arrow[r, "\nabla"] \arrow[rr, bend left=50, "D^{1,0}"] & T^*_{\mathbb{C}}M \otimes \mathcal{S} \arrow[u, "\mathrm{pr}_1"] \arrow[d, dashed, "\mathrm{pr}_2"] & \mathcal{S} \arrow[l, dashed, "\nabla"'] \arrow[dl, dashed, "\nabla^{0,1}"] \arrow[ll, dashed, bend left=50, "D^{0,1}"] \\
 & T^{0,1^*}M \otimes \mathcal{S} \arrow[ul, dashed, "Cl"] &
\end{tikzcd}
$$

As in [9] the Weitzenböck-type decomposition was described in the Fock picture and, besides, it was shown, that the operator $D^{1,0}$ involves only the creation part, while the operator $D^{0,1}$ only the annihilation part of the symplectic Clifford multiplication Cl.

In this thesis we are staying in the settings used in [4] to give an interpretation of the Laplacian $\mathcal{P}$. In order for the thesis to be relatively self-contained we divided it into four parts. In the beginning we shortly describe the basic structures, objects and constructions on symplectic vector spaces in order to provide a symplectic manifold with a structure of an almost Kähler manifold. After this we describe the canonical Hermitian connection on almost Kähler manifolds. It was the very first observation, although stated at the very end of the thesis, that the type of torsion of the canonical Hermitian connection yields primarily the Weitzenböck decomposition of $\mathcal{P}$ on almost Kähler manifolds.

In the second part we briefly recall the Lichnerowicz Laplacian Δ_{L} and its restriction to the symmetric algebra. We then introduce Laplacians $\Delta^{1,0}$ and $\Delta^{0,1}$ on the complex symmetric algebra arising from Δ_{L} after the complexification of the tangent bundle and complex bilinear extension of the canonical Hermitian connection, when considering an almost Kähler manifold.

The third chapter deals with the central extension of the metaplectic group Mp and its maximal compact subgroup MU^{c}. We discuss the existence and classification of Mp– and Mp^{c}–structures on almost symplectic manifolds and describe in the end the correspondence with $Spin$– resp. $Spin^{\mathrm{c}}$–structures.

In the last chapter, we first motivate at the beginning, why and how the fibers of the symplectic spinor bundle can be chosen to consist only of polynomials. This happens basically by passing in the Fock picture to the maximal compact subgroup of Mp^{c}, since, while the Lie group Mp^{c} does not leave the polynomials invariant, the restriction to MU^{c} not only leaves the space of polynomials invariant, but also preserves their degrees. Moreover, the action of MU^{c} turns out to be quite natural when identifying the polynomials with symmetric tensors, meaning, that the actions in the diagram below commute.

$$\begin{tikzcd}
MU^{\mathrm{c}} \times \mathcal{P}o\ell(V_j) \arrow[rr] \arrow[d, leftrightarrow, "\simeq"'] & & \mathcal{P}o\ell(V_j) \arrow[d, leftrightarrow, "\simeq"] \\
(U(V_j) \times \mathbb{S}^1) \times S^*(V_j^*) \arrow[rr] & & S^*(V_j^*)
\end{tikzcd}$$

Furthermore, since the Lie group MU^{c} is maximal compact in Mp^{c}, the symplectic spinor bundle $\mathcal{S}$ can be associated to the MU^{c} reduction and with the observation made on the action of MU^{c} on polynomials, we can omit the Hilbert space completion of the Seagal-Bargman space and focus on its dense subspace consisting of polynomials. The resulting Hilbert space bundle happens to be isomorphic to the tensor bundle $L \otimes S^{*,0}(M)$ for some Hermitian line bundle L over M, which only depends on the choice of the Mp^{c}–structure. After this, regarding a symplectic manifold as almost Kähler equipped with the canonical Hermitian connection we adjust appropriately the Clifford multiplication and give then an interpretation of the symplectic Dirac-Dolbeault operators and prove in the end the main theorem of this thesis, which can be stated as follows.

Theorem. *If the line bundle L is trivial, the operator $\mathcal{P}$ coincides with $-\Delta^{1,0}$.*

That is, we show, that by systematically using the maximal compact subgroup MU^{c} rather than all of Mp^{c}, we recover the operator $\mathcal{P}$ as an operator associated to the Hermitian structure in a very natural way, and whence gain some better insight into its significance.

Chapter 1

Preliminaries

1.1 Symplectic and complex vector spaces

This section deals with the Lie groups occurring on a symplectic vector space and the induced (Hermitian) inner product on the corresponding (complex) symmetric algebra. What follows in this section is partially based on Chapter I in [15].

Definition 1.1.1. A bilinear form Ω on an m−dimensional $\mathbb{R}$−vector space V is called *symplectic* iff:

i) Ω is skew-symmetric, i. e. $\Omega(u,v) = -\Omega(v,u)$ for all $u, v \in V$, and

ii) Ω is non-degenerate, i. e.

$$\Omega(u,v) = 0 \quad \text{for all} \quad u \in V \quad \Rightarrow \quad v = 0.$$

The pair (V,Ω) is referred to as a *symplectic vector space.*

Any finite-dimensional symplectic vector space (V,Ω) is of even dimension, that is $\dim V = 2n$ for some $n \in \mathbb{N}$, and admits a basis $\{e_1,\ldots,e_n, f_1,\ldots,f_n\}$, such that

$$\Omega(e_i,e_j) = 0 = \Omega(f_i,f_j) \qquad \text{and} \qquad \Omega(e_i,f_j) = \delta_{ij}$$

for all $1 \le i,j \le n$. Such a basis is called a *symplectic basis.* The corresponding dual basis $\{e^1,\ldots,e^n, f^1,\ldots,f^n\}$ is given with respect to Ω by

$$e^j(v) = -\Omega(f_j,v) \qquad \text{and} \qquad f^j(v) = \Omega(e_j,v). \tag{1.1}$$

A subspace W of V is referred to as *Lagrangian* if the restriction of Ω to W vanishes identically and $\dim W = \frac{1}{2}\dim V$.

Remark 1.1.2. *i)* The *standard symplectic vector space* is the Euclidean space $\mathbb{R}^{2n}$ with the skew-symmetric form Ω_0, which is represented by the matrix

$$\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} \in \mathbb{R}^{2n\times 2n}.$$

That is

$$\Omega_0(v,w) = v^{\intercal} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} w \qquad v,w \in \mathbb{R}^{2n}.$$

ii) For every symplectic vector space (V,Ω) of dimension $2n$ there exists an isomorphism $\Psi : (\mathbb{R}^{2n},\Omega_0) \longrightarrow (V,\Omega)$, such that $\Psi^*\Omega = \Omega_0$. This isomorphism is explicitly given by

$$\mathbb{R}^{2n} \ni \begin{pmatrix} z_1 \\ \vdots \\ z_{2n} \end{pmatrix} \longmapsto \sum_{j=1}^{n} z_j e_j + z_{n+j} f_j,$$

where $\{e_1,\dots,e_n,f_1,\dots,f_n\}$ is a symplectic basis for (V,Ω) as above. Such an isomorphism Ψ is called a *symplectic isomorphism.*

iii) An immediate consequence of *ii)* is, that a choice of a symplectic basis for (V,Ω) yields a symplectic isomorphism of $(\mathbb{R}^{2n},\Omega_0)$ to (V,Ω). This allows the symplectic bases to be considered as the symplectic isomorphisms of $(\mathbb{R}^{2n},\Omega_0)$ into (V,Ω).

The set of all automorphisms $F : (V,\Omega) \longrightarrow (V,\Omega)$ satisfying $F^*\Omega = \Omega$ defines a subgroup of $Gl(V)$, which will be denoted as $Sp(V) = Sp(V,\Omega)$, and is called the *symplectic group.* If we think of a symplectic basis for (V,Ω) as a symplectic isomorphism of (V,Ω) to $(\mathbb{R}^{2n},\Omega_0)$ as described in the remark, then the composition of maps defines a free and transitive action of $Sp(V,\Omega)$ on the set of all symplectic bases for (V,Ω).

1.1.1 Almost complex structures. A *complex structure* j on a real vector space V is a linear map $j : V \to V$, such that $j^2 = -I_V$. The pair (V,j) can also be viewed as a complex vector space, which will be for the sake of distinction denoted by V_j, where the multiplication by complex scalars is given by

$$(x + \mathrm{i}y)v := xv + yjv$$

for $v \in V$ and $x,y \in \mathbb{R}$. Some basic properties of (V,j) are listed below.

i) A real subspace U of V is a complex subspace of V_j if and only if $jU = U$.
ii) If $F \in Gl(V)$, then $F \in Gl(V_j)$ if and only if $[F,j] = 0$.
iii) If $\{v_1,\dots,v_n\}$ is a complex basis for V_j, then the set $\{v_1,\dots,v_n,jv_1,\dots,jv_n\}$ is a real basis for V.
iv) The Lie group $Gl(V_j)$ acts freely transitively on the set of all complex bases for V_j.

1.1.2 The connection between the symplectic forms and almost complex structures.

It is a well-known fact, that any even-dimensional vector space can be equipped with both a symplectic form and an almost complex structure (cf. [15], Proposition 4.1). Moreover, both of the structures can be adjusted to each other in the following sense

- any symplectic vector space (V,Ω) admits an $\Omega-$*compatible complex structure* j. That is

$$\Omega(jv,jw)=\Omega(v,w) \qquad \forall\ v,w\in V \quad \text{and}$$

$$\Omega(v,jv)>0 \qquad \forall\ v\in V\smallsetminus\{0\}.$$

  The set of all such almost complex structures on (V,Ω) will be denoted by $\mathfrak{J}(V,\Omega)$.
- any vector space V with a complex structure j admits a symplectic bilinear form Ω, such that j is compatible with Ω. The set of all such symplectic forms is denoted by $\mathfrak{S}(V,j)$.

The triple (V,Ω,j), where j is an $\Omega-$compatible complex structure on a symplectic vector space (V,Ω), can be given additional structures and can be treated as

- an Euclidean vector space (V,g) with the inner product

$$g(v,w):=\Omega(v,jw). \tag{1.2}$$

  The subgroup of elements in $Gl(V)$ preserving g is the orthogonal group $O(V)=O(V,g)$.
- a Hermitian vector space (V_j,h) with the Hermitian inner product

$$h(v,w):=\Omega(v,jw)-i\Omega(v,w). \tag{1.3}$$

  The subgroup of elements in $Gl(V_j)$ preserving h is the unitary group $U(V_j)=U(V_j,h)$.

Proposition 1.1.3 (Cf. Lemma 2.19 and Proposition 2.22 in [15])**.** *If j is an $\Omega-$compatible complex structure on a symplectic vector space (V,Ω) and g and h are defined as above, then*

$$Sp(V,\Omega)\cap Gl(V_j)=Sp(V,\Omega)\cap O(V,g)=O(V,g)\cap Gl(V_j)=U(V_j,h).$$

Moreover, the unitary group $U(V_j)$ is a maximal compact subgroup of both $Sp(V)$ and $Gl(V_j)$.

Remark 1.1.4. *i)* The upper relation among the three Lie groups can be interpreted as follows. Let $\{e_1,\dots,e_n\}$ be a unitary basis for V_j. Then we have

$$h(e_i,e_k)=\delta_{ik} \quad \overset{\text{def.}}{\Longleftrightarrow} \quad \Omega(e_i,je_k)=\delta_{ik} \quad \text{and} \quad \Omega(e_i,e_k)=0$$

$$\Longleftrightarrow \quad g(e_i,e_k)=\delta_{ik} \quad \text{and} \quad g(je_i,e_k)=0$$

for all $1\le i,k\le n$ and, since $\{e_1,\dots,e_n,je_1,\dots,je_n\}$ is a real basis for V, it follows, that $\{e_1,\dots,e_n,je_1,\dots,je_n\}$ is both symplectic and orthonormal.

ii) Since $\det F=1$ for any $F\in Sp(V)$, it follows, that $U(V_j)\subset SO(V,g)$.

Theorem 1.1.5 (Cartan-Iwasawa-Malcev theorem). *Every connected Lie group G contains at least one maximal compact subgroup K and it holds*

i) *K is connected.*

ii) *For any other maximal compact subgroup $\tilde{K} \subset G$ there is some $g \in G$, such that $g\tilde{K}g^{-1} = K$.*

iii) *The homogeneous space G/K is diffeomorphic to some Euclidean space $\mathbb{R}^n$.*

The last point from the Cartan-Iwasawa-Malcev theorem provides the following observation on the spaces $\mathfrak{J}(V,\Omega)$ and $\mathfrak{S}(V,j)$.

Proposition 1.1.6. *The spaces $\mathfrak{J}(V,\Omega)$ and $\mathfrak{S}(V,j)$ are contractible.*

Proof. The symplectic group acts transitively on $\mathfrak{J}(V,\Omega)$ by conjugation and, since the unitary group is an intersection of $Sp(V)$ with $Gl(V_j)$, it follows, that $U(V_j)$ is the isotropy group for this action. That is

$$\mathfrak{J}(V,\Omega) \cong Sp(V)/U(V_j).$$

On the other hand, the map

$$Gl(V_j) \times \mathfrak{S}(V,j) \longrightarrow \mathfrak{S}(V,j), \qquad (A,\Omega) \longmapsto A^{-1^*}\Omega$$

defines a transitive action with isotropy group isomorphic to $U(V_j)$. That is,

$$\mathfrak{S}(V,j) \cong Gl(V_j)/U(V_j).$$

The assertion follows by the Proposition 1.1.3 and Cartan-Iwasawa-Malcev theorem *iii*). □

1.1.3 The basis for V' and the corresponding dual basis. We continue dealing with the triple (V,Ω,j), where j is an Ω−compatible complex structure on the symplectic vector space (V,Ω), and the corresponding 2-forms g and h as defined in (1.2) resp. (1.3).

Let $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of V. The complex linear extension of j to $V_{\mathbb{C}}$ yields a splitting of $V_{\mathbb{C}}$ into $\pm\mathrm{i}$−eigenspaces of j

$$V_{\mathbb{C}} = V' \oplus V'',$$

where $jv = \mathrm{i}v$ for $v \in V'$ and $jw = -\mathrm{i}w$ for $w \in V''$. The corresponding decomposition of the complexified dual space we denote accordingly as

$$V_{\mathbb{C}}^* = V^{1,0} \oplus V^{0,1},$$

that is $V^{(1,0)} = V'^*$ and $V^{(0,1)} = V''^*$. Consider further the complex bilinear extension of the inner product g to $V_{\mathbb{C}}$, also denoted by g, and the canonical projection

$$\Phi : V \to V', \quad \Phi(v) := \frac{1}{2}(v - \mathrm{i}jv),$$

which is a $\mathbb{C}$–linear isomorphism if regarding it as a map from V_j into V'. Then it holds

$$g(\mathbf{v}, \mathbf{w}) = 0,$$

whenever $\mathbf{v}, \mathbf{w} \in V_{\mathbb{C}}$ are lying in the same eigenspace of j, and

$$g(\Phi(v), \overline{\Phi(w)}) = \frac{1}{2} h(v, w) \tag{1.4}$$

for all $v, w \in V$. This observation provides us on the one hand with a Hermitian inner product on $V_{\mathbb{C}}$ by defining $(\mathbf{v}, \mathbf{w}) := g(\mathbf{v}, \overline{\mathbf{w}})$. On the other hand, by setting

$$\mathbf{v}^\flat(\mathbf{w}) := g(\mathbf{v}, \mathbf{w}) \qquad \text{and} \qquad g(\tau^\#, \mathbf{w}) := \tau(\mathbf{w})$$

for $\mathbf{v}, \mathbf{w} \in V_{\mathbb{C}}$ and $\tau \in V_{\mathbb{C}}^*$, we conclude, that for all $1 \le m \le n$

$$\theta_m = 2\bar{\mathbf{u}}_m^\flat \quad \text{and} \quad \bar{\theta}_m = 2\mathbf{u}_m^\flat \qquad \Leftrightarrow \qquad \theta_m^\# = 2\bar{\mathbf{u}}_m \quad \text{and} \quad \bar{\theta}_m^\# = 2\mathbf{u}_m, \tag{1.5}$$

where

- $\{u_1, \ldots, u_n\}$ is a unitary basis for V_j, $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is the corresponding basis for V', that means $\mathbf{u}_k = \Phi(u_k)$ for all $1 \le k \le n$, and
- $\{\theta_1, \ldots, \theta_n\}$ is the corresponding dual basis for $V^{1,0}$.

Denote by $\Psi : V_j^* \to V^{1,0}$ the inverse of the dual map Φ^*, that is

$$\Psi(\phi) = \phi \circ \Phi^{-1} \qquad \forall\ \phi \in V_j^*. \tag{1.6}$$

$$\begin{tikzcd}
V_j \arrow[rrr, "\Phi"] \arrow[ddd, "h"'] & & & V' \arrow[ddd, "g"] \\
& v \arrow[r, mapsto] \arrow[d, mapsto] & \mathbf{v} \arrow[d, mapsto] & \\
& v^\flat \arrow[r, mapsto] & \bar{\mathbf{v}}^\flat & \\
V_j^* \arrow[rrr, "\Psi"'] & & & V^{1,0}
\end{tikzcd}$$

If we additionally denote by $\{\phi_1, \ldots, \phi_n\}$ the dual basis for V_j^* corresponding to $\{u_1, \ldots, u_n\}$, then

$$(\Psi\phi_k)(\mathbf{u}_i) = \phi_k(\Phi^{-1}\mathbf{u}_i) = \phi_k(u_i) = \delta_{ik} \quad \forall\ 1 \le i, k \le n.$$

So it holds

$$\Psi\phi_k = \theta_k \qquad \forall\ 1 \le k \le n. \tag{1.7}$$

1.1.4 The induced Hermitian inner product on $S^*(V_{\mathbb{C}})$. Denote by $\mathcal{T}V_j^*$ the tensor algebra of V_j^* and by $S^*(V_j^*) := \bigoplus_{q=0}^\infty S^q(V_j^*)$ its subspace consisting of symmetric tensors. The canonical projection $\mathrm{sym} : \bigotimes^q V_j^* \to S^q(V_j^*)$ is given by

$$\mathrm{sym}(\vartheta_1 \otimes \cdots \otimes \vartheta_q) = \sum_{\sigma \in \mathfrak{S}_q} \vartheta_{\sigma(1)} \otimes \cdots \otimes \vartheta_{\sigma(q)} =: \vartheta_1 \cdots \vartheta_q.$$

The *symmetric product* of two symmetric tensors $\zeta \in S^p(V_j^*)$ and $\psi \in S^q(V_j^*)$, defined as

$$\zeta \odot \psi := \mathrm{sym}(\zeta \otimes \psi) \in S^{p+q}(V_j^*), \tag{1.8}$$

provides the vector space $S^*(V_j^*)$ with a structure of a commutative algebra. That is, the symmetric product is commutative, associative and satisfies the distributional law. The Hermitian inner product on $S^*(V_j^*)$ is then induced by the Hermitian inner product on V_j^* and is given by

$$(\vartheta_1 \cdots \vartheta_p, \psi_1 \cdots \psi_q) = \begin{cases} 0, & \text{if } p \neq q \\ \sum_{\sigma \in \mathfrak{S}_p} \prod_{i=1}^p (\vartheta_{\sigma(i)}, \psi_i), & \text{if } p = q. \end{cases} \tag{1.9}$$

Consider now the rth symmetric power of the complexified dual vector space $V_{\mathbb{C}}^*$. The decomposition $V_{\mathbb{C}}^* = V^{1,0} \oplus V^{0,1}$ results in the decomposition

$$S^r(V_{\mathbb{C}}^*) = \bigoplus_{p+q=r} S^p(V^{1,0}) \otimes S^q(V^{0,1}) =: S^{p,q}(V).$$

The complex bilinear extension of the inner product g on V yields a symmetric form on $V_{\mathbb{C}}^*$, also denoted by g, by setting

$$g(\vartheta, \psi) = g(\vartheta^\#, \psi^\#)$$

for $\vartheta, \psi \in V_{\mathbb{C}}^*$, which similarly vanishes identically on $V^{1,0} \otimes V^{1,0}$ and $V^{0,1} \otimes V^{0,1}$, such that the induced Hermitian inner product on $V_{\mathbb{C}}^*$ is given by

$$(\vartheta, \psi) = g(\vartheta, \bar{\psi}). \tag{1.10}$$

This in turn provides the complex symmetric algebra $S^*(V_{\mathbb{C}})$ with a Hermitian inner product defined in the same manner as in (1.9).

Remark 1.1.7. Note, that for all $\mathbf{v} \in V_{\mathbb{C}}$ and $\zeta, \psi \in S^*(V_{\mathbb{C}})$ we have

$$(\mathbf{v} \lrcorner \zeta, \psi) = (\zeta, \bar{\mathbf{v}}^\flat \odot \psi).$$

1.1.5 The induced action of the unitary group on $S^{*,0}(V)$. The standard action of $U(V_j)$ on V_j induces the dual action of $U(V_j)$ on V_j^* given by

$$(k \cdot \phi)(v) = \phi(k^{-1} v),$$

for $k \in U(V_j)$ and $\phi \in V_j^*$. The corresponding action of $U(V_j)$ on the space of symmetric tensors on V_j, induced in turn by the action of $U(V_j)$ on the tensor algebra of V_j^*, is given by

$$(\rho(k)(\zeta))(v_1, \ldots, v_q) = \zeta(k^{-1} v_1, \ldots, k^{-1} v_q) \tag{1.11}$$

for $\zeta \in S^q(V_j^*)$.

Remark 1.1.8. If we use the identification of the symmetric tensors with homogeneous polynomials of degree q given by

$$f(z) = \zeta(z, \ldots, z)$$

for $\zeta \in S^q(V_j^*)$, $f \in \mathcal{H}_q(V_j)$ and $z \in V_j$, then *i*) the symmetric product (1.8) corresponds to the pointwise product of two functions and

ii) the action in (1.11) coincides with the action of $Gl(V_j)$ on the functions on V_j given by

$$(\varrho'(A)f)(z) := f(A^{-1}z) \qquad A \in Gl(V_j).$$

By using the isomorphism $\Phi : V_j \longrightarrow V'$ the unitary group $U(V_j)$ can be let act on the space of symmetric tensors on V'. For this purpose we simply continue the above calculation

$$\begin{aligned}
\zeta(k^{-1}v_1,\dots,k^{-1}v_q) &= \zeta(k^{-1}\Phi^{-1}(\Phi(v_1)),\dots,k^{-1}\Phi^{-1}(\Phi(v_q))) \\
&= \Psi^{-1}(\Psi\zeta)(k^{-1}\Phi^{-1}(\mathbf{v}_1),\dots,k^{-1}\Phi^{-1}(\mathbf{v}_q)) \\
&= (\Psi^{-1}\xi)(k^{-1}\Phi^{-1}(\mathbf{v}_1),\dots,k^{-1}\Phi^{-1}(\mathbf{v}_q)) \\
&\overset{(1.6)}{=} \xi((\Phi k\Phi^{-1})^{-1}(\mathbf{v}_1)\dots,(\Phi k\Phi^{-1})^{-1}(\mathbf{v}_q)),
\end{aligned}$$

where $\mathbf{v}_i := \Phi(v_i) \in V'$ for all $1 \le i \le n$ and $\xi := \Psi\zeta \in V^{1,0}$. This is exactly the behaviour one would expect by changing the spaces.

Lemma 1.1.9. *The induced group action of $U(V_j)$ on $S^{q,0}(V)$ is given by*

$$(\hat{\rho}(k)(\xi))(\mathbf{v}_1,\dots,\mathbf{v}_q) := \xi(K^{-1}\mathbf{v}_1,\dots,K^{-1}\mathbf{v}_q), \tag{1.12}$$

where $K := \Phi k \Phi^{-1}$.

1.2 The canonical Hermitian connection

If a manifold M admits a non-degenerate 2-form ω, then the pair (M,ω) is called *almost symplectic*, and if the 2-form ω is additionally closed, then the pair (M,ω) is referred to as a *symplectic manifold.*

Definition 1.2.1. An almost complex structure J on an almost symplectic manifold (M,ω) is called $\omega-$*compatible*, if it satisfies

$$\begin{aligned}
\omega(JX,JY) &= \omega(X,Y) && \forall\ X,Y \in TM \\
\omega(X,JX) &> 0 && \forall\ X \in TM \setminus \{0\}.
\end{aligned}$$

In the Corollary 3.2.7 we will show, that any almost symplectic manifold (M,ω) admits an $\omega-$compatible almost complex structure J. Thus, (M,ω) can be equipped by means of such J with a Riemannian metric g, obtained by setting

$$g(X,Y) := \omega(X,JY)$$

for $X, Y \in TM$. In particular, J is orthogonal with respect to g and the *fundamental form* of g is exactly the 2-form ω. Besides, the 2-form

$$h(X,Y) := g(X,Y) - i\omega(X,Y)$$

for $X, Y \in TM$ defines a Hermitian metric h on M. The corresponding unitary frame bundle $\mathcal{U}(M)$, which is by the same argumentation as in the Remark 1.1.2 i) a subbundle of both the symplectic frame bundle $\mathrm{Sp}(M,\omega)$ and the special orthogonal frame bundle $\mathrm{SO}(M,g)$, yields a principal $U(V_j)$–bundle over M.

An almost complex structure J is called *integrable* if its *Nijenhuis tensor* N vanishes identically, where

$$N(X,Y) = [JX, JY] - [X,Y] - J[X,JY] - J[JX,Y]$$

for vector fields X and Y on M.

Lemma 1.2.2. *i*) *The Nijenhuis tensor is of type* $(2,0)$*, that is*

$$N(JX,Y) = N(X,JY) = -JN(X,Y)$$

for all $X, Y \in TM$.

ii) *For a torsion-free connection* ∇ *on* M *the Nijenhuis tensor can be rewritten as*

$$N(X,Y) = (\nabla_{JX}J)(Y) - J(\nabla_X J)(Y) + J(\nabla_Y J)(X) - (\nabla_{JY}J)(X).$$

iii) *For a unitary connection* ∇ *on* M *the Nijenhuis tensor takes the form*

$$N(X,Y) = T(X,Y) + JT(JX,Y) + JT(X,JY) - T(JX,JY).$$

Theorem 1.2.3 (Newlander-Nirenberg)**.** *An almost complex structure* J *on a manifold* M *is integrable if and only if* M *admits local holomorphic coordinates for* J *around each point of* M.

1.2.1 The canonical Hermitian connection. A symplectic manifold, whose compatible almost complex structure is integrable, is a *Kähler manifold.* In this case the Levi-Civita connection ∇ of the Kähler metric g satisfies

$$\nabla g = \nabla\omega = 0, \quad T = 0 \quad \text{and} \quad \nabla J = 0.$$

On the other hand, if the almost complex structure is not integrable, it yields $\nabla J \neq 0$. However, the reduction of the group structure of M to $U(V_j)$ yields a connection, which by the Proposition 1.1.3 satisfies $\tilde{\nabla} J = 0$, $\tilde{\nabla} g = \tilde{\nabla}\omega = 0$. Among all such *unitary* connections there is an exceptional connection, which distinguishes itself from the others by the type of its torsion.

Definition 1.2.4. *i*) The connection $\tilde{\nabla}$ defined by

$$\tilde{\nabla}_X Y := \nabla_X Y - \frac{1}{2} J(\nabla_X J)(Y),$$

for $X, Y \in \Gamma(TM)$, is called the *canonical Hermitian connection.*

ii) If we denote by $\mathfrak{u}(V_j)^\perp$ the orthogonal complement of $\mathfrak{u}(V_j)$ as a subalgebra of $\mathfrak{so}(V)$, then the corrective term $\tau_X := -\frac{1}{2}J(\nabla_X J)$ defines a section $\tau \in \Gamma(T^*M \otimes \mathfrak{u}(V_j)^\perp)$, which is referred to as the *intrinsic torsion of the $U(V_j)$-structure on M (induced by J).*

To determine the type of the torsion $\tilde{T}(X,Y) = \tau_X Y - \tau_Y X$ of $\tilde{\nabla}$ we recall the correspondence between τ and the Nijenhuis tensor of J and $d\omega$. The torsion turns out to be of type $(2,0)$, and thus, the *torsion vector field* of $\tilde{\nabla}$ vanishes identically. In what follows, we base ourselves on the Section 2.2 of Chapter 10 in [5] by Paul-Andi Nagy.

We consider the image of $\nabla\omega$ under the skew-symmetrization map $\mathrm{alt}: T^*M \otimes \Lambda^2 \to \Lambda^3$, which can be expressed in two different ways. On the one hand, a direct calculation yields immediately $\mathrm{alt}(\nabla\omega) = \frac{1}{3}\, d\omega$ and, on the other hand

$$\nabla_X \omega = -\tau_X \omega, \tag{1.13}$$

which is to be understood in the following sense

$$\begin{aligned}(\nabla_X\omega)(Y,Z) &= (\tilde{\nabla}_X\omega)(Y,Z) + \omega(\tau_X Y, Z) + \omega(Y, \tau_X Z)\\ &= g(J\tau_X Y, Z) + g(JY, \tau_X Z)\\ &= 2g(J\tau_X Y, Z) = 2\omega(\tau_X Y, Z).\end{aligned}$$

To obtain another expression for $\mathrm{alt}(\nabla\omega)$, we introduce the tensor field $\mathcal{N}$ defined by

$$\mathcal{N}_X(Y,Z) := g(N(Y,Z),X).$$

Then it holds by Lemma 1.2.2

$$\mathcal{N}_{JX}(Y,Z) = \mathcal{N}_X(JY,Z) = \mathcal{N}_X(Y,JZ) \tag{1.14}$$

and the covariant derivative of ω can be now expressed in terms of the tensor $\mathcal{N}$ and the exterior differential of ω as follows

$$2(\nabla_X\omega)(Y,Z) = -\mathcal{N}_{JX}(Y,Z) + d\omega(X,Y,Z) - d\omega(X,JY,JZ). \tag{1.15}$$

Applying the skew-symmetrization to the upper equation yields

$$\begin{aligned}2\mathrm{alt}(\nabla\omega)(X,Y,Z) &= \frac{2}{3}((\nabla_X\omega)(Y,Z) - (\nabla_Y\omega)(X,Z) + (\nabla_Z\omega)(X,Y))\\ &= \frac{1}{3}(-\mathcal{N}_{JX}(Y,Z) + d\omega(X,Y,Z) - d\omega(X,JY,JZ)\\ &\quad + \mathcal{N}_{JY}(X,Z) - d\omega(Y,X,Z) + d\omega(Y,JX,JZ)\\ &\quad - \mathcal{N}_{JZ}(Y,Z) + d\omega(Z,X,Y) - d\omega(Z,JX,JY)).\end{aligned}$$

In order to interpret the occurring terms we introduce an operator acting on 3-forms

$$(\mathfrak{J}\alpha)(X,Y,Z) := \alpha(JX,Y,Z) + \alpha(X,JY,Z) + \alpha(X,Y,JZ)$$

and observe, that

$$\begin{aligned}
\mathfrak{J}(J^* d\omega)(X,Y,Z) &= J^* d\omega(JX,Y,Z) + J^* d\omega(X,JY,Z) + J^* d\omega(X,Y,JZ) \\
&= -d\omega(X,JY,JZ) - d\omega(JX,Y,JZ) - d\omega(JX,JY,Z) \\
\operatorname{alt}(\mathcal{N})(X,Y,Z) &= \frac{1}{6}(\mathcal{N}_X(Y,Z) + \mathcal{N}_Y(Z,X) + \mathcal{N}_Z(X,Y) \\
&\quad - \mathcal{N}_X(Z,Y) - \mathcal{N}_Y(X,Z) - \mathcal{N}_Z(Y,X)) \qquad (1.16) \\
&= \frac{1}{3}(\mathcal{N}_X(Y,Z) - \mathcal{N}_Y(X,Z) + \mathcal{N}_Z(X,Y)) \\
\mathfrak{J}(\operatorname{alt}(\mathcal{N}))(X,Y,Z) &= \operatorname{alt}(\mathcal{N})(JX,Y,Z) + \operatorname{alt}(\mathcal{N})(X,JY,Z) + \operatorname{alt}(\mathcal{N})(X,Y,JZ) \\
&\overset{(1.14)}{=} \mathcal{N}_{JX}(Y,Z) - \mathcal{N}_{JY}(X,Z) + \mathcal{N}_{JZ}(X,Y),
\end{aligned}$$

such that together with the equation (1.13) we obtain

$$\begin{aligned}
&\frac{2}{3}\, d\omega = -\frac{1}{3}\,\mathfrak{J}(\operatorname{alt}(\mathcal{N})) + d\omega + \frac{1}{3}\,\mathfrak{J}(J^* d\omega) \\
\Leftrightarrow \qquad & -d\omega = -\mathfrak{J}(\operatorname{alt}(\mathcal{N})) + \mathfrak{J}(J^* d\omega).
\end{aligned}$$

To proceed further, we recall the decomposition of $\Lambda^3 = \Lambda^3(T^*M)$ into the real invariant $U(V_j)$–modules (for the details of the decomposition cf. [19])

$$\Lambda^3 = \lambda^{3,0} \oplus \lambda^{2,1},$$

where

$$\begin{aligned}
\lambda^{3,0} &= \{\alpha \in \Lambda^3 \mid \alpha(JX,Y,Z) = \alpha(X,JY,Z) = \alpha(X,Y,JZ) \quad \forall\, X,Y,Z \in TM\} \\
\lambda^{2,1} &= \{\alpha \in \Lambda^3 \mid \alpha(JX,JY,Z) = \alpha(X,Y,Z) = -\alpha(X,JY,JZ) \quad \forall\, X,Y,Z \in TM\}.
\end{aligned}$$

Decomposing $d\omega \in \Lambda^3$ along the decomposition $\Lambda^3 = \lambda^{3,0} \oplus \lambda^{2,1}$ yields

$$\begin{aligned}
& -(d\omega)^{3,0} - (d\omega)^{2,1} = \mathfrak{J}(J^*(d\omega)^{3,0}) + \mathfrak{J}(J^*(d\omega)^{2,1}) - \mathfrak{J}(\operatorname{alt}(\mathcal{N})) \\
\Leftrightarrow \qquad & -(d\omega)^{3,0} = \mathfrak{J}(J^*(d\omega)^{3,0}) - \mathfrak{J}(\operatorname{alt}(\mathcal{N})),
\end{aligned}$$

since $\mathfrak{J}$ acts on $\lambda^{2,1}$ as J^*.

Since

$$\begin{aligned}\mathfrak{J}(J^*(d\omega)^{3,0})(X,Y,Z) &= J^*(d\omega)^{3,0}(JX,Y,Z) + J^*(d\omega)^{3,0}(X,JY,Z) + J^*(d\omega)^{3,0}(X,Y,JZ)\\ &= -(d\omega)^{3,0}(X,JY,JZ) - (d\omega)^{3,0}(JX,Y,JZ) - (d\omega)^{3,0}(JX,JY,Z)\\ &= 3(d\omega)^{3,0}(X,Y,Z)\end{aligned}$$

and $\mathrm{alt}(\mathcal{N}) \in \lambda^{3,0}$, it follows

$$4J^*(d\omega)^{3,0} = 3\mathrm{alt}(\mathcal{N}) \qquad \Leftrightarrow \qquad \mathrm{alt}(\mathcal{N}) = \frac{4}{3}J^*(d\omega)^{3,0}.$$

Decomposing the spaces $\Lambda^2 = \lambda^{2,0} + \lambda^{1,1}$ and Λ^3 into the (real) invariant $U(V_j)$−components and restricting the skew-symmetrization map to $T^*M \otimes \lambda^{2,0}$ we obtain by the Theorem 2.1 in [8] the following diagram.

$$\begin{tikzcd} T^*M \otimes \lambda^{2,0} \arrow[r, "\mathrm{alt}"] \arrow[d, equal, "\simeq_{U(V_j)}"'] & \Lambda^3 \arrow[d, equal, "\simeq_{U(V_j)}"] \\ \lambda^{3,0} \oplus \mathcal{W}_2 \oplus \lambda^{2,1} & \lambda^{3,0} \oplus \lambda^{2,1} \end{tikzcd}$$

The space $\mathcal{W}_2$ denotes whereby the kernel of the skew-symmetrization map (cf. [6], Lemma 2.1). So the tensor $\mathcal{N}$ splits by Schur's lemma as

$$\mathcal{N} = \hat{\mathcal{N}} + \frac{4}{3}J^*(d\omega)^{3,0} \quad \in \mathcal{W}_2 \oplus \lambda^{3,0} \tag{1.17}$$

with $\hat{\mathcal{N}} \in \mathcal{W}_2$. With the aid of (1.13), (1.15) and (1.17) we state the following theorem.

Theorem 1.2.5. *The intrinsic torsion $\tau \in T^*M \oplus \mathfrak{u}(V_j)^\perp$ of an almost Hermitian manifold M is entirely determined by $(d\omega, \hat{\mathcal{N}}) \in \Lambda^3 \oplus \mathcal{W}_2$ and vice versa. In particular,*

- *M is Hermitian iff $\tau \in \lambda^{2,1}$ and*
- *M is almost-Kähler iff $\tau \in \mathcal{W}_2$.*

If we assume M to be almost Kähler, i. e. $\tau \simeq \hat{\mathcal{N}}$, then by the equation (1.16) it holds

$$\begin{aligned} & g(\tau_Y(X),Z) = g(\tau_X(Y),Z) + g(\tau_Z(X),Y)\\ \Leftrightarrow \quad & g(\tilde{T}(X,Y),Z) = -g(\tau_Z(X),Y) \end{aligned}\tag{1.18}$$

and

$$g(\tau_{JX}(Y),Z) = g(-J\tau_X(Y),Z) = g(\tau_X(JY),Z) \tag{1.19}$$

so we obtain the following result.

Corollary 1.2.6. *If M is an almost-Kähler manifold, then the torsion $\tilde{T}$ of $\tilde{\nabla}$ is of type $(2,0)$ and*

$$\hat{\mathcal{N}}_X(Y,Z) = 4g(\tilde{T}(Y,Z),X) \qquad \textit{or equivalently} \qquad N(X,Y) = 4\tilde{T}(X,Y).$$

1.2.2 The curvature of the canonical connection. The following section is based on the Section 2 in [16]. We keep considering an almost Kähler manifold (M, ω, J). Denote by R the curvature tensor of the Levi-Civita connection ∇ and by $\tilde{R}$ the curvature of $\tilde{\nabla}$. A direct calculation then yields

$$\begin{aligned} \tilde{R}(X,Y) &= R(X,Y) + \mathrm{alt}(\tilde{\nabla}\tau)_{X,Y} - [\tau_X, \tau_Y] + \tau_{(\tau_X Y - \tau_Y X)} \\ &=: \underbrace{R(X,Y)}_{\in \mathfrak{so}(2n)} + \underbrace{\mathrm{alt}(\tilde{\nabla}\tau)_{X,Y}}_{\in \mathfrak{u}(n)^\perp} - \underbrace{\tau^2(X,Y)}_{\in \mathfrak{u}(n)} + \underbrace{\tau^T(X,Y)}_{\in \mathfrak{u}(n)^\perp}. \end{aligned} \tag{1.20}$$

Since $\tilde{R} \in \Lambda^2 \otimes \mathfrak{u}(V_j)$, its corresponding 4−tensor lies in

$$\Lambda^2 \otimes \lambda^{1,1} = (\lambda^{2,0} \oplus \lambda^{1,1}) \otimes \lambda^{1,1} = S^2(\lambda^{1,1}) \oplus \Lambda^2(\lambda^{1,1}) \oplus (\lambda^{2,0} \otimes \lambda^{1,1})$$

and it therefore splits accordingly as $\tilde{R} = \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3$ with

$$\tilde{R}_1 \in S^2(\lambda^{1,1}), \quad \tilde{R}_2 \in \Lambda^2(\lambda^{1,1}) \quad \text{and} \quad \tilde{R}_3 \in \lambda^{2,0} \otimes \lambda^{1,1}.$$

Consider next the terms occurring in the equation (1.20) depending on τ.

i) Since $\tilde{\nabla}$ is unitary and $\tau_X \in \lambda^{2,0}$, it follows $(\tilde{\nabla}_X \tau)_Y \in \lambda^{2,0}$. That means $\mathrm{alt}(\tilde{\nabla}\tau) \in \Lambda^2 \otimes \lambda^{2,0}$ and it splits along the decomposition $\Lambda^2 \otimes \lambda^{2,0} = (\lambda^{1,1} \otimes \lambda^{2,0}) \oplus (\lambda^{2,0} \otimes \lambda^{2,0})$ as

$$\mathrm{alt}(\tilde{\nabla}\tau) = \mathrm{alt}(\tilde{\nabla}\tau)_1 + \mathrm{alt}(\tilde{\nabla}\tau)_2.$$

ii) Since $\tau \in \mathcal{W}_2$, we have by the relation (1.19)

$$\tau^2 \in \lambda^{1,1} \otimes \lambda^{1,1} = S^2(\lambda^{1,1}) \oplus \Lambda^2(\lambda^{1,1}),$$

so τ^2 splits as

$$\tau^2 = S + A$$

with $S \in S^2(\lambda^{1,1})$ and $A \in \Lambda^2(\lambda^{1,1})$.

iii) Similarly, since $\tau \in \mathcal{W}_2$, we have by the relations (1.18) and (1.19) $\tau^T \in S^2(\lambda^{2,0})$.

	$S^2(\lambda^{1,1})$	$\oplus$	$S^2(\lambda^{2,0})$	$\oplus$	$\Lambda^2(\lambda^{1,1})$	$\oplus$	$\lambda^{2,0} \otimes \lambda^{1,1}$	$\lambda^{1,1} \otimes \lambda^{2,0}$
$\tilde{R}$	$\tilde{R}_1$			$+$	$\tilde{R}_2$	$+$	$\tilde{R}_3$	
$\mathrm{alt}(\tilde{\nabla}\tau)$			$\mathrm{alt}(\tilde{\nabla}\tau)_2$			$+$		$\mathrm{alt}(\tilde{\nabla}\tau)_1$
τ^2	S			$+$	A			
τ^T			τ^T					

Next we observe, that

$$\begin{aligned}\tilde{R}(X,Y,Z,W)-\tilde{R}(Z,W,X,Y)&=\mathrm{alt}(\tilde{\nabla}\tau)_{X,Y}(Z,W)-\tau^2(X,Y)(Z,W)\\&\quad-\mathrm{alt}(\tilde{\nabla}\tau)_{Z,W}(X,Y)+\tau^2(Z,W)(X,Y)\\&=\mathrm{alt}(\tilde{\nabla}\tau)_{X,Y}(Z,W)-\mathrm{alt}(\tilde{\nabla}\tau)_{Z,W}(X,Y)-2A(X,Y,Z,W).\end{aligned}$$

Since $\tilde{R}$ and A are elements of $\Lambda^2\otimes\lambda^{1,1}$, we obtain after projecting the equation onto $\lambda^{2,0}\otimes\lambda^{2,0}$ that $\mathrm{alt}(\tilde{\nabla}\tau)_2\in S^2(\lambda^{2,0})$. Moreover, if we project the upper equation onto $\Lambda^2(\lambda^{1,1})$ and $\lambda^{2,0}\otimes\lambda^{1,1}$ we obtain

$$\tilde{R}_2=-A\quad\text{and}\quad\tilde{R}_3=\mathrm{alt}(\tilde{\nabla}\tau)_1^\star$$

respectively. The corresponding 4-tensor of the Riemannian curvature tensor R of ∇ is an element of $S^2(\Lambda^2)$, which splits as

$$S^2(\Lambda^2)=S^2(\lambda^{1,1}\oplus\lambda^{2,0})=S^2(\lambda^{1,1})\oplus S^2(\lambda^{2,0})\oplus(\lambda^{2,0}\otimes\lambda^{1,1}),$$

and thus, R can be written as

$$R=\begin{pmatrix}R_1 & R_{12}\\ R_{12}^\star & R_2\end{pmatrix},$$

where

$$R_1\in S^2(\lambda^{1,1}),\quad R_2\in S^2(\lambda^{2,0}),\quad R_{12}\in\lambda^{1,1}\otimes\lambda^{2,0}.$$

We conclude by comparison the algebraic types of the occurring tensors in (1.20), that

$$\begin{aligned}&\tilde{R}_1=R_1-S\in S^2(\lambda^{1,1}), &&\tilde{R}_2=-A\in\Lambda^2(\lambda^{1,1}),\\&\tilde{R}_3=\mathrm{alt}(\tilde{\nabla}\tau)_1^\star\in\lambda^{2,0}\otimes\lambda^{1,1}, &&R_2-\mathrm{alt}(\tilde{\nabla}\tau)_2+\tau^T=0\in S^2(\lambda^{2,0}).\end{aligned}$$

Lemma 1.2.7. *With the upper notations it holds*

$$\tilde{R}=(R_1-S)-A+\mathrm{alt}(\tilde{\nabla}\tau)_1^\star.$$

Remark 1.2.8. If the tensor S additionally decomposed as $S=S^b+S^-$, where S^b satisfies the first Bianchi identity, then the component $\tilde{R}^K:=R_1-S^b=\tilde{R}_1+S^-$ of $\tilde{R}$ satisfies all the requirements of a *Kähler curvature*, that is an element of $S^2(\lambda^{1,1})$ satisfying the algebraic Bianchi identity.

Chapter 2

A short digression of Lichnerowicz Laplacian

The main purpose of this chapter is to introduce a Laplacian defined on the complexified symmetric algebra of an almost Kähler manifold using the canonical Hermitian connection. which was firstly defined by André Lichnerowicz in his paper [14] in the Riemannian setting.

2.1 Symmetric algebra of M

We begin by describing a natural Laplacian acting on the sections of the symmetric algebra of M as it was introduced in the original paper [14] by André Lichnerowicz. For this purpose we consider a Riemannian manifold (M, g). The Riemannian metric g on M induces analogously to (1.9) a Riemannian metric on $S^*(M) = \bigoplus_{q \geq 0} S^q(M)$, i. e., if $\varphi_{i_1 \dots i_q}$ and $\psi_{i_1 \dots i_q}$ denote the components of the tensor fields $\varphi, \psi \in S^*(M)$, then

$$\langle \varphi, \psi \rangle = \varphi^{j_1 \cdots j_q} \psi_{j_1 \dots j_q}, \tag{2.1}$$

where $\varphi^{j_1 \cdots j_q} = g^{i_1 j_1} \cdots g^{i_q j_q} \varphi_{i_1 \dots i_q}$ and g^{ij} denotes the components of the inverse matrix (g_{ij}). Similarly, a connection ∇ on TM induces a connection

$$\hat{\nabla} : \Gamma(S^*(M)) \to \Gamma(T^*M \otimes S^*(M)).$$

To describe $\hat{\nabla}$ locally, we observe first, that the induced connection ∇^* on T^*M is given by

$$(\nabla^*_X \vartheta)(Y) = X(\vartheta(Y)) - \vartheta(\nabla_X Y)$$

for $X, Y \in \Gamma(TM)$ und $\vartheta \in \Gamma(T^*M)$. Thus the induced connection $\nabla^{\otimes}$ on the tensor bundle $T^*M^{\otimes q}$ is given by

$$(\nabla^{\otimes}_X \zeta)(Y_1, \dots, Y_q) = X(\zeta(Y_1, \dots, Y_q)) - \sum_{\nu=1}^{q} \zeta(Y_1, \dots, \nabla_X Y_\nu, \dots, Y_q).$$

The restriction of ∇_X to $S^q(M)$ can be rewritten, if an element ζ of $S^q(M)$ is written in terms of some local frame as $\zeta = c_{i_1\dots i_q}\,\varphi_{i_1} \odot \dots \odot \varphi_{i_q}$ for $\varphi_{i_k} \in T^*M$. The covariant derivative of ζ in the direction X then reads

$$\hat{\nabla}_X\zeta = dc_{i_1\dots i_q}(X)\,\varphi_{i_1} \odot \cdots \odot \varphi_{i_q} + c_{i_1\dots i_q}\sum_{\nu=1}^{q}(\nabla^*_X\varphi_{i_\nu}) \odot \varphi_{i_1} \odot \cdots \odot \hat{\varphi}_{i_\nu} \odot \cdots \odot \varphi_{i_q}.$$

Since ζ depends only on the number of times each φ_k appears in the product, we may rewrite ζ as $\zeta = c_\alpha\,\varphi_1^{\alpha_1} \odot \cdots \odot \varphi_n^{\alpha_n}$, where $\varphi_\nu^{\alpha_\nu} := \varphi_\nu \odot \cdots \odot \varphi_\nu$, and obtain the following compact notation.

Lemma 2.1.1. *Let $\{e_1,\dots,e_n\}$ be a local frame for TM and $\{\varphi_1,\dots,\varphi_n\}$ be its dual frame. Writing $\zeta = c_\alpha\varphi^\alpha$ as above yields*

$$\hat{\nabla}\zeta = dc_\alpha \otimes \varphi^\alpha - \sum_{\nu,k,m=1}^{n}\Gamma^\nu_{m,k}\,\varphi_m \otimes \varphi_k \odot (e_\nu \lrcorner\, \zeta),$$

where $\Gamma^\nu_{m,k}$ denote the Christoffel symbols of ∇. The curvature tensor $\hat{R}$ of $\hat{\nabla}$ is given by

$$\hat{R}(e_j,e_k)\zeta = -\sum_{i,m=1}^{n} R^i_{j,k,m}\;\varphi_m \odot (e_i \lrcorner\, \zeta).$$

Remark 2.1.2. *i)* If ∇ is metric, then $\hat{\nabla}$ is also metric with respect to the induced metric.

ii) The linear part of $\hat{\nabla}$ is induced by the standard group action of $Gl(V)$ on $S^*(V^*)$

$$(k\cdot\zeta)(v_{i_1},\dots,v_{i_q}) = \zeta(k^{-1}v_{i_1},\dots,k^{-1}v_{i_q})$$

for $k \in Gl(V)$ and $\zeta \in S^*(V^*)$. (Cf. the action in (1.11).)

The L^2*−inner product* defined on the compactly supported sections of $S^q(M)$ is given by

$$\langle\varphi,\psi\rangle_{L^2} = \int_M \langle\varphi,\psi\rangle\;\mathrm{d}M. \tag{2.2}$$

Let $\hat{\nabla}$ be induced by a metric connection ∇ on M and $\varphi \in S^q(M)$ and $\psi \in T^*M \otimes S^q(M)$. Then we observe for an orthonormal frame parallel at a point

$$\begin{aligned}
\langle\hat{\nabla}\varphi,\psi\rangle &= \sum_{j=1}^{n}\left(\hat{\nabla}\varphi\right)_{j\,i_1\dots i_p}\;\psi^{j\,i_1\dots i_p}\\
&= \sum_{j=1}^{n}\left(\hat{\nabla}_{e_j}\varphi\right)_{i_1\dots i_p}(e_j \lrcorner\,\psi)^{i_1\dots i_p}\\
&= \sum_{j=1}^{n}\langle\hat{\nabla}_{e_j}\varphi, e_j \lrcorner\,\psi\rangle\\
&= \sum_{j=1}^{n} e_j\cdot\langle\varphi, e_i \lrcorner\,\psi\rangle - \sum_{j=1}^{n}\langle\varphi,\hat{\nabla}_{e_j} e_j \lrcorner\,\psi\rangle.
\end{aligned}$$

By integrating the last equation the first sum vanishes, since $\sum_{j=1}^{n} e_j\cdot\langle\varphi, e_j \lrcorner\,\psi\rangle = \mathrm{div}(Y)$ for $Y = \sum_{j=1}^{n}\langle\varphi, e_j \lrcorner\,\psi\rangle e_j$. We thus have shown the following lemma.

Lemma 2.1.3. *The formal adjoint of* $\hat{\nabla}^* : \Gamma(T^*M \otimes S^*(M)) \to \Gamma(S^*(M))$ *is given with respect to a local orthonormal frame parallel at a point by*

$$\hat{\nabla}^*\xi = -\sum_{j=1}^n e_j \lrcorner \hat{\nabla}_{e_j}\xi.$$

Corollary 2.1.4. *With respect to an orthonormal frame* $\{e_1, \dots, e_n\}$ *parallel at a point the Bochner Laplacian* $\hat{\nabla}^*\hat{\nabla}$ *of a metric connection is given by*

$$\hat{\nabla}^*\hat{\nabla}\xi = -\sum_{j=1}^n \hat{\nabla}_{e_j}(e_j \lrcorner \hat{\nabla}\xi) = -\sum_{j=1}^n \hat{\nabla}_{e_j}\hat{\nabla}_{e_j}\xi.$$

2.1.1 The Lichnerowicz Laplacian. The Laplacian introduced by André Lichnerowicz on the tensor algebra $\bigoplus T^*M^{\otimes n}$ should have generalized the Hodge-de Rham Laplacian when restricting to the exterior algebra. It turned out, that the restriction of the *Lichnerowicz Laplacian* to the symmetric algebra coincides with the commutator of the symmetrized connection with its formal adjoint (cf. [20], p 147). In what follows, we shortly describe the mentioned commutator.

The symmetrized connection is given as the composition $\delta := \mathrm{sym} \circ \hat{\nabla}$

$$S^q(T^*M) \xrightarrow{\hat{\nabla}} T^*M \otimes S^q(M) \xrightarrow{\mathrm{sym}} S^{q+1}(M),$$

which by the Lemma 2.1.1 can be expressed as follows.

Lemma 2.1.5 (Symmetrized connection)**.** *Let* $\{e_1, \dots, e_n\}$ *be a local frame for* TM *and denote by* $\{\varphi_1, \dots, \varphi_n\}$ *its dual frame. Writing* $\zeta = c_\alpha\varphi^\alpha$ *as above yields*

$$\delta\zeta = \sum_{k=1}^n \varphi_k \odot \hat{\nabla}_{e_k}\zeta. \tag{2.3}$$

Remark 2.1.6. If $\hat{\nabla}$ is induced by a metric connection, then the formal adjoint δ^* of δ coincides with ∇^*. To observe that in terms of a local orthonormal frame $\{e_1, \dots, e_n\}$ parallel at a point we have

$$\langle\delta\varphi, \zeta\rangle = \sum_{k=1}^n \langle\varphi_k \odot \hat{\nabla}_{e_k}\varphi, \zeta\rangle = \sum_{k=1}^n \langle\hat{\nabla}_{e_k}\varphi, e_k \lrcorner \zeta\rangle = \sum_{k=1}^n e_k \cdot \langle\varphi, e_k \lrcorner \zeta\rangle - \langle\varphi, e_k \lrcorner \hat{\nabla}_{e_k}\zeta\rangle$$

and, since $\sum_{k=1}^n e_k \cdot \langle\varphi, e_k \lrcorner \zeta\rangle = \mathrm{div}(Y)$ for $Y = \sum_{k=1}^n \langle\varphi, e_k \lrcorner \zeta\rangle e_k$ as above, it follows

$$\delta^*\zeta = \nabla^*\zeta = -\sum_{k=1}^n e_k \lrcorner \hat{\nabla}_{e_k}\zeta.$$

Lemma 2.1.7. *The principal symbol of* δ *at some* $x \in M$ *and* $\theta \in T_x^*M \setminus \{0\}$ *is given by*

$$ps(\delta)(x, \theta)\zeta_x = \theta \odot \zeta_x.$$

Proof. It is a short verification of the definition (cf. [23], p. 115). Since we deal with a real first-order differential operator, we choose a smooth function $f \in C^\infty(M)$, such that $df_x = \theta$ and observe with the Lemma 2.1.5, that

$$\begin{aligned} ps(\delta)(x,\theta)\zeta_x &= \sum_{k=1}^{n} \varphi_k \odot \hat{\nabla}_{e_k}((f - f(x))\zeta)_x \\ &= \sum_{k=1}^{n} \varphi_k \odot df_x(e_k)\zeta_x \\ &= \sum_{k=1}^{n} \theta(e_k)\,\varphi_k \odot \zeta_x = \theta \odot \zeta_x. \end{aligned}$$

□

The same calculation made for the formal ajoint of δ yields

$$ps(\delta^*)(x,\theta)\zeta_x = -\theta^\# \lrcorner\, \zeta_x$$

and, in particular, we have by the equation (2.4) in [23]

$$ps([\delta^*,\delta])(x,\theta)\zeta_x = -\theta^\# \lrcorner\, \theta \odot \zeta_x + \theta \odot \theta^\# \lrcorner\, \zeta_x = -g(\theta,\theta)\,\zeta_x,$$

which motivates the following definition.

Definition 2.1.8. The commutator $\Delta_L := [\delta^*, \delta]$ is called the *Lichnerowicz Laplacian.*

The Lichnerowicz Laplacian is a self-adjoint, elliptic and type preserving second order differential operator and has discrete eigenvalues with finite multiplicities on compact manifolds (cf. [3] or [2], p. 465).

Theorem 2.1.9. *It holds*

$$\Delta_L = \hat{\nabla}^*\hat{\nabla} - \mathcal{R},$$

where the linear part $\mathcal{R}$ depends only on the curvature of ∇ and is explicitly given by

$$\mathcal{R}(\zeta)_{i_1\dots i_p} = \sum_k g^{js}\,\mathrm{ric}_{i_k j}\,\zeta_{i_1\dots i_{k-1}\,s\,i_{k+1}\dots i_p} - \sum_{k\neq l} g^{js} g^{mt}\,R^m_{j i_k i_l}\,\zeta_{i_1\dots i_{k-1}\,s\,i_{k+1}\dots i_{l-1}\,t\,i_{l+1}\dots i_p}.$$

Proof. As before we choose an orthonormal frame parallel at a point and observe, that

$$\begin{aligned} \delta^*\delta\zeta - \delta\delta^*\zeta &= \sum_{j=1}^{n} -e_j \lrcorner\, \hat{\nabla}_{e_j}\delta\zeta - \varphi_j \odot \hat{\nabla}_{e_j}\delta^*\zeta \\ &= \sum_{j,k=1}^{n} -e_j \lrcorner\, \hat{\nabla}_{e_j}(\varphi_k \odot \hat{\nabla}_{e_k}\zeta) + \varphi_j \odot \hat{\nabla}_{e_j}(e_k \lrcorner\, \hat{\nabla}_{e_k}\zeta) \\ &= -\sum_{k=1}^{n} \hat{\nabla}_{e_k}\hat{\nabla}_{e_k}\zeta - \sum_{j,k=1}^{n} \varphi_k \odot e_j \lrcorner\, \hat{\nabla}_{e_j}\hat{\nabla}_{e_k}\zeta + \varphi_j \odot e_k \lrcorner\, \hat{\nabla}_{e_j}\hat{\nabla}_{e_k}\zeta \end{aligned}$$

$$= \hat{\nabla}^*\hat{\nabla}\zeta - \sum_{j,k=1}^{n} \varphi_k \odot e_j \lrcorner (\hat{\nabla}_{e_j}\hat{\nabla}_{e_k}\zeta - \hat{\nabla}_{e_k}\hat{\nabla}_{e_j}\zeta)$$

$$= \hat{\nabla}^*\hat{\nabla}\zeta - \sum_{j,k=1}^{n} \varphi_k \odot e_j \lrcorner \hat{R}(e_j, e_k)\zeta,$$

where the last equation holds by the choice of the frame. Write the linear part using the Lemma 2.1.1 out, then

$$\begin{aligned}
\mathcal{R}(\zeta) &= -\sum_{i,j,k,m=1}^{n} R^i_{jkm}\ \varphi_k \odot (e_j \lrcorner \varphi_m \odot (e_i \lrcorner \zeta)) \\
&= -\sum_{i,j,k,m=1}^{n} R^i_{jkm}\ \varphi_k \odot (\varphi_m \odot (e_j \lrcorner e_i \lrcorner \zeta)) - \sum_{i,j,k=1}^{n} R^i_{jkj}\ \varphi_k \odot (e_i \lrcorner \zeta) \\
&= -\sum_{i,j,k,m=1}^{n} R^i_{jkm}\ \varphi_k \odot (\varphi_m \odot (e_j \lrcorner e_i \lrcorner \zeta)) + \sum_{i,k=1}^{n} \mathrm{ric}_{ki}\ \varphi_k \odot (e_i \lrcorner \zeta)
\end{aligned}$$

and by writing $\zeta = \sum \zeta_{i_1\dots i_q}\, \varphi_{i_1} \odot \cdots \odot \varphi_{i_q}$ we obtain the desired expression for $\mathcal{R}(\zeta)$

$$\mathcal{R}(\zeta)_{j_1\cdots j_q} = \sum_{i=1}^{n}\sum_{s=1}^{q} \mathrm{ric}_{j_s i}\ \zeta_{j_1\cdots j_{s-1} i j_{s+1}\cdots j_q} - \sum_{i,k=1}^{n} \sum_{\substack{r,s=1\\ r\neq s}}^{q} R^k_{ij_sj_r}\ \zeta_{j_1\cdots j_{s-1} i j_{s+1}\cdots j_{r-1} k j_{r+1}\cdots j_q}.$$

□

Remark 2.1.10. The linear term of the Lichnerowicz Laplacian can also be written as

$$\begin{aligned}
\mathcal{R}(\zeta)(Y_{j_1}, \dots, Y_{j_q}) &= \sum_{s=1}^{q} \zeta(\mathrm{Ric}(Y_{i_s}), Y_{j_1}, \dots, \hat{Y}_{j_s}, \dots, Y_{j_q}) \\
&\quad - \sum_{j=1}^{n} \sum_{\substack{r,s=1\\ r\neq s}}^{q} \zeta(R(e_j, Y_{j_s})Y_{j_r}, e_j, Y_{j_1}, \dots, \hat{Y}_{j_s}, \dots, \hat{Y}_{j_r}, \dots, Y_{j_q}).
\end{aligned}$$

2.2 Complex symmetric algebra of M

Consider an almost symplectic manifold (M, ω) together with an ω−compatible almost complex structure J. The complex linear extension of J to the complexified tangent bundle $T_{\mathbb{C}}M$ yields a decomposition of $T_{\mathbb{C}}M$ into the direct sum of ±i−eigenbundles of J as $T_{\mathbb{C}}M = T'M \oplus T''M$. This in turn induces a decomposition of the complex cotangent bundle as $T^*_{\mathbb{C}}M = T^{1,0}M \oplus T^{0,1}M$, which results in the *type decomposition* of the complex symmetric algebra. That is, for some $r \in \mathbb{N}$ it holds

$$S^r(M, \mathbb{C}) = \bigoplus_{p+q=r} S^p(T^{1,0}M) \otimes S^q(T^{0,1}M) =: \bigoplus_{p+q=r} S^{p,q}(M). \tag{2.4}$$

Lemma 2.2.1. *i) The complex bilinear extension of the induced Riemannian metric on $S^*(M)$ to $S^*(M,\mathbb{C})$ yields by setting*

$$(\varphi,\psi) := \langle \varphi, \bar{\psi}\rangle \tag{2.5}$$

a Hermitian metric on $S^*(M,\mathbb{C})$ and a corresponding Hermitian L^2–product on the compactly supported sections of $S^*(M,\mathbb{C})$ as in (2.2). Moreover, the type decomposition (2.4) is orthogonal with respect to this Hermitian L^2–product[1]*.*

ii) If ∇ is a unitary connection on (M,ω,J), that is $\nabla J = 0$, then

1. *the complex bilinear extension of $\hat{\nabla}$ remains metric with respect to the induced Hermitian metric in (2.5)*
2. *∇_X respects the decomposition of $T_{\mathbb{C}}M = T'M \oplus T''M$ for all vector fields X on M.*

2.2.1 Symmetrized connection. Consider the complex bilinear extension of an arbitrary connection, also denoted as $\nabla : T_{\mathbb{C}}M \to T^*_{\mathbb{C}}M \otimes T_{\mathbb{C}}M$ and the corresponding induced connection $\hat{\nabla}$ on $S^*(M,\mathbb{C})$. It is a well-known fact, that the exterior derivative $d : \Lambda^*(M,\mathbb{C}) \longrightarrow \Lambda^{*+1}(M,\mathbb{C})$ splits because of the type decomposition

$$\Lambda^r(M,\mathbb{C}) = \bigoplus_{p+q=r} \Lambda^{p,q}(M)$$

for $r \in \mathbb{N}_0$ as $d = d^{1,0} + d^{2,-1} + d^{-1,2} + d^{0,1}$. In particular,

$$df = d^{1,0}f + d^{0,1}f \tag{2.6}$$

for all $f \in C^\infty(M)$. As already seen in the Section 1.2.1 the exterior derivative can be thought of as a multiple of the skew-symmetrization of a torsion-free connection. We describe next an analogous type decomposition of a symmetrized connection. To observe that consider the symmetrization of $\hat{\nabla}\xi$ for some $\xi \in S^{p,q}(M)$ with $p+q=r$

$$\delta\xi(X_0,\ldots,X_r) = \sum_{i=0}^{r} \nabla_{X_i}\xi(X_0,\ldots,\hat{X}_i,\ldots,X_r),$$

for $X_i \in T_{\mathbb{C}}M$ for all $0 \le i \le r$. Since on the symmetric tensors of rank one we have by (2.6)

$$\delta\xi \in S^{2,0}(M) \oplus S^{1,1}(M) \oplus S^{0,2}(M),$$

we obtain by induction and the Leibniz's rule the following observation.

Corollary 2.2.2. *The map $\delta : S^*(M,\mathbb{C}) \to S^{*+1}(M,\mathbb{C})$ splits as*

$$\delta = \delta^{1,0} + \delta^{2,-1} + \delta^{-1,2} + \delta^{0,1},$$

[1] Note, that each space $S^{p,q}(M)$ is an irreducible representation of $U(V_j)$.

where

$$\begin{tikzcd} S^{p,q} \arrow[r, bend right, "\delta^{1,0}"'] & S^{p+1,q} \arrow[l, dashed, bend right, "\delta^{1,0*}"'] \end{tikzcd} \qquad \begin{tikzcd} S^{p,q} \arrow[r, bend right, "\delta^{0,1}"'] & S^{p,q+1} \arrow[l, dashed, bend right, "\delta^{0,1*}"'] \end{tikzcd}$$

$$\begin{tikzcd} S^{p,q} \arrow[r, bend right, "\delta^{-1,2}"'] & S^{p-1,q+2} \arrow[l, dashed, bend right, "\delta^{-1,2*}"'] \end{tikzcd} \qquad \begin{tikzcd} S^{p,q} \arrow[r, bend right, "\delta^{2,-1}"'] & S^{p+2,q-1} \arrow[l, dashed, bend right, "\delta^{2,-1*}"'] \end{tikzcd}$$

2.2.2 Observation on the $(-1,2)$ and $(2,-1)$ parts of δ. For the first we observe, that for any symmetric tensor ξ and a function $f \in C^\infty(M)$ we have

$$\delta(f\xi) = df \odot \xi + f\delta\xi$$

The left hand side of the upper equation decomposes by types as in the Corollary (2.2.2), while the decomposition of the right hand side yields by (2.6)

$$d^{1,0}f \odot \xi + d^{0,1}f \odot \xi + f\delta^{1,0}\xi + f\delta^{2,-1}\xi + f\delta^{-1,2}\xi + f\delta^{0,1}\xi.$$

By comparing the types we obtain the following lemma.

Lemma 2.2.3. *The parts $\delta^{-1,2}$ and $\delta^{2,-1}$ are tensorial, that is*

$$\delta^{2,-1}(f\xi) = f\,\delta^{2,-1}\xi \quad \textit{and} \quad \delta^{-1,2}(f\xi) = f\,\delta^{-1,2}\xi$$

for all $f \in C^\infty(M)$.

Next we give explicit expressions of $\delta^{-1,2}$ and $\delta^{2,-1}$ first for a unitary and then for a torsion-free connection. We denote the decomposition of an $X \in T_{\mathbb{C}}M$ as $X = X' + X''$, with $X' \in T'M$ and $X'' \in T''M$, and consider a symmetric tensor $\xi \in S^{0,1}(M)$. Then $\delta^{2,-1}\xi \in S^{2,0}(M)$ and since $\delta^{1,0}\xi$ and $\delta^{0,1}\xi$ vanish on $T'M \otimes T'M$ we obtain

$$\begin{aligned}
(\delta^{2,-1}\xi)(X,Y) &= (\delta^{2,-1}\xi)(X',Y') \\
&= (\delta\xi)(X',Y') \\
&= (\hat{\nabla}_{X'}\xi)(Y') + (\hat{\nabla}_{Y'}\xi)(X') \\
&= X' \cdot \xi(Y') - \xi(\nabla_{X'}Y') + Y' \cdot \xi(X') - \xi(\nabla_{Y'}X') \\
&= -\xi(\nabla_{X'}Y' + \nabla_{Y'}X') \\
&= -\xi\big((\nabla_{X'}Y' + \nabla_{Y'}X')''\big).
\end{aligned}$$

Analogous calculation for $\delta^{-1,2}\xi$ and $\xi \in S^{1,0}(M)$ yields

$$\begin{aligned}(\delta^{-1,2}\xi)(X,Y) &= -\xi(\nabla_{X''}Y'' + \nabla_{Y''}X'')\\ &= -\xi\big((\nabla_{X''}Y'' + \nabla_{Y''}X'')'\big),\end{aligned}$$

which provides together with the Lemma 2.2.1 *ii)* 2. and the Leibniz's rule the following observation.

Lemma 2.2.4. *If the connection on M is chosen to be unitary, then $\delta^{-1,2} = \delta^{2,-1} = 0$.*

Remark 2.2.5. In particular, the $(-1,2)$ and $(2,-1)$ parts of the symmetrized connection does not provide any information about the integrability of J, if the connection is chosen to be unitary. Note also, that the statement of the lemma holds also true for the skew-symmetrization of a unitary connection.

For a not unitary connection we continue the above calculation and obtain

$$\begin{aligned}4(\nabla_{X'}Y' + \nabla_{Y'}X') &= \nabla_X Y - \nabla_{JX}JY + \nabla_Y X - \nabla_{JY}JX\\ &\quad - i(\nabla_X JY + \nabla_{JX}Y + \nabla_Y JX + \nabla_{JY}X).\end{aligned}$$

In particular, for a torsion-free connection a the straightforward calculation yields, because of $\nabla_X Y = \nabla_Y X + [X,Y]$,

$$4(\nabla_{X'}Y' + \nabla_{Y'}X')'' = -((\nabla_{JX}J)(Y) - J(\nabla_X J)(Y) + (\nabla_{JY}J)(X) - J(\nabla_Y J)(Y))''$$

$$4(\nabla_{X''}Y'' + \nabla_{Y''}X'')' = -((\nabla_{JX}J)(Y) - J(\nabla_X J)(Y) + (\nabla_{JY}J)(X) - J(\nabla_Y J)(X))'.$$

If we define

$$N^1(X,Y) := (\nabla_{JX}J)(Y) - J(\nabla_X J)(Y) \quad \text{and} \quad N^2(X,Y) := (\nabla_{JY}J)(X) - J(\nabla_Y J)(X),$$

such that $N = N^1 - N^2$. Then by setting

$$\tilde{N} := N^1 + N^2$$

it holds

$$(\nabla_{X'}Y' + \nabla_{Y'}X')'' = -\frac{1}{4}\tilde{N}(X,Y)''$$

and

$$(\nabla_{X''}Y'' + \nabla_{Y''}X'')' = -\frac{1}{4}\tilde{N}(X,Y)'$$

so that we proved the following result.

Lemma 2.2.6. *If ∇ is a torsion-free connection on M, then*

$$\frac{1}{4}\tilde{N}^*\xi = \begin{cases} \delta^{-1,2}\xi, & \text{if } \xi \in \Gamma(T^{1,0}M) \\ \delta^{2,-1}\xi, & \text{if } \xi \in \Gamma(T^{0,1}M). \end{cases}$$

Remark 2.2.7. The table lists the information one gets from the (skew-) symmetrization of the complex bilinear extension of a connection depending on its type.

connection ∇	skew-symmetrization	symmetrization
torsion-free	$d^{-1,2}, d^{2,-1} \sim N^*$	$\delta^{-1,2}, \delta^{2,-1} \sim \tilde{N}^*$
unitary	$(\mathrm{alt} \circ \nabla)^{-1,2}, (\mathrm{alt} \circ \nabla)^{2,-1} \equiv 0$	$\delta^{-1,2}, \delta^{2,-1} \equiv 0$

Proposition 2.2.8. *It holds*

i) $\tilde{N} \in S^2(M) \otimes TM$, *that is,* $\tilde{N}$ *is symmetric, and*
ii) $N = 0$ *if and only if* $\tilde{N} = 0$.

Proof. The assertion $i)$ follows by definition. For $ii)$ the direction „⇒“ holds trivially, while the opposite direction follows by considering the tensor

$$B(X,Y,Z) := \omega((\nabla_{JX}J)(Y) - J(\nabla_X J)(Y), Z) = \omega(N^1(X,Y), Z).$$

Since $N^1 = -N^2$ by assumption, B is skew-symmetric in the first two variables, i. e. $B(X,Y,Z) = -B(Y,X,Z)$. On the other hand, J and $\nabla_X J$ are anti-commuting skew-symmetric operators, so by skew-symmetry of ω the tensor B is symmetric in the last two variables, i. e. $B(X,Y,Z) = B(X,Z,Y)$. By taking the circular permutations we obtain

$$B(X,Y,Z) = -B(Y,Z,X) = B(Z,X,Y) = -B(X,Y,Z),$$

which implies $B = 0$ and hence $N^1 = 0$, since ω is non-degenerate, and the assertion follows by $i)$. □

Remark 2.2.9. A manifold M is Kähler if and only if the Levi-Civita connection satisfies

$$\delta^{-1,2} = \delta^{2,-1} = 0.$$

2.2.3 Frames and coframes for $T'M$ resp. $T^{1,0}M$. We keep considering the complexified tangent and cotangent bundles and their decompositions into $\pm$i−eigenbundles of J.

The complex bilinear extension of g to $T_{\mathbb{C}}M$, also denoted by g, satisfies $g(X,Y) = 0$, whenever $X, Y \in T'M$ or $X, Y \in T''M$. Identifying the complex vector bundle T_JM with $T'M$ by means of the canonical $\mathbb{C}$−linear isomorphism

$$\Phi : T_JM \longrightarrow T'M, \quad \Phi(X) := Z_X = \frac{1}{2}(X - \mathrm{i}JX) \tag{2.7}$$

we then have for all $X, Y \in T_J M$

$$g(Z_X, \bar{Z}_Y) = \frac{1}{2} h(X, Y). \tag{2.8}$$

So we deduce, if $\{e_1, \dots, e_n\}$ is a unitary frame for $T_J M$ and

$$Z_m := \Phi(e_m), \quad \bar{Z}_m := \overline{\Phi(e_m)}, \qquad 1 \leq m \leq n,$$

are the corresponding frames for $T'M$ resp. $T''M$, then the dual frames Z_j^* and $\bar{Z}_j^*$ for $T^{1,0}M$ resp. $T^{0,1}M$ are given by

$$Z_m^* = 2\bar{Z}_m^\flat \quad \text{and} \quad \bar{Z}_m^* = 2Z_m^\flat \qquad \Leftrightarrow \qquad (Z_m^*)^\# = 2\bar{Z}_m \quad \text{and} \quad (\bar{Z}_m^*)^\# = 2Z_m. \tag{2.9}$$

2.2.4 Curvature on $S^{q,0}(M)$ induced by a unitary connection on M. By linearity of curvature tensors we consider a pure symmetric tensor $\xi := Z_1^{*\alpha_1} \odot \cdots \odot Z_n^{*\alpha_n} \in \Gamma(S^{q,0}(M))$ to determine its local expression. So

$$\begin{aligned}
\hat{R}(X,Y)\xi &= \sum_{i=1}^n \alpha_i (R^*(X,Y)Z_i^*) \odot Z_1^{*\alpha_1} \odot \cdots \odot Z_i^{*\alpha_i - 1} \odot \cdots \odot Z_n^{*\alpha_n} \\
&= \sum_{i=1}^n (R^*(X,Y)Z_i^*) \odot (Z_i \lrcorner (Z_1^{*\alpha_1} \odot \cdots \odot Z_i^{*\alpha_i} \odot \cdots \odot Z_n^{*\alpha_n})) \\
&= \sum_{i=1}^n (R^*(X,Y)Z_i^*) \odot (Z_i \lrcorner \xi).
\end{aligned}$$

If the connection on the manifold is chosen to be unitary, then $R(X,Y)(T'M) \subseteq T'M$ for all $X, Y \in T_{\mathbb{C}}M$, thus we have for some $Z \in T'M$

$$(R^*(X,Y)Z_i^*)(Z) = -Z_i^*(R(X,Y)Z).$$

and using a unitary frame $\{e_1, \dots, e_n\}$ we can further write

$$R(e_j, e_k)(Z_m) = \sum_{l=1}^n R_{j,k,m'}^{l'} Z_l.$$

Then

$$Z_i^*(R(e_j, e_k)Z) = \sum_{m=1}^n z_m Z_i^*(\sum_{l=1}^n R_{j,k,m'}^{l'} Z_l) = \sum_{m=1}^n z_m R_{j,k,m'}^{i'} = \sum_{m=1}^n R_{j,k,m'}^{i'} Z_m^*(Z),$$

which proves the following lemma.

Lemma 2.2.10. *Let $\hat{R}$ be the curvature tensor on $S^{q,0}(M)$ induced by a unitary covariant derivative on M and $\{e_1, \dots, e_n\}$ be a unitary frame for $T_J M$. Then*

$$\hat{R}(e_j, e_k)\xi = -\sum_{i,m=1}^n R_{j,k,m'}^{i'} Z_m^* \odot Z_i \lrcorner \xi.$$

2.2.5 Lichnerowicz Laplacians associated to $\delta^{1,0}$ and $\delta^{0,1}$. We assume the manifold to be almost Kähler equipped with a unitary connection. According to the equation (2.3) we observe with regard to the Lemma 2.2.4, that because of

$$e_k = Z_k + \bar{Z}_k, \quad Je_k = i(Z_k - \bar{Z}_k) \quad \text{and} \quad e_k^* = \frac{1}{2}(Z_k^* + \bar{Z}_k^*), \quad (Je_k)^* = -\frac{i}{2}(Z_k^* - \bar{Z}_k^*)$$

for all $1 \le k \le n$, we have for all symmetric tensors ζ of type (p,q)

$$\begin{aligned}\delta\zeta &= \sum_{k=1}^{2n} \varphi_k \odot \hat{\nabla}_{e_k}\zeta \\ &= \frac{1}{2}\sum_{k=1}^{n}(Z_k^* + \bar{Z}_k^*) \odot \hat{\nabla}_{Z_k+\bar{Z}_k}\zeta + (Z_k^* - \bar{Z}_k^*) \odot \hat{\nabla}_{Z_k-\bar{Z}_k}\zeta \\ &= \sum_{k=1}^{n} Z_k^* \odot \hat{\nabla}_{Z_k}\zeta + \bar{Z}_k^* \odot \hat{\nabla}_{\bar{Z}_k}\zeta. \end{aligned} \tag{2.10}$$

By comparing the types $\sum_{k=1}^n Z_k^* \odot \hat{\nabla}_{Z_k}\zeta \in S^{p+1,q}(M)$ and $\sum_{k=1}^n \bar{Z}_k^* \odot \hat{\nabla}_{\bar{Z}_k}\zeta \in S^{p,q+1}(M)$ we have the following result.

Lemma 2.2.11. *For a unitary connection it holds*

$$\delta^{1,0}\zeta = \sum_{k=1}^{n} Z_k^* \odot \hat{\nabla}_{Z_k}\zeta \quad \textit{and} \quad \delta^{0,1}\zeta = \sum_{k=1}^{n} \bar{Z}_k^* \odot \hat{\nabla}_{\bar{Z}_k}\zeta.$$

Analogously, we determine the local expressions of the formal adjoint operators of $\delta^{1,0}$ resp. $\delta^{0,1}$

$$(\delta^{1,0})^* : S^{p,q}(M) \to S^{p-1,q}(M) \qquad \text{and} \qquad (\delta^{0,1})^* : S^{p,q}(M) \to S^{p,q-1}(M).$$

Using the equality $Z_k^* = 2\bar{Z}_k^\flat$ and $\bar{Z}_k^* = 2Z_k^\flat$ we obtain

$$\begin{aligned}(\delta\zeta,\psi) &= \sum_{k=1}^{n}(Z_k^* \odot \hat{\nabla}_{Z_k}\zeta,\psi) + (\bar{Z}_k^* \odot \hat{\nabla}_{\bar{Z}_k}\zeta,\psi) \\ &= 2\sum_{k=1}^{n}(\hat{\nabla}_{Z_k}\zeta, \bar{Z}_k \lrcorner \psi) + (\hat{\nabla}_{\bar{Z}_k}\zeta, Z_k \lrcorner \psi) \\ &= -2\sum_{k=1}^{n}(\zeta, \bar{Z}_k \lrcorner \hat{\nabla}_{Z_k}\psi) + (\zeta, Z_k \lrcorner \hat{\nabla}_{\bar{Z}_k}\psi) - Z_k \cdot (\zeta, \bar{Z}_k \lrcorner \psi) - \bar{Z}_k \cdot (\zeta, Z_k \lrcorner \psi).\end{aligned}$$

Since the last two terms equal to $\operatorname{div}(Y_1)$ and $\operatorname{div}(Y_2)$ for

$$Y_1 := 2\sum_{k=1}^{n}(\zeta, \bar{Z}_k \lrcorner \psi)Z_k \quad \text{and} \quad Y_2 := 2\sum_{k=1}^{n}(\zeta, Z_k \lrcorner \psi)\bar{Z}_k$$

it follows again by comparing the types the local expressions of the formal adjoint operators.

Lemma 2.2.12. *For a unitary connection it holds for a local unitary frame parallel at a point*

$$\delta^{1,0^*}\zeta = -2\sum_{k=1}^{n} Z_k \lrcorner \hat{\nabla}_{\bar{Z}_k}\zeta \quad \textit{and} \quad \delta^{0,1^*}\zeta = -2\sum_{k=1}^{n} \bar{Z}_k \lrcorner \hat{\nabla}_{Z_k}\zeta.$$

Lemma 2.2.13. *Let $\theta \in T_x^*M \setminus \{0\}$ for some $x \in M$ and $\zeta \in \Gamma(S^{p,q}(M))$. Then the principal symbols of $\delta^{1,0}$ and $\delta^{0,1}$ are given by*

$$ps(\delta^{1,0})(x,\theta)\zeta_x = \mathrm{i}\,\theta^{1,0} \odot \zeta_x \quad \text{and} \quad ps(\delta^{0,1})(x,\theta)\zeta_x = \mathrm{i}\,\theta^{0,1} \odot \zeta_x,$$

where $\theta^{1,0} := \sum_{k=1}^n \theta(Z_k)Z_k^*$ and $\theta^{0,1} := \sum_{k=1}^n \theta(\bar{Z}_k)\bar{Z}_k^*$ are the corresponding projections of θ onto $T^{1,0}M$ resp. $T^{0,1}M$.

Proof. Choose a function $f \in C^\infty(M)$, such that $df_x = \theta$. Then by definition we obtain

$$\begin{aligned} ps(\delta^{1,0})(x,\theta)\zeta_x = \mathrm{i}\delta^{1,0}(f - f(x)\zeta)_x &= \mathrm{i}\sum_{k=1}^n (Z_k^* \odot \hat{\nabla}_{Z_k}(f - f(x)\zeta))_x \\ &= \mathrm{i}\sum_{k=1}^n (Z_k^* \odot Z_k(f)\zeta)_x = \mathrm{i}\sum_{k=1}^n \theta(Z_k)Z_k^* \odot \zeta_x. \end{aligned}$$

The formula for $ps(\delta^{0,1})$ follows analogously. □

With the Remark 1.1.7 and above lemma we have

$$ps(\delta^{1,0^*})(x,\theta)\zeta_x = -\mathrm{i}(\theta^{0,1})^\# \lrcorner\, \zeta_x \qquad \text{and} \qquad ps(\delta^{0,1^*})(x,\theta)\zeta_x = -\mathrm{i}(\theta^{1,0})^\# \lrcorner\, \zeta_x.$$

In particular, we have

$$\begin{aligned} ps([\delta^{1,0}, \delta^{1,0^*}])(x,\theta)\zeta_x &= \theta^{1,0} \odot (\theta^{0,1})^\# \lrcorner\, \zeta_x - (\theta^{0,1})^\# \lrcorner\, \theta^{1,0} \odot \zeta_x \\ &= -g(\theta^{1,0}, \theta^{0,1})\,\zeta_x \\ &= -\frac{1}{2}g(\theta,\theta)\zeta_x \\ ps([\delta^{0,1}, \delta^{0,1^*}])(x,\theta)\zeta_x &= \theta^{0,1} \odot (\theta^{1,0})^\# \lrcorner\, \zeta_x - (\theta^{1,0})^\# \lrcorner\, \theta^{0,1} \odot \zeta_x \\ &= -g(\theta^{0,1}, \bar{\theta}^{1,0})\,\zeta_x \\ &= -\frac{1}{2}g(\theta,\theta)\zeta_x. \end{aligned}$$

Lemma 2.2.14. *The operators $\Delta^{1,0} := 2[\delta^{1,0}, \delta^{1,0^*}]$ and $\Delta^{0,1} := 2[\delta^{0,1}, \delta^{0,1^*}]$ define Laplacians on each $S^{p,q}(M)$.*

Corollary 2.2.15. *The Lichnerowicz Laplacians $\Delta^{1,0}$ and $\Delta^{0,1}$ can be locally written as*

$$\begin{aligned} \Delta^{1,0} &= 4\sum_{j=1}^n \hat{\nabla}_{\bar{Z}_j}\hat{\nabla}_{Z_j}\zeta + 4\sum_{j,k=1}^n Z_j^* \odot Z_k \lrcorner\, \hat{R}(\bar{Z}_k, Z_j)\zeta \\ \Delta^{0,1} &= 4\sum_{j=1}^n \hat{\nabla}_{Z_j}\hat{\nabla}_{\bar{Z}_j}\zeta + 4\sum_{j,k=1}^n \bar{Z}_j^* \odot \bar{Z}_k \lrcorner\, \hat{R}(Z_k, \bar{Z}_j)\zeta. \end{aligned}$$

Proof. The proof is a straightforward calculation using a unitary connection and a frame parallel at some point of M.

$$
\begin{aligned}
[\delta^{1,0}, \delta^{1,0^*}]\zeta &= \delta^{1,0}\delta^{1,0^*}\zeta - \delta^{1,0^*}\delta^{1,0}\zeta \\
&= 2\sum_{j,k=1}^{n} -Z_k^* \odot \hat{\nabla}_{Z_k}(Z_j \lrcorner \hat{\nabla}_{\bar{Z}_j}\zeta) + Z_k \lrcorner \hat{\nabla}_{\bar{Z}_k}(Z_j^* \odot \hat{\nabla}_{Z_j}\zeta) \\
&= 2\sum_{j=1}^{n} \hat{\nabla}_{\bar{Z}_j}\hat{\nabla}_{Z_j}\zeta + 2\sum_{j,k=1}^{n} Z_j^* \odot Z_k \lrcorner \hat{\nabla}_{\bar{Z}_k}\hat{\nabla}_{Z_j}\zeta - Z_k^* \odot Z_j \lrcorner \hat{\nabla}_{Z_k}\hat{\nabla}_{\bar{Z}_j}\zeta \\
&= 2\sum_{j=1}^{n} \hat{\nabla}_{\bar{Z}_j}\hat{\nabla}_{Z_j}\zeta + 2\sum_{j,k=1}^{n} Z_j^* \odot Z_k \lrcorner \hat{R}(\bar{Z}_k, Z_j)\zeta.
\end{aligned}
$$

The formula for $\Delta^{0,1}$ follows analogously. □

Remark 2.2.16. The Lichnerowicz Laplacians $\Delta^{1,0}$ and $\Delta^{0,1}$ can be locally also written as

$$
\begin{aligned}
\Delta^{1,0} &= -\hat{\nabla}^*\hat{\nabla}\zeta + 4\sum_{j,k=1}^{n} Z_j^* \odot Z_k \lrcorner \hat{R}(\bar{Z}_k, Z_j)\zeta - \mathrm{i}\sum_{j=1}^{n} \hat{R}(e_j, Je_j)\zeta \\
\Delta^{0,1} &= -\hat{\nabla}^*\hat{\nabla}\zeta + 4\sum_{j,k=1}^{n} \bar{Z}_j^* \odot \bar{Z}_k \lrcorner \hat{R}(Z_k, \bar{Z}_j)\zeta + \mathrm{i}\sum_{j=1}^{n} \hat{R}(e_j, Je_j)\zeta.
\end{aligned}
$$

Just observe for this purpose, that

$$
\begin{aligned}
4\sum_{j=1}^{n} \nabla^2\zeta(\bar{Z}_j, Z_j) &= 4\sum_{j=1}^{n}(\nabla_{\bar{Z}_j}\nabla_{Z_j} + \nabla_{\nabla_{\bar{Z}_j}Z_j})\zeta \\
&= \sum_{j=1}^{2n}(\nabla_{e_j}\nabla_{e_j} - \nabla_{\nabla_{e_j}e_j})\varphi - \mathrm{i}\sum_{j=1}^{n}(\nabla_{e_j}\nabla_{Je_j} - \nabla_{Je_j}\nabla_{e_j} - \nabla_{\nabla_{e_j}Je_j - \nabla_{Je_j}e_j})\zeta \\
&= \sum_{j=1}^{2n} \nabla^2\zeta(e_j, e_j) - \mathrm{i}\sum_{j=1}^{n}(\nabla_{e_j}\nabla_{Je_j} - \nabla_{Je_j}\nabla_{e_j} - \nabla_{T(e_j,Je_j)+[e_j,Je_j]})\zeta \\
&= -\hat{\nabla}^*\hat{\nabla}\varphi - \mathrm{i}\sum_{j=1}^{n} \hat{R}(e_j, Je_j)\zeta
\end{aligned}
$$

and analogously

$$
4\sum_{j=1}^{n} \nabla^2\zeta(Z_j, \bar{Z}_j) = -\hat{\nabla}^*\hat{\nabla}\varphi + \mathrm{i}\sum_{j=1}^{n} \hat{R}(e_j, Je_j)\zeta.
$$

Chapter 3

Metaplectic structures on manifolds

3.1 The group $Mp^{\mathrm{c}}(V,\Omega)$

In the following we keep considering a triple (V,Ω,j), where (V,Ω) is a symplectic vector space and j an Ω–compatible complex structure on V, and the induced 2-forms g and h (cf. the definitions in (1.2) and (1.3)). We will hence omit the reference of the triple in the notations, i. e. we shall write throughout this section e. g. Sp, Mp instead of $Sp(V,\Omega)$, $Mp(V,\Omega)$ etc.

As already mentioned in the first chapter, the unitary group $U(V_j)$ is a maximal compact subgroup of Sp. Thus, the symplectic group is connected and non-compact and the inclusion $\iota : U(V_j) \hookrightarrow Sp$ induces an isomorphism between the corresponding fundamental groups

$$\pi_1(Sp) \cong \pi_1(U(V_j)) \cong \mathbb{Z}. \tag{3.1}$$

Hence, there is up to isomorphism a unique connected space Mp, such that $\pi : Mp \xrightarrow{2:1} Sp$. Besides, Sp is a Lie group, that means Mp may be equipped with a unique structure of a Lie group, such that π becomes a Lie group homomorphism.

Definition 3.1.1. The Lie group Mp is called the *metaplectic group*.

Note, that since the Lie group Mp is a double cover of the symplectic group, the kernel of the covering map π is a subgroup of Mp consisting of two elements and can be hence identified with $\mathbb{Z}_2$, that is $\ker\pi := \{\pm\mathbb{I}\}$. The quotient of the direct product $Mp \times \mathbb{S}^1$ by the two-element subgroup generated by $(-\mathbb{I},-1)$ is a well-defined Lie group and will be denoted by Mp^{c}. That is

$$Mp^{\mathrm{c}} := Mp \times_{\mathbb{Z}_2} \mathbb{S}^1.$$

The following Lie group homomorphisms arise in a natural way

$$\sigma : Mp^{\mathrm{c}} \longrightarrow Sp, \quad \sigma([A,z]) := \pi(A)$$
$$\ell : Mp^{\mathrm{c}} \longrightarrow \mathbb{S}^1, \quad \ell([A,z]) := z^2$$
$$\iota : \mathbb{S}^1 \longrightarrow Mp^{\mathrm{c}}, \quad \iota(z) := [\mathbb{I}, z]$$

and yield the non-split short exact sequences

$$\begin{tikzcd}
1 \arrow[dr] & & & & 1 \\
& Mp \arrow[rr, dashed, "\pi"] \arrow[dr] & & Sp \arrow[ur] & \\
& & Mp^{\mathrm{c}} \arrow[ur, "\sigma"'] \arrow[dr, "\ell"] & & \\
& \mathbb{S}^1 \arrow[ur] \arrow[rr, dashed, "z \longmapsto z^2"'] & & \mathbb{S}^1 \arrow[dr] & \\
1 \arrow[ur] & & & & 1
\end{tikzcd} \tag{3.2}$$

On the other hand, the sequence of the double coverings

$$Mp \times \mathbb{S}^1 \longrightarrow Mp^{\mathrm{c}} \xrightarrow{p := \sigma \times \ell} Sp \times \mathbb{S}^1,$$

yields a sequence of injective group homomorphisms between the corresponding fundamental groups

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \pi_1(Mp^{\mathrm{c}}) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}.$$

Lemma 3.1.2. *The fundamental group of Mp^{c} is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.*

3.1.1 A Lie group homomorphism from $U(V_j)$ to Mp^{c}. We observe, that there is a Lie group homomorphism from $U(V_j)$ to Mp^{c}. The existence will essentially follow from the following proposition.

Proposition 3.1.3 (Lifting criterion (cf. [11], Proposition 1.33.))**.** *Let $p : \tilde{X} \longrightarrow X$ be a covering and $f : Y \longrightarrow X$ be a continuous map with Y path-connected and locally path-connected. Then there is a lift $\tilde{f} : Y \longrightarrow \tilde{X}$ of f if and only if* $\mathrm{im}(f_*) \subseteq \mathrm{im}(p_*)$.

We apply the lifting criterion to the injective Lie group homomorphism

$$f : U(V_j) \longrightarrow Sp \times \mathbb{S}^1, \quad f(k) := (k, \det k)$$

and the double covering

$$p : Mp^{\mathrm{c}} \longrightarrow Sp \times \mathbb{S}^1, \quad p([A,z]) := (\pi(A), z^2),$$

such that we obtain the following diagram

$$\begin{tikzcd} & Mp^c \arrow[d, "p"] \\ U(V_j) \arrow[r, "f"] & Sp \times \mathbb{S}^1. \end{tikzcd}$$

Now we observe, that the inclusion $U(V_j) \hookrightarrow Sp$ and $\det : U(V_j) \longrightarrow \mathbb{S}^1$ induce isomorphisms between the corresponding fundamental groups, thus

$$\operatorname{im}(f_*) = \left\{(m,m) \in \mathbb{Z}^2 \mid m \in \mathbb{Z}\right\} \subset \pi_1(Sp \times \mathbb{S}^1).$$

On the other hand, the fundamental group $\pi_1(Sp \times \mathbb{S}^1) \cong \pi_1(Sp) \oplus \pi_1(\mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ has exactly three non-conjugate subgroups of index 2

$$\mathbb{Z} \oplus 2\mathbb{Z}, \quad 2\mathbb{Z} \oplus \mathbb{Z}, \quad \left\{(m,n) \in \mathbb{Z}^2 \mid m+n \equiv 0 \mod 2\right\},$$

so there are up to isomorphism three non-equivalent double coverings of $Sp \times \mathbb{S}^1$, which can be listed directly, namely

$Sp \times \mathbb{S}^1$	$Mp \times \mathbb{S}^1$	Mp^c
$(g,z) \longmapsto (g,z^2)$	$(A,z) \longmapsto (\pi(A),z)$	$[A,z] \longmapsto (\pi(A),z^2)$

The first two coverings correspond to $\mathbb{Z} \oplus 2\mathbb{Z}$ and $2\mathbb{Z} \oplus \mathbb{Z}$ respectively, such that

$$\operatorname{im}(p_*) \;=\; \left\{(m,n) \in \mathbb{Z}^2 \mid m+n \equiv 0 \mod 2\right\}.$$

It follows, that $\operatorname{im}(f_*) \subseteq \operatorname{im}(p_*)$, thus the lifting criterion yields the existence of a desired lift $\tilde{f}$ of f to Mp^c.

Corollary 3.1.4. *There is a Lie group homomorphism $\tilde{f} : U(V_j) \longrightarrow Mp^c$, such that*

$$\sigma \circ \tilde{f} \;=\; \mathrm{id}_{U(V_j)}.$$

3.1.2 The double covering of the unitary group. To give an explicit description of $\tilde{f}$, we have to look by its construction at the preimage of $U(V_j) \subset Sp$ under the covering projection π, since it is involved in σ. We define

$$MU := \pi^{-1}(U(V_j)) \subset Mp$$

and state the next proposition.

Proposition 3.1.5. $MU \;\cong\; \{(k,\zeta) \in U(n) \times \mathbb{S}^1 \mid \zeta^2 = \det k\}.$

It is clear, that both of the spaces in the upper proposition are double covering spaces of the unitary group. That is, the assertion follows, once it is shown, that the both of the spaces are connected, where the connectedness of MU follows immediately from the next lemma.

Lemma 3.1.6. *Let $\rho : \tilde{X} \longrightarrow X$ be a regular $n-$covering and Y a path-connected and locally path-connected subset of X. If there is a lift $\tilde{\iota}$ of the inclusion $\iota : Y \hookrightarrow X$, then the preimage of Y by ρ is homeomorphic to the disjoint union of n copies of Y.*

Proof. For all $y \in Y$ there is exactly one $\tilde{y} \in \rho^{-1}(y)$ satisfying $\tilde{y} \in \mathrm{im}(\tilde{\iota})$. Thus the statement follows immediately by continuity and injectivity of $\tilde{\iota}$. □

Proof of the Proposition 3.1.5. Since the image of π_* is a proper subgroup of $\pi_1(Sp)$, while the map ι_* is by the equation (3.1) an isomorphism, there is by the lifting criterion no lift of the inclusion $\iota : U(V_j) \hookrightarrow Mp$. Thus, MU is path-connected.

To see, that $\{(k,\zeta) \in U(n) \times \mathbb{S}^1 \mid \zeta^2 = \det k\}$ is connected as well, we construct a path connecting all the possible elements with the identity $(I_n, 1)$. So let k be an element of $U(n)$ and $c : [0,1] \longrightarrow U(n)$ be a path with $c(0) = I_n$ and $c(1) = k$. The composition $\gamma := \det \circ c$ being a continuous path in $\mathbb{S}^1$ lifts under the double covering $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ to a unique path $\tilde{\gamma}$ in $\mathbb{S}^1$ starting at 1 and ending at some $\zeta_1 \in \mathbb{S}^1$ with $\{\zeta_1, \zeta_{-1}\} = \sqrt{\det k}$. That is,

$$\tilde{\gamma}(0) = 1, \quad \tilde{\gamma}(1) = \zeta_1 \quad \text{and} \quad \tilde{\gamma}(t)^2 = \gamma(t), \quad \forall t \in [0,1].$$

The point now is to find a path connecting (k, ζ_1) with (k, ζ_{-1}). Consider for this purpose a loop in $U(n)$ at k defined by

$$l : [0,1] \longrightarrow \mathbb{C}^{n\times n}, \quad l(t) := \begin{pmatrix} e^{2\pi i t} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot k.$$

Then it yields

$$l(0) = l(1) = k \quad \text{and} \quad \delta(t) := \det l(t) = e^{2\pi i t} \det k.$$

That means, δ is a loop in $\mathbb{S}^1$ at $\det k$, which is a representative of the generator of $\pi_1(\mathbb{S}^1)$. That means further, that any of the two lifts of δ under the double covering $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is not a loop, but a path connecting ζ_1 with ζ_{-1}. If $\tilde{\delta}$ is the lift starting at ζ_1, then the path product of $\tilde{\gamma}$ with $\tilde{\delta}$ yields a desired path. □

Having the description of MU the lift $\tilde{f} : U(V_j) \longrightarrow Mp^{\mathrm{c}}$ of f being a well-defined Lie group homomorphism can be now by the definition of f explicitly described as

$$\tilde{f}(k) = [(k,\xi),\xi],$$

where $k \in U(V_j)$ and $\xi^2 = \det k$.

3.1.3 A maximal compact subgroup of Mp^c. The Cartan-Iwasawa-Malcev theorem yields the existence of a maximal compact subgroup in Mp^c and with the next lemma we obtain a candidate for such a subgroup.

Lemma 3.1.7. *Let $\lambda : G \longrightarrow H$ be a surjective Lie group homomorphism between two Lie groups G and H, where H is non-compact. If $K \subset H$ is a maximal compact subgroup of H, then $\lambda^{-1}(K)$ is a maximal compact subgroup of G.*

Proof. Let G' be a compact subgroup of G such that

$$\lambda^{-1}(K) \subseteq G' \subseteq G. \tag{3.3}$$

Applying λ to (3.3), we obtain by surjectivity of λ

$$K = \lambda(\lambda^{-1}(K)) \subseteq \lambda(G') \subseteq \lambda(G) = H. \tag{3.4}$$

Since λ is a Lie group homomorphism, $\lambda(G')$ is a compact Lie subgroup of H and because K is maximal compact, it yields by (3.4)

$$\lambda(G') = K \qquad \text{or} \qquad \lambda(G') = H,$$

where $\lambda(G') \neq H$, since H is non-compact. So the only possible case remains $\lambda(G') = K$. Application of λ^{-1} yields

$$G' \subseteq \lambda^{-1}(\lambda(G')) = \lambda^{-1}(K),$$

so we get the inclusion $G' \subseteq \lambda^{-1}(K)$ and together with the inclusion in the equation (3.3) we obtain the equality $G' = \lambda^{-1}(K)$. □

Since $U(V_j)$ is maximal compact in Sp and $\sigma : Mp^c \longrightarrow Sp$ is a surjective Lie group homomorphism, the preimage of $U(V_j)$ under σ, denoted by

$$MU^c := \sigma^{-1}(U(V_j)) = \left\{[(k,\xi),z] \mid (k,\xi) \in MU,\, z \in \mathbb{S}^1\right\},$$

is by the previous lemma a maximal compact subgroup in Mp^c. The next corollary describes the structure of the Lie group MU^c.

Corollary 3.1.8. *There is an isomorphism $MU^c \cong U(V_j) \times \mathbb{S}^1$.*

Proof. Consider the map

$$\chi : MU^c \longrightarrow \mathbb{S}^1, \quad [(k,\xi),z] \longmapsto \xi^{-1}z.$$

Then χ is a well-defined Lie group homomorphism, which induces by definition of $\tilde{f}$ a short exact sequence

$$1 \longrightarrow U(V_j) \xrightarrow{\tilde{f}} MU^c \xrightarrow{\chi} \mathbb{S}^1 \longrightarrow 1,$$

which is left split by the Corollary 3.1.4. The splitting lemma provides an isomorphism between the Lie groups MU^{c} and $U(V_j) \times \mathbb{S}^1$ given by $\sigma \times \chi$. □

Remark 3.1.9. The inverse of $\sigma \times \chi$ is given by $\tilde{f} \cdot \iota$.

3.1.4 A subgroup of $Spin^{\mathrm{c}}$ isomorphic to MU^{c}. By the Remark 1.1.4 the unitary group $U(V_j)$ can be also thought of as a subgroup of the special orthogonal group $SO(V)$. The inclusion $U(V_j) \hookrightarrow SO(V)$ induces a surjective group homomorphism between the fundamental groups (cf. [18], pp. 263), therefore there is no lift of the inclusion to the double covering $\rho : Spin \longrightarrow SO(V)$. For this reason, the preimage of $U(V_j) \subset SO(V)$ under ρ, denoted by SpU, is path-connected and thus a double covering of $U(V_j)$ as well.

$$\begin{tikzcd}
Spin \arrow[d, "\rho", "2:1"'] & SpU \arrow[l, hook'] \arrow[dr, "\rho"'] & & MU \arrow[dl, "\pi"] \arrow[r, hook] & Mp \arrow[d, "\pi"', "2:1"] \\
SO(V) & & U(V_j) \arrow[ll, hook'] \arrow[rr, hook] & & Sp
\end{tikzcd}$$

As the fundamental group $\pi_1(U(V_j))$ is isomorphic to $\mathbb{Z}$ and there is only one subgroup in $\mathbb{Z}$ of index 2, the groups MU nad SpU are isomorphic as (connected) covering spaces of $U(V_j)$. Hence, each of them can be regarded as a Lie subgroup of both $Spin$ and Mp.

The Lie group $Spin^{\mathrm{c}}$, defined in complete analogy to Mp^{c} as

$$Spin^{\mathrm{c}} := Spin \times_{\mathbb{Z}_2} \mathbb{S}^1,$$

where $\mathbb{Z}_2$ is identified with $\langle(-\mathbf{I}, -1)\rangle$ and $\{\pm\mathbf{I}\} = \ker\rho$, possesses a subgroup isomorphic to MU^{c}. To determine this subgroup, consider the Lie group homomorphism $\lambda : Spin^{\mathrm{c}} \longrightarrow SO(V)$, which is defined analogous to σ. That is,

$$\lambda([A, z]) := \rho(A).$$

Then $SpU^{\mathrm{c}} := \lambda^{-1}(U(V_j))$ is the desired subgroup, i.e. $SpU^{\mathrm{c}} \cong MU^{\mathrm{c}}$. Furthermore, the quotient $Spin^{\mathrm{c}}/SpU^{\mathrm{c}}$ describes the space of all complex structures on V with fixed orientation, since

$$Spin^{\mathrm{c}}/SpU^{\mathrm{c}} \cong SO(V)/U(V_j).$$

Remark 3.1.10. The restriction of λ to SpU^{c} coincides with the restriction of σ to MU^{c}.

3.2 Reductions and extensions of structure groups

This section, which is largely based on the Section 2.5. in [1] and Chapter 2 in [12], intends to recall some basics on constructions of principal bundles over a given manifold. Let G and H be some Lie groups and $\lambda : G \longrightarrow H$ a Lie group homomorphism. A bundle map φ of a principal G-bundle $P \longrightarrow M$ into a principal H-bundle $Q \longrightarrow M$ is called $\lambda-$*equivariant* if for all $p \in P$ and $g \in G$ it holds

$$\varphi(p.g) = \varphi(p).\lambda(g).$$

3.2.1 Reduction of the structure group. Let $\pi_P : P \longrightarrow M$ be a principal G-bundle and $\lambda : H \longrightarrow G$ be a Lie group homomorphism. A $\lambda-$*reduction of* P is a pair (Q, f) consisting of a principal H-bundle $\pi_Q : Q \longrightarrow M$ and a smooth map $f : Q \longrightarrow P$ such that

i) $\pi_P \circ f = \pi_Q$ and

ii) f is $\lambda-$equivariant.

$$\begin{tikzcd} P \times G \arrow[r, dotted] & P \arrow[r, "\pi_P"] & M \\ Q \times H \arrow[u, "f\times\lambda"] \arrow[r, dotted] & Q \arrow[u, "f"] \arrow[ur, "\pi_Q"'] & \end{tikzcd}$$

These requirements for Q and f can be illustrated by the commutativity of each part of the diagram on the right. If $H \subseteq G$ is a Lie subgroup of G and λ is the inclusion map then a $\lambda-$reduction (Q, f) of P is simply referred to as *a reduction of P to H*.

By an appropriate choice of an open subset U of M, we can achieve the following identification

$$\pi_P^{-1}(U) \simeq U \times G \qquad \text{and} \qquad Q_U := \pi_Q^{-1}(U) \simeq U \times H.$$

$$\begin{tikzcd} \pi_P^{-1}(U) & \simeq & U \times G \\ \pi_Q^{-1}(U) \arrow[u, "f"] & \simeq & U \times H \arrow[u, "\mathrm{id}_U\times\lambda"'] \end{tikzcd}$$

That is, the map f can be locally thought of as $f\big|_{Q_U} \simeq \mathrm{id}_U \times \lambda$. As a consequence, the local properties of λ can be carried over to f, e. g.

- if λ is a covering map, then the same holds for f.
- if $H \subseteq G$ is a Lie subgroup of G and $\lambda = \iota$ is the inclusion map, the map $f : Q \longrightarrow P$ is then an injective immersion, i. e. $f(Q)$ is a submanifold of P.

Theorem 3.2.1 (Cf. Satz 2.16 in [1])**.** *Let G be a connected, non-compact Lie group and P be a principal G-bundle over a manifold M. Then P is reducible to every maximal compact subgroup $K \subset G$.*

Definition 3.2.2 (Isomorphic principal bundles)**.** Two principal G-bundles $\pi : P \to M$ and $\pi' : P' \to M$ over the same manifold M are said to be *isomorphic*, if there is a $G-$equivariant and fiber-preserving diffeomorphism $\Phi : P \longrightarrow P'$.

Theorem 3.2.3. *Let M be a manifold, G, H be some Lie groups, $\lambda : G \longrightarrow H$ be a Lie group homomorphism, $K \subset H$ a Lie subgroup of H and $L := \lambda^{-1}(K) \subset G$. Let further*

- $\pi_{P'} : P' \longrightarrow M$ *be a principal G-bundle,*
- $\pi_P : P \longrightarrow M$ *be a principal H-bundle,*
- $\pi_Q : Q \longrightarrow M$ *be a reduction of P to K and*
- $\varphi : P' \longrightarrow P$ *be a λ−equivariant bundle map.*

$$\begin{tikzcd}
P' \arrow[r, "\varphi"] \arrow[rr, bend left, "G"] & P \arrow[r, "H"] & M \\
Q' \arrow[u, dotted, hook] & Q \arrow[u, dotted, hook] \arrow[ur, "K"'] &
\end{tikzcd}$$

Then the restriction of $\pi_{P'}$ to $Q' := \varphi^{-1}(Q)$ is a principal L-bundle over M.

Proof. Let U be an open subset of M such that we obtain the diagram on the right. Since we can locally identify the map φ with $\mathrm{id}_U \times \lambda$, we obtain

$$\begin{tikzcd}
U \times G \arrow[r, phantom, "\simeq"] \arrow[rrr, bend left, "\mathrm{id}_U\times\lambda"] & \pi_{P'}^{-1}(U) \arrow[r, "\varphi"] & \pi_P^{-1}(U) \arrow[r, phantom, "\simeq"] & U \times H \\
 & & \pi_Q^{-1}(U) \arrow[u, hook] \arrow[r, phantom, "\simeq"] & U \times K. \arrow[u, hook, "\mathrm{id}_U\times\iota_K"']
\end{tikzcd}$$

$$\varphi^{-1}(\pi_Q^{-1}(U)) \simeq U \times L.$$

Therefore, Q' is locally trivial and because φ is fiber-preserving the fibers of Q' are especially given by

$$Q'_x = \varphi^{-1}(\pi_Q^{-1}(\{x\})) = \varphi^{-1}(Q_x).$$

Let $p \in Q'$. Then there is some $x \in M$ such that $p \in P'_x$. Since L is a subgroup of G, the action of L on P'_x is fiber-preserving, i. e. $p.l \in P'_x$ for all $l \in L$. Further it yields for all $l \in L$

$$\varphi(p.l) = \varphi(p).\lambda(l) \in Q_x \quad \Rightarrow \quad p.l \in \varphi^{-1}(Q_x) = Q'_x, \quad \forall l \in L.$$

So L acts fiber-preserving on the fibers of Q'.

Let next $p_1, p_2 \in Q'_x = \varphi^{-1}(Q_x)$. Then $\varphi(p_1), \varphi(p_2)$ are some elements of Q_x, so there exists an unique $k \in K$ such that $\varphi(p_2) = \varphi(p_1).k$. Regarding p_1 and p_2 as elements of P_x there exists an unique $g \in G$ such that $p_2 = p_1.g$. Applying φ yields

$$Q_x \ni \varphi(p_2) = \varphi(p_1).\lambda(g)$$

and we have $g \in \lambda^{-1}(k) \subset L$. So the action of L on Q'_x is transitive. Suppose next, there is some $l \in L$ such that $p_1 = p_1.l$. Again, if we think of p_1 as an element of P_x and l as an element of G it then yields $l = \mathbf{1}_G$. As a result we deduce that L acts fiber-preserving, freely and transitively on the fibers of Q'. □

Remark 3.2.4. If we let in the previous theorem the Lie group K be maximal compact in H then $L = \lambda^{-1}(K)$ is by the Lemma 3.1.7 maximal compact in G. That is, the principal G−bundle P' is reducible to L, where the reduction is given by $\varphi^{-1}(Q)$ with the obvious inclusion.

3.2.2 Extensions of principal fiber bundles. Next, we briefly sketch the inverse procedure to the reduction of the structure group. For any Lie group homomorphism $\lambda : H \longrightarrow G$ consider the action of H on G given by

$$H \times G \longrightarrow G, \qquad (h,g) \longmapsto h.g := \lambda(h)g$$

and form then for a principal H-bundle Q over M the associated fiber bundle

$$P := Q \times_{H,\lambda} G$$

over M with fiber G. Such fiber bundle P is then referred to as the λ*−extension of* Q.

Theorem 3.2.5 (Cf. [1], Satz 2.18)**.** *Let $\lambda : H \longrightarrow G$ be a Lie group homomorphism and Q be a principal H-bundle over M. Then it holds*

i) The λ−extension $P := Q \times_{H,\lambda} G$ of Q is a principal G-bundle over M.

ii) Let $f : Q \longrightarrow P$ be the map $f(q) = [q, \mathbf{1}_G]$. Then (Q, f) is a λ−reduction of P.

iii) Let P be a principal G-bundle over M and (Q, f) be a λ−reduction of P. Then P is isomorphic to the λ−extension of Q.

At the end we also observe, that if we let in the Theorem 3.2.5 the group H be a closed subgroup of G and consider the action of G on the homogeneous space G/H

$$G \times G/H \longrightarrow G/H, \qquad (g,[a]) \longmapsto [ga],$$

then there is a bundle isomorphism

$$P \times_G G/H \cong P/H, \qquad [p, g.H] \simeq (p.g).H. \tag{3.5}$$

Theorem 3.2.6 ([1], Satz 2.14)**.** *A principal G−bundle is reducible to a closed subgroup $H \subset G$ if and only if the associated fiber bundle $P \times_G G/H$ admits a global section.*

Corollary 3.2.7. *Let M be a manifold. Then it yields*

i) If M admits a non-degenerate 2-form ω, then it admits an ω-compatible almost complex structure J.

ii) If M admits an almost complex structure J, then it admits a non-degenerate 2-form ω compatible with J.

Proof. If M carries a non-degenerate 2-form ω, then the *symplectic frame bundle* $\mathrm{Sp}(M,\omega)$ yields an Sp−structure on (M,ω). A choice of an Ω-compatible complex structure j for (V,Ω) yields by the Theorem 3.2.1 a $U(V_j)$− structure on M. Applying the Theorem 3.2.6 to $\mathrm{Sp}(M,\omega)$ and $U(V_j)$ and the fact

$$Sp(V,\Omega)/U(V_j) \cong \mathcal{J}(V,\Omega)$$

we obtain an ω–compatible almost complex structure J as a global section of the fiber bundle

$$\mathrm{Sp}(M,\omega)\times_{Sp}(Sp/U(V_j)).$$

On the other hand, any almost complex manifold (M,J) admits a non-degenerate 2-form. Indeed, if g is an arbitrary Riemannian metric on M, define a 2-form g' on M as

$$g'(X,Y) := g(X,Y) + g(JX,JY), \qquad X,Y \in TM.$$

Then g' is a Riemannian metric on M as well, which additionally satisfies $J^*g' = g'$, and the corresponding fundamental form ω' on M defined as

$$\omega'(X,Y) := g'(JX,Y), \qquad X,Y \in TM,$$

is a non-degenerate 2-form, which satisfies the both conditions in the Definition 1.2.1. $\square$

3.2.3 Pullback bundle. If $\pi : E \longrightarrow M$ is a principal G–bundle over M and $f : N \longrightarrow M$ is a smooth map between two manifolds, then the *pullback of* E *by* f, defined as

$$f^*E := \{(x,e) \in N \times E \mid f(x) = \pi(e)\} \subset N \times E,$$

with the projection

$$\mathrm{pr}_1 : f^*E \longrightarrow M, \quad (x,e) \longmapsto x$$

is a principal G–bundle over M. The projection on the second factor yields a bundle map from f^*E to E. In particular, there is an important special case given by the smooth map

$$f = \Delta : M \longrightarrow M \times M, \quad x \mapsto (x,x)$$

and a principal $(G\times H)$–bundle $E = P \times Q$ over $M \times M$, where P and Q are principal G– resp. H–bundle over M. The pullback of $P \times Q$ by Δ, denoted simply by

$$P \times_M Q := \Delta^*(P \times Q),$$

is then a principal $(G \times H)$–bundle over M.

$$\begin{tikzcd} P\times_M Q \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & P\times Q \arrow[d] \\ M \arrow[r, "\Delta"'] & M\times M \end{tikzcd}$$

3.3 Existence and classification of the $Mp(V,\Omega)$-structures

For an almost symplectic manifold (M,ω) we define an Mp*-structure* as a pair (P,ϕ), where

- $P \longrightarrow (M,\omega)$ is a principal Mp-bundle over M and
- $\phi : P \longrightarrow \mathrm{Sp}(M,\omega)$ a $\pi-$equivariant bundle map.

$$\begin{tikzcd}
P \times Mp \arrow[rr, "\phi\times\pi"] \arrow[d, dotted] & & \mathrm{Sp}(M,\omega) \times Sp \arrow[d, dotted] \\
P \arrow[rr, "\phi"] \arrow[dr, "Mp"'] & & \mathrm{Sp}(M,\omega) \arrow[dl, "Sp"] \\
& (M,\omega) &
\end{tikzcd}$$

That means, that each part of the diagram on the right commutes. There is a topological obstruction for a manifold to admit an $Mp-$structure, which is related to the Spin-geometry.

Theorem 3.3.1. *An almost symplectic manifold admits a metaplectic structure if and only if it admits a spin structure, i. e. if and only if the second Stiefel-Whitney class $w_2(M)$ of M is trivial.*

Proof. Suppose, (M,ω) admits a metaplectic structure (P,ϕ). Then $\tilde{\mathcal{U}}(M)$ defined as the preimage of $\mathcal{U}(M)$ under ϕ is by the Theorem 3.2.3 a principal $MU-$bundle over M. Since the Lie group MU can be by the Section 3.1.4 thought of as a subgroup of $Spin$ the associated fiber bundle

$$P_{Spin} := \tilde{\mathcal{U}}(M) \times_{MU} Spin$$

is a principal $Spin-$bundle over M. Together with the $Spin-$equivariant bundle map

$$\vartheta : P_{Spin} \longrightarrow \mathrm{SO}(M), \qquad \vartheta([\tilde{u}, B]) = \phi(\tilde{u}).\rho(B),$$

for $\tilde{u} \in \tilde{\mathcal{U}}(M)$ and $B \in Spin$, the pair (P_{Spin}, ϑ) is a $Spin-$structure on M.

$$\begin{tikzcd}
P_{Spin} \arrow[d, dashed, "\vartheta"] & \tilde{\mathcal{U}}(M) \arrow[l, dashed, hook'] \arrow[r, hook] \arrow[d, "2:1"'] & P \arrow[d, "\phi"'] \\
\mathrm{SO}(M) & \mathcal{U}(M) \arrow[l, hook'] \arrow[r, hook] & \mathrm{Sp}(M,\omega)
\end{tikzcd}$$

The very same argumentation and construction applied to the preimage of $\mathcal{U}(M)$ under ϑ yield an $Mp-$structure on M. □

It is not hard to see, that isomorphic $Mp-$structures induce isomorphic $Spin-$structures over M and vice versa. Indeed, if there are two isomorphic $Mp-$structures (P,ϕ) and (P',ϕ') on M, i. e. there is an $Mp-$equivariant and fiber-preserving diffeomorphism $\Phi : P \longrightarrow P'$. Then the restriction of Φ to $\tilde{\mathcal{U}}(M)$ yelds a well-defined isomorphism between $\tilde{\mathcal{U}}(M)$ and $\phi'^{-1}(\mathcal{U}(M))$. So the induced $Spin$–structures are isomorphic as well. By the same argumentation we conclude, that isomorphic $Spin$–structures induce isomorphic Mp–structures on (M,ω). We state this result in the following corollary.

Corollary 3.3.2. *There is an one-to-one correspondence*

$$\left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{metaplectic structures on } M\end{array}\right\} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{c}\textit{Isomorphism classes of}\\ \textit{spin structures on } M\end{array}\right\}.$$

3.4 Existence and classification of the $Mp^c(V,\Omega)$–structures

We define the Mp^c*–structure* on (M,ω) in complete analogy to the Mp–structure as a pair (P,φ), where

- $P \longrightarrow (M,\omega)$ is a principal Mp^c-bundle over M and
- $\varphi : P \longrightarrow \mathrm{Sp}(M,\omega)$ a σ–equivariant bundle map.

Theorem 3.4.1. *A manifold admits an* Mp^c*–structure if and only if it admits an almost complex structure.*

Proof. The structure group of any almost complex manifold can be reduced to the unitary group, since any complex vector bundle admits a Hermitian metric. By using the map $\tilde{f}$ from the Corollary 3.1.4 the Mp^c–structure is obtained by the associated principal Mp^c–bundle

$$\mathcal{U}(M) \times_{U(V_j),\tilde{f}} Mp^c$$

and the map $\phi : [q,A] \longmapsto q.\sigma(A)$.

For the converse, if M admits an Mp^c–structure, it admits by definition an almost symplectic form and the assertion follows by the Corollary 3.2.7. □

3.4.1 Reductions of the Mp^c–structure to MU^c. Let (M,ω) be an almost symplectic manifold, J an ω–compatible almost complex structure on M and $\mathcal{U}(M)$ the corresponding unitary frame bundle over M. Assume further, there is a principal $\mathbb{S}^1$-bundle $\mathcal{L} \longrightarrow M$. Then the pullback bundle

$$\mathcal{U}(M) \times_M \mathcal{L} = \Delta^*(\mathcal{U}(M) \times \mathcal{L})$$

of $\mathcal{U}(M) \times \mathcal{L}$ under the diagonal map $\Delta : M \longrightarrow M \times M,\ x \mapsto (x,x)$ is a principal $(U(V_j) \times \mathbb{S}^1)$ -bundle over M. By identifying $U(V_j) \times \mathbb{S}^1$ with MU^c via the Corollary 3.1.8 as a subgroup of Mp^c, the extension

$$(\mathcal{U}(M) \times_M \mathcal{L}) \times_{MU^c} Mp^c$$

is a principal Mp^{c}-bundle over M and $\mathcal{U}(M)\times_M\mathcal{L}$ is by the Theorem 3.2.5 its reduction to the maximal compact subgroup.

$$\begin{tikzcd}
(\mathcal{U}(M)\times_M\mathcal{L})\times_{MU^{c}} Mp^{c} \arrow[r, "\varphi"] \arrow[rr, bend left, "Mp^{c}"] & \mathrm{Sp}(M,\omega) \arrow[r, "Sp"] & (M,\omega) \\
(\mathcal{U}(M)\times_M\mathcal{L}) \arrow[u, hook] \arrow[r] \arrow[urr, bend right, "MU^{c}"'] & \mathcal{U}(M) \arrow[u, hook] \arrow[ur, "U(V_j)"] & \mathcal{L} \arrow[u, "\mathbb{S}^1"']
\end{tikzcd}$$

The map $\varphi:(\mathcal{U}(M)\times_M\mathcal{L})\times_{MU^{c}} Mp^{c}\longrightarrow \mathrm{Sp}(M,\omega)$, given by

$$\varphi:[(u,r),A]\longmapsto u.\sigma(A),$$

yields an Mp^{c}–structure on M. On the other hand, since MU^{c} is maximal compact in Mp^{c}, any principal Mp^{c}–bundle $P\longrightarrow M$ is reducible to MU^{c}. That is, there is a principal MU^{c}–bundle $Q\longrightarrow M$ such that

$$P\cong Q\times_{MU^{c}} Mp^{c}. \tag{3.6}$$

In the following, we will see, that any such reduction Q is isomorphic to the pullback bundle $\mathcal{U}(M)\times_M\mathcal{L}$ for some principal $\mathbb{S}^1$–bundle $\mathcal{L}$ over M.

Denote by $\mathfrak{Mp}^{c}$ the set of all isomorphism classes of Mp^{c}–structures on (M,ω). Since the extension of two isomorphic principal MU^{c}–bundles to Mp^{c} by injection result in two isomorphic principal Mp^{c}–bundles, we consider the two maps

$$\left\{\begin{matrix}\text{Isomorphism classes of}\\ \text{principal } \mathbb{S}^1\text{-bundles over } (M,\omega)\end{matrix}\right\} \underset{\Upsilon}{\overset{\Lambda}{\rightleftarrows}} \mathfrak{Mp}^{c}$$

by defining

$$\begin{aligned}
\Lambda:\quad & [\mathcal{L}] &\longmapsto\quad & [((\mathcal{U}(M)\times_M\mathcal{L})\times_{MU^{c}} Mp^{c},\varphi)]\\
\Upsilon:\quad & [(P,\phi)] &\longmapsto\quad & [Q\times_{MU^{c},\chi}\mathbb{S}^1],
\end{aligned}$$

where P is realized as the extension of some principal MU^{c}–bundle $Q\longrightarrow M$ as in (3.6).

Let $\mathcal{L}\in[\mathcal{L}]$ be a representative and consider the image of $\mathcal{L}$ under $\Upsilon\circ\Lambda$. Define then the map between these two principal $\mathbb{S}^1$–bundles by

$$f:(\mathcal{U}(M)\times_M\mathcal{L})\times_{MU^{c},\chi}\mathbb{S}^1\longrightarrow\mathcal{L},\qquad [(u,r),\zeta]\longrightarrow\zeta r. \tag{3.7}$$

Then f is obviously smooth and fiber-preserving. Moreover, f is well-defined, that is, if $(k,\lambda)\in MU^{c}$, then for any other representative of the class $[(u,r),\zeta]$ it holds

$$f([(u,r)\cdot(k,\lambda),\chi((k,\lambda)^{-1})\zeta])=f([(u\cdot k,\lambda r),\lambda^{-1}\zeta])=\zeta r.$$

Besides, f is $\mathbb{S}^1$–equivariant

$$f([(u,r),\zeta]\cdot\lambda) = f([(u,r),\lambda\zeta])$$
$$= \lambda\zeta r = f([(u,r),\zeta])\cdot\lambda$$

for $\lambda \in \mathbb{S}^1$. As an $\mathbb{S}^1$–equivariant bundle map between two principal $\mathbb{S}^1$–bundles f is an isomorphism, therefore $\Upsilon \circ \Lambda = \mathrm{id}$.

On the other hand, the image of a representative (P,φ) under Υ is the principal $\mathbb{S}^1$-bundle $Q \times_{MU^c,\chi} \mathbb{S}^1$, where Q is a reduction of P to $U(V_j)$, such that $P \cong Q \times_{MU^c} Mp^c$. Recall next, since we can think of Q as a submanifold of P and of $\mathcal{U}(M)$ as a submanifold of $\mathrm{Sp}(M,\omega)$, the restriction of φ to Q is by definition σ–equivariant, i. e.

$$\phi(p\cdot(k,\lambda)) = \phi(p)\cdot\sigma(k,\lambda) = \phi(p)\cdot k$$

for all $(k,\lambda) \in MU^c$. Using additionally the embedding of Q into $Q \times_{MU^c,\chi} \mathbb{S}^1$ via $q \longmapsto [q,1]$ we define the map

$$f' : Q \longrightarrow (\mathcal{U}(M) \times_M (Q \times_{MU^c,\chi} \mathbb{S}^1)), \qquad q \longmapsto (\phi(q),[q,1]),$$

which is clearly well-defined, fiber-preserving and smooth. Furthermore, f' is MU^c–equivariant, since

$$f'(q.(k,\lambda)) = (\phi(q.(k,\lambda)),[q.(k,\lambda),1])$$
$$= (\phi(q).k,[q,\chi(k,\lambda)])$$
$$= (\phi(q).k,[q,\lambda])$$
$$= (\phi(q).k,[q,1].\lambda)$$
$$= (\phi(q),[q,1]).(k,\lambda) = f'(q).(k,\lambda).$$

So as an MU^c–equivariant bundle map between two principal MU^c–bundles f' is an isomoprhism, which implies, that the map Υ is the right inverse for Λ.

Remark 3.4.2 (Cf. [23], pp. 103-105). Any differentiable complex line bundle over M is uniquely determined by its first *Chern class* in the second *cohomology group* $H^2(M,\mathbb{Z})$ and vice versa.

Thinking of line bundles as elements of $H^2(M,\mathbb{Z})$ yields the following theorem.

Theorem 3.4.3. *Any almost symplectic manifold (M,ω) admits an Mp^c–structure and there is an one-to-one correspondence*

$$H^2(M,\mathbb{Z}) \quad \overset{1:1}{\longleftrightarrow} \quad \mathfrak{Mp}^c.$$

Remark 3.4.4. In view of the Remark 3.2.4 we note, that any reduction of Mp^c–structure (P,φ) to MU^c is of the form:

$$\varphi^{-1}(\mathcal{U}(M)) \cong \mathcal{U}(M) \times_M \mathcal{L}$$

for some principal $\mathbb{S}^1$–bundle $\mathcal{L}$ over (M,ω). In particular, there are two distinguished Mp^c–structures:

i) if the principal $\mathbb{S}^1$–bundle $\mathcal{L}$ is chosen to be trivial, then

$$\mathcal{U}(M) \times_M \mathcal{L} \cong \mathcal{U}(M)$$

and it follows

$$\begin{aligned} P &\cong (\mathcal{U}(M) \times_M \mathcal{L}) \times_{MU^c} Mp^c \\ &\cong \mathcal{U}(M) \times_{U(V_j),\tilde{f}} Mp^c. \end{aligned}$$

So under the above correspondence the trivial principal $\mathbb{S}^1$–bundle corresponds to the unitary frame bundle. This Mp^c–structure is also referred to as the *canonical* one.

ii) if M admits a symplectic form ω with *integral cohomology class*, then there is a (holomorphic) line bundle $\mathbb{L}$ with $c_1(\mathbb{L}) = [\omega]$.

3.4.2 The action of $H^2(M,\mathbb{Z})$ on $\mathfrak{Mp}^c$. Let (P,φ) be an Mp^c–structure on M and $\mathcal{L}$ be the corresponding line bundle over M. By identifying $\ker\sigma \subset \mathcal{Z}(Mp^c)$ with $\mathbb{S}^1$ it yields

$$Mp^c \times_{\ker\sigma} \mathbb{S}^1 \;\cong\; Mp^c, \qquad [[A,z]_{\mathbb{Z}_2},\xi]_{\ker\sigma} \;\simeq\; [A,z\xi]_{\mathbb{Z}_2}. \tag{3.8}$$

Lemma 3.4.5. *Let $\mathbb{L}$ be a Hermitian line bundle over M and $\mathbb{L}^{(1)}$ the corresponding principal $\mathbb{S}^1$–bundle, then*

$$(P \times_M \mathbb{L}^{(1)})/_{\ker\sigma}$$

is a principal Mp^c–bundle over M, which corresponds to the line bundle $\mathcal{L} \otimes \mathbb{L}$.

Proof. We identify by the equation (3.5)

$$(P \times_M \mathbb{L}^{(1)})/_{\ker\sigma} \overset{(3.8)}{\cong} (P \times_M \mathbb{L}^{(1)}) \times_{Mp^c \times \mathbb{S}^1} Mp^c$$

and define

$$F : (P \times_M \mathbb{L}^{(1)}) \times_{Mp^c \times \mathbb{S}^1} Mp^c \;\longrightarrow\; (\mathcal{U}(M) \times_M (\mathcal{L} \otimes \mathbb{L})^{(1)}) \times_{MU^c} Mp^c$$

with the mapping rule

$$F : [([(u,l),A],\zeta),B] \longmapsto [(u,l\otimes\zeta),AB].$$

F is obviously smooth and fiber-preserving. Moreover, it is well-defined, since for all $(k,\xi) \in MU^c$ and $C \in Mp^c$ one has

$$[([(u,l),A],\zeta),B] \;=\; [([(u,l).(k,\xi),(k,\xi)^{-1}.A],\zeta),B]$$

$$\begin{aligned}
&= [([(u.k,\xi l),(k,\xi)^{-1}.A],\zeta),B] \\
[([(u,l),A],\zeta),B] &= [([(u,l),A],\zeta).(C,\xi),(C,\xi)^{-1}.B] \\
&= [([(u,l),AC],\zeta\xi),\xi^{-1}C^{-1}B]
\end{aligned}$$

and

$$\begin{aligned}
F([([(u.k,\xi l),(k,\xi)^{-1}.A],\zeta),B]) &= [(u.k,\xi l\otimes\zeta),(k,\xi)^{-1}AB] \\
&= [(u,l\otimes\zeta).(k,\xi),(k,\xi)^{-1}AB] \\
&= [(u,l\otimes\zeta),AB] \\
F([([(u,l),AC],\zeta\xi),\xi^{-1}C^{-1}B]) &= [(u,l\otimes\zeta\xi),AC\xi^{-1}C^{-1}B] \\
&= [(u,l\otimes\zeta).\xi,\xi^{-1}AB] \\
&= [(u,l\otimes\zeta),AB].
\end{aligned}$$

Further, F is Mp^{c}–equivariant, since for all $C\in Mp^{\mathrm{c}}$ it yields

$$\begin{aligned}
F([([(u,l),A],\zeta),B].C) &= F([([(u,l),A],\zeta),BC]) \\
&= [(u,l\otimes\zeta),ABC] \\
&= [(u,l\otimes\zeta),AB].C \\
&= F([([(u,l),A],\zeta),B]).C.
\end{aligned}$$

□

Having this correspondence we describe the induced action of $H^2(M,\mathbb{Z})$ on $\mathfrak{Mp}^{\mathrm{c}}$.

Corollary 3.4.6. *Let $P\longrightarrow M$ be a principal Mp^{c}–bundle and $\mathbb{L}\longrightarrow M$ a Hermitian line bundle over M. If we think of $\mathbb{L}$ as an element of $H^2(M,\mathbb{Z})$, then the action*

$$\begin{aligned}
H^2(M,\mathbb{Z})\times\mathfrak{Mp}^{\mathrm{c}} &\longrightarrow \mathfrak{Mp}^{\mathrm{c}} \\
(\mathbb{L},P) &\longmapsto (P\times_M\mathbb{L}^{(1)})/_{\ker\sigma}
\end{aligned}$$

is simply transitive.

Proof. Let $\mathcal{L}$ be the corresponding line bundle of P. Then the statement follows immediately by the Theorem 3.4.3 and Lemma 3.4.5, since

$$(P\times_M\mathbb{L}^{(1)})/_{\ker\sigma}\simeq\mathcal{L}\otimes\mathbb{L}\simeq c_1(\mathcal{L})+c_1(\mathbb{L}).$$

□

3.4.3 ***Mp^c– and $Spin^c$–structures.*** Let (P,φ) be an Mp^c–structure over (M,ω). Denote as usual by Q the reduction of P to MU^c. Then by identifying MU^c with $SpU^c \subset Spin^c$ we obtain a $Spin^c$–structure given by the principal $Spin^c$–bundle

$$E := Q \times_{SpU^c} Spin^c$$

and the $Spin^c$–equivariant bundle map

$$\psi : E \longrightarrow SO(V), \qquad \psi([q,A]) := \varphi(q).\lambda(A).$$

On the other hand, any $Spin^c$–structure (E,ψ) on an almost complex manifold is by the Theorem 3.2.6 reducible to SpU^c. That is

$$E \cong Q \times_{SpU^c} Spin^c$$

for a principal SpU^c–bundle $Q \longrightarrow M$. Realizing SpU^c as a subgroup MU^c of Mp^c yields an Mp^c–structure (P,φ) given by the principal Mp^c–bundle

$$P := Q \times_{MU^c} Mp^c$$

and the Mp^c–equivariant bundle map

$$\varphi : P \longrightarrow Sp(V), \qquad \varphi([q,A]) := \psi(q).\sigma(A).$$

This observation leads to the following lemma.

Lemma 3.4.7. *On an almost complex manifold (M,J) there is an one-to-one correspondence between the $Spin^c$– and Mp^c–structures.*

Remark 3.4.8. The previous lemma does not hold without the assumption on the manifold to be almost complex, since the class of manifolds admitting a $Spin^c$–structure is bigger, than the class of manifolds admitting an almost complex structure and thus an almost symplectic structure. For example, $\mathbb{S}^4$ admits as a four dimensional manifold a $Spin^c$–structure (cf. [22]), while the only spheres admitting an almost complex structure are $\mathbb{S}^2$ and $\mathbb{S}^6$.

Chapter 4

The symplectic Dirac operator and the corresponding Laplacian

In this last chapter we describe the symplectic Dirac-Dolbeault operators when acting on the dense subbundle consisting of polynomial-valued spinor fields. In order to do so, we describe first the identification of this subbundle with the tensor bundle $L \otimes S^{*,0}(M)$ and adjust then the symplectic Clifford multiplication. After this, we describe the geometric interpretation of the symplectic Dirac-Dolbeault operators and their commutator. We keep considering a complex structure j on (V, Ω) and an almost complex structure J on (M, ω) as compatible with Ω resp. ω and the naturally arising (Hermitian) inner product resp. metric on V resp. M.

4.1 Symplectic spinor bundle

We begin with the description of, how a *symplectic spinor space*, that is a vector space carrying a representation of the metaplectic group, can be obtained. The most common way to do so is to consider the *Heisenberg group* $H(V, \Omega)$, that is a Lie group $V \times \mathbb{R}$ with the multiplication

$$(v, t)(w, s) := (v + w, t + s - \frac{1}{2}\Omega(v, w)),$$

and its unitary irreducible representation, which is known to be either one-dimensional or infinite-dimensional depending thereby on some parameter $\lambda \in \mathbb{R}$.

If $\mathfrak{r}$ denotes an irreducible representation of $H(V, \Omega)$, then λ appears by Schur's lemma when restricting $\mathfrak{r}$ to the center $\{0\} \times \mathbb{R}$ of $H(V, \Omega)$ and is therefore referred to as the *central parameter* of $\mathfrak{r}$. The Stone-von Neumann theorem (cf. [7], (1.59) Theorem) gives a complete classification of all unitary irreducible representations of a Heisenberg group with $\lambda \neq 0$. It states, that *any unitary irreducible representations of the Heisenberg group on a complex separable Hilbert space with the same non-zero central parameter are unitarily equivalent.*

For the explicit description of this classification one usually uses the Schrödinger representations on $L^2(W)$ for a Lagrangian subspace W of (V,Ω). That is, if V is written as $V = W \oplus U$, then

$$(\mathfrak{r}_\lambda(v,t)f)(x) := e^{i\lambda t}\, e^{\frac{\langle 2x+\lambda w, u\rangle}{2}} f(x+\lambda w),$$

where $v = w + u$ with $w \in W$ and $u \in U$(cf. [7], Section 3 in Chapter 1). For our purposes it is more convenient to realize the symplectic spinor space as a *Segal-Bargmann space*, denoted by $\mathcal{F}_c$, instead of $L^2(W)$. These are defined as follows

$$\begin{aligned} f \in \mathcal{F}_c \quad :\Leftrightarrow \quad & 1) \quad f : V_j \to \mathbb{C} \text{ entire and} \\ & 2) \quad \|f\|_c := (\pi c)^{-n} \int_{V_j} |f(z)|^2\, e^{-\frac{\|z\|^2}{c}}\, dz < \infty, \end{aligned}$$

where $c > 0$ and $\dim_{\mathbb{C}}(V_j) = n$. The change of the Hilbert spaces is justified by the isometries $B_c : L^2(W) \to \mathcal{F}_c$ called the *Bargmann transforms* (cf. [7], Section 6 in Chapter 1). On this space the unitary irreducible representations of the Heisenberg group corresponding to $\lambda = -\frac{2}{c}$ translates to the unitary multiple of the *unitarized translations* on $\mathcal{F}_c$ (cf. [10], Section 4.1). That is

$$(\mathfrak{r}_c(v,t)f)(z) := e^{-\frac{2it}{c}}\, e^{\frac{(2z-v,v)}{2c}}\, f(z-v).$$

One of the advantages of using a Segal-Bargmann space is that the set of all monomials on V_j is a complete orthogonal basis for any $\mathcal{F}_c$ with $c > 0$. In particular, the Segal-Bargmann spaces are Hilbert space completions of the polynomial ring $\mathcal{P}o\ell(V_j)$ and, if we identify the polynomials with the symmetric tensors $S^*(V_j^*)$, it follows, that each $\mathcal{F}_c$ is a Fock space with respect to its Hermitian inner product.

We fix in what follows a unitary irreducible representation $\mathfrak{r}$ of $H(V,\Omega)$ on some Segal-Bargmann space $\mathcal{F}_c$ with the central parameter λ. Then the group Mp^{c} appears in this context by observing, that the action of the symplectic group on $H(V,\Omega)$, given by

$$\tau_g : H(V,\Omega) \to H(V,\Omega), \quad \tau_g(v,t) := (gv,t)$$

for $g \in Sp$, preserves its center. Hence the composition $\mathfrak{r}^g := \mathfrak{r} \circ \tau_g$ defines another unitary representation of $H(V,\Omega)$ on $\mathcal{F}_c$, which remains irreducible and has the same central parameter λ. That means, there is by the Stone-von Neumann theorem a unitary operator $U_g \in \mathbf{U}(\mathcal{F}_c)$, such that

$$U_g \circ \mathfrak{r}^g = \mathfrak{r} \circ U_g. \tag{4.1}$$

By Schur's lemma the operator U_g is determined up to a scalar multiple by an element of the circle group. The collection of all possible $(g, U_g) \in Sp \times \mathbf{U}(\mathcal{F}_c)$ satisfying (4.1) defines a Lie group isomorphic to Mp^{c} (cf. [12], Section 2.2). The natural representation of Mp^{c} on $\mathcal{F}_c$, obtained by the projection on the second factor

$$\varrho : Mp^{\mathrm{c}} \longrightarrow \mathbf{U}(\mathcal{F}_c), \qquad (g, U_g) \longmapsto U_g$$

is surely unitary and faithful, however not irreducible (cf. [21], 4.15 Corollary). Its restriction to Mp was explicitly calculated for the generators of Sp (cf. [7], Chapter 4, Sections 2 and 3), which reveals, that the subspace $\mathcal{P}o\ell(V_j) \subset \mathcal{F}_c$ is not preserved under the action of Mp. Besides, the action of MU involves a square root of a complex number

$$U_k f(z) = \det^{-\frac{1}{2}}(k) f(k^{-1}z),$$

which needs some additional clarification. This ambiguity caused by the square root can be bypassed by passing to the group Mp^{c} and the fact, that any representation of Mp extends to a representation of Mp^{c} by setting

$$\varrho([A,\lambda])f = \lambda\varrho(A)f$$

for all $[A,\lambda] \in Mp^{\mathrm{c}}$. Then by the identification of MU^{c} with $U(V_j)\times\mathbb{S}^1$ as in the Corollary 3.1.8, the action of MU^{c} is given by

$$(\varrho(k,U_k)f)(z) = \lambda f(k^{-1}z), \tag{4.2}$$

where (k,λ) is the corresponding element of (k,U_k) in $U(V_j)\times\mathbb{S}^1$.

Remark 4.1.1. This formula reveals two crucial properties of the MU^{c}–action on polynomials. First it not only leaves $\mathcal{P}o\ell(V_j)$ invariant, but also respects its decomposition by degrees, and secondly, it coincides according to the Remark 1.1.8 with the induced action of the unitary group $U(V_j)$ together with the the obvious action of $\mathbb{S}^1$ when identifying $\mathcal{P}o\ell(V_j)$ with $S^*(V_j^*)$. This observation leads to the following lemma.

Lemma 4.1.2. *The diagram below commutes.*

$$\begin{tikzcd} MU^{\mathrm{c}}\times\mathcal{P}o\ell(V_j) \arrow[r, "\varrho"] \arrow[d, leftrightarrow, "\simeq"'] & \mathcal{P}o\ell(V_j) \arrow[d, leftrightarrow, "\simeq"] \\ (U(V_j)\times\mathbb{S}^1)\times S^*(V_j^*) \arrow[r, "\chi\cdot(\rho\circ\sigma)"] & S^*(V_j^*) \end{tikzcd} \tag{4.3}$$

The *space of smooth vectors* $\mathcal{F}_c^\infty$ *of* ϱ, this is a subspace of $\mathcal{F}_c$, on which the Lie algebra representation ϱ_* of $\mathfrak{mp}^{\mathrm{c}}$ is well-defined, is dense in $\mathcal{F}_c$ and contains $\mathcal{P}o\ell(V_j)$ (cf. [13], Theorem 3.15). To obtain the description of the action of the Lie algebra $\mathfrak{mu}^{\mathrm{c}}$ of MU^{c} we identify $\mathfrak{mu}^{\mathrm{c}}$ with $\mathfrak{u}(V_j)\oplus\mathfrak{u}(1)$ via the isomorphism $\sigma_*\times\chi_*$ from the Corollary 3.1.8. That means, for each $\Xi\in\mathfrak{mu}^{\mathrm{c}}$, which we identify with $(\sigma_*(\Xi),\chi_*(\Xi)) := (\kappa,\mu)$, we consider the curve $(\exp(t\kappa),\exp(t\mu))$ and differentiate the equation (4.2) at $t=0$. Then we obtain

$$\begin{aligned} (\varrho_*(\Xi)f)(z) &= \frac{d}{dt}\Big|_{t=0}(\varrho(\exp(t(\kappa,\mu)))f)(z) \\ &= \frac{d}{dt}\Big|_{t=0}\exp(t\mu)f(\exp(t\kappa)^{-1}z) \\ &= \mu f(z) - (\partial_z f)(\kappa z) \end{aligned} \tag{4.4}$$

for all $f\in\mathcal{F}_c^\infty$.

4.1.1 Symplectic spinor bundle. Let (P,φ) be an Mp^{c}–structure on (M,ω) determined by a line bundle L. For a fixed $c>0$ the *symplectic spinor bundle* can be defined as the associated Hilbert space bundle

$$\tilde{\mathcal{S}} := P \times_{Mp^{\mathrm{c}},\varrho} \mathcal{F}^{\infty}$$

and with the aid of an ω–compatible almost complex structure J on M the symplectic spinor bundle can also be written as

$$\tilde{\mathcal{S}} = Q \times_{MU^{\mathrm{c}},\varrho} \mathcal{F}^{\infty},$$

where Q is a reduction of P to MU^{c}. As mentioned above, the space of polynomials $\mathcal{P}o\ell(V_j)$ is invariant under the restriction of ϱ to MU^{c} and is a dense subspace of $\mathcal{F}^{\infty}$, which allows to consider the well-defined subbundle

$$\mathcal{S} := Q \times_{MU^{\mathrm{c}},\varrho} \mathcal{P}o\ell(V_j)$$

lying dense in $\tilde{\mathcal{S}}$. If we additionally decompose $\mathcal{P}o\ell(V_j)$ by degrees as

$$\mathcal{P}o\ell(V_j) = \bigoplus_{q=0}^{\infty} \mathcal{H}_q(V_j),$$

where $\mathcal{H}_q(V_j)$ is the space of homogeneous polynomials of degree q on V_j, then by the Remark 4.1.1 we obtain a well-defined vector bundle

$$\mathcal{S}_q := Q \times_{MU^{\mathrm{c}},\varrho} \mathcal{H}_q(V_j). \tag{4.5}$$

Being a complex vector bundle, it follows further by writing the restriction of ϱ to MU^{c} as $\varrho = \chi \cdot (\varrho' \circ \sigma)$ (cf. the notation in the Remark 1.1.8 *ii*) and Corollary 3.1.8)

$$\begin{aligned}
\mathcal{S}_q &= Q \times_{MU^{\mathrm{c}},\chi\cdot(\varrho'\circ\sigma)} (\mathbb{C} \otimes_{\mathbb{C}} \mathcal{H}_q(V_j)) \\
&\cong (Q \times_{MU^{\mathrm{c}},\chi} \mathbb{C}) \otimes_{\mathbb{C}} (Q \times_{MU^{\mathrm{c}},\varrho'\circ\sigma} \mathcal{H}_q(V_j)) \\
&\cong (\mathcal{L} \times_{\mathbb{S}^1} \mathbb{C}) \otimes (\mathcal{U}(M) \times_{U(V_j),\varrho'} \mathcal{H}_q(V_j)).
\end{aligned}$$

Recall, that the line bundle $\mathcal{L} \times_{\mathbb{S}^1} \mathbb{C}$ occurring above is isomorphic to the line bundle L, which we chose for constructing the principal Mp^{c}–bundle over M (cf. the Section 3.4.1 and, in particular, the bundle isomorphism in (3.7)).

Finally, if we identify

$$\mathcal{H}_q(V_j) \cong S^q(V_j^*) \cong S^q(V^{1,0}) = S^{q,0}(V)$$

by the isomorphism Φ and let the unitary group $U(V_j)$ act on $S^{*,0}(V)$ as described in the Section 1.1.5, then it yields

$$\mathcal{U}(M) \times_{U(V_j),\varrho'} \mathcal{H}_q(V_j) \cong \mathcal{U}(M) \times_{U(V_j),\hat{\rho}} S^q(V^{1,0}) \cong S^{q,0}(M).$$

Lemma 4.1.3. *There is a bundle isomorphism*

$$\mathcal{S} \cong L \otimes S^{*,0}(M).$$

Proof. To obtain the assertion we continue the above calculation

$$\begin{aligned}\mathcal{S} &= Q \times_{MU^c,\varrho} \mathcal{P}o\ell(V_j) = \bigoplus_{q=0}^{\infty} Q \times_{MU^c,\varrho} \mathcal{H}_q(V_j)\\ &\cong \bigoplus_{q=0}^{\infty} L \otimes S^{q,0}(M) = L \otimes \bigoplus_{q=0}^{\infty} S^{q,0}(M)\\ &= L \otimes S^{*,0}(M).\end{aligned}$$

□

Remark 4.1.4. Thinking of $\mathcal{S}$ as in the above lemma provides the following basic observations.

1) If the line bundle L is chosen as
   i) holomorphic, then $\mathcal{S}$ is holomorphic, provided J is a complex structure on M.
   ii) trivial, then $\mathcal{S}$ is isomorphic to the symmetric algebra of $T^{1,0}M$.
2) A natural covariant derivative on $\mathcal{S}$ is obtained by choosing a covariant derivative ∇ on TM, which induces a covariant derivative $\hat{\nabla}$ on $S^{*,0}(M)$, and a covariant derivative ∇' on the line bundle L. By defining

$$\nabla := \nabla' \otimes \mathbf{1} + \mathbf{1} \otimes \hat{\nabla}$$

   we obtain a covariant derivative on $\mathcal{S}$.

4.2 Symplectic Clifford multiplication

By taking the complexification of V and the complex bilinear extension of the inner product we transfer the symmetric product and the contraction to the symmetric tensors on $V^{1,0}$ by defining the linear maps

$$\boldsymbol{c}, \boldsymbol{a} : V \longrightarrow \mathrm{End}(S^{*,0}(V)), \qquad \boldsymbol{c}(v)\zeta := \bar{\mathbf{v}}^\flat \odot \zeta \quad \text{and} \quad \boldsymbol{a}(v)\zeta := \mathbf{v} \lrcorner \zeta,$$

where $\mathbf{v} = \frac{1}{2}(v - \mathrm{i}jv) \in V'$ as defined in the Section 1.1.3. These operators satisfy the following properties:

i) Since for any $v \in V$ we have $\Phi(jv) = \mathrm{i}\Phi(v) = \mathrm{i}\mathbf{v}$, the operator $\boldsymbol{c}$ is j−antilinear and $\boldsymbol{a}$ is j−linear. That is

$$\boldsymbol{c}(jv) = -\mathrm{i}\boldsymbol{c}(v) \quad \text{and} \quad \boldsymbol{a}(jv) = \mathrm{i}\boldsymbol{a}(v).$$

ii) For all $k \in U(V_j)$ it yields

$$\begin{aligned} \boldsymbol{c}(kv)(k\cdot\zeta) &= k\cdot(\boldsymbol{c}(v)\zeta) \\ \boldsymbol{a}(kv)(k\cdot\zeta) &= k\cdot(\boldsymbol{a}(v)\zeta). \end{aligned} \tag{4.6}$$

iii) The commutation relations

$$[\boldsymbol{c}(v),\boldsymbol{c}(w)] = [\boldsymbol{a}(v),\boldsymbol{a}(w)] = 0 \quad \text{and} \quad [\boldsymbol{a}(v),\boldsymbol{c}(w)] = \frac{1}{2}h(v,w)\mathrm{id}.$$

iv) With respect to the induced Hermitian inner product on $S^{*,0}(V)$ it holds

$$(\boldsymbol{a}(v)\zeta,\zeta') = (\zeta,\boldsymbol{c}(v)\zeta') \tag{4.7}$$

for all $v \in V$ (cf. Remark 1.1.7).

Define for a symplectic vector space (V,Ω) the *symplectic Clifford algebra*, denoted by $\mathrm{Cl}(V)$, to be a unital associative algebra over $\mathbb{C}$ generated by V with the relations

$$v\cdot w - w\cdot v = \mathrm{i}\,\Omega(v,w)\mathbf{1}$$

for all $v,w \in V$. If $\mathrm{Cl}(V)$ is additionally equipped with the Lie bracket

$$[a,b] := a\cdot b - b\cdot a$$

for $a,b \in \mathrm{Cl}(V)$, then $(\mathrm{Cl}(V),[\cdot,\cdot])$ becomes a Lie algebra. The linear map

$$cl : V \longrightarrow \mathrm{End}(S^{*,0}(V)), \qquad cl(v) := \boldsymbol{c}(v) - \boldsymbol{a}(v) \tag{4.8}$$

satisfies then for all $v,w \in V$

$$\begin{aligned} [cl(v),cl(w)] &= cl(v)cl(w) - cl(w)cl(v) \\ &= [\boldsymbol{a}(w),\boldsymbol{c}(v)] + [\boldsymbol{c}(w),\boldsymbol{a}(v)] \\ &= \frac{1}{2}h(w,v) - \frac{1}{2}h(v,w) \\ &= \mathrm{i}\,\Omega(v,w). \end{aligned}$$

That is, cl defines a *(symplectic) Clifford multiplication* (cf. [17], Definition 10.1.4).

To define the (symplectic) Clifford multiplication on $\mathcal{S}$, we write by Lemma 1.1.9

$$TM = Q\times_{MU^{\mathrm{c}},\sigma} V \quad \text{and} \quad \mathcal{S} = Q\times_{MU^{\mathrm{c}},\chi\cdot\hat{\rho}\circ\sigma} S^{*,0}(V),$$

that is, any $\xi \in \Gamma(\mathcal{S})$ is of the form $\xi = [q,\zeta]$. Write additionally $X = [q,v] \in \Gamma(TM)$ and define the operators C and A by

$$\begin{aligned} C : TM\otimes\mathcal{S} &\longrightarrow \mathcal{S}, \quad X\otimes\xi \longmapsto [q,\boldsymbol{c}(v)\zeta] = \bar{Z}_X^\flat \odot \xi \\ A : TM\otimes\mathcal{S} &\longrightarrow \mathcal{S}, \quad X\otimes\xi \longmapsto [q,\boldsymbol{a}(v)\zeta] = Z_X \lrcorner\, \xi. \end{aligned}$$

Then C and A are globally well-defined operators, since for all $(\lambda,k)\in MU^{c}$ we have by the equation (4.6)

$$\begin{aligned}C\big([q\cdot(\lambda,k),k^{-1}v]\big)\big([q\cdot(\lambda,k),\lambda^{-1}(k^{-1}\cdot\zeta)]\big)&=\big[q\cdot(\lambda,k),\lambda^{-1}\boldsymbol{c}(k^{-1}\cdot v)(k^{-1}\cdot\zeta)\big]\\&=\big[q\cdot(\lambda,k),\lambda^{-1}k^{-1}\cdot(\boldsymbol{c}(v)\zeta)\big]\\&=[q,\boldsymbol{c}(v)\zeta]\\&=C([q,v])([q,\zeta])\end{aligned}$$

and analogously

$$A\big([q\cdot(\lambda,k),k^{-1}\cdot v]\big)\big([q\cdot(\lambda,k),\lambda^{-1}(k^{-1}\cdot\zeta)]\big)=A([q,v])([q,\zeta]).$$

Lemma 4.2.1. *The bundle map $Cl:=C-A$ defines a (symplectic) Clifford multiplication.*

Remark 4.2.2. The operators A and C acting on holomorphic symmetric tensors inherit the properties $i)-iv)$ from the operators $\boldsymbol{a}$ and $\boldsymbol{c}$ listed above. That is

i) $C(JX)=-\mathrm{i}\,C(X)$ and $A(JX)=\mathrm{i}\,A(X)$.

ii) the commutation relations

$$[C(X),C(Y)]=[A(X),A(Y)]=0\quad\text{and}\quad[A(X),C(Y)]=\frac{1}{2}(X,Y)\mathrm{id}.\tag{4.9}$$

iii) the operators A and C are adjoint to each other with respect to the induced Hermitian inner product on $\mathcal{S}$. That is

$$(C(X)\xi,\xi')=(\xi,A(X)\xi')\tag{4.10}$$

for any $X\in TM$.

4.2.1 Parallelity of the Clifford multiplication. We observe next, that the operators C and A are parallel. Since by the definition of $\mathcal{S}$ the covariant derivative on M has to be chosen as unitary, we obtain for all $X\in\Gamma(TM)$ and $\xi\in\Gamma(\mathcal{S})$. Then it yields

$$\nabla_X(C(e_j)\xi)=\nabla_X(\bar{Z}_j^\flat)\odot\xi+\bar{Z}_j^\flat\odot\nabla_X\xi$$

and on the other hand we have

$$C(\nabla_Xe_j)\xi=\overline{Z_{\nabla_Xe_j}}^\flat\odot\xi=(\nabla_X\bar{Z}_j)^\flat\odot\xi$$

and

$$C(e_j)\nabla_X\xi=\bar{Z}_j^\flat\odot\nabla_X\xi.$$

By observing, that for all $Y \in T^{1,0}M$ we have

$$\begin{aligned}(\nabla_X \bar{Z}_j^\flat)(Y) &= X(\bar{Z}_j^\flat(Y)) - \bar{Z}_j^\flat(\nabla_X Y) = X(\bar{Z}_j^\flat(Y)) - \langle \nabla_X Y, \bar{Z}_j\rangle \\ &= X(\bar{Z}_j^\flat(Y)) - X \cdot \langle Y, \bar{Z}_j\rangle + \langle Y, \nabla_X \bar{Z}_j\rangle) = (\nabla_X \bar{Z}_j)^\flat(Y).\end{aligned}$$

we conclude, that

$$\nabla_X(C(e_k)\xi) = C(\nabla_X e_k)\xi + C(e_k)\nabla_X \xi$$

for all $X \in \Gamma(TM)$. Similarly, the operator A satisfies

$$\begin{aligned}\nabla_X(A(e_j)\xi) = \nabla_X(Z_j \lrcorner \xi) &= (\nabla_X Z_j) \lrcorner \xi + Z_j \lrcorner \nabla_X \xi \\ &= A(\nabla_X e_j)\xi + A(e_j)\nabla_X \xi\end{aligned}$$

since ∇ is unitary and $\mathbb{C}$−linear

$$Z_{\nabla_X e_j} = \frac{1}{2}(\nabla_X e_j - iJ\nabla_X e_j) = \nabla_X Z_j.$$

Lemma 4.2.3. *The Clifford multiplication as well as the operators A and C are parallel with respect to a covariant derivative induced by a unitary connection on M.*

4.3 The Laplacian induced by a symplectic Dirac operator

We begin this section with the definition of the *symplectic Dirac operators* on an almost symplectic manifold (M,ω), describe then its splitting into the *symplectic Dirac-Dolbeault operators* when decomposing the complexified tangent bundle by means of an almost complex structure and give at the end the mentioned relation to the Lichnerowicz Laplacian.

Definition 4.3.1 (Symplectic Dirac operator)**.** The *symplectic Dirac operator D* is defined as the composition of the maps

$$D : \Gamma(\mathcal{S}) \xrightarrow{\nabla} \Gamma(T^*M \otimes \mathcal{S}) \cong_\omega \Gamma(TM \otimes \mathcal{S}) \xrightarrow{Cl} \Gamma(\mathcal{S}).$$

If the tangent bundle TM is identified with T^*M by means of g instead of ω, the arising operator is called the *J-twisted symplectic Dirac operator*

$$\tilde{D} : \Gamma(\mathcal{S}) \xrightarrow{\nabla} \Gamma(T^*M \otimes \mathcal{S}) \cong_g \Gamma(TM \otimes \mathcal{S}) \xrightarrow{Cl} \Gamma(\mathcal{S}).$$

The two different ways of the identification of the tangent bundle with the cotangent bundle, give rise to the two different symplectic Dirac operators. To describe them locally we use a special local frame $\{e_1, \ldots, e_{2n}\}$ for TM, where $\{e_1, \ldots, e_n\}$ is a local unitary frame for T_JM and $e_{n+k} = Je_k$ for all $1 \le k \le n$, and refer to it also as *unitary* (see also the Remark 1.1.4 *i*)).

To keep the notation short the Clifford multiplication will be simply denoted by a dot $\cdot$.

Proposition 4.3.2. *Let $\{e_1, \dots, e_{2n}\}$ be a local unitary frame for TM. Then it yields*

$$D\xi = \sum_{j=1}^{2n} e_j \cdot \nabla_{Je_j}\xi = -\sum_{j=1}^{2n} Je_j \cdot \nabla_{e_j}\xi \qquad \text{and} \qquad \tilde{D}\xi = \sum_{j=1}^{2n} e_j \cdot \nabla_{e_j}\xi.$$

As already described in the Section 2.2 the decomposition of $T_{\mathbb{C}}M$ into the eigenbundles of J and the complex linear extension of ∇ yields the decomposition

$$\nabla^{1,0}_X := \nabla_{Z_X} \quad \text{and} \quad \nabla^{0,1}_X := \nabla_{\bar{Z}_X},$$

where $2Z_X = X - \mathrm{i}JX$ as in the Definition 2.7. This results in the decomposition of the symplectic Dirac operator as

$$D^{1,0}\xi := -\sum_{j=1}^{2n} Je_j.\nabla^{1,0}_{e_j}\xi \qquad \text{and} \qquad D^{0,1}\xi := -\sum_{j=1}^{2n} Je_j.\nabla^{0,1}_{e_j}\xi.$$

Definition 4.3.3. These operators are called the *symplectic Dirac-Dolbeault operators.*

Proposition 4.3.4. *The symplectic Dirac-Dolbeault operators satisfy*

$$D = D^{1,0} + D^{0,1} \quad \text{and} \quad \tilde{D} = \mathrm{i}(D^{0,1} - D^{1,0}). \tag{4.11}$$

Proof. The first equation is obvious. The second equation follows essentially by the independence of ∇ of the chosen local frame. That is, if $\{e_1, \dots, e_{2n}\}$ is a local unitary frame for TM, then $\{Je_1, \dots, Je_{2n}\}$ is a local unitary frame for TM as well and thus

$$\begin{aligned}
\tilde{D}\xi &= \sum_{j=1}^{2n} Je_j \cdot \nabla_{Je_j}\xi = \sum_{j=1}^{2n} Je_j \cdot (\nabla^{1,0}_{Je_j}\xi + \nabla^{0,1}_{Je_j}\xi) \\
&= \sum_{j=1}^{2n} Je_j \cdot (\nabla_{Z_{Je_j}}\xi + \nabla_{\bar{Z}_{Je_j}}\xi) = \sum_{j=1}^{2n} Je_k \cdot (\nabla_{\mathrm{i}Z_j}\xi + \nabla_{-\mathrm{i}\bar{Z}_j}\xi) \\
&= \sum_{j=1}^{2n} Je_j \cdot (\mathrm{i}\nabla^{1,0}_{e_j}\xi - \mathrm{i}\nabla^{0,1}_{e_j}\xi) = \mathrm{i}\sum_{j=1}^{2n} Je_j \cdot (\nabla^{1,0}_{e_j}\xi) - \mathrm{i}\sum_{j=1}^{2n} Je_j \cdot (\nabla^{0,1}_{e_j}\xi) \\
&= -\mathrm{i}D^{1,0}\xi + \mathrm{i}D^{0,1}\xi.
\end{aligned}$$

□

Next we observe, that the operator $D^{1,0}$ can be expressed only in terms of C and the operator $D^{0,1}$ only in terms of A and the corresponding projections. That is, for all $\xi \in \Gamma(\mathcal{S})$ we have

$$\begin{aligned}
D^{1,0}\xi &= -\frac{1}{2}\sum_{j=1}^{2n} Je_j \cdot \nabla_{(e_j - \mathrm{i}Je_j)}\xi = -\frac{1}{2}(\sum_{j=1}^{2n} Je_j \cdot \nabla_{e_j}\xi - \mathrm{i}Je_j \cdot \nabla_{Je_j}\xi) \\
&= \frac{1}{2}(\sum_{j=1}^{2n} e_j \cdot \nabla_{Je_j}\xi + \mathrm{i}Je_j \cdot \nabla_{Je_j}\xi)
\end{aligned}$$

$$= \frac{1}{2}\Big(\sum_{j=1}^{2n}(C(e_j) - A(e_j) + C(e_j) + A(e_j))\nabla_{Je_j}\xi\Big)$$

$$= \sum_{j=1}^{2n} C(e_j)\nabla_{Je_j}\xi.$$

In addition it holds

$$\sum_{j=1}^{2n} C(e_j)\nabla_{Je_j}\xi = -\sum_{j=1}^{2n} C(Je_j)\nabla_{e_j}\xi = i\sum_{j=1}^{2n} C(e_j)\nabla_{e_j}\xi = \sum_{j=1}^{2n} C(e_j)\nabla_{\mathrm{i}e_j}\xi$$

so we finally obtain

$$D^{1,0}\xi = \frac{1}{2}\Big(\sum_{j=1}^{2n} C(e_j)\nabla_{Je_j}\xi + \sum_{j=1}^{2n} C(e_j)\nabla_{\mathrm{i}e_j}\xi\Big)$$

$$= \frac{1}{2}\Big(\sum_{j=1}^{2n} C(e_j)\nabla_{\mathrm{i}(e_j - \mathrm{i}Je_j)}\xi\Big) = \sum_{j=1}^{2n} C(e_j)\nabla^{1,0}_{Je_j}\xi = 2\mathrm{i}\sum_{j=1}^{n} C(e_j)\nabla^{1,0}_{e_j}\xi.$$

With the same argumentation and calculation we obtain for all $\xi \in \Gamma(\mathcal{S})$

$$D^{0,1}\xi = -\sum_{j=1}^{2n} A(Je_j)\nabla^{0,1}_{e_j}\xi = -2i\sum_{j=1}^{n} A(e_j)\nabla^{0,1}_{e_j}\xi.$$

The obtained expressions of the symplectic Dirac-Dolbeault operators in the above lemma allows to formulate the following theorem.

Theorem 4.3.5. *i) The operator $D^{1,0}$ coincides with the symmetrization of $\nabla^{1,0}$, that is*

$$D^{1,0} = \mathrm{i}\,\mathrm{sym}\circ\nabla^{1,0}.$$

ii) If $d\omega = 0$ and ∇ is the canonical Hermitian connection on M, the symplectic Dirac-Dolbeault operators are formal adjoint to each other with respect to the induced Hermitian L^2-product on the compactly supported sections of $\mathcal{S}$.

Proof. The first statement follows from the previous calculation and the definition of C, namely

$$D^{1,0}\xi = 2\mathrm{i}\sum_{j=1}^{n} C(e_j)\nabla^{1,0}_{e_j}\xi = \mathrm{i}\sum_{j=1}^{n} 2\bar{Z}_j^\flat \odot \nabla_{Z_j}\xi = \mathrm{i}\sum_{j=1}^{n} Z_j^* \odot \nabla_{Z_j}\xi.$$

for $\xi \in \Gamma(\mathcal{S})$. For the second statement we observe, that by definition of A we have

$$D^{0,1}\xi = -2\mathrm{i}\sum_{j=1}^{n} A(e_j)\nabla^{0,1}_{e_j}\xi = -2\mathrm{i}\sum_{j=1}^{n} Z_j \lrcorner \nabla_{\bar{Z}_j}\xi,$$

and a straight forward calculation yields

$$(D^{1,0}\xi, \psi)_{L^2} = (\xi, D^{0,1}\psi)_{L^2} + \int_M \mathrm{div}(Y)\ \mathrm{d}M$$

with $Y = \sum_{j=1}^n (\xi, Z_j \lrcorner \psi) Z_j$. Thus the assertion follows by the Corollary 2.2.7 in [9]. $\square$

A short straightforward verification of the definition analogous to that made in the Lemma 2.2.13 yields the following lemma.

Lemma 4.3.6. *Let $(x,\theta) \in T_x^*M \smallsetminus \{0\}$ and $\xi_x \in \mathcal{S}_x$. Then the principal symbol of $D^{1,0}$ is given by*

$$ps(D^{1,0})(x,\theta)\xi_x = \mathrm{i}\,\theta^{1,0} \odot \xi_x, \tag{4.12}$$

where $\theta^{1,0} = \sum_{j=1}^n \theta(Z_j) Z_j^*$ is the projection of θ onto $T^{1,0}M$.

Having the principal symbol of $D^{1,0}$ the principal symbols of $D, \tilde{D}$ and $D^{0,1}$ are then obtained by the equation (4.11) and the Theorem 4.3.5 *i*). In particular, we have

$$ps(D^{0,1})(x,\theta)\xi_x = ps(D^{1,0})^*(x,\theta)\xi_x = -\mathrm{i}(\theta^{0,1})^{\#} \lrcorner\, \xi_x,$$

such that the commutation relation (4.9), ensures the operator $\mathcal{P} := 2\,[D^{1,0}, D^{0,1}]$ to define a Laplacian. It was shown in [9], that on Kähler manifolds the operator $\mathcal{P}$ satisfies the Weitzenböck formula. The following theorem shows, that for the canonical Hermitian connection the Weitzenböck formula also holds on almost Kähler manifolds.

Theorem 4.3.7. *With respect to the canonical Hermitian connection on an almost Kähler manifold the Laplacian $\mathcal{P}$ satisfies the Weitzenböck formula and, if L is trivial, then $\mathcal{P}$ coincides with the Lichnerowicz Laplacian $-\Delta^{1,0}$.*

Proof. Denote by $\nabla = \nabla' \otimes 1 + 1 \otimes \hat{\nabla}$ a covariant derivative on $L \otimes S^{*,0}(M)$ as mentioned in the Remark 4.1.4 2), where $\hat{\nabla}$ is induced by the canonical Hermitian connection on M. Then for a local unitary frame parallel at a point we obtain on the one hand

$$\begin{aligned}
D^{1,0}D^{0,1}(\lambda\otimes\varphi) &= -2\mathrm{i}D^{1,0}\Big(\sum_{j=1}^n Z_j \lrcorner\, \nabla_{\bar{Z}_j}(\lambda\otimes\varphi)\Big) = -2\mathrm{i}D^{1,0}\Big(\sum_{j=1}^n Z_j \lrcorner\, (\nabla'_{\bar{Z}_j}\lambda\otimes\varphi + \lambda\otimes\hat{\nabla}_{\bar{Z}_j}\varphi)\Big)\\
&= -2\mathrm{i}\sum_{j=1}^n D^{1,0}(\nabla'_{\bar{Z}_j}\lambda\otimes Z_j \lrcorner\,\varphi) + \mathrm{i}D^{1,0}(\lambda\otimes\delta^{1,0^*}\varphi)\\
&= 2\sum_{j,k=1}^n Z_k^*\odot\nabla_{Z_k}(\nabla'_{\bar{Z}_j}\lambda\otimes Z_j\lrcorner\,\varphi) - \sum_{k=1}^n Z_k^*\odot\nabla_{Z_k}(\lambda\otimes\delta^{1,0^*}\varphi)\\
&= 2\sum_{j,k=1}^n \nabla'_{Z_k}\nabla'_{\bar{Z}_j}\lambda\otimes Z_k^*\odot Z_j\lrcorner\,\varphi + 2\sum_{j=1}^n \nabla'_{\bar{Z}_j}\lambda\otimes\delta^{1,0}(Z_j\lrcorner\,\varphi)\\
&\qquad - \sum_{k=1}^n \nabla'_{Z_k}\lambda\otimes Z_k^*\odot\delta^{1,0^*}\varphi - \lambda\otimes\delta^{1,0}\delta^{1,0^*}\varphi\\
&= 2\sum_{j,k=1}^n \nabla'_{Z_k}\nabla'_{\bar{Z}_j}\lambda\otimes Z_k^*\odot Z_j\lrcorner\,\varphi + 2\sum_{j=1}^n \nabla'_{\bar{Z}_j}\lambda\otimes(-\hat{\nabla}_{Z_j}\varphi + Z_j\lrcorner\,\delta^{1,0}\varphi)\\
&\qquad - \sum_{k=1}^n \nabla'_{Z_k}\lambda\otimes Z_k^*\odot\delta^{1,0^*}\varphi - \lambda\otimes\delta^{1,0}\delta^{1,0^*}\varphi
\end{aligned}$$

and on the other hand

$$\begin{aligned}
D^{0,1}D^{1,0}(\lambda\otimes\varphi) &= \mathrm{i}D^{0,1}\Big(\sum_{k=1}^{n}\nabla'_{Z_k}\lambda\otimes Z_k^*\odot\varphi+\lambda\otimes Z_k^*\odot\hat{\nabla}_{Z_k}\varphi)\Big)\\
&= \mathrm{i}\sum_{k=1}^{n}D^{0,1}(\nabla'_{Z_k}\lambda\otimes Z_k^*\odot\varphi)+\mathrm{i}D^{0,1}(\lambda\otimes\delta^{1,0}\varphi)\\
&= 2\sum_{j,k=1}^{n}Z_j\lrcorner\nabla_{\bar{Z}_j}(\nabla'_{Z_k}\lambda\otimes Z_k^*\odot\varphi)+2\sum_{j=1}^{n}Z_j\lrcorner\nabla_{\bar{Z}_j}(\lambda\otimes\delta^{1,0}\varphi)\\
&= 2\sum_{j,k=1}^{n}\nabla'_{\bar{Z}_j}\nabla'_{Z_k}\lambda\otimes Z_j\lrcorner Z_k^*\odot\varphi-\sum_{k=1}^{n}\nabla'_{Z_k}\lambda\otimes\delta^{1,0^*}(Z_k^*\odot\varphi)\\
&\qquad+2\sum_{j=1}^{n}\nabla'_{\bar{Z}_j}\lambda\otimes Z_j\lrcorner\delta^{1,0}\varphi-\lambda\otimes\delta^{1,0^*}\delta^{1,0}\varphi\\
&= 2\sum_{j=1}^{n}\nabla'_{\bar{Z}_j}\nabla'_{Z_j}\lambda\otimes\varphi+2\sum_{j,k=1}^{n}\nabla'_{\bar{Z}_j}\nabla'_{Z_k}\lambda\otimes Z_k^*\odot Z_j\lrcorner\varphi\\
&\qquad-\sum_{k=1}^{n}\nabla'_{Z_k}\lambda\otimes(-2\hat{\nabla}_{\bar{Z}_k}\varphi+Z_k^*\odot\delta^{1,0^*}\varphi)\\
&\qquad+2\sum_{j=1}^{n}\nabla'_{\bar{Z}_j}\lambda\otimes Z_j\lrcorner\delta^{1,0}\varphi-\lambda\otimes\delta^{1,0^*}\delta^{1,0}\varphi.
\end{aligned}$$

Denote the curvature of ∇ by $\mathfrak{R}$ then we obtain further

$$\begin{aligned}
\mathcal{P}(\lambda\otimes\varphi) &= 2[D^{1,0},D^{0,1}](\lambda\otimes\varphi)\\
&= -4\sum_{j=1}^{n}\nabla'_{\bar{Z}_j}\nabla'_{Z_j}\lambda\otimes\varphi-4\sum_{j,k=1}^{n}R'(\bar{Z}_j,Z_k)\lambda\otimes Z_k^*\odot Z_j\lrcorner\varphi\\
&\qquad-4\sum_{j=1}^{n}\nabla'_{\bar{Z}_j}\lambda\otimes\hat{\nabla}_{Z_j}\varphi+\nabla'_{Z_j}\lambda\otimes\hat{\nabla}_{\bar{Z}_j}\varphi-\lambda\otimes\Delta^{1,0}\varphi\\
&= -4\sum_{j=1}^{n}\nabla'_{\bar{Z}_j}\nabla'_{Z_j}\lambda\otimes\varphi-4\sum_{j,k=1}^{n}R'(\bar{Z}_j,Z_k)\lambda\otimes Z_k^*\odot Z_j\lrcorner\varphi\\
&\qquad-4\sum_{j=1}^{n}\nabla'_{\bar{Z}_j}\lambda\otimes\hat{\nabla}_{Z_j}\varphi+\nabla'_{Z_j}\lambda\otimes\hat{\nabla}_{\bar{Z}_j}\varphi\\
&\qquad-4\sum_{j=1}^{n}\lambda\otimes\hat{\nabla}_{\bar{Z}_j}\hat{\nabla}_{Z_j}\varphi-4\sum_{j,k=1}^{n}\lambda\otimes Z_k^*\odot Z_j\lrcorner\hat{R}(\bar{Z}_j,Z_k)\varphi\\
&= -4\sum_{j=1}^{n}\nabla_{\bar{Z}_j}\nabla_{Z_j}(\lambda\otimes\varphi)-4\sum_{j,k=1}^{n}Z_k^*\odot Z_j\lrcorner\mathfrak{R}(\bar{Z}_j,Z_k)(\lambda\otimes\varphi)\\
&= \nabla^*\nabla(\lambda\otimes\varphi)-4\sum_{j,k=1}^{n}Z_k^*\odot Z_j\lrcorner\mathfrak{R}(\bar{Z}_j,Z_k)(\lambda\otimes\varphi)+\mathrm{i}\sum_{j=1}^{n}\mathfrak{R}(e_j,Je_j)(\lambda\otimes\varphi).
\end{aligned}$$

The comparison with the Theorem 2.2.15 completes the proof. □

Remark 4.3.8. *i*) If the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ is decomposed by types as described in the Lemma 1.2.7

$$\tilde{R} = \underbrace{(R_1 - S)}_{\in S^2(\lambda^{1,1})} - \underbrace{A}_{\in\Lambda^2(\lambda^{1,1})} + \underbrace{\mathbf{a}(\tilde{\nabla}\tau)_1^\star}_{\in\lambda^{2,0}\otimes\lambda^{1,1}},$$

then the linear part of $\mathcal{P}$ involving $\tilde{\nabla}\tau$ vanishes identically, simply because of its type, and moreover it holds

$$\begin{aligned}\sum_{i,j,k,m=1}^{n} A^{i'}_{\bar{k}'j'm'} Z_j^* \odot Z_m^* \odot Z_k \lrcorner Z_i \lrcorner \varphi &= -\sum_{i,j,k,m=1}^{n} A^{\bar{j}'}_{m'\bar{i}'\bar{k}'} Z_j^* \odot Z_m^* \odot Z_k \lrcorner Z_i \lrcorner \varphi \\ &= -\sum_{i,j,k,m=1}^{n} A^{k'}_{\bar{i}'m'j'} Z_j^* \odot Z_m^* \odot Z_k \lrcorner Z_i \lrcorner \varphi \\ &= -\sum_{i,j,k,m=1}^{n} A^{i'}_{\bar{k}'j'm'} Z_j^* \odot Z_m^* \odot Z_i \lrcorner Z_k \lrcorner \varphi.\end{aligned}$$

Since additionally the Ricci tensor is symmetric, we conclude, that the terms of $\tilde{R}$ occurring in the linear part of $\mathcal{P}$ involve only R_1 and S.

ii) For the proof of the the Weitzenböck formula of $\mathcal{P}$ we could have used the expression of $\mathcal{P}$ calculated in the Proposition 10 in [4], which reads

$$\mathcal{P}\varphi = -\Delta\varphi + \nabla_{J\tau}\varphi + \mathrm{i}\sum_{j,k=1}^{2n} Je_j \cdot e_k \cdot (\mathfrak{R}(e_j, e_k) - \nabla_{\tilde{\mathrm{T}}(e_j,e_k)})\varphi.$$

Then with respect to the canonical Hermitian connection on an almost Kähler manifold the term involving τ vanishes by definition. Further on, by using additionally the definition of the Clifford multiplication the operator $\mathcal{P}$ can be written as

$$\mathcal{P}\varphi = -\Delta\varphi + \sum_{j,k=1}^{2n} (C(e_j)A(e_k) - A(e_j)C(e_k))(\mathfrak{R}(e_j, e_k) - \nabla_{\tilde{\mathrm{T}}(e_j,e_k)})\varphi. \tag{4.13}$$

To see, that the term involving the torsion vanishes for the canonical Hermitian connection as well, we observe, that

$$\begin{aligned}\sum_{j,k=1}^{2n} C(e_j)A(e_k)\nabla_{\tilde{\mathrm{T}}(e_j,e_k)}\varphi &= \sum_{j,k=1}^{n} C(e_j)A(e_k)\nabla_{\tilde{\mathrm{T}}(e_j,e_k)}\varphi + C(Je_j)A(e_k)\nabla_{\tilde{\mathrm{T}}(Je_j,e_k)}\varphi \\ &\quad + C(e_j)A(Je_k)\nabla_{\tilde{\mathrm{T}}(e_j,Je_k)}\varphi + C(Je_j)A(Je_k)\nabla_{\tilde{\mathrm{T}}(Je_j,Je_k)}\varphi \\ &= \sum_{j,k=1}^{n} C(e_j)A(e_k)\nabla_{\tilde{\mathrm{T}}(e_j,e_k)}\varphi + \mathrm{i}C(e_j)A(e_k)\nabla_{J\tilde{\mathrm{T}}(e_j,e_k)}\varphi \\ &\quad - \mathrm{i}C(e_j)A(e_k)\nabla_{J\tilde{\mathrm{T}}(e_j,e_k)}\varphi - C(e_j)A(e_k)\nabla_{\tilde{\mathrm{T}}(e_j,e_k)}\varphi \\ &= 0\end{aligned}$$

and with the same calculation the sum $\sum_{j,k=1}^{2n}(C(e_j)A(e_k) - A(e_j)C(e_k))\nabla_{\tilde{T}(e_j,e_k)}$ vanishes as well.

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