New Diffuse Approximations of the Willmore Energy, the Mean Curvature Flow, and the Willmore Flow

Dissertation
zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften
(Dr. rer. nat.)

Der Fakultät für Mathematik
der Technischen Universität Dortmund
vorgelegt von

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im April 2023

Dissertation

New Diffuse Approximations of the Willmore Energy, the Mean Curvature Flow, and the Willmore Flow

Fakultät für Mathematik

Technische Universität Dortmund

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Tag der mündlichen Prüfung: 12.06.2023

Abstract

In this thesis we derive a higher order diffuse approximation of the Willmore energy from contributions by Karali and Katsoulakis [*J. Differential Equations*, 2007], who studied a diffuse approximation of mean curvature flow. We prove Γ–convergence in smooth limit points for the sum of diffuse perimeter and the higher order diffuse Willmore energy in dimensions 2 and 3.

Moreover, we prove the convergence on arbitrary time intervals towards weak solutions of mean curvature flow.

We also consider a gradient-free diffuse approximation of the Willmore energy in the sense of Γ–convergence which we derive from a gradient-free diffuse approximation of the perimeter by Amstutz and Van Goethem [*Interfaces Free Bound.*, 2012]. We prove the $\limsup$–property for the Γ–convergence towards a multiple of the Willmore energy.

In addition, we consider L^2-type gradient flows of both diffuse Willmore energies, and give an asymptotic convergence result. Formally these constitute diffuse approximations of mean curvature flow and Willmore flow. In a restricted class of diffuse phase-field evolutions, we prove that these gradient flows convergence towards rescaled mean curvature flow and rescaled Willmore flow, respectively.

Acknowledgement

I want to thank my supervisor Prof. Dr. Matthias Röger for all the support throughout my time as a PhD student. He helped me any time I was struggling.
Furthermore, my gratitude belongs to all of the (former) members of the Analysis group and the Biomathematics group during my time at the TU Dortmund including, but not limited to, Prof. Dr. Ben Schweizer, Prof. Dr. Peter Bella, Prof. em. Dr. Winfried Kaballo, JProf. Dr. Mathias Schäffner, PD Dr. Andreas Rätz, Dr. Peter Furlan, Dr. Michael Kniely, Dr. Jörg Sawollek, Dr. Roberta Marziani, Dr. Christopher Irving, Dr. Yohanes Tjandrawidjaja, Dr. Carsten Zwilling, Dr. Lisa Wahlers, Dr. Nils Dabrock, Dr. Maik Urban, Dr. Klaas Poelstra, Dr. Florian Oschmann, M.Sc. Jakob Fuchs, M.Sc. Daniel Trampe, B. Sc. Tim Schubert, B.Sc. Friederike Lemming, and Insa-Marie Schneider for many fruitful (mathematical) discussions, excellent suggestions, and thrilling games of darts.
A special thank you goes to Jakob for his support with the corrections during my last semester. I would also like to thank all non-academic staff, in particular the secretaries Fiona Drees, Saskia Orlok, Sarah Kuhlmann, Janine Textor without whom the 6^{th} floor would be in trouble.
As a side note I would like to point out that the quotes on pages IV and 193 are in no way directed at anyone, but rather self-deprecatingly express what doing mathematical research sometimes felt like to me.
I am grateful to all of my former teachers, in particular Bettina Dombrowski-Hunsche, Renate Sennholz, Heiko Hörmeyer, Sonja Jacobs, Dr. Martin Brückner, Dr. Markus Tendahl, Paul Steffen, and Inge Levin all of whom promoted me in some form.
Additionally, I am grateful to my fellow students from my Bachelor and Master studies Mirko Jakubzik, Katharina Bata, and Nicole Hufnagel for their friendship and advice.

I thank all of my friends (MtG-group, SH-group and many more) and family for their support and motivation, in particular my parents Bea and Dieter who have helped me in troubled times and always encouraged my wildest ideas and my brother Marius who has always been there for me. Finally I am thankful to Larissa for her love, patience, and support.
Without them, I would not have made it to this point.

This thesis is dedicated to my grandparents, all of whom have helped me become the person I am today.

"To witness secrets sealed, one must endure the harshest punishment."[1]

[1] quoted from a lore tablet in the *Path of Pain* in the video game *Hollow Knight* developed and published by Team Cherry in 2017

Contents

List of symbols

$\mathbb{N}$	$\{1, 2, \dots\}$ natural numbers
$\mathbb{N}_0$	$\{0, 1, 2, \dots\}$ natural numbers including 0
$\mathbb{N}^{\geq m}$	$\{m, m+1, m+2, \dots\}$ natural numbers greater than or equal to $m \in \mathbb{N}$
n	$n \in \mathbb{N}$ dimension of $\mathbb{R}^n$
$\widehat{\mathbb{R}}$	$\mathbb{R} \cup \{+\infty\}$
Ω	$\emptyset \neq \Omega \subseteq \mathbb{R}^n$ open set with varying additional assumptions but never empty
$\partial^* A$	essential boundary of A
$A^\perp$	orthogonal complement of a set or vector
$\mathbin{\dot\cup}$	disjoint union
$A \Subset B$	A is compact in B
$\mathcal{L}^n$	n-dimensional Lebsgue measure
$\mathcal{H}^{n-1}$	$(n-1)$-dimensional Hausdorff measure
a.e.	almost every
$\mu \llcorner A$	restriction of the measure μ to the set A
$D_\mu \nu$	measure derivative of ν with respect to μ in the sense of Radon-Nikodym
$\fint_\Omega f \, \mathrm{d}\mu$	normalized integral $\frac{1}{\mu(\Omega)} \int_\Omega f \, \mathrm{d}\mu$
C	positive constants whose exact value does not matter
$\varepsilon \to 0$	$\varepsilon \searrow 0$
δ_{kj}	Kronecker-δ
e_j	j-th canonical base vector on R^n
ν_Γ	inner normal vector of a compact hypersurface
ν_Ω	outer normal vector of an open set with Lipschitz boundary
$T_p\Gamma$	tangent space of a hypersurface or approximate tangent space of a $(n-1)$-rectifable set at $p \in \Gamma$
$G(n,k)$	Grassmannian
${}^{\oplus}G(n,k)$	oriented Grassmannian
$\mathbb{V}_k(\Omega)$	k-varifolds on Ω
${}^{\oplus}\mathbb{V}_k(\Omega)$	oriented k-varifolds on Ω
δV	first variation of a varifold or functional
$\lVert V \rVert$	mass measure of a varifold or norm of a linear mapping

$\|f\|_Y$	norm of f in Y
$\ker(f)$	kernel of f
$\mathrm{range}(f)$	range of f
$f^{\#}g$	pullback function
$f_{\#}\mu$	pushforward measure
$\partial_j^h f(x)$	discrete partial derivative $\frac{f(x+he_j)-f(x)}{h}$ in direction of e_j
∂_ν	normal derivative
∇_Γ	tangential derivative on a hypersurface Γ
Δ_Γ	Laplace-Beltrami operator on a hypersurface Γ
$*$	convolution operation
∇_{L^2}	L^2-gradient of a functional
$\hookrightarrow$	continuous embedding
$\overset{c}{\hookrightarrow}$	compact embedding
E'	dual of E
$\langle \cdot, \cdot \rangle_{E'}$	dual product on $E \times E'$
$\langle \cdot \vert \cdot \rangle_E$	scalar product on $E \times E$
$BV(\Omega; Y)$	functions of bounded variation on Ω with values in Y
$L^p(\Omega, \mu; Y)$	equivalent classes of functions f defined on Ω with values in Y such that $\vert f\vert^p$ is integrable with respect to μ
$W^{k,p}(\Omega; Y)$	Sobolev space of functions on Ω with values in Y and with k weak derivatives in $L^p(\Omega; Y)$
$H^k(\Omega; Y)$	$W^{k,2}(\Omega; Y)$
$C^k(\Omega; Y)$	space of k-times differentiable functions on Ω with continuous k^{th}-derivative and values in Y
$C^{k,\alpha}(\Omega; Y)$	space of functions that are k-times differentiable and the k^{th} derivative is α-Hölder continuous with values in Y
$Z_{\mathrm{loc}}(\Omega; Y)$	space of all functions that are in $Z(\Omega; Y)$ for all $K \Subset \Omega$
$Z_c(\Omega; Y)$	space of functions in $Z(\Omega; Y)$ with compact support
$Z_0(\Omega; Y)$	closure of $Z_c(\Omega; Y)$ with respect to the $Z(\Omega; Y)$-norm
$Z_b(\Omega; Y)$	space of bounded functions in $Z(\Omega; Y)$

1 Introduction

The Willmore energy $\mathcal{W}(\Gamma)$ of a C^2-hypersurface $\Gamma \subseteq \mathbb{R}^n$ with $n \in \mathbb{N}^{\geq 2}$ is defined as

$$\mathcal{W}(\Gamma) \coloneqq \int_\Gamma |\vec{H}|^2 \, \mathrm{d}\mathcal{H}^{n-1}. \tag{1.1}$$

Here $\vec{H}$ denotes the mean curvature vector. Functionals like $\mathcal{W}$ have been investigated for more than two centuries, in fact already Poisson [Poi14] in 1814 and Germain [Ger21] in 1821 have discussed curvature based energies. The name Willmore energy has been used since the early 2000s as an acknowledgment of the contributions of Thomas Willmore [Wil65, Wil93]. During the 20$^{\text{th}}$ century the Willmore energy has appeared in many different important works such as [Tho24, GG29, Can70, Hel73].
The Willmore energy is closely related to the perimeter and its gradient flow. The perimeter of a set E with smooth boundary can be defined as

$$\mathcal{P}(E) \coloneqq \mathcal{H}^{n-1}(\partial E).$$

The perimeter is connected to the mean curvature vector via

$$\nabla_{L^2}\mathcal{P} = -\vec{H}. \tag{1.2}$$

The mean curvature vector $\vec{H}$ points in the direction of the steepest area descent. Next we consider the gradient flow induced by $\mathcal{P}$, the mean curvature flow.

The mean curvature flow is one of the most prominent geometric flows and has been studied extensively in the past decades. We consider an evolution of open sets $(E(t))_{t\in(0,T)}$ with smooth boundaries $\Gamma_t \coloneqq \partial E(t)$. We say that the surfaces $(\Gamma_t)_{t\in(0,T)}$ evolve by mean curvature flow if for all $t \in (0,T)$ we have

$$\vec{\mathcal{V}}(t) = \vec{H}_t, \tag{1.3}$$

where $\mathcal{V}(t)$ is the velocity of the evolution. This is well-described with the image that the velocity of each point coincides with the direction of the steepest area descent. For convex sets this results in a shrinking motion, giving rise to a singularity as the surface shrinks to a single point.

The perimeter is decreasing along solutions of the mean curvature flow and we can quantify this with the Willmore energy. We have

$$\partial_t \mathcal{P}(E(t)) = -\int_{\Gamma_t} \vec{H}_t \cdot \vec{\mathcal{V}}(t) \, \mathrm{d}\mathcal{H}^{n-1} = -\int_{\Gamma_t} |\vec{H}_t|^2 \, \mathrm{d}\mathcal{H}^{n-1} = -\mathcal{W}(\Gamma_t). \tag{1.4}$$

This is called the energy-dissipation. This relation can be used to define a weak formulation of the mean curvature flow which we will discuss later.
In 1975 De Giorgi and Franzoni [DGF75] published a conjecture about a diffuse approximation of the perimeter in the sense of Γ–convergence; see Section 2.3. If $\Omega \subseteq \mathbb{R}^n$ is the ambient space with sufficient regularity the authors consider a version of the Van der Waals-Cahn-Hilliard energy from physics

$$\mathcal{P}_\varepsilon(u) := \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\Big)\,\mathrm{d}\mathcal{L}^n, \quad u \in H^1(\Omega)$$

where W is a suitable potential. Shortly after the conjecture was published, the papers by Modica and Mortola from 1977 [MM77] and 1987 [Mod87] proved that

$$\mathcal{P}_\varepsilon \overset{\Gamma(L^1(\Omega))}{\longrightarrow} c_0\mathcal{P} \quad \text{as} \quad \varepsilon \to 0 \quad \text{with} \quad c_0 := \int_{-1}^{1} \sqrt{2W}\,\mathrm{d}\mathcal{L}^1;$$

see Theorem 2.4.2. Since then many authors have conducted analysis based on the result or in style of the result from Modica and Mortola, for instance [AB98, AVG12].

The Modica-Mortola Theorem establishes a diffuse approximation of the perimeter thus it is only consequent to use (1.2) and consider $-\nabla_{L^2}\mathcal{P}_\varepsilon = -\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)$ as a diffuse mean curvature. This leads to possible diffuse approximations of other energies based on the mean curvature (vector). The first to come up with possible diffuse approximations based on this was De Giorgi. He posed several open questions in 1991, among them he considered (see Conjecture 4 in [DG91]) the diffuse energy

$$\mathcal{W}_\varepsilon^{\mathrm{dG}}(u) := \int_\Omega \Big[1 + \Big| -\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\Big|^2\Big]\Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\Big)\,\mathrm{d}\mathcal{L}^n. \tag{1.5}$$

The second factor in (1.5) is the integrand of the diffuse perimeter, thus it is a natural question to ask if these terms combined converge towards a linear combination of the perimeter and the Willmore energy. In a recent paper [BFP22] the authors show that the $\Gamma(L^1(\Omega))$–limit of $\mathcal{W}_\varepsilon^{\mathrm{dG}}$ in (1.5) is a multiple of the perimeter and that the Willmore energy does not appear.

Another idea is to replicate the energy-dissipation for $\varepsilon > 0$. We have

$$\nabla_{L^2}\mathcal{P}_\varepsilon(u) = -\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)$$

and thus the gradient flow of $\mathcal{P}_\varepsilon$ is described by the Allen-Cahn equation

$$-\varepsilon\partial_t u_\varepsilon = -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon). \tag{1.6}$$

Let u_ε be a solution to (1.6) with suitable boundary conditions, then we obtain that

$$\partial_t\mathcal{P}_\varepsilon(u_\varepsilon) = \int_\Omega \Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\partial_t u_\varepsilon\,\mathrm{d}\mathcal{L}^n = -\int_\Omega \frac{1}{\varepsilon}\Big| -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big|^2\,\mathrm{d}\mathcal{L}^n. \tag{1.7}$$

A comparison with (1.4) suggests that the expression on the right-hand side could be a diffuse Willmore energy. The first to consider this diffuse expression were Bellettini and

Paolini who called this a modified version of this De Giorgi conjecture. They considered the energy

$$\mathcal{W}_\varepsilon(u) := \int_\Omega \frac{1}{\varepsilon}\Big| -\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\Big|^2 \,\mathrm{d}\mathcal{L}^n. \tag{1.8}$$

In [BP93] the authors prove that the modified functional suffices the lim sup–property of

$$\mathcal{W}_\varepsilon \overset{\Gamma(L^1(\Omega))}{\longrightarrow} c_0\mathcal{W} \quad \text{as} \quad \varepsilon \to 0.$$

The next major milestone was the $\Gamma(L^1) - \liminf$ estimate for the modified de Giorgi approximation. In 2006 Röger and Schätzle proved in [RS06] the lim inf–estimate for

$$\mathcal{P}_\varepsilon + \mathcal{W}_\varepsilon \overset{\Gamma(L^1(\Omega))}{\longrightarrow} c_0(\mathcal{P} + \mathcal{W})$$

in smooth points and low dimensions, i.e. $n \in \{2,3\}$. This was done by incorporating techniques from geometric measure theory which have already been used in other publications such as [Ilm93] and a blow up inspired by [HT00].

In 2007 the paper [KK07] by Karali and Katsoulakis considered a combination of the Allen-Cahn and the Cahn-Hilliard equation

$$-\varepsilon\partial_t u = (-\varepsilon^2\Delta + \mathrm{Id})\Big(-\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)\Big). \tag{1.9}$$

Given a family of surfaces evolving under mean curvature flow, the authors construct a family of classical solutions to (1.9) which converge as $\varepsilon \to 0$ towards the modified indicator function of the original family which evolves by mean curvature flow. This may seem surprising at first because it means that the higher order term contributes on the same order as the terms from the Allen-Cahn equation. For the asymptotic expansion (2.4.5) this can be explained by the fact that the factor ε^2 cancels out the $1/\varepsilon$ terms from the chain rule in the lowest order.

Since (1.9) is a diffuse version of mean curvature flow we consider its energy-dissipation related to the diffuse perimeter. Let u_ε be a solution to (1.9) with suitable boundary conditions, then setting $H_\varepsilon := -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)$ we have

$$\partial_t\mathcal{P}_\varepsilon(u_\varepsilon) = \int_\Omega \Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\partial_t u_\varepsilon \,\mathrm{d}\mathcal{L}^n = -\int_\Omega \Big(\frac{1}{\varepsilon}|H_\varepsilon|^2 + \varepsilon\big|\nabla H_\varepsilon\big|^2\Big)\,\mathrm{d}\mathcal{L}^n. \tag{1.10}$$

The comparison to (1.4) and (1.7) suggests that the functional $\mathcal{W}_\varepsilon^{\mathrm{KK}} : H^3(\Omega) \longrightarrow [0,\infty]$ with

$$\mathcal{W}_\varepsilon^{\mathrm{KK}}(w) := \int_\Omega \Big[\frac{1}{\varepsilon}\Big| -\varepsilon\Delta w + \frac{1}{\varepsilon}W'(w)\Big|^2 + \varepsilon\Big|\nabla\Big(-\varepsilon\Delta w + \frac{1}{\varepsilon}W'(w)\Big)\Big|^2\Big]\,\mathrm{d}\mathcal{L}^n$$

is a good candidate for a diffuse Willmore energy. In Chapter 4, see Theorem 4.3.1 we identify the possible $\Gamma(\mathcal{L}^1(\Omega))$–limit of $\mathcal{W}_\varepsilon^{\mathrm{KK}}$ as $c_0\sigma\mathcal{W}$ and prove the lim sup–property of

$$\mathcal{W}_\varepsilon^{\mathrm{KK}} \overset{\Gamma(L^1(\Omega))}{\longrightarrow} c_0\sigma\mathcal{W}. \tag{1.11}$$

Here $\sigma > 1$ is a constant which can be calculated from the double-well potential. The fact $\sigma > 1$ also means that the higher order term actually contributes in the limit. The harder problem of establishing a lower bound is addressed in Chapter 5, see Theorems 5.2.5 and 5.1.1, where we prove the $\liminf$–estimate of

$$\mathcal{P}_\varepsilon + \mathcal{W}_\varepsilon^{\mathrm{KK}} \overset{\Gamma(L^1(\Omega))}{\longrightarrow} c_0(\mathcal{P} + \sigma\mathcal{W}). \tag{1.12}$$

For the proof we utilize a blow-up, similar to the proof by Röger and Schätzle. The hardest challenge was to overcome the non-locality of a solution operator.
A further motivation to consider the $\liminf$–estimate is that the inequality is useful for the proof that solutions to (1.9) converge towards varifold solutions to mean curvature flow. In Chapter 6, see Theorems 6.3.5, 6.3.17 and 6.3.16 we prove that weak solutions to (1.9) with suitable boundary and initial conditions converge in a suitable sense towards a De Giorgi type varifold solution for rescaled mean curvature flow; see Definition 2.5.3.

For the proof we need another blow-up. In contrast to the blow-up in Chapter 5 there is no issue of non-locality, however the blow-up is done with the additional parameters t, δ which introduces additional complications. In particular the time-dependency is difficult because we can not use the blow-up argument in time-space. Instead we use results from [MR08] to get the proper convergences of the measures involved.

Then the remaining properties of De Giorgi type varifold solution for rescaled mean curvature flow can be proven. It remains an open problem to show that the varifold is a solution to mean curvature flow in the Brakke sense, to which we give a partial result.

In 2012 Amstutz and Van Goethem presented a gradient-free diffuse approximation of the perimeter in [AVG12].

$$\mathcal{P}_\varepsilon^{\mathrm{AG}}(u) \coloneqq \inf_{v \in H^1(\Omega)} \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla v|^2 + \frac{1}{2\varepsilon}(u-v)^2 + \frac{1}{2\varepsilon}W(u)\Big)\,\mathrm{d}\mathcal{L}^n. \tag{1.13}$$

The expression “gradient-free” refers to the fact that no derivative is applied towards the argument u of the functional. The approximation is based on the two variable diffuse perimeter approximation by Solci and Vitali [SV03]. For a particular double-well potential W Amstutz and Van Goethem prove that

$$\mathcal{P}_\varepsilon^{\mathrm{AG}} \overset{\Gamma(L^1(\Omega))}{\longrightarrow} c_{\mathrm{AG}}\mathcal{P}.$$

As before we can derive a diffuse Willmore energy by considering the energy-dissipation. The L^2-gradient of $\mathcal{P}_\varepsilon^{\mathrm{AG}}$ is given by

$$\nabla_{L^2}\mathcal{P}_\varepsilon^{\mathrm{AG}}(u) = \frac{1}{\varepsilon}\Big(u + \frac{1}{2}W'(u) - \overline{u}_\varepsilon\Big). \tag{1.14}$$

Here $\overline{u}_\varepsilon$ is a weak solution to

$$-\varepsilon^2\Delta\overline{u}_\varepsilon + \overline{u}_\varepsilon = u_\varepsilon \tag{1.15}$$

with suitable boundary conditions. Let u_ε be a solution to

$$-\varepsilon\partial_t u_\varepsilon = \frac{1}{\varepsilon}\Big(u_\varepsilon + \frac{1}{2}W'(u_\varepsilon) - \overline{u}_\varepsilon\Big) \tag{1.16}$$

with suitable boundary conditions. The energy-dissipation is given by

$$\begin{aligned}\partial_t \mathcal{P}_\varepsilon^{\mathrm{AG}}(u_\varepsilon) &= \int_\Omega \frac{1}{\varepsilon}\Big(u_\varepsilon + \frac{1}{2}W'(u_\varepsilon) - \overline{u}_\varepsilon\Big)\partial_t u_\varepsilon \,\mathrm{d}\mathcal{L}^n \\ &= -\int_\Omega \frac{1}{\varepsilon^3}\Big|u_\varepsilon + \frac{1}{2}W'(u_\varepsilon) - \overline{u}_\varepsilon\Big|^2 \,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

This suggests that

$$\mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon) \coloneqq \int_\Omega \frac{1}{\varepsilon^3}\Big|u_\varepsilon + \frac{1}{2}W'(u_\varepsilon) - \overline{u}_\varepsilon\Big|^2 \,\mathrm{d}\mathcal{L}^n$$

is a good candidate to be a diffuse Willmore energy. We remark that no gradients appear (explicitly) in the functional, instead we have to deal with the non-local solution operator $u \mapsto \overline{u}_\varepsilon$ associated to (1.15).

In Chapter 3, see Theorem 3.4.1, we prove the lim sup–property of

$$\mathcal{W}_\varepsilon^{\mathrm{AG}} \overset{\Gamma(L^1(\Omega))}{\longrightarrow} c_{\mathrm{AG}}\mathcal{P}. \tag{1.17}$$

The ansatz is to assume that, in a small neighborhood of the surface, we can expand u_ε as a power series in ε with coefficient profile functions which have to be determined. Then we expand all of the functions, operators, and the functional itself by powers of ε and minimize each order, finding the optimal choices for the profile functions in the process. The main difficulty here is, that we have to prove that the non-local solution operator $(-\varepsilon^2\Delta + \mathrm{Id})^{-1}$ preserves the property that such a local expansion exists. We introduce spaces of functions which decay exponentially away from the surface with respect to the modified distant functions to justify this. We will also prove a lim inf–estimate for this very specific class of functions with exponential decay, however the general case remains open.

Next, we will briefly discuss the Willmore flow, which is the L^2-gradient flow induced by the Willmore energy. Let $(\Gamma_t)_{t\in(0,T)}$ be a family of evolving surfaces with mean curvature H_t, normal velocity $\mathcal{V}(t,\cdot)$, and the second fundamental form $\mathrm{I\!I}(t,\cdot)$. As was proven in Sections 7.4 - 7.5 in [Wil93] by Willmore the family evolves by Willmore flow if

$$\mathcal{V} = -\Delta_\Gamma H + \frac{1}{2}H^3 - H|\mathrm{I\!I}|^2, \tag{1.18}$$

where Δ_Γ is the Laplace-Beltrami operator on Γ_t. The Willmore flow is a fourth order geometric evolution law, which introduces several additional challenges in the analysis of the flow. We refer to the fundamental contributions [Sim01, KS01, KS02, KS04].

The Willmore flow can be approximated by a phase field approximation in the following sense: we consider the L^2-gradient flow induced by the diffuse Willmore energy (1.8), i.e.

$$-\varepsilon\partial_t u_\varepsilon = \frac{1}{\varepsilon}\Big(-\varepsilon\Delta + \frac{1}{\varepsilon}W''(u_\varepsilon)\,\mathrm{Id}\Big)\Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big). \tag{1.19}$$

In their paper [LM00] from 2000, Loretti and March considered asymptotic expansions of (1.19) and concluded that in a formal sense the solutions to (1.19) converge towards solutions of the Willmore flow as $\varepsilon \to 0$. They employed a formal asymptotic ansatz involving expanding u_ε in powers of ε. Based on this idea Wang published similar results in 2008; see [Wan08].

Convergence proofs based on asymptotic expansion techniques are known for the standard diffuse approximation of mean curvature and Willmore flow; see [dMS90] and [FL21].

The last type of results in this thesis is similar for the Amstutz-Van Goethem approximation and the Karali-Katsoulakis approximation. In both cases we constructed diffuse approximations of the Willmore energy. We use these functionals to construct formal diffuse approximations of the Willmore flow by considering the induced gradient flows of the diffuse approximations. The L^2-gradient flow of $\mathcal{W}_\varepsilon^{\mathrm{KK}}$ is given by

$$-\varepsilon\partial_t u_\varepsilon = \frac{2}{\varepsilon^2}\Big(- \varepsilon^2\Delta + W''(u_\varepsilon)\Big)(-\varepsilon^2\Delta + \mathrm{Id})\Big(- \varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big). \tag{1.20}$$

Similarly we can consider the L^2-gradient flow of $\mathcal{W}_\varepsilon^{\mathrm{AG}}$, i.e.

$$-\varepsilon\partial_t u_\varepsilon = \frac{2}{\varepsilon^3}\Big(1 + \frac{1}{2}W''(u_\varepsilon) - (-\varepsilon^2\Delta + \mathrm{Id})^{-1}\Big)\Big(u_\varepsilon + \frac{1}{2}W'(u_\varepsilon) - \overline{u}_\varepsilon\Big). \tag{1.21}$$

We may expect that the diffuse flows both converge in the sharp interface limit $\varepsilon \to 0$ to the Willmore flow. We discuss the convergence of (1.20) in Chapter 4, see Theorem 4.4.4, and the convergence of (1.21) in Chapter 3, see Theorem 3.5.4, both under rather restrictive assumptions by considering asymptotic expansions. For the proof we will follow the approach by Loreti and March [LM00] and Wang [Wan08]. However in our setting the operators that define the gradient-free approximation are different from the standard case, and the derivation of the convergence property is much more involved.

The results of Chapter 3 have already been published in the paper "'Gradient-free' diffuse approximations of the Willmore functional and Willmore flow" of N. Dabrock, M. Röger, and myself, published in *Asymptot. Anal.* in 2022. The paper contains an additional section discussing numerical simulations done by N. Dabrock but the analysis is not as detailed as in Chapter 3 of this thesis.

The outline for the thesis is as follows. In the second chapter we point out the mathematical foundations. In Chapter 3 we construct a recovery sequence for (1.17) and prove a formal approximation of the Willmore flow by considering (1.21). In Chapter 4 we construct a recovery sequence for (1.11) and prove a formal approximation of the Willmore flow by considering (1.20). In Chapter 5 we prove the $\liminf$–property of (1.12). In Chapter 6 we construct De Giorgi type varifold solutions for rescaled mean curvature flow by considering the limit $\varepsilon \to 0$ in (1.9) and give a partial result for the Brakke flow.

2 Preliminaries

In this chapter we give a brief overview over the relevant terms and important previous results. We start with notations most commonly used throughout the thesis followed by the definitions of the central objects of this thesis i.e. the Willmore energy, the perimeter, the Willmore flow, and the mean curvature flow. In the later sections we introduce the basic terms of geometric measure theory such as functions of bounded variation, rectifiable sets, and varifolds. One short section is dedicated to Γ–convergence and in the last section we present weak formulations of mean curvature flow.
The following notations will be applied throughout the entire thesis.

- $n \in \mathbb{N}$ will always denote the dimension of the surrounding space $\mathbb{R}^n$. Usually $\Omega \subseteq \mathbb{R}^n$ will be a non-empty open set with varying additional properties.
- Every limit for ε will only occur for positive ε, we will simply write $\varepsilon \to 0$ instead of $\varepsilon \searrow 0$. Furthermore we will refer to objects indicated by $\varepsilon > 0$ as sequences, even if the index set is not countable. A subsequence in this context will be an actual sequence, meaning that it is indicated by a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $0 < \varepsilon_k \to 0$. Usually we will not relabel subsequences.
- Constants are usually denoted by $C > 0$ and may change from line to line without introducing a new variable if the value of C is not important. Sometimes we denote the dependencies of C in parentheses.
- We try to avoid double parentheses as much as possible, thus we will write $C^0[-1,1)$ instead of $C^0([-1,1))$ for the space of continuous functions on the interval $[-1,1)$. Similar for other spaces, we write $L^2(0,T;C^0(\Gamma))$ instead of $L^2((0,T);C^0(\Gamma))$ for the Bochner space of L^2-functions defined on $(0,T)$ with values in $C^0(\Gamma)$.

2.1 Some terms from differential geometry

In this section we only give definitions for the objects which are relevant for this thesis. For the sake of introducing the basic terminology and theory of (sub-)manifolds we follow and adapt the content of [Jos17]. In the entire thesis when restricting to the smooth case we only discuss embedded and compact hypersurfaces in $\mathbb{R}^n$. Thus we can simplify the more general definition.

In the following we consider a smooth hypersurface $\Gamma \subseteq \mathbb{R}^n$ with a Riemannian metric, i.e. a smooth family of scalar products $g = (g_p)_{p\in\Gamma}$ where g_p is defined on the tangent space $T_p\Gamma$. We will denote a vector basis of $T_p\Gamma$ by $e_1, \dots, e_{n-1}$. More details can be found in the appendix.

Definition 2.1.1 (Differential operators on hypersurfaces)**.**
Let Γ be a C^1-hypersurface and let $f \in C^1(\Gamma)$. For $p \in \Gamma$ let $e_1(p), \dots, e_{n-1}(p)$ be a basis of $T_p\Gamma$ as in (8.1.1)*. We define the gradient*

$$\mathrm{grad}_\Gamma(f) \coloneqq \nabla_\Gamma f \coloneqq \sum_{j,k=1}^{n-1} g^{jk} e_k \partial_j f.$$

Let $Z = \sum_{j=1}^{n-1} Z_j e_j \in C^1(\Gamma; \mathbb{R}^n)$ be a vector field on Γ, then we define the divergence

$$\mathrm{div}_\Gamma Z \coloneqq \nabla_\Gamma \cdot Z \coloneqq \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n-1} e_j(\sqrt{g} Z_j) = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n-1} e_j\Big(\sqrt{g} g^{jk} \langle Z, \partial_k \rangle_{T_p\Gamma}\Big)$$

Lastly if Γ is a C^2-hypersurface and $f \in C^2(\Gamma)$ we define the Laplace-Beltrami operator

$$\Delta_\Gamma \coloneqq \mathrm{div}_\Gamma \nabla_\Gamma f = \frac{1}{\sqrt{g}} \sum_{j,k=1}^{n-1} e_j\Big(\sqrt{g} g^{jk} e_k f\Big).$$

Definition 2.1.2 (Second fundamental form)**.**
Let Γ be a C^2-hypersurface and $p \in \Gamma$. The second fundamental form of Γ at p is the map

$$S : T_p\Gamma \times T_p\Gamma^\perp \longrightarrow T_p\Gamma, \quad S(X, \nu) \coloneqq \Pi_{T_p\Gamma} D_X \nu, \quad X \in T_p\Gamma, \quad \nu \in T_p\Gamma^\perp$$

with the orthogonal projection $\Pi_{T_p\Gamma} : \mathbb{R}^n \longrightarrow T_p\Gamma$. By Lemma 8.1.6 the directional derivative only depends on ν and X and thus S is well-defined.

In the general case of manifolds that are not necessarily embedded we would need the concept of Levi-Civita-connection and parallel transport in order to compare vectors from different tangent spaces, however we can use the surrounding algebraic structure, i.e. the directional derivative and the scalar product on $\mathbb{R}^n$ instead.

Lemma 2.1.3 (Second fundamental form and mean curvature vector)**.**
Let Γ be a C^2-hypersurface, $p \in \Gamma$ and $\nu \in T_p\Gamma^\perp$. The bilinear form

$$L_\nu : T_p\Gamma \times T_p\Gamma \longrightarrow \mathbb{R}, \quad L_\nu(X, Y) \coloneqq \langle S(X, \nu) | Y \rangle_{T_p\Gamma}$$

is symmetric and thus its eigenvalues $\kappa_1(p), \dots, \kappa_{n-1}(p)$ are real. They are called principal curvatures of Γ at p in direction ν. We define the mean curvature and the mean curvature vector

$$H_{\Gamma,\nu}(p) \coloneqq \sum_{j=1}^{n-1} \kappa_j(p) \quad \textit{and} \quad \vec{H}_\Gamma(p) \coloneqq H_{\Gamma,\nu}(p)\nu.$$

When the choice of normal ν or the surface Γ is clear from the context either or both are omitted from the notation. We call L_ν and its representation with respect to the standard basis the second fundamental form. It is also commonly referred to as Weingartenmapping.

Note that the mean curvature vector is independent from ν however the scalar mean curvature switches its sign under the change of orientation $\nu \mapsto -\nu$.

Next we examine how the total area of a hypersurface changes with a perturbation. Let Γ be a C^2-hypersurface and $X \in C^2(\Gamma; \mathbb{R}^n)$, then there exists $\delta > 0$ such that

$$\Gamma + rX \coloneqq \{p + rX(p) \mid p \in \Gamma\}$$

is a C^2-hypersurface for $|r| < \delta$ by the Implicit Function Theorem.

Theorem 2.1.4 (First variation of the surface of a hypersurface).
Let Γ *be an oriented* C^2*-hypersurface and* $X \in C^1(\Gamma; \mathbb{R}^n)$*. Then we have*

$$\partial_r \mathcal{H}^{n-1}(\Gamma + rX)|_{r=0} = -\int_\Gamma X \cdot \vec{H} \, \mathrm{d}\mathcal{H}^{n-1}. \tag{2.1.1}$$

This implies in particular that surfaces are stationary with respect to small perturbations if and only if the mean curvature vanishes.
Having discussed H we can introduce one of the central objects of this thesis.

Definition 2.1.5 (Willmore energy of smooth hypersurfaces).
Let $\Gamma \subseteq \mathbb{R}^n$ *be a smooth hypersurface without boundary. We define the Willmore energy of* Γ

$$\mathcal{W}(\Gamma) := \int_\Gamma |\vec{H}|^2 \, \mathrm{d}\mathcal{H}^{n-1}.$$

This is the definition for surfaces, later we will consider the Willmore energy defined on function spaces; see Definition 2.4.6.

Next we move from the static setting to dynamics. We consider two different geometric flows, namely the mean curvature flow and the Willmore flow. We need to define velocity and normal velocity in advance.

Definition 2.1.6 ((Normal) velocity).
Let $(\Gamma_t)_{t>0}$ *be a family of evolving surfaces without boundary that can be parametrized over a fixed surface* Γ_0 *without boundary, i.e., for* $t > 0$ *there exists an immersion* $\Phi_t : \Gamma_0 \longrightarrow \Gamma_t$*. At a given time* $t \in (0, T)$ *the velocity vector at* $\vec{\mathcal{V}}(t, \cdot) : \Gamma_t \longrightarrow \mathbb{R}^n$ *of the evolution is given by*

$$\vec{\mathcal{V}}(t, \cdot) := [\partial_t \Phi_t] \circ \Phi_t^{-1}.$$

If $\nu(t, \cdot)$ *is a given unit normal of* Γ_t *we also define the normal velocity with respect to* $\nu(t, \cdot)$ *as the function* $\mathcal{V}_\nu(t, \cdot) : \Gamma_t \longrightarrow \mathbb{R}$

$$\mathcal{V}_\nu(t, \cdot) := \vec{\mathcal{V}} \cdot \nu(t, \cdot).$$

If the chosen normal vector is clear from the context it is omitted from the notation and we simply write $\mathcal{V} = \mathcal{V}_\nu$*.*

Now we can introduce the mean curvature flow.

Definition 2.1.7 (Mean curvature flow).
Let $(\Gamma_t)_{t \in (0,T)}$ *be an evolving family of smooth hypersurfaces without boundary with velocity* $\vec{\mathcal{V}}$*. The family evolves by mean curvature flow if*

$$\vec{\mathcal{V}}(t, \cdot) = \vec{H}_t. \tag{2.1.2}$$

If we can apply the chain rule we get the energy-dissipation

$$\partial_t \mathcal{H}^{n-1}(\Gamma_t) = -\int_{\Gamma_t} \vec{\mathcal{V}} \cdot \vec{H}_t \, \mathrm{d}\mathcal{H}^{n-1} = -\int_{\Gamma_t} |\vec{H}_t|^2 \, \mathrm{d}\mathcal{H}^{n-1} = -\mathcal{W}(\Gamma_t) \leq 0. \tag{2.1.3}$$

Since the velocity points in the direction of the mean curvature vector which itself points in the direction of steepest area descent the mean curvature flow is usually associated with a shrinking motion. This is true especially for convex sets as the next theorem shows.

Theorem 2.1.8 (Huisken (1984))**.**
Let $n \geq 2$ and assume that Γ_0 is uniformly convex, i.e., the eigenvalues of its second fundamental form are strictly positive everywhere. Then (2.1.2) has a smooth solution on a finite time interval $[0, T)$ and the surfaces converge towards a single point as $t \to T$.

For non-convex sets the evolution can be more complicated. In Chapter 3 in [Eck04] examples are given for surfaces that break apart under mean curvature flow, examples for other singularities can be found in [Ton19]. Surfaces are expected to develop singularities in a finite time under mean curvature flow.

At last we introduce the concept of Willmore flow.

Definition 2.1.9 (Willmore flow)**.**
Let $(\Gamma_t)_{t>0}$ be an evolving family of smooth hypersurfaces without boundary with normal velocity $\mathcal{V}$ and second fundamental from $\mathrm{I\!I}_t$. The family evolves by Willmore flow if

$$\mathcal{V} = -\Delta_{\Gamma_t} H_t + \frac{1}{2} H_t^3 - H_t |\mathrm{I\!I}_t|^2, \tag{2.1.4}$$

where Δ_{Γ_t} is the Laplace-Beltrami operator on Γ_t.

The Willmore flow is a rather new field of research, the most important contributions are the papers from Kuwert and Schätzle [KS01, KS02] and the paper from Simonett [Sim01]. The L^2-gradient of the Willmore energy is given by

$$\nabla_{L^2} \mathcal{W} = -\Delta_{\Gamma_t} H_t + \frac{1}{2} H_t^3 - H_t |\mathrm{I\!I}_t|^2;$$

see [Wil65]. The Willmore energy is decreasing along solutions of the Willmore flow.

In the following we introduce a new coordinate system in a neighborhood of a smooth hypersurface. These coordinates will be very useful for the construction of recovery sequences for phase-field approximations of the Willmore energy in Chapters 3 and 4.

Definition 2.1.10 (Normal and tangential coordinates)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be open and E a domain $E \Subset \Omega$ with C^4-boundary $\Gamma := \partial E$. We write $\nu_\Gamma(y)$ for the inner normal of Γ at $y \in \Gamma$. Since Γ is compact there exists $0 < \delta < 1$ such that the following hold.

- *The orthogonal projection $\Pi : \{|d| < 5\delta\} \longrightarrow \Gamma$ with $x \mapsto \Pi(x) =: y(x) \in \Gamma$ is well-defined with*

$$\sup_{y \in \Gamma} \max_{j=1}^{n-1} |k_j(y)| < \frac{1}{5\delta},$$

  where k_j denote the principle curvatures of Γ for $j \in \{1, \ldots, n-1\}$.

- *For $\varepsilon > 0$ the coordinate transformation*

$$\Psi_\varepsilon : \left(-\frac{5\delta}{\varepsilon}, \frac{5\delta}{\varepsilon} \right) \times \Gamma \longrightarrow \{|d| < 5\delta\} \Subset \Omega, \quad \Psi_\varepsilon(z, y) := y + \varepsilon z \nu_\Gamma(y) \tag{2.1.5}$$

  is a well-defined C^3-diffeomorphism. We write $\omega := \{|d| < 5\delta\}$.

The first claim is true because Γ has C^2 regularity, meaning the principle curvatures are continuous functions on a compact set. By definition we get for $x \in \omega$

$$\Psi_\varepsilon\Big(\frac{d(x)}{\varepsilon}, y(x)\Big) = x.$$

We now follow [LorettiMarch2000] and calculate for $x \in \omega$

$$\begin{aligned}\Delta d(x) &= \sum_{j=1}^{n-1} \frac{k_j(y)}{1 + k_j(y)d(x)} = \sum_{j=1}^{n-1}\sum_{l=0}^{\infty}(-1)^l k_j(y)^{1+l} d(x)^l \\ &= \sum_{l=0}^{\infty}(-1)^l d(x)^l \sum_{j=1}^{n-1} k_j(y)^{1+l} =: H(y) - d(x)|\mathrm{I\!I}(y)|^2 + g^R(x).\end{aligned}$$

This is an absolute convergent powerseries in ω. The error term can be estimated as follows: there exists $C(\Gamma) > 0$ such that for all $x \in \omega$

$$|g^R(x)| \leq \varepsilon^2 C(\Gamma) z^2. \tag{2.1.6}$$

Possibly lowering the value of $\delta > 0$ we can assume that in $(-\frac{5\delta}{\varepsilon}, \frac{5\delta}{\varepsilon}) \times \Gamma$

$$2\varepsilon \geq \det(D\Psi_\varepsilon(z, y)) = \varepsilon - \varepsilon^2 z H(y) + \varepsilon^3 z^2 R(z, y) \geq \frac{\varepsilon}{2}, \tag{2.1.7}$$

where $R : \mathbb{R} \times \Gamma \to \mathbb{R}$ is uniformly bounded. The coordinates are displayed in Figure 2.1.

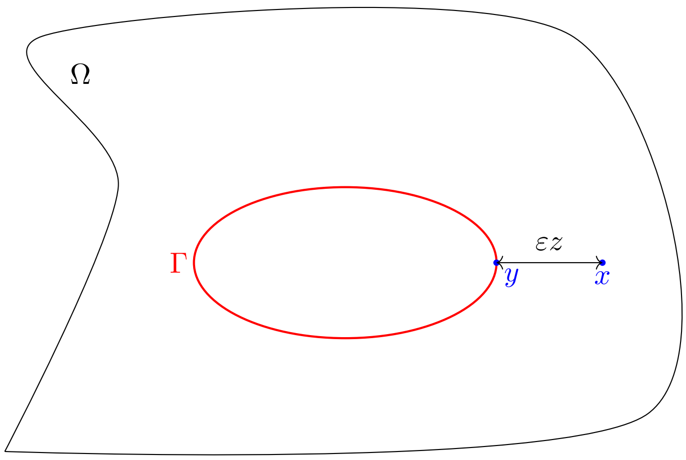

Figure 2.1: Visualization of the geometry and coordinates.

Notations 2.1.11 (Geometry)**.**
We can represent a function $u : \omega \longrightarrow \mathbb{R}$ *as*

$$\tilde{u} : \Big(-\frac{5\delta}{\varepsilon}, \frac{5\delta}{\varepsilon}\Big) \times \Gamma \longrightarrow \mathbb{R} \quad \textit{with} \quad \tilde{u}(z, y) := u(\Psi_\varepsilon(z, y)).$$

It is convenient to extend $\tilde{u}$ *to a function* U *that is constant in normal directions, i.e.*

$$U : \Big(-\frac{5\delta}{\varepsilon}, \frac{5\delta}{\varepsilon}\Big) \times \omega \longrightarrow \mathbb{R} \quad \textit{with} \quad U(z, x) := \tilde{u}(z, y)$$

for all $x \in \omega$ with $\Pi_\Gamma(x) = y$ and all $z \in (-5\delta/\varepsilon, 5\delta/\varepsilon)$.

From [LM00, Wan08] we recall that

$$\nabla u = \frac{1}{\varepsilon} U' \nabla d + \nabla U, \qquad \Delta u = \frac{1}{\varepsilon^2} U'' + \frac{\Delta d}{\varepsilon} U' + \Delta U, \tag{2.1.8}$$

$$|\nabla d| = 1, \quad \nabla d \cdot \nabla_\Gamma = 0, \quad \Delta d(x) = H(y) - \varepsilon z |\mathrm{I\!I}|^2(y) + \varepsilon^2 |z|^2 R^H_\varepsilon(x), \tag{2.1.9}$$

where $|R^H_\varepsilon| \le C(\Gamma, \delta)$ in ω. We will often write $u(z, x)$ instead of $U(z, x)$ or $\tilde{u}(z, y)$ if it is clear from the context what is meant.

2.2 Geometric measure theory

In this section we introduce functions of bounded variation, the perimeter, and the Willmore energy. We also consider varifolds. Varifolds are very relevant for us as they are in some sense a relaxation of the term (hyper-)surface to a setting with lower regularity and better compactness properties. This is comparable to the step from classical derivatives and the C^k-spaces to weak derivatives and Sobolev spaces.

The basic properties of Radon measures can be found in Section 8.2 in the appendix. Here we recall two important theorems about Radon measures; see [AFP00].

Theorem 2.2.1 (Riesz's representation Theorem)**.**
Let Ω be a locally compact and separable metric space and $m \in \mathbb{N}$.
(i) Let $L \in C^0_c(\Omega; \mathbb{R}^m)'$ then there exist a $\mathbb{R}^m$-valued Radon measure μ and a $\mathbb{R}^m$-valued function $f \in L^1(\Omega, \mu; \mathbb{R}^m)$ such that for all $\eta \in C^0_c(\Omega; \mathbb{R}^m)$

$$\langle \eta, L \rangle_{C^0_c(\Omega;\mathbb{R}^m)'} = \int_\Omega f \cdot \,\mathrm{d}\mu.$$

*(ii) Let $L \in C^0_0(\Omega; \mathbb{R}^m)'$ then there exist a **finite** $\mathbb{R}^m$-valued Radon measure μ and a $\mathbb{R}^m$-valued function $f \in L^1(\Omega, \mu; \mathbb{R}^m)$ such that for all $\eta \in C^0_0(\Omega; \mathbb{R}^m)$*

$$\langle \eta, L \rangle_{C^0_0(\Omega;\mathbb{R}^m)'} = \int_\Omega f \cdot \,\mathrm{d}\mu \quad \text{and} \quad \|L\| = |\mu|(\Omega).$$

We will identify the continuous linear form with the measure and thus simply write $\mu \in C^0_c(\Omega; Y)'$ or $\mu \in C^0_0(\Omega; Y)'$ respectively. Note $C^0_0(\Omega; Y)' \hookrightarrow C^0_c(\Omega; Y)'$. As a dual space of $C^0_c(\Omega; \mathbb{R}^m)'$ or $C^0_0(\Omega; \mathbb{R}^m)'$ we can consider the weak* convergence for Radon measures and finite Radon measures. It has good compactness properties, as is stated in the next theorem.

Theorem 2.2.2 (Compactness of Radon measures)**.**
Let Ω be a locally compact and separable metric space.
(i) Let $(\mu_k)_{k\in\mathbb{N}} \in C^0_c(\Omega)'$ be a sequence of Radon measures on Ω such that for all $K \Subset \Omega$ it holds $\sup_{k\in\mathbb{N}} \mu_k(K) < \infty$. Then there exists a subsequence $(\mu_{k_j})_{j\in\mathbb{N}}$ and a Radon measure $\mu \in C^0_c(\Omega)'$ such that

$$\mu_{k_j} \xrightarrow{w^*} \mu \quad \text{as} \quad j \to \infty \quad \text{in} \quad C^0_c(\Omega)'.$$

(ii) Let $(\mu_k)_{k\in\mathbb{N}} \in C_0^0(\Omega)'$ be a sequence of finite Radon measures on Ω such that it holds $\sup_{k\in\mathbb{N}} \mu_k(\Omega) < \infty$. Then there exists a subsequence $(\mu_{k_j})_{j\in\mathbb{N}}$ and a finite Radon measure $\mu \in C_0^0(\Omega)'$ such that

$$\mu_{k_j} \xrightarrow{w^*} \mu \quad as \quad j \to \infty \quad in \quad C_0^0(\Omega)'.$$

Additionally it holds

$$\mu(\Omega) \leq \liminf_{k\to\infty} \mu_k(\Omega).$$

Remark.

- *This is a specialized version of Alaoglu's Theorem, which is also called Banach-Alaoglu or Alaoglu-Bourbaki Theorem and can be found in [Kab14, Thm. 8.6].*
- *If Ω is an open subset of $\mathbb{R}^n$ then Ω is locally compact and separable.*

Next we introduce the concept of bounded variation. We follow [AFP00] and [EG15]. In one dimension this can be explained without measure theory and can be reduced to monotone functions. However in higher dimensions more abstract concepts are necessary.

Definition 2.2.3 (Functions of bounded variation)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in L^1(\Omega)$. f is said to have bounded variation in Ω, i.e. $f \in BV(\Omega)$, if

$$\sup\left\{ \int_\Omega f\nabla \cdot \phi \,\mathrm{d}\mathcal{L}^n \;\middle|\; \phi \in C_c^1(\Omega;\mathbb{R}^n),\ \|\phi\|_{C^0(\Omega;\mathbb{R}^n)} \leq 1 \right\} < \infty.$$

We can characterize the derivatives of functions of bounded variation with Radon measures.

Theorem 2.2.4 (Structure theorem for BV functions)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in BV(\Omega)$. Then there exists a unique finite $\mathbb{R}^n$-valued Radon measure ∇f such that for all $\phi \in C_c^1(\Omega)$

$$-\int_\Omega f \cdot \nabla\phi \,\mathrm{d}\mathcal{L}^n = \int_\Omega \phi \cdot \,\mathrm{d}\nabla f.$$

By Theorem 2.2.4 and the Riesz representation Theorem $\nabla f \in C_0^0(\Omega;\mathbb{R}^n)'$ is well-defined. The following proposition shows that this theorem can also be used to define functions of bounded variation as the properties of the theorem and the definition are equivalent.

Proposition 2.2.5.
Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in L^1(\Omega)$. Then $f \in BV(\Omega)$ if and only if $\nabla f \in C_0^0(\Omega;\mathbb{R}^n)'$. In that case we have

$$|\nabla f|(\Omega) = \sup\left\{ \int_\Omega f\nabla \cdot \phi \,\mathrm{d}\mathcal{L}^n \;\middle|\; \phi \in C_c^1(\Omega;\mathbb{R}^n),\ \|\phi\|_{C^0(\Omega;\mathbb{R}^n)} \leq 1 \right\}$$

and the mapping $BV(\Omega) \ni f \mapsto |\nabla f|(\Omega)$ is lower semicontinuous with respect to the $L^1(\Omega)$-topology.

The expression

$$\|\eta\|_{BV(\Omega)} := \|\eta\|_{L^1(\Omega)} + |\nabla\eta|(\Omega), \quad \eta \in BV(\Omega)$$

is a norm on $BV(\Omega)$ and with this norm $BV(\Omega)$ is a Banach space. Another important property of $BV(\Omega)$ is its compact embedding into $L^1(\Omega)$.

Theorem 2.2.6 (Compactness in $BV(\Omega)$).
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary and assume $(f_j)_{j\in\mathbb{N}}$ is a bounded sequence in $BV(\Omega)$. Then there exists $f \in BV(\Omega)$ such that up to a subsequence

$$f_j \longrightarrow f \quad \textit{as} \quad j \to \infty \quad \textit{in} \quad L^1(\Omega).$$

In other words the embedding $BV(\Omega) \hookrightarrow L^1(\Omega)$ is compact. Combined with the lower semicontinuity of $f \mapsto |\nabla f|(\Omega)$ we conclude the following corollary.

Corollary 2.2.7.
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary and assume $(f_j)_{j\in\mathbb{N}}$ is a sequence in $BV(\Omega)$ with $f_j \longrightarrow f$ in $L^1(\Omega)$ and $\liminf_{j\to\infty} |\nabla f_j|(\Omega) < \infty$. Then it follows $f \in BV(\Omega)$ and

$$|\nabla f|(\Omega) \leq \liminf_{j\to\infty} |\nabla f_j|(\Omega).$$

Functions of bounded variation satisfy a generalized version of Gauß's Divergence Theorem 8.3.6.

Theorem 2.2.8 (Trace Theorem for BV-functions).
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary. Then there exists a bounded linear mapping

$$T : BV(\Omega) \longrightarrow L^1(\partial\Omega; \mathcal{H}^{n-1})$$

such that for all $f \in BV(\Omega)$ and all $\phi \in C^1(\Omega; \mathbb{R}^n)$

$$\int_\Omega f \nabla \cdot \phi \, \mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} \phi \cdot \nu T f \, \mathrm{d}\mathcal{H}^{n-1} - \int_\Omega \phi \cdot \, \mathrm{d}\nabla f,$$

where ν is the outer unit normal of $\partial\Omega$.

As a slight abuse of notation we usually write f for Tf. With these preparations we are ready to define the perimeter.

Definition 2.2.9 (Perimeter).
Let $\Omega \subset \mathbb{R}^n$ be open, then we define $\mathcal{P}_\Omega : L^1(\Omega) \longrightarrow [0, \infty]$ by

$$\mathcal{P}_\Omega(u) := \begin{cases} \frac{1}{2}|\nabla u|(\Omega), & \textit{if } u \in BV(\Omega, \{\pm 1\}) \\ \\ +\infty, & \textit{else.} \end{cases} \tag{2.2.1}$$

As a slight abuse of notation we sometimes write $\mathcal{P}(E)$ instead of $\mathcal{P}(2\chi_E - 1)$. If it is clear from the context the index Ω will be omitted from the notation and we simply write $\mathcal{P}$. By this definition $BV(\Omega; \{\pm 1\})$ is the set which contains the sets of finite perimeter in Ω, where we associate to a set E of finite perimeter the rescaled characteristic function $u := 2\chi_E - 1$.

By definition if χ_E has bounded variation then E has finite perimeter. We can represent the perimeter with the Hausdorff measure. Therefore we need to define the essential boundary first.

Definition 2.2.10 (Essential boundary)**.**
Let $E \subseteq \mathbb{R}^n$ be a Borelset. The essential boundary $\partial^* E$ is defined as

$$\partial^* E := \Big\{ x \in \mathbb{R}^n \mid \limsup_{\rho \to 0} \rho^{-n} \mathcal{L}^n(B_\rho(x) \cap E) > 0 \text{ and } \limsup_{\rho \to 0} \rho^{-n} \mathcal{L}^n(B_\rho(x) \setminus E) > 0 \Big\}.$$

If $E \subseteq \mathbb{R}^n$ has smooth boundary, then for all $x \in \partial^* E$ both limits equal $1/2$.

Theorem 2.2.11.
Let $\Omega \subseteq \mathbb{R}^n$ be open and $E \subseteq \Omega$ have finite perimeter. Then it holds

$$\mathcal{P}_\Omega(2\chi_E - 1) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega).$$

Next we define measure-function pairs and their weak convergence, which was presented in [Hut86].

Definition 2.2.12 (Measure-function pair)**.**
Let $m, n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ be open, $\mu \in C_c^0(\Omega)'$ be a Radon measure on Ω and $f \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$. Then (μ, f) is a measure-function pair over Ω with values in $\mathbb{R}^m$.

Definition 2.2.13 (Weak convergence of measure-function pairs)**.**
Assume that for all $k \in \mathbb{N}$ $(\mu, f), (\mu_k, f_k)$ are measure-function pairs over Ω with values in $\mathbb{R}^m$ and that $\mu_k \xrightarrow{w^*} \mu$ in $C_c^0(\Omega)'$. We say that $(\mu_k, f_k)_{k \in \mathbb{N}}$ converges to (μ, f) in the weak sense if

$$f_k \mu_k \xrightarrow{w^*} f\mu \quad \text{as} \quad k \to \infty \quad \text{in} \quad C_c^0(\Omega; \mathbb{R}^m)'.$$

In that case we write $(\mu_k, f_k) \xrightarrow{w} (\mu, f)$.

The central part of this theory is the following compactness result (we give a special version which fits our purposes).

Theorem 2.2.14 (Compactness Theorem for measure-function pairs)**.**
Assume $(\mu_k, f_k)_{k \in \mathbb{N}}$ is a sequence of measure-function pairs over Ω with values in $\mathbb{R}^m$ and that $\mu_k \xrightarrow{w^*} \mu$ in $C_c^0(\Omega)'$ for some Radon measure $\mu \in C_c^0(\Omega)'$. Assume that there exists $\Lambda > 0$ such that we have for all $k \in \mathbb{N}$

$$\int_\Omega |f_k|^2 \, \mathrm{d}\mu_k \leq \Lambda.$$

Then there exists $f \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$ such that up to a subsequence we have $(\mu_k, f_k) \xrightarrow{w} (\mu, f)$ as $k \to \infty$.

Next we introduce varifolds. The idea behind the concept is to find a “weaker”definition for surfaces while maintaining a concept of tangent spaces. We follow the introductions in [Sim83] and [FX06].

Definition 2.2.15 (Grassmannian)**.**
Let $n \in \mathbb{N}$ and $k \in \{1, \dots, n-1\}$. We write $G(n, k)$ for the set of all k-dimensional linear subspaces of $\mathbb{R}^n$ and ${}^{\oplus}G(n, k)$ for the set of all k-dimensional oriented linear subspaces of

$\mathbb{R}^n$. We can identify a linear subspace E with the orthogonal projection onto E. This way we can define a scalar product on $G(n,k)$ and ${}^{\oplus}G(n,k)$

$$P : Q \coloneqq \operatorname{tr}(P^T Q) = \sum_{j,k=1}^{n} P_{j,k} Q_{k,j} \quad \textit{for} \quad P, Q \in G(n,k) \quad \textit{or} \quad P, Q \in {}^{\oplus}G(n,k).$$

In the following we identify subspaces with the orthogonal projection onto them. Most relevant will be the case $k = n-1$ where we can construct P from a normal vector ν of the hypersurface

$$P = \operatorname{Id} - \nu \otimes \nu.$$

Furthermore, we can identify ${}^{\oplus}G(n, n-1) \cong \mathbb{S}^{n-1}$.

Let $U \subseteq \mathbb{R}^n$ be open and $k \in \{1, \dots, n-1\}$. We define $G_k(U) \coloneqq U \times G(n,k)$ and ${}^{\oplus}G_k(U) \coloneqq U \times {}^{\oplus}G(n,k)$.

If not specified otherwise we will always consider unoriented subspaces.

Definition 2.2.16 (Countably k-rectifiable sets)**.**
A set $M \subseteq \mathbb{R}^n$ is called countably k-rectifiable ($k \in \{1, \dots, n-1\}$) if

$$M \subseteq \bigcup_{j=0}^{\infty} M_j$$

where $\mathcal{H}^k(M_0) = 0$ and $M_j = F_j(\mathbb{R}^k)$ for $j \in \mathbb{N}$ and Lipschitz continuous functions $F_j : \mathbb{R}^k \longrightarrow \mathbb{R}^n$.

In fact we can represent k-rectifiable sets in a smoother way.

Lemma 2.2.17.
A set $M \subseteq \mathbb{R}^n$ is countably k-rectifiable if and only if there exists $N_0 \subseteq \mathbb{R}^n$ with $\mathcal{H}^k(N_0) = 0$ and if for all $j \in \mathbb{N}$ there exists a k-dimensional C^1-submanifold N_j in $\mathbb{R}^n$ such that

$$M \subseteq \bigcup_{j=0}^{\infty} N_j. \tag{2.2.2}$$

As mentioned before a concept of tangent spaces is important for the theory. To define the approximate tangent space we first introduce some notation.

Definition 2.2.18 (Pushforward and pullback)**.**
Given $r > 0$ and $x \in \mathbb{R}^n$ we introduce the function $\zeta_{x,r} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $\zeta_{x,r}(y) \coloneqq \frac{y-x}{r}$. We write $\zeta_r \coloneqq \zeta_{0,r}$. Furthermore we define the pushforward. Let μ be a measure on $\Omega \subseteq \mathbb{R}^n$, then we define for any set A in the associated σ-algebra on $\zeta_{x,r}(\Omega)$

$$\zeta_{x,r,\#}\mu(A) \coloneqq \mu(\zeta_{x,r}^{-1}(A)), \quad \zeta_{x,r,\#}\mu \in C_c^0(\zeta_{x,r}(\Omega))'.$$

Analogously we define for any $\eta \in C_c^0(\zeta_{x,r}(\Omega))$ the pullback

$$\zeta_{x,r}^{\#}\eta \coloneqq \eta \circ \zeta_{x,r} \in C_c^0(\Omega).$$

With these notations we have

$$\int_{\zeta_{x,r}(\Omega)} \eta \,\mathrm{d}\zeta_{x,r,\#}\mu = \langle \eta, \zeta_{x,r,\#}\mu\rangle_{C_c^0(\zeta_{x,r}(\Omega))'} = \langle \zeta_{x,r}^{\#}\eta, \mu\rangle_{C_c^0(\Omega)'} = \int_\Omega \zeta_{x,r}^{\#}\eta \,\mathrm{d}\mu.$$

If $\mu = \mathcal{H}^k$ for some $k \in \{1, \dots, n-1\}$ then a coordinate transformation yields that

$$\int_\Omega \zeta_{x,r}^{\#}\eta \,\mathrm{d}\mathcal{H}^k = r^k \int_{\zeta_{x,r}(\Omega)} \eta \,\mathrm{d}\mathcal{H}^k \tag{2.2.3}$$

and thus $\zeta_{x,r,\#}\mathcal{H}^k = r^k\mathcal{H}^k$.

Definition 2.2.19 (Approximate tangent space)**.**
Let $k \in \{1, \dots, n-1\}$ and $M \subseteq \mathbb{R}^n$ be $\mathcal{H}^k$-measurable with $\mathcal{H}^k(M \cap K) < \infty$ for all compact sets $K \Subset \mathbb{R}^n$. $P \in G(n,k)$ is called approximate tangent space for M at a given point $x \in M$ if for all $\eta \in C_c^0(\mathbb{R}^n)$:

$$\lim_{r\to 0} \int_{\zeta_{x,r}(M)} \eta \,\mathrm{d}\mathcal{H}^k = \int_P \eta \,\mathrm{d}\mathcal{H}^k. \tag{2.2.4}$$

By (2.2.3) *this is equivalent to*

$$\lim_{r\to 0} \frac{1}{r^k} \int_\Omega \zeta_{x,r}^{\#}\eta \,\mathrm{d}\mathcal{H}^k = \int_P \eta \,\mathrm{d}\mathcal{H}^k.$$

In that case we write $T_xM = P$.

Remark.

- *If such a P exists it is unique and will be noted as $P = T_xM$ in analogy to the classical tangent spaces of manifolds.*
- *If N_j is a submanifold as in* (2.2.2) *for some $j \in \mathbb{N}$ we get*

$$T_xN_j = T_xM \quad \textit{for} \quad \mathcal{H}^k - \textit{a.e. } x \in M \cap N_j.$$

Definition 2.2.20 (Approximate tangent space with multiplicity)**.**
Let $k \in \{1, \dots, n-1\}$ and $M \subseteq \mathbb{R}^n$ be $\mathcal{H}^k$-measurable and $\theta \in L^1_{\mathrm{loc}}(M, \mathcal{H}^k)$ be non-negative. $P \in G(n,k)$ is called approximate tangent space with respect to the multiplicity $\theta(x)$ for M at a given point $x \in M$ if for all $\eta \in C_c^0(\mathbb{R}^n)$:

$$\lim_{r\to 0} \int_{\zeta_{x,r}(M)} \eta(y)\theta(x+ry) \,\mathrm{d}\mathcal{H}^k(y) = \theta(x) \int_P \eta \,\mathrm{d}\mathcal{H}^k. \tag{2.2.5}$$

By (2.2.3) *this is equivalent to*

$$\lim_{r\to 0} \frac{1}{r^k} \int_\Omega \theta\zeta_{x,r}^{\#}\eta \,\mathrm{d}\mathcal{H}^k = \theta(x) \int_P \eta \,\mathrm{d}\mathcal{H}^k.$$

In that case we write $T_xM = P$.

The following theorem shows that approximate tangent spaces are directly linked to countably rectifiable sets.

Theorem 2.2.21.
Let $M \subseteq \mathbb{R}^n$ be a $\mathcal{H}^k$-measurable set with $\mathcal{H}^k(M \cap K) < \infty$ for all compact sets $K \subseteq \mathbb{R}^n$. Then M is countably k-rectifiable if and only if there exists a non-negative $\theta \in L^1_{\mathrm{loc}}(M, \mathcal{H}^k)$ such that at $\mathcal{H}^k$-a.e. $x \in M$ the approximate tangent space T_xM with multiplicity $\theta(x)$ exists.

Definition 2.2.22 ((unoriented) Varifold)**.**
Let $U \subseteq \mathbb{R}^n$ be open and $k \in \{1, \dots, n-1\}$, a (unoriented) k-varifold V on U is a Radon measure on $U \times G(n,k)$, we write $V \in \mathbb{V}_k(U) := C_c^0(U \times G(n,k))'$. For $V \in \mathbb{V}_k(U)$ we define the weight measure $\|V\|$ which is the measure on U defined by

$$\langle \phi, \|V\| \rangle_{C_c^0(U)'} := \int_{G_k(U)} \phi(z) \, \mathrm{d}V(z,S) \quad \textit{for all } \phi \in C_c^0(U).$$

The term varifold and unoriented varifold will be used as synonyms while we will clearly state oriented varifold whenever it comes up. The difference comes from the space where the measures are defined. The subspaces in $G(n,k)$ can be seen as unoriented. If $k = n-1$ we can also define Radon measures on the space of oriented subspaces, identifying the spaces with the normal. This leads to the following definition.

Definition 2.2.23 (oriented Varifold)**.**
Let $U \subseteq \mathbb{R}^n$ be open and $k \in \{1, \dots, n-1\}$, an oriented varifold ${}^{\oplus}V$ on U is a Radon measure on $U \times {}^{\oplus}G(n,k)$, we write $V \in {}^{\oplus}\mathbb{V}_k(U) := C_c^0(U \times {}^{\oplus}G(n,k))'$. In the case $k = n-1$ we simply write $\mathbb{V}_{n-1}(U) := C_c^0(U \times \mathbb{S}^{n-1})'$

For all $k \in \{1, \dots, n-1\}$ there exists a projection ${}^{\oplus}G(n,k) \longrightarrow G(n,k)$ which simply forgets the orientation (an oriented subspace is a subspace after all). Thus every oriented varifold can also be seen as an unoriented varifold. The definition of the weight measure is also valid for oriented varifolds.

Example (Rectifiable varifolds)**.**
Let $U \subseteq \mathbb{R}^n$ be open $k \in \{1, \dots, n-1\}$ and $M \subseteq U$ countably $(n-1)$-rectifiable. We can define a varifold in a natural way: For all $\phi \in C_c^0(G_{n-1}(U))$

$$\langle \phi, V_M \rangle_{C_c^0(G_{n-1}(U))'} := \int_{G_{n-1}(U)} \phi \, \mathrm{d}V_M := \int_M \phi(x, T_xM) \, \mathrm{d}\mathcal{H}^{n-1}(x).$$

The weight measure is given by its action on $\phi \in C_c^0(U)$:

$$\langle \phi, \|V_M\| \rangle_{C_c^0(U)'} := \int_{G_{n-1}(U)} \phi(x) \, \mathrm{d}V_M = \int_M \phi(x) \, \mathrm{d}\mathcal{H}^{n-1}(x).$$

Varifolds induced this way by a $(n-1)$-rectifiable set are much more concrete than a general varifold and are easier to handle. Thus we define the term of a rectifiable varifold motivated by this example.

Definition 2.2.24 (Rectifiable and integral varifolds)**.**
Let $U \subseteq \mathbb{R}^n$ be open and $V \in \mathbb{V}_k(U)$. V is called rectifiable if there exists a countably k-rectifiable and $\mathcal{H}^k$-measurable set $M \subseteq U$ and a non-negative function $\theta \in L^1_{\mathrm{loc}}(M, \mathcal{H}^k)$ such that

$$V = \theta \mathcal{H}^k \llcorner M \otimes \delta_{T_\cdot M},$$

which means that for all $\phi \in C_c^0(G_k(U))$ we have

$$\langle \phi, V \rangle_{C_c^0(G_k(U))'} = \int_{G_k(U)} \phi(x,P)\,\mathrm{d}V(x,P) = \int_M \phi(x, T_x M)\theta(x)\,\mathrm{d}\mathcal{H}^k(x).$$

V is called an integral varifold if V is rectifiable and $\theta(x) \in \mathbb{N}$ for $\mathcal{H}^k$−a.e. $x \in M$.

Next we proceed by defining the first variation of a varifold. This is motivated by the connection between the first variation of submanifolds and its mean curvature. We want to define weak mean curvature vectors similarly.

Definition 2.2.25 (First variation)**.**
Let $U \subset \mathbb{R}^n$ be open, $V \in \mathbb{V}_k(U)$ and $X \in C_c^1(U;\mathbb{R}^n)$. We define

$$\langle X, \delta V \rangle_{C_c^0(U)'} := \int_{G_k(U)} P : DX(x)\,\mathrm{d}V(x,P).$$

Example. *If V is rectifiable with $\|V\| = \theta\mathcal{H}^k \llcorner M$ we get for $X \in C_c^1(U;\mathbb{R}^n)$*

$$\langle X, \delta V \rangle_{C_c^0(U)'} = \int_U \mathrm{div}_M X \,\mathrm{d}\|V\| = \int_M \mathrm{div}_M X\theta\,\mathrm{d}\mathcal{H}^k.$$

We use the analogy to (2.1.1) from the theory of hypersufaces in order to define a weak mean curvature vector.

Definition 2.2.26 (Generalized mean curvature)**.**
A rectifiable varifold $V \in \mathbb{V}_k(U)$ has the generalized mean curvature vector $\vec{H} = \vec{H}_V$ if for all $X \in C_c^1(U;\mathbb{R}^n)$ we have

$$\langle X, \delta V \rangle_{C_c^0(U)'} = -\int_U X \cdot \vec{H}\,\mathrm{d}\|V\|.$$

2.3 Introduction to Γ–convergence

Next we explain the sense in which we will approximate curvature based energies. We use a general concept of "convergence of functionals" introduced by De Giorgi and Franzoni in [DGF75]. The motivation for the definition of Γ–convergence is to find a convergence for extended real-valued functionals which forces minima and minimizers to converge alongside the functional under mild assumptions.

Let X be a metric space and for $j \in \mathbb{N}$ let $F, F_j : X \longrightarrow \widehat{\mathbb{R}}$ be functionals with values in $\widehat{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$. We define $F_j \xrightarrow{\Gamma(X)} F$ as $j \to \infty$ if for all $x \in X$:

$$\text{For all sequences } x_j \to x \text{ in } X : \liminf_{j\to\infty} F_j(x_j) \geq F(x) \qquad (\Gamma - \inf)$$

$$\text{there exists a sequence } x_j^* \to x \text{ in } X : \limsup_{j\to\infty} F_j(x_j^*) \leq F(x). \qquad (\Gamma - \sup)$$

The definition is made such that if $F_\varepsilon \overset{\Gamma(X)}{\longrightarrow} F$ as $\varepsilon \to 0$ in (X,d) we get in many cases

$$\lim_{\varepsilon\to 0} \min\{F_\varepsilon(x) \mid x \in X\} = \min\{F(x) \mid x \in X\}$$

and all cluster points of $(x_\varepsilon)_{\varepsilon>0}$ with $x_\varepsilon \in \operatorname*{argmin}_{x\in X} F_\varepsilon(x)$ for all $\varepsilon > 0$ are minimizers of F. In this sense the functional F is well approximated as the convergence of minima and minimizers is very useful.

The following theorem, which corresponds to Theorem 12.1.1. in [ABM14] precises the previous remark.

Theorem 2.3.1 (Stability of minima).
Let (X,d) be a metric space and for $j \in \mathbb{N}$ let $F_j, F : X \longrightarrow \widehat{\mathbb{R}}$ be functionals with $F_j \overset{\Gamma(X)}{\longrightarrow} F$ in X. Let $x_j \in X$ be such that $F_n(x_j) \leq \inf_X F_j + \delta_j$ with $0 \leq \delta_j \to 0$. Assume that $\{x_j\}_{j\in\mathbb{N}}$ is relative compact, then every cluster point $\overline{x}$ of $\{x_j\}_{j\in\mathbb{N}}$ is a minimizer of F and

$$\lim_{j\to\infty} \inf_X F_j = F(\overline{x}).$$

This implies the desired convergence of minima and minimizers. More information on Γ–convergence can be found in [Bra02] or [DM93].

2.4 Phase-field approximations

In this section we list the diffuse approximations relevant for this thesis. As mentioned in the introduction the first important result is the approximation of the perimeter.

Definition 2.4.1 (Cahn-Hilliard energy).
Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded with Lipschitz boundary and $W \in C(\mathbb{R})$ with $W \geq 0$ and $\{W = 0\} = \{\pm 1\}$. We define the diffuse perimeter $\mathcal{P}_\varepsilon : L^1(\Omega) \longrightarrow [0,\infty]$, with

$$\mathcal{P}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega \left(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\right) \mathrm{d}\mathcal{L}^n, & \text{if } u \in H^1(\Omega) \cap L^4(\Omega) \\ +\infty, & \text{else.} \end{cases}$$

Theorem 2.4.2 (Modica-Mortola).
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary and W as in Definition 2.4.1. Then we have

$$\mathcal{P}_\varepsilon \overset{\Gamma}{\longrightarrow} c_0\mathcal{P} \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad L^1(\Omega)$$

for $c_0 = \int_{-1}^1 \sqrt{2W}\,\mathrm{d}\mathcal{L}^1$.

In addition to the Γ–convergence result we also have a compactness result for sequences with bounded energy. The following theorem can be found in [Alb00] or [Leo13].

Theorem 2.4.3 (Compactness for sequences with bounded energy).
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with Lipschitz boundary and $W(r) := (1-r^2)^2$. Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^1(\Omega)$ with

$$\sup_{\varepsilon>0} \mathcal{P}_\varepsilon(u_\varepsilon) \leq \Lambda.$$

for some $\Lambda > 0$. Then there exists a limit function $u \in BV(\Omega; \{\pm 1\})$ such that up to a subsequence we have

$$u_\varepsilon \longrightarrow u \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad c_0 \mathcal{P}(u) \leq \liminf_{\varepsilon \to 0} \mathcal{P}_\varepsilon(u_\varepsilon).$$

Building upon results from [SV03] Amstutz and Van Goethem considered a gradient-free approximation of the perimeter.

Definition 2.4.4 (Diffuse gradient-free perimeter by Amstutz and Van Goethem)**.**
We define the gradient-free diffuse perimeter by $\mathcal{P}_\varepsilon^{\mathrm{AG}} : L^1(\Omega) \longrightarrow [0, \infty]$,

$$\mathcal{P}_\varepsilon^{\mathrm{AG}}(u) := \inf_{v \in H^1(\Omega)} \int_\Omega \Big(\frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{2\varepsilon}(u - v)^2 + \frac{1}{2\varepsilon} W(u) \Big) \,\mathrm{d}\mathcal{L}^n.$$

This infimum is attained for $v = \overline{u}_\varepsilon$, where $\overline{u}_\varepsilon \in H^1(\Omega)$ is the unique solution to the Euler-Lagrange equation

$$-\varepsilon^2 \Delta \overline{u}_\varepsilon + \overline{u}_\varepsilon = u \quad \text{in} \quad \Omega \tag{2.4.1}$$

$$\partial_\nu \overline{u}_\varepsilon = 0 \quad \text{on} \quad \partial\Omega. \tag{2.4.2}$$

This leads to the representation

$$\begin{aligned} \mathcal{P}_\varepsilon^{\mathrm{AG}}(u) &= \inf_{v \in H^1(\Omega)} \int_\Omega \Big(\frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{2\varepsilon}(u - v)^2 + \frac{1}{2\varepsilon} W(u) \Big) \,\mathrm{d}\mathcal{L}^n \\ &= \int_\Omega \Big(\frac{\varepsilon}{2} |\nabla \overline{u}_\varepsilon|^2 + \frac{1}{2\varepsilon}(u - \overline{u}_\varepsilon)^2 + \frac{1}{2\varepsilon} W(u) \Big) \,\mathrm{d}\mathcal{L}^n \\ &= \int_\Omega \frac{1}{2\varepsilon} \Big(u(u - \overline{u}_\varepsilon) + W(u) \Big) \,\mathrm{d}\mathcal{L}^n. \end{aligned} \tag{2.4.3}$$

The resulting diffuse energy is well-defined and finite if only $W(u) \in L^1(\Omega)$ in contrast to the Cahn-Hilliard energy, which requires H^1-regularity because the gradient is shifted to the auxiliary function $\overline{u}_\varepsilon$.

In Theorem 3.7 of [AVG12] the authors state their approximation result, we adjusted the wells of W.

Theorem 2.4.5 (Gradient-free approximation of the perimeter)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded set with Lipschitz boundary and assume $W(r) = 1 - r^2$ for $r \in \mathbb{R}$. Then it holds, writing $c_{\mathrm{AG}} := \int_{-1}^1 (1 + \frac{1}{2} W'') \sqrt{W + \frac{1}{4}(W')^2} \,\mathrm{d}\mathcal{L}^1$

$$\mathcal{P}_\varepsilon^{\mathrm{AG}} \overset{L^1(\Omega;[-1,1])}{\longrightarrow} c_{\mathrm{AG}} \mathcal{P}.$$

Next we give the results for approximations of the Willmore energy.
First we introduce the Willmore energy as a functional on $L^1(\Omega)$.

Definition 2.4.6 (Willmore energy as a functional)**.**
Let $\Omega \subset \mathbb{R}^n$ be open, we define $\mathcal{W} : L^1(\Omega) \longrightarrow [0, \infty]$ such that

$$\mathcal{W}(u) := \begin{cases} \displaystyle\int_{\Omega \cap \partial E} |H_{\partial E}|^2 \,\mathrm{d}\mathcal{H}^{n-1}, & \text{if } u = 2\chi_E - 1 \text{ and } \partial E \text{ is a } C^2\text{-hypersurface} \\ +\infty, & \text{else.} \end{cases} \tag{2.4.4}$$

Defining $\mathcal{W}$ on functions instead of sets has the advantage that $L^1(\Omega)$ is a Banach space, i.e. it has an algebraic and a topologic structure that we can use to construct approximations in the sense of Γ–convergence.

Bellettini and Paolini [BP93] proved the lim sup–condition for an approximation of the Willmore energy. Let $\Omega \subseteq \mathbb{R}^n$ be open, $W(r) := (1-r^2)^2$ for $r \in \mathbb{R}$ and let $u \in H^2(\Omega)$. Then we define $H_\varepsilon = H_\varepsilon(u) := -\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)$ and the standard diffuse Willmore energy by Bellettini and Paolini $\mathcal{W}_\varepsilon : L^1(\Omega) \longrightarrow [0,\infty]$ with

$$\mathcal{W}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{\varepsilon}\big|H_\varepsilon(u)\big|^2 \,\mathrm{d}\mathcal{L}^n, & \text{if } u \in H^2(\Omega) \cap L^6(\Omega) \\ +\infty, & \text{else.} \end{cases}$$

Theorem 2.4.7 (Bellettini-Paolini)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be open and assume $E \subseteq \Omega$ has C^2-boundary Γ. Then there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ such that $u_\varepsilon \longrightarrow u$ in $L^1(\Omega)$ and

$$\limsup_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon) \le c_0 \mathcal{W}(u)$$

for $u := 2\chi_E - 1$.

The central idea of their proof was to reduce to a one dimensional problem. First they assumed that u_ε can be represented as a polynomial in ε close to the surface, i.e.

$$u_\varepsilon(x) = U_0\Big(\frac{d(x)}{\varepsilon}, y\Big) + \varepsilon U_1\Big(\frac{d(x)}{\varepsilon}, y\Big) + \dots \tag{2.4.5}$$

Here $d(x) := \operatorname{sdist}(x,\Gamma) \in \mathbb{R}$ is the signed distance from x to the surface Γ and $y \in \Gamma$ is the orthogonal projection from $x \in \Omega$ onto Γ, which is well-defined in a small neighborhood of Γ. For more on the coordinate system see Definition 2.1.10. Then $\mathcal{W}_\varepsilon(u_\varepsilon)$ can be expanded as a polynomial in ε as well, and the profile functions $U_0, U_1, \dots$ are chosen in way to minimize each order of $\mathcal{W}_\varepsilon(u_\varepsilon)$.

It will turn out that the tangential coordinate y is far less important after minimizing than the normal coordinate, thus this process reduces the n-dimensional coordinate of x to a real number $\operatorname{sdist}(x,\Gamma)$. This method will serve as a prototype for proving the $\Gamma(L^1) - \limsup$ estimate in different models.

The lim inf–estimate in smooth limit points and small dimensions was proven by Röger and Schätzle [RS06]. The authors considered the sum of perimeter and Willmore energy.

Theorem 2.4.8 (Röger-Schätzle)**.**
Let $n \in \{2,3\}$, $\Omega \subseteq \mathbb{R}^n$ be open and assume $E \subseteq \Omega$ with C^2-boundary in Ω. Consider a sequence $(u_\varepsilon)_{\varepsilon>0}$ with $u_\varepsilon \longrightarrow u$ in $L^1_{\mathrm{loc}}(\Omega)$, then it holds

$$c_0\big(\mathcal{P}(u) + \mathcal{W}(u)\big) \le \liminf_{\varepsilon\to 0} \big(\mathcal{P}_\varepsilon(u_\varepsilon) + \mathcal{W}_\varepsilon(u_\varepsilon)\big)$$

for $u := 2\chi_E - 1$.

Given a sequence $u_\varepsilon \longrightarrow u$ in $L^1(\Omega)$ the authors consider the diffuse perimeter and diffuse Willmore energy as Radon measures

$$\begin{aligned} \mu_\varepsilon &:= \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon}W(u)\Big)\mathcal{L}^n \llcorner \Omega \in C_c^0(\Omega)', \\ \xi_\varepsilon &:= \Big(\frac{\varepsilon}{2}|\nabla u|^2 - \frac{1}{\varepsilon}W(u)\Big)\mathcal{L}^n \llcorner \Omega \in C_c^0(\Omega)', \\ \text{and} \quad \alpha_\varepsilon &:= \frac{1}{\varepsilon}\Big| - \varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big|^2 \mathcal{L}^n \llcorner \Omega \in C_c^0(\Omega)'. \end{aligned}$$

With the diffuse normal vector $\nu_\varepsilon := \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ on $\{\nabla u_\varepsilon \neq 0\}$ they also define the varifold $V_\varepsilon := \mu_\varepsilon \otimes \nu_\varepsilon^\perp$ such that for all $\eta \in C_c^0(\mathbb{R}^n \times G(n, n-1))$

$$\langle \eta, V_\varepsilon \rangle_{C_c^0(\mathbb{R}^n \times G(n,n-1))'} = \int\limits_{\Omega \times G(n,n-1)} \eta(x, S)\, \mathrm{d}V(x, S) := \int\limits_\Omega \eta(x, \nu_\varepsilon(x)^\perp)\, \mathrm{d}\mu_\varepsilon(x).$$

The authors assume without loss of generality that $\liminf_{\varepsilon \to 0}\big(\mathcal{P}_\varepsilon(u_\varepsilon) + \mathcal{W}(u_\varepsilon)\big) < \infty$, which yields compactness for $(\mu_\varepsilon)_{\varepsilon>0}, (\xi_\varepsilon)_{\varepsilon>0}$ and $(\alpha_\varepsilon)_{\varepsilon>0}$. Thus there exist finite Radon measures $\mu, \xi, \alpha \in C_0^0(\Omega)'$, a varifold $V \in C_c^0(\Omega \times G(n, n-1))'$, such that up to a subsequence we have as $\varepsilon \to 0$

$$\begin{aligned} \mu_\varepsilon \xrightarrow{w^*} \mu, \quad \xi_\varepsilon \xrightarrow{w^*} \xi, \quad \alpha_\varepsilon \xrightarrow{w^*} \alpha \quad \text{in} \quad C_0^0(\Omega)', \\ \text{and} \quad V_\varepsilon \xrightarrow{w^*} V \quad \text{in} \quad C_c^0(\Omega \times G(n, n-1))'. \end{aligned}$$

Then the authors prove that $\xi = 0$ and that V is $(n-1)$-rectifiable, which are also achieved in [Ilm93] or [PT98], though with different context and proofs. However the critical and new component for the proof of the lim inf–property is the integrality of $\frac{1}{c_0}V$. This has been a question of big interest as in both of the publications [Ilm93] and [PT98], the integrality of the limit varifold is explicitly mentioned as an open question. A reason for that is the following. From the $(n-1)$-rectifiability we have

$$\|V\| = \mu = \tilde{\theta}\mathcal{H}^{n-1} \llcorner \Gamma$$

for some $\mathcal{H}^{n-1}$-measurable function $\tilde{\theta} : \Gamma \longrightarrow \mathbb{R}$. Knowing that $\frac{1}{c_0}V$ is integral implies $\frac{1}{c_0}\tilde{\theta}(y) \in \mathbb{N}$ for all $y \in \Gamma$ and in particular that $\tilde{\theta} \geq c_0$. Thus we can estimate

$$c_0\mathcal{H}^{n-1} \llcorner \Gamma \leq \mu$$

which is the key to the lim inf–estimate as it connects back to the Willmore energy after discussing μ and its properties. The proof of integrality employs the blow-up method and is inspired by a proof from Hutchinson and Tonegawa in [HT00].

Having proven the integrality of $\frac{1}{c_0}V$, the core argument of their proof starts with the observation that for all $\eta \in C_c^1(\Omega; \mathbb{R}^n)$ we have

$$\int_\Omega \eta \cdot \nabla u_\varepsilon \Big(- \varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\, \mathrm{d}\mathcal{L}^n \longrightarrow \int_\Omega \eta \cdot \vec{H}_V\, \mathrm{d}\mu \quad \text{as} \quad \varepsilon \to 0. \tag{2.4.6}$$

Then the authors estimate

$$\int_\Omega \eta \cdot \vec{H}_V \,\mathrm{d}\mu \leq \liminf_{\varepsilon\to 0} \left[\int_\Omega \varepsilon |\eta|^2 \left|\nabla u_\varepsilon\right|^2 \mathrm{d}\mathcal{L}^n \int_\Omega \frac{1}{\varepsilon} |H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \right]^{\frac{1}{2}} \tag{2.4.7}$$
$$= \sqrt{\langle |\eta|^2, \mu\rangle_{C_c^0(\Omega)'}} \liminf_{\varepsilon\to 0} \sqrt{\alpha_\varepsilon(\Omega)}.$$

Taking the supremum over $\eta \in C_c^1(\Omega;\mathbb{R}^n)$ with $\int_\Omega |\eta|^2 \,\mathrm{d}\mu \leq 1$ yields

$$\|\vec{H}_V\|_{L^2(\Omega,\mu;\mathbb{R}^n)} \leq \liminf_{\varepsilon\to 0} \sqrt{\alpha_\varepsilon(\Omega)},$$

and thus

$$c_0 \mathcal{W}(u) = c_0 \int_\Omega \left|\vec{H}_V\right|^2 \mathrm{d}\mathcal{H}^{n-1} \leq \int_\Omega \left|\vec{H}_V\right|^2 \mathrm{d}\mu \tag{2.4.8}$$
$$\leq \liminf_{\varepsilon\to 0} \int_\Omega \frac{1}{\varepsilon} \Big| -\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon)\Big|^2 \,\mathrm{d}\mathcal{L}^n \leq \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon).$$

A more detailed presentation of the results from [RS06] can be found in Theorem 5.2.3, the central proof of the paper is shortly outlined in a remark at the end of Section 5.2.

2.5 Varifold solutions to mean curvature flow

There are several concepts for weak solutions for the mean curvature flow, of which the first and most prominent is the Brakke flow. In his book [Bra78] from 1978, Brakke presents the results of his dissertation from 1975. He introduced a weak formulation of mean curvature flow with the idea to generalize the energy-dissipation. Here we present a different but equivalent definition, which was proposed by Tonegawa in [Ton19].

Assumption 2.5.1.
Let $\Omega \subseteq \mathbb{R}^n$ be open, $k \in \{1,\dots,n-1\}$ and $T, \sigma > 0$. Assume $(V_t)_{t\in[0,T)}$ in $\mathbb{V}_k(\Omega)$ is a family of integral varifolds and for a.e. $t \in [0,T)$ V^t has a weak mean curvature vector $\vec{H}_t$, we define $\vec{H}(t) := \vec{H}_t$. Furthermore, it holds $\vec{H} \in L^2_{\mathrm{loc}}([0,T)\times\Omega, \mathcal{L}^1 \otimes (\|V^t\|)_{t\in[0,T)};\mathbb{R}^n)$. The family is bounded in the following sense: for all $K \Subset \Omega$ and all $t \in [0,T)$ we have

$$\sup_{s\in[0,t]} \|V^s\|(K) < \infty.$$

Definition 2.5.2 (Brakke flow)**.**
Let Assumptions 2.5.1 hold. Then $(V_t)_{t\in[0,T)}$ is moving by mean curvature flow in the sense of Brakke if for all non-negative $\psi \in C_c^1[0,T)$, $\eta \in C_c^1(\Omega)$ and all $0 \leq t_1 < t_2 < T$

$$\psi(t) \int_\Omega \eta \,\mathrm{d}\|V_t\| \Big|_{t=t_1}^{t_2} \leq -\sigma \int_{t_1}^{t_2} \psi \int_\Omega \eta \left|\vec{H}_t\right|^2 \mathrm{d}\|V^t\| \,\mathrm{d}t + \sigma \int_{t_1}^{t_2} \psi \int_\Omega \nabla\eta \cdot \vec{H}_t \,\mathrm{d}\|V_t\| \,\mathrm{d}t$$
$$+ \int_{t_1}^{t_2} \psi' \int_\Omega \eta \,\mathrm{d}\|V_t\| \,\mathrm{d}t. \tag{2.5.1}$$

The identity (2.5.1) is an inequality because this way sudden losses of k-dimensional are accounted for. As mentioned in the introduction singularities can not only occur in evolutions by mean curvature flow, they are even expected to occur.

The original definition from Brakke uses an upper difference quotient fro the time derivative instead of a weak derivative as we explain in the following. Let $V := (V^t)_{t>0}$ be a family of varifolds in Ω with mean curvature vectors $\vec{H}_t$ for $t > 0$. Then V is a solution to mean curvature flow in the sense of Brakke if for all $\psi \in C_c^1(\Omega)$ with $\psi \geq 0$ and all $t > 0$ we have

$$\overline{\partial}_t \int_\Omega \psi \, \mathrm{d}\|V^t\| \leq -\int_\Omega \psi |\vec{H}_t|^2 \, \mathrm{d}\|V^t\| + \int_{G_{n-1}(\Omega)} \vec{H}_t(x) \cdot S^\perp \nabla \psi(x) \, \mathrm{d}V^t(x, S).$$

Here $\overline{\partial}_t$ is the upper partial derivative and is the lim sup of the difference quotient. By far the most important contribution with respect to phase-field approximations is Ilmanen's paper [Ilm93]. Ilmanen starts with solutions to the Allen-Cahn equation

$$-\varepsilon \partial_t u_\varepsilon = -\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) \tag{2.5.2}$$

for a double-well potential W and defines

$$\begin{aligned} \mu_\varepsilon^t &:= \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon(t,\cdot)|^2 + \frac{1}{\varepsilon} W'(u_\varepsilon(t,\cdot))\Big)\mathcal{L}^n \\ \xi_\varepsilon^t &:= \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon(t,\cdot)|^2 - \frac{1}{\varepsilon} W'(u_\varepsilon(t,\cdot))\Big)\mathcal{L}^n \\ V_\varepsilon^t &:= \mu_\varepsilon^t \otimes \nabla u_\varepsilon(t,\cdot)^\perp \end{aligned}$$

such that for all $\eta \in C_c^0(G_{n-1}(\mathbb{R}^n))$,

$$\langle \eta, V_\varepsilon^t \rangle_{C_c^0(G_{n-1}(\mathbb{R}^n))'} = \int_{G_{n-1}(\mathbb{R}^n)} \eta(x, S) \, \mathrm{d}V(x, S) := \int_{\mathbb{R}^n} \eta(x, \nabla u_\varepsilon(t, x)^\perp) \, \mathrm{d}\mu_\varepsilon^t(x).$$

Important steps in the paper are the use of the monotonicity formula by Huisken [Hui90], the proof that there exists a subsequence ($\varepsilon \to 0$) such that $\mu_\varepsilon \xrightarrow{w^*} \mu^t$ in $C_c^0(\mathbb{R}^n)'$ for all $t > 0$, and that $\xi_\varepsilon^t \xrightarrow{w^*} 0$. The author then proves that there exists a limit varifold, which is a solution to mean curvature flow in the sense of Brakke. After Theorem 2.4.8 was proven in [RS06], Ilmanen's proof was significantly shortened in small dimensions by Sato [Sat08].

Another type of varifold solutions for mean curvature flow was recently proposed in [HL21]. This concept considers evolving varifolds with mean curvature vector and normal velocity. The key requirement is to characterize the motion law in form of an optimal energy-dissipation inequality. If $(\Gamma_t)_{t \in [0,T)}$ is a family of smooth surfaces evolving by mean curvature flow we get by the chain rule

$$\partial_t \mathcal{H}^{n-1}(\Gamma_t) = \int_{\Gamma_t} \vec{\mathcal{V}} \cdot \vec{H}_t \, \mathrm{d}\mathcal{H}^{n-1} \geq -\frac{1}{2} \int_{\Gamma_t} |\mathcal{V}|^2 \, \mathrm{d}\mathcal{H}^{n-1} - \frac{1}{2} \int_{\Gamma_t} |\vec{H}_t|^2 \, \mathrm{d}\mathcal{H}^{n-1}. \tag{2.5.3}$$

Thus if the inequality holds with "$\leq$" we have the energy-dissipation equality of mean curvature flow. The idea behind the De Giorgi formulation is to *define* this optimal energy-dissipation inequality with "$\leq$" as the motion law for mean curvature flow, even in absence of a chain rule.

Definition 2.5.3 (De Giorgi type varifold solutions for rescaled mean curvature flow)**.** *Let ${}^{\oplus}V = \mathcal{L}^1 \otimes ({}^{\oplus}V^t)_{t\in[0,T)}$ be a family of oriented varifolds in Ω such that for all $\phi \in L^1(0,T;C_0^0(\Omega \times \mathbb{S}^{n-1}))$ the map $(0,T) \ni t \longmapsto \int_{\Omega\times\mathbb{S}^{n-1}} \phi(t,\cdot,\cdot)\,\mathrm{d}^{\oplus}V^t$ is measurable. We consider a family $(E(t))_{t>0}$ of open subsets of Ω with finite perimeter in Ω such that the associated indicator function $u(t,\cdot) := 2\chi_{E(t)} - 1$, $t > 0$, satisfies $u \in L^\infty(0,T;BV(\Omega;\{\pm 1\}))$. Let $c_0 > 0$ be a surface tension constant and $\sigma > 0$ be a time rescaling factor. We call ${}^{\oplus}V_0$ the initial oriented varifold. Let $u_0 \in BV(\Omega;\{\pm1\})$ be an initial phase indicator function, we call the pair $({}^{\oplus}V, u)$ a De Giorgi type varifold solution for the rescaled mean curvature flow $\mathcal{V} = \sigma H$ with initial data $({}^{\oplus}V^0, u_0)$ if the following hold.*

(*a*) *For $t \in (0,T)$ we write $\mu^t := \|{}^{\oplus}V^t\|$ for the weight measure and $\mu := \mathcal{L}^1 \otimes (\mu^t)_{t\in[0,T)}$. We require the existence of $\mathcal{V} \in L^2(0,T;L^2(\Omega,\mu^t))$ encoding a generalized normal velocity in the sense of*

$$\frac{c_0}{2}\int_{\Omega_\tau} u\partial_t\phi\,\mathrm{d}\mathcal{L}^{n+1} - \int_{\Omega_\tau} \mathcal{V}\phi\,\mathrm{d}\mu = \frac{c_0}{2}\int_{\Omega} u(\tau,\cdot)\phi(\tau,\cdot)\,\mathrm{d}\mathcal{L}^n - \frac{c_0}{2}\int_{\Omega} u_0\phi(0,\cdot)\,\mathrm{d}\mathcal{L}^n \tag{a}$$

for a.e. $\tau \in (0,T)$ and every $\phi \in C_c^\infty([0,T)\times\Omega))$.

(*b*) *We require the existence of $\vec{H} \in L^2(0,T;L^2(\Omega,\mu^t;\mathbb{R}^n))$ encoding a generalized mean curvature vector by*

$$\int_{\Omega_T} \vec{H}\cdot\eta\,\mathrm{d}\mu = -\int\limits_{\Omega_T\times\mathbb{S}^{n-1}} (\mathrm{Id} - s\otimes s) : D\eta(t,x)\,\mathrm{d}^{\oplus}V(t,x,s) \tag{b}$$

for all $\eta \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^n)$.

(*c*) *A sharp energy-dissipation principle in form of*

$$\mu^\tau(\Omega) + \frac{\sigma}{2}\int_{\Omega_\tau} |\vec{H}|^2\,\mathrm{d}\mu + \frac{1}{2\sigma}\int_{\Omega_\tau} |\mathcal{V}|^2\,\mathrm{d}\mu \le \mu^0(\Omega) \tag{c}$$

for a.e. $\tau \in (0,T)$.

(*d*) *For a.e. $t \in (0,T)$ and all $\eta \in C_c^\infty(\Omega;\mathbb{R}^n)$ we have*

$$\frac{c_0}{2}\int_{\Omega} \eta(x)\cdot\,\mathrm{d}\nabla u(t,x) = \int\limits_{\Omega\times\mathbb{S}^{n-1}} \eta(x)\cdot s\,\mathrm{d}^{\oplus}V^t(x,s). \tag{d}$$

Note that there are a few differences in the definition from the source because we consider the space $[0,T)\times\Omega$ instead of $[0,\infty)\times\mathbb{R}$ and the interface function u takes values in $\{\pm1\}$ instead of $\{0,1\}$ which is the reason for the additional factor $\frac{1}{2}$ in (a) and (d). Also we added the additional parameter $\sigma > 0$ which does not exist in the original definition. Note that in the case of any smooth evolution of smooth surfaces we have, writing $H(t) := \vec{H}_t\cdot\nu$ where ν is the inner normal of the surface

$$
\begin{aligned}
\partial_t \mu^t(\Omega) &= -\int_\Omega H(t,\cdot)\mathcal{V}(t,\cdot)\,\mathrm{d}\mu^t \\
&= -\frac{\sigma}{2}\int_\Omega |H(t,\cdot)|^2\,\mathrm{d}\mu^t - \frac{1}{2\sigma}\int_\Omega \left|\mathcal{V}(t,\cdot)\right|^2\mathrm{d}\mu^t + \frac{1}{2\sigma}\int_\Omega \left|\sigma H(t,\cdot) - \mathcal{V}(t,\cdot)\right|^2\mathrm{d}\mu^t.
\end{aligned}
$$

We conclude for all $\tau \in (0,T)$

$$
\mu^\tau(\Omega) + \frac{\sigma}{2}\int_{\Omega_\tau} |H(t,\cdot)|^2 \mathrm{d}\mu + \frac{1}{2\sigma}\int_{\Omega_\tau}\left|\mathcal{V}(t,\cdot)\right|^2 \mathrm{d}\mu = \mu^0(\Omega) + \frac{1}{2\sigma}\int_{\Omega_\tau}\left|\sigma H(t,\cdot) - \mathcal{V}(t,\cdot)\right|^2 \mathrm{d}\mu.
$$

Hence, if (c) holds for a smooth evolution of smooth surfaces then the classical energy-dissipation equality and

$$
\sigma H(t,x) = \mathcal{V}(t,x)
$$

hold for μ-a.e. $(t,x) \in \Omega_T$.

2.6 Other weak formulations of the mean curvature flow

In the mean curvature flow sets are expected to develop singularities in finite times. For instance for boundaries of non-convex sets the surfaces can break apart into multiple surfaces; see Chapter 3 in [Eck04]. Such singularities have been studied by Huisken [Hui90]. Classical solutions rely on parametrizations over a fixed surface, however when the topology (in a geometric way) changes, as described above this is no longer possible. Thus classical solutions stop existing at singularities, which makes weak solutions and possible weak formulations much more interesting.

In the 1980's the notion of viscosity solutions was established. Starting with the works of Crandall and Lions in [CL83] and [Lio83] this allows for a weak formulation of mean curvature flow. Other important contributions are the papers from Chen, Giga, and Goto, mainly [CGG91]; see also the references therein. For us the framework of viscosity solutions is unsuitable owing to the lack of comparison principle for fourth-order PDEs.

In 1992 a discretization scheme to approximate the mean curvature flow was established called BMO (Bence-Merriman-Osher) or Thresholding scheme; see [MBO92]. The setup is inspired by the Allen-Cahn equation (2.5.2) where the double-well potential W has its wells at ± 1. We start with a given set E_0. For each time step we encode E_k with a $\{\pm 1\}$-valued characteristic function. We imitate the effect of the Laplacian by convolving the function with a fundamental solution of the heat equation. Motivated by the forcing term in the Allen-Cahn equation we set E_{k+1} as the super-level set of 0. Other important contributions to the study of the scheme can be found in [Law93, ELO15, LO16].

Another solution concept that should be mentioned, even though it is not connected to our research is the method of De Giorgi's barriers using functions which map point of time onto sets whose signed distance function is a solution to the heat equation. The concept is centered around the inclusion principle for mean curvature flow . For a precise definition we refer to [Bel13], where the concept is explained in detail. Important contributions to this theory include the paper [BP95] from Bellettini and Paolini from

1995, its errata [BP02] from 2002, and the paper [BN00] from Bellettini and Novaga from 2000.

3 A gradient-free approximation of the Willmore energy based on the Amstutz-Van Goethem model

In this chapter we prove the Γ–lim sup estimate for a new, “gradient free”approximation based on the article [AVG12] by Amstutz and Van Goethem from 2012. They consider a different diffuse perimeter than the standard Cahn-Hilliard energy from 2.4.1, motivated by a two-variable energy studied by Solci and Vitali in [SV03]. The energy in [AVG12] is given by

$$\mathcal{P}_\varepsilon^{\mathrm{AG}}(u) = \inf_{v \in H^1(\Omega)} \int_\Omega \Big(\frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{2\varepsilon}(u-v)^2 + \frac{1}{2\varepsilon} W(u) \Big) \,\mathrm{d}\mathcal{L}^n,$$

where W is a double-well potential. In [AVG12] the authors consider $W_{\mathrm{AG}}(r) := r(1-r)$ for $r \in [-1,1]$, however we pose different assumptions on W which excludes W_{AG}; see Assumptions 3.1.1. The infimum is attained for $v = \overline{u}_\varepsilon$ where $\overline{u}_\varepsilon \in H^1(\Omega)$ is a solution to

$$-\varepsilon^2 \Delta \overline{u}_\varepsilon + \overline{u}_\varepsilon = u \quad \text{in} \quad \Omega \tag{3.0.1}$$

$$\partial_\nu u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \tag{3.0.2}$$

which yields

$$\begin{aligned} \mathcal{P}_\varepsilon^{\mathrm{AG}}(u) &= \int_\Omega \Big(\frac{\varepsilon}{2} |\nabla \overline{u}_\varepsilon|^2 + \frac{1}{2\varepsilon}(u-\overline{u}_\varepsilon)^2 + \frac{1}{2\varepsilon} W(u) \Big) \,\mathrm{d}\mathcal{L}^n \\ &= \int_\Omega \frac{1}{2\varepsilon} \Big(u(u-\overline{u}_\varepsilon) + W(u) \Big) \,\mathrm{d}\mathcal{L}^n. \end{aligned}$$

The solution operator to $-\varepsilon^2\Delta + \mathrm{Id}$ with Neumann boundary conditions is linear and self-adjoint. The L^2-gradient of $\mathcal{P}_\varepsilon^{\mathrm{AG}}$ therefore is given by

$$\nabla_{L^2} \mathcal{P}_\varepsilon^{\mathrm{AG}}(u) = \frac{1}{\varepsilon} \Big(u + \frac{1}{2} W'(u) - \overline{u}_\varepsilon \Big) =: H_\varepsilon^{\mathrm{AG}},$$

which can be seen as a *diffuse mean curvature.* This suggests the formal Willmore energy approximation

$$\mathcal{W}_\varepsilon^{\mathrm{AG}}(u) := \int_\Omega \frac{1}{\varepsilon^3} \Big| u + \frac{1}{2} W'(u) - \overline{u}_\varepsilon \Big|^2 \mathrm{d}\mathcal{L}^n, \tag{3.0.3}$$

which is the main object we study in this chapter. The additional factor ε^{-1} in $\mathcal{W}_\varepsilon^{\text{AG}}$ accounts for the small volume of the transition layer region.

The second topic we consider in this chapter is convergence for the gradient flows of $\mathcal{P}_\varepsilon^{\text{AG}}$ and $\mathcal{W}_\varepsilon^{\text{AG}}$. Under rather restrictive assumptions on the approximations we prove that up to a factor we get that the gradient flow of $\mathcal{P}_\varepsilon^{\text{AG}}$ converges to mean curvature flow and $\mathcal{W}_\varepsilon^{\text{AG}}$ converges to the Willmore flow, again up to a rescaling in time. The content of this chapter (in a shorter version) has already been published in [DKR22].

3.1 Preparations

Assumption 3.1.1 (on Ω and W)**.**
Throughout this chapter we assume $n \in \mathbb{N}$ and $\Omega \subseteq \mathbb{R}^n$ is an open, bounded set with Lipschitz boundary and $\varepsilon > 0$.

Furthermore we assume for the double-well potential

- $W \in C^m(\mathbb{R})$, $m \in \mathbb{N}^{\geq 4}$.
- $W \geq 0$, $\{W = 0\} = \{\pm 1\}$, $W''(\pm 1) > 0$.
- $1 + \frac{1}{2}W'' > 0$ *in* $[-1, 1]$.
- W *has at least linear growth at* $\pm\infty$.

We associate W with the mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(r) \coloneqq r + \frac{1}{2}W'(r)$.

Remark.
We have $f'(r) = 1+\frac{1}{2}W''(r) \geq \gamma$ for some $\gamma > 0$ and all $r \in [-1, 1]$. Thus $f \in C^{m-1}[-1, 1]$ is strictly increasing and we further obtain that $f : [-1, 1] \to [-1, 1]$ is one-to-one and that f has an inverse function $f^{-1} \in C^{m-1}[-1, 1]$ such that for all $r \in (-1, 1)$

$$(f^{-1})'(r) = \frac{1}{f'(f^{-1}(r))} \leq \frac{1}{\gamma} \quad \text{and} \quad |(f^{-1})''(r)| = \frac{|f''(f^{-1}(r))|}{|f'(f^{-1}(r))|^3} \leq \frac{\|W'''\|_{C^0[-1,1]}}{\gamma^3}. \tag{3.1.1}$$

Since we have $f'(r) \geq \gamma$ for all $r \in [-1, 1]$ and f' is continuous on $\mathbb{R}$ there exists an open interval $U \supseteq [-1, 1]$ such that f is strictly increasing on U. Thus we can consider the derivatives of f and f^{-1} on the closure $[-1, 1]$.

If W is an even function then f and f^{-1} are odd.

The conditions in the Assumption 3.1.1 cover a large class of admissible double-well potentials, such as the standard quartic double-well potential $W(r) = \frac{1}{4}(1 - r^2)^2$ that is most often used in simulations. On the other hand the particular choice $W(r) = 1 - r^2$, $r \in [-1, 1]$, with locally constant linear growth outside $[-1, 1]$ in [AVG12] is not allowed, since in this case f would be constant in $(-1, 1)$ and is not C^1-regular on $\mathbb{R}$.

For many diffuse approximations the study of the optimal transition between the pure phases on the real line is key for understanding its behavior, see for example [Alb00]

and the references therein. As for the Cahn–Hilliard approximation $\mathcal{P}_\varepsilon$ and Willmore functional $\mathcal{W}_\varepsilon$ we expect that typical small-energy configurations for $\mathcal{P}_\varepsilon^{\mathrm{AG}}$ and $\mathcal{W}_\varepsilon^{\mathrm{AG}}$ are quasi one-dimensional and can be constructed from an optimal transition profile and the rescaled signed distance from the zero-level set.
To characterize the optimal profile associated to $\mathcal{P}_\varepsilon^{\mathrm{AG}}$ we consider the following minimization problem on the real line. We fix a suitable class of real functions

$$\mathcal{M} := \{u \in L^\infty(\mathbb{R}) \,:\, \operatorname*{ess-lim\,sup}_{r\to-\infty} u(r) < 0,\ \operatorname*{ess-lim\,inf}_{r\to\infty} u(r) > 0\}.$$

Moreover we define for $u \in \mathcal{M}$, $v \in H^1_{\mathrm{loc}}(\mathbb{R})$ with $\lim_{x\to\pm\infty} v(x) = \pm 1$ the energies

$$\mathcal{G}_\varepsilon^{\mathbb{R}}(u,v) := \int_{\mathbb{R}} \frac{1}{2}\big(\varepsilon (v')^2 + \frac{1}{\varepsilon}(u-v)^2 + \frac{1}{\varepsilon} W(u)\big)\,\mathrm{d}\mathcal{L}^1,$$

$$\mathcal{E}_\varepsilon(u) := \inf\Big\{\mathcal{G}_\varepsilon^{\mathbb{R}}(u,v) \,:\, v \in H^1_{\mathrm{loc}}(\mathbb{R}),\ \lim_{r\to\pm\infty} v(r) = \pm 1\Big\}.$$

By rescaling we see that the minimization problem can be reduced to the case $\varepsilon = 1$ and we write $\mathcal{E} = \mathcal{E}_1$, $\mathcal{G} = \mathcal{G}_1^{\mathbb{R}}$ in what follows. Looking at the proof in [BP93] for the $\limsup$ condition of Γ-convergence from $\mathcal{W}_\varepsilon^{\mathrm{AC}}$ to the Willmore energy we are interested in information about one-dimensional minimizing profiles of $\mathcal{E}$ with limit behavior as in $\mathcal{M}$.

Before we turn to the theorem on the one-dimensional profiles we prove that an auxiliary function has the properties of a double-well potential on $[-1,1]$.

Lemma 3.1.2 (Modified double-well potential)**.**
Let W, f, U be as in Assumptions 3.1.1. Then the function

$$W_* : U \longrightarrow \mathbb{R}, \quad W_*(r) := \frac{1}{4} W'(f^{-1}(r))^2 + W(f^{-1}(r))$$

has the following properties

- $W_* \in C^m(U)$ *and* $W_* \geq 0$,
- $\{W_* = 0\} = \{\pm 1\}$,
- $W_*'(r) = W'(f^{-1}(r)) = 2(r - f^{-1}(r))$,
- $\sqrt{W_*}$ *is Lipschitz and* $\left|\partial\sqrt{W_*(r)}\right| \leq 1$.

In particular W_* can be considered as the restriction of a double-well potential $\hat{W} : \mathbb{R} \longrightarrow \mathbb{R}$ to U.

Proof. $W_* \geq 0$ is clear and from the regularity of W we have $W_* \in C^{m-1}(U)$. To determine $\{W_* = 0\}$, we calculate for $r \in U$

$$\begin{aligned} W_*(r) = 0 &\Longleftrightarrow W'(f^{-1}(r)) = 0 \quad \text{and} \quad W'(f^{-1}(r)) = 0 \\ &\Longleftrightarrow f^{-1}(r) = \pm 1 \Longleftrightarrow r = \pm 1. \end{aligned}$$

Let $r \in U$, then we have for the derivative

$$\begin{aligned}
W_*'(r) &= \frac{1}{2} W'(f^{-1}(r)) \cdot W''(f^{-1}(r)) \cdot (f^{-1})'(r) + W'(f^{-1}(r)) \cdot (f^{-1})'(r) \\
&= W'(f^{-1}(r)) \cdot (f^{-1})'(r) \cdot \left(1 + \frac{1}{2} W''(f^{-1}(r))\right) \\
&= W'(f^{-1}(r)) \cdot (f^{-1})'(r) \cdot f'(f^{-1}(r)) = W'(f^{-1}(r)) = 2(r - f^{-1}(r)),
\end{aligned}$$

which implies $W_*' \in C^{m-1}(U)$ hence $W_* \in C^m(U)$. At last we prove that $\sqrt{W_*}$ is Lipschitz. We have for any $r \in U$

$$\left|\partial \sqrt{W_*(r)}\right| = \frac{|W_*'(r)|}{2\sqrt{W_*(r)}} \leq \frac{|W'(f^{-1}(r))|}{2 \cdot \frac{1}{2}|W'(f^{-1}(r))|} \leq 1. \qquad \square$$

Theorem 3.1.3 (Optimal profile)**.**
Let W be as in Remark 3.1.1. Every minimizer of $\mathcal{E}$ lies in $C^{m-1}(\mathbb{R})$. There exists a unique minimizer q_0 of $\mathcal{E}$ that satisfies $q_0(0) = 0$. This minimizer is determined by

$$q_0 = f^{-1}(\overline{q}_0), \tag{3.1.2}$$

where $\overline{q}_0 \in C^{m+1}(\mathbb{R})$ is the unique solution to

$$\overline{q}_0' = \sqrt{W_*(\overline{q}_0)} \quad \textit{with} \quad \overline{q}_0(0) = f(0). \tag{3.1.3}$$

We also have

- $1 < q_0, \overline{q}_0 < 1$,
- $q_0' > 0$, $\overline{q}_0' > 0$,
- $\lim_{r \to \pm\infty} q_0(r) = \pm 1 = \lim_{r \to \pm\infty} \overline{q}_0(r)$

and

$$-\overline{q}_0'' + \overline{q}_0 = q_0 \quad \textit{in} \quad \mathbb{R}. \tag{3.1.4}$$

Remark. *We use the simpler first order ODE* (3.1.3) *to define $\overline{q}_0$ and show that it solves equation* (3.1.4) *instead of the other way around, even though it would be more natural to use* (3.1.4) *as definition.*
We also remark that a priori we consider functions with values in $\mathbb{R}$ instead of $(-1, 1)$. In the proof below that we can also restrict the minimization to functions with values in $[-1, 1]$ and obtain that the optimal profile takes it values only in $(-1, 1)$. Since the diffuse Willmore flow is of fourth order and does not satisfy a maximum principle, we cannot guarantee that evolutions take values only in $(-1, 1)$. In particular, the behavior of the double-well potential on $\mathbb{R}$ matters for the analysis below.

Proof of Theorem 3.1.3. We start with $(u, v) \in L^\infty(\mathbb{R}) \times H^1_{\mathrm{loc}}(\mathbb{R})$ and assume that (u, v) is a minimizer of $\mathcal{G}$ thus u is a minimizer of $\mathcal{E}$. In the following we deduce necessary properties of u and v which we can then use to prove that these minimizers exist and satisfy the claims of the theorem.

First we can assume that the integral is finite. This implies $u \in L^4(\mathbb{R})$ and $v \in H^1(\mathbb{R})$. Next we restrict the function values of u, v to $[-1,1]$. We write $P_{[-1,1]} : \mathbb{R} \longrightarrow [-1,1]$ for the projection

$$P_{[-1,1]}(r) := \begin{cases} \operatorname{sgn}(r) & \text{if} \quad |r| \geq 1, \\ r & \text{if} \quad |r| < 1. \end{cases}$$

For $v \in H^1(\mathbb{R})$ we have from standard theory $P_{[-1,1]}v \in H^1(\mathbb{R})$ and

$$\nabla P_{[-1,1]}v = \chi_{\{|v|<1\}} \nabla v, \quad |P_{[-1,1]}u - P_{[-1,1]}v| \leq |u - v| \quad \text{and} \quad W(P_{[-1,1]}u) \leq W(u).$$

It follows

$$\mathcal{G}(P_{[-1,1]}u, P_{[-1,1]}v) \leq \mathcal{G}(u, v)$$

which implies $-1 \leq u, v \leq 1$.

Next we are looking for a formula to calculate u from v and vice versa. Let $x \in \mathbb{R}$ and $v(x) \in [-1,1]$, we minimize $(u(x) - v(x))^2 + W(u(x))$ pointwise in $u(x) \in [-1,1]$. The existence of a minimizer is guaranteed because $[-1,1]$ is compact and the considered function is continuous, it is even C^2. If the minimizing $u(x)$ lies in $(-1,1)$ then the choice $u(x) = f^{-1}(v(x))$ is optimal because of

$$\begin{aligned} 0 &\overset{!}{=} \partial_{u(x)}\Big((u(x) - v(x))^2 + W(u(x))\Big) = 2(u(x) - v(x)) + W'(u(x)) \\ \Longleftrightarrow v(x) &= u(x) + \frac{1}{2}W'(u(x)) = f(u(x)) \end{aligned}$$

and

$$\partial^2_{u(x)}\Big((u(x) - v(x))^2 + W(u(x))\Big) = 2 + W''(u(x)) = 2f'(u(x)) > 0.$$

The analysis of the boundary shows that $u(x) = \pm 1$ can only be optimal when $v(x) = \pm 1$, but this is also covered by $u(x) = f^{-1}(v(x))$. It follows $u = f^{-1}(v)$.

Plugging $u = f^{-1}(v)$ hence $v - f^{-1}(v) = \frac{1}{2}W'(u)$ into the energy leads to

$$\begin{aligned} \mathcal{E}(f^{-1}(v)) &= \frac{1}{2}\int_{\mathbb{R}} ((v')^2 + (f^{-1}(v) - v)^2 + W(f^{-1}(v)))\, \mathrm{d}\mathcal{L}^1 \\ &= \frac{1}{2}\int_{\mathbb{R}} ((v')^2 + \frac{1}{4}W'(f^{-1}(v))^2 + W(f^{-1}(v)))\, \mathrm{d}\mathcal{L}^1 \\ &= \frac{1}{2}\int_{\mathbb{R}} ((v')^2 + W_*(v))\, \mathrm{d}\mathcal{L}^1. \end{aligned}$$

Knowing that W_* is a double-well potential we can proceed in the following as in section 3a of [Alb00]: we estimate with the so-called Modica-Mortola trick with the Young-inequality

$$\mathcal{E}(f^{-1}(v)) \geq \int_{\mathbb{R}} v' \sqrt{W_*(v)}\, \mathrm{d}\mathcal{L}^1.$$

In the Young-inequality there is equality if and only if $v' = \pm\sqrt{W_*(v)}$. The condition $\lim_{r\to\pm\infty} v(r) = \pm 1$ in the definition of $\mathcal{E}$ can not be satisfied by a decreasing function which implies

$$v' = \sqrt{W_*(v)}. \tag{3.1.5}$$

Since $\sqrt{W_*}$ is Lipschitz on $U \supseteq [-1,1]$ we get that for any initial value $v_0 \in U$ there exists a unique the solution to the inital value problem (3.1.5) by the Picard-Lindelöf Theorem. Furthermore we get from the Lipschitz condition that every solution to an initial value problem of (3.1.5) with $v_0 \in U$ can be extended to a function on $\mathbb{R}$ by Theorem 2.5.6 in [Aul04]. Thus we can always the initial value $v(0) = v_0$. If $v_0 = \pm 1$ then the solution is constant because of $W_*(\pm 1) = 0$. By the uniqueness of the solution we get $|v| < 1$ if $|v_0| < 1$. Since $u = f^{-1}(v)$ and $u \in \mathcal{M}$ it follows $|v(0)| < 1$.

Since v is increasing and bounded we have the existence of $\lim_{r\to\infty} v(r) = \sup_{r\in\mathbb{R}} v(r)$. From the ODE it follows that $\lim_{r\to\infty} v'(r)$ exists. This has to be 0, because otherwise $v(r)$ could not be bounded as $r \to \infty$. From (3.1.5) we can conclude that $\lim_{r\to\infty} v(r) = 1$, since W_* has no other zeroes in $(-1,1]$, $v_0 \in (-1,1)$, and v is strictly increasing. The same argument can be applied for $r \to -\infty$. For u this implies $\lim_{r\to\pm\infty} u(r) = \lim_{r\to\pm 1} f^{-1}(v(r)) = \pm 1$.

This also shows that the initial value for v can not satisfy $|v_0| > 1$ because this solution satisfies $|v| > 1$ by the uniqueness of the solution. Since $W_* \in C^m(U)$ we get from standard regularity theory for ODE's $v \in C^{m+1}(\mathbb{R})$ and the respective minimizer $u = f^{-1}(v) \in C^{m-1}(\mathbb{R})$.

We conclude that any minimizer u of $\mathcal{E}$ and thus any minimizer (u,v) of $\mathcal{G}$ is characterized by $v = f(u)$ and v is a solution to (3.1.5) with $|v_0| < 1$. From $\lim_{r\to\pm\infty} v(r) = \pm 1$ and the uniqueness of the solution to the initial value problem we get, that all of the minimizers are the same up to a shift of the argument.

Now we consider the additional condition $u(0) = 0$. This translates into the condition $v(0) = f(0) \in (-1,1)$ for v. By the previous argumentation the existence of the unique minimizer with $u(0) = 0$ is guaranteed by the Picard-Lindelöf Theorem and we denote these functions with $q_0 := u$ and $\overline{q}_0 := v$. We also have $q_0' = (f^{-1})'(\overline{q}_0)\overline{q}_0' > 0$ and thus q_0 is strictly increasing as well.

Now we can verify that q_0 and $\overline{q}_0$ solve (3.1.4). Owing to $\overline{q}_0' > 0$ we get

$$(\overline{q}_0')^2 = W_*(\overline{q}_0), \quad \text{thus} \quad 2\overline{q}_0'\overline{q}_0'' = W_*'(\overline{q}_0)\overline{q}_0' \quad \text{and} \quad \overline{q}_0'' = \frac{1}{2}W_*'(\overline{q}_0) = \overline{q}_0 - f^{-1}(\overline{q}_0).$$

This yields

$$-\overline{q}_0'' + \overline{q}_0 = q_0. \qquad \square$$

Next we prove even better decay for the profile functions $q_0, \overline{q}_0$, and their derivatives as $r \to \pm\infty$.

Lemma 3.1.4 (Exponential decay of $q_0, \overline{q}_0$, and derivatives)**.**
There exist $R, C, c > 0$ such that for all $|r| > R$ we have

$$\begin{aligned} 1 - e^{-c|r|} \le |q_0(r)| &< 1, & 1 - e^{-c|r|} \le |\overline{q}_0(r)| &< 1, \\ 0 < q_0'(r) &\le Ce^{-c|r|}, & 0 \le \overline{q}_0'(r) &\le Ce^{-c|r|}, \\ |q_0''(r)| &\le Ce^{-c|r|}, \quad \textit{and} & |\overline{q}_0''(r)| &\le e^{-c|r|}. \end{aligned}$$

It follows $q_0 - \operatorname{sgn}, \overline{q}_0 - \operatorname{sgn} \in L^2(\mathbb{R})$, and $q_0', \overline{q}_0' \in H^1(\mathbb{R})$.

Proof. Since $W_*(1) = W_*'(1) = 0$ and $W_*''(1) > 0$ we can Taylor-expand W_* as follows, defining $0 < c < \frac{1}{2}$ such that $2c^2 := \frac{W_*''(1)}{2} = \frac{\frac{1}{2}W''(1)}{1+\frac{1}{2}W''(1)}$

$$\overline{q}_0' = \sqrt{2c^2(1-\overline{q}_0)^2 + \mathcal{O}((1-\overline{q}_0)^3)} \quad (\overline{q}_0 \to 1). \tag{3.1.6}$$

There exists $\tau > 0$ such that $1 - \tau \le \overline{q}_0 \le 1$ implies $|\mathcal{O}((1-\overline{q}_0)^3)| \le c^2(1-\overline{q}_0)^2$. We chose $R_1 > 0$ such that for all $r > R_1$ we have $1 - \tau \le \overline{q}_0(r) \le 1$ and estimate

$$\begin{aligned} (\overline{q}_0(r) - 1)' &\ge \sqrt{c^2(1-\overline{q}_0(r))^2} \quad && \text{for} \quad r > R_1 \\ \implies (1 - \overline{q}_0(r))' &\le -c(1-\overline{q}_0(r)) && \text{for} \quad r > R_1. \end{aligned}$$

Now we can use the Gronwall Lemma and $\lim_{r\to\infty} \overline{q}_0(r) = 1$ to obtain for $r > R_1$

$$(1-\overline{q}_0)(r) \le e^{-cr} \quad \text{and thus} \quad \overline{q}_0(r) \ge 1 - e^{-cr}.$$

Since $\overline{q}_0(r) < 1$ we have the claimed exponential convergence. Plugging this into (3.1.6) we find for $r > R_1$

$$0 < \overline{q}_0'(r) \le \sqrt{4c^2(1-\overline{q}_0(r))^2} = 2c(1-\overline{q}_0(r)) \le 2ce^{-cr}.$$

We can transfer this decay to q_0. Let $r > R_1$, owing to the monotonicity of f we get

$$1 > q_0(r) = f^{-1}(\overline{q}_0(r)) \ge f^{-1}(1 - e^{-cr})$$

Now we Taylor-expand f^{-1}

$$f^{-1}(1-e^{-cr}) = 1 - (f^{-1})'(1)e^{-cr} + \mathcal{O}(e^{-2cr}).$$

Since $(f^{-1})'(1) < 1$ by $(f^{-1})'(1) = \frac{1}{1+\frac{1}{2}W''(1)}$ and $W''(1) > 0$ we can find $R > R_1$ such that for $r > R$ we have

$$1 > q_0(r) \ge 1 - e^{-cr}.$$

For $\overline{q}_0''$ we use (3.1.4) and get for $r > R$

$$\overline{q}_0'' = \overline{q}_0 - q_0 \begin{cases} \le 1 - (1 - e^{-cr}) = e^{-cr} \\ \ge 1 - e^{-cr} - 1 = -e^{-cr} \end{cases} \quad \text{and thus} \quad |\overline{q}_0''| \le e^{-cr}.$$

For q_0' we calculate for $r > R_1$

$$0 < q_0'(r) = (f^{-1})'(\overline{q}_0(r))\overline{q}_0' \le \frac{2c}{\gamma}e^{-cr}.$$

At last we estimate for $r > R$ using (3.1.1)

$$\begin{aligned} |q_0''(r)| &\le |(f^{-1})''(\overline{q}_0(r))||\overline{q}_0'(r)|^2 + |(f^{-1})'(\overline{q}_0(r))\overline{q}_0''(r)| \\ &\le \frac{2c\|W'''\|_{C^0[-1,1]}}{\gamma^3} e^{-2cr} + \frac{1}{\gamma} e^{-cr} \le C(W) e^{-cr}. \end{aligned}$$

The estimates can be done the same way for $r \to -\infty$. $\square$

The following two constants will be relevant for our approximations.

Corollary 3.1.5 (Double-well potential depending constants)**.**
The constants

$$c_{\mathrm{AG}} \coloneqq \min_{\mathcal{M}} \mathcal{E} = \int_{\mathbb{R}} |\overline{q}_0'|^2 \,\mathrm{d}\mathcal{L}^1 \quad \textit{and} \quad \sigma_{\mathrm{AG}} \coloneqq \frac{c_{\mathrm{AG}}}{\|q_0'\|^2_{L^2(\mathbb{R})}}, \tag{3.1.7}$$

are in terms of the double-well potential characterized by

$$c_{\mathrm{AG}} = \int_{-1}^{1} \Big(1 + \frac{1}{2} W''\Big) \sqrt{W + \frac{1}{4}(W')^2} \,\mathrm{d}\mathcal{L}^1 \quad \textit{and} \quad \frac{c_{\mathrm{AG}}}{\sigma_{\mathrm{AG}}} = \int_{-1}^{1} \frac{\sqrt{W + \frac{1}{4}(W')^2}}{1 + \frac{1}{2}W''} \,\mathrm{d}\mathcal{L}^1. \tag{3.1.8}$$

Proof. We get equation (3.1.8) from

$$\|\overline{q}_0'\|^2_{L^2(\mathbb{R})} = \int_{-1}^{1} W_*^{\frac{1}{2}} \,\mathrm{d}\mathcal{L}^1 = \int_{-1}^{1} f' \sqrt{W + \frac{1}{4}(W')^2} \,\mathrm{d}\mathcal{L}^1$$

and

$$\|q_0'\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \frac{|\overline{q}_0'|^2}{|f' \circ f^{-1}|^2} \,\mathrm{d}\mathcal{L}^1 = \int_{-1}^{1} \frac{W_*^{\frac{1}{2}}}{f'} \,\mathrm{d}\mathcal{L}^1 = \int_{-1}^{1} \frac{\sqrt{W + \frac{1}{4}(W')^2}}{1 + \frac{1}{2}W''} \,\mathrm{d}\mathcal{L}^1. \qquad \square$$

For a special choice of W the constants are calculated in the appendix; see Section 8.3. Motivated by the equations occurring in the minimizing of the one-dimensional energy we are looking for a solution operator with "good" properties. Given a function u we solve for v with

$$-v'' + v = u \quad \text{with} \quad v'(\pm\infty) = 0$$

in a suitable sense. A possible ansatz would be to consider $u \in L^2(\mathbb{R})$ and to use the Lax-Milgram Theorem to find solutions in $H^1(\mathbb{R})$ or even $H^2(\mathbb{R})$. However the functions we considered before are not in $L^2(\mathbb{R})$ as they approach ± 1 at $\pm\infty$. So we need a solution operator that works on $L^\infty(\mathbb{R})$ as well. We get this by considering the Green's function of the ODE and the induced convolution operator. This leads to the following definition.

Definition 3.1.6 (Solution operator on $\mathbb{R}$)**.**
We define $\boldsymbol{A}_0 : L^2(\mathbb{R}) + L^\infty(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) + L^\infty(\mathbb{R})$,

$$\boldsymbol{A}_0 w \coloneqq \boldsymbol{A}_0(w) \coloneqq J_1 * w. \tag{3.1.9}$$

where $J_1 : \mathbb{R} \longrightarrow \mathbb{R}$ *is the Green's function of* $-\partial^2 + \mathrm{Id}$ *in* $\mathbb{R}$

$$J_1(r) \coloneqq \frac{1}{2} e^{-|r|} \quad \textit{for} \quad r \in \mathbb{R};$$

see Theorem 6.23 in [LL01].

This is well-defined for $w \in L^2(\mathbb{R}) + L^\infty(\mathbb{R})$ and restricted to $L^2(\mathbb{R})$ this coincides with the operator constructed from Lax-Milgram, as we will prove in Proposition 3.1.8. Before we can prove properties of $\mathbf{A}_0$ we collect a few properties of J_1.

Lemma 3.1.7 (Properties of J_1)**.**
The function $J_1 : \mathbb{R} \longrightarrow \mathbb{R}$ with $J_1(r) = \frac{1}{2}e^{-|r|}$ has the following properties

- $J_1 \in C_b^0(\mathbb{R})$ *and* $J_1 \geq 0$,
- $J_1 \in W^{1,1}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$,
- $\|J_1\|_{L^1(\mathbb{R})} = 1 = \|J_1'\|_{L^1(\mathbb{R})}$ *and* $\|J_1\|_{L^2(\mathbb{R})} = \frac{1}{2} = \|J_1'\|_{L^2(\mathbb{R})}$.

Proof. $J_1 \geq 0$ and $J_1 \in C_b^0(\mathbb{R})$ is clear. We calculate the weak derivative. Let $\psi \in C_c^\infty(\mathbb{R})$ then we have

$$\begin{aligned}
-\int_{\mathbb{R}} \psi' J_1 \,\mathrm{d}\mathcal{L}^1 &= -\frac{1}{2}\int_{-\infty}^0 \psi'(r)e^r \,\mathrm{d}r - \frac{1}{2}\int_0^\infty \psi'(r)e^{-r} \,\mathrm{d}r \\
&= -\frac{1}{2}\psi(0) + \frac{1}{2}\int_{-\infty}^0 \psi(r)e^r \,\mathrm{d}r + \frac{1}{2}\psi(0) - \frac{1}{2}\int_0^\infty \psi(r)e^{-r} \,\mathrm{d}r \\
&= \int_{\mathbb{R}} \psi(r)\frac{1}{2}\operatorname{sgn}(-r)e^{-|r|} \,\mathrm{d}r.
\end{aligned}$$

For $r \in \mathbb{R}$ we conclude $J_1'(r) = \frac{1}{2}\operatorname{sgn}(-r)e^{-|r|}$ in the weak sense. We have $|J_1'| = J_1$ and $J_1, J_1' \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ from the exponential decay, which implies the second claim. Since $|J_1'| = J_1$ it suffices for the last claim to calculate $\|J_1\|_{L^1(\mathbb{R})}$ and $\|J_1\|_{L^2(\mathbb{R})}$. We have

$$\|J_1\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |J_1(r)| \,\mathrm{d}r = \int_{\mathbb{R}} \frac{1}{2}e^{-|r|} \,\mathrm{d}r = \int_0^\infty e^{-r} \,\mathrm{d}r = 1$$

and

$$\|J_1\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} |J_1(r)|^2 \,\mathrm{d}r = \int_{\mathbb{R}} \frac{1}{4}e^{-2|r|} \,\mathrm{d}r = \frac{1}{2}\int_0^\infty e^{-2r} \,\mathrm{d}r = \frac{1}{4}. \qquad \square$$

Proposition 3.1.8 (One-dimensional solution operator)**.**
The operator $\boldsymbol{A}_0 : L^2(\mathbb{R}) + L^\infty(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) + L^\infty(\mathbb{R})$ satisfies

(1) $\boldsymbol{A}_0(L^2(\mathbb{R})) \subseteq H^2(\mathbb{R}) \cap C_b^1(\mathbb{R})$, $\boldsymbol{A}_0(L^\infty(\mathbb{R})) \subseteq W^{2,\infty}(\mathbb{R}) \cap C_b^1(\mathbb{R})$ *and* $\boldsymbol{A}_0(C_b^0(\mathbb{R})) \subseteq C_b^2(\mathbb{R})$.

(2) If $u \in L^2(\mathbb{R})$ then $\boldsymbol{A}_0(u)$ is the unique solution to

$$(-\partial^2 + \mathrm{Id})\boldsymbol{A}_0 u = u \quad \text{a.e. in } \mathbb{R} \text{ and in the weak sense.}$$

(3) $\boldsymbol{A}_0|_{L^2(\mathbb{R})} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ *is linear, continuous, self-adjoint, and positive.*

(4) $\boldsymbol{A}_0|_{L^\infty(\mathbb{R})} : L^\infty(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$ *is linear, continuous and if $\lim_{r\to\infty} u(r) = y$ for some $u \in L^\infty(\mathbb{R})$ and some $y \in \mathbb{R}$ then we have $\lim_{r\to\infty} \boldsymbol{A}_0 u(r) = y$. The same is true for $r \to -\infty$.*

(5) If $h \in C_b^1(\mathbb{R})$ we have $\partial \boldsymbol{A}_0 h = \boldsymbol{A}_0 h'$.

Proof. We start with proving (1) and (2). Given $u \in L^2(\mathbb{R})$ we can find a unique solution $v \in H^1(\mathbb{R})$ using the Lax-Milgram Theorem such that

$$\forall \psi \in H^1(\mathbb{R}) : \int_{\mathbb{R}} \psi' v' \,\mathrm{d}\mathcal{L}^1 + \int_{\mathbb{R}} \psi v \,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} \psi u \,\mathrm{d}\mathcal{L}^1. \tag{3.1.10}$$

From $v'' = v - u \in L^2(\mathbb{R})$ and we get $v \in H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$. Furthermore we have $J_1 * u \in H^2(\mathbb{R})$ from Theorem 5.18 in [Kab14]. Since J_1 is the Green's function of the ODE can conclude with to the uniqueness of the solution $v = \mathbf{A}_0 u$. We get with the standard properties of convolutions and Lemma 3.1.7

$$\begin{aligned} \mathbf{A}_0 u &= J_1 * u, \quad |\mathbf{A}_0 u(r)| \le \|J_1\|_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R})} = \frac{1}{4} \|u\|_{L^2(\mathbb{R})} \quad \text{for all} \quad r \in \mathbb{R}, \\ \mathbf{A}_0 u' &= J_1' * u, \quad |\mathbf{A}_0 u'(r)| \le \|J_1'\|_{L^2(\mathbb{R})} \|u\|_{L^2(\mathbb{R})} = \frac{1}{4} \|u\|_{L^2(\mathbb{R})} \quad \text{for all} \quad r \in \mathbb{R} \end{aligned}$$

and thus $\mathbf{A}_0 u \in C_b^1(\mathbb{R})$.

$\mathbf{A}_0(L^\infty(\mathbb{R})) \subseteq W^{2,\infty}(\mathbb{R})$ is shown in [LL01]. To show $\mathbf{A}_0(L^\infty(\mathbb{R})) \subseteq C_b^1(\mathbb{R})$ we estimate similar as before. Let $u \in L^\infty(\mathbb{R})$ then we have

$$\begin{aligned} \mathbf{A}_0 u &= J_1 * u, \quad |\mathbf{A}_0 u(r)| \le \|J_1\|_{L^1(\mathbb{R})} \|u\|_{L^\infty(\mathbb{R})} = \|u\|_{L^\infty(\mathbb{R})} \quad \text{for all} \quad r \in \mathbb{R}, \\ \mathbf{A}_0 u' &= J_1' * u, \quad |\mathbf{A}_0 u'(r)| \le \|J_1'\|_{L^1(\mathbb{R})} \|u\|_{L^\infty(\mathbb{R})} = \|u\|_{L^\infty(\mathbb{R})} \quad \text{for all} \quad r \in \mathbb{R} \end{aligned}$$

and thus $\mathbf{A}_0 u \in C_b^1(\mathbb{R})$.

If $u \in C_b^0(\mathbb{R})$ then we get $\mathbf{A}_0(u) \in C^2(\mathbb{R})$ from standard ODE regularity theory by considering the ODE locally. $\mathbf{A}_0 u, \mathbf{A}_0 u' \in L^\infty(\mathbb{R})$ follow from the previous case. $\mathbf{A}_0 u'' \in L^\infty(\mathbb{R})$ follows from the ODE, in fact we have $\mathbf{A}_0 u'' = \mathbf{A}_0 u - u \in C_b^0(\mathbb{R})$.

For (3) and (4) the linearity is clear. To estimate the norm we test the ODE with the solution v itself. With a Young estimate and a partial integration we get

$$\begin{aligned} \int_{\mathbb{R}} |\mathbf{A}_0 u'|^2 \,\mathrm{d}\mathcal{L}^1 + \int_{\mathbb{R}} |\mathbf{A}_0 u|^2 \,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} u \mathbf{A}_0 u \,\mathrm{d}\mathcal{L}^1 \le \frac{1}{2} \int_{\mathbb{R}} |u|^2 \,\mathrm{d}\mathcal{L}^1 + \frac{1}{2} \int_{\mathbb{R}} |\mathbf{A}_0 u|^2 \,\mathrm{d}\mathcal{L}^1 \\ \Longrightarrow \|\mathbf{A}_0 u\|_{L^2(\mathbb{R})} \le \|u\|_{L^2(\mathbb{R})}. \end{aligned}$$

Thus $\mathbf{A}_0 : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is linear and continuous. From (3.1.10) we get that it is also self-adjoint and positive.

If u satisfies $\lim_{r\to\infty} u(r) = y$ for some $y \in \mathbb{R}$ we get with the Dominated Convergence Theorem

$$\lim_{r\to\infty} \mathbf{A}_0 u(r) = \lim_{r\to\infty} \int_{\mathbb{R}} J_1(s) u(r-s) \,\mathrm{d}s = y \int_{\mathbb{R}} J_1(s) \,\mathrm{d}s = y.$$

Same for $r \to -\infty$.
(5) follows immediately from standard theory about parameter integrals. $\square$

Next we prove that if u has exponential decay at $\pm\infty$, so does $\mathbf{A}_0(u)$.

Lemma 3.1.9 (Further properties of $\mathbf{A}_0$).
(a) Let $u \in L^\infty(\mathbb{R})$ and assume there exist $C, c > 0$ such that

$$|u(r)| \leq Ce^{-c|r|} \quad \textit{for all} \quad r \in \mathbb{R}, \tag{3.1.11}$$

then there exist $C_1, c_1 > 0$ with

$$|\boldsymbol{A}_0 u(r)| \leq C_1 e^{-c_1|r|} \quad \textit{for all} \quad r \in \mathbb{R}.$$

(b) Let $u \in L^2(\mathbb{R})$ be uniformly continuous, then

$$\lim_{r \to \pm\infty} u(r) = 0.$$

In particular we get that for $v \in L^\infty(\mathbb{R})$

$$\lim_{r \to \pm\infty} \boldsymbol{A}_0 v(r) = 0$$

and for $v \in C_b^0(\mathbb{R})$

$$\lim_{r \to \pm\infty} \boldsymbol{A}_0 v(r) = 0 = \lim_{r \to \pm\infty} \boldsymbol{A}_0 v'(r).$$

(c) Let $a_\pm \in \mathbb{R}$ and define we define $\tilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\tilde{\alpha} := a_- \chi_{(-\infty,0)} + a_+ \chi_{[0,\infty)}$$

Then we have $\alpha := \boldsymbol{A}_0 \tilde{\alpha} \in C_b^1(\mathbb{R})$ and

$$\alpha = a_{\mathrm{sgn}(-r)} \frac{e^{-|r|}}{2} + a_{\mathrm{sgn}(r)} \Big(1 - \frac{1}{2} e^{-|r|}\Big) \quad \textit{for} \quad r \neq 0$$

and $\alpha(0) = \frac{a_- + a_+}{2}$.

Proof. (a) Without loss of generality we can assume $C = 2$ and $c = 1$, otherwise we consider the function $\widetilde{u}(r) := \frac{2}{C} u(\frac{r}{c})$. We analyse the behavior for $r > 0$, by definition we have

$$\begin{aligned}
|\mathbf{A}_0 u(r)| &\leq \int_{\mathbb{R}} J_1(r-s)|u(s)| \, \mathrm{d}s \leq \int_{\mathbb{R}} \frac{1}{2} e^{-|r-s|} 2e^{-|s|} \, \mathrm{d}s \\
&= \int_{-\infty}^0 e^{-(r-s)} e^s \, \mathrm{d}s + \int_0^r e^{-(r-s)} e^{-s} \, \mathrm{d}s + \int_r^\infty e^{-(s-r)} e^{-s} \, \mathrm{d}s \\
&= e^{-r} \int_{-\infty}^0 e^{2s} \, \mathrm{d}s + re^{-r} + e^r \int_r^\infty e^{-2s} \, \mathrm{d}s \\
&= \frac{1}{2} e^{-r} + \underbrace{re^{-\frac{1}{2}r}}_{\leq \frac{2}{e} \leq 1} e^{-\frac{1}{2}r} + \frac{1}{2} e^{-r} \leq 2e^{-\frac{1}{2}r}.
\end{aligned}$$

The same can be done for $r < 0$.

(b) Assume not $\lim_{r \to \infty} u(r) = 0$. Then there exist $\varepsilon > 0$ and a sequence $(x_k)_{k \in \mathbb{N}}$ in $\mathbb{R}$ with $x_k \to \infty$ for all $k \in \mathbb{N}$ such that

$$|u(x_k)| \geq \varepsilon \quad \text{for all} \quad k \in \mathbb{N}.$$

Since $x_k \nearrow \infty$ we can choose a subsequence $(x_{k_j})_{j\in\mathbb{N}}$ such that for all $j \in \mathbb{N}$: $x_{k_{j+1}} > x_{k_j}+2$. u is uniformly continuous thus there exists $0 < \delta < 1$ such that

$$|u(r)| \geq \frac{\varepsilon}{2} \quad \text{for all} \quad r \in (x_{k_j} - 2\delta, x_{k_j} + 2\delta) \quad \text{and all} \quad j \in \mathbb{N}.$$

This leads to a contradiction because of

$$\infty > \int_{\mathbb{R}} |u|^2 \, \mathrm{d}\mathcal{L}^1 \geq \sum_{j=1}^{\infty} \int_{x_{k_j}-2\delta}^{x_{k_j}+2\delta} |u|^2 \, \mathrm{d}\mathcal{L}^1 \geq \sum_{j=1}^{\infty} \delta\varepsilon^2.$$

We prove the last remarks. If $v \in L^\infty(\mathbb{R})$ then $\mathbf{A}_0 v \in C_b^1(\mathbb{R})$ by Proposition 3.1.8. The Mean Value Theorem implies $\mathbf{A}_0 v$ is Lipschitz and hence uniformly continuous. If $v \in C_b^0(\mathbb{R})$ then $\mathbf{A}_0 v \in C_b^2(\mathbb{R})$ by Proposition 3.1.8 and thus $\mathbf{A}_0 v, \mathbf{A}_0 v' \in C_b^1(\mathbb{R})$ and the claim follows.

For the proof of (c) we calculate the convolution. Since $\tilde{\alpha} \in L^\infty(\mathbb{R})$ we have $\alpha \in C_b^1(\mathbb{R})$ by Proposition 3.1.8. We get for $r > 0$

$$\begin{aligned}
\alpha(r) &= \frac{1}{2}\int_{\mathbb{R}} e^{-|r-s|}\tilde{\alpha}(s)\,\mathrm{d}s = \frac{a_-}{2}\int_{-\infty}^{0} e^{-|r-s|}\,\mathrm{d}s + \frac{a_+}{2}\int_{0}^{\infty} e^{-|r-s|}\,\mathrm{d}s \\
&= \frac{a_-}{2}\int_{-\infty}^{0} e^{s-r}\,\mathrm{d}s + \frac{a_+}{2}\int_{0}^{r} e^{s-r}\,\mathrm{d}s + \frac{a_+}{2}\int_{r}^{\infty} e^{r-s}\,\mathrm{d}s \\
&= \frac{a_-}{2} e^{s-r}\Big|_{-\infty}^{0} + \frac{a_+}{2} e^{s-r}\Big|_{0}^{r} - \frac{a_+}{2} e^{r-s}\Big|_{r}^{\infty} \\
&= \frac{a_-}{2} e^{-r} + \frac{a_+}{2}(1 - e^{-r}) + \frac{a_+}{2} = a_- \frac{e^{-r}}{2} + a_+\Big(1 - \frac{1}{2}e^{-r}\Big)
\end{aligned}$$

We calculate similar for $r < 0$

$$\begin{aligned}
\alpha(r) &= \frac{1}{2}\int_{\mathbb{R}} e^{-|r-s|}\tilde{\alpha}(s)\,\mathrm{d}s = \frac{a_-}{2}\int_{-\infty}^{0} e^{-|r-s|}\,\mathrm{d}s + \frac{a_+}{2}\int_{0}^{\infty} e^{-|r-s|}\,\mathrm{d}s \\
&= \frac{a_-}{2}\int_{-\infty}^{r} e^{s-r}\,\mathrm{d}s + \frac{a_-}{2}\int_{r}^{0} e^{r-s}\,\mathrm{d}s + \frac{a_+}{2}\int_{0}^{\infty} e^{r-s}\,\mathrm{d}s \\
&= \frac{a_-}{2} e^{s-r}\Big|_{-\infty}^{r} - \frac{a_-}{2} e^{r-s}\Big|_{r}^{0} - \frac{a_+}{2} e^{r-s}\Big|_{0}^{\infty} \\
&= \frac{a_-}{2} + \frac{a_-}{2}(1 - e^{r}) + \frac{a_+}{2} e^{r} = a_-\Big(1 - \frac{1}{2}e^{r}\Big) + \frac{a_+}{2} e^{r}.
\end{aligned}$$

Hence the claimed representation follows. □

We also need the solution operator on open sets of $\mathbb{R}^n$.

Lemma 3.1.10 (Solution operator on bounded open sets)**.**
Given $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ open, bounded with Lipschitz-boundary, $u \in H^1(\Omega)'$ and $\varepsilon > 0$ there exists a unique solution $\mathcal{A}_\varepsilon u := \overline{u}_\varepsilon \in H^1(\Omega)$ such that

$$-\varepsilon^2 \Delta \overline{u}_\varepsilon + \overline{u}_\varepsilon = u \quad \text{in} \quad \Omega \tag{3.1.12}$$

$$\partial_\nu \overline{u}_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \tag{3.1.13}$$

in the weak sense. The operator $\mathcal{A}_\varepsilon : H^1(\Omega)' \longrightarrow H^1(\Omega)$ is linear and continuous.

If Ω has C^2-boundary and $u \in L^2(\Omega)$ then we get $\mathcal{A}_\varepsilon u \in H^2(\Omega)$. As $H^2(\Omega) \hookrightarrow L^2(\Omega)$ we can also consider $\mathcal{A}_\varepsilon$ as an operator on $L^2(\Omega)$ without introducing a different notation. $\mathcal{A}_\varepsilon : L^2(\Omega) \longrightarrow L^2(\Omega)$ is linear, continuous, self-adjoint, and positive. We have for all $\psi, \phi \in H^1(\Omega)'$

$$\langle \mathcal{A}_\varepsilon \phi, \psi \rangle_{H^1(\Omega)'} = \langle \mathcal{A}_\varepsilon \psi, \phi \rangle_{H^1(\Omega)'}. \tag{3.1.14}$$

Proof. Let $\varepsilon > 0$ and $u \in H^1(\Omega)'$, we consider the weak formulation of the PDE. By definition we are looking for $\overline{u}_\varepsilon \in H^1(\Omega)$ such that for all $\psi \in H^1(\Omega)$

$$\varepsilon^2 \int_\Omega \nabla v \cdot \nabla \psi \, \mathrm{d}\mathcal{L}^n + \int_\Omega v\psi \, \mathrm{d}\mathcal{L}^n = \langle \psi, u \rangle_{H^1(\Omega)'}. \tag{3.1.15}$$

The bilinear form on the left-hand side is symmetric and positive definite as $\varepsilon > 0$ is fixed. Thus it induces an equivalent scalar product $(\cdot \, , \, \cdot)$ on $H^1(\Omega)$. Since the scalar product is equivalent to the standard scalar product on $H^1(\Omega)$ the space $H^1(\Omega)$ equipped with $(\cdot \, , \, \cdot)$ is a Hilbert space. By Riesz's representation Theorem there exists a unique solution $\overline{u}_\varepsilon =: \mathcal{A}_\varepsilon u \in H^1(\Omega)$ to (3.1.15) such that for all $\psi \in H^1(\Omega)$ we have $\langle \psi, u \rangle_{H^1(\Omega)'} = (\psi, \overline{u}_\varepsilon)$. It follows

$$\sqrt{(\overline{u}_\varepsilon, \overline{u}_\varepsilon)} \leq \|u\|_{H^1(\Omega)'} \quad \text{and in particular} \quad \|\mathcal{A}_\varepsilon u\|_{L^2(\Omega)} = \|\overline{u}_\varepsilon\|_{L^2(\Omega)} \leq \|u\|_{H^1(\Omega)'}. \tag{3.1.16}$$

In the case of higher regularity we get $\overline{u}_\varepsilon \in H^2(\Omega)$ by Friedrich's Theorem.

As an operator on $L^2(\Omega)$ the linearity of $\mathcal{A}_\varepsilon$ follows from the uniqueness of $\overline{u}_\varepsilon$ and the bilinearity of $(\cdot \, , \, \cdot)$. The continuity was already shown in (3.1.16). We prove that $\mathcal{A}_\varepsilon$ is self-adjoint. Let $u, w \in L^2(\Omega)$, define $\psi := \mathcal{A}_\varepsilon u$ and $\phi := \mathcal{A}_\varepsilon w$. The partial integration combined with the boundary data allows to calculate

$$\begin{aligned}\int_\Omega u \mathcal{A}_\varepsilon w \, \mathrm{d}\mathcal{L}^n &= \int_\Omega \phi(-\varepsilon^2 \Delta + \mathrm{Id})\psi \, \mathrm{d}\mathcal{L}^n = \int_\Omega \left(\varepsilon^2 \nabla\psi \cdot \nabla\phi + \phi\psi\right) \mathrm{d}\mathcal{L}^n \\ &= \int_\Omega \psi(-\varepsilon^2 \Delta + \mathrm{Id})\phi \, \mathrm{d}\mathcal{L}^n = \int_\Omega w \mathcal{A}_\varepsilon u \, \mathrm{d}\mathcal{L}^n.\end{aligned}$$

The positivity of $\mathcal{A}_\varepsilon$ follows by considering $u = w$ in this calculation. Using the definition of $\mathcal{A}_\varepsilon$ we get for all $\psi, \phi \in H^1(\Omega)'$

$$\begin{aligned}\langle \mathcal{A}_\varepsilon \psi, \phi \rangle_{H^1(\Omega)'} &= \langle \mathcal{A}_\varepsilon \psi, -\varepsilon^2 \Delta \mathcal{A}_\varepsilon \phi + \mathcal{A}_\varepsilon \phi \rangle_{H^1(\Omega)'} = \int_\Omega \left(\varepsilon^2 \nabla \mathcal{A}_\varepsilon \psi \cdot \nabla \mathcal{A}_\varepsilon \phi + \mathcal{A}_\varepsilon \psi \mathcal{A}_\varepsilon \phi\right) \mathrm{d}\mathcal{L}^n \\ &= \langle \mathcal{A}_\varepsilon \phi, -\varepsilon^2 \Delta \mathcal{A}_\varepsilon \psi + \mathcal{A}_\varepsilon \psi \rangle_{H^1(\Omega)'} = \langle \mathcal{A}_\varepsilon \phi, \psi \rangle_{H^1(\Omega)'}\end{aligned}$$
□

In addition to these properties we prove a maximum principle for $\mathcal{A}_\varepsilon$.

Lemma 3.1.11 (Maximum principle)**.**
Let $\varepsilon > 0$, Ω as in Lemma 3.1.10 and $u \in L^2(\Omega)$ with $u \leq R$ a.e. in Ω for some $R \in \mathbb{R}$. Then the function $\overline{u}_\varepsilon := \mathcal{A}_\varepsilon u$ also satisfies $\overline{u}_\varepsilon \leq R$ a.e. in Ω. This remains correct if "$\leq$" is replaced with "$\geq$" in the statement.

Proof. The function $\overline{u}_\varepsilon - R$ satisfies

$$\begin{aligned} -\varepsilon^2\Delta(\overline{u}_\varepsilon - R) + \overline{u}_\varepsilon - R = u - R \quad &\text{in} \quad \Omega \\ \partial_\nu \overline{u}_\varepsilon = 0 \quad &\text{on} \quad \partial\Omega \end{aligned}$$

in the weak sense. We test with $(\overline{u}_\varepsilon - R)_+ \in H^1(\Omega)$ and get

$$\begin{aligned} &\int_\Omega \Big(\varepsilon^2\nabla(\overline{u}_\varepsilon - R)\cdot\nabla(\overline{u}_\varepsilon - R)_+ + (\overline{u}_\varepsilon - R)(\overline{u}_\varepsilon - R)_+\Big)\,\mathrm{d}\mathcal{L}^n = \int_\Omega (u-R)(\overline{u}_\varepsilon - R)_+\,\mathrm{d}\mathcal{L}^n \\ &\qquad\Longrightarrow \int_\Omega \Big(\varepsilon^2\big|\nabla(\overline{u}_\varepsilon - R)_+\big|^2 + |(\overline{u}_\varepsilon - R)_+|^2\Big)\,\mathrm{d}\mathcal{L}^n \le 0. \end{aligned}$$

Thus we get $\overline{u}_\varepsilon \le R$ a.e. in Ω. This carries over to lower bounds as well, assume $u \ge R$ a.e. for some $R \in \mathbb{R}$, then

$$-\varepsilon^2\Delta(-\overline{u}_\varepsilon) + (-\overline{u}_\varepsilon) = -u \le -R \quad \text{and hence} \quad -\overline{u}_\varepsilon \le -R. \qquad \square$$

Next we consider a Fredholm operator as a preparation. It appears in the linearization of the first variation of the diffuse perimeter on $\mathbb{R}$.

Lemma 3.1.12 ($\mathbf{L}_0$ is Fredholm)**.**
We consider the function f from the Assumptions 3.1.1, the optimal profile q_0 from Theorem 3.1.3 and the one-dimensional solution operator $\boldsymbol{A}_0$ from Definition 3.1.6. The operator

$$\boldsymbol{L}_0 : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \quad \boldsymbol{L}_0 \coloneqq f'(q_0)\,\mathrm{Id} - \boldsymbol{A}_0 \tag{3.1.17}$$

is a Fredholm operator with index 0.

The proof uses a clever splitting of $\mathbf{L}_0$, which was to our knowledge first introduced in the proof of Lemma 5.3. in [BFRW97]. The idea is to write $\mathbf{L}_0$ as the sum of an isomorphism and a compact operator. We adapt the method from $L^\infty(\mathbb{R})$ to $L^2(\mathbb{R})$.

Proof. We choose a function $\alpha \in C(\mathbb{R})$ such that

- There exist $c, R > 0$ such that $\big|\alpha(\pm r) - f'(\pm 1)\big| \le e^{-c|r|}$ for all $|r| > R$.
- There exist $1 < m < M$ such that $m \le \alpha \le M$.

If W is even we can use $\alpha(x) \coloneqq f'(1) + e^{-x^2}$. In the general case we define $\tilde{\alpha} : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\tilde{\alpha} \coloneqq f'(-1)\chi_{(-\infty,0)} + f'(1)\chi_{[0,\infty)} \quad \text{and} \quad \alpha \coloneqq \mathbf{A}_0\tilde{\alpha}.$$

From Lemma 3.1.9 (c) we get

$$\alpha(r) = f'(\operatorname{sgn}(-r))\frac{e^{-|r|}}{2} + f'(\operatorname{sgn}(r))\Big(1 - \frac{1}{2}e^{-|r|}\Big) \quad \text{for} \quad r \neq 0$$

and $\alpha(0) = \frac{f'(-1)+f'(1)}{2}$. From this representation we can immediately conclude that the exponential convergence towards $f'(\pm 1)$ as $r \to \pm\infty$ is satisfied. Since the function is a convex combination of $f'(\pm 1)$ and $f'(\pm 1) = 1 + \frac{1}{2}W''(\pm 1) > 1$ we also get the second condition.

We rewrite $\mathbf{L}_0(w)$ for $w \in L^2(\mathbb{R})$

$$\mathbf{L}_0(w) = f'(q_0)\big(\mathbf{L}^{(1)}(w) + \mathbf{L}^{(2)}(w)\big) \quad \text{with}$$
$$\mathbf{L}^{(1)}(w) := w - \frac{1}{\alpha} J_1 * w \quad \text{and} \quad \mathbf{L}^{(2)}(w) := \Big(\frac{1}{\alpha} - \frac{1}{f'(q_0)}\Big) J_1 * w.$$

Owing to $0 < \gamma \leq f'(q_0(r)) \leq C < \infty$ for all $r \in \mathbb{R}$ we get that if the operator $\mathbf{L}^{(1)} + \mathbf{L}^{(2)}$ is Fredholm then $\mathbf{L}_0$ is also a Fredholm operator with the same index. We start by showing that $\mathbf{L}^{(1)}$ is an isomorphism and prove that $\mathbf{L}^{(2)}$ is compact.

For $w \in L^2(\mathbb{R})$ with $\|w\|_{L^2(\mathbb{R})} \leq 1$ we get from Lemma 3.1.7

$$\Big\|\frac{1}{\alpha} J_1 * w\Big\|_{L^2(\mathbb{R})} \leq \frac{1}{m}\|J_1\|_{L^1(\mathbb{R})}\|w\|_{L^2(\mathbb{R})} \leq \frac{1}{m} < 1.$$

Therefore $\mathbf{L}^{(1)} = \operatorname{Id} - \frac{1}{\alpha}\mathbf{A}_0 : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bijection with continuous inverse operator, given by the corresponding Neumann series. It follows that $\mathbf{L}^{(1)}$ is an isomorphism and hence a Fredholm operator with index 0. If we can prove that $\mathbf{L}^{(2)} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a compact operator the proof is finished. We can write $\mathbf{L}^{(2)}$ as an integral operator and calculate the Hilbert-Schmidt norm

$$\int_{\mathbb{R}} \Big|\frac{1}{f'(q_0(r))} - \frac{1}{\alpha(r)}\Big|^2 \int_{\mathbb{R}} \frac{1}{4} e^{-2|r-s|} \,\mathrm{d}s \,\mathrm{d}r \leq \frac{1}{4m^2\gamma^2} \int_{\mathbb{R}} |f'(q_0) - \alpha|^2 \,\mathrm{d}r.$$

To ensure that the last integral exists we check that $f'(q_0) - f'(\pm 1)$ has exponential decay. Here it is sufficient to have that f' is Lipschitz on $[-1, 1]$. This is satisfied because of $f'' = \frac{1}{2} W''' \in C^0(\mathbb{R})$. Thus we get with $R, c > 0$ from Lemma 3.1.4 for $r > R$

$$|f'(q_0(\pm r)) - f'(\pm 1)| \leq \|f'\|_{C^1[-1,1]} \Big| \pm 1 - q_0(\pm r)\Big| \leq \frac{\|W'''\|_{C^0[-1,1]}}{2} e^{-cr}.$$

So $L^{(2)}$ is Hilbert-Schmidt and hence compact. □

We can even provide more information on the kernel of $\mathbf{L}_0$. It follows from equations (3.1.2) and (3.1.4) that $\mathbf{L}_0(q_0') = 0$. The next lemma shows, that the kernel is a one-dimensional subspace.

Lemma 3.1.13 **(**$\mathbf{L}_0$ has a one-dimensional kernel**).**
The operator $\boldsymbol{L}_0$ *has a one-dimensional kernel, more precisely*

$$\ker(\boldsymbol{L}_0) = \operatorname{span}(q_0') \tag{3.1.18}$$

and $\boldsymbol{L}_0 : \{q_0'\}^\perp \to \{q_0'\}^\perp$ *is an isomorphism.*

We follow and adapt the proof of Lemma 5.3 in [BFRW97].

Proof. It follows from $f(q_0) = \mathbf{A}_0 q_0$ that $w = q_0'$ is a solution to the equation $\mathbf{L}_0(w) = 0$. In order to get that every other solution is a multiple of q_0' we follow the line of argumentation from Lemma 5.3 in [BFRW97]. Because of

$$J_1 * q_0 = \mathbf{A}_0 q_0 = f(q_0) \quad \text{we have} \quad J_1 * q_0' = f'(q_0) q_0'.$$

Assume $\mathbf{L}_0(w) = 0$ for some $w \in L^2(\mathbb{R}) \setminus \{0\}$. Owing to the assumptions on f we know $f'(q_0) \geq \gamma$ for some $\gamma > 0$. The equation

$$w = \frac{J_1 * w}{f'(q_0)}. \tag{3.1.19}$$

implies higher regularity. Since $w \in L^2(\mathbb{R})$ we have from Proposition 3.1.8 $J_1 * w = \mathbf{A}_0 w \in C_b^1(\mathbb{R})$ thus $w \in C_b^1(\mathbb{R})$. By rescaling we can assume w to have a positive value somewhere on $\mathbb{R}$.

For $\beta \in \mathbb{R}$ we define $w_\beta := \beta w + q_0'$ and

$$\overline{\beta} := \sup\{\beta < 0 | \; \exists r \in \mathbb{R} : w_\beta(r) < 0\}.$$

Since our goal will be to show $w_{\overline{\beta}} \equiv 0$, it is useful to consider $\inf_{\mathbb{R}} w_\beta$. Let $\beta < \overline{\beta}$, then we know $\inf_{\mathbb{R}} w_\beta < 0$ by definition of $\overline{\beta}$. We claim that there exists $\xi_\beta \in \mathbb{R}$, such that $\inf_{\mathbb{R}} w_\beta = w_\beta(\xi_\beta)$. If not, without loss of generality there exists a sequence $(r_j)_{j\in\mathbb{N}}$ in $\mathbb{R}$, such that $r_j \longrightarrow \infty$ and $w_\beta(r_j) \longrightarrow \inf_{\mathbb{R}} w_\beta < 0$. We conclude with $\frac{1}{2}W''(q_0) = f'(q_0) - 1$ and (3.1.19)

$$\begin{aligned} 0 > \frac{1}{2}W''(1) \cdot \inf_{\mathbb{R}} w_\beta &= \lim_{j\to\infty} \frac{1}{2}W''(q_0(r_j))w_\beta(r_j) = \lim_{j\to\infty} \left[J_1 * w_\beta - w_\beta\right](r_j) \\ &= \lim_{j\to\infty} \left[(J_1 * w_\beta)(r_j) - \inf_{\mathbb{R}} w_\beta\right] \geq 0 \end{aligned}$$

which is a contradiction. It follows $\inf_{\mathbb{R}} w_\beta = w_\beta(\xi_\beta)$. This yields

$$\frac{1}{2}W''(q_0(\xi_\beta))w_\beta(\xi_\beta) = (J_1 * w_\beta - w_\beta)(\xi_\beta) > 0 \text{ so } W''(q_0(\xi_\beta)) < 0. \tag{3.1.20}$$

We can deduce that $(\xi_\beta)_{\beta<\overline{\beta}}$ can be found in the interval where $W''(q_0)$ is negative. Since $W''(\pm 1) > 0$ and $q_0(r) \longrightarrow \pm 1$ as $r \to \pm\infty$, this interval is bounded. Therefore we can extract a subsequence such that $\xi_\beta \longrightarrow \overline{\xi}$ as $\beta \to \overline{\beta}$. Since w, q_0' are bounded we know $w_\beta \longrightarrow w_{\overline{\beta}}$ as $\beta \to \overline{\beta}$ uniformly. We conclude

$$w_{\overline{\beta}}(\overline{\xi}) \longleftarrow w_\beta(\xi_\beta) = \inf_{\mathbb{R}} w_\beta \longrightarrow 0 \quad \text{as} \quad \beta \to \overline{\beta}.$$

Since $f' \neq 0$ it follows with (3.1.19)

$$0 = (J_1 * w_{\overline{\beta}})(\overline{\xi}) = \int_{\mathbb{R}} \underbrace{J_1(\overline{\xi} - s)}_{>0} \underbrace{w_{\overline{\xi}}(s)}_{\geq 0} \, \mathrm{d}s \Longrightarrow w_{\overline{\xi}} \equiv 0. \qquad \square$$

For the construction of the recovery sequence we are concerned with the minimization of the functional Ξ from the following lemma.

Lemma 3.1.14 (Existence of q_1)**.**
Let $\boldsymbol{L}_0$ be as in Lemma 3.1.12, $\boldsymbol{A}_0$ as in Definition 3.1.6 and $\overline{q}_0$ as in Theorem 3.1.3. Then for every minimizer $w_* \in L^2(\mathbb{R})$ of the functional $\Xi : L^2(\mathbb{R}) \longrightarrow \mathbb{R}$ with

$$\Xi(w) := \int_{\mathbb{R}} \left|\boldsymbol{L}_0(w) - \boldsymbol{A}_0\overline{q}_0'\right|^2 \mathrm{d}\mathcal{L}^1 \quad \textit{for} \quad w \in L^2(\mathbb{R}), \tag{3.1.21}$$

we have $w_* \in C_b^{m-2}(\mathbb{R})$. There exists a unique minimizer $q_1 \in L^2(\mathbb{R}) \cap C_b^{m-2}(\mathbb{R})$ that satisfies $q_1(0) = 0$. It is determined by the equation

$$\boldsymbol{L}_0(q_1) - \boldsymbol{A}_0(\overline{q}_0') = -\sigma_{\mathrm{AG}} q_0', \tag{3.1.22}$$

where σ_{AG} is the constant from (3.1.7). Furthermore we get that for $\lambda \in \mathbb{R}$ with $\lambda \neq \sigma_{\mathrm{AG}}$ there exists no $u \in L^2(\mathbb{R})$ such that

$$\boldsymbol{L}_0(u) - \boldsymbol{A}_0(\overline{q}_0') = -\lambda q_0'.$$

Proof. First we prove the existence of a minimizer with the direct method from the calculus of variations. We have $\Xi \geq 0$ and $L^2(\mathbb{R})$ is reflexive. Since $\mathbf{L}_0$ is a Fredholm operator it has a closed and convex range, in particular range($\mathbf{L}_0$) is weakly closed in $L^2(\mathbb{R})$. Ξ is the distance between an element of range($\mathbf{L}_0$) and $\mathbf{A}_0\overline{q}_0'$ in $L^2(\mathbb{R})$ and thus it is weakly lower semi-continuous. It follows that there exists $w_* \in L^2(\mathbb{R})$ such that

$$\|\mathbf{L}_0(w_*) - \mathbf{A}_0\overline{q}_0'\|_{L^2(\mathbb{R})}^2 = \inf_{L^2(\mathbb{R})} \Xi.$$

Let $w_* \in L^2(\mathbb{R})$ be any minimizer of Ξ thus w_* is a solution to the Euler-Lagrange equation

$$\begin{aligned} \mathbf{L}_0\big(\mathbf{L}_0(w_*) - \mathbf{A}_0\overline{q}_0'\big) &= 0 \qquad \text{and thus} \\ \mathbf{L}_0(w_*) - \mathbf{A}_0\overline{q}_0' &= \lambda q_0' \quad \text{for some} \quad \lambda \in \mathbb{R}, \end{aligned}$$

by Lemma 3.1.13. Rearranging yields

$$w_* = \frac{1}{f'(q_0')}\Big(\mathbf{A}_0\overline{q}_0' + \lambda q_0' + \mathbf{A}_0 w_*\Big).$$

We can apply a bootstrap argument. Proposition 3.1.8 yields $\mathbf{A}_0 w_* \in C_b^1(\mathbb{R})$ thus we get $w_* \in C^1(\mathbb{R})$. This improves the regularity of the right-hand side and by iteration we get $w_* \in C_b^{m-2}(\mathbb{R})$ which is the regularity of f' and q_0'. Now that we have continuity it is well-defined to discuss the additional condition $w(0) = 0$.

From $\mathbf{L}_0 : \{q_0'\}^\perp \longrightarrow \{q_0'\}^\perp$ we know that the Euler-Lagrange equation can only be solved if

$$\lambda q_0' + \mathbf{A}_0\overline{q}_0' \in \{q_0'\}^\perp.$$

Thus we calculate

$$0 \stackrel{!}{=} \langle q_0' | \lambda q_0' + \mathbf{A}_0(\overline{q}_0')\rangle_{L^2(\mathbb{R})} \Longleftrightarrow \lambda = -\frac{\|\overline{q}_0'\|_{L^2(\mathbb{R})}^2}{\|q_0'\|_{L^2(\mathbb{R})}^2} = -\sigma_{\mathrm{AG}}$$

with $\sigma_{\mathrm{AG}} > 0$ as in (3.1.7).

At last we next prove that there a exists a unique minimizer $q_1 \in L^2(\mathbb{R}) \cap C_b^{m-2}(\mathbb{R})$ of Ξ that satisfies $q_1(0) = 0$. From the Euler-Lagrange equation and the previous argument we get that every minimizer w_* of Ξ is a solution to

$$\mathbf{L}_0(w_*) = \sigma_{\mathrm{AG}} q_0' + \mathbf{A}_0\overline{q}_0'.$$

$\mathbf{L}_0 : \{q_0'\}^\perp \longrightarrow \{q_0'\}^\perp$ is an isomorphism by Lemma 3.1.13, thus there exists a unique $\hat{w}_* \in \{q_0'\}^\perp$ such that $\mathbf{L}_0(\hat{w}_*) = \sigma_{\mathrm{AG}} q_0' + \mathbf{A}_0 \overline{q}_0'$.
Since $\ker(\mathbf{L}_0) = \mathrm{span}(q_0')$ the minimizer can only be unique up to adding $\tau q_0'$ for $\tau \in \mathbb{R}$. We know $q_0'(0) > 0$ and thus there exists a unique $\tau_* \in \mathbb{R}$ such that

$$q_1(0) = \hat{w}_*(0) + \tau_* q_0'(0) = 0. \qquad \square$$

The last preparation we need for the Γ–$\limsup$ construction is the exponential decay of q_1, q_1'.

Lemma 3.1.15 (Exponential decay)**.**
There exist $R, C, c > 0$ such that for all $|r| > R$ we have

$$\begin{aligned} |q_1(r)| &\le Ce^{-c|r|}, & |q_1'(r)| &\le Ce^{-c|r|}, \\ |\hat{q}_1(r)| &\le Ce^{-c|r|}, \quad \textit{and} & |\hat{q}_1'(r)| &\le Ce^{-c|r|}. \end{aligned}$$

Proof. We consider the auxiliary function $\hat{q}_1 \in C_b^m(\mathbb{R}) \cap L^2(\mathbb{R})$

$$\hat{q}_1 \coloneqq \mathbf{A}_0(q_1 + \overline{q}_0'). \tag{3.1.23}$$

Combining this with (3.1.22) we get

$$q_1 = \frac{1}{f'(q_0)}(\hat{q}_1 - \sigma_{\mathrm{AG}} q_0'). \tag{3.1.24}$$

Since $\hat{q}_1 \in C_b^m(\mathbb{R}) \cap L^2(\mathbb{R})$ we get from Lemma 3.1.9

$$\lim_{r \to \pm\infty} \hat{q}_1(r) = 0 = \lim_{r \to \pm\infty} \hat{q}_1'(r). \tag{3.1.25}$$

We observe that $\xi \coloneqq (\overline{q}_0 - 1, \overline{q}_0', \hat{q}_1, \hat{q}_1')$ is a solution to the ODE system

$$\begin{aligned} \xi_1' &= \xi_2, \\ \xi_2' &= \xi_1 + 1 - f^{-1}(\xi_1 + 1), \\ \xi_3' &= \xi_4, \\ \xi_4' &= \xi_3 - \xi_2 - \frac{1}{f'(f^{-1}(\xi_1 + 1))}\Big(\xi_3 - \frac{\sigma_{\mathrm{AG}}\xi_2}{f'(f^{-1}(\xi_1 + 1))}\Big). \end{aligned}$$

Lemma 3.1.4 combined with (3.1.25) yield

$$|\xi(r)| \longrightarrow 0 \quad \text{as} \quad r \to \infty.$$

We recall $2c^2 = \frac{\frac{1}{2}W''(1)}{1+\frac{1}{2}W''(1)} = 1 - (f^{-1})'(1) \in (0,1)$ from the proof of Lemma 3.1.4. The linearization of the right-hand side of the ODE at $\xi = 0$ is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\frac{1}{2}W''(1)}{1+\frac{1}{2}W''(1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{\sigma_{\mathrm{AG}}}{(1+\frac{1}{2}W''(1))^2} - 1 & \frac{\frac{1}{2}W''(1)}{1+\frac{1}{2}W''(1)} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \sigma_{\mathrm{AG}}[1-2c^2]^2 - 1 & 2c^2 & 0 \end{pmatrix}$$

Owing to the block structure we get the Eigenvalues $\pm\sqrt{2}c$ with algebraic multiplicity 2 and geometric multiplicity 1 respectively. We compute a Jordan-Decomposition and get

$$\begin{pmatrix} -\sqrt{2}c & 1 & 0 & 0 \\ 0 & -\sqrt{2}c & 0 & 0 \\ 0 & 0 & \sqrt{2}c & 1 \\ 0 & 0 & 0 & \sqrt{2}c \end{pmatrix}.$$

We conclude that the stationary point 0 is hyperbolic. We already know $\xi(r) \longrightarrow 0$ as $r \to \infty$ and thus the solution is on the stable manifold. It follows from stable manifold theory that the convergence is exponential; see the remark on page 115 in [Per96]. The exponential convergence of q_1, q_1' follow from the representation

$$q_1 = \frac{1}{f'(q_0)}\big(- \sigma_{\mathrm{AG}} q_0' + \hat{q}_1\big), \tag{3.1.26}$$

$$q_1' = \frac{1}{f'(q_0)}\big(- \sigma_{\mathrm{AG}} q_0'' + \hat{q}_1'\big) + \frac{f''(q_0)q_0'}{|f'(q_0)|^2}\big(- \sigma_{\mathrm{AG}} q_0' + \hat{q}_1\big), \tag{3.1.27}$$

and the previous estimates from Lemma 3.1.4. Similar for $r \to -\infty$. □

3.2 Formal identification of a candidate for the Γ–limit and for a recovery sequence

With the preparations from the last section we can start with the key objects. We adapt the concept presented in [BP93] where the classical Γ–lim sup estimate was shown. We concentrate on a small neighborhood of the given surface as in Definition 2.1.10, formally expand $\mathcal{W}_\varepsilon^{\mathrm{KK}}$ in powers of ε and minimize each order. The idea to do an asymptotic expansion u_ε in powers of ε is not new, it was presented in [LM00] and considered by [Wan08].

Definition 3.2.1 (Diffuse Willmore energy)**.** *We define the gradient-free diffuse Willmore energy* $\mathcal{W}_\varepsilon^{\mathrm{AG}} : L^2(\Omega) \longrightarrow [0, \infty]$

$$\mathcal{W}_\varepsilon^{\mathrm{AG}}(u) := \int_\Omega \frac{1}{\varepsilon^3}\Big|f(u) - \mathcal{A}_\varepsilon u\Big|^2 \,\mathrm{d}\mathcal{L}^n. \tag{3.2.1}$$

and the diffuse mean curvature

$$H_\varepsilon^{\mathrm{AG}}(u) := \nabla_{L^2}\mathcal{P}_\varepsilon^{\mathrm{AG}} = \frac{1}{\varepsilon}\Big(u - \overline{u}_\varepsilon + \frac{1}{2}W'(u)\Big) = \frac{1}{\varepsilon}\Big(f(u) - \mathcal{A}_\varepsilon u\Big)$$

Let $E \Subset \Omega$ be open with smooth boundary $\Gamma := \partial E$, we write $u := 2\chi_E - 1$. For the lim sup condition of Γ-convergence we have to construct an approximation $(u_\varepsilon)_{\varepsilon>0}$ of u in $L^1(\Omega)$ such that

$$\limsup_{\varepsilon\to 0} \mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon) \le c_{\mathrm{AG}}\sigma_{\mathrm{AG}}\mathcal{W}(u).$$

We recall the geometry from Figure 2.1 and the coordinates for $x \in \omega = \{|d| < 5\delta\}$. The idea behind the construction is visualized in Figure 3.1.

$$x = \Psi_\varepsilon(z, y) = y + \varepsilon z \nu_\Gamma(y) \quad \text{with} \quad (z, y) \in \mathbb{R} \times \Gamma$$

from Definition 2.1.10. We use the ansatz

$$u_\varepsilon(x) = U_0(z,x) + \varepsilon U_1(z,x), \tag{3.2.2}$$

where U_0, U_1 are profile functions with properties specified below. As in 2.1.11 we consider profile functions that are constant in normal direction. We pose the following conditions on the profile functions:

- $U_0 \in C^0(\mathbb{R} \times \omega)$ with $U_0(0,x) = 0$ and $U_0(\cdot,x) - \operatorname{sgn} \in L^2(\mathbb{R})$ for all $x \in \omega$.
- $U_1 \in H^1(\mathbb{R}; C(\omega))$ with $U_1(0,x) = 0$ for all $x \in \omega$.

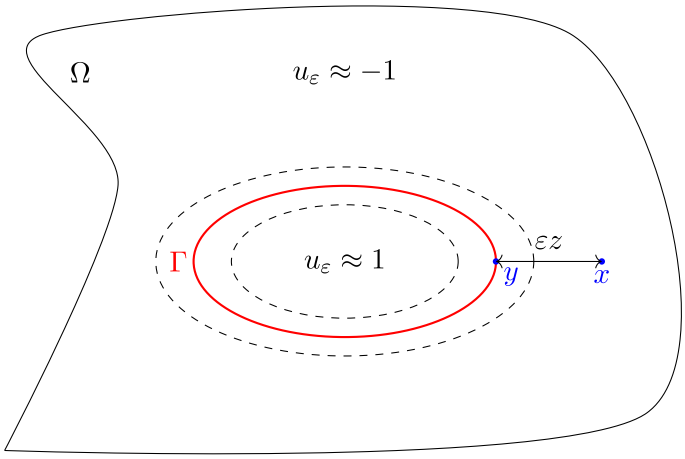

Figure 3.1: Visualization of the geometry and coordinates.

We write $\partial U_j = U_j'$ for the z-derivative and $\nabla_\Gamma U_j$ for the tangential gradient with respect to the y variable. In addition we assume an expansion for the corresponding solution $\overline{u}_\varepsilon := \mathcal{A}_\varepsilon u_\varepsilon$ of the PDE (3.0.1) of the form

$$\overline{u}_\varepsilon(x) \approx V_0(z,x) + \varepsilon V_1(z,x)$$

with similar properties as the expansion for u_ε. We plug this formally into (3.0.1) and use the expansion of the differential operator in the new coordinates (2.1.9)-(2.1.8). We get at each point $x \in \omega$

$$\begin{aligned} U_0 + \varepsilon U_1 = u_\varepsilon = (-\varepsilon^2\Delta + \mathrm{Id})\overline{u}_\varepsilon &= (-\partial^2 + \mathrm{Id} - \varepsilon \Delta d \partial - \varepsilon^2 \Delta_\Gamma)(V_0 + \varepsilon V_1) \\ &= (-\partial^2 + \mathrm{Id} - \varepsilon H \partial)(V_0 + \varepsilon V_1) + \mathcal{O}(\varepsilon^2) \\ &= -V_0'' + V_0 + \varepsilon(-V_1'' + V_1 - HV_0') + \mathcal{O}(\varepsilon^2). \end{aligned}$$

We conclude formally

$$-V_0'' + V_0 = U_0 \quad \text{and} \quad -V_1'' + V_1 = U_1 + HV_0'. \tag{3.2.3}$$

In the next step we expand $\mathcal{W}_\varepsilon^{\mathrm{AG}}$ by powers of ε and choose U_0, U_1 by minimizing the functionals in each order. We neglect the integral over $\Omega \setminus \omega$ and use a Taylor expansion

$$\mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon) \approx \int_\omega \frac{1}{\varepsilon^3} |f(u_\varepsilon) - \overline{u}_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n = \int_\omega \frac{1}{\varepsilon^3} \Big| f(U_0) + \varepsilon f'(U_0)U_1 - V_0 - \varepsilon V_1 \Big|^2 \,\mathrm{d}\mathcal{L}^n.$$

For the transformation $x = \Psi_\varepsilon(z, y)$ we have $|\det(D\Psi_\varepsilon)| = \varepsilon + \mathcal{O}(\varepsilon^2) \approx \varepsilon$ from (2.1.7)

$$\mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon) \approx \int_\Gamma \int_{-\frac{5\delta}{\varepsilon}}^{\frac{5\delta}{\varepsilon}} \left| \frac{1}{\varepsilon}(f(U_0) - V_0) + f'(U_0)U_1 - V_1 \right|^2 \mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1}$$
$$\approx \int_\Gamma \int_{\mathbb{R}} \left| \frac{1}{\varepsilon}(f(U_0) - V_0) + f'(U_0)U_1 - V_1 \right|^2 \mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1}$$

We will minimize this expression on each order of ε-powers. We fix $y \in \omega$, then first term to minimize is

$$\int_{\mathbb{R}} |f(U_0(\cdot, y) - V_0(\cdot, y)|^2 \mathrm{d}\mathcal{L}^1$$

The lowest possible value is 0 and this is achieved with the choice $U_0 = q_0$ and thus $V_0 = \overline{q}_0$. In particular U_0 is independent from the second variable. We get

$$\mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon) \approx \int_\Gamma \int_{\mathbb{R}} \left| f'(q_0)U_1 - V_1 \right|^2 \mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1}.$$

We plug in $V_1 = \mathbf{A}_0(U_1 + H\overline{q}_0')$ and get

$$\mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon) \approx \int_\Gamma \int_{\mathbb{R}} \left| (f'(q_0) - \mathbf{A}_0)U_1 - H\mathbf{A}_0\overline{q}_0' \right|^2 \mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1}.$$

We see that if $H = 0$ the optimal choice is $U_1 = 0$ and thus the set $\{y \in \Gamma \mid H(y) = 0\}$ has no impact on the value of the integral. For all $y \in \Gamma$ with $H(y) \neq 0$ we write $U_1 = H\frac{U_1}{H}$ and get

$$\mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon) \approx \int_{\{H\neq 0\}} |H|^2 \int_{\mathbb{R}} \left| \Big(f'(q_0) - \mathbf{A}_0\Big)\frac{U_1}{H} - \mathbf{A}_0\overline{q}_0' \right|^2 \mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1}. \tag{3.2.4}$$

Here we minimize the inner integral again. For $y \in \{H \neq 0\}$ fixed the term $\frac{1}{H(y)}$ is just a factor independent from z. We consider

$$\Xi : L^2(\mathbb{R}) \longrightarrow [0, \infty], \quad \Xi(w) := \int_{\mathbb{R}} \left| \mathbf{L}_0(w) - \mathbf{A}_0\overline{q}_0' \right|^2 \mathrm{d}\mathcal{L}^1,$$

In Lemma 3.1.14 we have proven the existence of a unique minimizer q_1 that satisfies $q_1(0) = 0$. Since $U_1(0, x) = 0$ for all $x \in \omega$ was a condition for U_1 we get

$$U_1 = Hq_1 \quad \text{and} \quad \mathbf{L}_0(q_1) - \mathbf{A}_0\overline{q}_0' = -\sigma_{\mathrm{AG}}q_0'.$$

We conclude that the minimum is

$$\Xi(q_1) = c_{\mathrm{AG}}\sigma_{\mathrm{AG}}. \tag{3.2.5}$$

Combining this with (3.2.4) we get that for the choices $U_0 = q_0,\ U_1 = Hq_1$ we obtain

$$\mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon) \approx c_{\mathrm{AG}}\sigma_{\mathrm{AG}} \int_\Gamma H^2 \,\mathrm{d}\mathcal{H}^{n-1} = c_{\mathrm{AG}}\sigma_{\mathrm{AG}}\mathcal{W}(u). \tag{3.2.6}$$

The right-hand side in this equation characterizes our Γ–limit candidate of $(\mathcal{W}_\varepsilon^{\mathrm{AG}})_{\varepsilon>0}$. We get the $\limsup$-condition of Γ-convergence if '$\leq$' is proven rigorously.

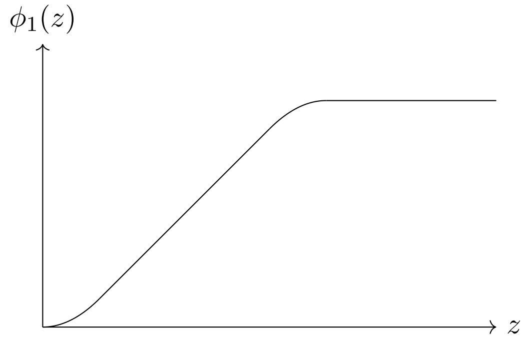

Figure 3.2: Plot of ϕ_1

3.3 Γ–$\liminf$ estimate for a specific class of functions

In this section we construct a suitable class of functions which are constant ± 1 far away from the surface and has an asymptotic expansion $u_\varepsilon = U_0 + \varepsilon U_1 + \ldots$ as in (3.2.2) close to the surface. In between we interpolate with a cut-off function. The difficulty is to do the construction in a way such that the family $(\overline{u}_\varepsilon)_{\varepsilon>0}$ with $\overline{u}_\varepsilon := \mathcal{A}_\varepsilon u_\varepsilon$ also has an asymptotic expansion $\overline{u}_\varepsilon = V_0 + \varepsilon V_1 + \ldots$ as in (3.2.2) close to the surface. This is not trivial because the normal and tangential coordinates are not well-defined on Ω, just in a neighborhood of Γ and the solution operator is nonlocal, meaning that all values from u_ε in Ω influence the solution $\overline{u}_\varepsilon$.

Assumption 3.3.1 (General assumptions).
Let Ω, W as in Assumptions 3.1.1. Let $E \Subset \Omega$ with C^4-boundary $\Gamma := \partial E$, recall $u = 2\chi_E - 1$, $d = \operatorname{sdist}(\cdot, \Gamma)$ and $\omega = \{|d| < 5\delta\} \Subset \Omega$ with $\delta > 0$ sufficiently small such that the coordinate transformation Ψ_ε from Definition 2.1.10 for $\varepsilon > 0$ is well-defined on ω.

In the following we introduce a modification of the signed distance function with a cut-off function.

Definition 3.3.2 (Modified distance and cut-off functions).
Choose an odd and increasing function $\phi_1 \in C^\infty(\mathbb{R})$ with

$$\phi_1'(0) = 0, \quad 0 < \phi_1(z) \le \frac{9}{10} z, \quad 0 \le \phi_1'(z) \le 1 \qquad \textit{for all } z \in (0,\infty),$$
$$\phi_1(z) = \begin{cases} z - \frac{1}{4}, & \textit{if } z \in (\frac{1}{2}, 2) \\ 2, & \textit{if } z \in (\frac{5}{2}, \infty). \end{cases}$$

We set $\phi_\delta(z) := \delta\phi_1(\frac{z}{\delta})$ for $z \in \mathbb{R}$ and define a modification of the signed distance function (being constant $\pm 2\delta$ outside $\{|d| < \frac{5}{2}\delta\}$) by

$$d_\delta := \phi_\delta \circ d \in C^4(\Omega).$$

Finally, choose an even and on $(0,\infty)$ decreasing function $\eta_1 \in C_c^\infty(\mathbb{R})$ with

$$0 \le \eta_1 \le 1, \quad |\eta_1'| \le 2, \quad \eta_1 = \begin{cases} 1 & \textit{in } [0,3], \\ 0 & \textit{in } [4,\infty) \end{cases}$$

and define the cut-off function

$$\eta_\delta(x) := \eta_1\Big(\frac{d(x)}{\delta}\Big) \quad \textit{for all } x \in \Omega.$$

We remark that $\eta_\delta \in C^4(\Omega)$ *since* η_δ *has support in* $\{|d| \le 4\delta\}$.

We next define spaces of functions that are exponentially controlled in terms of the modified distance function d_δ far away from Γ.

Definition 3.3.3. *For* $\Lambda, \mu > 0$ *we consider*

$$X_\delta^{\mu,\Lambda}(\Omega) := \Big\{ w \in L^\infty(\Omega) \mid \operatorname*{ess-sup}_{x\in\Omega} |e^{\mu|d_\delta(x)|}w(x)| \le \Lambda \Big\}$$

and

$$X^{\mu,\Lambda}(\mathbb{R};\Gamma) := \Big\{ w \in L^\infty(\mathbb{R}\times\omega) \mid \operatorname*{ess-sup}_{(z,x)\in\mathbb{R}\times\omega} |e^{\mu|z|}w(z,x)| \le \Lambda, \\ w(z,\cdot) \textit{ is constant in normal direction}\}$$

and set $X(\mathbb{R};\Gamma) := \bigcup_{\mu,\Lambda>0} X^{\mu,\Lambda}(\mathbb{R};\Gamma)$.

Note that for $0 < \mu_1 < \mu_2$ and any $\Lambda > 0$ we have $X_\delta^{\mu_2,\Lambda}(\Omega) \subseteq X_\delta^{\mu_1,\Lambda}(\Omega)$. Next we define a suitable class of phase field approximations that have an expansion in powers of ε close to Γ and are constant ± 1 far away from Γ.

Assumption 3.3.4 (Additional assumptions for lim inf–estimate)**.**
We assume 3.3.1 and let $K \in \mathbb{N}_0$. *Consider a family* $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *that can be represented as follows: There exist* $0 < \mu < 1$, $\Lambda > 0$, *and profile functions* u_j, $j = 0,\dots,K$, *such that for all* $0 < \varepsilon < \varepsilon_0$

$$u_\varepsilon = \eta_\delta u_\varepsilon^{\text{in}} + (1-\eta_\delta)\operatorname{sgn}(d) + \varepsilon^{K+1}R_\varepsilon^u \quad \textit{in } \Omega, \tag{3.3.1}$$

$$u_\varepsilon^{\text{in}} = \Big(\sum_{j=0}^{K} \varepsilon^j u_j\Big) \circ \Psi_\varepsilon^{-1} \quad \textit{in } \{|d| < 4\delta\}, \tag{3.3.2}$$

and such that the following properties hold:

1. *The profile functions* $u_j \in C^0(\mathbb{R}\times\omega)$, $u_j = u_j(z,x)$ *are* C^4*-regular with respect to the* x*-variable and satisfy*

$$u_0 - \operatorname{sgn} \in X(\mathbb{R};\Gamma),\ u_j \in X(\mathbb{R};\Gamma) \quad \textit{for} \quad 1 \le j \le K,$$
$$|\nabla_x u_j|,\ \Delta_x u_j, |\nabla_x \Delta_x u_j|,\ \Delta_x^2 u_j \in X(\mathbb{R};\Gamma) \quad \textit{for} \quad 0 \le j \le K.$$

2. *The remainder satisfies* $R_\varepsilon^u \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega)$ *for all* $0 < \varepsilon < \varepsilon_0$.

Finally we assume that there are height functions h_j, $j = 0,\dots,K-1$ *such that*

$$y \mapsto y + \varepsilon\Big(\sum_{j=0}^{K-1} \varepsilon^j h_j(y) + \varepsilon^K R_\varepsilon^H(y)\Big)\nu_\Gamma(y), \quad y \in \Gamma \tag{3.3.3}$$

is a C^4*-diffeomorphism onto* $\{u_\varepsilon = 0\}$ *with* $\sup_{\varepsilon>0} \|R_\varepsilon^H\|_{C^4(\Gamma)} < \infty$.

Lemma 3.3.5 (Convergence towards u).
Consider $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ as in Assumption 3.3.4. Then we have $u_\varepsilon \longrightarrow u$ in $L^p(\Omega)$ for all $1 \le p < \infty$.

Proof. Note that $u = 2\chi_E - 1 = \operatorname{sgn}(d)$. We fix $\Lambda, \mu, \delta > 0$ such that $R^u_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega)$ and $u_0 - \operatorname{sgn}$, $u_j \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)$ for all $1 \le j \le K$. Using the representation of u_ε and (2.1.7) we have

$$\begin{aligned}
\int_\Omega |u_\varepsilon - u|^p \,\mathrm{d}\mathcal{L}^n &= \int_\Omega \Big|\eta_\delta(u_\varepsilon^{\mathrm{in}} - \operatorname{sgn}(d)) + \varepsilon^{K+1} R^u_\varepsilon\Big|^p \mathrm{d}\mathcal{L}^n \\
&\le C(p) \int_{\{|d|<4\delta\}} |u_\varepsilon^{\mathrm{in}} - \operatorname{sgn}(d)|^p \,\mathrm{d}\mathcal{L}^n + \varepsilon^{K+1} C(p) \int_\Omega |R^u_\varepsilon|^p \,\mathrm{d}\mathcal{L}^n \\
&\le C(p) \int_\Gamma \int_{-\frac{4\delta}{\varepsilon}}^{\frac{4\delta}{\varepsilon}} \varepsilon \Big| \sum_{j=0}^K \varepsilon^j u_j - \operatorname{sgn}\Big|^p \mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} + \varepsilon^{K+1} C(p) \Lambda^p \mathcal{L}^n(\Omega).
\end{aligned}$$

We use the bounds for the profile functions to further estimate the right-hand side and deduce that for some $\Lambda > 0$ and some $\mu \in (0,1)$

$$\begin{aligned}
\int_\Omega |u_\varepsilon - u|^p \,\mathrm{d}\mathcal{L}^n &\le C(\Gamma, p) \sum_{j=0}^K \varepsilon^{j+1} \int_{\mathbb{R}} e^{-p\mu|t|} \Lambda^p \,\mathrm{d}t + \varepsilon^{K+1} C(\Lambda, \Omega, p) \\
&\le C(\Gamma, \Lambda, p, \mu) \sum_{j=0}^K \varepsilon^{j+1} + \varepsilon^{K+1} C(\Lambda, \Omega, p). \qquad \square
\end{aligned}$$

Next we show that $(f(u_\varepsilon))_{0<\varepsilon<\varepsilon_0}$ also has an appropriate expansion. For the proof we need the following lemma.

Lemma 3.3.6. *Let $a < b$ and $\tilde f \in C^2(U)$, for an open set $U \supseteq [a,b]$ and define for $0 \le \lambda \le 1$*

$$g(\lambda) := \tilde f((1-\lambda)a + \lambda b) - ((1-\lambda)\tilde f(a) + \lambda \tilde f(b)).$$

Then

$$|g(\lambda)| \le C \|\tilde f''\|_{C^0[a,b]} \lambda(1-\lambda)(a-b)^2 \tag{3.3.4}$$

holds.

Proof. We have $g(0) = g(1) = 0$. Taylor expansions give

$$\begin{aligned}
g(\lambda) &= \tilde f'(a)(b-a)\lambda + \frac12 \tilde f''(\xi_1)(b-a)^2\lambda^2 - \lambda(\tilde f(b) - \tilde f(a)) \\
&= \tilde f'(a)(b-a)\lambda + \frac12 \tilde f''(\xi_1)(b-a)^2\lambda^2 - \lambda \tilde f'(a)(b-a) - \frac12 \lambda \tilde f''(\xi_2)(b-a)^2 \\
&= \frac12 (b-a)^2 \lambda \Big(\tilde f''(\xi_1)\lambda - \tilde f''(\xi_2)\Big)
\end{aligned}$$

for some $\xi_1, \xi_2 \in (a,b)$. Similar we have

$$\begin{aligned}
g(\lambda) &= \tilde f'(b)(b-a)(\lambda-1) + \frac12 \tilde f''(\xi_3)(b-a)^2(1-\lambda)^2 - (\lambda-1)(\tilde f(b) - \tilde f(a)) \\
&= \frac12 (b-a)^2 (1-\lambda) \Big(\tilde f''(\xi_3)(1-\lambda) - \tilde f''(\xi_4)\Big)
\end{aligned}$$

for some $\xi_3, \xi_4 \in (a,b)$. Multiplying the first equality by $1-\lambda$, the second by λ and adding up yields the desired estimate. $\square$

Lemma 3.3.7.
Consider $K \leq 2$ and $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ as in Assumption 3.3.4. Let $\tilde{f} \in C^2(U)$ for an open set $U \supseteq [-1,1]$ then the family $\big(\tilde{f}(u_\varepsilon)\big)_{0<\varepsilon<\tilde{\varepsilon}_0}$ can be represented as

$$\tilde{f}(u_\varepsilon) = \eta_\delta \tilde{f}(u_\varepsilon)^{\text{in}} + (1-\eta_\delta)\operatorname{sgn} d + \varepsilon^{K+1} R_\varepsilon \tag{3.3.5}$$

with $\tilde{f}(u_\varepsilon)^{\text{in}} \circ \Psi_\varepsilon = \sum_{j=0}^{K} \varepsilon^j F_j$,

$$F_0 = \tilde{f}(u_0), \quad F_1 = \tilde{f}'(u_0)u_1, \quad F_2 = \frac{1}{2}\tilde{f}''(u_0)(u_1)^2 + \tilde{f}'(u_0)u_2.$$

Moreover, $F_0 - \operatorname{sgn}, F_1, F_2 \in X(\mathbb{R};\Gamma)$, $R_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon}, C(\tilde{f})\Lambda}(\Omega)$ holds.

Proof. We proof the claim for $K = 2$. Choose $\tilde{\varepsilon}_0 < \varepsilon_0$ such that $u_\varepsilon(x) \in U$ for all $x \in \Omega$. Choose μ, Λ such that $u_0, u_1, u_2 \in X^{\mu,\Lambda}(\mathbb{R})$, $R_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}$. We first obtain

$$\tilde{f}(u_\varepsilon) = \tilde{f}\big(\eta_\delta u_\varepsilon^{\text{in}} + (1-\eta_\delta)\operatorname{sgn} d + \varepsilon^3 R_\varepsilon^u\big) = \tilde{f}\big(\eta_\delta u_\varepsilon^{\text{in}} + (1-\eta_\delta)\operatorname{sgn} d\big) + \varepsilon^3 R_\varepsilon^{(1)}, \tag{3.3.6}$$

with $|R_\varepsilon^{(1)}| \leq C(\tilde{f})|R_\varepsilon^u|$. Since $\tilde{f} \in C^2(U)$ with $[-1,1] \subseteq U$ Lemma 3.3.6 yields

$$\big|\tilde{f}(\eta_\delta u_\varepsilon^{\text{in}} + (1-\eta_\delta)\operatorname{sgn} d) - \eta_\delta \tilde{f}(u_\varepsilon^{\text{in}}) - (1-\eta_\delta)\operatorname{sgn} d\big| \leq C(\tilde{f})\eta_\delta(1-\eta_\delta)(u_\varepsilon^{\text{in}} - \operatorname{sgn} d)^2. \tag{3.3.7}$$

Another Taylor expansion implies that in $\{\eta_\delta > 0\}$

$$\Big|\tilde{f}(u_\varepsilon^{\text{in}}) - \Big(\tilde{f}(u_0) + \varepsilon \tilde{f}'(u_0)u_1 + \varepsilon^2\Big(\frac{1}{2}\tilde{f}''(u_0)(u_1)^2 + \tilde{f}'(u_0)u_2\Big)\Big) \circ \Psi_\varepsilon^{-1}\Big| \leq \varepsilon^3 C(\tilde{f}) R_\varepsilon^{(2)} \tag{3.3.8}$$

for $R_\varepsilon^{(2)}$ with $R_\varepsilon^{(2)} \leq (|u|_1 + |u_2| + |u_3|)^2 \circ \Psi_\varepsilon^{-1}$. From (3.3.6)-(3.3.8) we conclude the desired representation (3.3.5) with

$$|R_\varepsilon| \leq C(\tilde{f})\Big[R_\varepsilon^{(1)} + \eta_\delta(1-\eta_\delta)(u_\varepsilon^{\text{in}} - \operatorname{sgn} d)^2 + (|u|_1 + |u_2| + |u_3|)^2 \circ \Psi_\varepsilon\Big].$$

Since $|u_\varepsilon^{\text{in}} - \operatorname{sgn} d| \leq \big(|u_0 - \operatorname{sgn}| + |u_1| + |u_2|\big) \circ \Psi_\varepsilon$ we deduce that $R_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}$. Finally, $F_0 - \operatorname{sgn}, F_1, F_2 \in X(\mathbb{R};\Gamma)$ follows from $|\tilde{f}(u_0) - \operatorname{sgn}| = |\tilde{f}(u_0) - \tilde{f}(\operatorname{sgn})| \leq C(\tilde{f})|u_0 - \operatorname{sgn}|$ and the assumptions on u_0, u_1, u_2. ☐

Below we will only need orders $K \leq 2$. The key observation at this point is that the solution operator $\mathcal{A}_\varepsilon$ conserves the expansion properties. For the proof we need a few preparations. First we extend the solution operator $\mathbf{A}_0$ from 3.1.8 to functions defined on $\mathbb{R} \times \omega$.

Lemma 3.3.8.
The convolution operator $\boldsymbol{A}_0 w = J_1 * w$ *can be extended. For* $w \in L^\infty(\mathbb{R}\times\omega)$ *we define*

$$\boldsymbol{A}_0 w(z,x) := \int_{\mathbb{R}} J_1(z-\zeta) w(\zeta,x) d\zeta = [J_1 * w(\cdot,x)](z).$$

It has the following properties:

1. *If* $w \in C^{j_1}(\mathbb{R}; C^{j_2}(\omega))$, $j_1, j_2 \in \mathbb{N}_0$ *then* $\boldsymbol{A}_0 w \in C^{j_1+2}(\mathbb{R}; C^{j_2}(\omega))$.
2. *If* $w \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)$ *for some* $\Lambda > 0$ *and* $\mu \in (0,1)$ *we get* $\boldsymbol{A}_0 w, (\boldsymbol{A}_0 w)', (\boldsymbol{A}_0 w)'' \in X^{\mu,\tilde{\Lambda}}(\mathbb{R};\Gamma)$ *for some* $\tilde{\Lambda} = \tilde{\Lambda}(\Lambda,\mu)$.

Proof. The first claim follows from standard theory of parameter dependent integrals. For the second claim we estimate

$$\begin{aligned}\left|e^{\mu|z|}\mathbf{A}_0 w(z,x)\right| &\le \int_{\mathbb{R}} J_1(\zeta) e^{\mu|z|}\left|w(z-\zeta,y)\right| \mathrm{d}\zeta \le \Lambda \int_{\mathbb{R}} J_1(\zeta) e^{\mu|z|} e^{-\mu|z-\zeta|}\,\mathrm{d}\zeta \\ &\le \Lambda \int_{\mathbb{R}} J_1(\zeta) e^{\mu|\zeta|}\,\mathrm{d}\zeta = \frac{\Lambda}{2}\int_{\mathbb{R}} e^{-(1-\mu)|\zeta|}\,\mathrm{d}\zeta = \frac{\Lambda}{1-\mu}.\end{aligned}$$

The estimate for $(\mathbf{A}_0 w)'$ follows similarly since $J_1' \in L^\infty(\mathbb{R})$ also decays exponentially. Finally, these properties also yield the decay of $(\mathbf{A}_0 w)'' = (\mathbf{A}_0 w) - w$. ☐

The next lemma contains the key argument why the exponential control from u_ε carries over to $\overline{u}_\varepsilon$. We consider the PDE that is solved by the phase-field function multiplied with an exponential term of the form introduced in the Definition 3.3.3.

Lemma 3.3.9.
Let $\Lambda > 0$, $\delta \in (0,1)$, *and* $\mu \in (0,1)$ *be given. There exists* $\varepsilon_0 = \varepsilon_0(\delta,\mu,\Gamma) > 0$ *with the following property: Let* $0 < \varepsilon < \varepsilon_0$, $\tilde{u}_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega)$ *be given and assume* $\tilde{v}_\varepsilon \in C^2(\Omega)\cap C^1(\overline{\Omega})$ *solves*

$$\begin{aligned}(-\varepsilon^2\Delta + \mathrm{Id})\tilde{v}_\varepsilon &= \tilde{u}_\varepsilon \quad \text{in} \quad \Omega \\ \partial_\nu \tilde{v}_\varepsilon &= 0 \quad\ \ \text{on} \quad \partial\Omega.\end{aligned}$$

Then we have $\tilde{v}_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\tilde{\Lambda}}(\Omega)$ *for* $\tilde{\Lambda} = \frac{2}{1-\mu^2}\Lambda$.

Proof. We obtain that the function $Z := e^{\frac{\mu}{\varepsilon}|d_\delta|}\tilde{v}_\varepsilon$ satisfies

$$\begin{aligned}-\varepsilon^2\Delta Z + 2\mu\varepsilon\nabla|d_\delta|\cdot\nabla Z + (1+\mu\varepsilon\Delta|d_\delta| - \mu^2|\nabla|d_\delta||^2)Z &= e^{\frac{\mu}{\varepsilon}|d_\delta|}\tilde{u}_\varepsilon \quad \text{in} \quad \Omega \\ \partial_\nu Z &= 0 \qquad\qquad \text{on} \quad \partial\Omega.\end{aligned}$$

Choose $\varepsilon_0 > 0$ sufficiently small such that

$$\inf_\Omega \left(1+\mu\varepsilon\Delta|d_\delta| - \mu^2|\nabla|d_\delta||^2\right) \ge \frac{1-\mu^2}{2} > 0.$$

This is possible because of

$$|\nabla|d_\delta|| = |\phi_\delta'(d)\nabla d| = \left|\phi_1'\Big(\frac{d}{\delta}\Big)\right| \le 1 \quad \text{and} \quad |\Delta|d_\delta|| \le C(\eta_1,\delta,\|d\|_{C^2(\Omega)}).$$

We have $d \in C_b^2(\Omega)$ because $\Gamma \Subset \Omega$ is a C^4-hypersurface.

Assume that $M := \max_{\overline{\Omega}} Z > \tilde{\Lambda} := \frac{2}{1-\mu^2}\Lambda$ and observe that in $\{Z > \tilde{\Lambda}\}$

$$-\varepsilon^2\Delta(Z-\tilde{\Lambda}) + 2\mu\varepsilon\nabla|d_\delta| \cdot \nabla(Z-\tilde{\Lambda}) + \frac{1-\mu^2}{2}(Z-\tilde{\Lambda}) \leq e^{\frac{\mu}{\varepsilon}|d_\delta|}\tilde{u}_\varepsilon - \Lambda \leq 0 \qquad (3.3.9)$$

since $\tilde{u}_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega)$. If the maximum of Z is attained at a point $x_0 \in \partial\Omega$ we choose an open ball $B \subset \Omega \cap \{Z > \tilde{\Lambda}\}$ with $\overline{B} \cap \partial\Omega = \{x_0\}$. We deduce from the Hopf-Lemma and $\nabla Z \cdot \nu_\Omega = 0$ that $Z = M$ in B holds. This implies that the maximum of Z is always attained in Ω, which yields by (3.3.9) that $Z \leq \tilde{\Lambda}$, contradicting the assumption. Similarly we obtain $-Z \leq \tilde{\Lambda}$. □

The next lemma shows a *quasi-locality* of the operator $-\varepsilon^2\Delta + \mathrm{Id}$ if applied to functions that are exponentially close to ± 1 away from the interface.

Lemma 3.3.10.
Let $\mu, \Lambda > 0$, $0 < \delta < \frac{1}{2}$, and a cut-off function η_1 as in Definition 3.3.2 be given. Then there exists $\varepsilon_0 = \varepsilon_0(\delta, \Gamma, \eta_1) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and any $w_\varepsilon \in C^2(\omega)$ the following property holds: If for all $x \in \{|d| \geq 3\delta\}$

$$|\varepsilon\nabla w_\varepsilon(x)| \leq \Lambda e^{-\frac{\mu}{\varepsilon}|d_\delta(x)|} \quad \text{and} \quad |w_\varepsilon(x) - \operatorname{sgn}(d(x))| \leq \Lambda e^{-\frac{\mu}{\varepsilon}|d_\delta(x)|},$$

then there exists $R_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega)$ such that

$$\begin{aligned}(-\varepsilon^2\Delta + \mathrm{Id})&(\eta_\delta w_\varepsilon + (1-\eta_\delta)\operatorname{sgn}(d)) \\ &= \eta_\delta \cdot (-\varepsilon^2\Delta + \mathrm{Id})w_\varepsilon + (1-\eta_\delta)\operatorname{sgn}(d) + \chi_{\{|d|\geq 3\delta\}}R_\varepsilon. \qquad (3.3.10)\end{aligned}$$

Proof. We calculate

$$\begin{aligned}&(-\varepsilon^2\Delta + \mathrm{Id})(\eta_\delta w_\varepsilon + (1-\eta_\delta)\operatorname{sgn}(d)) \\ &= \eta_\delta \cdot (-\varepsilon^2\Delta + \mathrm{Id})w_\varepsilon + (1-\eta_\delta)\operatorname{sgn}(d) - 2\varepsilon^2\nabla w_\varepsilon \nabla\eta_\delta - \varepsilon^2\Delta\eta_\delta \cdot (w_\varepsilon - \operatorname{sgn}(d)).\end{aligned}$$

For the last two terms we obtain

$$2\varepsilon^2|\nabla w_\varepsilon \nabla\eta_\delta| \leq \frac{4\varepsilon^2}{\delta}\chi_{\{|d|\geq 3\delta\}}|\nabla w_\varepsilon| \leq \frac{4\varepsilon\Lambda}{\delta}\chi_{\{|d|\geq 3\delta\}}e^{-\frac{\mu}{\varepsilon}|d_\delta|}$$

and

$$|\varepsilon^2\Delta\eta_\delta \cdot (w_\varepsilon - \operatorname{sgn}(d))| \leq \frac{\varepsilon^2\Lambda C(\eta_1,\Gamma)}{\delta^2}\chi_{\{|d|\geq 3\delta\}}e^{-\frac{\mu}{\varepsilon}|d_\delta|}.$$

Choosing $\varepsilon_0 \leq \min\{\frac{1}{8}, (2C(\eta_1,\Gamma))^{-\frac{1}{2}}\}\delta$ yields the claim. □

We need a corresponding statement for functions that are defined in terms of the inner variables.

Lemma 3.3.11.
There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and any $w \in C^2(\mathbb{R} \times \omega)$ the following holds: Assume

$$w - \operatorname{sgn},\ \partial_z w,\ \nabla_x w \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)$$

and define $w_\varepsilon^{\mathrm{in}} := w \circ \Psi_\varepsilon^{-1} \in C^2(\omega)$. Then there exist $R_\varepsilon^w \in X_\delta^{\frac{\tilde{\mu}}{\varepsilon},2\Lambda}(\Omega)$ such that (3.3.10) holds for $w_\varepsilon = w_\varepsilon^{\mathrm{in}}$.

Proof. We observe that $|d_\delta(x)| \le \frac{2}{3}|d(x)|$ in $\{|d| \ge 3\delta\}$ and deduce in $\{3\delta \le |d| \le 5\delta\}$

$$\varepsilon\big|\nabla w_\varepsilon^{\text{in}}(x)\big| \le \big|\partial_z w(z,x)\big| + \varepsilon\big|(\nabla_x w)(z,x)\big| \le 2\Lambda e^{-\mu|z|} \le 2\Lambda e^{-\frac{3\mu d_\delta(x)}{2\varepsilon}}$$

and

$$|w_\varepsilon^{\text{in}}(x) - \operatorname{sgn}(d(x))| \le \Lambda e^{-\mu|z|} \le \Lambda e^{-\frac{3\mu d_\delta(x)}{2\varepsilon}}.$$

The claim then follows from Lemma 3.3.10. □

Now can prove that $\mathcal{A}_\varepsilon$ preserves the properties listed in Assumptions 3.3.4.

Proposition 3.3.12.
Consider $K \le 2$ and $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ as in Assumption 3.3.4. Then the family $(\overline{u}_\varepsilon)_{0<\varepsilon<\varepsilon_0}$, $\overline{u}_\varepsilon = \mathcal{A}_\varepsilon u_\varepsilon$ has an analogue representation, meaning that there exist $\tilde{\Lambda} > 0$ and profile functions $v_j \in C^2(\mathbb{R}\times\omega)$, $j = 0,\dots,K$ such that v_j are C^4 with respect to $x \in \omega$ and such that for all $0 < \varepsilon < \varepsilon_0$

$$\overline{u}_\varepsilon = \eta_\delta \overline{u}_\varepsilon^{\text{in}} + (1-\eta_\delta)\operatorname{sgn}(d) + \varepsilon^{K+1} R_\varepsilon^v \qquad \text{in } \Omega, \tag{3.3.11}$$

$$\overline{u}_\varepsilon^{\text{in}} = \left(\sum_{j=0}^{K} \varepsilon^j v_j\right) \circ \Psi_\varepsilon^{-1} \qquad \text{in } \{|d| < 4\delta\}, \tag{3.3.12}$$

with $R_\varepsilon^v \in X_\delta^{\frac{\mu}{\varepsilon},\tilde{\Lambda}}(\Omega)$ for all $0 < \varepsilon < \varepsilon_0$.
The profile functions are given by

$$v_0(z,x) = \boldsymbol{A}_0 u_0(z,x) \tag{3.3.13}$$
$$v_1(z,x) = \boldsymbol{A}_0\big(u_1(z,x) + H(y)v_0'(z,x)\big) \tag{3.3.14}$$
$$v_2(z,x) = \boldsymbol{A}_0\big(u_2(z,x) + H(y)v_1'(z,x) + (\Delta_x - z|\mathrm{I\!I}|^2(y)\partial_z)v_0(z,x)\big) \tag{3.3.15}$$

for $z \in \mathbb{R}$, $x \in \omega$, $y = \Pi_\Gamma(x)$. Moreover,

$$v_0 - \operatorname{sgn},\ v_1, v_2 \in X(\mathbb{R};\Gamma) \qquad \textit{and} \qquad \partial_z v_j,\ |\nabla_x v_j|,\ \Delta_x v_j \in X(\mathbb{R};\Gamma)\ \ \textit{for } j = 0,1,2.$$

Proof. Assume $K = 2$, let $0 < \mu < 1$, $\Lambda > 0$, and $u_0, u_1, u_2 \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)$ as in Assumption 3.3.4. Using the representation for Δd from (2.1.9) we obtain in $\{|d| < 3\delta\}$

$$\begin{aligned}-\varepsilon^2\Delta + \mathrm{Id} &= -\partial_z^2 + \mathrm{Id} - \varepsilon\Delta d\,\partial_z - \varepsilon^2\Delta_x \\ &= -\partial_z^2 + \mathrm{Id} - \varepsilon H\partial_z - \varepsilon^2(\Delta_x - z|\mathrm{I\!I}|^2\partial_z) - \varepsilon^3|z|^2 R_\varepsilon^H\partial_z,\end{aligned}$$

and

$$\begin{aligned}&\Big(u_\varepsilon^{\text{in}} - (-\varepsilon^2\Delta + \mathrm{Id})\overline{u}_\varepsilon^{\text{in}}\Big) \circ \Psi_\varepsilon^{-1} \\ &\qquad = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 - (-\partial_z^2 + \mathrm{Id})v_0 - \varepsilon\big((-\partial_z^2 + \mathrm{Id})v_1 - H\partial_z v_0\big) \\ &\qquad\quad - \varepsilon^2\big((-\partial_z^2 + \mathrm{Id})v_2 - H\partial_z v_1 - (\Delta_x - z|\mathrm{I\!I}|^2\partial_z)v_0\big) + \varepsilon^3 R_\varepsilon^v.\end{aligned} \tag{3.3.16}$$

The equations (3.3.13)-(3.3.15) are then equivalent to the property, that the expression in (3.3.16) vanishes up to order $\mathcal{O}(\varepsilon^3)$. We then obtain

$$(-\varepsilon^2\Delta + \mathrm{Id})\overline{u}_\varepsilon^{\text{in}} = u_\varepsilon^{\text{in}} + \varepsilon^3 R_\varepsilon^v \circ \Psi_\varepsilon, \tag{3.3.17}$$

with

$$
\begin{aligned}
R_\varepsilon^v \circ \Psi_\varepsilon = &-|z|^2 R_\varepsilon^H \partial_z v_0 + H\partial_z v_2 + (\Delta_x - z|\mathrm{II}|^2\partial_z)v_1 \\
&- \varepsilon|z|^2 R_\varepsilon^H \partial_z v_1 + \varepsilon(\Delta_x - z|\mathrm{II}|^2\partial_z)v_2 - \varepsilon^2|z|^2 R_\varepsilon^H \partial_z v_2.
\end{aligned}
\tag{3.3.18}
$$

It remains to show that the profile functions v_0, v_1, v_2, their derivatives and the error term R_ε^v have the claimed exponential control. In this proof Λ may change from line to line but will always be independent of ε.

We first observe from Lemma 3.3.8 and $(\mathbf{A}_0 \operatorname{sgn}) = \operatorname{sgn}(z)(1 - e^{-|z|})$ that v_0 inherits the exponential decay to ± 1 from u_0, since

$$
v_0 - \operatorname{sgn} = \mathbf{A}_0(u_0 - \operatorname{sgn}) + (\mathbf{A}_0 \operatorname{sgn} - \operatorname{sgn}) \in X^{\mu,\Lambda}(\mathbb{R};\Gamma).
$$

Lemma 3.3.8 also yields $|\nabla v_0| = |\mathbf{A}_0 \nabla u_0| \le \mathbf{A}_0|\nabla u_0| \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)$, $\partial_z v_0 \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)$, $\Delta v_0 = \mathbf{A}_0 \Delta u_0 \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)$, and $v_0 \in C^2(\mathbb{R}\times\omega)$ is C^4-regular with respect to x.

For v_1 Lemma 3.3.8 yields

$$
v_1 = \mathbf{A}_0(u_1 + H\partial_z v_0) \in C^2(\mathbb{R}\times\omega) \cap X^{\mu,\Lambda}(\mathbb{R};\Gamma),
$$

v_1 is C^4 with respect to $x \in \omega$ and $\partial_z v_1 \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)$. Using $\mathbf{A}_0\partial_z^2 = -\mathbf{A}_0(-\partial_z^2 + \mathrm{Id}) + \mathbf{A}_0 = \mathbf{A}_0 - \mathrm{Id}$ we get in addition

$$
|\nabla_x v_1| = \Big|\mathbf{A}_0\Big(\nabla_x u_1 + H\partial_z^2 v_0\Big)\Big| \le \mathbf{A}_0|\nabla_x u_1| + |H||(\mathbf{A}_0 - \mathrm{Id})v_0| \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)
$$

because $|\nabla u_1| \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)$ by assumption, the previous results, and Lemma 3.3.8. Similar we have

$$
\Delta_x v_1 = \mathbf{A}_0\Big(\Delta_x u_1 + H\Delta\partial_z v_0\Big) = \mathbf{A}_0\Delta u_1 + H\partial_z \mathbf{A}_0 \Delta_x v_0 \in X^{\mu,\Lambda}(\mathbb{R};\Gamma)
$$

by assumptions for u_1, the previous results, and Lemma 3.3.8.

Owing to $X^{\mu,\Lambda}(\mathbb{R};\Gamma) \subseteq X^{\frac{10\mu}{11},\Lambda}(\mathbb{R};\Gamma)$, $\mathrm{II} \in C_b^2(\omega)$, and $|z v_0 e^{\frac{10\mu}{11}|z|}| = |z e^{-\frac{\mu}{11}|z|}||v_0 e^{\mu|z|}| \le \Lambda$ we have for u_2

$$
v_2 = \mathbf{A}_0\Big(u_2 + H\partial_z v_1 + (\Delta_x - z|\mathrm{II}|^2\partial_z)v_0\Big) \in X^{\frac{10}{11}\mu,\Lambda}(\mathbb{R};\Gamma),
$$

with $v_2 \in C^2(\mathbb{R}\times\omega)$, v_2 is C^4 with respect to $x \in \omega$ and $\partial_z v_2 \in X^{\frac{10}{11}\mu,\Lambda}(\mathbb{R};\Gamma)$ by Lemma 3.3.8. In addition we have

$$
\begin{aligned}
|\nabla_x v_2| &\le |\mathbf{A}_0\nabla_x u_2| + |H\partial_z \mathbf{A}_0 \nabla_x v_1| + |\mathbf{A}_0 \nabla_x(\Delta_x - z|\mathrm{II}|^2\partial_z)v_0| \in X^{\frac{10}{11}\mu,\Lambda}(\mathbb{R};\Gamma) \quad \text{and} \\
\Delta_x v_2 &= |\mathbf{A}_0\Delta_x u_2| + |H\partial_z \mathbf{A}_0 \Delta_x v_1| + |\mathbf{A}_0 \Delta_x(\Delta_x - z|\mathrm{II}|^2\partial_z)v_0| \in X^{\frac{10}{11}\mu,\Lambda}(\mathbb{R};\Gamma).
\end{aligned}
$$

From the asymptotic control of v_0, v_1, v_2 and their derivatives we conclude from (3.3.18) $R_\varepsilon^v \circ \Psi_\varepsilon \in X^{\frac{10}{11}\mu,\Lambda}(\mathbb{R};\Gamma)$. Since $\phi_1(z) \le \frac{9}{10}z$ for all $z \ge 0$ by Definition 3.3.2 we deduce $|d_\delta| \le \frac{9}{10}|d|$ and therefore obtain $R_\varepsilon^v \in X_\delta^{\mu,\Lambda}(\Omega)$.

The previous observations show that $w \coloneqq v_0 + \varepsilon v_1 + \varepsilon^2 v_2$ satisfies the assumptions of Lemma 3.3.11 with μ replaced by $\frac{10}{11}\mu$. Applying the lemma to $w_\varepsilon^{\mathrm{in}} = \overline{u}_\varepsilon^{\mathrm{in}}$ and using (3.3.17) we therefore obtain for some $R_\varepsilon^w \in X_\delta^{\frac{15\mu}{11\varepsilon},\Lambda}(\Omega)$

$$
\begin{aligned}
(-\varepsilon^2\Delta + \mathrm{Id})&\big(\eta_\delta \overline{u}_\varepsilon^{\mathrm{in}} + (1-\eta_\delta)\operatorname{sgn}(d)\big) \\
&= \eta_\delta \cdot (-\varepsilon^2\Delta + \mathrm{Id})\overline{u}_\varepsilon^{\mathrm{in}} + (1-\eta_\delta)\operatorname{sgn}(d) + \chi_{\{|d|\geq 3\delta\}} R_\varepsilon^w \\
&= \eta_\delta u_\varepsilon^{\mathrm{in}} + \varepsilon^3 \eta_\delta R_\varepsilon^v + (1-\eta_\delta)\operatorname{sgn}(d) + \chi_{\{|d|\geq 3\delta\}} R_\varepsilon^w \\
&= u_\varepsilon + \varepsilon^3 R_\varepsilon,
\end{aligned}
$$

where $R_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega)$ due to the stronger exponential decay of R_ε^w.
Since $\nabla\big(\eta_\delta \overline{u}_\varepsilon^{\mathrm{in}} + (1-\eta_\delta)\operatorname{sgn}(d)\big)\cdot \nu_\Omega = 0$ at $\partial\Omega$ we deduce from Lemma 3.3.9

$$
\overline{u}_\varepsilon = \eta_\delta \overline{u}_\varepsilon^{\mathrm{in}} + (1-\eta_\delta)\operatorname{sgn}(d) + \varepsilon^3 \tilde{R}_\varepsilon,
$$

with $\tilde{R}_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\tilde{\Lambda}}(\Omega)$. ☐

The next theorem proves a lower bound estimate for phase-field approximations that satisfy the Assumptions 3.3.4.

Theorem 3.3.13 (Γ–lim inf estimate for special class of function)**.**
We assume 3.3.1 and let $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ satisfy the Assumptions 3.3.4. Then we have

$$
c_{\mathrm{AG}}\sigma_{\mathrm{AG}}\mathcal{W}(u) \leq \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon). \tag{3.3.19}
$$

Proof. Let $\overline{u}_\varepsilon = \mathcal{A}_\varepsilon(u_\varepsilon)$, recall $H_\varepsilon^{\mathrm{AG}} = \frac{1}{\varepsilon}\big(f(u_\varepsilon) - \mathcal{A}_\varepsilon u_\varepsilon\big)$. By Assumption 3.3.4, Proposition 3.3.12, and Lemma 3.3.7 we deduce in $\{|d| < 2\delta\}$ for ε sufficiently small

$$
\begin{aligned}
\varepsilon H_\varepsilon^{\mathrm{AG}} &= \big(f(u_\varepsilon) - \overline{u}_\varepsilon\big) \\
&= \eta_\delta f(u_\varepsilon)^{\mathrm{in}} + (1-\eta_\delta)\operatorname{sgn} d - \big(\eta_\delta \overline{u}_\varepsilon^{\mathrm{in}} + (1-\eta_\delta)\operatorname{sgn}(d)\big) + \varepsilon^{K+1} R_\varepsilon \\
&= \eta_\delta\big(f(u_\varepsilon)^{\mathrm{in}} - \overline{u}_\varepsilon^{\mathrm{in}}\big) + \varepsilon^{K+1} R_\varepsilon,
\end{aligned} \tag{3.3.20}
$$

and in particular for $K = 0$

$$
\varepsilon H_\varepsilon^{\mathrm{AG}} = \eta_\delta \cdot \big(f(u_0) - \mathbf{A}_0 u_0\big) \circ \Psi_\varepsilon^{-1} + \varepsilon R_\varepsilon.
$$

Together with (2.1.7) we deduce

$$
\begin{aligned}
\varepsilon^2 \mathcal{W}_\varepsilon(u_\varepsilon) = &\int_\Omega \frac{1}{\varepsilon}\eta_\delta^2\big(f(u_0) - \mathbf{A}_0 u_0\big)^2 \circ \Psi_\varepsilon^{-1} \,\mathrm{d}\mathcal{L}^n \\
&+ 2\int_\Omega \eta_\delta\big(f(u_0) - \mathbf{A}_0 u_0\big) \circ \Psi_\varepsilon^{-1} R_\varepsilon \,\mathrm{d}\mathcal{L}^n + \varepsilon \int_\Omega R_\varepsilon^2 \,\mathrm{d}\mathcal{L}^n \\
\geq &\int_\Gamma \int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} \frac{1}{2}\big(f(u_0) - \mathbf{A}_0 u_0\big)^2 \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} \\
&- C\varepsilon \int_\Gamma \int_{-\infty}^{\infty} \big|f(u_0) - \mathbf{A}_0 u_0\big| \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} \\
\geq &\int_\Gamma \int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} \frac{1}{2}\big(f(u_0) - \mathbf{A}_0 u_0\big)^2 \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} - C\varepsilon,
\end{aligned} \tag{3.3.21}
$$

where we have used, that $f(u_0) - \operatorname{sgn}$ and $\mathbf{A}_0 u_0 - \operatorname{sgn}$ both decay exponentially at $\pm\infty$. In order to prove (3.3.19) it is sufficient to consider the case $\liminf_{\varepsilon\to 0} \varepsilon^2 \mathcal{W}_\varepsilon(u_\varepsilon) = 0$. This implies $f(u_0) = \mathbf{A}_0 u_0$, that is $u_0(\cdot, x) = q_0(\cdot - z_0(y))$ and $v_0(\cdot, x) = \overline{q}_0(\cdot - z_0(y))$ with $y = \Pi_\Gamma(x)$. The condition (3.3.3) implies that $z_0(y) = h_0(y)$.

With $u_0(\cdot, x) = q_0(\cdot - h_0(y))$ and $v_0(\cdot, x) = \overline{q}_0(\cdot - h_0(y))$ we deduce from Proposition 3.3.12, Lemma 3.3.7, and (3.3.20) with $K = 1$

$$H_\varepsilon^{\mathrm{AG}}(x) = \eta_\delta(x)\Big[f'[q_0(z - h_0(y))]u_1(z, x) - \mathbf{A}_0\big[u_1(z, x) + H(y)\overline{q}_0'(z - h_0(y))\big]\Big] + \varepsilon R_\varepsilon(x). \tag{3.3.22}$$

and by similar calculations as above

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon) \\
&\geq \int_\Gamma \int_{-\frac{3\delta}{\varepsilon} - h_0(y)}^{\frac{3\delta}{\varepsilon} - h_0(y)} \big|(f'(q_0) - \mathbf{A}_0)u_1(\cdot + h_0(y), x) - H(y)\mathbf{A}_0(\overline{q}_0')\big|^2 \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1}(y) &(3.3.23)\\
&\geq \int_\Gamma \int_{-\infty}^{\infty} \big|(f'(q_0) - \mathbf{A}_0)u_1(\cdot + h_0(y), x) - H(y)\mathbf{A}_0(\overline{q}_0')\big|^2 \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1}(y). &(3.3.24)
\end{aligned}$$

If $H(y) = 0$ the inner integral is minimized by $u_1(\cdot + h_0(y), x) \equiv 0$. Therefore, to prove a lower bound, we can assume $u_1(z + h_0(y), x) = H(y)\tilde{u}_1(z, x)$ and compute

$$\liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon) \geq \int_\Gamma |H(y)|^2 \inf_{w(\cdot,y)\in L^2(\mathbb{R})} \Xi(w(\cdot, y)) \,\mathrm{d}\mathcal{H}^{n-1}(y) \geq c_{\mathrm{AG}}\sigma_{\mathrm{AG}} \int_\Gamma |H|^2 \,\mathrm{d}\mathcal{H}^{n-1}, \tag{3.3.25}$$

with the functional Ξ as in Lemma 3.1.14. We obtain that for every minimizer $\tilde{u}_1$ of Ξ there exists $\alpha \in \mathbb{R}$ such that $\tilde{u}_1 = q_1 + \alpha q_0'$. This proves (3.3.19). Finally, we can determine α from condition (3.3.3), which implies

$$\begin{aligned}
0 &= u_\varepsilon(y + \varepsilon(h_0(y) + \varepsilon h_1(y) + \varepsilon^2 R_\varepsilon^h(y))\nu(y)) \\
&= q_0(\varepsilon h_1(y) + \varepsilon^2 R_\varepsilon^h(y)) + \varepsilon\big(q_1(\varepsilon h_1(y) + \varepsilon^2 R_\varepsilon^h(y)) + \alpha q_0'(\varepsilon h_1(y) + \varepsilon^2 R_\varepsilon^h(y))\big) + \mathcal{O}(\varepsilon^2) \\
&= \varepsilon q_0'(0)(h_1(y) + \alpha) + \mathcal{O}(\varepsilon^2)
\end{aligned}$$

and therefore $\alpha = -h_1(y)$. □

The proof shows that equality in (3.3.19) can only be attained if $u_0(z, x) = q_0(\cdot - h_0(y))$ and $u_1(z, x) = H(y)q_1(z - h_0(y)) - h_1(y)q_0'(z - h_0(y))$. By Theorem 3.1.3 we have that $q_0 - \operatorname{sgn}, \overline{q}_0 - \operatorname{sgn}$, q_0' all decay exponentially at $\pm\infty$. Combining the Lemmata 3.3.7, 3.1.15, and 3.3.8 shows that q_1, $\overline{q}_0'$ also decay exponentially at $\pm\infty$.

We therefore obtain as a candidate for a recovery sequence $(u_\varepsilon)_{\varepsilon>0}$

$$u_\varepsilon = \eta_\delta u_\varepsilon^{\mathrm{in}} + (1 - \eta_\delta)\operatorname{sgn}(d), \quad u_\varepsilon^{\mathrm{in}}(\cdot, x) = q_0 + \varepsilon H(y)q_1. \tag{3.3.26}$$

3.4 Rigorous proof of the Γ–$\limsup$ estimate

In this section we do the constructive part of the Γ–convergence statement. We use the previous computations from the asymptotic expansion of approximations and the successive minimization of the energy order as orientation for the rigorous proof.
We use the previous computations and the candidate (3.3.26).

Theorem 3.4.1 (lim sup estimate for "gradient-free" Willmore approximation)**.**
We assume 3.3.1. Then there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$ such that $u_\varepsilon \longrightarrow u$ in $L^1(\Omega)$ and

$$\lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon) = c_{\mathrm{AG}}\sigma_{\mathrm{AG}}\mathcal{W}(u).$$

Proof. We will use the Ansatz (3.3.26) and let

$$u_\varepsilon(x) = \eta_\delta(x)\big(q_0(z) + \varepsilon H(y)q_1(z)\big) + \big(1-\eta_\delta(x)\big)\operatorname{sgn}(d(x)),$$

where $x = \Psi_\varepsilon(z,y)$. The convergence towards u was already shown in Lemma 3.3.5. We deduce from Proposition 3.3.12 that $\overline{u}_\varepsilon := \mathcal{A}_\varepsilon u_\varepsilon$ can be represented as

$$\overline{u}_\varepsilon(x) = \eta_\delta(x)\big(\overline{q}_0(z) + \varepsilon\overline{q}_1(z,x)\big) + (1-\eta_\delta(x))\operatorname{sgn} z + \varepsilon^2 R_\varepsilon(x)$$

with $\overline{q}_0 = \mathbf{A}_0 q_0$ as in (3.3.13) and

$$v_1(\cdot,x) = H(y)\overline{q}_1 \quad \text{for any} \quad x\in\omega,\ y=\Pi_\Gamma(x), \quad \overline{q}_1 = \mathbf{A}_0(q_1+\overline{q}_0'), \tag{3.4.1}$$

as introduced before Lemma 3.1.15. Moreover, we have $\sup_{\varepsilon>0}\sup_{x\in\Omega}|R_\varepsilon(x)| \le C$ and $\overline{q}_1, \overline{q}_1'$ decay exponentially at $\pm\infty$ by Lemma 3.3.8. We deduce from equations (3.3.22) and (3.1.22).

$$\begin{aligned} H_\varepsilon^{\mathrm{AG}}(x) &= \eta_\delta(x)H(y)\big(f'(q_0)q_1 - \mathbf{A}_0(q_1+\overline{q}_0')\big)(z) + \varepsilon R_\varepsilon(x) \\ &= -\eta_\delta(x)\sigma_{\mathrm{AG}}H(y)q_0'(z) + \varepsilon R_\varepsilon(x). \end{aligned}$$

By similar calculations as above this implies

$$\begin{aligned} \mathcal{W}_\varepsilon(u_\varepsilon) &= \int_\Omega \frac{1}{\varepsilon}\eta_\delta^2\big|\sigma_{\mathrm{AG}}q_0'\big|^2 H^2\circ\Psi_\varepsilon^{-1}\,\mathrm{d}\mathcal{L}^n + 2\int_\Omega \eta_\delta(\sigma_{\mathrm{AG}}q_0')H\circ\Psi_\varepsilon^{-1}R_\varepsilon\,\mathrm{d}\mathcal{L}^n + \varepsilon\int_\Omega R_\varepsilon^2\,\mathrm{d}\mathcal{L}^n \\ &\le \sigma_{\mathrm{AG}}^2\int_\Gamma H^2\,\mathrm{d}\mathcal{H}^{n-1}\int_{-\infty}^{\infty}|q_0'|^2\,\mathrm{d}\mathcal{L}^1 + C\mathcal{H}^{n-1}(\Gamma)\varepsilon\int_{-\infty}^{\infty}|q_0'|(z)\,\mathrm{d}\mathcal{L}^1 + \varepsilon C\mathcal{L}^n(\Omega) \\ &\le c_{\mathrm{AG}}\sigma_{\mathrm{AG}}\int_\Gamma H^2\,\mathrm{d}\mathcal{H}^{n-1} + C\varepsilon. \end{aligned}$$

This yields $\limsup_{\varepsilon\to0}\mathcal{W}_\varepsilon(u_\varepsilon) \le c_{\mathrm{AG}}\sigma_{\mathrm{AG}}\mathcal{W}(u)$ and together with (3.3.19) the recovery sequence property. ☐

3.5 Diffuse gradient flows in the AG model

In this section we consider the dynamic of evolving surfaces $\Gamma(t)$. We assume that there exists a phase-field function $(u_\varepsilon)_{\varepsilon>0}$ which has an expansion with respect to $\Gamma(t)$ and is a solution to either the rescaled gradient flow of the diffuse perimeter (3.5.2) or the gradient flow of the rescaled diffuse Willmore energy (3.5.3). Our goal is to show that the surfaces evolve by mean curvature flow or Willmore flow respectively. We refer to Definitions 2.1.7 and 2.1.9 for the formulation of mean curvature flow and Willmore flow.

We already know the L^2-gradient of $\mathcal{P}_\varepsilon^{\mathrm{AG}}$, in fact we have

$$\nabla_{L^2}\mathcal{P}_\varepsilon^{\mathrm{AG}} = \frac{1}{\varepsilon}\Big(-\overline{u}_\varepsilon + u + \frac{1}{2}W'(u)\Big) = H_\varepsilon^{\mathrm{AG}}.$$

For the L^2-gradient of $\mathcal{W}_\varepsilon^{\mathrm{AG}}$ we get for all test functions η

$$\langle \eta, \delta\mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon)\rangle_{C_c^\infty(\Omega)'} = \int_\Omega \frac{2}{\varepsilon^2}\eta(f'(u_\varepsilon)\operatorname{Id} - \mathcal{A}_\varepsilon)H_\varepsilon^{\mathrm{AG}} \,\mathrm{d}\mathcal{L}^n$$

and thus

$$\nabla_{L^2}\mathcal{W}_\varepsilon^{\mathrm{AG}}(u) = \frac{2}{\varepsilon^2}\Big(1 + \frac{1}{2}W''(u) - \mathcal{A}_\varepsilon\Big)H_\varepsilon^{\mathrm{AG}} = \frac{2}{\varepsilon^2}\Big(f'(u) - \mathcal{A}_\varepsilon\Big)H_\varepsilon^{\mathrm{AG}}. \tag{3.5.1}$$

Now we can formulate the gradient flow equations for the diffuse mean curvature flow

$$\varepsilon\partial_t u_\varepsilon = -H_\varepsilon^{\mathrm{AG}} \tag{3.5.2}$$

and diffuse Willmore flow

$$\varepsilon\partial_t u_\varepsilon = -\frac{2}{\varepsilon^2}(f'(u_\varepsilon)\operatorname{Id} - \mathcal{A}_\varepsilon)H_\varepsilon^{\mathrm{AG}}. \tag{3.5.3}$$

We write

$$L_\varepsilon := f'(u_\varepsilon)\operatorname{Id} - \mathcal{A}_\varepsilon.$$

Assumption 3.5.1 (Set evolution)**.**
Consider a continuous evolution of open sets $(E(t))_{t\in[0,T]}$ in Ω with associated signed distance function $d : \Omega_T \to \mathbb{R}$, $d(\cdot,t) = \operatorname{dist}(\cdot, \Omega\setminus E(t)) - \operatorname{dist}(\cdot, E(t))$, phase boundaries $\Gamma(t) := \partial E(t)$ for $t \in [0,T]$ and $\Omega_T := \Omega \times [0,T]$.
We assume the following properties:

1. *$\Gamma(t)$ is a C^4-regular hypersurface for all $t \in [0,T]$.*
2. *$\bigcup_{t\in[0,T]} E(t) \Subset \Omega$.*
   With this assumption we can choose $\delta > 0$ sufficiently small such that for all $t \in [0,T]$ the projections $\Pi_{\Gamma(t)} : \{|d(\cdot,t)| < 5\delta\} \longrightarrow \Gamma(t)$ are well defined and set
   $$\omega_T := \{(x,t) \in \Omega_T \;:\; |d(x,t)| < 5\delta\}.$$
3. *$d \in C_b^1(\omega_T)$ and $D_x^\gamma d \in C_b^0(\omega_T)$ for all $\gamma \in \mathbb{N}_0^n$ with $|\gamma| \leq 4$.*

Let $\Psi_\varepsilon(\cdot,t)$, $t \in [0,T]$ denote the parametrization that are defined according to (2.1.5) *with Γ replaced by $\Gamma(t)$.*

We extend the definition of functions that are exponentially decaying to the time-dependent case and set

$$X_\delta^{\mu,\Lambda}(\Omega_T) := \Big\{u \in L^\infty(\Omega_T) \;:\; \operatorname*{ess-sup}_{x\in\Omega_T} |e^{\mu|d_\delta(x)|}u(x,t)| \leq \Lambda\Big\} \tag{3.5.4}$$

and

$$X(\mathbb{R};\Gamma_T) := \Big\{u \in L^\infty(\mathbb{R}\times\omega_T) \,|\, \exists\Lambda,\mu > 0 : \operatorname*{ess-sup}_{(z,x,t)\in\mathbb{R}\times\omega_T} |e^{\mu|z|}u(z,x,t)| \leq \Lambda, \\ u(z,\cdot,t) \text{ is constant in normal direction}\Big\}. \tag{3.5.5}$$

We consider the modified distance functions d_δ and the cut-off functions η_δ as defined in Assumption 3.3.2 and introduce classes of phase field evolutions that we will consider in the following.

Assumption 3.5.2 (Phase field evolution)**.**
Let $(E(t))_{t\in[0,T]}$ *be a continuous evolution of sets in* Ω*, the signed distance function* d*, and* $\delta > 0$ *as in Assumption 3.5.1 be given. Consider an evolution of smooth phase fields* $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$*. We assume that there exist* $\mu \in (0,1)$*,* $\Lambda > 0$*, and profile functions* $u_j : \mathbb{R} \times \omega_T \to \mathbb{R}$*,* $j \in \{0,1,2\}$*, such that for all* $0 < \varepsilon < \varepsilon_0$ *and all* $t \in [0,T]$

$$u_\varepsilon(\cdot,t) = \eta_\delta(\cdot,t)u_\varepsilon^{\mathrm{in}}(\cdot,t) + (1-\eta_\delta(\cdot,t))\operatorname{sgn}(d(\cdot,t)) + \varepsilon^3 R_\varepsilon \quad \text{in } \Omega_T, \tag{3.5.6}$$

$$u_\varepsilon^{\mathrm{in}}(\cdot,t) = \Big(\sum_{j=0}^{2} \varepsilon^j u_j(\cdot,t)\Big) \circ \Psi_\varepsilon^{-1} \quad \text{in } \{|d(\cdot,t)| < 4\delta\}, \tag{3.5.7}$$

and such that the following properties hold:

1. *The profile functions* $u_j \in C^0(\mathbb{R}\times\omega_T)$*,* $u_j = u_j(z,x,t)$ *satisfy* $u_j(z,\cdot) \in C_b^1(\omega_T)$*,* $D_x^\gamma u_j(z,\cdot) \in C_b^0(\omega_T)$ *for all* $\gamma \in \mathbb{N}_0^n$ *with* $|\gamma| \le 4$

   $u_0 - \operatorname{sgn} \in X(\mathbb{R};\Gamma_T)$*,* u_j*,* $|\nabla_x u_j|$*,* $\Delta_x u_j$*,* $|\nabla_x \Delta_x u_j|$*,* $\Delta_x^2 u_j \in X(\mathbb{R};\Gamma_T)$ *for* $j \in \{1,2\}$*.*

2. *The remainder satisfies* $R_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega_T)$ *for all* $0 < \varepsilon < \varepsilon_0$*.*

Moreover, we assume that

$$\{u_\varepsilon(\cdot,t) = 0\} = \Gamma(t) \quad \text{for all } t \in [0,T],\ 0 < \varepsilon < \varepsilon_0 \tag{3.5.8}$$

and that

$$\mathcal{W}_\varepsilon^{\mathrm{AG}}(u_\varepsilon(\cdot,0)) + \mathcal{P}_\varepsilon^{\mathrm{AG}}(u_\varepsilon(\cdot,0)) \le C \tag{3.5.9}$$

for all $0 < \varepsilon < \varepsilon_0$*. We want to highlight operators which only refer to the z-variable and thus we write them in bold letter, such as* $\boldsymbol{L}_0$ *and* $\boldsymbol{A}_0$ *from* (3.1.17) *and Definition 3.1.6.*

We have chosen in (3.5.8) for a more restrictive setting than in the static case. We could also have allowed for an offset between the zero level set of $u_\varepsilon(\cdot,t)$ and $\Gamma(t)$ as in (3.3.3). For simplicity we restrict ourselves to (3.5.8) but allow an additional contribution of order ε in the gradient flow equations; see (3.5.15) and (3.5.10) below.

Theorem 3.5.3 (Convergence towards the mean curvature flow)**.**
Consider a sequence of evolutions of smooth phase fields $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *as in Assumption 3.5.2, satisfying an asymptotic expansion* (3.5.6)*-*(3.5.7) *with respect to an evolution* $(E(t))_{t\in[0,T]}$ *of sets in* Ω*. Assume that* u_ε *satisfies*

$$-\varepsilon\partial_t u_\varepsilon = \frac{1}{\varepsilon}\big(f(u_\varepsilon) - \mathcal{A}_\varepsilon(u_\varepsilon)\big) + \varepsilon R_\varepsilon, \tag{3.5.10}$$

with $\sup\limits_{0<\varepsilon<\varepsilon_0} \|R_\varepsilon\|_{C^0(\overline{\Omega_T})} \le C$*.*
Then $(\Gamma(t))_{t\in[0,T]}$ *evolves by the rescaled mean curvature flow*

$$\mathcal{V} = \sigma_{\mathrm{AG}} H \tag{3.5.11}$$

with σ_{AG} *as defined in* (3.1.7)*.*

Proof. We consider the equation (3.5.10), expand both sides of and evaluate the identity order by order. To identify the evolution law in the limit $\varepsilon \to 0$ it is sufficient to consider the region $\{|d| < 2\delta\}$, in which $\eta_\delta \equiv 1$.
We in particular use that the right-hand side of equation (3.5.10) is in this region to the relevant orders already determined by the inner expansion with respect to ε of u_ε: even though $\mathcal{A}_\varepsilon u_\varepsilon$ and depend on the values of u_ε in the whole set Ω, applying Proposition 3.3.12 shows that we only need the inner expansion of u_ε to determine the relevant contributions in $\{|d| < 2\delta\}$.

For the left-hand side of (3.5.10) we obtain in $\{|d| < 2\delta\}$ from (3.5.6), (3.5.17), and the definition of $\mathcal{V}$

$$-\varepsilon\partial_t u_\varepsilon = -\varepsilon \sum_{j=0}^{2} \Big(\partial_t u_j + \frac{1}{\varepsilon}\partial_z u_j \partial_t d\Big) + \mathcal{O}(\varepsilon) = -\partial_z u_0 \partial_t d + \mathcal{O}(\varepsilon) = -q_0' \mathcal{V} + \mathcal{O}(\varepsilon). \tag{3.5.12}$$

We expand the right-hand side and deduce from Proposition 3.3.12, Lemma 3.3.7, and (3.3.20) that in $\{|d| < 2\delta\}$

$$H_\varepsilon^{\mathrm{AG}}(x,t) = H_0(z,x,t) + \varepsilon R_\varepsilon^H(x,t) \tag{3.5.13}$$

with $R_\varepsilon^H \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega_T)$ and H_0 is characterized as follows: Firstly, by (3.3.22)

$$H_0 = f'(q_0)u_1 - \mathbf{A}_0(u_1 + H\overline{q}_0') = \mathbf{L}_0(u_1) - \mathcal{A}_1(q_0), \tag{3.5.14}$$

where $\mathcal{A}_1 = H\mathbf{A}_0^2\partial_z$ and where here and below $H_\varepsilon^{\mathrm{AG}}, R_\varepsilon$ are evaluated in (x,t), q_0 in z, u_j in (z,x,t) and H in (y,t) with $y = \Pi_{\Gamma(t)}x$.

Now we consider the ε^{-1}-order gives and get

$$0 = f(u_0) - \mathbf{A}_0 u_0$$

and thus $u_0 = q_0$. We further expand the right-hand side of (3.5.10) and get in $\{|d| < 2\delta\}$

$$\begin{aligned}\frac{1}{\varepsilon}(f(u_\varepsilon) - \mathcal{A}_\varepsilon(u_\varepsilon)) &= f'(q_0)u_1 - \mathbf{A}_0(u_1) - \mathcal{A}_1(q_0) + \mathcal{O}(\varepsilon)\\ &= \mathbf{L}_0(u_1) - \mathcal{A}_1(q_0) + \mathcal{O}(\varepsilon).\end{aligned}$$

Equating this with the expansion in (3.5.12) we get by testing with $q_0' \in \ker(\mathbf{L}_0)$

$$-\mathcal{V}\int_{\mathbb{R}} |q_0'|^2 \,\mathrm{d}\mathcal{L}^1 = -\int_{\mathbb{R}} q_0' \mathcal{A}_1(q_0)\,\mathrm{d}\mathcal{L}^1$$

Taking the defining integrals (3.1.7) and $\mathcal{A}_1 = H\mathbf{A}_0^2\partial_z$ into account we get the evolution by mean curvature

$$\mathcal{V} = \sigma_{\mathrm{AG}} H. \qquad \square$$

Theorem 3.5.4 (Convergence towards the Willmore flow).
Consider a sequence of evolutions of smooth phase fields $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *as in Assumption 3.5.2, satisfying an asymptotic expansion* (3.5.6)-(3.5.7) *with respect to an evolution* $(E(t))_{t\in[0,T]}$ *of sets in* Ω. *Assume that* u_ε *satisfies*

$$-\varepsilon\partial_t u_\varepsilon = \frac{2}{\varepsilon^2}\Big(f'(u_\varepsilon)\,\mathrm{Id} - \mathcal{A}_\varepsilon\Big)\Big(\frac{f(u_\varepsilon) - \mathcal{A}_\varepsilon(u_\varepsilon)}{\varepsilon}\Big) + \varepsilon R_\varepsilon, \tag{3.5.15}$$

with $\sup\limits_{0<\varepsilon<\varepsilon_0} \|R_\varepsilon\|_{C^0(\overline{\Omega_T})} \leq C$. *Then* $(\Gamma(t))_{t\in[0,T]}$ *evolves by the rescaled Willmore flow*

$$\mathcal{V} = 2\sigma_{\mathrm{AG}}^2\big(- \Delta_\Gamma H - H|\mathrm{II}|^2 + \frac{1}{2}H^3\big) \tag{3.5.16}$$

with σ_{AG} *as defined in* (3.1.7).

Proof. Similar to the proof for the other evolution we will expand both sides of (3.5.15) and evaluate the identity order by order. To identify the evolution law in the limit $\varepsilon \to 0$ it is sufficient to consider the region $\{|d| < 2\delta\}$ as before. We in particular use that the right-hand side of equation (3.5.15) is in this region to the relevant orders already determined by the inner expansion with respect to ε of u_ε: Even though $\overline{u}_\varepsilon = \mathcal{A}_\varepsilon u_\varepsilon$ and $\mathcal{A}_\varepsilon H_\varepsilon^{\mathrm{AG}}$ depend on the values of u_ε in the whole of Ω, applying Proposition 3.3.12 and Lemma 3.3.7 shows that we only need the inner expansion of u_ε to determine the relevant contributions in $\{|d| < 2\delta\}$.

To expand the right-hand side of (3.5.15) we first consider $H_\varepsilon^{\mathrm{AG}}$. Since under the flow (3.5.15) the energy $\mathcal{W}_\varepsilon^{\mathrm{AG}}$ decreases with time and by (3.5.9) we obtain that $\mathcal{W}_\varepsilon(u_\varepsilon(\cdot,t))$ is uniformly bounded. By the calculations in the proof of Theorem 3.3.13, see (3.3.21), we therefore deduce that

$$u_0(z,x,t) = q_0(z) \quad \text{for all } (x,t) \in \omega_T. \tag{3.5.17}$$

We deduce from Proposition 3.3.12, Lemma 3.3.7, and (3.3.20) that in $\{|d| < 2\delta\}$

$$H_\varepsilon^{\mathrm{AG}}(x,t) = H_0(z,x,t) + \varepsilon H_1(z,x,t) + \varepsilon^2 H_2(z,x,t) + \varepsilon^3 R_\varepsilon^H(x,t) \tag{3.5.18}$$

with $R_\varepsilon^H \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega_T)$ and H_0, H_1 characterized as follows: Firstly, by (3.5.14) we have

$$H_0 = \mathbf{L}_0(u_1) - \mathcal{A}_1(q_0),$$

where $\mathcal{A}_1 = H\mathbf{A}_0^2\partial_z$ and where $H_\varepsilon^{\mathrm{AG}}, R_\varepsilon$ are evaluated in (x,t), q_0 in z, u_j in (z,x,t) and H in (y,t) with $y = \Pi_{\Gamma(t)}x$.
Secondly, we derive from (3.3.20) with $K = 2$ and Proposition 3.3.12, Lemma 3.3.7

$$\begin{aligned} H_1 &= \Big(\frac{1}{2}f''(u_0)(u_1)^2 + f'(u_0)u_2\Big) - \mathbf{A}_0\big(u_2 + Hv_1' + (\Delta_x - z|\mathrm{II}|^2\partial_z)v_0\big) \\ &= \mathbf{L}_0(u_2) + \frac{1}{2}f''(u_0)u_1^2 - \mathcal{A}_1(u_1) - \mathcal{A}_2(u_0), \end{aligned} \tag{3.5.19}$$

where

$$\begin{aligned} &\mathcal{A}_2 = \Delta_x\mathbf{A}_0^2 - |\mathrm{II}|^2\mathbf{A}_0 z\partial_z\mathbf{A}_0 + H^2\mathbf{A}_0^3\partial_z^2 \quad \text{and} \\ &\mathbf{L}_0 = f'(q_0)\,\mathrm{Id} - \mathbf{A}_0 \quad \text{as defined in} \quad (3.1.17). \end{aligned} \tag{3.5.20}$$

The function H_2 in (3.5.18) belongs to $X(\mathbb{R};\Gamma_T)$ and is C^2-regular with respect to the x variable. We will see below that this term is not relevant for the identification of the evolution law, thus we will not include its precise characterization.

We next consider the action of $L_\varepsilon = f'(u_\varepsilon)\,\mathrm{Id} - \mathcal{A}_\varepsilon$ on $H_\varepsilon^{\mathrm{AG}}$. Since we are only interested in the values in the region $\{|d| < 2\delta\}$ we can use a Taylor expansion and the representation of u_ε in this region for the local term $f'(u_\varepsilon)\,\mathrm{Id}$.
For the application of $\mathcal{A}_\varepsilon$ to $H_\varepsilon^{\mathrm{AG}}$ we use analogue arguments as in Proposition 3.3.12 with the following difference. We only have C^2-regularity with respect to the x-variable of the profile functions that represent $H_\varepsilon^{\mathrm{AG}}$. Therefore we obtain only C^0-regularity with respect to the x-variable for the profile functions representing $\mathcal{A}_\varepsilon H_\varepsilon^{\mathrm{AG}}$. Therefore, following the analogue computation as in the proof of Proposition 3.3.12 we deduce

$$L_\varepsilon(H_\varepsilon^{\mathrm{AG}}) = \mathbf{L}_0(H_0) + \varepsilon\big(L_1(H_0) + \mathbf{L}_0(H_1)\big) + \varepsilon^2\big(L_2(H_0) + L_1(H_1) + \mathbf{L}_0(H_2)\big) + \varepsilon^3 R_\varepsilon \quad (3.5.21)$$

and

$$L_1 = f''(q_0)u_1\,\mathrm{Id} - \mathcal{A}_1, \quad L_2 = f''(q_0)u_2\,\mathrm{Id} + \frac{1}{2}f'''(q_0)u_1^2\,\mathrm{Id} - \mathcal{A}_2. \quad (3.5.22)$$

and $R_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega_T)$.

Now we expand the equation (3.5.15). For the left-hand side we obtain in $\{|d| < 2\delta\}$ from (3.5.12)

$$\varepsilon\partial_t u_\varepsilon = \partial_z u_0 \partial_t d + \mathcal{O}(\varepsilon) \;=\; q_0'\mathcal{V} + \mathcal{O}(\varepsilon).$$

We next consider the right-hand side of evolution (3.5.15). By (3.5.18)-(3.5.19) we obtain

$$\begin{aligned}\varepsilon^{-2}L_\varepsilon(H_\varepsilon^{\mathrm{AG}}) = {} & \varepsilon^{-2}\mathbf{L}_0(H_0) + \varepsilon^{-1}\big(\mathbf{L}_0(H_1) + L_1(H_0)\big) \\ & + \big(\mathbf{L}_0(H_2) + L_1(H_1) + L_2(H_0)\big) + \mathcal{O}(\varepsilon). \end{aligned} \quad (3.5.23)$$

To order ε^{-2} we deduce from equations (3.5.15) and (3.5.12) that $\mathbf{L}_0(H_0) = 0$, which is by (3.5.14) equivalent to

$$0 = \mathbf{L}_0(\mathbf{L}_0(u_1) - H\mathbf{A}_0^2(q_0')).$$

In addition we have the condition $u_1(0) = 0$. Comparing this to Lemma 3.1.14 and in particular equation (3.1.22) we deduce $u_1 = Hq_1$ and $H_0 = -H\sigma_{\mathrm{AG}}q_0'$. In particular,

$$\begin{aligned} L_1 &= H\mathbf{L}_1, & \mathcal{A}_1 &= H\mathbf{A}_1, \\ \mathbf{L}_1 &= f''(q_0)q_1\,\mathrm{Id} - \mathbf{A}_0^2\partial_z, & \mathbf{A}_1 &= \mathbf{A}_0^2\partial_z, \end{aligned}$$

where $\mathbf{L}_1$ and $\mathbf{A}_1$ only depend on z. From (3.5.15) we conclude that also the contribution of order ε^{-1} of the right-hand side in (3.5.15) vanishes, thus

$$0 = L_1(H_0) + L_0(H_1) = -\sigma_{\mathrm{AG}}H^2\mathbf{L}_1(q_0') + \mathbf{L}_0(H_1). \quad (3.5.24)$$

We will now proceed to the crucial order ε^0 in equation (3.5.15). We test the corresponding equation with q_0' and integrate with respect to the variable z. We get by formulas (3.5.15) and (3.5.12)

$$-\frac{1}{2}\|q_0'\|_{L^2(\mathbb{R})}^2\mathcal{V} = \int_{\mathbb{R}} q_0'(L_1(H_1) + L_2(H_0))\,\mathrm{d}\mathcal{L}^1. \quad (3.5.25)$$

For the second term on the right-hand side of equation (3.5.25) we use

$$\begin{aligned} L_2(H_0) &= (f''(q_0)u_2 + \frac{1}{2}f'''(q_0)u_1^2 - \mathcal{A}_2)(-H\sigma_{\mathrm{AG}}q_0') \\ &= -\sigma_{\mathrm{AG}}Hf''(q_0)q_0'u_2 - \frac{\sigma_{\mathrm{AG}}}{2}H^3 f'''(q_0)q_0'q_1^2 + \sigma_{\mathrm{AG}}\mathcal{A}_2(Hq_0'). \end{aligned} \tag{3.5.26}$$

We obtain the following commutator rule, with $[A,B] \coloneqq AB - BA$ for operators A, B

$$[\partial, \mathbf{L}_0](w) = \big(\mathbf{L}_0(w)\big)' - \mathbf{L}_0(w') = f''(q_0)q_0'w \quad \text{for all} \quad w \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$$

and rewrite equation (3.5.19)

$$\mathbf{L}_0(u_2) = w \quad \text{with} \quad w \coloneqq H_1 - H^2\frac{1}{2}f''(q_0)q_1^2 + H^2\mathbf{A}_1(q_1) + \mathcal{A}_2(q_0). \tag{3.5.27}$$

The commutator helps us to generate $\mathbf{L}_0$ in front of u_2 in the right-hand side of equation (3.5.26) so we can apply equation (3.5.27)

$$\int_{\mathbb{R}} q_0' f''(q_0)q_0'u_2 \,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} q_0'(w' - \mathbf{L}_0(u_2')) \,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} q_0'w' \,\mathrm{d}\mathcal{L}^1.$$

Together with equation (3.5.26) we obtain

$$\begin{aligned} \int_{\mathbb{R}} q_0' L_2(H_0) \,\mathrm{d}\mathcal{L}^1 &= -\sigma_{\mathrm{AG}}H \int_{\mathbb{R}} q_0'w' \,\mathrm{d}\mathcal{L}^1 - \sigma_{\mathrm{AG}} \int_{\mathbb{R}} q_0' \Big(\frac{1}{2}H^3 f'''(q_0)q_0'q_1^2 - \mathcal{A}_2(Hq_0')\Big) \,\mathrm{d}\mathcal{L}^1 \\ &= -\sigma_{\mathrm{AG}}H \int_{\mathbb{R}} q_0' \Big(H_1 - H^2\frac{1}{2}f''(q_0)q_1^2 + H^2\mathbf{A}_1(q_1) + \mathcal{A}_2(q_0)\Big)' \,\mathrm{d}\mathcal{L}^1 \\ &\quad - \sigma_{\mathrm{AG}} \int_{\mathbb{R}} q_0' \Big(\frac{1}{2}H^3 f'''(q_0)q_0'q_1^2 - \mathcal{A}_2(Hq_0')\Big) \,\mathrm{d}\mathcal{L}^1. \\ &= \sigma_{\mathrm{AG}}H \int_{\mathbb{R}} \Big(q_0''H_1 - q_0'\partial_z\mathcal{A}_2 q_0\Big) \,\mathrm{d}\mathcal{L}^1 \\ &\quad + \sigma_{\mathrm{AG}}H^3 \int_{\mathbb{R}} q_0' \Big(f''(q_0)q_1q_1' + \frac{1}{2}f'''(q_0)q_0'q_1^2 - \mathbf{A}_1(q_1')\Big) \,\mathrm{d}\mathcal{L}^1 \\ &\quad - \sigma_{\mathrm{AG}} \int_{\mathbb{R}} q_0' \Big(\frac{1}{2}H^3 f'''(q_0)q_0'q_1^2 - \mathcal{A}_2(Hq_0')\Big) \,\mathrm{d}\mathcal{L}^1. \\ &= \sigma_{\mathrm{AG}}H \int_{\mathbb{R}} q_0''H_1 \,\mathrm{d}\mathcal{L}^1 + \sigma_{\mathrm{AG}}H^3 \int_{\mathbb{R}} q_0' \Big(f''(q_0)q_1q_1' - \mathbf{A}_1 q_1'\Big) \,\mathrm{d}\mathcal{L}^1 \\ &\quad + \sigma_{\mathrm{AG}} \int_{\mathbb{R}} q_0' [\mathcal{A}_2, H\partial_z](q_0) \,\mathrm{d}\mathcal{L}^1. \end{aligned} \tag{3.5.28}$$

Next we calculate the commutator

$$[\mathcal{A}_2, H\partial_z](w)$$

for $w \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$. Since w is independent from x we have $\Delta_x w = 0$ and thus

$$\begin{aligned} [\mathcal{A}_2, H\partial_z](w) = \mathcal{A}_2(H\partial_z w) - H\partial_z\mathcal{A}_2 w &= \Delta_x H\mathbf{A}_0^2 w' - H|\mathrm{I\!I}|^2\mathbf{A}_0(z\mathbf{A}_0 w'') \\ &\quad + H^3\mathbf{A}_0^3 w''' + H|\mathrm{I\!I}|^2\mathbf{A}_0\Big(\partial_z(z\mathbf{A}_0 w')\Big) - H^3\mathbf{A}_0^3 w''' \\ &= \Delta_x H\mathbf{A}_0^2 w' + H|\mathrm{I\!I}|^2\mathbf{A}_0^2 w' = (\Delta_x H + H|\mathrm{I\!I}|^2)\mathbf{A}_1 w. \end{aligned}$$

Since the functions in ω are constant in normal direction we can replace Δ_x it with the Laplace-Beltrami operator on $\Gamma(t)$. We write $\Delta_\Gamma = \Delta_{\Gamma(t)}$ for simpler notation.

We get with $\int_{\mathbb{R}} q_0' \mathbf{A}_1 q_0 \, \mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} |\mathbf{A}_0 q_0'|^2 \, \mathrm{d}\mathcal{L}^1 = c_{\mathrm{AG}}$ and the definition of $\mathbf{L}_1$

$$\begin{aligned} \int_{\mathbb{R}} q_0' L_2(H_0) \, \mathrm{d}\mathcal{L}^1 = \sigma_{\mathrm{AG}} H \int_{\mathbb{R}} q_0'' H_1 \, \mathrm{d}\mathcal{L}^1 + \sigma_{\mathrm{AG}} H^3 \int_{\mathbb{R}} q_0' \mathbf{L}_1(q_1') \, \mathrm{d}\mathcal{L}^1 \\ + c_{\mathrm{AG}} \sigma_{\mathrm{AG}} (\Delta_\Gamma H + H |\mathrm{II}|^2). \end{aligned} \tag{3.5.29}$$

By differentiating formula (3.1.22) we have

$$-\sigma_{\mathrm{AG}} q_0'' = f''(q_0) q_0' q_1 + f'(q_0) q_1' - \mathbf{A}_0(q_1') - \mathbf{A}_1(q_0') = \mathbf{L}_1(q_0') + \mathbf{L}_0(q_1'). \tag{3.5.30}$$

The next tool we need is the commutator $[\mathbf{A}_0, z]$. Since the operators are defined on $L^2(\mathbb{R})$ we need to make sure, that the functions multiplied with z are still in $L^2(\mathbb{R})$. Let $h \in C_b^0(\mathbb{R}) \cap L^2(\mathbb{R})$ with exponential decay. This implies $z \mapsto zh(z) \in L^2(\mathbb{R})$ and $\mathbf{A}_0 h, \mathbf{A}_0 h', \mathbf{A}_0 h''$ all have exponential decay. From Proposition 3.1.8 we have $\mathbf{A}_0(zh), \mathbf{A}_0 h \in C_b^2(\mathbb{R}) \cap L^2(\mathbb{R})$ and we get

$$\begin{aligned} -z\mathbf{A}_0 h'' + z\mathbf{A}_0 h &= zh \\ -2\mathbf{A}_0 h' - \mathbf{A}_0(zh'') + \mathbf{A}_0(zh) &= -\mathbf{A}_0(zh)'' + \mathbf{A}_0(zh) = zh. \end{aligned}$$

Equating and rearranging yields

$$\begin{aligned} \mathbf{A}_0(zh) - z\mathbf{A}_0 h &= 2\mathbf{A}_0 h' + \mathbf{A}_0(zh'') - z\mathbf{A}_0 h'' \\ \iff [\mathbf{A}_0, z](-h'' + h) &= 2\mathbf{A}_0 h' \end{aligned}$$

We conclude by replacing h with $\mathbf{A}_0 h$ the following commutator rule

$$[\mathbf{A}_0, z](h) = 2\mathbf{A}_0^2 h' = 2\mathbf{A}_1 h.$$

Since q_0' satisfies the conditions for h we can apply the commutator rule together with $\mathbf{L}_0(q_0') = 0$ and get

$$0 = z\mathbf{L}_0(q_0') = f'(q_0) z q_0' - z\mathbf{A}_0(q_0') = \mathbf{L}_0(zq_0') + 2\mathbf{A}_1(q_0'). \tag{3.5.31}$$

Before moving to the final calculations we need the anti-symmetric part of $\mathbf{L}_1$. Since $\mathbf{A}_1$ is anti-symmetric because of the negative sign in the partial integration formula we have for $w_1, w_2 \in L^2(\mathbb{R})$

$$\begin{aligned} \int_{\mathbb{R}} (w_1 \mathbf{L}_1(w_2) - w_2 \mathbf{L}_1(w_1)) \, \mathrm{d}\mathcal{L}^1 &= \int_{\mathbb{R}} \Big(w_1 (f''(q_0) q_1 w_2 - w_1 \mathbf{A}_1 w_2 \Big) \, \mathrm{d}\mathcal{L}^1 \\ &\quad - \int_{\mathbb{R}} \Big(w_2 f''(q_0) q_1 w_1 - w_2 \mathbf{A}_1 w_1 \Big) \, \mathrm{d}\mathcal{L}^1 \\ &= \int_{\mathbb{R}} 2 w_2 \mathbf{A}_1(w_1) \, \mathrm{d}\mathcal{L}^1. \end{aligned} \tag{3.5.32}$$

We consider the sum of the contributions from the first terms of the right-hand side of formulas (3.5.25) and (3.5.29) to deduce

$$
\begin{aligned}
H \int_{\mathbb{R}} \Big(q_0' \mathbf{L}_1(H_1) + \sigma_{\mathrm{AG}} q_0'' H_1 \Big) \,\mathrm{d}\mathcal{L}^1 &\overset{(3.5.30)}{=} H \int_{\mathbb{R}} \Big(q_0' \mathbf{L}_1(H_1) - H_1 \big(\mathbf{L}_1(q_0') + \mathbf{L}_0(q_1') \big) \Big) \,\mathrm{d}\mathcal{L}^1 \\
&\overset{(3.5.32)}{=} H \int_{\mathbb{R}} H_1 \Big(2\mathbf{A}_1(q_0') - \mathbf{L}_0(q_1') \Big) \,\mathrm{d}\mathcal{L}^1 \\
&\overset{(3.5.31)}{=} H \int_{\mathbb{R}} H_1 \mathbf{L}_0(-z q_0' - q_1') \,\mathrm{d}\mathcal{L}^1 \\
&\overset{(3.5.24)}{=} -\sigma_{\mathrm{AG}} H^3 \int_{\mathbb{R}} (z q_0' + q_1') \mathbf{L}_1(q_0') \,\mathrm{d}\mathcal{L}^1. \qquad (3.5.33)
\end{aligned}
$$

Plugging the results from equations (3.5.33) and (3.5.29) into the identity (3.5.25) we obtain

$$
-\mathcal{V} = 2\sigma_{\mathrm{AG}}^2 (\Delta_\Gamma H + H|\mathrm{II}|^2 + \frac{\kappa_1}{c_{\mathrm{AG}}} H^3),
$$

with

$$
\begin{aligned}
\kappa_1 &= \int_{\mathbb{R}} \Big(q_0' \mathbf{L}_1(q_1') - (z q_0' + q_1') \mathbf{L}_1(q_0') \Big) \,\mathrm{d}\mathcal{L}^1 \\
&\overset{(3.5.32)}{=} \int_{\mathbb{R}} \Big(2 q_1' \mathbf{A}_1(q_0') - z q_0' \mathbf{L}_1(q_0') \Big) \,\mathrm{d}\mathcal{L}^1 \\
&\overset{(3.5.31)}{=} - \int_{\mathbb{R}} z q_0' (\mathbf{L}_1(q_0') + \mathbf{L}_0(q_1')) \,\mathrm{d}\mathcal{L}^1 \overset{(3.5.30)}{=} -\frac{c_{\mathrm{AG}}}{2},
\end{aligned}
$$

which proves the Willmore-flow equation. □

4 A higher order approximation of the Willmore energy based on the Karali-Katsoulakis model

In this chapter we consider a new diffuse Willmore energy, motivated by the contributions of Karali and Katsoulakis [KK07]. They considered a combination of surface diffusion and ad/de-sorption, modelled by

$$-\varepsilon\partial_t u_\varepsilon = \big(-\varepsilon^2\Delta + \mathrm{Id}\big)\Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big), \tag{4.0.1}$$

which is a cross-over between the Allen-Cahn and the Cahn-Hilliard equation. Here W is a double-well potential, as before. The PDE (4.0.1) has gradient flow structure as the right-hand side is the gradient of the standard diffuse perimeter $\mathcal{P}_\varepsilon$ from Definition 2.4.1 with respect to the metric induced by $(\phi,\psi) \mapsto \int_\Omega \phi\mathcal{A}_\varepsilon\psi \,\mathrm{d}\mathcal{L}^n$ where $\mathcal{A}_\varepsilon = \big(-\varepsilon^2\Delta + \mathrm{Id}\big)^{-1}$ is the solution operator from Lemma 3.1.10. In the case of smooth solutions we can apply the chain rule and obtain for solutions of (4.0.1) (with suitable boundary conditions)

$$\begin{aligned}\partial_t\mathcal{P}_\varepsilon(u_\varepsilon) &= \partial_t\int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\,\mathrm{d}\mathcal{L}^n = \int_\Omega \Big(-\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\partial_t u_\varepsilon\,\mathrm{d}\mathcal{L}^n \\ &= -\int_\Omega \frac{1}{\varepsilon}H_\varepsilon\big(-\varepsilon^2\Delta + \mathrm{Id}\big)H_\varepsilon\,\mathrm{d}\mathcal{L}^n = -\int_\Omega \Big(\frac{1}{\varepsilon}|H_\varepsilon|^2 + \varepsilon\big|\nabla H_\varepsilon\big|^2\Big)\,\mathrm{d}\mathcal{L}^n. \end{aligned} \tag{4.0.2}$$

We write $H_\varepsilon := H_\varepsilon(u_\varepsilon) := -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)$ for the diffuse mean curvature. To interpret this identity we compare it to the sharp interface setting and the standard approximation of the perimeter. If a family of surfaces is evolving by mean curvature flow we have $\partial_t\mathcal{P} = -\mathcal{W}$. Here the Willmore energy appears on the right-hand side of the energy-dissipation .

If w_ε is a solution to a formulation of diffuse mean curvature flow, i.e.

$$-\varepsilon\partial_t w_\varepsilon = -\varepsilon\Delta w_\varepsilon + \frac{1}{\varepsilon}W'(w_\varepsilon)$$

we get $\partial_t\mathcal{P}_\varepsilon = -\mathcal{W}_\varepsilon$. Here the energy-dissipation features the standard diffuse Willmore energy. With this background we expect the right-hand side of (4.0.2) to be a new diffuse Willmore energy. This is the motivation for us to investigate whether the diffuse functional $\mathcal{W}_\varepsilon^{\mathrm{KK}} : H^3(\Omega) \longrightarrow [0,\infty]$ with

$$\mathcal{W}_\varepsilon^{\mathrm{KK}}(u) := \int_\Omega \Big(\frac{1}{\varepsilon}|H_\varepsilon(u)|^2 + \varepsilon\big|\nabla H_\varepsilon(u)\big|^2\Big)\,\mathrm{d}\mathcal{L}^n \tag{4.0.3}$$

is converging towards a multiple of the Willmore energy in the sense of Γ–convergence with respect to the $L^1(\Omega)$-topology. We start with the Γ–lim sup estimate in this chapter and handle the Γ–lim inf estimate in Chapter 5. It is important to mention that we obtain a larger factor in front of the Willmore energy in the limit compared to the standard Willmore approximation. Thus the higher order term contributes on the same ε-scale as the classical $\int_\Omega \frac{1}{\varepsilon}|H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n$ term.

Another motivation to consider the Γ–convergence is the fact that the diffuse Willmore energy (4.0.3) appears in diffuse formulations of the Brakke flow (see Definition 2.5.2) or the De Giorgi type varifold solutions for rescaled mean curvature flow (see Definition 2.5.3), i.e. the Γ–convergence helps prove that solutions to (4.0.1) converge towards solutions for mean curvature flow in a suitable sense.

4.1 Preparations

We introduce the notations for this chapter and prove a few lemmata that we need below.

Assumption 4.1.1 (and Notations).
In this chapter we assume $\Omega \subseteq \mathbb{R}^n$ is open. We consider the standard double-well potential $W(r) := (1-r^2)^2$ for $r \in \mathbb{R}$, the induced optimal profile $q_0 \in C^1(\mathbb{R})$ which solves

$$q_0' = \sqrt{2W(q_0)} \quad \text{and} \quad q_0(0) = 0. \tag{4.1.1}$$

In addition to that we define $\overline{q}_0 := \boldsymbol{A}_0 q_0$ such that

$$-\overline{q}_0'' + \overline{q}_0 = q_0 \quad \text{in} \quad \mathbb{R} \quad \text{and} \quad \overline{q}_0(x) \longrightarrow \pm 1 \quad \text{as} \quad x \to \pm\infty.$$

The important constants in this model are given by

$$c_0 := \int_{\mathbb{R}} |q_0'|^2 \,\mathrm{d}\mathcal{L}^1 \quad \text{and} \quad \sigma := \frac{c_0}{\displaystyle\int_{\mathbb{R}} q_0' \overline{q}_0' \,\mathrm{d}\mathcal{L}^1}. \tag{4.1.2}$$

The existence of the integrals is proved in the next lemmata. In this model we consider the Cahn–Hilliard energy $\mathcal{P}_\varepsilon : L^1(\Omega) \longrightarrow [0,\infty]$, also called standard diffuse perimeter introduced in 2.4.1

$$\mathcal{P}_\varepsilon(u) := \begin{cases} \displaystyle\int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{\varepsilon} W(u)\Big) \,\mathrm{d}\mathcal{L}^n, & \text{if } u \in H^1(\Omega) \cap L^4(\Omega) \\ +\infty, & \text{else.} \end{cases} \tag{4.1.3}$$

Lemma 4.1.2 (Properties of q_0).
The optimal profile q_0 is given by

$$q_0(r) = \tanh(\sqrt{2} r).$$

It holds

- $q_0 \in C^\infty(\mathbb{R})$, $q_0'' = W'(q_0)$, *and* $q_0' > 0$ *on* $\mathbb{R}$.
- $\lim_{r \to \pm\infty} q_0(r) = \pm 1$ *for all* $r \in \mathbb{R}$.

- $|q_0(r) - \operatorname{sgn}(r)| \leq 2e^{-2|r|}$ *for* $r \in \mathbb{R}$.
- $q_0 - \operatorname{sgn} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.
- *There exist* $C, c > 0$ *such that for all* $j \in \{1, 2, 3\}$ *and all* $r \in \mathbb{R}$ *we have* $|q_0^{(j)}(r)| \leq Ce^{-c|r|}$.
- $q_0', q_0'', q_0''' \in L^2(\mathbb{R})$ *and* $q_0' \in H^2(\mathbb{R})$.
- q_0 *is odd.*

Most of these properties can be proven for more general double well potentials, see [BNN15], however the formulation of the lemma is sufficient for this thesis.

Proof. Differentiating tanh we get for all $r \in \mathbb{R}$

$$\tanh'(r) = \frac{\cosh^2(r) - \sinh^2(r)}{\cosh^2(r)} = \frac{1}{\cosh^2(r)} = 1 - \tanh^2(r).$$

It follows that $q_0 = \tanh(\sqrt{2}\cdot)$ is indeed the unique solution to (4.1.1). From there we conclude by squaring and differentiating again

$$2q_0'q_0'' = 2W(q_0)q_0' \quad \text{and thus} \quad q_0'' = W'(q_0).$$

We also have for all $r > 0$

$$q_0(r) = \frac{e^r - e^{-r}}{e^r + e^{-r}} = 1 - \frac{2e^{-2r}}{1 + e^{-2r}} \quad \text{and thus} \quad |q_0(r) - 1| \leq 2e^{-2r}.$$

We can proceed similar for $r < 0$. Thus the second and third claim are proven. The exponential decay immediately implies the fourth claim. For the last claims we need to transfer the exponential decay from $q_0 - \operatorname{sgn}$ to its derivative. We use (4.1.1) and estimate for all $r \in \mathbb{R}$ with the third property

$$0 < q_0'(r) = \sqrt{2W(q_0(r))} = \sqrt{2}|1 - q_0(r)^2| = \sqrt{2}|1 + q_0(r)||1 - q_0(r)| \leq 4\sqrt{2}e^{-2|r|}.$$

Thus we get $q_0' \in L^2(\mathbb{R})$. For the second derivative we use $q_0'' = W''(q_0)$, the previous estimate and get for all $r \in \mathbb{R}$

$$|q_0''(r)| = |W'(q_0(r))| = 4|q_0(r)||1 - q_0(r)^2| \leq 16e^{-2|r|}.$$

We get $q_0'' \in L^2(\mathbb{R})$. Lastly we prove a similar estimate for the third derivative. Let $r \in \mathbb{R}$, then we have

$$|q_0'''| = |W''(q_0(r))|q_0' = |12q_0^2 - 4|q_0' \leq 32\sqrt{2}e^{-2|r|}.$$

It follows $q_0''' \in L^2(\mathbb{R})$ and thus $q_0' \in H^2(\mathbb{R})$. Since tanh is odd so is q_0. $\square$

With the properties of $\mathbf{A}_0$ we can transfer most of these properties to $\overline{q}_0$.

Lemma 4.1.3 (Properties of $\overline{q}_0$)**.**
The function $\overline{q}_0 = \boldsymbol{A}_0 q_0$ *has the following properties*

- $\overline{q}_0 \in C^\infty(\mathbb{R})$ *and* $\overline{q}_0' > 0$ *on* $\mathbb{R}$*.*
- $\lim\limits_{r \to \pm\infty} \overline{q}_0(r) = \pm 1$ *for all* $r \in \mathbb{R}$*.*
- *There exist* $C, c > 0$ *such that* $|\overline{q}_0(r) - \operatorname{sgn}(r)| \leq Ce^{-c|r|}$ *for* $r \in \mathbb{R}$*.*
- $\overline{q}_0 - \operatorname{sgn} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$*.*
- *There exist* $C, c > 0$ *such that for all* $j \in \{1, 2, 3\}$ *and all* $r \in \mathbb{R}$ *we have* $|\overline{q}_0^{(j)}(r)| \leq Ce^{-c|r|}$*.*
- $\overline{q}_0', \overline{q}_0'', \overline{q}_0''' \in L^2(\mathbb{R})$ *and* $\overline{q}_0' \in H^2(\mathbb{R})$*.*
- $\overline{q}_0$ *is odd.*

Proof. From $q_0 \in C^\infty(\mathbb{R})$ and the fact that $\mathbf{A}_0$ is a convolution operator and thus only improves the regularity, see 3.1.8, we get $\overline{q}_0 \in C^\infty(\mathbb{R})$. We get $\overline{q}_0' > 0$ from (5) in Proposition 3.1.8 which implies $\overline{q}_0' = J_1 * q_0'$. The limit as $r \to \pm\infty$ follows from the limit of q_0 and (4) from Proposition 3.1.8. The next claim follows from (a) and (c) in Lemma 3.1.9 and

$$\overline{q}_0 - \operatorname{sgn} = \mathbf{A}_0(q_0 - \operatorname{sgn}) + \mathbf{A}_0 \operatorname{sgn} - \operatorname{sgn}.$$

The exponential decay of $\overline{q}_0 - \operatorname{sgn}$ yields $\overline{q}_0 - \operatorname{sgn} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. The exponential decay of $\overline{q}_0', \overline{q}_0'', \overline{q}_0'''$ follows from the exponential decay of q_0', q_0'', q_0''' and (a) in Lemma 3.1.9. It follows that $\overline{q}_0', \overline{q}_0'', \overline{q}_0''' \in L^2(\mathbb{R})$ and $\overline{q}_0' \in H^2(\mathbb{R})$. $\overline{q}_0$ is odd because q_0 is and this carries over because of the explicit representation of $\mathcal{A}_0$ as the convolution operator induced by J_1. □

It is typical for asymptotic constructions that a Fredholm operator appears in the relevant ε-scale. This is also the case here and thus we prove suitable properties of the operator that we need to consider.

Lemma 4.1.4 ($\mathbf{T}_0$ is Fredholm)**.**
The operator

$$\boldsymbol{T}_0 : H^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \quad \boldsymbol{T}_0 \coloneqq -\partial^2 + W''(q_0) \operatorname{Id} \tag{4.1.4}$$

is a Fredholm operator with index 0.

Proof. We have

$$\begin{aligned}\mathbf{T}_0 &= -\partial^2 + W''(q_0) = 8\Big(-\frac{1}{8}\partial^2 + \operatorname{Id} + \frac{1}{8}W''(q_0) - \operatorname{Id}\Big)\\ &= 8\Big[\operatorname{Id} + \Big(\frac{1}{8}W''(q_0) - \operatorname{Id}\Big)\Big(-\frac{1}{8}\partial^2 + \operatorname{Id}\Big)^{-1}\Big]\Big(-\frac{1}{8}\partial^2 + \operatorname{Id}\Big)\end{aligned}$$

The operator $\big(-\frac{1}{8}\partial^2 + \operatorname{Id}\big) : H^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is an isomorphism (which can be shown with the Lax-Milgram Theorem), thus it suffices to show that

$$\Big(\frac{1}{8}W''(q_0) - \operatorname{Id}\Big)\Big(-\frac{1}{8}\partial^2 + \operatorname{Id}\Big)^{-1} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$$

is a compact operator. We prove that it is a Hilbert-Schmidt operator which implies the compactness. Since $(-\partial^2 + \mathrm{Id})^{-1}$ can be represented as a convolution operator we can use this representation to characterize $\big(-\frac{1}{8}\partial^2 + \mathrm{Id}\big)^{-1}$. The Greens function of $-\frac{1}{8}\partial^2 + \mathrm{Id}$ is given by

$$\tilde{J}(r) := \frac{1}{\sqrt{8}} J_1(\sqrt{8}r) = \frac{1}{4\sqrt{2}} e^{-2\sqrt{2}|r|},$$

for $r \in \mathbb{R}$, where J_1 is the Greens function of $(-\partial^2 + \mathrm{Id})$ from (3.1.9). We calculate the Hilbert-Schmidt norm using $W''(r) = 8 + 12(r^2 - 1)$ and $|1 - q^2(r)| \leq 4e^{-2|r|}$ from the proof of Lemma 4.1.2

$$\begin{aligned}
\int_{\mathbb{R}} \Big|\frac{1}{8} W''(q_0(r)) - 1\Big|^2 \int_{\mathbb{R}} \frac{1}{32} e^{-4\sqrt{2}|r-s|} \, \mathrm{d}s \, \mathrm{d}r &= \int_{\mathbb{R}} \Big|\frac{3}{2}(q_0(r)^2 - 1)\Big|^2 \int_{\mathbb{R}} \frac{1}{32} e^{-4\sqrt{2}|s|} \, \mathrm{d}s \, \mathrm{d}r \\
&= \frac{9}{64} \int_{\mathbb{R}} |1 - q_0(r)^2|^2 \, \mathrm{d}r \int_0^\infty e^{-4\sqrt{2}s} \, \mathrm{d}s \\
&\leq \frac{9}{16\sqrt{2}} \int_{\mathbb{R}} e^{-4|r|} \, \mathrm{d}r = \frac{9}{32\sqrt{2}} < \infty.
\end{aligned}$$

Since compact perturbations of the identity are Fredholm operators with index 0 [Alt12, Thm. 9.8] the proof is complete. □

We can even provide more information on the kernel of $\mathbf{T}_0$. It follows from Lemma 4.1.2 that $\mathbf{T}_0(q_0') = 0$. The next Lemma shows, that the kernel is a one-dimensional subspace thus $\ker(\mathbf{T}_0) = \mathrm{span}(q_0')$.

Lemma 4.1.5 ($\mathbf{T}_0$ has a one-dimensional kernel)**.**
The operator $\boldsymbol{T}_0 : H^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ *has a one-dimensional kernel, more precisely*

$$\ker(\boldsymbol{T}_0) = \mathrm{span}(q_0') \tag{4.1.5}$$

and $\boldsymbol{T}_0 : \{q_0'\}^\perp \to \{q_0'\}^\perp$ *is an isomorphism.*

The proof is inspired by Lemma 5.3 in [BFRW97] and similar to the one in Lemma 3.1.13.

Proof. Owing to $2W(q_0) = |q_0'|^2$ we have $q_0' \in \ker(\mathbf{T}_0)$. Since $H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$ we know for any $w \in H^2(\mathbb{R})$

$$\mathbf{T}_0(w) = 0 \iff w'' = W''(q_0)w \in C^1(\mathbb{R})$$

making $w \in C^3(\mathbb{R})$. By possibly multiplying with (-1) we can assume $w(x) > 0$ for some $x \in \mathbb{R}$. For $\beta \in \mathbb{R}$ we define $w_\beta := \beta w + q_0'$ and

$$\overline{\beta} := \sup\{\beta < 0 \big| \, \exists x \in \mathbb{R} : w_\beta(x) < 0\}.$$

Since our goal will be to show $w_{\overline{\beta}} \equiv 0$, it is useful to consider $\inf_{\mathbb{R}} w_\beta$. For $\beta < \overline{\beta}$ there exists $\xi_\beta \in \mathbb{R}$, such that $w_\beta(\xi_\beta) = \inf_{\mathbb{R}} w_\beta < 0$. This is true because $w' \in H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ yields that w is uniformly continuous, implying $\lim_{r \to \pm\infty} w(r) = 0$ by Lemma 3.1.9. Since w_β is C^2 and has a local mimimum at ξ_β we get

$$W''(q_0(\xi_\beta)) w_\beta(\xi_\beta) = w_\beta''(\xi_\beta) \geq 0 \text{ so } W''(q_0(\xi_\beta)) \leq 0. \tag{4.1.6}$$

We can deduce that $(\xi_\beta)_{\beta<\overline{\beta}}$ can be found in the interval where $W''(q_0)$ is non-positive. Since $W''(\pm 1) > 0$ and $q_0(x) \longrightarrow \pm 1$ as $x \to \pm\infty$, this interval is bounded. Therefore we can extract a subsequence such that $\xi_\beta \longrightarrow \overline{\xi}$ as $\beta \to \overline{\beta}$. Since w is bounded we know $w_\beta \longrightarrow w_{\overline{\beta}}$ as $\beta \to \overline{\beta}$ uniformly. So

$$w_{\overline{\beta}}(\overline{\xi}) \overset{\beta\to\overline{\beta}}{\longleftarrow} w_\beta(\xi_\beta) = \inf_{\mathbb{R}} w_\beta \overset{\beta\to\overline{\beta}}{\longrightarrow} 0.$$

Due to the definition of $\overline{\beta}$ we know $w_{\overline{\beta}} \geq 0$, making $\overline{\xi}$ a local minimum, thus $w'_{\overline{\beta}}(\overline{\xi}) = 0$. Collecting everything we now have $\mathbf{T}_0(w_{\overline{\beta}}) \equiv 0$, $w_{\overline{\beta}}(\overline{\xi}) = 0$ and $w'_{\overline{\beta}}(\overline{\xi}) = 0$. This violates the uniqueness part of the Picard-Lindelöf Theorem, unless $w_{\overline{\beta}} \equiv 0$ which is what we wanted to prove. Picard-Lindelöf is applicable to $\mathbf{T}_0$ because it is a linear differential operator with non-constant but smooth coefficients.

Since $\mathbf{T}_0$ is a Fredholm operator with index 0, $\ker(\mathbf{T}_0) = \operatorname{span}(q_0')$, and $\mathbf{T}_0$ is self-adjoint we conclude that $\mathbf{T}_0 : \{q_0\}^\perp \longrightarrow \{q_0\}^\perp$ is an isomorphism. $\square$

For the following lemma recall σ from (4.1.2).

Lemma 4.1.6 (Existence and properties of q_1)**.**
The functional $\Xi : H^3(\mathbb{R}) \longrightarrow [0,\infty]$

$$\Xi(w) := \int_{\mathbb{R}} \Big(\Big| \boldsymbol{T}_0 w - q_0' \Big|^2 + \Big| \big(\boldsymbol{T}_0 w - q_0' \big)' \Big|^2 \Big) \,\mathrm{d}\mathcal{L}^1$$

has a unique minimizer q_1 *on* $\{w \in H^3(\mathbb{R}) \mid w(0) = 0\}$ *which is also a minimizer on* $H^3(\Omega)$. q_1 *is determined by*

$$\boldsymbol{T}_0(q_1) = -\sigma\overline{q}_0' + q_0' \quad \textit{with} \quad q_1(0) = 0. \tag{4.1.7}$$

The minimal value is given by

$$\min_{H^3(\Omega)} \Xi = \Xi(q_1) = c_0\sigma. \tag{4.1.8}$$

Furthermore we get that for $\lambda \in \mathbb{R}$ *with* $\lambda \neq \sigma$ *there exists no* $u \in H^3(\mathbb{R})$ *such that*

$$\boldsymbol{T}_0(u) = -\lambda\overline{q}_0' + q_0'.$$

Note that the condition $w(0) = 0$ is well posed because of $H^3(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$.

Proof. From Lemma 4.1.5 we have $\ker(\mathbf{T}_0) = \operatorname{span}(q_0')$. We calculate for any $\tilde{\sigma} \in \mathbb{R}$

$$\int_{\mathbb{R}} q_0' \big(-\tilde{\sigma}\overline{q}_0' + q_0' \big) \,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} \big(-\tilde{\sigma} q_0 \overline{q}_0' + |q_0'|^2 \big) \,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} \big(-\tilde{\sigma} q_0 \overline{q}_0' + |q_0'|^2 \big) \,\mathrm{d}\mathcal{L}^1.$$

We conclude by (4.1.2) that $-\tilde{\sigma}\overline{q}_0' + q_0' \in \{q_0'\}^\perp$ if and only if $\tilde{\sigma} = \sigma$. By Lemma 4.1.5 thus there exists $\hat{q}_1 \in \{q_0'\}^\perp \subseteq H^2(\mathbb{R})$ such that

$$-\hat{q}_1'' + W''(q_0)\hat{q}_1 = \mathbf{T}_0(\hat{q}_1) = -\tilde{\sigma}\overline{q}_0' + q_0'$$

if and only if $\tilde{\sigma} = \sigma$. In that case $\hat{q}_1$ with this property is unique because $\mathbf{T}_0$ is an isomorphism. Rearranging yields

$$\hat{q}_1'' = \sigma \overline{q}_0' - q_0' + W''(q_0)\hat{q}_1. \tag{4.1.9}$$

Since $\hat{q}_1 \in H^2(\mathbb{R}) \hookrightarrow C^1(\mathbb{R})$ we get that the right-hand side lays in $C^1(\mathbb{R})$, thus $\hat{q}_1 \in C^3(\mathbb{R})$ by the left-hand side. This implies that the right-hand side lays in $C^3(\mathbb{R})$ thus $\hat{q}_1 \in C^5(\mathbb{R})$. By bootstrapping this way we get $\hat{q}_1 \in C^\infty(\mathbb{R})$. Furthermore we know from the exponential decay of $q_0', \overline{q}_0'$, and $\hat{q}_1 \in H^2(\mathbb{R})$ that the right-hand side lays in $H^1(\mathbb{R})$ thus $\hat{q}_1 \in H^3(\mathbb{R})$.

Since $q_0'(0) > 0$ we can find $\lambda \in \mathbb{R}$ such that

$$q_1 \coloneqq \hat{q}_1 + \lambda q_0'$$

satisfies $q_1(0) = 0$ and keeps all of the other properties of $\hat{q}_1$. From (4.1.7) it follows

$$(-\partial^2 + \mathrm{Id})(\mathbf{T}_0(q_1) - q_0') = -\sigma q_0'. \tag{4.1.10}$$

Since $\mathbf{T}_0(q_1) - q_0' = -\sigma \overline{q}_0'$ we can calculate with a partial integration

$$\begin{aligned}
\Xi(q_1) &= \int_{\mathbb{R}} \left(\left| \mathbf{T}_0 q_1 - q_0' \right|^2 + \left| (\mathbf{T}_0 q_1 - q_0')' \right|^2 \right) \mathrm{d}\mathcal{L}^1 \\
&= \int_{\mathbb{R}} (\mathbf{T}_0 q_1 - q_0')(-\partial^2 + \mathrm{Id})(\mathbf{T}_0 q_1 - q_0') \,\mathrm{d}\mathcal{L}^1 \\
&= -\sigma \int_{\mathbb{R}} (\mathbf{T}_0 q_1 - q_0') q_0' \,\mathrm{d}\mathcal{L}^1 = c_0 \sigma.
\end{aligned}$$

We show that q_1 is a minimizer of Ξ by calculating for any $w \in H^3(\mathbb{R})$

$$\begin{aligned}
\Xi(w) = \Xi(q_1) &+ \int_{\mathbb{R}} \mathbf{T}_0(w - q_1)(-\partial^2 + \mathrm{Id})\mathbf{T}_0(w - q_1) \,\mathrm{d}\mathcal{L}^1 \\
&+ 2 \int_{\mathbb{R}} \mathbf{T}_0(w - q_1)(-\partial^2 + \mathrm{Id})(\mathbf{T}_0(q_1) - q_0') \,\mathrm{d}\mathcal{L}^1.
\end{aligned}$$

The last term vanishes due to (4.1.10) and Lemma 4.1.5 leaving us with

$$\begin{aligned}
\Xi(w) &= \Xi(q_1) + \int_{\mathbb{R}} \mathbf{T}_0(w - q_1)(-\partial^2 + \mathrm{Id})\mathbf{T}_0(w - q_1) \,\mathrm{d}\mathcal{L}^1 \\
&= \Xi(q_1) + \int_{\mathbb{R}} \left(\left| \mathbf{T}_0(w - q_1) \right|^2 + \left| \partial \mathbf{T}_0(w - q_1) \right|^2 \right) \mathrm{d}\mathcal{L}^1.
\end{aligned}$$

It follows (4.1.8) and the uniqueness of q_1 up to adding $\ker(\mathbf{T}_0) = \operatorname{span}(q_0')$. The condition $q_1(0) = 0$ makes it unique. ☐

Before we can move on to the lim sup statement and its proof we need to show the exponential decay of the profile functions and their derivatives. For $q_0, \dots, q_0'''$ and $\overline{q}_0, \dots, \overline{q}_0'''$ we already established an exponential decay in the Lemmata 4.1.2 and 4.1.3.

Lemma 4.1.7 (Exponential decay)**.**
There exist $C, c > 0$ such that for all $r \in \mathbb{R}$ we have

$$\begin{aligned}
|q_1(r)| &\le Ce^{-c|r|}, & |q_1'(r)| &\le Ce^{-c|r|}, \\
|q_1''(r)| &\le Ce^{-c|r|}, \quad \textit{and} & |q_1'''(r)| &\le Ce^{-c|r|}.
\end{aligned}$$

Proof. We start with the analysis of the behavior as $r \to \infty$. We observe that the vector $\xi := (q_0 - 1, q_0', \mathbf{A}_0(q_0) - 1, \mathbf{A}_0(q_0'), q_1, q_1')$ is a solution to the ODE system

$$\begin{aligned}
\xi_1' &= \xi_2, \\
\xi_2' &= W'(1+\xi_1), \\
\xi_3' &= \xi_4, \\
\xi_4' &= -\xi_1 + \xi_3, \\
\xi_5' &= \xi_6, \\
\xi_6' &= -\xi_2 + \sigma\xi_4 + W''(1+\xi_1)\xi_5.
\end{aligned}$$

From $H^3(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and (4.1.9) we conclude $q_1'' \in C_b^0(\mathbb{R})$. This implies that q_1' is uniformly continuous and thus by Lemma 3.1.9 we get $\lim_{r\to\pm\infty} q_1'(r) = 0$. It also follows $\lim_{r\to\pm\infty} q_1(r) = 0$ and thus we get $\lim_{r\to\infty} \xi(r) = 0$. Writing $c := \sqrt{W''(1)}$ the linearization of the right-hand side of the ODE at $\xi = 0$ is given by

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
c^2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & \sigma & c^2 & 0
\end{pmatrix}$$

For the calculation of the eigenvalues we use that the matrix is a lower triangular matrix if we consider the 2×2 block structure. We get the eigenvalues $\pm c$ with algebraic multiplicity 2 and ± 1 with algebraic multiplicity 1. Thus the stationary point $\xi = 0$ is hyperbolic. We already know that $|\xi|$ vanishes at ∞. From the stable manifold theory, see for example [Per96, p. 115], we get that the solution approaches the stationary state exponentially. This works the same way for $r \to -\infty$. We can transfer the result to q_1'' and q_1''' because of (4.1.9). $\square$

4.2 Formal identification of a candidate for the Γ–limit and for a recovery sequence

We proceed as in Chapter 3. Motivated by the diffuse energy-dissipation (4.0.2) we introduce the candidate for new diffuse Willmore energy.

Definition 4.2.1 (Definition of the diffuse Willmore energy)**.**
Recall $H_\varepsilon := H_\varepsilon(u) := -\varepsilon\Delta u + \frac{1}{\varepsilon}W'(u)$*, we define* $\mathcal{W}_\varepsilon^{\mathrm{KK}} : L^1(\Omega) \longrightarrow [0,\infty]$

$$\mathcal{W}_\varepsilon^{\mathrm{KK}}(u) := \begin{cases} \displaystyle\int_\Omega \Big(\frac{1}{\varepsilon}|H_\varepsilon(u)|^2 + \varepsilon\big|\nabla H_\varepsilon(u)\big|^2\Big)\,\mathrm{d}\mathcal{L}^n, & \textit{if } u \in H^3(\Omega)\cap L^6(\Omega) \\[2ex] +\infty, & \textit{else.} \end{cases} \tag{4.2.1}$$

We use the notations and assumptions from Assumptions 4.1.1 and the coordinates from Definition 2.1.10. To get an idea how to construct the recovery sequence associated to the lim sup property we will do some formal calculations and use them as a motivation for the rigorous proof. The method of asymptotic expansions was already presented by

Loreti and March [LM00] and considered by Wang [Wan08].

We use the ansatz for u_ε

$$u_\varepsilon(x) = U_0(z,x) + \varepsilon U_1(z,x) \tag{4.2.2}$$

as in Notations 2.1.11. We pose the following conditions on our functions:

- $U_0 \in C^0(\mathbb{R}\times\omega)$ with $U_0(0,x)=0$ and $U_0(\cdot,x) - \operatorname{sgn} \in L^2(\mathbb{R})$ for all $x\in\omega$.
- $U_1 \in H^1(\mathbb{R}; C(\omega))$ with $U_1(0,x)=0$ for all $x\in\omega$.
- For all $z\in\mathbb{R}$ and all $j\in\{0,1\}$ we have that $U_j(z,\cdot)$ is constant in normal direction.

The concept for the recovery sequence is visualized in Figure 4.1.

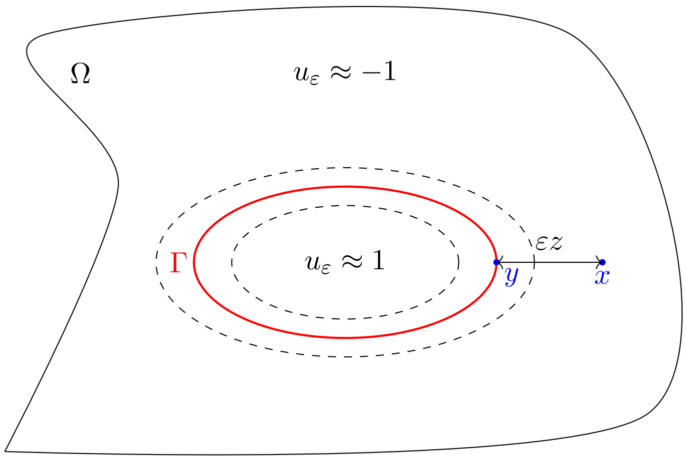

Figure 4.1: Visualization of the geometry and coordinates.

In the first condition sgn refers to the z-variable. We write $\partial_z U_j = V_j'$ for the z-derivative and $\nabla_\Gamma U_j$ for the tangential y-derivative.
We formally expand H_ε in the new coordinates and get by (2.1.8)

$$\begin{aligned}
H_\varepsilon &= \Big(-\frac{1}{\varepsilon}\partial_z^2 - H\partial_z - \varepsilon(\Delta_x - z|\mathrm{II}|^2\partial_z)\Big)[U_0+\varepsilon U_1] + \frac{1}{\varepsilon}W'(U_0) + W''(U_0)U_1 + \mathcal{O}(\varepsilon)\\
&= \frac{1}{\varepsilon}\big(-U_0'' + W'(U_0)\big) + \big(-U_1'' + W''(U_0)U_1 - HU_0'\big) + \mathcal{O}(\varepsilon)\\
&=: \frac{1}{\varepsilon}H_{-1} + H_0 + \mathcal{O}(\varepsilon).
\end{aligned} \tag{4.2.3}$$

We take a look of the lowest order in $\mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon)$. We plug in the expansion and consider $\{|d|<3\delta\}$ instead of Ω. We get with the coordinate transformation Ψ_ε from Definition 2.1.10

$$\mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon) = \int_\Gamma\int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} \frac{1}{\varepsilon^2}\Big(\Big|H_{-1}+\varepsilon H_0\Big|^2 + \Big|\varepsilon\nabla H_{-1} + \varepsilon^2\nabla H_0\Big|^2\Big)\,\mathrm{d}\mathcal{L}^1\,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{O}(\varepsilon).$$

The lowest order term $|H_{-1}|^2$ is minimized by $U_0 = q_0$, which implies $H_{-1} = 0$ and thus

$$\begin{aligned}
\mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon) &= \int_\Gamma \int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} \Big(|H_0|^2 + \varepsilon^2 \big|\nabla H_0\big|^2\Big) \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{O}(\varepsilon) \\
&= \int_\Gamma \int_{-\frac{3\delta}{\varepsilon}}^{\frac{3\delta}{\varepsilon}} \Big(|\mathbf{T}_0 U_1 - H q_0'|^2 + \big|\partial_z(\mathbf{T}_0 U_1 - H q_0')\big|^2\Big) \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{O}(\varepsilon).
\end{aligned}$$

We can see that at points $y \in \Gamma$ with $H(y) = 0$ the optimal choice is $U_1(\cdot, y) = 0$ and there is no contribution to the integral. Thus we can reduce the integral to the set $\{H \neq 0\}$. We get

$$\mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon) \leq \int_{\{H\neq 0\}} |H|^2 \int_\mathbb{R} \Big(\Big|\mathbf{T}_0 \frac{U_1}{H} - q_0'\Big|^2 + \Big|\partial_z\Big(\mathbf{T}_0 \frac{U_1}{H} - q_0'\Big)\Big|^2\Big) \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} + \mathcal{O}(\varepsilon).$$

Thus we want to minimize the functional

$$\Xi(w) = \int_\mathbb{R} \Big(\Big|\mathbf{T}_0(w) - q_0'\Big|^2 + \Big|\partial_z\Big(\mathbf{T}_0(w) - q_0'\Big)\Big|^2\Big) \,\mathrm{d}\mathcal{L}^1, \;\; w \in H^3(\mathbb{R}).$$

from Lemma 4.1.6. Ξ is minimized for $U_1 = q_1 H$ by Lemma 4.1.6. Inserting this back into $\mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon)$ we get on a formal level

$$\mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon) = c_0 \sigma \mathcal{W}(u) + \mathcal{O}(\varepsilon).$$

4.3 Rigorous proof of the $\Gamma - \limsup$ estimate

In this section we construct the recovery sequence associated to the lim sup property of $\mathcal{W}_\varepsilon^{\mathrm{KK}} \xrightarrow{\Gamma} c_0\sigma\mathcal{W}$ with respect to $L^1(\Omega)$-topology. The process is motivated by the formal calculations from the previous section. To handle the transition from the set ω which is close to Γ to the rest of Ω we can work with the same cut-off function as in the Γ–lim sup construction for the AG-model.

Theorem 4.3.1 (lim sup estimate for Willmore approximation)**.**
Let Ω, W as in Assumptions 4.1.1. Let $E \Subset \Omega$ with $\partial E \in C^5$. There exists $(u_\varepsilon)_{\varepsilon>0}$ such that $u_\varepsilon \longrightarrow u$ in $L^1(\Omega)$ as $\varepsilon \to 0$ and

$$\lim_{\varepsilon\to 0} \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon) \leq c_0\sigma\mathcal{W}(u),$$

with $u \coloneqq 2\chi_E - 1$.

Proof. We choose an even and on $(0,\infty)$ decreasing function $\eta_1 \in C_c^\infty(\mathbb{R})$ with

$$0 \leq \eta_1 \leq 1, \quad |\eta_1'|, |\eta_1''|, |\eta_1'''| \leq C(\eta_1), \quad \eta_1 = \begin{cases} 1 & \text{in } [0,3], \\ 0 & \text{in } [4,\infty) \end{cases}$$

and define the cut-off function

$$\eta_\delta(x) \coloneqq \eta_1\Big(\frac{d(x)}{\delta}\Big) \quad \text{ for all } x \in \Omega.$$

We remark that $\eta_\delta \in C^4(\Omega)$ since η_δ has support in $\{|d| < 4\delta\}$.

Now we can define the recovery sequence

$$\begin{aligned} u_\varepsilon(x) &\coloneqq \eta_\delta(x)\Big(q_0(z) + \varepsilon H(y) q_1(z)\Big) + (1 - \eta_\delta(x)) \operatorname{sgn}(d(x)), \\ &= u(x) + \eta_\delta(x)\Big(q_0(z) + \varepsilon H(y) q_1(z) - \operatorname{sgn}(z)\Big) \end{aligned}$$

with $x = \Psi_\varepsilon(z, y)$ as in the coordinates from Definiton 2.1.10. Note $u(x) = \operatorname{sgn}(\operatorname{sdist}(z))$ For shorter notation we will drop the arguments (x, y, z) from now on.

We start with the proof of $u_\varepsilon \to u$ in $L^1(\Omega)$. For $\delta > 0$ from Definition 2.1.10 we have

$$\begin{aligned} \int_\Omega |u_\varepsilon - u| \,\mathrm{d}\mathcal{L}^n &= \int_{\{|d|<4\delta\}} |u_\varepsilon - u| \,\mathrm{d}\mathcal{L}^n = \int_\Gamma \int_{-\frac{4\delta}{\varepsilon}}^{\frac{4\delta}{\varepsilon}} \varepsilon \eta_\delta \Big| q_0 + \varepsilon H q_1 - \operatorname{sgn}(d) \Big| \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} \\ &\le \int_\Gamma \int_\mathbb{R} \varepsilon \Big| q_0 + \varepsilon H q_1 - \operatorname{sgn}(d) \Big| \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} \\ &\le \int_\Gamma \int_\mathbb{R} \varepsilon \Big| q_0 - \operatorname{sgn}(d) \Big| \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} + \int_\Gamma \int_\mathbb{R} \varepsilon^2 |H q_1| \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{H}^{n-1} \\ &\le \varepsilon C(\Gamma) \|q_0 - \operatorname{sgn}\|_{L^1(\mathbb{R})} + \varepsilon^2 C(\Gamma) \|q_1\|_{L^1(\mathbb{R})} \|H\|_{L^1(\Gamma)}. \end{aligned}$$

The respective integrals exist because Γ is compact, $H \in C^0(\Gamma)$ and the exponential decay from $q_0 - \operatorname{sgn}, q_1$. Next we calculate H_ε and ∇H_ε and get

$$\begin{aligned} u_\varepsilon &= u + \eta_\delta(q_0 + \varepsilon H q_1 - u) \\ \nabla u_\varepsilon &= \eta_\delta'(q_0 - u + \varepsilon H q_1)\nabla d + \frac{\eta_\delta}{\varepsilon}(q_0' + \varepsilon H q_1')\nabla d + \varepsilon \eta_\delta q_1 \nabla_\Gamma H \\ H_\varepsilon &= -\varepsilon(\eta_\delta'' + \eta_\delta' \Delta d)(q_0 - u + \varepsilon H q_1) - (2\eta_\delta' + \eta_\delta \Delta d)(q_0' + \varepsilon H q_1') \\ &\quad - \frac{\eta_\delta}{\varepsilon}(q_0'' + \varepsilon H q_1'') - \varepsilon^2 \eta_\delta q_1 \Delta_\Gamma H + \frac{1}{\varepsilon} W'(u_\varepsilon) \\ \nabla H_\varepsilon &= -\frac{\eta_\delta}{\varepsilon^2}(q_0''' + \varepsilon H q_1''')\nabla d - \frac{1}{\varepsilon}(3\eta_\delta' + \eta_\delta \Delta d)(q_0'' + \varepsilon H q_1'')\nabla d \\ &\quad - \big[(3\eta_\delta'' + 2\eta_\delta' \Delta d)\nabla d + \eta_\delta \nabla \Delta d\big](q_0' + \varepsilon H q_1') \\ &\quad - \varepsilon\big[(\eta_\delta''' + \eta_\delta'' \Delta d)\nabla d + \eta_\delta' \nabla \Delta d\big](q_0 - u + \varepsilon H q_1) \\ &\quad - \big[\eta_\delta q_1'' + \varepsilon(2\eta_\delta' + \eta_\delta \Delta d) q_1' + \varepsilon^2(\eta_\delta'' + \eta_\delta' \Delta d) q_1\big]\nabla_\Gamma H \\ &\quad - \varepsilon(\eta_\delta q_1' + \varepsilon \eta_\delta' q_1)\nabla d \Delta_\Gamma H - \varepsilon^2 \eta_\delta q_1 \nabla_\Gamma \Delta_\Gamma H \\ &\quad + \frac{1}{\varepsilon} W''(u_\varepsilon)\big[\eta_\delta'(q_0 - u + \varepsilon H q_1)\nabla d + \frac{\eta_\delta}{\varepsilon}(q_0' + \varepsilon H q_1')\nabla d + \varepsilon \eta_\delta q_1 \nabla_\Gamma H\big]. \end{aligned}$$

Now we can proceed to show the lim sup property. We split the integral (note $H_\varepsilon = 0$ on $\{|d| \ge 4\delta\}$)

$$\mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon) = \int_{\{|d|<3\delta\}} \frac{1}{\varepsilon}\Big(|H_\varepsilon|^2 + \varepsilon^2 |\nabla H_\varepsilon|^2\Big) \,\mathrm{d}\mathcal{L}^n + \int_{\{3\delta \le |d| < 4\delta\}} \frac{1}{\varepsilon}\Big(|H_\varepsilon|^2 + \varepsilon^2 |\nabla H_\varepsilon|^2\Big) \,\mathrm{d}\mathcal{L}^n.$$

On $\{|d| < 3\delta\}$ we have $\eta_\delta = 1$ and thus $u_\varepsilon = q_0 + \varepsilon H q_1$ which means that the formal calculations from the previous section can be applied to the first integral. In fact we have

$$\int_{\{|d|<3\delta\}} \frac{1}{\varepsilon}\Big(|H_\varepsilon|^2 + \varepsilon^2 |\nabla H_\varepsilon|^2\Big) \,\mathrm{d}\mathcal{L}^n \le \Xi(q_1) \int_\Gamma |H|^2 \,\mathrm{d}\mathcal{H}^{n-1} = c_0 \sigma \mathcal{W}(u).$$

Thus it remains to show, that the integral over $\{3\delta \le |d| < 4\delta\}$ does not contribute in the limit. From $H \in C^3(\Gamma)$ we have $\Delta H \in C^1(\Gamma)$ and $\nabla_\Gamma \Delta_\Gamma H \in C^0(\Gamma; \mathbb{R}^n)$. Since Γ is compact every continuous function on Γ is bounded. Furthermore we have $q_0, q_1 \in C_b^0(\mathbb{R})$ and thus $|u_\varepsilon| \le R$ for some $R > 0$ independent of ε on Ω. Thus we estimate with the Mean-Value Theorem $\left|W'(u_\varepsilon)\right|^2 = \left|W'(u_\varepsilon) - W'(u)\right|^2 \le \|W''\|^2_{C^0[-R,R]} |u_\varepsilon - u|^2$ and get

$$|H_\varepsilon|^2 \le \frac{1}{\varepsilon^2} C(\Gamma, \eta_\delta) \Big(|q_0 - \operatorname{sgn}|^2 + \sum_{j=1}^{2} |q_0^{(j)}|^2 + \sum_{j=0}^{2} |q_1|^2 \Big).$$

For the estimate of $|\nabla H_\varepsilon|^2$ we also need $d \in C_b^3(\Omega)$ and $|W''(u_\varepsilon)| \le \|W''\|_{C^0[-R,R]} \le C$. We get

$$\left|\varepsilon \nabla H_\varepsilon\right|^2 \le \frac{1}{\varepsilon^2} C(\Gamma, \eta_\delta) \Big(|q_0 - \operatorname{sgn}|^2 + \sum_{j=1}^{3} |q_0^{(j)}|^2 + \sum_{j=0}^{3} |q_1^{(j)}|^2 \Big).$$

On the set $\{3\delta \le |d| < 4\delta\}$ we have $|z| \ge \frac{3\delta}{\varepsilon} \to \infty$ as $\varepsilon \to 0$. From Lemma 4.1.7 we know that all of the terms have exponential decay as $|z| \to \infty$. There exist $\varepsilon_0, \lambda > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and all $x \in \{3\delta \le |d| < 4\delta\}$ we have

$$|H_\varepsilon(x)|^2 \le \frac{1}{\varepsilon^2} C(\Gamma, \eta_\delta) e^{-\frac{3\delta\lambda}{\varepsilon}} \quad \text{and} \quad \left|\varepsilon \nabla H_\varepsilon(x)\right|^2 \le \frac{1}{\varepsilon^2} C(\Gamma, \eta_\delta) e^{-\frac{3\delta\lambda}{\varepsilon}}.$$

It follows that the integral vanishes

$$\int\limits_{\{3\delta \le |d| < 4\delta\}} \frac{1}{\varepsilon} \big(|H_\varepsilon|^2 + \varepsilon^2 \left|\nabla H_\varepsilon\right|^2 \big) \, \mathrm{d}\mathcal{L}^n \le \frac{1}{\varepsilon^3} C(\Gamma, \delta, \eta_\delta) e^{-\frac{3\lambda\delta}{\varepsilon}} \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0. \qquad \square$$

4.4 Diffuse gradient flows in KK model

In this section we consider the following rescaled gradient flows of the diffuse Willmore and perimeter functional.

$$-\varepsilon \partial_t u_\varepsilon = (-\varepsilon^2 \Delta + \mathrm{Id}) H_\varepsilon \tag{4.4.1}$$

$$-\varepsilon \partial_t u_\varepsilon = \frac{2}{\varepsilon^2} \big(-\varepsilon^2 \Delta + W''(u_\varepsilon) \big) (-\varepsilon^2 \Delta + \mathrm{Id}) H_\varepsilon, \tag{4.4.2}$$

where $H_\varepsilon = -\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon)$ denotes the diffuse mean curvature. The PDE (4.4.2) is a gradient flow because of

$$\nabla_{L^2} \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon) = \frac{2}{\varepsilon} \big(-\varepsilon^2 \Delta + W''(u_\varepsilon) \big) (-\varepsilon^2 \Delta + \mathrm{Id}) H_\varepsilon.$$

We will prove convergence towards mean curvature flow respective Willmore flow in a formal sense, as in the previous Chapter 3. For the Formulations of mean curvature and Willmore flow see Definitions 2.1.7 and 2.1.9.
The justification of phase field approximations of geometric evolution laws has a long history. Our analysis closely follows the formal analysis in [LM00]; see also [Wan08, BMO15] and [RR21]. We formulate here assumptions under which the derivation is rigorous. This however does not give a general convergence proof, since the assumed properties need to be verified for a phase field approximation. Complete convergence proofs based on asymptotic expansion techniques are known for the standard diffuse approximation of mean curvature and Willmore flow; see [dMS90] and [FL21].

Assumption 4.4.1 (Set evolution)**.**
Consider a continuous evolution of open sets $(E(t))_{t\in[0,T]}$ *in* Ω *with associated signed distance function* $d : \Omega_T \to \mathbb{R}$, $d(\cdot,t) = \operatorname{dist}(\cdot, \Omega \setminus E(t)) - \operatorname{dist}(\cdot, E(t))$, *phase boundaries* $\Gamma(t) := \partial E(t)$ *for* $t \in [0,T]$ *and We assume the following properties:*

1. $\Gamma(t)$ *is a* C^5*-regular hypersurface for all* $t \in [0,T]$.
2. $\bigcup_{t\in[0,T]} E(t) \Subset \Omega$. *With this assumption we can choose* $\delta > 0$ *sufficiently small such that for all* $t \in [0,T]$ *the projections* $\Pi_{\Gamma(t)} : \{|d(\cdot,t)| < 5\delta\} \longrightarrow \Gamma(t)$ *are well defined and set*
$$\omega_T := \{(x,t) \in \Omega \times [0,T] \,:\, |d(t,x)| < 5\delta\}.$$
3. $d \in C_b^1(\omega_T)$ *and* $D_x^\gamma d \in C_b^0(\omega_T)$ *for all* $\gamma \in \mathbb{N}_0^n$ *with* $|\gamma| \le 4$.

Let $\Psi_\varepsilon(\cdot,t)$, $t \in [0,T]$ *denote the parametrization that are defined according to* (2.1.5) *with* Γ *replaced by* $\Gamma(t)$.

We consider the cut-off function η_δ from the proof of Theorem 4.3.1 and introduce classes of phase field evolutions that we will consider in the following. We use the function spaces of exponentially decaying functions from (3.5.4) and (3.5.5). Recall the coordinate transformation $\Psi_\varepsilon(z,y) = y + \varepsilon z \nu_\Gamma(y)$ from Definition 2.1.10.

Assumption 4.4.2 (Phase field evolution)**.**
Let $(E(t))_{t\in[0,T]}$ *be a continuous evolution of sets in* Ω, *the signed distance function* d *and* $\delta > 0$ *as in Assumption 4.4.1 be given. Consider an evolution of smooth phase fields* $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$. *We assume that there exist* $\mu \in (0,1)$, $\Lambda > 0$, *and profile functions* $u_j : \mathbb{R} \times \omega_T \to \mathbb{R}$ *for* $j \in \{0,1,2\}$, *such that for all* $0 < \varepsilon < \varepsilon_0$ *and all* $t \in (0,T)$

$$u_\varepsilon(\cdot,t) = \eta_\delta u_\varepsilon^{\mathrm{in}}(\cdot,t) + (1-\eta_\delta)\operatorname{sgn}(d) + \varepsilon^3 R_\varepsilon \quad \textit{in } \Omega, \tag{4.4.3}$$
$$u_\varepsilon^{\mathrm{in}}(\cdot,t) = \Big(\sum_{j=0}^{2} \varepsilon^j u_j\Big) \circ \Psi_\varepsilon^{-1} \quad \textit{in} \quad \{|d| < 4\delta\}, \tag{4.4.4}$$

and such that the following properties hold:

1. *The profile functions* $u_j \in C^0(\mathbb{R} \times \omega_T)$, $u_j = u_j(z,x,t)$ *satisfy* $u_j(z,\cdot,\cdot) \in C_b^1(\omega_T)$, $D_x^\gamma u_j(z,\cdot,\cdot) \in C_b^0(\omega_T)$ *for all* $\gamma \in \mathbb{N}_0^n$ *with* $|\gamma| \le 4$
$$u_0 - \operatorname{sgn} \in X(\mathbb{R};\Gamma_T),\ u_j,\ |\nabla_x u_j|,\ \Delta_x u_j \in X(\mathbb{R};\Gamma_T) \quad \textit{for} \quad j \in \{1,2\}.$$
2. *The remainder satisfies* $R_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega_T)$ *for all* $0 < \varepsilon < \varepsilon_0$.

The spaces $X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega_T), X(\mathbb{R};\Gamma_T)$ *have been introduced in 3.5.1. Moreover, we assume that*

$$\{u_\varepsilon(\cdot,t) = 0\} = \Gamma(t) \quad \textit{for all} \quad t \in [0,T], \quad 0 < \varepsilon < \varepsilon_0, \tag{4.4.5}$$

and that
$$\mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon(\cdot,0)) + \mathcal{P}_\varepsilon(u_\varepsilon(\cdot,0)) \le C \tag{4.4.6}$$
for all $0 < \varepsilon < \varepsilon_0$.

Theorem 4.4.3 (Convergence towards the mean curvature flow).
Consider a sequence of evolutions of smooth phase fields $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ as in Assumption 4.4.2, satisfying an asymptotic expansion (4.4.3)*-*(4.4.4) *with respect to an evolution $(E(t))_{t\in[0,T]}$ of sets in Ω. Assume that u_ε satisfies*

$$-\varepsilon\partial_t u_\varepsilon = \big(-\varepsilon^2\Delta + \mathrm{Id}\big)\Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big) + \varepsilon R_\varepsilon, \tag{4.4.7}$$

with $\sup_{0<\varepsilon<\varepsilon_0}\|R_\varepsilon\|_{C^0(\overline{\Omega_T})} \leq C$ *then* $(\Gamma(t))_{t\in[0,T]}$ *evolves by the rescaled mean curvature flow*

$$\mathcal{V} = \sigma H \tag{4.4.8}$$

with σ as in (4.1.2).

Proof. We expand both sides of (4.4.7) and evaluate the identity order by order. To identify the evolution law in the limit $\varepsilon \to 0$ it is sufficient to consider the region $\{|d| < 2\delta\}$, in which $\eta_\delta \equiv 1$. We deduce from Lemma 3.3.7

$$W'(u_\varepsilon(x,t)) = W'(u_0(z,x,t)) + \varepsilon W''(u_0(z,x,t))u_1 + \varepsilon^2 R^W_\varepsilon(x,t)$$

with $R^W_\varepsilon \in X^{\frac{\mu}{\varepsilon},\Lambda}_\delta(\Omega_T)$. Next we expand H_ε as in (4.2.3) and get

$$\begin{aligned} H_\varepsilon = -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon(x,t)) &= \varepsilon^{-1}\big(-u_0'' + W'(u_0)\big) + \mathbf{T}_0 u_1 - Hq_0' + \varepsilon R^H_\varepsilon(z,x,t) \\ &=: \varepsilon^{-1}H_{-1}(z,x,t) + H_0(z,x,t) + \varepsilon R^H_\varepsilon(z,x,t) \end{aligned} \tag{4.4.9}$$

with $R^H_\varepsilon \in X^{\frac{\mu}{\varepsilon},\Lambda}_\delta(\Omega_T)$. We expand the evolution (4.4.7). For the left-hand side we obtain in $\{|d| < 2\delta\}$

$$-\varepsilon\partial_t u_\varepsilon = -\varepsilon\sum_{j=0}^{2}\Big(\partial_t u_j + \frac{1}{\varepsilon}\partial_z u_j\partial_t d\Big) + \mathcal{O}(\varepsilon) = -\partial_z u_0\partial_t d + \mathcal{O}(\varepsilon) = -u_0'\mathcal{V} + \mathcal{O}(\varepsilon). \tag{4.4.10}$$

The ε^{-1}-order of the evolution (4.4.7) yields

$$0 = (-\partial^2 + \mathrm{Id})H_{-1} \quad \text{thus} \quad H_{-1} = 0,$$

because H_{-1} is bounded by the assumptions on u_0. We conclude $u_0'' = W'(u_0)$ and thus $u_0 = q_0$. The next order yields in $\{|d| < 2\delta\}$

$$\begin{aligned} \big(-\varepsilon^2\Delta + \mathrm{Id}\big)H_\varepsilon &= \big(-\partial_z^2 + \mathrm{Id}\big)H_0 + \mathcal{O}(\varepsilon) \\ &= \big(-\partial_z^2 + \mathrm{Id}\big)\big(-\mathbf{T}_0(u_1) - Hq_0'\big) + \mathcal{O}(\varepsilon). \end{aligned}$$

Equating this with the expansion of $-\varepsilon\partial_t u_\varepsilon$ done in (4.4.10) we get by testing with $\overline{q}_0'$

$$-\mathcal{V}\int_{\mathbb{R}} q_0'\overline{q}_0'\,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} q_0'\big(-\mathbf{T}_0(u_1) - Hq_0'\big)\,\mathrm{d}\mathcal{L}^1 = -H\int_{\mathbb{R}} |q_0'|^2\,\mathrm{d}\mathcal{L}^1.$$

With (4.1.2) we get that the evolution evolves by mean curvature flow

$$\mathcal{V} = \sigma H.$$

□

We can prove a similar result for the gradient flow of the diffuse Willmore energy.

Theorem 4.4.4 (Convergence towards the Willmore flow)**.**
Consider a sequence of evolutions of smooth phase fields $(u_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ as in Assumption 4.4.2, satisfying an asymptotic expansion (4.4.3)-(4.4.4) *with respect to an evolution $(E(t))_{t\in[0,T]}$ of sets in Ω. Assume that u_ε satisfies*

$$-\frac{\varepsilon}{2}\partial_t u_\varepsilon = \frac{1}{\varepsilon^2}\Big(-\varepsilon^2\Delta + W''(u_\varepsilon)\Big)\big(-\varepsilon^2\Delta + \mathrm{Id}\big)\Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big) + \varepsilon R_\varepsilon, \quad (4.4.11)$$

with $\sup_{0<\varepsilon<\varepsilon_0} \|R_\varepsilon\|_{C^0(\overline{\Omega_T})} \leq C$*. Then $(\Gamma(t))_{t\in[0,T]}$ evolves by the rescaled Willmore flow*

$$\mathcal{V} = 2\sigma\Big(-\Delta_{\Gamma_t}H - H|\mathrm{I\!I}|^2 + \frac{1}{2}H^3\Big) \quad (4.4.12)$$

with σ as in (4.1.2).

Proof. We expand both sides of (4.4.11) and evaluate the identity order by order. To identify the evolution law in the limit $\varepsilon \to 0$ it is sufficient to consider the region $\{|d| < 2\delta\}$, in which $\eta_\delta \equiv 1$. We deduce from Lemma 3.3.7

$$\begin{aligned} W'(u_\varepsilon(x,t)) = {} & W'(u_0(z,x,t)) + \varepsilon W''(u_0(z,x,t))u_1 \\ & + \varepsilon^2\Big[W''(u_0(z,x,t)) + \frac{1}{2}W''(u_0(z,x,t))u_1^2\Big] + \varepsilon^3 R_\varepsilon^W(x,t) \end{aligned}$$

with $R_\varepsilon^W \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega_T)$. Next we expand H_ε as in (4.2.3) and get

$$\begin{aligned} H_\varepsilon &= -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon(x,t)) = \varepsilon^{-1}\big(-u_0'' + W'(u_0)\big) + \mathbf{T}_0 u_1 - Hu_0' + \varepsilon R_\varepsilon^H(x,t) \\ &=: \varepsilon^{-1}H_{-1}(z,x,t) + H_0(z,x,t) + \mathcal{O}(\varepsilon). \end{aligned} \quad (4.4.13)$$

Expanding $\mathcal{W}_\varepsilon^{\mathrm{KK}}$ by orders of ε we get

$$\begin{aligned} \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon) &= \frac{1}{\varepsilon^2}\int_{-\frac{2\delta}{\varepsilon}}^{\frac{2\delta}{\varepsilon}}\int_\Gamma \Big(|H_{-1}|^2 + |H_{-1}'|^2\Big)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\mathcal{L}^1 + \mathcal{O}(1) \\ &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}}\int_\Gamma \big(|H_{-1}|^2 + |H_{-1}'|^2\big)\,\mathrm{d}\mathcal{H}^{n-1}\,\mathrm{d}\mathcal{L}^1 + \mathcal{O}(1). \end{aligned}$$

Since under the flow (4.4.11) the energy $\mathcal{W}_\varepsilon^{\mathrm{KK}}$ decreases with time and by (4.4.6) we obtain that $\mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon(\cdot,t))$ is uniformly bounded, which implies

$$0 = H_{-1} = -u_0'' + W'(u_0) \quad \text{and thus} \quad u_0(z,x,t) = q_0(z) \quad \text{for all} \quad (x,t) \in \omega_T.$$

Now we can expand the next order of H_ε with less effort because terms like $\Delta_{\Gamma_t}u_0$ vanish. We get

$$\begin{aligned} H_\varepsilon &= \mathbf{T}_0 u_1 - Hq_0' + \varepsilon\Big(\mathbf{T}_0 u_2 + z|\mathrm{I\!I}|^2 q_0' - Hu_1' + \frac{1}{2}W''(q_0)u_1^2\Big) + \varepsilon^2 R_\varepsilon^H(x,t) \\ &=: H_0(z,x,t) + \varepsilon H_1(z,x,t) + \varepsilon^2 R_\varepsilon^H(x,t). \end{aligned} \quad (4.4.14)$$

with $R_\varepsilon^H \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega_T)$. Since the left-hand side of the evolution (4.4.11) is the same as in the gradient flow of the diffuse mean curvature flow up to a factor of $\frac{1}{2}$ we get from (4.4.10) in $\{|d| < 2\delta\}$

$$-\frac{\varepsilon}{2}\partial_t u_\varepsilon = -\frac{1}{2}u_0'\mathcal{V} + \mathcal{O}(\varepsilon). \tag{4.4.15}$$

To expand the right-hand side of (4.4.11) we define the operators $T_\varepsilon := -\varepsilon^2\Delta + W''(u_\varepsilon)$, $D_\varepsilon := -\varepsilon^2\Delta + \mathrm{Id}$ and expand

$$\mathbf{D}_0 := -\partial_z^2 + \mathrm{Id}, \quad D_1 := -H\partial_z, \quad D_2 := -\Delta_{\Gamma_t} + z|\mathrm{II}|^2\partial_z,$$
$$T_1 := W'''(u_0)u_1 - H\partial_z \quad \text{and} \quad T_2 := W'''(u_0)u_2 + \frac{1}{2}W^{(4)}(u_0)u_1^2 - \Delta_{\Gamma_t} + z|\mathrm{II}|^2\partial_z.$$

The expansion of D_ε has already been done in (2.1.8), (2.1.9). We plug this into the right-hand side of the evolution and expand in $\{|d| < 2\delta\}$

$$T_\varepsilon D_\varepsilon H_\varepsilon = \mathbf{T}_0\mathbf{D}_0 H_0 + \varepsilon\big(T_1\mathbf{D}_0H_0 + \mathbf{T}_0D_1H_0 + \mathbf{T}_0\mathbf{D}_0H_1\big) + \mathcal{O}(\varepsilon^2).$$

Since on the left-hand side of the evolution (4.4.11) the lowest order is ε^0 and there is a factor ε^{-2} in front of the operators on the right-hand side we know that the lowest two orders have to vanish. We conclude

$$0 = \mathbf{T}_0\big[\big(-\partial_z^2 + \mathrm{Id}\big)\big(\mathbf{T}_0u_1 - Hq_0'\big)\big].$$

From Lemma 4.1.5 it follows that there exists $\lambda \in \mathbb{R}$ such that

$$(-\partial_z^2 + \mathrm{Id})\big(\mathbf{T}_0u_1 - Hq_0'\big) = \lambda q_0'. \tag{4.4.16}$$

If $H(\Pi x) = 0$ for $x \in \omega$ then $\lambda = 0$ and $q_1(z,x,t) = 0$ are solutions. If $H(\Pi x) \neq 0$ we get with the solution operator $\mathcal{A}_0$ from Proposition 3.1.8

$$\mathbf{T}_0\frac{u_1}{H} - q_0' = \frac{\lambda}{H}\overline{q}_0'.$$

From Lemma 4.1.6 we get that this equation can only be solved if $\lambda = -\sigma H$. Thus we get

$$u_1 = Hq_1 \quad \text{and thus} \quad H_0 = -\sigma H\overline{q}_0'. \tag{4.4.17}$$

This also covers the case $H(\Pi x) = 0$.

Considering the ε^{-1}-order of the evolution (4.4.11) we get

$$0 = \mathbf{T}_0\big[\mathbf{D}_0H_1 + D_1H_0\big] + T_1\mathbf{D}_0H_0 \quad \text{and thus} \quad \mathbf{T}_0\big[\mathbf{D}_0H_1 + D_1H_0\big] = H\sigma T_1q_0'. \tag{4.4.18}$$

For the next calculations we recall the commutator of two operators $[A,B] = AB - BA$. We have for $h \in H^3(\mathbb{R})$

$$\begin{aligned}[\partial_z, \mathbf{T}_0](h) = \partial_z\mathbf{T}_0(h) - \mathbf{T}_0(h') &= -h''' + W''(q_0)h' + W'''(q_0)q_0'h - \big(-h''' + W''(q_0)h'\big)\\ &= W'''(q_0)q_0'h.\end{aligned} \tag{4.4.19}$$

We calculate with the product rule

$$\mathbf{T}_0(zq_0') = -\partial_z(q_0' + zq_0'') + W'''(q_0)zq_0' = -2q_0'' + z\mathbf{T}_0(q_0') = -2q_0''. \tag{4.4.20}$$

With these preparations we are ready to consider the ε^0-order of the evolution (4.4.11). Since the lower orders vanished with the right choices for u_0, u_1 we have

$$T_\varepsilon D_\varepsilon H_\varepsilon = \varepsilon^2\Big(T_2\mathbf{D}_0H_0 + T_1(D_1H_0 + T_1\mathbf{D}_0H_1) + \mathbf{T}_0(*)\Big) + \varepsilon^3 R_\varepsilon$$

with $R_\varepsilon \in X_\delta^{\frac{\mu}{\varepsilon},\Lambda}(\Omega_T)$. To calculate this precisely we would need to consider the ε^2-order in the expansion of H_ε. However we will test the evolution with q_0' thus all of the terms with $\mathbf{T}_0$ will vanish because $\mathbf{T}_0$ is self-adjoint and $\mathbf{T}_0(q_0') = 0$. Thus ε^2-order of (4.4.11) is given by

$$-\frac{1}{2}q_0'\mathcal{V} = \mathbf{T}_0(*) + T_1\left[\mathbf{D}_0H_1 + D_1H_0\right] + T_2\mathbf{D}_0H_0.$$

Testing this equation with q_0' we get

$$\begin{aligned}
-\frac{c_0}{2}\mathcal{V} &= \int_{\mathbb{R}} q_0'\left[T_1(\mathbf{D}_0H_1 + D_1H_0) + T_2\mathbf{D}_0H_0\right] \mathrm{d}\mathcal{L}^1 \\
&= \int_{\mathbb{R}} q_0'\big(-H\partial_z + W'''(q_0)Hq_1\big)(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1 \\
&\quad + \int_{\mathbb{R}} q_0'\left[W'''(q_0)u_2 + \frac{1}{2}W^{(4)}(q_0)H^2q_1^2 - \Delta_{\Gamma_t} + z|\mathrm{II}|^2\partial_z\right]\big(-H\sigma q_0'\big) \mathrm{d}\mathcal{L}^1.
\end{aligned}$$

We split the first term and apply a partial integration on the first summand. We get

$$\begin{aligned}
-\frac{c_0}{2}\mathcal{V} &= \int_{\mathbb{R}} Hq_0''(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1 + \int_{\mathbb{R}} HW'''(q_0)q_0'q_1(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1 \\
&- \int_{\mathbb{R}} H\sigma q_0' \underbrace{W'''(q_0)q_0'u_2}_{=[\partial_z,\mathbf{T}_0](u_2)} \mathrm{d}\mathcal{L}^1 - \int_{\mathbb{R}} \sigma q_0'\left[\frac{H^3}{2}W^{(4)}(q_0)q_0'q_1^2 - q_0'\Delta_{\Gamma_t}H + z|\mathrm{II}|^2Hq_0''\right] \mathrm{d}\mathcal{L}^1.
\end{aligned} \tag{4.4.21}$$

Now we isolate some of the terms to calculate them. We start with the second term on the right-hand side of (4.4.21). We use that $\mathbf{T}_0$ is self-adjoint and (4.4.19) to get

$$\begin{aligned}
\int_{\mathbb{R}} HW'''(q_0)q_0'q_1(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1 &= \int_{\mathbb{R}} H\Big(\partial_z\mathbf{T}_0(q_1) - \mathbf{T}_0(q_1')\Big)(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1 \\
= \int_{\mathbb{R}} H(\mathbf{T}_0(q_1))'(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1 &- \int_{\mathbb{R}} Hq_1'\mathbf{T}_0(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1.
\end{aligned} \tag{4.4.22}$$

By (4.4.14) and (4.4.17) we have $H\mathbf{T}_0(q_1) = H_0 + Hq_0'$. Together with (4.4.18) and $D_1 = -H\partial_z$ this yields

$$\begin{aligned}
&\int_{\mathbb{R}} H(\mathbf{T}_0(q_1))'(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1 - \int_{\mathbb{R}} Hq_1'\mathbf{T}_0(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1 \\
&\quad = \int_{\mathbb{R}} (H_0 + q_0')'(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1 - \int_{\mathbb{R}} \sigma H^2q_1'T_1q_0' \mathrm{d}\mathcal{L}^1 \\
&\quad = \int_{\mathbb{R}} H_0'(\mathbf{D}_0H_1 - HH_0') \mathrm{d}\mathcal{L}^1 + \int_{\mathbb{R}} Hq_0''(\mathbf{D}_0H_1 + D_1H_0) \mathrm{d}\mathcal{L}^1 - \int_{\mathbb{R}} \sigma H^2q_1'T_1q_0' \mathrm{d}\mathcal{L}^1.
\end{aligned} \tag{4.4.23}$$

For the first term of the second line of (4.4.21) we use the representation from (4.4.14) $\mathbf{T}_0u_2 = H_1 + H^2q_1' - z|\mathrm{II}|^2q_0' - \frac{1}{2}W'''(q_0)H^2q_1^2$, (4.4.19), $\mathbf{T}_0(q_0') = 0$ and that $\mathbf{T}_0$ is self-adjoint. We get

$$
\begin{aligned}
-\int_{\mathbb{R}} H\sigma q_0' W'''(q_0) q_0' u_2 \,\mathrm{d}\mathcal{L}^1 &= -\int_{\mathbb{R}} H\sigma q_0' \Big(\partial_z \mathbf{T}_0(u_2) - \mathbf{T}_0(u_2')\Big) \,\mathrm{d}\mathcal{L}^1 \\
&= -\int_{\mathbb{R}} H\sigma q_0' \Big(H_1 + H^2 q_1' - z|\mathrm{II}|^2 q_0' - \frac{1}{2} W'''(q_0) H^2 q_1^2\Big)' \,\mathrm{d}\mathcal{L}^1 \\
&= -\int_{\mathbb{R}} \sigma q_0' \Big(H H_1' + H^3 q_1'' - |\mathrm{II}|^2 H q_0' - z|\mathrm{II}|^2 H q_0'' \\
&\quad - \frac{H^3}{2} W^{(4)}(q_0) q_0' q_1^2 - H^3 W'''(q_0) q_1 q_1'\Big) \,\mathrm{d}\mathcal{L}^1. \qquad (4.4.24)
\end{aligned}
$$

Now we plug (4.4.23) and (4.4.24) into (4.4.21). We rearrange the terms and get

$$
\begin{aligned}
-\frac{c_0}{2}\mathcal{V} = &\int_{\mathbb{R}} 2q_0''(\mathbf{D}_0 H_1 + D_1 H_0) H \,\mathrm{d}\mathcal{L}^1 - \int_{\mathbb{R}} \sigma H^2 q_1' T_1 q_0' \,\mathrm{d}\mathcal{L}^1 \qquad (4.4.25) \\
&+ \int_{\mathbb{R}} H_0'(\mathbf{D}_0 H_1 - H H_0) \,\mathrm{d}\mathcal{L}^1 - \int_{\mathbb{R}} \sigma q_0' \Big(H H_1' + H^3 q_1'' - H^3 W'''(q_0) q_1 q_1'\Big) \,\mathrm{d}\mathcal{L}^1 \\
&\qquad (4.4.26) \\
&+ c_0 \sigma(\Delta_{\Gamma_t} H + H|\mathrm{II}|^2). \qquad (4.4.27)
\end{aligned}
$$

We again isolate some of the integrals for further calculations. We start with the first term on the right-hand side of (4.4.25). We apply (4.4.20), the fact that $\mathbf{T}_0$ is self-adjoint and (4.4.18). This yields

$$
\begin{aligned}
\int_{\mathbb{R}} 2q_0''(\mathbf{D}_0 H_1 + D_1 H_0) H \,\mathrm{d}\mathcal{L}^1 &= -\int_{\mathbb{R}} \mathbf{T}_0(z q_0')(\mathbf{D}_0 H_1 + D_1 H_0) H \,\mathrm{d}\mathcal{L}^1 \\
&= -\int_{\mathbb{R}} \sigma H^2 z q_0' T_1 q_0' \,\mathrm{d}\mathcal{L}^1.
\end{aligned}
$$

Next we use the definition of T_1, (4.4.19), a partial integration and the fact that $\mathbf{T}_0$ is self-adjoint. We get

$$
\begin{aligned}
-\int_{\mathbb{R}} \sigma H^2 z q_0' T_1 q_0' \,\mathrm{d}\mathcal{L}^1 &= -\int_{\mathbb{R}} \sigma H^3 z q_0' (W'''(q_0) q_1 - \partial_z) q_0' \,\mathrm{d}\mathcal{L}^1 \\
&= -\int_{\mathbb{R}} \sigma H^3 z q_0' (\partial_z \mathbf{T}_0(q_1) - \mathbf{T}_0(q_1')) \,\mathrm{d}\mathcal{L}^1 + \int_{\mathbb{R}} \sigma H^3 z q_0' q_0'' \,\mathrm{d}\mathcal{L}^1 \\
&= \int_{\mathbb{R}} \sigma H^3((q_0' + z q_0'')\mathbf{T}_0(q_1) - q_1' \mathbf{T}_0(-z q_0')) \,\mathrm{d}\mathcal{L}^1 \\
&\quad - \frac{1}{2}\int_{\mathbb{R}} \sigma H^3 |q_0'|^2 \,\mathrm{d}\mathcal{L}^1.
\end{aligned}
$$

Now we use (4.4.20), $\mathbf{T}_0(q_0') = 0$ and $\mathbf{T}_0 q_1 = q_0' - \sigma \overline{q}_0'$. This yields

$$
\begin{aligned}
\int_{\mathbb{R}} \sigma H^3((q_0' + z q_0'')\mathbf{T}_0(q_1) - q_1' \mathbf{T}_0(-z q_0')) \,\mathrm{d}\mathcal{L}^1 - \frac{1}{2}\int_{\mathbb{R}} \sigma H^3 |q_0'|^2 \,\mathrm{d}\mathcal{L}^1 \\
= \int_{\mathbb{R}} \sigma H^3 \Big(z q_0'' \mathbf{T}_0(q_1) - 2 q_0'' q_1'\Big) \,\mathrm{d}\mathcal{L}^1 - \frac{1}{2} c_0 \sigma H^3 \\
= \int_{\mathbb{R}} \sigma H^3 \Big(z q_0'' q_0' - \sigma z q_0'' \overline{q}_0' - 2 q_0'' q_1'\Big) \,\mathrm{d}\mathcal{L}^1 - \frac{1}{2} c_0 \sigma H^3. \qquad (4.4.28)
\end{aligned}
$$

Next we consider the second integral on the right-hand side of (4.4.25). Here we just plug in the definition of T_1 and get

$$-\int_{\mathbb{R}} \sigma H^2 q_1' T_1 q_0' \,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} \sigma H^3 \Big(q_1' q_0'' - W'''(q_0) q_0' q_1 q_1' \Big) \,\mathrm{d}\mathcal{L}^1. \tag{4.4.29}$$

For the first term in (4.4.26) we use a partial integration and that $\mathbf{D}_0$ is self-adjoint. We get with $H_0 = -\sigma H \overline{q}_0'$ from (4.4.17)

$$\begin{aligned}
\int_{\mathbb{R}} H_0' (\mathbf{D}_0 H_1 - H H_0') \,\mathrm{d}\mathcal{L}^1 &= \int_{\mathbb{R}} \Big(- H_1' \mathbf{D}_0 H_0 + H H_0'' H_0 \Big) \,\mathrm{d}\mathcal{L}^1 \\
&= \int_{\mathbb{R}} H \Big(\sigma H_1' q_0' - (-H_0'' + H_0 - H_0) H_0 \Big) \,\mathrm{d}\mathcal{L}^1 \\
&= \int_{\mathbb{R}} H \Big(\sigma H_1' q_0' - H_0 \mathbf{D}_0 H_0 + H_0^2 \Big) \,\mathrm{d}\mathcal{L}^1 \\
&= \int_{\mathbb{R}} \sigma H \Big(H_1' q_0' - \sigma H^2 q_0' \overline{q}_0' + \sigma H^2 |\overline{q}_0'|^2 \Big) \,\mathrm{d}\mathcal{L}^1.
\end{aligned}$$

Applying the definition of σ yields

$$\int_{\mathbb{R}} \sigma H \Big(H_1' q_0' - \sigma H^2 q_0' \overline{q}_0' + \sigma H^2 |\overline{q}_0'|^2 \Big) \,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} \sigma H \Big(H_1' q_0' - H^2 |q_0'|^2 + \sigma H^2 |\overline{q}_0'|^2 \Big) \,\mathrm{d}\mathcal{L}^1. \tag{4.4.30}$$

We plug (4.4.28)-(4.4.30) into (4.4.25)-(4.4.27). The terms containing $\sigma H^3 q_0'' q_1'$ and $\sigma H^3 q_0' q_1''$ combine to 0 after a partial integration, the terms $\sigma H^3 W'''(q_0) q_0' q_1 q_1'$ cancel each other out and the same is true for $\sigma H q_0' H_1'$. We get

$$-\frac{c_0}{2}\mathcal{V} = c_0 \sigma \Big(\Delta_{\Gamma_t} H + H |\mathrm{II}|^2 - \frac{1}{2} H^3 \Big) + \sigma H^3 \int_{\mathbb{R}} \Big(z q_0'' q_0' - |q_0'|^2 - \sigma z q_0'' \overline{q}_0' + \sigma |\overline{q}_0'|^2 \Big) \,\mathrm{d}\mathcal{L}^1.$$

Thus the proof is finished if we can prove that the last integral vanishes. We use $q_0'' = -\overline{q}_0^{(4)} + \overline{q}_0''$ and calculate with a partial integration

$$\begin{aligned}
&\int_{\mathbb{R}} \Big(z q_0'' q_0' - |q_0'|^2 - \sigma z q_0'' \overline{q}_0' + \sigma |\overline{q}_0'|^2 \Big) \,\mathrm{d}\mathcal{L}^1 \\
&\qquad = \int_{\mathbb{R}} \Big(-\frac{3}{2} |q_0'|^2 + \sigma |\overline{q}_0'|^2 - \sigma z (-\overline{q}_0^{(4)} + \overline{q}_0'') \overline{q}_0' \Big) \,\mathrm{d}\mathcal{L}^1
\end{aligned}$$

Thus the claim is proven if the remaining integral vanishes. We calculate with a few partial integrations

$$\begin{aligned}
&\int_{\mathbb{R}} \Big(-\frac{3}{2} |q_0'|^2 + \sigma |\overline{q}_0'|^2 - \sigma z (-\overline{q}_0^{(4)} + \overline{q}_0'') \overline{q}_0' \Big) \,\mathrm{d}\mathcal{L}^1 \\
&\qquad = \int_{\mathbb{R}} \Big(-\frac{3}{2} |q_0'|^2 + \sigma |\overline{q}_0'|^2 - \sigma z \overline{q}_0'' \overline{q}_0' + \sigma z \overline{q}_0^{(4)} \overline{q}_0' \Big) \,\mathrm{d}\mathcal{L}^1 \\
&\qquad = \int_{\mathbb{R}} \Big(-\frac{3}{2} |q_0'|^2 + \frac{3}{2} \sigma |\overline{q}_0'|^2 - \sigma \overline{q}_0''' (\overline{q}_0' + z \overline{q}_0'') \Big) \,\mathrm{d}\mathcal{L}^1 \\
&\qquad = \frac{3}{2} \int_{\mathbb{R}} \Big(\sigma \Big(|\overline{q}_0''|^2 + |\overline{q}_0'|^2 \Big) - |q_0'|^2 \Big) \,\mathrm{d}\mathcal{L}^1.
\end{aligned}$$

The last formula we need for the Willmore flow is

$$\int_{\mathbb{R}} |q_0'|^2 \,\mathrm{d}\mathcal{L}^1 = \sigma \int_{\mathbb{R}} \left(|\overline{q}_0''|^2 + |\overline{q}_0'|^2 \right) \mathrm{d}\mathcal{L}^1,$$

which comes from testing $-\overline{q}_0'' + \overline{q}_0 = q_0$ with $\overline{q}_0''$ and using $\sigma \int_{\mathbb{R}} q_0' \overline{q}_0' \,\mathrm{d}\mathcal{L}^1 = c_0$. We finally get

$$-\mathcal{V} = 2\sigma \Big(\Delta_{\Gamma_t} H + H |\mathrm{II}|^2 - \frac{1}{2} H^3 \Big). \qquad \square$$

5 Liminf estimate for the diffuse Willmore approximation based on the Karali-Katsoulakis model

In this chapter we will prove the lim inf estimate of the sum of the standard diffuse perimeter energy as discussed in [MM77] and a new diffuse Willmore energy introduced in Chapter 4 and based on [KK07]. We modify and adapt ideas and concepts from [RS06, HT00]. A main ingredient will be to characterize the w^*-limit of a modified diffuse surface measure.

5.1 $\Gamma-\liminf$ estimate for a new Willmore approximation

In this chapter we consider $n \in \{2,3\}$ and assume that $\Omega \subseteq \mathbb{R}^n$ is open and bounded with C^2-boundary. We recall the functions from 4.1.1, among them the double-well potential $W(r) := (1-r^2)^2$ for $r \in \mathbb{R}$, the induced optimal profile $q_0(r) = \tanh(\sqrt{2}r)$ and $\overline{q}_0 = \mathbf{A}_0 q_0$. q_0 solves

$$q_0' = \sqrt{2W(q_0)} \quad \text{and} \quad q_0(0) = 0. \tag{5.1.1}$$

$\overline{q}_0$ is characterized by

$$-\overline{q}_0'' + \overline{q}_0 = q_0.$$

We also recall the constants

$$c_0 = \int_{\mathbb{R}} |q_0'|^2 \, \mathrm{d}\mathcal{L}^1 \quad \text{and} \quad \sigma = \frac{c_0}{\displaystyle\int_{\mathbb{R}} q_0' \overline{q}_0' \, \mathrm{d}\mathcal{L}^1}. \tag{5.1.2}$$

The following theorem is the lim inf estimate for the approximation $\mathcal{W}_\varepsilon^{\mathrm{KK}} \xrightarrow{\Gamma} c_0 \sigma \mathcal{W}$. We also prove a more general version; see Theorem 5.2.5.

Theorem 5.1.1 (lim inf estimate for a new Willmore approximation).
Let $E \subseteq \Omega$ with $\partial E \cap \Omega \in C^2$. For any sequence $(u_\varepsilon)_{\varepsilon > 0}$ with $u_\varepsilon \longrightarrow u = 2\chi_E - 1$ in $L^1_{\mathrm{loc}}(\Omega)$ we get

$$c_0\big(\mathcal{P}(u) + \sigma \mathcal{W}(u)\big) \leq \liminf_{\varepsilon \to 0} \Big(\mathcal{P}_\varepsilon(u_\varepsilon) + \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon)\Big). \tag{5.1.3}$$

Remark.

- *Because of the Modica Mortola Theorem 2.4.2 we get*

$$c_0\mathcal{P}(u) \le \liminf_{\varepsilon\to 0} \mathcal{P}_\varepsilon(u_\varepsilon). \tag{5.1.4}$$

If the right hand side is finite we can conclude $u \in BV(\Omega;\{\pm 1\})$ thus E has finite perimeter.

- *We can prove*

$$c_0\sigma\mathcal{W}(u) \le \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon)$$

for any sequence $(u_\varepsilon)_{\varepsilon>0}$ with bounded diffuse perimeter $(\mathcal{P}_\varepsilon(u_\varepsilon))_{\varepsilon>0}$. A suitable formulation for that is to consider the $\liminf$*-inequality for the sum $\mathcal{P}_\varepsilon + \mathcal{W}_\varepsilon$ because the bound on the diffuse perimeter will then follow from the assumption that the* $\liminf$ *is finite.*

- *Together with the* $\limsup$ *estimate in Theorem 4.3.1 and the Modica-Mortola Theorem 2.4.2 this implies*

$$\Gamma(\mathcal{L}^1(\Omega)) - \lim_{\varepsilon\to 0}\Big(\mathcal{P}_\varepsilon + \mathcal{W}_\varepsilon^{\mathrm{KK}}\Big) = c_0\Big(\mathcal{P} + \sigma\mathcal{W}\Big) \tag{5.1.5}$$

on the set of smooth limit points and for $n \le 3$.

- *Recall $G_{n-1}(\Omega) := \Omega \times G(n, n-1)$ and $\mathbb{V}_{n-1}(\Omega) := C_c^0(G_{n-1}(\Omega))'$ from Section 2.2.*

Theorem 5.1.1 is a corollary of a more general result (see Theorem 5.2.5) and will be proven later.

5.2 Measure-theoretic formulation of the main theorem and preparations

Definition 5.2.1 (Diffuse Radon measures and varifolds)**.**
For a sequence $(u_\varepsilon)_{\varepsilon>0}$ in $H^3(\Omega)$ we introduce the diffuse area measure μ_ε, the discrepancy measure ξ_ε, the standard diffuse Willmore measure α_ε and add the new diffuse Willmore measure κ_ε

$$\mu_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\mathcal{L}^n \llcorner \Omega, \tag{5.2.1}$$

$$\xi_\varepsilon := \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\mathcal{L}^n \llcorner \Omega, \tag{5.2.2}$$

$$\alpha_\varepsilon := \frac{1}{\varepsilon}|H_\varepsilon|^2\mathcal{L}^n \llcorner \Omega, \tag{5.2.3}$$

$$\text{and} \quad \kappa_\varepsilon := \Big(\frac{1}{\varepsilon}|H_\varepsilon|^2 + \varepsilon|\nabla H_\varepsilon|^2\Big)\mathcal{L}^n \llcorner \Omega, \tag{5.2.4}$$

$$\text{with} \quad H_\varepsilon := -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon). \tag{5.2.5}$$

The names stem from the obvious connections to the respective functionals

$$\mu_\varepsilon(\Omega) = \mathcal{P}_\varepsilon(u_\varepsilon)$$
$$\alpha_\varepsilon(\Omega) = \mathcal{W}_\varepsilon(u_\varepsilon)$$
$$\kappa_\varepsilon(\Omega) = \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon).$$

For each $\varepsilon > 0$ *we fix a Borel-measurable function* $\nu_\varepsilon : \Omega \longrightarrow \mathbb{S}^{n-1}$ *with* $\nu_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ *on* $\{\nabla u_\varepsilon \neq 0\}$, $\nu_\varepsilon = e_1$ *on* $\{\nabla u_\varepsilon = 0\}$ *and define a* $(n-1)$*-varifold* $V_\varepsilon := \mu_\varepsilon \otimes \nu_\varepsilon^\perp \in \mathbb{V}_{n-1}(\Omega)$ *by*

$$\int_{G_{n-1}(\Omega)} \phi(x,S)\,\mathrm{d}V_\varepsilon(x,S) := \int_\Omega \phi(x,\nu_\varepsilon(x)^\perp)\,\mathrm{d}\mu_\varepsilon(x) \quad \text{for} \quad \phi \in C_c^0(G_{n-1}(\Omega)). \tag{5.2.6}$$

Corollary 5.2.2 (Limit measures and varifold)**.**
Let $\Lambda > 0$ *such that*

$$\liminf_{\varepsilon\to 0} \left(\mu_\varepsilon(\Omega) + \kappa_\varepsilon(\Omega)\right) \leq \Lambda.$$

Then there exist a subsequence which realizes the $\liminf$*, a function* $u \in BV(\Omega; \{\pm 1\})$*, finite Radon measures* $\mu, \xi, \alpha, \kappa \in C_c^0(\Omega)'$*, and a varifold* $V \in \mathbb{V}_{n-1}(\Omega)$ *such that*

$$u_\varepsilon \longrightarrow u \quad in \quad L^1(\Omega), \tag{5.2.7}$$
$$\mu_\varepsilon \xrightarrow{w^*} \mu \quad in \quad C_0^0(\Omega)', \tag{5.2.8}$$
$$\xi_\varepsilon \xrightarrow{w^*} \xi, \quad \alpha_\varepsilon \xrightarrow{w^*} \alpha \quad in \quad C_0^0(\Omega)', \tag{5.2.9}$$
$$V_\varepsilon \xrightarrow{w^*} V \quad in \quad \mathbb{V}_{n-1}(\Omega), \tag{5.2.10}$$
$$and \quad \kappa_\varepsilon \xrightarrow{w^*} \kappa \quad in \quad C_0^0(\Omega)'. \tag{5.2.11}$$

Proof. By Theorems 2.4.2 and 2.4.3 we conclude that there exists $u \in BV(\Omega; \{\pm 1\})$ and a subsequence such that $u_\varepsilon \longrightarrow u$ in $L^1(\Omega)$. The claims (5.2.8)-(5.2.11) follow immediately from Theorem 2.2.2. ☐

We highlight important results from [RS06] which will be used in the proof of the $\liminf$-estimate.

Theorem 5.2.3 (Key results from [RS06])**.**
Let $\Lambda > 0$ *with*

$$\liminf_{\varepsilon\to 0} \left(\mu_\varepsilon(\Omega) + \alpha_\varepsilon(\Omega)\right) \leq \Lambda$$

and assume (5.2.7)-(5.2.10)*. Then it holds*

(*i*) $\mu \geq \frac{c_0}{2}|\nabla u|$.

(*ii*) V *is a rectifiable* $(n-1)$*-varifold with weak mean curvature vector* $\vec{H}_V \in L^2(\Omega, \mu; \mathbb{R}^n)$ *and* $|\vec{H}_V|^2 \mu \leq \alpha$.

(*iii*) *The discrepancy measure vanishes in the limit, i.e.* $\xi = 0$.

(*iv*) $c_0^{-1}V$ *is an integral* $(n-1)$*-varifold and thus*

$$\mu = c_0\theta\mathcal{H}^{n-1} \llcorner \Gamma,$$

where Γ *is a countably* $(n-1)$*-rectifiable set and* $\theta : \Gamma \longrightarrow \mathbb{N}$ *is* $\mathcal{H}^{n-1}$*-measurable.*

(*v*) $\limsup_{\rho\to 0} \rho^{1-n}\mu(B_\rho(x)) < \infty$ *for all* $x \in \Omega$.

Proof. (*i*) follows from [MM77], see also [Mod87]. The claims (*ii*)-(*v*) follow from Theorems 4.1, Proposition 4.5, Proposition 4.9, and Theorem 5.1 in [RS06]. □

Remark.

- *In* (*i*) ∇u *is the derivative in the sense of Radon measures which exists because* u *has bounded variation because of* (5.1.4)*.*
- *The* σ *used in [RS06] corresponds to* c_0 *here, not* σ*.*
- *The space* $L^2_{\mathrm{loc}}(\Omega, \mu; \mathbb{R}^n)$ *is the* L^2_{loc} *space of the measure* μ *for functions (more precisely equivalence classes of functions) defined on* Ω *with values in* $\mathbb{R}^n$*.*

Lemma 5.2.4 (Approximation of the first variation)**.**
Let the assumptions from Theorem 5.2.3 hold and let $\eta \in C^1_c(\Omega; \mathbb{R}^n)$*, then we have*

$$\int_\Omega \vec{H}_V \cdot \eta \,\mathrm{d}\mu = \lim_{\varepsilon\to 0} \int_\Omega \nabla u_\varepsilon \cdot \eta H_\varepsilon \,\mathrm{d}\mathcal{L}^n.$$

Proof. From Proposition 4.10 in [RS06] we have

$$\langle \eta, \delta V_\varepsilon \rangle_{C^0_c(\Omega)'} = -\int_\Omega \nabla u_\varepsilon \cdot \eta H_\varepsilon \,\mathrm{d}\mathcal{L}^n + \int_\Omega \nu_\varepsilon \cdot D\eta\nu_\varepsilon \,\mathrm{d}\xi_\varepsilon.$$

The second term vanishes as $\varepsilon \to 0$ because of $|\nu_\varepsilon| = 1$ and (*iii*) from Theorem 5.2.3. For the term on the left-hand side we have $\|V_\varepsilon\| = \mu_\varepsilon$ by construction and thus

$$\|V\| \xleftarrow{w^*} \|V_\varepsilon\| = \mu_\varepsilon \xrightarrow{w^*} \mu \quad \text{in} \quad C^0_c(\Omega)'.$$

Combined with (5.2.10) we get

$$\langle \eta, \delta V_\varepsilon \rangle_{C^0_c(\Omega)'} \longrightarrow \langle \eta, \delta V \rangle_{C^0_c(\Omega)'} = -\int_\Omega \vec{H}_V \cdot \eta \,\mathrm{d}\mu$$

which finishes the proof. □

With these notations and preparations we can formulate the general, measure-theoretic formulation of the lim inf estimate.

Theorem 5.2.5 (Measure control of diffuse Willmore energy)**.**
Let $\Lambda > 0$ *such that*

$$\liminf_{\varepsilon\to 0} \left(\mu_\varepsilon(\Omega) + \kappa_\varepsilon(\Omega)\right) \le \Lambda \tag{5.2.12}$$

for some $\Lambda > 0$ *and assume* (5.2.8)-(5.2.11)*. Then we have*

$$\sigma \left|\vec{H}_V\right|^2 \mu \le \kappa, \tag{5.2.13}$$

in the sense of Borel measures.

The proof of Theorem 5.2.5 needs a lot of preparations and is presented in the last section of this chapter. In particular we conclude the following Corollary, which will be useful in Chapter 6.

Corollary 5.2.6.
Let $(u_\varepsilon)_{\varepsilon>0}$ *be a sequence in* $H^3(\Omega)$ *with* (5.2.8), (5.2.10), *and*

$$\limsup_{\varepsilon\to 0} \mu_\varepsilon(\Omega) < \infty. \tag{5.2.14}$$

Then we conclude

$$\sigma \int_\Omega |\vec{H}_V|^2 \,\mathrm{d}\mu \le \liminf_{\varepsilon\to 0} \kappa_\varepsilon(\Omega). \tag{5.2.15}$$

Assuming Theorem 5.2.5 is proven we can derive Corollary 5.2.6 and Theorem 5.1.1.

Prood of Corollary 5.2.6.
If the right hand-side of (5.2.15) is infinite there is nothing to prove. If it is finite we can assume (5.2.12) by (5.2.14). Thus the claims from Corollary 5.2.2 and Theorem 5.2.3 hold. Take any $\phi \in C_c^0(\Omega)$ with $0 \le \phi \le 1$, then by Theorem 5.2.5 we have that

$$\sigma \int_\Omega \phi |\vec{H}_V|^2 \,\mathrm{d}\mu = \left\langle \phi, \sigma |\vec{H}_V|^2 \mu \right\rangle_{C_c^0(\Omega)'} \le \langle \phi, \kappa \rangle_{C_c^0(\Omega)'} = \liminf_{\varepsilon\to 0} \langle \phi, \kappa_\varepsilon \rangle_{C_c^0(\Omega)'} \le \liminf_{\varepsilon\to 0} \kappa_\varepsilon(\Omega).$$

Taking the supremum over $\phi \in C_c^0(\Omega)$ with $0 \le \phi \le 1$ results in

$$\sigma \int_\Gamma |\vec{H}_V|^2 \,\mathrm{d}\mu \le \liminf_{\varepsilon\to 0} \kappa_\varepsilon(\Omega) \tag{5.2.16}$$

by the regularity of Radon measures. □

The proof of Theorem 5.1.1 is also based on (5.2.16), we just need to rewrite or estimate the terms at the ends of the inequality.

Proof of Theorem 5.1.1.
If the right hand-side of (5.1.3) is infinite there is nothing to prove. Thus we can assume (5.2.12) and the claims from Corollary 5.2.2 and Theorem 5.2.3 hold. As in the proof of Corollary 5.2.6 we conclude (5.2.16). We deduce by (iv) from Theorem 5.2.3

$$c_0\sigma \int_\Omega |\vec{H}_V|^2 \,\mathrm{d}\mathcal{H}^{n-1} \le \sigma \int_\Gamma |\vec{H}_V|^2 \,\mathrm{d}\mu \le \liminf_{\varepsilon\to 0} \kappa_\varepsilon(\Omega) = \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon).$$

In the following we connect the weak mean curvature vector $\vec{H}_V$ of the varifold V with the mean curvature vector $\vec{H}_E$ of the hypersurface $\partial E \cap \Omega$. Owing to the C^2-regularity we have $\partial E \cap \Omega = \partial^* E \cap \Omega$, where $\partial^* E$ is the essential boundary defined in Definition 2.2.10. Since $E \subseteq \Omega$ and because of the C^2-regularity at every point of $\partial E \cap \Omega$ there exists an inner normal ν_E.
To compare the mean curvature vectors we need to compare the respective varifolds. Define $V_E := (\mathcal{H}^{n-1} \llcorner \partial E \cap \Omega) \otimes \nu_E^\perp \in \mathbb{V}_{n-1}(\Omega)$. With (i) from Theorem 5.2.3 we can apply Corollary 4.3 from [Sch09] which yields $\vec{H}_V = \vec{H}_E$ and thus we get with (5.2.16)

$$c_0\sigma\mathcal{W}(u) = c_0\sigma \int_{\partial E\cap\Omega} |\vec{H}_E|^2 \,\mathrm{d}\mathcal{H}^{n-1} \le \sigma \int_\Omega |\vec{H}_V|^2 \,\mathrm{d}\mu \le \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon).$$

Since we also have the $\liminf$ estimate for $\mathcal{P}_\varepsilon \longrightarrow c_0\mathcal{P}$ from the Modica-Mortola Theorem 2.4.2 the claim follows. □

Before starting with the technical preparations for the proof of Theorem 5.2.5 we present the central notion to motivate why we need to examine the measures in the next sections. The rigorous proof is done in Section 5.5, here we outline the idea in a formal way.

Remark.
Let us first recall the arguments from [RS06] for the proof of the $\liminf$ *estimate for the standard approximation and replicate the idea for our approximation.*

We use the dual representation of the $L^2(\mu)$*-norm of* $\vec{H}_V$*. Let* $\eta \in C^1_c(\Omega;\mathbb{R}^n)$ *with*

$$\int_\Omega |\eta|^2 \,\mathrm{d}\mu \le 1. \tag{5.2.17}$$

Then we apply Lemma 5.2.4 and get that

$$\int_\Omega \vec{H}_V \cdot \eta \,\mathrm{d}\mu = \lim_{\varepsilon\to 0} \int_\Omega \nabla u_\varepsilon \cdot \eta H_\varepsilon \,\mathrm{d}\mathcal{L}^n, \tag{5.2.18}$$

where $H_\varepsilon = -\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon)$*. We use the Cauchy-Schwarz inequality and get*

$$\begin{aligned}\int_\Omega \vec{H}_V \cdot \eta \,\mathrm{d}\mu &\le \lim_{\varepsilon\to 0}\left[\int_\Omega \varepsilon|\eta|^2 |\nabla u_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \int_\Omega \frac{1}{\varepsilon}|H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n\right]^{\frac{1}{2}} \\ &\le \sqrt{\limsup_{\varepsilon\to 0}\langle |\eta|^2, \mu_\varepsilon + \xi_\varepsilon\rangle_{C^0_c(\Omega)'} \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon)} \\ &= \underbrace{\sqrt{\langle |\eta|^2, \mu\rangle_{C^0_c(\Omega)'}}}_{\le 1} \sqrt{\liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon)}.\end{aligned}$$

We take the supremum over all η *with* (5.2.17)*. We get by (iv) from Theorem 5.2.3*

$$c_0 \int_\Gamma |\vec{H}_V|^2 \,\mathrm{d}\mathcal{H}^{n-1} \le \int_\Omega |\vec{H}_V|^2 \,\mathrm{d}\mu \le \liminf_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon).$$

Applying Corollary 4.3 from [Sch09] ensures $\vec{H}_V = \vec{H}_E$ *and the proof for the standard approximation is finished.*

However our approximation is different because H_ε *is replaced by* $(-\varepsilon^2\Delta + \mathrm{Id})H_\varepsilon$*. Thus we need to modify the above argument and introduce* $\overline{u}_\varepsilon$ *as the solution to*

$$\begin{aligned} -\varepsilon^2 \Delta \overline{u}_\varepsilon + \overline{u}_\varepsilon &= u_\varepsilon \quad \textit{in} \quad \Omega \\ \partial_\nu \overline{u}_\varepsilon &= 0 \quad\ \ \textit{on} \quad \partial\Omega. \end{aligned}$$

We will show that

$$\int_\Omega \vec{H}_V \cdot \eta \,\mathrm{d}\mu = \lim_{\varepsilon\to 0} \int_\Omega \nabla \overline{u}_\varepsilon \cdot \eta (-\varepsilon^2\Delta + \mathrm{Id}) H_\varepsilon \,\mathrm{d}\mathcal{L}^n$$

and can apply the Cauchy-Schwarz estimate for the inner product induced by the differential operator. This results in

$$\int_\Omega \vec{H}_V \cdot \eta \,\mathrm{d}\mu \le \liminf_{\varepsilon\to 0}\left[\int_\Omega \varepsilon \eta \cdot \nabla \overline{u}_\varepsilon \big(-\varepsilon^2\Delta + \mathrm{Id}\big)(\eta\cdot\nabla\overline{u}_\varepsilon) \,\mathrm{d}\mathcal{L}^n\right.$$

$$\cdot \int_\Omega \frac{1}{\varepsilon} H_\varepsilon(-\varepsilon^2\Delta + \mathrm{Id}) H_\varepsilon \,\mathrm{d}\mathcal{L}^n \Bigg]^{\frac{1}{2}}.$$

Further manipulations yield

$$\lim_{\varepsilon\to 0} \int_\Omega \varepsilon\eta \cdot \nabla \overline{u}_\varepsilon \big(-\varepsilon^2\Delta + \mathrm{Id} \big)(\eta \cdot \nabla \overline{u}_\varepsilon) \,\mathrm{d}\mathcal{L}^n = \lim_{\varepsilon\to 0} \int_\Omega \varepsilon(\eta\cdot\nabla\overline{u}_\varepsilon)(\eta\cdot\nabla u_\varepsilon)\,\mathrm{d}\mathcal{L}^n$$

and thus

$$\int_\Omega \vec{H}_V \cdot \eta \,\mathrm{d}\mu \le \liminf_{\varepsilon\to 0} \left[\kappa_\varepsilon(\Omega) \int_\Omega |\eta|^2 \varepsilon |\nabla \overline{u}_\varepsilon| |\nabla u_\varepsilon| \,\mathrm{d}\mathcal{L}^n \right]^{\frac{1}{2}}.$$

Taking the supremum over η with (5.2.17) *will result in the Willmore energy on the left-hand side. The desired term $\kappa_\varepsilon(\Omega)$ is already on the right-hand side so the key for this proof is to deal with the remaining integral. We do so by examining the measure $\varepsilon|\nabla\overline{u}_\varepsilon||\nabla u_\varepsilon|\mathcal{L}^n$ and its weak*-limit.*

5.3 Uniform bounds for the modified phase fields

Definition 5.3.1 (Modified phase field and area measure)**.**
Assume that $u_\varepsilon \in C^3(\Omega)$ satisfies (5.2.12) *for some $\Lambda > 0$. We define $\overline{u}_\varepsilon \in H^1(\Omega)$ as the solution to*

$$-\varepsilon^2\Delta\overline{u}_\varepsilon + \overline{u}_\varepsilon = u_\varepsilon \quad \text{in} \quad \Omega \tag{5.3.1}$$

$$\partial_\nu \overline{u}_\varepsilon = 0 \quad \text{on} \quad \partial\Omega. \tag{5.3.2}$$

We also define the Radon measure

$$\vartheta_\varepsilon := \varepsilon|\nabla u_\varepsilon||\nabla\overline{u}_\varepsilon|\mathcal{L}^n \llcorner \Omega \in C_c^0(\Omega)'. \tag{5.3.3}$$

From elliptic regularity theory we get $\overline{u}_\varepsilon \in C^5(\Omega) \cap H^5(\Omega)$. We start with an estimate that makes use of the boundary condition for $\overline{u}_\varepsilon$.

Lemma 5.3.2.
Consider $\Omega, u_\varepsilon, \overline{u}_\varepsilon$ as in Definition 5.3.1. Then we have for all $\varepsilon > 0$

$$\int_{\{|\overline{u}_\varepsilon|>1\}} \Big[\varepsilon|\nabla\overline{u}_\varepsilon| + \frac{1}{2\varepsilon}(|\overline{u}_\varepsilon| - 1)^2\Big] \,\mathrm{d}\mathcal{L}^n \le \int_{\{|u_\varepsilon|>1\}} \frac{1}{2\varepsilon} W(u_\varepsilon)\,\mathrm{d}\mathcal{L}^n \tag{5.3.4}$$

Proof. We first obtain from (5.3.1) that

$$-\varepsilon\Delta(\overline{u}_\varepsilon - 1) + \frac{1}{\varepsilon}(\overline{u}_\varepsilon - 1) = \frac{1}{\varepsilon}(u_\varepsilon - 1) \tag{5.3.5}$$

Testing this equation with $(\overline{u}_\varepsilon - 1)_+$ and using (5.3.2) yields

$$\int_\Omega \Big[\varepsilon\nabla(\overline{u}_\varepsilon - 1)\cdot\nabla(\overline{u}_\varepsilon - 1)_+ + \frac{1}{\varepsilon}(\overline{u}_\varepsilon - 1)(\overline{u}_\varepsilon - 1)_+\Big]\,\mathrm{d}\mathcal{L}^n$$
$$= \int_\Omega \frac{1}{\varepsilon}(u_\varepsilon - 1)(\overline{u}_\varepsilon - 1)_+ \,\mathrm{d}\mathcal{L}^n \le \int_\Omega \frac{1}{\varepsilon}(u_\varepsilon - 1)_+(\overline{u}_\varepsilon - 1)_+ \,\mathrm{d}\mathcal{L}^n$$

$$\implies \int\limits_{\{\overline{u}_\varepsilon>1\}} \left[\varepsilon\left|\nabla\overline{u}_\varepsilon\right|^2 + \frac{1}{\varepsilon}(\overline{u}_\varepsilon - 1)^2\right] \mathrm{d}\mathcal{L}^n$$

$$\leq \int\limits_{\{\overline{u}_\varepsilon>1\}} \frac{1}{2\varepsilon}(\overline{u}_\varepsilon - 1)^2 \,\mathrm{d}\mathcal{L}^n + \int\limits_{\{u_\varepsilon>1\}} \frac{1}{2\varepsilon}(u_\varepsilon - 1)^2 \,\mathrm{d}\mathcal{L}^n$$

$$\implies \int\limits_{\{\overline{u}_\varepsilon>1\}} \left[\varepsilon\left|\nabla\overline{u}_\varepsilon\right|^2 + \frac{1}{2\varepsilon}(\overline{u}_\varepsilon - 1)^2\right] \mathrm{d}\mathcal{L}^n$$

$$\leq \int\limits_{\{u_\varepsilon>1\}} \frac{1}{2\varepsilon}(u_\varepsilon^2 - 1)^2 \,\mathrm{d}\mathcal{L}^n = \int\limits_{\{u_\varepsilon>1\}} \frac{1}{2\varepsilon}W(u_\varepsilon) \,\mathrm{d}\mathcal{L}^n$$

Similar we deduce

$$\int\limits_{\{\overline{u}_\varepsilon<-1\}} \left[\varepsilon\left|\nabla\overline{u}_\varepsilon\right|^2 + \frac{1}{2\varepsilon}(\overline{u}_\varepsilon + 1)^2\right] \mathrm{d}\mathcal{L}^n \leq \int\limits_{\{u_\varepsilon<-1\}} \frac{1}{2\varepsilon}W(u_\varepsilon) \,\mathrm{d}\mathcal{L}^n.$$

Adding both estimates yields (5.3.4). $\square$

From (5.2.12) we get with a Cauchy-Schwarz estimate

$$\begin{aligned}\int_\Omega |u_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n = \int_\Omega (|u_\varepsilon|^2 - 1) \,\mathrm{d}\mathcal{L}^n + C(\Omega) &\leq \left[C(\Omega)\int_\Omega W(u_\varepsilon)\,\mathrm{d}\mathcal{L}^n\right]^{\frac{1}{2}} + C(\Omega) \\ &\leq C(\Omega)(1+\sqrt{\varepsilon\Lambda}) \leq C(\Omega,\Lambda).\end{aligned} \tag{5.3.6}$$

With (5.3.4) we can deduce similarly

$$\begin{aligned}\int_\Omega |\overline{u}_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \leq C(\Omega) + \int\limits_{\{|\overline{u}_\varepsilon|>1\}} (|\overline{u}_\varepsilon| - 1 + 1)^2 \,\mathrm{d}\mathcal{L}^n &\leq C(\Omega) + \int\limits_{\{|\overline{u}_\varepsilon|>1\}} 2(|\overline{u}_\varepsilon| - 1)^2 \,\mathrm{d}\mathcal{L}^n \\ \leq C(\Omega)(1+2\varepsilon\Lambda) \leq C(\Omega,\Lambda)\end{aligned} \tag{5.3.7}$$

Both of these estimates yield uniform bounds as $\varepsilon \to 0$. In the following we will rely on (5.3.7) and prove results that hold independent from (5.3.2).

Lemma 5.3.3.
Consider $u \in C^3(\Omega)$ with (5.2.12) and assume that $\overline{u}_\varepsilon \in H^3(\Omega)$ satisfies (5.3.1) and (5.3.7). Then we have for any $\eta \in C_c^1(\Omega)$ and all $\varepsilon > 0$

$$\begin{aligned}\int_\Omega \left(\varepsilon\eta^2\left|\nabla(\overline{u}_\varepsilon - u_\varepsilon)\right|^2 + \frac{1}{\varepsilon}\eta^2|\overline{u}_\varepsilon - u_\varepsilon|^2\right) \mathrm{d}\mathcal{L}^n \\ \leq \int_\Omega \left(10\varepsilon\left|\nabla\eta\right|^2(\overline{u}_\varepsilon - u_\varepsilon)^2 + 4\varepsilon\eta^2\left|\nabla u_\varepsilon\right|^2\right) \mathrm{d}\mathcal{L}^n.\end{aligned} \tag{5.3.8}$$

Furthermore we have for all $j \in \{1,\dots,n\}$

$$\int_\Omega \left(\varepsilon^3\eta^2\left|\nabla\partial_j\overline{u}_\varepsilon\right|^2 + \varepsilon\eta^2|\partial_j\overline{u}_\varepsilon|^2\right) \mathrm{d}\mathcal{L}^n \leq \int_\Omega \left(4\varepsilon^3|\partial_j\overline{u}_\varepsilon|^2\left|\nabla\eta\right|^2 + \varepsilon\eta^2|\partial_j u_\varepsilon|^2\right) \mathrm{d}\mathcal{L}^n. \tag{5.3.9}$$

Proof. We get from (5.3.1)

$$-\varepsilon\Delta(\overline{u}_\varepsilon - u_\varepsilon) + \frac{1}{\varepsilon}(\overline{u}_\varepsilon - u_\varepsilon) = \varepsilon\Delta u_\varepsilon.$$

By testing with $\eta^2(\overline{u}_\varepsilon - u_\varepsilon)$ we get

$$\begin{aligned}\int_\Omega \Big(\varepsilon\nabla(\overline{u}_\varepsilon - u_\varepsilon)\cdot\nabla[\eta^2(\overline{u}_\varepsilon - u_\varepsilon)] + \frac{1}{\varepsilon}\eta^2(\overline{u}_\varepsilon - u_\varepsilon)^2\Big)\,\mathrm{d}\mathcal{L}^n \\ = -\int_\Omega \varepsilon\nabla u_\varepsilon\cdot\nabla[\eta^2(\overline{u}_\varepsilon - u_\varepsilon)]\,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

Rearranging and applying the Young-inequality yield

$$\begin{aligned}&\int_\Omega\Big(\varepsilon\eta^2\left|\nabla(\overline{u}_\varepsilon - u_\varepsilon)\right|^2 + \frac{1}{\varepsilon}\eta^2(\overline{u}_\varepsilon - u_\varepsilon)^2\Big)\,\mathrm{d}\mathcal{L}^n \\ &\quad = -\int_\Omega\Big(2\sqrt{\frac{\varepsilon}{4}}\eta\nabla(\overline{u}_\varepsilon - u_\varepsilon)\cdot\sqrt{4\varepsilon}(\overline{u}_\varepsilon - u_\varepsilon)\nabla\eta + 2\sqrt{\frac{\varepsilon}{4}}\eta\nabla(\overline{u}_\varepsilon - u_\varepsilon)\cdot\sqrt{\varepsilon}\eta\nabla u_\varepsilon \\ &\qquad + 2\sqrt{\varepsilon}\eta\nabla u_\varepsilon\cdot\sqrt{\varepsilon}(\overline{u}_\varepsilon - u_\varepsilon)\nabla\eta\Big)\,\mathrm{d}\mathcal{L}^n \\ &\quad \le \int_\Omega\Big(\frac{\varepsilon}{2}\eta^2\left|\nabla(\overline{u}_\varepsilon - u_\varepsilon)\right|^2 + 5\varepsilon\left|\nabla\eta\right|^2(\overline{u}_\varepsilon - u_\varepsilon)^2 + 2\varepsilon\eta^2\left|\nabla u_\varepsilon\right|^2\Big)\,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

By absorbing the good term on the right-hand side we get (5.3.8). For the proof of (5.3.9) we deduce from (5.3.1) for $j \in \{1, \dots, n\}$

$$-\varepsilon^2\Delta\partial_j\overline{u}_\varepsilon + \partial_j\overline{u}_\varepsilon = \partial_j u_\varepsilon$$

and compute by testing with $\varepsilon\eta^2\partial_j\overline{u}_\varepsilon$

$$\int_\Omega\Big(\varepsilon^3\nabla\partial_j\overline{u}_\varepsilon\cdot\nabla[\eta^2\partial_j\overline{u}_\varepsilon] + \varepsilon\eta^2(\partial_j\overline{u}_\varepsilon)^2\Big)\,\mathrm{d}\mathcal{L}^n = \int_\Omega \varepsilon\eta^2\partial_j\overline{u}_\varepsilon\partial_j u_\varepsilon\,\mathrm{d}\mathcal{L}^n.$$

Rearranging and applying the Young-inequality yield

$$\begin{aligned}&\int_\Omega\Big(\varepsilon^3\eta^2\left|\nabla\partial_j\overline{u}_\varepsilon\right|^2 + \varepsilon\eta^2(\partial_j\overline{u}_\varepsilon)^2\Big)\,\mathrm{d}\mathcal{L}^n \\ &\qquad = \int_\Omega\Big(-2\sqrt{\frac{\varepsilon^3}{2}}\eta\nabla\partial_j\overline{u}_\varepsilon\cdot\sqrt{2\varepsilon^3}\partial_j\overline{u}_\varepsilon\nabla\eta + 2\sqrt{\frac{\varepsilon}{2}}\eta\partial_j\overline{u}_\varepsilon\sqrt{\frac{\varepsilon}{2}}\eta\partial_j u_\varepsilon\Big)\,\mathrm{d}\mathcal{L}^n \\ &\qquad \le \int_\Omega\Big(\frac{\varepsilon^3}{2}\eta^2\left|\nabla\partial_j\overline{u}_\varepsilon\right|^2 + 2\varepsilon^3(\partial_j\overline{u}_\varepsilon)^2\left|\nabla\eta\right|^2 + \frac{\varepsilon}{2}\eta^2(\partial_j\overline{u}_\varepsilon)^2 + \frac{\varepsilon}{2}\eta^2(\partial_j u_\varepsilon)^2\Big)\,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

Absorbing the good terms from the right-hand side proves (5.3.9). □

Corollary 5.3.4.
Let $u_\varepsilon \in C^3(\Omega)$ with (5.2.12) for some $\Lambda > 0$ and assume that $\overline{u}_\varepsilon \in H^1(\Omega)$ satisfies (5.3.1) and (5.3.7). For any open set $\Omega_0 \Subset \Omega$ there exists $C = C(\Omega, \Omega_0) > 0$ such that for all $\varepsilon > 0$ we have

$$\int_{\Omega_0} \left(\varepsilon |\nabla \overline{u}_\varepsilon|^2 + \frac{1}{\varepsilon}(\overline{u}_\varepsilon - u_\varepsilon)^2 + \frac{1}{\varepsilon}\overline{W}(\overline{u}_\varepsilon) \right) \mathrm{d}\mathcal{L}^n \le C(1+\Lambda), \tag{5.3.10}$$

where $\overline{W}(r) := \min\{(1-r)^2, (1+r)^2\}$ for $r \in \mathbb{R}$. Also we have for any $j \in \{1, \dots, n\}$

$$\int_{\Omega_0} \varepsilon(\partial_j \overline{u}_\varepsilon)^2 \,\mathrm{d}\mathcal{L}^n \le \int_{\Omega} \varepsilon(\partial_j u_\varepsilon)^2 \,\mathrm{d}\mathcal{L}^n + \varepsilon^2 C(1+\Lambda) \tag{5.3.11}$$

$$\textit{and} \quad \int_{\Omega_0} \left(\varepsilon^3 |D^2 \overline{u}_\varepsilon|^2 + \varepsilon |\nabla \overline{u}_\varepsilon|^2 \right) \mathrm{d}\mathcal{L}^n \le C. \tag{5.3.12}$$

Proof. We start by estimating for $r \in \mathbb{R}$ and $s > 0$

$$\begin{aligned}(r-1)^2 &= (r-s)^2 + 2(r-s)(s-1) + (s-1)^2 \\ &= (r-s)^2 + 2\frac{r-1}{\sqrt{2}}\sqrt{2}(s-1) - (s-1)^2 \le (r-s)^2 + \frac{1}{2}(r-1)^2 + (s-1)^2\end{aligned}$$

We can absorb the term $\frac{1}{2}(r-1)^2$ and get

$$\frac{1}{2}(r-1)^2 \le (r-s)^2 + (s-1)^2 \le (r-s)^2 + (s-1)^2(s+1)^2 = (r-s)^2 + W(s).$$

It follows

$$\overline{W}(r) \le (r-1)^2 \le 2(r-s)^2 + 2W(s).$$

We proceed similar for $s < 0$ and get for all $s, r \in \mathbb{R}$

$$\overline{W}(r) \le 2(r-s)^2 + 2W(s). \tag{5.3.13}$$

Given $\Omega_0 \Subset \Omega$ we can choose $\eta \in C_c^1(\Omega)$ such that $\eta = 1$ on Ω_0. Then we get from first applying and (5.3.13) followed by (5.3.8)

$$\begin{aligned}&\int_{\Omega_0} \left(\varepsilon |\nabla \overline{u}_\varepsilon|^2 + \frac{1}{\varepsilon}(\overline{u}_\varepsilon - u_\varepsilon)^2 + \frac{1}{\varepsilon}\overline{W}(\overline{u}_\varepsilon) \right) \mathrm{d}\mathcal{L}^n \\ &\qquad \le \int_{\Omega} \left(\varepsilon \eta^2 |\nabla(\overline{u}_\varepsilon - u_\varepsilon + u_\varepsilon)|^2 + \frac{3}{\varepsilon}\eta^2(\overline{u}_\varepsilon - u_\varepsilon)^2 + \frac{2}{\varepsilon}\eta^2 W(u_\varepsilon) \right) \mathrm{d}\mathcal{L}^n \\ &\qquad \le \int_{\Omega} \left(30\varepsilon |\nabla \eta|^2 (\overline{u}_\varepsilon - u_\varepsilon)^2 + 2\varepsilon\eta^2 |\nabla u_\varepsilon|^2 + 12\varepsilon\eta^2 |\nabla u_\varepsilon|^2 + \frac{2}{\varepsilon}\eta^2 W(u_\varepsilon) \right) \mathrm{d}\mathcal{L}^n \\ &\qquad \le \int_{\Omega} \left(60\varepsilon |\nabla \eta|^2 \overline{u}_\varepsilon^2 + 60\varepsilon |\nabla \eta|^2 u_\varepsilon^2 + 14\varepsilon\eta^2 |\nabla u_\varepsilon|^2 + \frac{2}{\varepsilon}\eta^2 W(u_\varepsilon) \right) \mathrm{d}\mathcal{L}^n \\ &\qquad \le C(\eta) \int_{\Omega} \varepsilon(\overline{u}_\varepsilon^2 + u_\varepsilon^2) \,\mathrm{d}\mathcal{L}^n + C(\eta)\mu_\varepsilon(\Omega).\end{aligned}$$

Note that η is only dependent on Ω_0, Ω. Applying (5.3.6), (5.3.7), and (5.2.12) yields (5.3.10).

To prove (5.3.11) we choose and open set Ω_1 such that $\Omega_0 \Subset \Omega_1 \Subset \Omega$. Then we apply (5.3.10) to Ω_1 and get

$$\int_{\Omega_1} \varepsilon\left|\nabla\overline{u}_\varepsilon\right|^2 \,\mathrm{d}\mathcal{L}^n \leq C(1+\Lambda).$$

We choose $\eta \in C^1(\Omega_1)$ such that $\eta = 1$ on Ω_1 and use (5.3.9). This yields

$$\begin{aligned}\int_{\Omega_0} \varepsilon(\partial_j\overline{u}_\varepsilon)^2 \,\mathrm{d}\mathcal{L}^n = \int_{\Omega_1} \varepsilon\eta^2(\partial_j\overline{u}_\varepsilon)^2 \,\mathrm{d}\mathcal{L}^n &\leq \int_{\Omega_1} \left(4\varepsilon^3(\partial_j\overline{u}_\varepsilon)^2\left|\nabla\eta\right|^2 + \varepsilon\eta^2(\partial_j u_\varepsilon)^2\right) \mathrm{d}\mathcal{L}^n \\ &\leq \int_{\Omega} \varepsilon\eta^2(\partial_j u_\varepsilon)^2 \,\mathrm{d}\mathcal{L}^n + \varepsilon^2 C(1+\Lambda).\end{aligned}$$

To prove (5.3.12) we use (5.3.9). First we choose $\eta \in C^1_c(\Omega)$ with $\eta = 1$ on Ω_0. then we get by summing over j in (5.3.9)

$$\begin{aligned}\int_{\Omega_0} \left(\varepsilon^3\left|D^2\overline{u}_\varepsilon\right|^2 + \varepsilon\left|\nabla\overline{u}_\varepsilon\right|^2\right) \mathrm{d}\mathcal{L}^n &\leq \int_{\Omega} \left(\varepsilon^3\eta^2\left|D^2\overline{u}_\varepsilon\right|^2 + \varepsilon\eta^2\left|\nabla\overline{u}_\varepsilon\right|^2\right) \mathrm{d}\mathcal{L}^n \\ &\leq \int_{\Omega} \left(4\varepsilon^3\left|\nabla\overline{u}_\varepsilon\right|^2\left|\nabla\eta\right|^2 + \varepsilon\eta^2\left|\nabla u_\varepsilon\right|^2\right) \mathrm{d}\mathcal{L}^n \\ &\leq C(\eta)\int_{\Omega} \left(\varepsilon^3\left|\nabla\overline{u}_\varepsilon\right|^2 + \varepsilon\left|\nabla u_\varepsilon\right|^2\right) \mathrm{d}\mathcal{L}^n\end{aligned}$$

By (5.3.10) and (5.2.12) we deduce (5.3.12). □

Corollary 5.3.5 (Convergence and absolute continuity)**.**
Let $u_\varepsilon \in C^3(\Omega)$ with (5.2.12) *for some $\Lambda > 0$ and assume that $\overline{u}_\varepsilon \in H^1(\Omega)$ satisfies* (5.3.1) *and* (5.3.7) *for all $\varepsilon > 0$. There exists a finite Radon measure $\vartheta \in C^0_c(\Omega)'$ such that up to a subsequence we have*

$$\vartheta_\varepsilon \xrightarrow{w^*} \vartheta \quad \textit{as} \quad \varepsilon \to 0 \quad \textit{in} \quad C^0_0(\Omega)'. \tag{5.3.14}$$

Moreover, $\vartheta \ll \mu$ holds.

Proof. Let $\eta \in C^0_0(\Omega)$, by Young's inequality we get

$$\langle\eta^2,\vartheta_\varepsilon\rangle_{C^0_0(\Omega)'} = \int_\Omega \eta^2 \,\mathrm{d}\vartheta_\varepsilon \leq \frac{1}{2}\int_\Omega \eta^2\varepsilon\left|\nabla u_\varepsilon\right|^2 \mathrm{d}\mathcal{L}^n + \frac{1}{2}\int_\Omega \eta^2\varepsilon\left|\nabla\overline{u}_\varepsilon\right|^2 \mathrm{d}\mathcal{L}^n \leq C(\eta,\Lambda)$$

by 5.3.4 and (5.2.12). From Theorem 2.2.2 we get the existence of a subsequence such that (5.3.14) holds. More precisely we get from (5.3.7) and (5.3.10) for $\eta \in C^1_c(\Omega)$

$$\begin{aligned}\langle\eta^2,\vartheta\rangle_{C^0_0(\Omega)'} \longleftarrow \langle\eta^2,\vartheta_\varepsilon\rangle_{C^0_0(\Omega)'} = \int_\Omega \eta^2 \,\mathrm{d}\vartheta_\varepsilon &\leq \frac{1}{2}\int_\Omega \eta^2\varepsilon\left|\nabla u_\varepsilon\right|^2 \mathrm{d}\mathcal{L}^n + \frac{1}{2}\int_\Omega \eta^2\varepsilon\left|\nabla\overline{u}_\varepsilon\right|^2 \mathrm{d}\mathcal{L}^n \\ &\leq 3\langle\eta^2,\mu_\varepsilon\rangle_{C^0_c(\Omega)'} + \int_\Omega \eta^2\varepsilon\left|\nabla(\overline{u}_\varepsilon - u_\varepsilon)\right|^2 \mathrm{d}\mathcal{L}^n \\ &\leq 7\langle\eta^2,\mu_\varepsilon\rangle_{C^0_c(\Omega)'} + \int_\Omega 10\varepsilon\left|\nabla\eta\right|^2(\overline{u}_\varepsilon - u_\varepsilon)^2 \,\mathrm{d}\mathcal{L}^n \\ &\leq 7\langle\eta^2,\mu_\varepsilon\rangle_{C^0_c(\Omega)'} + \varepsilon C(\eta)\int_\Omega (\overline{u}_\varepsilon^2 + u_\varepsilon^2) \,\mathrm{d}\mathcal{L}^n \\ &\leq 7\langle\eta^2,\mu_\varepsilon\rangle_{C^0_c(\Omega)'} + \varepsilon C(\eta,\Omega,\Lambda) \longrightarrow 7\langle\eta^2,\mu\rangle_{C^0_0(\Omega)'}.\end{aligned}$$

It follows $\vartheta \ll \mu$ by Lemma 8.2.6. □

Lemma 5.3.6.
Let $u_\varepsilon \in C^3(\Omega)$ with (5.2.12) for some $\Lambda > 0$ and assume that $\overline{u}_\varepsilon \in H^1(\Omega)$ satisfies (5.3.1) and (5.3.7). Then for any $\eta \in C_c^1(\Omega)$ we have for all $\varepsilon > 0$

$$\int\limits_{\{|\overline{u}_\varepsilon|>1\}} \Big(\varepsilon\eta^2\big|\nabla(|\overline{u}_\varepsilon|-1)\big|^2 + \frac{1}{\varepsilon}\eta^2(|\overline{u}_\varepsilon|-1)^2\Big)\,\mathrm{d}\mathcal{L}^n \tag{5.3.15}$$
$$\leq \int\limits_{\{|u_\varepsilon|>1\}} \frac{1}{\varepsilon}\eta^2 W'(u_\varepsilon)^2\,\mathrm{d}\mathcal{L}^n + \int\limits_{\{|\overline{u}_\varepsilon|>1\}} 4\varepsilon|\nabla\eta|^2(|\overline{u}_\varepsilon|-1)^2\,\mathrm{d}\mathcal{L}^n.$$

Furthermore for any $\Omega_0 \Subset \Omega$ and $k \in \mathbb{N}$ there exist $C(\Omega_0,\Omega,k) > 0$ and $\varepsilon_0(k,\Omega_0,\Omega)$ such that for all $\varepsilon \in (0,\varepsilon_0)$ we have

$$\int_{\Omega_0} \frac{1}{\varepsilon}(|\overline{u}_\varepsilon|-1)_+^2\,\mathrm{d}\mathcal{L}^n \leq C(\varepsilon^2\alpha_\varepsilon(\Omega) + \varepsilon^{2k-1}). \tag{5.3.16}$$

Proof. We start by proving a localized version of Lemma 5.3.2. Let $\eta \in C_c^1(\Omega)$, then we get by testing (5.3.5) with $\eta^2(\overline{u}_\varepsilon - 1)_+$

$$\int_\Omega \Big(\varepsilon\nabla(\overline{u}_\varepsilon-1)\cdot\nabla[\eta^2(\overline{u}_\varepsilon-1)_+] + \frac{1}{\varepsilon}\eta^2(\overline{u}_\varepsilon-1)(\overline{u}_\varepsilon-1)_+\Big)\,\mathrm{d}\mathcal{L}^n$$
$$= \int_\Omega \frac{1}{\varepsilon}\eta^2(u_\varepsilon-1)(\overline{u}_\varepsilon-1)_+\,\mathrm{d}\mathcal{L}^n.$$

Rearranging and applying a Young inequality yield

$$\begin{aligned}
&\int_\Omega \Big(\varepsilon\eta^2\big|\nabla(\overline{u}_\varepsilon-1)_+\big|^2 + \frac{1}{\varepsilon}\eta^2(\overline{u}_\varepsilon-1)_+^2\Big)\,\mathrm{d}\mathcal{L}^n\\
&\quad\leq \int_\Omega 2\frac{1}{\sqrt{2\varepsilon}}\eta(\overline{u}_\varepsilon-1)_+\frac{1}{\sqrt{2\varepsilon}}\eta(u_\varepsilon-1)_+\,\mathrm{d}\mathcal{L}^n\\
&\qquad-\int_\Omega 2\sqrt{\frac{\varepsilon}{2}}\eta\nabla(\overline{u}_\varepsilon-1)_+\cdot\sqrt{2\varepsilon}(\overline{u}_\varepsilon-1)_+\nabla\eta\,\mathrm{d}\mathcal{L}^n\\
&\quad\leq \int_\Omega \frac{1}{2\varepsilon}\eta^2(\overline{u}_\varepsilon-1)_+^2\,\mathrm{d}\mathcal{L}^n + \int_\Omega \frac{1}{2\varepsilon}\eta^2(u_\varepsilon-1)_+^2\,\mathrm{d}\mathcal{L}^n\\
&\qquad+\int_\Omega \frac{\varepsilon}{2}\eta^2\big|\nabla(\overline{u}_\varepsilon-1)_+\big|^2\,\mathrm{d}\mathcal{L}^n + \int_\Omega 2\varepsilon|\nabla\eta|^2(\overline{u}_\varepsilon-1)_+^2\,\mathrm{d}\mathcal{L}^n.
\end{aligned}$$

We can absorb the good terms and get

$$\int_\Omega \Big(\varepsilon\eta^2\big|\nabla(\overline{u}_\varepsilon-1)_+\big|^2 + \frac{1}{\varepsilon}\eta^2(\overline{u}_\varepsilon-1)_+^2\Big)\,\mathrm{d}\mathcal{L}^n$$
$$\leq \int_\Omega \frac{1}{\varepsilon}\eta^2(u_\varepsilon-1)_+^2\,\mathrm{d}\mathcal{L}^n + \int_\Omega 4\varepsilon|\nabla\eta|^2(\overline{u}_\varepsilon-1)_+^2\,\mathrm{d}\mathcal{L}^n.$$

Estimating $(u_\varepsilon-1)^2 \leq 16u_\varepsilon^2(u_\varepsilon-1)^2(u_\varepsilon+1)^2 = W'(u_\varepsilon)^2$ on $\{u_\varepsilon > 1\}$ yields

$$\int\limits_{\{\overline{u}_\varepsilon>1\}} \Big(\varepsilon\eta^2\big|\nabla(\overline{u}_\varepsilon-1)\big|^2 + \frac{1}{\varepsilon}\eta^2(\overline{u}_\varepsilon-1)^2\Big)\,\mathrm{d}\mathcal{L}^n$$
$$\leq \int\limits_{\{u_\varepsilon>1\}} \frac{1}{\varepsilon}\eta^2 W'(u_\varepsilon)^2\,\mathrm{d}\mathcal{L}^n + \int\limits_{\{\overline{u}_\varepsilon>1\}} 4\varepsilon|\nabla\eta|^2(\overline{u}_\varepsilon-1)^2\,\mathrm{d}\mathcal{L}^n.$$

We proceed similar on $\{\overline{u}_\varepsilon < -1\}$ and add the integrals to get (5.3.15).

For any two sets $V_1 \Subset V_2 \subseteq \Omega$ we derive by choosing a cut-off function $\eta \in C^1_c(V_2)$ with $\eta = 1$ on V_1 and $|\nabla\eta| \le \frac{2}{r}$ for $r := \operatorname{dist}(V_1, \mathbb{R}^n \setminus V_2)$ that

$$\int_{V_1} \frac{1}{\varepsilon}\eta^2(|\overline{u}_\varepsilon| - 1)^2_+ \,\mathrm{d}\mathcal{L}^n \le \int_{V_2} \frac{1}{\varepsilon}(|u_\varepsilon| - 1)^2_+ \,\mathrm{d}\mathcal{L}^n + \frac{16\varepsilon^2}{r^2}\int_{V_2} \frac{1}{\varepsilon}(|\overline{u}_\varepsilon| - 1)^2_+ \,\mathrm{d}\mathcal{L}^n \tag{5.3.17}$$

Now we choose open sets $\Omega_1, \dots, \Omega_k$ such that

$$\Omega_0 \Subset \Omega_1 \Subset \dots \Subset \Omega_k \Subset \Omega_{k+1} := \Omega$$

with $\operatorname{dist}(\Omega_j, \mathbb{R}^n \setminus \Omega_{j+1}) \le \frac{C}{k}\operatorname{dist}(\Omega_0, \mathbb{R}^n \setminus \Omega)$ for all $j \in \{0, 1, \dots, k\}$. Using iteratively (5.3.17) and (5.3.7) yields

$$\begin{aligned}\int_{\Omega_0} \frac{1}{\varepsilon}(|\overline{u}_\varepsilon| - 1)^2_+ \,\mathrm{d}\mathcal{L}^n &\le k\int_{\Omega_k} \frac{1}{\varepsilon}(|u_\varepsilon| - 1)^2_+ \,\mathrm{d}\mathcal{L}^n + \frac{16\varepsilon^{2k}}{r^{2k}}\int_{\Omega} \frac{1}{\varepsilon}(|\overline{u}_\varepsilon| - 1)^2_+ \,\mathrm{d}\mathcal{L}^n \\ &\le k\int_{\Omega_k} \frac{1}{\varepsilon}(|u_\varepsilon| - 1)^2_+ \,\mathrm{d}\mathcal{L}^n + C(k, \Omega_0, \Omega)\varepsilon^{2k-1}. \end{aligned} \tag{5.3.18}$$

Since $(1 - |r|)^2_+ \le CW'(r)^2\chi_{\{|r|\ge1\}}$ we can apply Propositions 3.5 and 3.6 from [RS06] which implies that for all $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned}\int_{\Omega_k} \frac{1}{\varepsilon}(|u_\varepsilon| - 1)^2_+ \,\mathrm{d}\mathcal{L}^n &\le C(k, \Omega_0, \Omega)\varepsilon\int_{\Omega} |H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n + C(k, \Omega_0, \Omega)\varepsilon^{2k}\int_{\Omega_k} \frac{1}{\varepsilon}W'(u_\varepsilon)^2 \,\mathrm{d}\mathcal{L}^n \\ &\le C(k, \Omega_0, \Omega)\varepsilon\int_{\Omega} |H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n + C'(k, \Omega_0, \Omega)\varepsilon^{2k-1}.\end{aligned}$$

Together with (5.3.18) this yields (5.3.16). □

5.4 Characterization of ϑ

To identify ϑ we apply the blow-up method as in [RS06], see also [HT00].

Theorem 5.4.1.
Consider $(u_\varepsilon)_{\varepsilon>0}$ in $C^3(\Omega)$ and assume that (5.2.12) *holds for some $\Lambda > 0$ and restrict to a subsequence such that* (5.2.8)-(5.2.11) *and* (5.3.14) *hold. Then it holds*

$$\vartheta = \frac{1}{\sigma}\mu.$$

The proof is done throughout the entire section. First we introduce the notations for the proof and reduce the claims without loss of generality. Recall $\Gamma = \operatorname{supp}(\mu)$ as introduced in Theorem 5.2.3.

Lemma 5.4.2 (Good points)**.**
Let $(u_\varepsilon)_{\varepsilon>0}$ as in Definition 5.2.1, assume (5.2.8)-(5.2.12) *and* (5.3.14)*. Then we have for μ-a.e. $x \in \Omega$*

- $B_{2\rho_0}(x) \Subset \Omega$ *for some* $\rho_0 = \rho_0(x) > 0$,
- x *is a* μ*-Lebesgue point of* $D_\mu \vartheta$,
- $\limsup_{\rho \to 0} \rho^{1-n} \mu(B_\rho(x)) < \infty$,
- $\kappa(\{x\}) = 0$,
- *the approximate tangent space* $T_x\Gamma$ *exists,*
- *there exist* $\theta(x) \in \mathbb{N}$ *and* $S_x \in G(n, n-1)$ *such that* $T_x\mu = c_0\theta(x)S_x$.

Proof. For x we can find a ρ_0 as described because Ω is open. We know from Corollary 5.3.5 and the Radon-Nikodym Theorem 8.2.5 $D_\mu\vartheta \in L^1(\Omega, \mu)$ and $\vartheta = D_\mu\vartheta\mu$. In particular μ-a.e. $x \in \Omega$ is a μ-Lebesgue point of $D_\mu\vartheta$ by Theorem 8.3.5. Furthermore, by (5.2.12) we get $\limsup_{n\to\infty} \rho^{1-n}\mu(B_\rho(x)) < \infty$ from (v) in Theorem 5.2.3.

The fourth condition is true for a cocountable subset of Ω because κ is a finite Radon measure on Ω It follows that κ can at most have a countable set of atoms.

The fifth condition is satisfied by μ-a.e. $x \in \Omega$ because by (5.2.12) and Theorem 5.2.3 V is a rectifiable $(n-1)$-varifold and $\mu = c_0\theta\mathcal{H}^{n-1} \llcorner \Gamma$. The last point stems from the fact that $\frac{1}{c_0}V$ is integral, see Theorem 5.2.3, which implies that for μ-a.e. $x \in \Gamma$ the multiplicity $\theta(x)$ is a natural number. $\square$

Recall $\zeta_{x,\rho}(y) = \frac{y-x}{\rho}$ for $\rho > 0$ and $y \in \mathbb{R}^n$, the pullback $\zeta^{\#}_{\rho,x}$ and the pushforward $\zeta_{\rho,x,\#}$ from Definition 2.2.18. In the following we fix a good point $x \in \operatorname{supp}(\mu)$ and $\rho_0 > 0$ such that the properties in Lemma 5.4.2 hold. Set $\theta := \theta(x)$. Without loss of generality we can assume $x = 0$ and $S := S_0 = \mathbb{R}^{n-1} \times \{0\}$ for the proof of Theorem 5.4.1. In fact this is possible because we consider $\zeta_{x,\rho,\#}\mu$ in the following and ζ shifts x to 0 anyways, the assumption $x = 0$ simply translates into the expression $\zeta_{\rho,0,\#}\mu$ instead of $\zeta_{\rho,x,\#}\mu$. We write $\zeta_{\rho,\#}\mu := \zeta_{\rho,0,\#}\mu$. We get $S = \mathbb{R}^{n-1} \times \{0\}$ with an orthogonal coordinate transformation in the integrals where S appears.

Lemma 5.4.3.
Let $(u_\varepsilon)_{\varepsilon>0}$ as in Definition 5.2.1, assume (5.2.8)-(5.2.12) *and* (5.3.14)*. Then there exist sequences $(\rho_j)_{j\in\mathbb{N}}$ and $(\varepsilon_j)_{j\in\mathbb{N}}$ with $0 < \rho_j < \rho_0$ for all $j \in \mathbb{N}$ such that as $j \to \infty$ we have*

$$\varepsilon_j \to 0, \quad \rho_j \to 0, \tag{5.4.1}$$

$$\frac{\varepsilon_j}{\rho_j} \to 0, \quad \frac{\varepsilon_j^2}{\rho_j^{n+1}} \to 0, \tag{5.4.2}$$

$$\rho_j^{1-n}\zeta_{\rho_j,\#}\mu_{\varepsilon_j} \xrightarrow{w^*} c_0\theta\mathcal{H}^{n-1} \llcorner S \quad \textit{in} \quad C_c^0(B_{16}(0))', \tag{5.4.3}$$

$$\textit{and} \quad \rho_j^{1-n}\zeta_{\rho_j,\#}\vartheta_{\varepsilon_j} \xrightarrow{w^*} D_\mu\vartheta(0)c_0\theta\mathcal{H}^{n-1} \llcorner S \quad \textit{in} \quad C_c^0(B_{16}(0))', \tag{5.4.4}$$

$$\textit{and for all } j \in \mathbb{N} \quad \kappa_{\varepsilon_j}(B_\rho(0)) \le \kappa(B_{2\rho}(0)) + \rho_j^{n-2} \quad \textit{for} \quad \rho_j \le \rho \le \rho_0. \tag{5.4.5}$$

Proof. Let $(\rho_j)_{j\in\mathbb{N}}$ be a decreasing sequence with $\rho_1 < \rho_0$ and $\rho_j \to 0$ as $j \to \infty$. By the definition of the approximate tangent space we have

$$\rho_j^{1-n}\zeta_{\rho_j,\#}\mu \xrightarrow{w^*} c_0\theta\mathcal{H}^{n-1} \llcorner S \quad \text{in} \quad C_c^0(B_{16}(0))'.$$

Since 0 is a μ-Lebesgue point of $D_\mu\vartheta$ we get by Lemma 8.2.7 that

$$\rho_j^{1-n}\zeta_{\rho_j,\#}\vartheta = \rho_j^{1-n}\zeta_{\rho_j,\#}D_\mu\vartheta\mu \xrightarrow{w^*} D_\mu\vartheta(0)c_0\theta\mathcal{H}^{n-1} \llcorner S \quad \text{in} \quad C_c^0(B_{16}(0))'.$$

Using that the weak*-topology on bounded subsets of $C_c^0(B_{16}(0))'$ is metrizable, (5.2.8), (5.3.14), and

$$\begin{aligned}
\rho_j^{1-n}\zeta_{\rho_j,\#}\mu_\varepsilon &= \rho_j^{1-n}\zeta_{\rho_j,\#}(\mu_{\varepsilon_j} - \mu) + \rho_j^{1-n}\zeta_{\rho_j,\#}\mu \\
\rho_j^{1-n}\zeta_{\rho_j,\#}\vartheta_\varepsilon &= \rho_j^{1-n}\zeta_{\rho_j,\#}(\vartheta_\varepsilon - \vartheta) + \rho_j^{1-n}\zeta_{\rho_j,\#}\vartheta
\end{aligned}$$

we can choose a subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$ dependent on $(\rho_j)_{j\in\mathbb{N}}$ such that (5.4.1)-(5.4.4) hold. Finally by possibly lowering the value of ε_j, we obtain for all $l \in \mathbb{N}_0$ with $2^{-l}\rho_0 > \rho_j$

$$\kappa_{\varepsilon_j}(B_{2^{-l}\rho_0}(0)) \le \kappa(\overline{B_{2^{-l}\rho_0}(0)}) + \rho_j^{n-2} \le \kappa(B_{2^{-l+1}\rho_0}(0)) + \rho_j^{n-2}.$$

We deduce for any $\rho_j \le \rho \le \rho_0$ and $l \in \mathbb{N}_0$ such that $\rho \in (2^{-l-1}\rho_0, 2^{-l}\rho_0)$

$$\kappa_{\varepsilon_j}(B_\rho(0)) \le \kappa_{\varepsilon_j}(B_{2^{-l}\rho_0}(0)) \le \kappa(\overline{B_{2^{-l}\rho_0}(0)}) + \rho_j^{n-2} \le \kappa(B_{2\rho_0}(0)) + \rho_j^{n-2}.$$

Thus (5.4.5) holds as well. □

Proposition 5.4.4 (Properties of the rescaled functions and measures).
Let $(u_\varepsilon)_{\varepsilon>0}$ as in Definition 5.2.1, assume (5.2.8)-(5.2.12) *and* (5.3.14). *We set $\tilde{\varepsilon}_j := \frac{\varepsilon_j}{\rho_j}$ and define for $x \in B_{\frac{\rho_0}{\rho_j}}(0)$*

$$\begin{aligned}
\tilde{u}_{\tilde{\varepsilon}_j}(x) &:= u_{\varepsilon_j}(\rho_j x), \qquad \hat{u}_{\tilde{\varepsilon}_j}(x) := \overline{u}_{\varepsilon_j}(\rho_j x), \quad \tilde{H}_{\tilde{\varepsilon}_j}(x) := \rho_j H_{\varepsilon_j}(\rho_j x), \\
\tilde{\nu}_{\tilde{\varepsilon}_j}(x) &:= \frac{\nabla\tilde{u}_{\tilde{\varepsilon}_j}(x)}{|\nabla\tilde{u}_{\tilde{\varepsilon}_j}(x)|} \quad \textit{for} \quad \nabla\tilde{u}_{\tilde{\varepsilon}_j}(x) \neq 0, \quad \textit{and} \quad \tilde{\nu}_{\tilde{\varepsilon}_j}(x) := e_1 \quad \textit{else.}
\end{aligned}$$

Moreover we set

$$\tilde{\mu}_{\tilde{\varepsilon}_j} := \Big(\frac{\tilde{\varepsilon}_j}{2}|\nabla\tilde{u}_{\tilde{\varepsilon}_j}|^2 + \frac{1}{\tilde{\varepsilon}_j}W(\tilde{u}_{\tilde{\varepsilon}_j})\Big)\mathcal{L}^n \llcorner B_{\frac{\rho_0}{\rho_j}}(0), \tag{5.4.6}$$

$$\tilde{\xi}_{\tilde{\varepsilon}_j} := \Big(\frac{\tilde{\varepsilon}_j}{2}|\nabla\tilde{u}_{\tilde{\varepsilon}_j}|^2 - \frac{1}{\tilde{\varepsilon}_j}W(\tilde{u}_{\tilde{\varepsilon}_j})\Big)\mathcal{L}^n \llcorner B_{\frac{\rho_0}{\rho_j}}(0), \tag{5.4.7}$$

$$\tilde{\alpha}_{\tilde{\varepsilon}_j} := \frac{1}{\tilde{\varepsilon}_j}\tilde{H}_{\tilde{\varepsilon}_j}^2\mathcal{L}^n \llcorner B_{\frac{\rho_0}{\rho_j}}(0), \tag{5.4.8}$$

$$\tilde{\vartheta}_{\tilde{\varepsilon}_j} := \tilde{\varepsilon}_j|\nabla\tilde{u}_{\tilde{\varepsilon}_j}||\nabla\hat{u}_{\tilde{\varepsilon}_j}|\mathcal{L}^n \llcorner B_{\frac{\rho_0}{\rho_j}}(0), \tag{5.4.9}$$

$$\tilde{\kappa}_{\tilde{\varepsilon}_j} := \Big(\frac{1}{\tilde{\varepsilon}_j}|\tilde{H}_{\tilde{\varepsilon}_j}|^2 + \tilde{\varepsilon}_j|\nabla\tilde{H}_{\tilde{\varepsilon}_j}|^2\Big)\mathcal{L}^n \llcorner B_{\frac{\rho_0}{\rho_j}}(0), \tag{5.4.10}$$

$$\textit{and} \quad \tilde{V}_{\tilde{\varepsilon}_j} := \tilde{\mu}_{\tilde{\varepsilon}_j} \otimes \tilde{\nu}_{\tilde{\varepsilon}_j}^\perp \in \mathbb{V}_{n-1}\Big(B_{\frac{\rho_0}{\rho_j}}(0)\Big). \tag{5.4.11}$$

Then it holds

$$\tilde{\varepsilon}_j \to 0 \quad as \quad j \to \infty, \tag{5.4.12}$$

$$-\tilde{\varepsilon}_j \Delta \tilde{u}_{\tilde{\varepsilon}_j} + \frac{1}{\tilde{\varepsilon}_j} W'(\tilde{u}_{\tilde{\varepsilon}_j}) = \tilde{H}_{\tilde{\varepsilon}_j} \quad in \quad B_{\frac{\rho_0}{\rho_j}}(0), \tag{5.4.13}$$

$$-\tilde{\varepsilon}_j^2 \Delta \hat{u}_{\tilde{\varepsilon}_j} + \hat{u}_{\tilde{\varepsilon}_j} = \tilde{u}_{\tilde{\varepsilon}_j} \quad in \quad B_{\frac{\rho_0}{\rho_j}}(0), \tag{5.4.14}$$

and with $j \to \infty$ we have

$$\rho_j^{1-n} \zeta_{\rho_j,\#} \mu_{\varepsilon_j} = \tilde{\mu}_{\tilde{\varepsilon}_j} \xrightarrow{w^*} c_0 \theta \mathcal{H}^{n-1} \llcorner S, \tag{5.4.15}$$

$$\rho_j^{1-n} \zeta_{\rho_j,\#} \vartheta_{\varepsilon_j} = \tilde{\vartheta}_{\tilde{\varepsilon}_j} \xrightarrow{w^*} c_0 \theta D_\mu \vartheta(0) \mathcal{H}^{n-1} \llcorner S, \tag{5.4.16}$$

$$\tilde{\alpha}_{\tilde{\varepsilon}_j} \xrightarrow{w^*} 0, \quad and \quad \tilde{\kappa}_{\tilde{\varepsilon}_j} \xrightarrow{w^*} 0 \tag{5.4.17}$$

in $C_c^0(B_{16}(0))'$. Furthermore there exist $\tilde{V} \in \mathbb{V}_{n-1}(B_{15}(0))$ such that up to a subsequence we have as $j \to \infty$

$$\tilde{V}_{\tilde{\varepsilon}_j} \xrightarrow{w^*} \tilde{V} \quad in \quad \mathbb{V}_{n-1}(B_{15}(0)). \tag{5.4.18}$$

Finally,

$$\int_{B_8(0)} \Big(\tilde{\varepsilon}_j |\nabla \hat{u}_{\tilde{\varepsilon}_j}|^2 + \frac{1}{\tilde{\varepsilon}_j} (\tilde{u}_{\tilde{\varepsilon}_j} - \hat{u}_{\tilde{\varepsilon}_j})^2 \Big) \, \mathrm{d}\mathcal{L}^n \le C(\Omega, \Lambda). \tag{5.4.19}$$

Proof. (5.4.12) follows directly from (5.4.1). For (5.4.13) we calculate

$$-\tilde{\varepsilon}_j \Delta \tilde{u}_{\tilde{\varepsilon}_j} + \frac{1}{\tilde{\varepsilon}_j} W'(\tilde{u}_{\tilde{\varepsilon}_j}) = -\frac{\varepsilon_j}{\rho_j} \rho_j^2 \Delta u_{\varepsilon_j}(\rho_j \cdot) + \frac{\rho_j}{\varepsilon_j} W'(u_{\varepsilon_j}(\rho_j \cdot)) = \rho_j H_{\varepsilon_j}(\rho_j \cdot) = \tilde{H}_{\tilde{\varepsilon}_j}.$$

We get (5.4.14) from

$$-\tilde{\varepsilon}_j^2 \Delta \hat{u}_{\tilde{\varepsilon}_j} + \hat{u}_{\tilde{\varepsilon}_j} = -\varepsilon_j^2 \Delta \overline{u}_{\varepsilon_j}(\rho_j \cdot) + \overline{u}_{\varepsilon_j}(\rho_j \cdot) = u_{\varepsilon_j}(\rho_j \cdot) = \tilde{u}_{\tilde{\varepsilon}_j}.$$

(5.4.15) follows from (5.4.3) and the following calculation. Let $j_0 \in \mathbb{N}$ such that $16 < \frac{\rho_0}{\rho_j}$ for all $j \ge j_0$ and let $\eta \in C_c^0(B_{16}(0))$, then we have

$$\begin{aligned}
\rho_j^{1-n} \langle \eta, \zeta_{\rho_j,\#} \mu_{\varepsilon_j} \rangle_{C_c^0(B_{16}(0))'} &= \rho_j^{1-n} \langle \zeta_{\rho_j}^{\#} \eta, \mu_{\varepsilon_j} \rangle_{C_c^0(\Omega)'} = \rho_j^{1-n} \int_\Omega \eta \circ \zeta_{\rho_j} \, \mathrm{d}\mu_{\varepsilon_j} \\
&= \int_{B_{16\rho_j}(0)} \rho_j^{1-n} \eta\Big(\frac{x}{\rho_j}\Big) \Big(\frac{\varepsilon_j}{2} |\nabla u_{\varepsilon_j}(x)|^2 + \frac{1}{\varepsilon_j} W(u_{\varepsilon_j}(x)) \Big) \, \mathrm{d}x \\
&= \int_{B_{16}(0)} \rho_j \eta(x) \Big(\frac{\varepsilon_j}{2} |\nabla u_{\varepsilon_j}(\rho_j x)|^2 + \frac{1}{\varepsilon_j} W(u_{\varepsilon_j}(\rho_j x)) \Big) \, \mathrm{d}x \\
&= \int_{B_{16}(0)} \eta(x) \Big(\frac{\varepsilon_j / \rho_j}{2} |\rho_j \nabla u_{\varepsilon_j}(\rho_j x)|^2 + \frac{1}{\varepsilon_j / \rho_j} W(u_{\varepsilon_j}(\rho_j x)) \Big) \, \mathrm{d}x \\
&= \int_{B_{16}(0)} \eta \Big(\frac{\tilde{\varepsilon}_j}{2} |\nabla \tilde{u}_{\tilde{\varepsilon}_j}|^2 + \frac{1}{\tilde{\varepsilon}_j} W(\tilde{u}_{\tilde{\varepsilon}_j}) \Big) \, \mathrm{d}\mathcal{L}^n = \langle \eta, \tilde{\mu}_{\tilde{\varepsilon}_j} \rangle_{C_c^0(B_{16}(0))'}.
\end{aligned}$$

For (5.4.16) we use (5.4.4) and calculate for any $\eta \in C_c^0(B_{16}(0))$

$$\begin{aligned}
\rho_j^{1-n}\langle \eta, \zeta_{\rho_j,\#}\vartheta_{\varepsilon_j}\rangle_{C_c^0(B_{16}(0))'} &= \rho_j^{1-n}\langle \zeta_{\rho_j}^{\#}\eta, \vartheta_{\varepsilon_j}\rangle_{C_c^0(\Omega)'} = \rho_j^{1-n}\int\limits_{\Omega} \eta\circ\zeta_{\rho_j}\,\mathrm{d}\vartheta_{\varepsilon_j}\\
&= \int\limits_{B_{16\rho_j}(0)} \rho_j^{1-n}\eta\Big(\frac{x}{\rho_j}\Big)\varepsilon_j|\nabla u_{\varepsilon_j}(x)||\nabla \overline{u}_{\varepsilon_j}(x)|\,\mathrm{d}x\\
&= \int\limits_{B_{16}(0)} \rho_j\eta(x)\varepsilon_j|\nabla u_{\varepsilon_j}(\rho_j x)||\nabla \overline{u}_{\varepsilon_j}(\rho_j x)|\,\mathrm{d}x\\
&= \int\limits_{B_{16}(0)} \eta(x)\frac{\varepsilon_j}{\rho_j}|\rho_j\nabla u_{\varepsilon_j}(\rho_j x)||\rho_j\nabla \overline{u}_{\varepsilon_j}(\rho_j x)|\,\mathrm{d}x\\
&= \int\limits_{B_{16}(0)} \eta\tilde{\varepsilon}_j|\nabla\tilde{u}_{\tilde{\varepsilon}_j}||\nabla\hat{u}_{\tilde{\varepsilon}_j}|\,\mathrm{d}\mathcal{L}^n = \langle\eta,\tilde{\vartheta}_{\tilde{\varepsilon}_j}\rangle_{C_c^0(B_{16}(0))'}.
\end{aligned}$$

For the proof of (5.4.17) we choose $j_1 \in \mathbb{N}$ such that $32 < \frac{\rho_0}{\rho_j}$ for all $j \geq j_1$. Then we have

$$\begin{aligned}
\tilde{\kappa}_{\tilde{\varepsilon}_j}(B_{16}(0)) &= \int_{B_{16}(0)} \Big(\frac{\rho_j}{\varepsilon_j}\big|\rho_j H_{\varepsilon_j}(\rho_j\cdot)\big|^2 + \frac{\varepsilon_j}{\rho_j}\big|\rho_j^2\nabla H_{\varepsilon_j}(\rho_j\cdot)\big|^2\Big)\,\mathrm{d}\mathcal{L}^n\\
&= \rho_j^{3-n}\int_{B_{16\rho_j}(0)} \Big(\frac{1}{\varepsilon_j}\big|H_{\varepsilon_j}\big|^2 + \varepsilon_j\big|\nabla H_{\varepsilon_j}\big|^2\Big)\,\mathrm{d}\mathcal{L}^n\\
&= \rho_j^{3-n}\kappa_{\varepsilon_j}(B_{16\rho_j}(0)) \leq \rho_j^{3-n}\big(\kappa(B_{32\rho_j}(0) + \rho_j^{n-2}\big).
\end{aligned}$$

In the last step comes from (5.4.5). Since $\kappa(\{0\}) = 0$, $n \leq 3$, and $\tilde{\alpha}_{\tilde{\varepsilon}_j} \leq \tilde{\kappa}_{\tilde{\varepsilon}_j}$ the claim follows.

(5.4.18) follows from

$$\|\tilde{V}_{\tilde{\varepsilon}_j}\|(B_{15}(0)) = \tilde{\mu}_{\tilde{\varepsilon}_j}(B_{15}(0)) \leq C \quad \text{for all } j \in \mathbb{N}$$

because $(\tilde{\mu}_{\tilde{\varepsilon}_j})_{j\in\mathbb{N}}$ is weakly*-convergent in $C_c^0(B_{16}(0))'$ and Theorem 2.2.2.

At last we prove (5.4.19). We will reverse the previous coordinate transformation to get back from $\tilde{u}_{\tilde{\varepsilon}_j}$ to u_{ε_j} and apply the estimates from Lemma 5.3.3. We choose $j \geq j_0$ and a test function $\eta \in C_c^1(B_{16}(0))$ such that $0 \leq \eta \leq 1$ and $\eta = 1$ on $B_8(0)$. We calculate with the coordinate transformation $\rho_j x \mapsto x$ and (5.3.8)

$$\begin{aligned}
&\int\limits_{B_8(0)} \Big(\tilde{\varepsilon}_j\big|\nabla\hat{u}_{\tilde{\varepsilon}_j}\big|^2 + \frac{1}{\tilde{\varepsilon}_j}(\tilde{u}_{\tilde{\varepsilon}_j} - \hat{u}_{\tilde{\varepsilon}_j})^2\Big)\,\mathrm{d}\mathcal{L}^n\\
&\qquad\leq \int\limits_{B_{16}(0)} \eta^2\Big(\tilde{\varepsilon}_j\big|\nabla\hat{u}_{\tilde{\varepsilon}_j}\big|^2 + \frac{1}{\tilde{\varepsilon}_j}(\tilde{u}_{\tilde{\varepsilon}_j} - \hat{u}_{\tilde{\varepsilon}_j})^2\Big)\,\mathrm{d}\mathcal{L}^n\\
&\qquad= \int\limits_{B_{16\rho_j}(0)} \rho_j^{1-n}\eta^2\Big(\frac{x}{\rho_j}\Big)\Big(\varepsilon_j\big|\nabla\overline{u}_{\varepsilon_j}(x)\big|^2 + \frac{1}{\varepsilon_j}(u_{\varepsilon_j}(x) - \overline{u}_{\varepsilon_j}(x))^2\Big)\,\mathrm{d}x\\
&\qquad\leq \int\limits_{B_{16\rho_j}(0)} 10\rho_j^{1-n}\Big(\eta^2\Big(\frac{x}{\rho_j}\Big)\varepsilon_j\big|\nabla u_{\varepsilon_j}(x)\big|^2 + \frac{\varepsilon_j}{\rho_j^2}\Big|\nabla\eta\Big(\frac{x}{\rho_j}\Big)\Big|^2(u_{\varepsilon_j} - \overline{u}_{\varepsilon})^2\Big)\,\mathrm{d}x.
\end{aligned}$$

We apply (5.3.10) from Corollary 5.3.4 and transform $x \mapsto \rho_j x$ back and use (5.4.2). We get

$$\int\limits_{B_8(0)} \Big(\tilde{\varepsilon}_j |\nabla \hat{u}_{\tilde{\varepsilon}_j}|^2 + \frac{1}{\tilde{\varepsilon}_j}(\tilde{u}_{\tilde{\varepsilon}_j} - \hat{u}_{\tilde{\varepsilon}_j})^2\Big)\, \mathrm{d}\mathcal{L}^n$$
$$\leq \int\limits_{B_{16\rho_j}(0)} 20\rho_j^{1-n}\eta^2\Big(\frac{x}{\rho_j}\Big)\, \mathrm{d}\mu_\varepsilon + \frac{\varepsilon_j^2}{\rho_j^{n+1}} C(\eta) \int\limits_{B_{\rho_0}(0)} \frac{1}{\varepsilon_j}(u_{\varepsilon_j} - \overline{u}_\varepsilon)^2\, \mathrm{d}\mathcal{L}^n$$
$$\leq \int\limits_{B_{16}(0)} 20\eta^2\, \mathrm{d}\tilde{\mu}_{\tilde{\varepsilon}_j} + \frac{\varepsilon_j^2}{\rho_j^{n+1}} C(\eta, \Omega, \Lambda, \rho_0) \longrightarrow 20\langle \eta^2, c_0\theta\mathcal{H}^{n-1} \llcorner S\rangle_{C_c^0(B_{16}(0))'}.$$

Since this is convergent as $j \to \infty$ there exists $C(\Omega, \Lambda) > 0$ such that (5.4.19) holds. $\square$

In order to prove Theorem 5.4.1 it is therefore sufficient to prove the following statement and apply it with $\Omega, (u_{\varepsilon_j})_{j\in\mathbb{N}}, (\overline{u}_{\varepsilon_j})_{j\in\mathbb{N}}$ replaced with $B_8(0), (\tilde{u}_{\tilde{\varepsilon}_j})_{j\in\mathbb{N}}, (\hat{u}_{\tilde{\varepsilon}_j})_{j\in\mathbb{N}}$ (the rescaled functions and measures also satisfy the assumptions of Theorem 5.4.1).

Proposition 5.4.5.
Assume $(u_\varepsilon)_{\varepsilon>0}$ as in Theorem 5.4.1 with $B_4(0) \Subset \Omega$, modified phase fields $(\overline{u}_\varepsilon)_{\varepsilon>0}$ that satisfy (5.3.1) *and*

$$\int_\Omega \Big(\frac{\varepsilon}{2}|\nabla \overline{u}_\varepsilon|^2 + \frac{1}{\varepsilon}(\overline{u}_\varepsilon - u_\varepsilon)^2\Big)\, \mathrm{d}\mathcal{L}^n \leq \overline{\Lambda} \tag{5.4.20}$$

for some $\overline{\Lambda} > 0$. Consider Radon measures $\mu_\varepsilon, \kappa_\varepsilon, \vartheta_\varepsilon \in C_c^0(\Omega)'$ satisfying (5.2.1)-(5.2.5) *and* (5.3.3)*, varifolds $V_\varepsilon \in C_c^0(\Omega \times G(n.n-1))'$ with* (5.2.6)*, a subsequence $\varepsilon \to 0$, finite Radon measures μ, κ, ϑ, and a limit varifold V such that* (5.2.8)-(5.2.11) *and* (5.3.14) *hold on $B_4(0)$. In addition assume that*

$$\mu = c_0\theta\mathcal{H}^{n-1} \llcorner S \quad \textit{for some} \quad \theta \in \mathbb{N},\ S \in G(n, n-1), \quad \textit{and} \quad \alpha = 0 = \kappa.$$

Then we have

$$\vartheta = \frac{1}{\sigma}\mu.$$

We prepare the proof of Proposition 5.4.5 with the following generalization of Proposition 5.5 in [RS06].

Proposition 5.4.6.
For all $\tau, \delta \in (0, 1)$ and $\Lambda > 0$ there exist $\omega = \omega(\delta, \tau, \Lambda) > 0$ and $L = L(\delta, \tau) \in (1, \infty)$ such that the following holds: Let the assumptions from Proposition 5.4.5 be satisfied with $\Omega = B_{4L\varepsilon}(0)$ and further assume

$$|u_\varepsilon(0)| \leq 1 - \tau, \tag{5.4.21}$$

$$|\xi_\varepsilon|(B_{4L\varepsilon}(0)) + \int_{B_{4L\varepsilon}(0)} \varepsilon|\nabla u_\varepsilon|^2\sqrt{1 - |\nu_{\varepsilon,n}|^2}\, \mathrm{d}\mathcal{L}^n \leq \omega(4L\varepsilon)^{n-1}, \tag{5.4.22}$$

$$\int_{B_{4L\varepsilon}} \Big(\varepsilon\sum_{l=1}^{n-1} |\partial_l \overline{u}_\varepsilon|^2 + \frac{1}{\varepsilon}(|\overline{u}_\varepsilon| - 1)_+^2\Big)\, \mathrm{d}\mathcal{L}^n \leq \omega(4L\varepsilon)^{n-1}, \tag{5.4.23}$$

with $\nu_{\varepsilon,n} \coloneqq e_n \cdot \nu_\varepsilon$, *and*

$$\mu_\varepsilon(B_{4L\varepsilon}(0)) \leq \Lambda(4L\varepsilon)^{n-1}, \tag{5.4.24}$$
$$\kappa_\varepsilon(B_{4L\varepsilon}(0)) \leq \Lambda(4L\varepsilon)^{n-3}. \tag{5.4.25}$$

Then we also have, writing $(0,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$

$$|u_\varepsilon(0,t)| \geq 1 - \frac{\tau}{2} \quad \textit{for all} \quad L\varepsilon \leq |t| \leq 3L\varepsilon, \tag{5.4.26}$$
$$\left| \frac{1}{\omega_{n-1}(L\varepsilon)^{n-1}} \mu_\varepsilon(B_{L\varepsilon}(0)) - c_0 \right| \leq \delta, \tag{5.4.27}$$
$$\left| \int_{-L\varepsilon}^{L\varepsilon} \frac{1}{\varepsilon} W(u_\varepsilon(0,t))\,\mathrm{d}t - \frac{c_0}{2} \right| \leq \delta, \tag{5.4.28}$$
$$\left| \int_{-L\varepsilon}^{L\varepsilon} \Big(\varepsilon |\nabla u_\varepsilon||\nabla \overline{u}_\varepsilon| - \frac{2}{\varepsilon\sigma} W(u_\varepsilon) \Big)(0,t)\,\mathrm{d}t \right| \leq \delta. \tag{5.4.29}$$

Here ω_m *is defined by* $\mathcal{L}^m(B_1(0)) = \omega_m$ *for* $m \in \mathbb{N}$.

Proof. We follow the proof of Proposition 5.5 from [RS06]. The existence of ω, L such that the statements (5.4.26)-(5.4.28) hold have already been proved there. In the following be possibly lower the value of ω and increase the value of L, which maintains (5.4.26)-(5.4.28).

We prove in the following that we can assume $\varepsilon = 1$ without loss of generality. In fact since ε is fixed, by rescaling $x \longmapsto \varepsilon x$ and defining $\mathbf{u}(x) \coloneqq u_\varepsilon(\varepsilon x)$ for $x \in B_L(0)$, we can drop the index. For the claims (5.4.26)-(5.4.28) this has already been done in [RS06], we prove it for the remaining expression in (5.4.29)

$$\int_{-L\varepsilon}^{L\varepsilon} \varepsilon |\nabla u_\varepsilon||\nabla \overline{u}_\varepsilon|(0,t)\,\mathrm{d}t = \int_{-L}^{L} \varepsilon |\nabla u_\varepsilon||\nabla \overline{u}_\varepsilon|(0,\varepsilon t)\varepsilon\,\mathrm{d}t = \int_{-L}^{L} |\nabla \mathbf{u}||\nabla \overline{\mathbf{u}}|(0,t)\,\mathrm{d}t.$$

with $-\Delta\overline{\mathbf{u}} + \overline{\mathbf{u}} = \mathbf{u}$.
We recall that by Lemma 4.1.2 and the definitions of c_0, σ in Assumptions 4.1.1 we have

- $|q_0| < 1$ and $q_0' > 0$,
- $\lim_{z \to \pm\infty} q_0(z) = \pm 1$,
- $\int_{\mathbb{R}} \frac{1}{2}|q_0'|^2\,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} W(q_0)\,\mathrm{d}\mathcal{L}^1 = \frac{c_0}{2}$,
- $\int_{\mathbb{R}} |q_0'||\overline{q}_0'|\,\mathrm{d}\mathcal{L}^1 = \frac{c_0}{\sigma}$.

For a given $a \in \mathbb{R}$ we define

$$q_a(t) \coloneqq q_0(t+a), \quad \overline{q}_a(t) \coloneqq \overline{q}_0(t+a) \quad \text{for} \quad t \in \mathbb{R}, \quad \text{and} \quad Q_a(x) \coloneqq q_a(x_n) \quad \text{for} \quad x \in \mathbb{R}^n,$$

and claim that we can choose $L(\tau, \delta)$ sufficiently large such that: If

$$|q_0(a)| \leq 1 - \tau, \quad -\overline{q}_0'' + \overline{q}_0 = q_a \quad \text{with} \quad |\overline{q}| \leq 1$$

then

$$|Q_a(0,t)| \geq 1 - \frac{\tau}{3} \quad \text{for all} \quad L \leq |t| \leq 3L, \tag{5.4.30}$$

$$\left| \frac{1}{\omega_{n-1} L^{n-1}} \int_{B_L(0)} \left(\frac{1}{2} |\nabla Q_a|^2 + W(Q_a) \right) \mathrm{d}\mathcal{L}^n - c_0 \right| \leq \frac{\delta}{2}, \tag{5.4.31}$$

$$\left| \int_{-L}^{L} W(Q_a(0,t)) \, \mathrm{d}t - \frac{c_0}{2} \right| \leq \frac{\delta \sigma}{6}, \tag{5.4.32}$$

$$\left| \int_{-L}^{L} |q_a'||\overline{q}'|(0,t) \, \mathrm{d}t - \frac{c_0}{\sigma} \right| \leq \frac{\delta}{3}. \tag{5.4.33}$$

The first three properties are guaranteed by [RS06]. For the fourth identity we use $|q_0(a)| \leq 1 - \tau$ and thus $|a| \leq q_0^{-1}(1-\tau)$. Furthermore we have $\overline{q} = \mathbf{A}_0 q_a = \overline{q}_a$ because the difference $\overline{q} - \overline{q}_a$ is a bounded solution to the homogeneous equation $-w'' + w = 0$ and thus vanishes. We conclude

$$\int_{-L}^{L} |q_a'||\overline{q}'|(0,t) \, \mathrm{d}t = \int_{-L}^{L} |q_a'||\overline{q}_a'| \, \mathrm{d}\mathcal{L}^1 = \int_{-L-a}^{L-a} |q_0'||\overline{q}_0'| \, \mathrm{d}\mathcal{L}^1 \longrightarrow \frac{c_0}{\sigma}.$$

Since we have a uniform bound on $|a|$ only dependent on τ we can choose $L(\tau, \delta) > 1$ independent from a such that (5.4.33) holds.

Since H is bounded in $H^1(B_{4L}(0))$ by (5.4.25) we conclude by inner elliptic regularity theory similar as in [RS06]

$$\|u\|_{H^3(B_{\frac{7L}{2}}(0))} \leq C(\Lambda, L). \tag{5.4.34}$$

We proceed by a contradiction argument, adapting [RS06]. Assume the claim is wrong then there exists a sequence $(u_k)_{k \in \mathbb{N}}$ with $\omega_k \to 0$ as $k \to \infty$ and for any $k \in \mathbb{N}$ there exist functions $u_k, \overline{u}_k, H_k$ satisfying the assumptions of Proposition 5.4.5 with $\varepsilon = 1$, $\Omega = B_{4L}(0)$, and satisfying the properties (5.4.21)-(5.4.25) but violating (5.4.29).

By inner elliptic regularity theory, see Theorem 2 in §6.3 in [Eva10], there exists $C > 0$ such that

$$\|\overline{u}\|_{H^5(B_{3L}(0))} \leq C \Big(\|\overline{u}\|_{L^2(B_{\frac{7L}{2}}(0))} + \|u\|_{H^3(B_{\frac{7L}{2}}(0))} \Big).$$

Thus we get

$$\|\overline{u}_k\|_{H^5(B_{3L}(0))} \leq C(\Lambda, L) \Big(\|u_k\|_{H^3(B_{\frac{7L}{2}}(0)} + \|\overline{u}_k\|_{L^2(B_{\frac{7L}{2}}(0)} \Big) \leq C(\Lambda, L). \tag{5.4.35}$$

Because of (5.4.34) and (5.4.35) we can find $u \in H^3(B_{3L}(0))$, $\overline{u} \in H^5(B_{3L}(0))$, and $H \in H^1(B_{3L}(0))$ such that we have up to a subsequence as $k \to \infty$

$$u_k \xrightarrow{w} u \quad \text{in} \quad H^3(B_{3L}(0)), \quad \overline{u}_k \xrightarrow{w} \overline{u} \quad \text{in} \quad H^5(B_{2L}(0)), \tag{5.4.36}$$

$$\text{and} \quad H_k \xrightarrow{w} H \quad \text{in} \quad H^1(B_{3L}(0)). \tag{5.4.37}$$

By the compact Sobolev embedding $H^3(B_{3L}(0)) \overset{c}{\hookrightarrow} C^1(\overline{B_{3L}(0)})$ as $n \leq 3$ hence

$$u_k \longrightarrow u \quad \text{and} \quad \nabla u_k \longrightarrow \nabla u \quad \text{uniformly in} \quad B_{3L}(0). \tag{5.4.38}$$

Similar we deduce from (5.4.35), (5.4.36)

$$\overline{u}_k \longrightarrow \overline{u} \quad \text{in} \quad C^2(\overline{B_{3L}(0)}). \tag{5.4.39}$$

As in the proof of Proposition 5.5 in [RS06], letting $x = (y,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we get

$$u(y,t) = u_0(t) \quad \text{for all} \quad (y,t) \in B_{3L}(0),$$

where $u_0 = \pm q_{t_0}$ with t_0 determined by $u(0)$. Since a reflection $(y, x_n) \mapsto (y, -x_n)$ does neither affect the assumptions nor the conclusions of the proposition we can assume $u_0 = +q_{t_0}$ without loss of generality.

Next we obtain from (5.4.23) and $\omega_k \to 0$ that $\overline{u}(y,t) = \overline{u}_0(t)$ and $|\overline{u}_0| \leq 1$ for a suitable function $\overline{u}_0 : \mathbb{R} \longrightarrow \mathbb{R}$. Now we show with (5.4.33) that for large k (5.4.29) holds, in fact we have

$$\begin{aligned}
&\left| \int_{-L}^{L} \Big(|\nabla u_k||\nabla \overline{u}_k| - \frac{2}{\sigma} W(u_k) \Big)(0,t)\, \mathrm{d}t \right| \\
&\qquad \leq \int_{-L}^{L} \Big| |\nabla u_k||\nabla \overline{u}_k| - |\nabla u||\nabla \overline{u}| \Big|(0,t)\, \mathrm{d}\mathcal{L}^1 + \left| \int_{-L}^{L} |\nabla u||\nabla \overline{u}|(0,t)\, \mathrm{d}\mathcal{L}^1 - \frac{c_0}{\sigma} \right| \\
&\qquad\quad + \frac{2}{\sigma} \left| \frac{c_0}{2} - \int_{-L}^{L} W(u_k(0,t))\, \mathrm{d}\mathcal{L}^1 \right|.
\end{aligned}$$

The last two terms are estimated by (5.4.32) and (5.4.33). For the first integral we use (5.4.38) and (5.4.39) to choose $k_0 \in \mathbb{N}$ large enough such that for all $k \geq k_0$ we have

$$\|u_k - u\|_{C^1(\overline{B_L(0)})} + \|\overline{u}_k - \overline{u}\|_{C^1(\overline{B_L(0)})} \leq \frac{\delta}{6RL}$$

with $R := \sup_{k \in \mathbb{N}} \Big(\|u_k\|_{C^1(\overline{B_L(0)})} + \|\overline{u}_k\|_{C^1(\overline{B_L(0)})} \Big)$. The supremum is finite because converging sequences are bounded. Thus we get

$$\begin{aligned}
&\left| \int_{-L}^{L} \Big(|\nabla u_k||\nabla \overline{u}_k| - \frac{2}{\sigma} W(u_k) \Big)(0,t)\, \mathrm{d}t \right| \\
&\qquad \leq \int_{-L}^{L} |\nabla u_k||\nabla \overline{u}_k - \nabla \overline{u}|(0,t)\, \mathrm{d}\mathcal{L}^1 + \int_{-L}^{L} |\nabla \overline{u}||\nabla u_k - \nabla u|(0,t)\, \mathrm{d}\mathcal{L}^1 + \frac{\delta}{3} + \frac{2}{\sigma} \cdot \frac{\sigma\delta}{6} \\
&\qquad \leq 2RL \|\overline{u}_k - \overline{u}\|_{C^1(\overline{B_L(0)})} + 2RL \|u_k - u\|_{C^1(\overline{B_L(0)})} + \frac{2\delta}{3} \leq 2RL \cdot \frac{\delta}{6RL} + \frac{2\delta}{3} = \delta.
\end{aligned}$$

Thus for $k \geq k_0$ (5.4.29) holds, a contradiction to our assumption. $\square$

Proof of Proposition 5.4.5. We assume that $x = 0$ is a good point in the sense of Lemma 5.4.2 and $S = \mathbb{R}^{n-1} \times \{0\}$. Let $\Pi : \mathbb{R}^n \longrightarrow S$ be the orthogonal projection. We use the representation $x = (y,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and denote by $\nabla' = \nabla_y$ the horizontal gradient. By Theorem 5.2.3 the limit of V of V_ε is given by $V = c_0 \theta \mathcal{H}^{n-1} \llcorner S \otimes \delta_S$. Convergence as varifolds yields in particular

$$\lim_{\varepsilon \to 0} \int_{B_4(0)} \varepsilon |\nabla u_\varepsilon|^2 \sqrt{1 - \nu_{\varepsilon,n}^2}\, \mathrm{d}\mathcal{L}^n = 0.$$

Moreover by (5.4.20) we can apply Corollary 5.3.4 and conclude from (5.3.11) and varifold convergence

$$
\begin{aligned}
\int_{B_3(0)} \varepsilon\left|\nabla'\overline{u}_\varepsilon\right|^2 \,\mathrm{d}\mathcal{L}^n &\leq \int_{B_4(0)} \varepsilon|\nabla' u_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n + \varepsilon^2 C(\Lambda) && (5.4.40)\\
&\leq \int_{B_4(0)} (1-|\nu_{\varepsilon,n}|^2)\varepsilon\left|\nabla u_\varepsilon\right|^2 \,\mathrm{d}\mathcal{L}^n + \varepsilon^2 C(\Lambda) && (5.4.41)\\
&\leq \int_{B_4(0)} \sqrt{1-|\nu_{\varepsilon,n}|^2}\varepsilon\left|\nabla u_\varepsilon\right|^2 \,\mathrm{d}\mathcal{L}^n + \varepsilon^2 C(\Lambda) \longrightarrow 0 \quad \text{as} \quad \varepsilon \to 0. && (5.4.42)
\end{aligned}
$$

Furthermore by (5.3.16) for $k = 1$ we get

$$
\int_{B_3(0)} \frac{1}{\varepsilon}(|\overline{u}_\varepsilon| - 1)_+^2 \,\mathrm{d}\mathcal{L}^n \leq C\varepsilon\Big(\varepsilon\alpha_\varepsilon(\Omega) + 1\Big) \longrightarrow 0. \tag{5.4.43}
$$

By the proof of Proposition 5.2 in [RS06, page 711] for any $\delta > 0$ there exist $\omega_0, \varepsilon_0, \tau_0 > 0$, all depending on δ such that for any $0 < \omega < \omega_0$, any $0 < \tau < \tau_0$ and any $0 < \varepsilon < \varepsilon_0$ the following two properties hold:

(1)

$$
\int_{\{|u_\varepsilon|\geq 1-\tau\}\cap B_4(0)} \frac{1}{\varepsilon} W'(u_\varepsilon)^2 \,\mathrm{d}\mathcal{L}^n \leq \delta \quad \text{and} \quad \mu_\varepsilon\Big(\{|u_\varepsilon| \geq 1-\tau\} \cap B_4(0)\Big) \leq 3\delta. \tag{5.4.44}
$$

(2) For the set

$$
\begin{aligned}
A_\varepsilon := \Big\{x \in B_1(0) \;\Big|\; & \;|u_\varepsilon(x)| \leq 1-\tau,\\
& \forall \varepsilon \leq \rho \leq 3 : |\xi_\varepsilon|(B_\rho(x)) + \textstyle\int_{B_\rho(x)} \varepsilon|\nabla u_\varepsilon|^2\sqrt{1-\nu_{\varepsilon,n}^2} \leq \omega\rho^{n-1},\\
& \quad \text{and} \quad \alpha_\varepsilon(B_\rho(x)) \leq \omega\rho^{\frac{1}{2}}\Big\}
\end{aligned}
$$

we have

$$
\mu_\varepsilon(B_1(0) \setminus A_\varepsilon) \leq 4\delta. \tag{5.4.45}
$$

We now define a subset of A_ε with additional "good properties",

$$
\begin{aligned}
A'_\varepsilon := A_\varepsilon \cap \Big\{x \in B_1(0) \mid \forall \rho \in [\varepsilon, 3] : \int_{B_\rho(x)} \Big(\varepsilon\left|\nabla'\overline{u}_\varepsilon\right|^2 + \frac{1}{\varepsilon}(|\overline{u}_\varepsilon| - 1)_+^2\Big) \,\mathrm{d}\mathcal{L}^n \leq \omega\rho^{n-1}\\
\text{and} \quad \kappa_\varepsilon(B_\rho(x)) \leq \omega\rho^{\frac{1}{2}}\Big\}.
\end{aligned}
$$

We show that the complement in A'_ε is small. For all $x \in A_\varepsilon \setminus A'_\varepsilon$ there exists $\rho_x \in \big(0, \frac{1}{2}\big)$ such that $B_{2\rho_x}(x) \subseteq B_1(0)$. It follows

$$
A_\varepsilon \setminus A'_\varepsilon \subseteq \bigcup_{x\in A_\varepsilon\setminus A'_\varepsilon} \overline{B_{\rho_x}(x)}.
$$

By Besicovitch's covering Theorem there exist $N \in \mathbb{N}$ only dependent on n and sets $D_1, \ldots, D_N \subseteq B_1(0)$ such that for fixed $k \in \{1, \ldots, N\}$ the collections

$$
\Big\{\overline{B_{\rho_x}(x)} \mid x \in D_k\Big\}
$$

are disjoint and

$$A_\varepsilon \setminus A'_\varepsilon \subseteq \bigcup_{k=1}^{N} \dot{\bigcup_{x\in D_k}} \overline{B_{\rho_x}(x)}.$$

Since for all $k \in \{1,\dots,N\}$ the union $\bigcup_{x\in D_k} \overline{B_{\rho_x}(x)} \subseteq B_1(0)$ is disjoint it follows that

$$\omega_n \sum_{x\in D_k} \rho_x^n = \sum_{x\in D_k} \mathcal{L}^n\Big(\overline{B_{\rho_x}(x)}\Big) = \mathcal{L}^n\Big(\dot{\bigcup_{x\in D_k}} \overline{B_{\rho_x}(x)}\Big) \le \mathcal{L}^n(B_1(0)) = \omega_n < \infty.$$

The sum is convergent and thus D_k has to be at most countable. We conclude

$$A_\varepsilon \setminus A'_\varepsilon \subseteq \bigcup_{k=1}^{N} \dot{\bigcup_{j\in\mathbb{N}}} \overline{B_{\rho_{k,j}}(x_{k,j})}. \tag{5.4.46}$$

Since $x_{k,j} \in A_\varepsilon \setminus A'_\varepsilon$ we have for all k,j that there exists $\varepsilon \le \rho_{k,j} \le 3$ such that

$$\int_{B_{\rho_{k,j}}(x_{k,j})} \Big(\varepsilon|\nabla'\overline{u}_\varepsilon|^2 + \frac{1}{\varepsilon}(|\overline{u}_\varepsilon|-1)_+^2\Big)\,\mathrm{d}\mathcal{L}^n > \omega\rho_{k,j}^{n-1}$$

or

$$\kappa_\varepsilon(B_{\rho_{k,j}}(x_{k,j})) > \omega\rho_{k,j}^{\frac{1}{2}}.$$

Since $x_{k,j} \in A_\varepsilon$ we can use $\alpha_\varepsilon(B_\rho(x_{k,j})) \le \omega\rho^{\frac{1}{2}}$ for all $\varepsilon \le \rho \le 3$ and (5.4.44). We deduce from Proposition 4.7 in [RS06] that

$$\mu_\varepsilon(\overline{B_{\rho_{k,j}}(x_{k,j})}) \le C\rho_{k,j}^{n-1}.$$

We then obtain by (5.4.46)

$$\begin{aligned}
\mu_\varepsilon(A_\varepsilon \setminus A'_\varepsilon) &\le C\sum_{k=1}^{N}\sum_{j\in\mathbb{N}} \rho_{k,j}^{n-1} \\
&\le \frac{C}{\omega}\sum_{k=1}^{N}\sum_{j\in\mathbb{N}} \int_{B_{\rho_{k,j}}(x_{k,j})} \Big(\varepsilon|\nabla'\overline{u}_\varepsilon|^2 + \frac{1}{\varepsilon}(|\overline{u}_\varepsilon|-1)_+^2\Big)\,\mathrm{d}\mathcal{L}^n \\
&\quad + C\omega^{2(1-n)}\kappa_\varepsilon(B_4(0))^{2(n-1)-1}\sum_{k=1}^{N}\sum_{j\in\mathbb{N}} \kappa_\varepsilon(B_{\rho_{k,j}}(x_{k,j})) \\
&\le \frac{CN}{\omega}\int_{B_4(0)} \Big(\varepsilon|\nabla'\overline{u}_\varepsilon|^2 + \frac{1}{\varepsilon}(|\overline{u}_\varepsilon|-1)_+^2\Big)\,\mathrm{d}\mathcal{L}^n + \omega^{2(1-n)}N\kappa_\varepsilon(B_4(0))^{2(n-1)} \le \delta
\end{aligned} \tag{5.4.47}$$

for ε sufficiently small, where we have used $n \in \{2,3\}$, (5.4.42), (5.4.43), and $\kappa_\varepsilon \xrightarrow{w^*} 0$.

By the definition of A_ε for all $x \in A'_\varepsilon$ we can apply Proposition 5.4 from [RS06] with $N=1$ and deduce (5.4.24) with (with 0 replaced by x). Together with the definition of A'_ε we obtain that we can apply Proposition 5.4.6 for all $x \in A'_\varepsilon$. By page 713 in [RS06] this yields that for all $y \in S \cap B_1(0)$ there exist $\mathbb{N} \ni K = K(y) \leq \theta$ and $t_1(y), \ldots, t_K(y) \in \mathbb{R}$ with

$$A_\varepsilon \cap \Pi^{-1}(y) \subseteq \{y\} \times \bigcup_{l=1}^{K} (t_l(y) - L\varepsilon, t_l(y) + L\varepsilon).$$

We now fix an arbitrary $\eta \in C_c^1(B_1(0))$ and deduce from $\xi_\varepsilon \xrightarrow{w^*} 0$, (5.4.45), and (5.4.48)

$$\begin{aligned} \limsup_{\varepsilon \to 0} & \left| \int_{B_1(0)} \eta \, \mathrm{d}\vartheta_\varepsilon - \frac{1}{\sigma} \int_{B_1(0)} \eta \, \mathrm{d}\mu_\varepsilon \right| \\ & \leq \limsup_{\varepsilon \to 0} \left| \int_{A'_\varepsilon} \eta \, \mathrm{d}\vartheta_\varepsilon - \frac{2}{\sigma} \int_{A'_\varepsilon} \eta \frac{1}{\varepsilon} W(u_\varepsilon) \, \mathrm{d}\mathcal{L}^n \right| + C\delta \|\eta\|_{C^0(\overline{B_1(0)})} \end{aligned} \tag{5.4.48}$$

for some $C > 0$. Furthermore we obtain

$$\begin{aligned} & \left| \int_{A'_\varepsilon} \eta \, \mathrm{d}\vartheta_\varepsilon - \frac{2}{\sigma} \int_{A'_\varepsilon} \eta \frac{1}{\varepsilon} W(u_\varepsilon) \, \mathrm{d}\mathcal{L}^n \right| \\ & \quad = \left| \int_{\Pi(A'_\varepsilon)} \sum_{l=1}^{K(y)} \int_{t_l(y) - L\varepsilon}^{t_l(y) + L\varepsilon} \eta(y,t) \Big(\varepsilon |\nabla u_\varepsilon| |\nabla \overline{u}_\varepsilon| - \frac{2}{\sigma\varepsilon} W(u_\varepsilon) \Big)(y,t) \, \mathrm{d}t \, \mathrm{d}y \right. \\ & \quad \leq \left| \int_{\Pi(A'_\varepsilon)} \sum_{l=1}^{K(y)} |\eta(y,t_j)| \int_{t_l(y) - L\varepsilon}^{t_l(y) + L\varepsilon} \Big(\varepsilon |\nabla u_\varepsilon| |\nabla \overline{u}_\varepsilon| - \frac{2}{\sigma\varepsilon} W(u_\varepsilon) \Big)(y,t) \, \mathrm{d}t \, \mathrm{d}y \right. \\ & \qquad + C \sup_{\substack{(y,s),(y,t) \in B_1(0) \\ |t-s| < L\varepsilon}} |\eta(y,t) - \eta(y,s)| \Big(\vartheta_\varepsilon(B_1(0)) + \mu_\varepsilon(B_1(0)) \Big). \end{aligned}$$

For the first term we can apply (5.4.29). For the second term we use that $\eta \in C_c^1(B_1(0))$ is uniformly continuous, thus for ε sufficiently small we have for all s, t with $|t - s| < L\varepsilon$ that $|\eta(y,t) - \eta(y,s)| < \delta$. We conclude

$$\begin{aligned} & \left| \int_{A'_\varepsilon} \eta \, \mathrm{d}\vartheta_\varepsilon - \frac{2}{\sigma} \int_{A'_\varepsilon} \eta \frac{1}{\varepsilon} W(u_\varepsilon) \, \mathrm{d}\mathcal{L}^n \right| \\ & \qquad \leq \|\eta\|_{C^0(B_1(0))} \int_{\Pi(A'_\varepsilon)} \theta\delta \, \mathrm{d}y + C\delta(\Lambda + \overline{\Lambda}). \end{aligned}$$

Hence we conclude with (5.4.48)

$$\limsup_{\varepsilon \to 0} \left| \int_{B_1(0)} \eta \, \mathrm{d}\vartheta_\varepsilon - \frac{1}{\sigma} \int_{B_1(0)} \eta \, \mathrm{d}\mu_\varepsilon \right| \leq C(\Lambda, \overline{\Lambda}, \eta, \theta)\delta.$$

Since $\delta > 0$ and $\eta \in C_c^1(B_1(0))$ were arbitrary we deduce

$$\vartheta = \lim_{\varepsilon \to 0} \vartheta_\varepsilon = \lim_{\varepsilon \to 0} \frac{1}{\sigma} \mu_\varepsilon = \frac{1}{\sigma} \mu. \qquad \square$$

5.5 Main proof of the $\liminf$-estimate

With the results from the previous section we can give a rigorous proof of Theorem 5.2.5. We start by proving a lemma which allows us to neglect some error terms as $\varepsilon \to 0$.

Lemma 5.5.1 (Error estimates)**.**
Let $(u_\varepsilon)_{\varepsilon>0}$ be as in Definition 5.2.1 and assume (5.2.12). Let $\eta \in C_c^2(\Omega;\mathbb{R}^n)$ and $\psi \in C_c^2(\Omega)$. Then the following identities hold

$$\lim_{\varepsilon\to 0}\int_\Omega H_\varepsilon \eta\cdot\nabla(-\varepsilon^2\Delta+\mathrm{Id})\overline{u}_\varepsilon \,\mathrm{d}\mathcal{L}^n = \lim_{\varepsilon\to 0}\int_\Omega \nabla\overline{u}_\varepsilon\cdot\eta(-\varepsilon^2\Delta+\mathrm{Id})H_\varepsilon\,\mathrm{d}\mathcal{L}^n \quad (i)$$

$$\limsup_{\varepsilon\to 0}\int_\Omega \varepsilon\big[\nabla\overline{u}_\varepsilon\cdot\eta\big](-\varepsilon^2\Delta+\mathrm{Id})\big[\nabla\overline{u}_\varepsilon\cdot\eta\big]\,\mathrm{d}\mathcal{L}^n = \limsup_{\varepsilon\to 0}\int_\Omega \varepsilon\big[\nabla\overline{u}_\varepsilon\cdot\eta\big]\big[\nabla u_\varepsilon\cdot\eta\big]\,\mathrm{d}\mathcal{L}^n \quad (ii)$$

$$\liminf_{\varepsilon\to 0}\int_\Omega\left(\frac{1}{\varepsilon}\big|H_\varepsilon\psi\big|^2+\varepsilon\big|\nabla(H_\varepsilon\psi)\big|^2\right)\mathrm{d}\mathcal{L}^n = \liminf_{\varepsilon\to 0}\int_\Omega|\psi|^2\,\mathrm{d}\kappa_\varepsilon. \quad (iii)$$

Proof. For (i) we have

$$\begin{aligned}\int_\Omega H_\varepsilon\eta\cdot\nabla(-\varepsilon^2\Delta+\mathrm{Id})\overline{u}\,\mathrm{d}\mathcal{L}^n &= \int_\Omega \nabla\overline{u}_\varepsilon\cdot(-\varepsilon^2\Delta+\mathrm{Id})[\eta H_\varepsilon]\,\mathrm{d}\mathcal{L}^n\\ &= \int_\Omega\nabla\overline{u}_\varepsilon\cdot\Big[-\varepsilon^2H_\varepsilon\Delta\eta-2\varepsilon^2 D\eta\nabla H_\varepsilon+\eta(-\varepsilon^2\Delta+\mathrm{Id})H_\varepsilon\Big]\,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

The claim is that the first and second term in $[\,\cdot\,]$ vanish in the integral as $\varepsilon\to 0$, so we estimate

$$\begin{aligned}&\left|\int_\Omega\varepsilon^2\nabla\overline{u}_\varepsilon\cdot(H_\varepsilon\cdot\Delta\eta+2D\eta\nabla H_\varepsilon)\,\mathrm{d}\mathcal{L}^n\right|\\ &\qquad\le \|\eta\|_{C^2(\Omega)}\varepsilon^2\int_{\mathrm{supp}(\eta)}\varepsilon^{-\frac12}|H_\varepsilon|\varepsilon^{\frac12}\big|\nabla\overline{u}_\varepsilon\big|\,\mathrm{d}\mathcal{L}^n\\ &\qquad\quad+2\|\eta\|_{C^1(\Omega)}\varepsilon\int_{\mathrm{supp}(\eta)}\varepsilon^{\frac12}|\nabla H_\varepsilon|\varepsilon^{\frac12}\big|\nabla\overline{u}_\varepsilon\big|\,\mathrm{d}\mathcal{L}^n\\ &\qquad\le\|\eta\|_{C^2(\Omega)}\varepsilon^2\left[\int_\Omega\frac{1}{\varepsilon}|H_\varepsilon|^2\,\mathrm{d}\mathcal{L}^n\int_{\mathrm{supp}(\eta)}\varepsilon\big|\nabla\overline{u}_\varepsilon\big|^2\,\mathrm{d}\mathcal{L}^n\right]^{\frac12}\\ &\qquad\quad+2\|\eta\|_{C^1(\Omega)}\varepsilon\left[\int_\Omega\varepsilon\big|\nabla H_\varepsilon\big|^2\,\mathrm{d}\mathcal{L}^n\int_\Omega\varepsilon\big|\nabla u_\varepsilon\big|^2\,\mathrm{d}\mathcal{L}^n\right]^{\frac12}\end{aligned}$$

By (5.2.12) and (5.3.10) we get for $0<\varepsilon<1$

$$\left|\int_\Omega\varepsilon^2\nabla\overline{u}_\varepsilon\cdot(H_\varepsilon\cdot\Delta\eta+2D\eta\nabla H_\varepsilon)\,\mathrm{d}\mathcal{L}^n\right|\le C(\Lambda,\eta)\varepsilon.$$

For (ii) we have

$$\begin{aligned}&\int_\Omega\varepsilon\big[\nabla\overline{u}_\varepsilon\cdot\eta\big](-\varepsilon^2\Delta+\mathrm{Id})\big[\nabla\overline{u}_\varepsilon\cdot\eta\big]\,\mathrm{d}\mathcal{L}^n\\ &\qquad=\int_\Omega\varepsilon\big[\nabla\overline{u}_\varepsilon\cdot\eta\big]\Big[-\varepsilon^2\nabla\overline{u}_\varepsilon\cdot\Delta\eta-2\varepsilon^2D\overline{u}_\varepsilon:D\eta+\eta\cdot\nabla(-\varepsilon^2\Delta+\mathrm{Id})\overline{u}_\varepsilon\Big]\,\mathrm{d}\mathcal{L}^n\end{aligned}$$

Because of (5.3.1) the claim is that the first and second term in $[\,\cdot\,]$ vanish in the integral as $\varepsilon \to 0$. We estimate

$$
\begin{aligned}
&\left| \int_\Omega \varepsilon^3 (\nabla \overline{u}_\varepsilon \cdot \eta)\big(D^2 \overline{u}_\varepsilon : D\eta + \nabla \overline{u}_\varepsilon \cdot \Delta \eta\big) \,\mathrm{d}\mathcal{L}^n \right| \\
&\qquad\qquad \le 2\|\eta\|_{C^1(\Omega)} \|\eta\|_{C^0(\Omega)} \varepsilon \int_{\operatorname{supp}(\eta)} \varepsilon^{\frac12} |\nabla \overline{u}_\varepsilon| \varepsilon^{\frac32} |D^2 \overline{u}_\varepsilon| \,\mathrm{d}\mathcal{L}^n \\
&\qquad\qquad\quad + \|\eta\|_{C^2(\Omega)} \|\eta\|_{C^0(\Omega)} \varepsilon^2 \int_{\operatorname{supp}(\eta)} \varepsilon |\nabla \overline{u}_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \\
&\qquad\qquad \le 2\|\eta\|_{C^1(\Omega)} \|\eta\|_{C^0(\Omega)} \varepsilon \left[\int_{\operatorname{supp}(\eta)} \varepsilon |\nabla \overline{u}_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \int_{\operatorname{supp}(\eta)} \varepsilon^3 |D^2 \overline{u}_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \right]^{\frac12} \\
&\qquad\qquad\quad + 2\Lambda \|\eta\|_{C^2(\Omega)} \|\eta\|_{C^0(\Omega)} \varepsilon^2.
\end{aligned}
$$

By (5.3.12) we get

$$
\left| \int_\Omega \varepsilon^3 (\nabla \overline{u}_\varepsilon \cdot \eta)\big(\nabla \overline{u}_\varepsilon \cdot \Delta \eta + 2 D^2 \overline{u}_\varepsilon : D\eta\big) \,\mathrm{d}\mathcal{L}^n \right| \le C(\eta, \Lambda)\varepsilon.
$$

For (iii) we calculate

$$
\begin{aligned}
\int_\Omega \left(\frac{1}{\varepsilon} |H_\varepsilon \psi|^2 + \varepsilon |\nabla (H_\varepsilon \psi)|^2 \right) \mathrm{d}\mathcal{L}^n &= \int_\Omega \left(\frac{1}{\varepsilon} |H_\varepsilon \psi|^2 + \varepsilon |H_\varepsilon \nabla \psi + \psi \nabla H_\varepsilon|^2 \right) \mathrm{d}\mathcal{L}^n \\
= \int_\Omega \left(\frac{1}{\varepsilon} |H_\varepsilon \psi|^2 + \varepsilon \Big[|H_\varepsilon \nabla \psi|^2 + 2 H_\varepsilon \psi \nabla H_\varepsilon \cdot \nabla \psi + |\psi \nabla H_\varepsilon|^2 \Big] \right) \mathrm{d}\mathcal{L}^n.&
\end{aligned}
$$

Because of (5.2.4) the claim is that the first and second term in $[\,\cdot\,]$ vanish in the integral as $\varepsilon \to 0$. We estimate for $0 < \varepsilon < 1$

$$
\begin{aligned}
&\int_\Omega \varepsilon \Big(|H_\varepsilon \nabla \psi|^2 + 2 \frac{1}{\sqrt{\varepsilon}} |H_\varepsilon \nabla \psi| \sqrt{\varepsilon} |\nabla H_\varepsilon \psi| \Big) \,\mathrm{d}\mathcal{L}^n \\
&\quad \le \|\psi\|^2_{C^1(\Omega)} \varepsilon^2 \int_\Omega \frac{1}{\varepsilon} |H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n + \int_\Omega |H_\varepsilon \nabla \psi|^2 \,\mathrm{d}\mathcal{L}^n + \int_\Omega \varepsilon^2 |\nabla H_\varepsilon \psi|^2 \,\mathrm{d}\mathcal{L}^n \\
&\quad \le \|\psi\|^2_{C^1(\Omega)} \Lambda \varepsilon^2 + \|\psi\|^2_{C^1(\Omega)} \varepsilon \int_\Omega \frac{1}{\varepsilon} |H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n + \|\psi\|^2_{C^0(\Omega)} \varepsilon \int_\Omega \varepsilon |\nabla H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \\
&\quad \le 3\Lambda \|\psi\|^2_{C^1(\Omega)} \varepsilon.
\end{aligned}
$$

□

Next we estimate the first variation of the varifold V.

Proposition 5.5.2 (Estimate for first variation)**.**
Let $(u_\varepsilon)_{\varepsilon>0}$ *as in Definition 5.2.1 and assume* (5.2.12). *Let* $\eta \in C^2_c(\Omega; \mathbb{R}^n)$ *and* $\psi \in C^2_c(\Omega)$ *with* $0 \le \psi \le 1$ *and* $\psi = 1$ *on* $\operatorname{supp}(\eta)$. *Then we have*

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \int_\Omega H_\varepsilon \eta \cdot \nabla u_\varepsilon \,\mathrm{d}\mathcal{L}^n \qquad (5.5.1)\\
&\quad \le \left[\limsup_{\varepsilon \to 0} \int_\Omega |\eta|^2 \varepsilon |\nabla \overline{u}_\varepsilon| |\nabla u_\varepsilon| \,\mathrm{d}\mathcal{L}^n \liminf_{\varepsilon \to 0} \int_\Omega |\psi|^2 \left[\frac{1}{\varepsilon} |H_\varepsilon|^2 + \varepsilon |\nabla H_\varepsilon|^2 \right] \mathrm{d}\mathcal{L}^n \right]^{\frac12}.
\end{aligned}
$$

Before proving this proposition we improve the regularity of u_ε without loss of generality.

Lemma 5.5.3 (Regularity of u_ε)**.**
Let the assumptions from Proposition 5.5.2 hold. For the proof of (5.5.1) *we can assume* $u_\varepsilon \in C^3(\Omega)$ *without loss of generality.*

Proof. Since $C^\infty(\Omega)$ is dense in $H^3(\Omega)$, $u_\varepsilon \in H^3(\Omega)$, and $H^3(\Omega) \hookrightarrow C^1(\overline{\Omega})$ we can find a sequence $(\phi_k)_{k\in\mathbb{N}}$ in $C^\infty(\Omega)$ such that

$$\phi_k \longrightarrow u_\varepsilon \quad \text{as} \quad k \to \infty \quad \text{in} \quad H^3(\Omega) \quad \text{and in} \quad C^1(\overline{\Omega}).$$

From the definition of the $H^3(\Omega)$-norm we get $\Delta\phi_k \longrightarrow \Delta u_\varepsilon$ and $\nabla\Delta\phi_k \longrightarrow \nabla\Delta u_\varepsilon$ in $L^2(\Omega)$. Since $\phi_k \longrightarrow u_\varepsilon$ in $C^1(\overline{\Omega})$ and W is a polynomial we also get $W'(\phi_k) \longrightarrow W'(u_\varepsilon)$ and $W''(\phi_k)\nabla\phi_k \longrightarrow W''(u_\varepsilon)\nabla u_\varepsilon$ in $L^2(\Omega)$. This implies

$$\mathcal{W}_\varepsilon^{\mathrm{KK}}(\phi_k) \longrightarrow \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon) \quad \text{as} \quad k \to \infty.$$

We choose $k(\varepsilon) \in \mathbb{N}$ such that

$$\left|\mathcal{W}_\varepsilon^{\mathrm{KK}}(\phi_k) - \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon)\right| < \varepsilon.$$

This implies that in the third integral in (5.5.1) we can replace u_ε with $\phi_{k(\varepsilon)}$. This also works for the first integral because $\nabla\phi_k \longrightarrow \nabla u_\varepsilon$ in $L^2(\Omega)$ and we have from the properties above $H_k \longrightarrow H_\varepsilon$ as $k \to \infty$ in $L^2(\Omega)$ with $H_k := -\varepsilon\Delta\phi_k + \frac{1}{\varepsilon}W'(\phi_k)$.
For the middle integral we have to consider $\overline{\phi}_k := \mathcal{A}_\varepsilon\phi_k$. Since $\phi_k \longrightarrow u_\varepsilon$ in $L^2(\Omega)$ and $\mathcal{A}_\varepsilon$ is a bounded linear operator on $L^2(\Omega)$ we also get $\overline{\phi}_k \longrightarrow \overline{u}_\varepsilon$ in $L^2(\Omega)$. Thus we can replace u_ε with $\phi_{k(\varepsilon)}$ in all three integrals without changing the value of the limits. It follows that we can assume $u_\varepsilon \in C^3(\Omega)$ (even C^∞) without loss of generality for the proof of Proposition 5.5.2. □

Proof of Proposition 5.5.2.
Let Ω, η, ψ be as in the assumptions. Then we have by definition of $\overline{u}_\varepsilon$

$$\lim_{\varepsilon\to 0}\int_\Omega H_\varepsilon\eta\cdot\nabla u_\varepsilon \,\mathrm{d}\mathcal{L}^n = \lim_{\varepsilon\to 0}\int_\Omega H_\varepsilon\eta\cdot\nabla(-\varepsilon^2\Delta + \mathrm{Id})\overline{u}_\varepsilon \,\mathrm{d}\mathcal{L}^n.$$

The existence of the limit on the left-hand side is guaranteed by Lemma 5.2.4. In the next step we want to shift the differential operator on H_ε. This is correct without any error terms because of (i) from Lemma 5.5.1. We get with the specific choice of ψ

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega H_\varepsilon\eta\cdot\nabla u_\varepsilon \,\mathrm{d}\mathcal{L}^n &= \lim_{\varepsilon\to 0}\int_\Omega \nabla\overline{u}_\varepsilon\cdot\eta(-\varepsilon^2\Delta + \mathrm{Id})H_\varepsilon \,\mathrm{d}\mathcal{L}^n \\ &= \lim_{\varepsilon\to 0}\int_\Omega \nabla\overline{u}_\varepsilon\cdot\eta(-\varepsilon^2\Delta + \mathrm{Id})\psi H_\varepsilon \,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

We want to apply a Cauchy-Schwarz estimate with the inner product induced by the differential operator in the middle of the integral. However this is only a scalar product on function spaces whose functions allow for an a partial integration. This is satisfied here because the test functions make all of the involved functions vanish on the boundary. Thus we get

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega \nabla\overline{u}_\varepsilon\cdot\eta(-\varepsilon^2\Delta+\mathrm{Id})[H_\varepsilon\psi]\,\mathrm{d}\mathcal{L}^n\\
&\leq \liminf_{\varepsilon\to 0}\Bigg[\underbrace{\int_\Omega \varepsilon\big[\nabla\overline{u}_\varepsilon\cdot\eta\big](-\varepsilon^2\Delta+\mathrm{Id})\big[\nabla\overline{u}_\varepsilon\cdot\eta\big]\,\mathrm{d}\mathcal{L}^n}_{=:I_\varepsilon^{(1)}}\underbrace{\int_\Omega\frac{1}{\varepsilon}H_\varepsilon\psi(-\varepsilon^2\Delta+\mathrm{Id})[H_\varepsilon\psi]\,\mathrm{d}\mathcal{L}^n}_{=:I_\varepsilon^{(2)}}\Bigg]^{\frac{1}{2}}\\
&\leq\Big[\limsup_{\varepsilon\to 0}I_\varepsilon^{(1)}\liminf_{\varepsilon\to 0}I_\varepsilon^{(2)}\Big]^{\frac{1}{2}}.
\end{aligned}
\tag{5.5.2}
$$

To estimate $I_\varepsilon^{(1)}$ we use (ii) from Lemma 5.5.1 and get

$$
\begin{aligned}
\limsup_{\varepsilon\to 0}I_\varepsilon^{(1)}&=\limsup_{\varepsilon\to 0}\int_\Omega\varepsilon\big[\nabla u_\varepsilon\cdot\eta\big]\big[\nabla\overline{u}_\varepsilon\cdot\eta\big]\,\mathrm{d}\mathcal{L}^n=\limsup_{\varepsilon\to 0}\int_\Omega\eta\otimes\eta:\varepsilon\nabla\overline{u}_\varepsilon\otimes\nabla u_\varepsilon\,\mathrm{d}\mathcal{L}^n\\
&\leq\limsup_{\varepsilon\to 0}\int_\Omega|\eta|^2\varepsilon|\nabla\overline{u}_\varepsilon||\nabla u_\varepsilon|\,\mathrm{d}\mathcal{L}^n.
\end{aligned}
$$

For $I_\varepsilon^{(2)}$ we do a partial integration and use (iii) from Lemma 5.5.1

$$
\begin{aligned}
\liminf_{\varepsilon\to 0}I_\varepsilon^{(2)}&=\liminf_{\varepsilon\to 0}\int_\Omega\left(\frac{1}{\varepsilon}\big|H_\varepsilon\psi\big|^2+\varepsilon\big|\nabla(H_\varepsilon\psi)\big|^2\right)\mathrm{d}\mathcal{L}^n\\
&=\liminf_{\varepsilon\to 0}\int_\Omega|\psi|^2\left(\frac{1}{\varepsilon}\big|H_\varepsilon\big|^2+\varepsilon\big|\nabla H_\varepsilon\big|^2\right)\mathrm{d}\mathcal{L}^n.
\end{aligned}
$$
□

This variational estimate is the central notion for the proof of Theorem 5.2.5.

Proof of Theorem 5.2.5.
Let $\eta\in C_c^2(\Omega,\mu;\mathbb{R}^n)$ with $\|\eta\|_{L^2(\Omega,\mu;\mathbb{R}^n)}\leq 1$ and let $\psi\in C_c^2(\Omega)$ with $0\leq\psi\leq 1$ and $\psi=1$ on $\operatorname{supp}(\eta)$. From Lemma 5.2.4 and Proposition 5.5.2 we have

$$
\begin{aligned}
\int_\Omega\vec{H}_V\cdot\eta\,\mathrm{d}\mu&=\lim_{\varepsilon\to 0}\int_\Omega H_\varepsilon\eta\cdot\nabla u_\varepsilon\,\mathrm{d}\mathcal{L}^n\leq\left[\limsup_{\varepsilon\to 0}\int_\Omega|\eta|^2\,\mathrm{d}\vartheta_\varepsilon\liminf_{\varepsilon\to 0}\int_\Omega|\psi|^2\,\mathrm{d}\kappa_\varepsilon\right]^{\frac{1}{2}}\\
&=\left[\int_\Omega|\eta|^2\,\mathrm{d}\vartheta\int_\Omega|\psi|^2\,\mathrm{d}\kappa\right]^{\frac{1}{2}}.
\end{aligned}
$$

Now we need the result $\vartheta=\frac{1}{\sigma}\mu$ from Theorem 5.4.1. Together with $\|\eta\|_{L^2(\Omega,\mu;\mathbb{R}^n)}\leq 1$ we get

$$
\int_\Omega\vec{H}_V\cdot\eta\,\mathrm{d}\mu\leq\sqrt{\frac{1}{\sigma}\kappa(\Omega)}.
$$

We conclude by taking the supremum over all $\eta\in C_c^2(\Omega;\mathbb{R}^n)$ with $\|\eta\|_{L^2(\Omega,\mu;\mathbb{R}^n)}\leq 1$. Since V is $(n-1)$-rectifiable $C_c^1(\Omega;\mathbb{R}^n)$ is dense in $L^2(\Omega,\mu;\mathbb{R}^n)$ by Lemma 7.4. in [Ilm94]. $C_c^2(\Omega;\mathbb{R}^n)$ is dense in $C_c^1(\Omega;\mathbb{R}^n)$ and thus the supremum yields the dual representation of the $L^2(\Omega,\mu;\mathbb{R}^n)$–norm. Hence we get

$$
\int_\Omega\sigma\big|\vec{H}_V\big|^2\,\mathrm{d}\mu\leq\kappa(\Omega)\quad\text{and thus}\quad\Big[\sigma\big|\vec{H}_V\big|^2\mu\Big](\Omega)\leq\kappa(\Omega).
$$
□

6 Convergence towards mean curvature flow of solutions of the Karali-Katsoulakis equation

In this chapter we prove the existence of weak solutions to the Karali-Katsoulakis equation

$$-\varepsilon\partial_t u_\varepsilon = (-\varepsilon^2\Delta + \mathrm{Id})\Big(-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big) \tag{6.0.1}$$

for $\varepsilon > 0$ and discuss convergence results as $\varepsilon \to 0$. This equation to our knowledge was first considered by Karali and Katsoulakis in their paper [KK07] from 2007. The authors start with a classical solution to mean curvature flow and prove that there exist solutions to (6.0.1) that converge towards the given classical solution of mean curvature flow.

Since classical solutions of the mean curvature flow can cease to exist at singularities such as topology changes, they can not be long time solutions in general. Here the concept of weak solutions is advantageous as they allow for singularities.

As explained in the introduction of Chapter 4, the PDE (6.0.1) has gradient flow structure as the right-hand side is the gradient of the diffuse perimeter 2.4.1 with respect to the metric induced by the solution operator $\mathcal{A}_\varepsilon$ from Lemma 3.1.10. Thus $\mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot))$ will decrease in time if u_ε solves (6.0.1).

6.1 Existence of solutions for the diffuse equation

We will apply an approximation method based on contributions from De Giorgi and put together by Almgren-Taylor-Wang in their paper [ATW93] in 1993. It has also been used in [LS95] and [JKO98]. Recently it was applied in [KL21] where it was proven that solutions of the Allen-Cahn equation converge towards a De Giorgi varifold type solutions for mean curvature flow; see Definition 2.5.3. The general approach to the proof of existence is classical. We define functions that solve a discretized version of the equation, similar to the Euler method. Then we use a priori estimates to generate compactness in a suitable way, combined with an Aubin-Lions-Dubinskii embedding. The last step is to prove that limit points of the constructed sequence solve the equation. The general approach is well-known, and thouroughly described in [Sch13] for the example of the heat-equation.

Notations 6.1.1.
We use the definitions and notations from section 5.1 regarding the double-well potential

W, the profiles $q_0, \overline{q}_0$, and the constants c_0, σ. Let $T > 0$, $\varepsilon > 0$, $n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^n$ open and bounded with C^3-boundary and outer unit normal field ν. We write $\Omega_T := (0,T) \times \Omega$. Let $u_{0,\varepsilon} \in H^1(\Omega) \cap L^4(\Omega)$.

Writing $H_\varepsilon := -\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon)$ we consider

$$
\begin{aligned}
-\varepsilon \partial_t u_\varepsilon &= (-\varepsilon^2 \Delta + \mathrm{Id}) H_\varepsilon && \text{in} && \Omega_T, && (6.1.1)\\
u_\varepsilon(0,\cdot) &= u_{0,\varepsilon} && \text{in} && \Omega, && (6.1.2)\\
\partial_\nu u_\varepsilon &= 0 && \text{on} && (0,T) \times \partial\Omega, && (6.1.3)\\
\partial_\nu H_\varepsilon &= 0 && \text{on} && (0,T) \times \partial\Omega. && (6.1.4)
\end{aligned}
$$

To define all occuring terms in the weak formulation we demand the following. We call $u_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)') \cap L^3(0,T;L^3(\Omega))$ a weak solution to (6.1.1)-(6.1.4) if for all $\phi \in C_c^\infty([0,T) \times \Omega)$ we have

$$
\begin{aligned}
\int_\Omega \varepsilon \phi(0,x) \mathcal{A}_\varepsilon u_{0,\varepsilon}(x) \,\mathrm{d}x + \int_0^T \int_\Omega \varepsilon \mathcal{A}_\varepsilon u_\varepsilon \partial_t \phi \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 \qquad (6.1.5)\\
= \int_0^T \int_\Omega \Big(\varepsilon \nabla u_\varepsilon \cdot \nabla \phi + \frac{1}{\varepsilon} W'(u_\varepsilon) \phi \Big) \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1.
\end{aligned}
$$

Here $\mathcal{A}_\varepsilon = (-\varepsilon^2 \Delta + \mathrm{Id})^{-1} : H^1(\Omega)' \longrightarrow H^1(\Omega)$ is the solution operator introduced in Lemma 3.1.10. The Neumann boundary condition for H_ε is encoded in $H_\varepsilon = -\varepsilon \mathcal{A}_\varepsilon \partial_t u_\varepsilon$ and Definition 3.1.10 of $\mathcal{A}_\varepsilon$. Recall that bv (3.1.14) we have for all $\psi, \phi \in H^1(\Omega)'$

$$
\langle \mathcal{A}_\varepsilon \phi, \psi \rangle_{H^1(\Omega)'} = \langle \mathcal{A}_\varepsilon \psi, \phi \rangle_{H^1(\Omega)'}.
$$

The following Theorem is the main result of this section. It provides the first long-term existence result for weak solutions of (6.1.1)-(6.1.4).

Theorem 6.1.2 (Existence and regularity of solutions)**.**
There exists a weak solution u_ε to (6.1.1)-(6.1.4) *with the additional regularity*

$$
\begin{aligned}
u_\varepsilon \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;H^1(\Omega)') \cap C^0([0,T];L^2(\Omega))\\
\cap L^\infty(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^4(\Omega)) \cap L^6(0,T;L^6(\Omega))
\end{aligned}
$$

and

$$
\lim_{t \to 0} u_\varepsilon(t,\cdot) = u_{0,\varepsilon} \quad \text{in} \quad L^2(\Omega).
$$

Additionally, if $n \leq 3$

- $u_\varepsilon \in L^2(0,T;H^3(\Omega)) \cap C^0([0,T];H^1(\Omega))$.
- $\lim_{t \to 0} \mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) = \mathcal{P}_\varepsilon(u_{0,\varepsilon})$.
- *The Cahn-Hilliard energy of the solution as a function in time*

$$
(0,T) \ni t \longmapsto \mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) = \int_\Omega \Big(\frac{\varepsilon}{2} |\nabla u_\varepsilon(t,\cdot)|^2 + \frac{1}{\varepsilon} W(u_\varepsilon(t,\cdot)) \Big) \,\mathrm{d}\mathcal{L}^n
$$

lies in $W^{1,1}(0,T)$.

- *For a.e. $t \in (0,T)$ we have the energy-dissipation*

$$\mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) + \int_0^t \int_\Omega \Big(\frac{1}{\varepsilon}|H_\varepsilon|^2 + \varepsilon\big|\nabla H_\varepsilon\big|^2\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 = \mathcal{P}_\varepsilon(u_{\varepsilon,0}). \tag{6.1.6}$$

Since $\varepsilon > 0$ is fixed in this section we will not always denote if objects are dependent on ε.

The proof of Theorem 6.1.2 consists of several steps and is done throughout the remainder of this section. First we consider a time discretization and prove that solutions to the discretized equation exist. Starting with $u^{(0)} := u_{0,\varepsilon} \in H^1(\Omega) \cap L^4(\Omega)$ from (6.1.2) we obtain time steps iteratively as described in the following lemma.

Lemma 6.1.3 (Existence of minimizers for discretized energy)**.**
Given parameters $\varepsilon, h > 0$, and a function $u^{(k)} \in H^1(\Omega)$ for $k \in \mathbb{N}_0$ there exists a minimizer $u^{(k+1)} \in H^1(\Omega) \cap L^4(\Omega)$ of $\mathcal{E} : H^1(\Omega) \longrightarrow [0,\infty]$,

$$\mathcal{E}(u) := \int_\Omega \Big(\frac{\varepsilon}{2}\big|\nabla u\big|^2 + \frac{1}{\varepsilon}W(u)\Big)\,\mathrm{d}\mathcal{L}^n + \frac{\varepsilon}{2h}\int_\Omega (u - u^{(k)})\mathcal{A}_\varepsilon(u - u^{(k)})\,\mathrm{d}\mathcal{L}^n. \tag{6.1.7}$$

$u^{(k+1)}$ is a weak solution to

$$\begin{aligned} -\varepsilon\mathcal{A}_\varepsilon\Big(\frac{u^{(k+1)} - u^{(k)}}{h}\Big) &= -\varepsilon\Delta u^{(k+1)} + \frac{1}{\varepsilon}W'(u^{(k+1)}) \quad && \text{in} \quad \Omega \\ \partial_\nu u^{(k+1)} &= 0 && \text{on} \quad \partial\Omega. \end{aligned} \tag{6.1.8}$$

Proof. We use the direct method from the calculus of variations. Since all of the terms are non-negative the infimum exists in $\mathbb{R}$. Let $(u_j)_{j\in\mathbb{N}}$ be a minimizing sequence. Since the sequence $(\mathcal{E}(u_j))_{j\in\mathbb{N}}$ is converging there exists $\Lambda > 0$ such that $\sup_j \mathcal{E}(u_j) \le \Lambda$. Since $\varepsilon > 0$ is fixed we immediately get that $(u_j)_{j\in\mathbb{N}}$ is bounded in $H^1(\Omega)$. Since $W(t) = (1-t^2)^2$ there exists $R_0 > 1$ such that $\frac{1}{2}t^4 \le W(t)$ for all $|t| > R_0$. This yields

$$\begin{aligned} \int_\Omega |u_j|^4\,\mathrm{d}\mathcal{L}^n &= \int_{\{|u_j|\le R_0\}} |u_j|^4\,\mathrm{d}\mathcal{L}^n + \int_{\{|u_j|>R_0\}} |u_j|^4\,\mathrm{d}\mathcal{L}^n \\ &\le R_0^4\mathcal{L}^n(\Omega) + 2\int_\Omega W(u_j)\,\mathrm{d}\mathcal{L}^n \le R_0^4\mathcal{L}^n(\Omega) + 2\varepsilon\mathcal{E}(u_j) \le C(\Omega,\Lambda). \end{aligned} \tag{6.1.9}$$

Thus $(u_j)_{j\in\mathbb{N}}$ is bounded in $L^4(\Omega)$. Both $H^1(\Omega)$ and $L^4(\Omega)$ are reflexive thus we can find limit functions $u_* \in H^1(\Omega),\ \tilde{u}_* \in L^4(\Omega)$ such that up to a subsequence we have as $j \to \infty$

$$u_j \xrightarrow{w} u_* \quad \text{in} \quad H^1(\Omega) \quad \text{and} \quad u_j \xrightarrow{w} \tilde{u}_* \quad \text{in} \quad L^4(\Omega).$$

Both of these convergences imply weak convergence in $L^2(\Omega)$ so by uniqueness of weak limits we have $u_* = \tilde{u}_* \in H^1(\Omega) \cap L^4(\Omega)$. The weak lower semi-continuity of the norms (both H^1 and L^4) imply

$$\mathcal{P}_\varepsilon(u_*) \le \liminf_{j\to\infty} \mathcal{P}_\varepsilon(u_j). \tag{6.1.10}$$

To prove that u_* is a minimizer of $\mathcal{E}$ we want to replace $\mathcal{P}_\varepsilon$ with $\mathcal{E}$ in (6.1.10). Therefore we need an analogous estimate for the second integral in (6.1.7). We get this by using that $\mathcal{A}_\varepsilon : L^2(\Omega) \longrightarrow L^2(\Omega)$ is a compact operator, thus it turns weak convergence into strong convergence. Now we have the $L^2(\Omega)$-product of the weakly convergent sequence $(u_j - u^{(k)})_{j\in\mathbb{N}}$ (k is fixed) with the strongly convergent sequence $(\mathcal{A}_\varepsilon(u_j - u^{(k)}))_{j\in\mathbb{N}}$ which gives convergence towards the product of the limits. These arguments combined result in

$$\inf_{H^1(\Omega)} \mathcal{E} \leq \mathcal{E}(u_*) \leq \liminf_{j\to\infty} \mathcal{E}(u_j) = \inf_{H^1(\Omega)} \mathcal{E}.$$

This implies that u_* is a minimizer of $\mathcal{E}$ and we define $u^{(k+1)} := u_* \in H^1(\Omega) \cap L^4(\Omega)$. Now that we have a minimizer we consider the Euler-Lagrange equation. For any $\eta \in H^1(\Omega)$ we have

$$\begin{aligned} 0 &= \partial_\delta \mathcal{E}(u^{(k+1)} + \delta\eta)\Big|_{\delta=0} \\ &= \int_\Omega \Big(\varepsilon \nabla u^{(k+1)} \cdot \nabla \eta + \frac{1}{\varepsilon} W'(u^{(k+1)})\eta\Big) \,\mathrm{d}\mathcal{L}^n + \frac{\varepsilon}{h} \int_\Omega \eta \mathcal{A}_\varepsilon (u^{(k+1)} - u^{(k)}) \,\mathrm{d}\mathcal{L}^n. \end{aligned} \tag{6.1.11}$$

For the last term we used that $\mathcal{A}_\varepsilon$ is self-adjoint in $L^2(\Omega)$. It follows that $u^{(k+1)}$ is a weak solution to (6.1.8). □

We define $H^{(k)} := -\Delta u^{(k)} + \frac{1}{\varepsilon} W'(u^{(k)}) \in H^1(\Omega)'$ for $k \in \mathbb{N}_0$. By (6.1.8) and the properties of $\mathcal{A}_\varepsilon$ it follows that $H^{(k)}$ has better regularity, in fact $H^{(k)} \in H^1(\Omega)$. From $(H^{(k)})_{k\in\mathbb{N}}$ and $(u^{(k)})_{k\in\mathbb{N}}$ we construct approximate solutions defined on Ω_T.

Definition 6.1.4.
Let $h > 0$ and $t \in [0, T)$ then there exists a unique $k \in \mathbb{N}_0$ such that $t \in [hk, h(k+1))$. We define $\overline{u}_h, \overline{H}_h : \Omega_T \longrightarrow \mathbb{R}$ by

$$\overline{u}_h(t,x) := \left\{ \begin{array}{ll} u^{(k)}(x), & \text{if} \quad h(k+1) < T \\ 0 & \text{if} \quad h(k+1) \geq T \end{array} \right\} \quad \text{for} \quad t \in [hk, h(k+1))$$

$$\text{and} \quad \overline{H}_h(t,x) := \left\{ \begin{array}{ll} H^{(k)}(x), & \text{if} \quad h(k+1) < T \\ 0 & \text{if} \quad h(k+1) \geq T \end{array} \right\} \quad \text{for} \quad t \in [hk, h(k+1)).$$

We also define the piecewise affine functions in time $u_h, H_h : \Omega_T \longrightarrow \mathbb{R}$

$$u_h(t,x) = \left(1 - \frac{t-hk}{h}\right) u^{(k)}(x) + \frac{t-hk}{h} u^{(k+1)}(x) \quad \text{for} \quad t \in [hk, h(k+1)), \tag{6.1.12}$$

$$H_h(t,x) = \left(1 - \frac{t-hk}{h}\right) H^{(k)}(x) + \frac{t-hk}{h} H^{(k+1)}(x) \quad \text{for} \quad t \in [hk, h(k+1)). \tag{6.1.13}$$

By definition of u_h we get for every $x \in \Omega$ and $t \in (0, T-h)$

$$\partial_t u_h(t,x) = \partial_t^h \overline{u}_h(t,x) \quad \text{and} \quad \partial_t H_h(t,x) = \partial_t^h \overline{H}_h(t,x),$$

where ∂_t^h is the discrete partial derivative defined by

$$\partial_t^h \phi(t,x) := \frac{\phi(t+h,x) - \phi(t,x)}{h} \quad \text{for any real valued function } \phi \text{ and } h \neq 0.$$

Note that $\overline{H}_h = -\varepsilon \mathcal{A}_\varepsilon \partial_t^h \overline{u}_h = -\varepsilon \Delta \overline{u}_h + \frac{1}{\varepsilon} W'(\overline{u}_h)$ by (6.1.8). Since $u^{(k)}, H^{(k)} \in H^1(\Omega)$ for all $k \in \mathbb{N}$ we have

$$\overline{u}_h \in L^1(0,T;H^1(\Omega)), \qquad u_h \in C^0([0,T];H^1(\Omega)),$$
$$\overline{H}_h \in L^1(0,T;H^1(\Omega)), \quad \text{and} \quad H_h \in C^0([0,T];H^1(\Omega)).$$

Furthermore we obtain that

$$\partial_\nu u_h = 0 = \partial_\nu \overline{u}_h \quad \text{and} \quad \partial_\nu H_h = 0 = \partial_\nu \overline{H}_h \quad \text{on} \quad \partial\Omega$$

because of $\partial_\nu u^{(k)} = 0$ on $\partial\Omega$ for all $k \in \mathbb{N}$ and $\overline{H}_h = -\varepsilon \mathcal{A}_\varepsilon \partial_t^h \overline{u}_h$.

For all $\eta \in H^1(\Omega)$ it follows from (6.1.11) and $\partial_\nu \overline{u}_h = 0$

$$-\int_\Omega \varepsilon \partial_t u_h \mathcal{A}_\varepsilon \eta \, \mathrm{d}\mathcal{L}^n = \int_\Omega \Big(\varepsilon \nabla \overline{u}_h \cdot \nabla \eta + \frac{1}{\varepsilon} W'(\overline{u}_h)\eta\Big) \, \mathrm{d}\mathcal{L}^n = \int_\Omega \eta \overline{H}_h \, \mathrm{d}\mathcal{L}^n. \tag{6.1.14}$$

We want to extend this identity such that it also holds for $\eta \in H^1(\Omega)'$. To prove that let $\eta \in H^1(\Omega)$, we embed η into $H^1(\Omega)'$ by defining

$$\langle \phi, \eta \rangle_{H^1(\Omega)'} := \int_\Omega \phi \eta \, \mathrm{d}\mathcal{L}^n \quad \text{for all } \phi \in H^1(\Omega).$$

In this sense we have that $H^1(\Omega)$ is dense in $H^1(\Omega)'$. Note that with this identification we can not use Riesz' representation Theorem for Hilbert spaces as we used the $L^2(\Omega)$-scalar product instead of the $H^1(\Omega)$-scalar product. Because of this density, $\overline{H}_h \in H^1(\Omega)$, and $\mathcal{A}_\varepsilon : H^1(\Omega)' \longrightarrow H^1(\Omega)$ we get that (6.1.14) can be extended to hold for $\eta \in H^1(\Omega)'$, i.e. we have for all $\eta \in H^1(\Omega)'$

$$-\Big\langle \varepsilon \mathcal{A}_\varepsilon \partial_t u_h, \eta \Big\rangle_{H^1(\Omega)'} = -\int_\Omega \varepsilon \partial_t u_h \mathcal{A}_\varepsilon \eta \, \mathrm{d}\mathcal{L}^n = \langle \overline{H}_h, \eta \rangle_{H^1(\Omega)'}. \tag{6.1.15}$$

Lemma 6.1.5 (Precompactness of $(\overline{u}_h)_{h>0}$)**.**
Assume $(u_h)_{h>0}, (\overline{u}_h)_{h>0}, (\overline{H}_h)_{h>0}$ are the one-parameter families of functions constructed above. Then there exist

$$u_\varepsilon \in L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)') \cap L^\infty(0,T;L^4(\Omega)), \quad H_\varepsilon \in L^2(0,T;H^1(\Omega))$$

such that we have up to a subsequence as $h \to 0$

$$u_h \longrightarrow u_\varepsilon \qquad \text{in} \quad L^3(0,T;L^3(\Omega)) \tag{6.1.16}$$
$$\text{and} \quad \overline{H}_h \xrightarrow{w} H_\varepsilon \qquad \text{in} \quad L^2(0,T;H^1(\Omega)), \tag{6.1.17}$$
$$\text{with} \quad H_\varepsilon = -\varepsilon \Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) \quad \text{in} \quad L^2(0,T;H^1(\Omega)').$$

Proof. Since $u^{(k)}$ is admissible for the minimizing problem for $u^{(k+1)}$ in (6.1.7) we get

$$\mathcal{P}_\varepsilon(u^{(k+1)}) + \frac{\varepsilon}{2h} \int_\Omega (u^{(k+1)} - u^{(k)}) \mathcal{A}_\varepsilon (u^{(k+1)} - u^{(k)}) \, \mathrm{d}\mathcal{L}^n \le \mathcal{P}_\varepsilon(u^{(k)}).$$

With a telescope sum argument and $u^{(0)} = u_{0,\varepsilon}$ we obtain for any $j \in \mathbb{N}$

$$\mathcal{P}_\varepsilon(u^{(j+1)}) + \frac{\varepsilon}{2h} \sum_{k=0}^{j} \int_\Omega (u^{(k+1)} - u^{(k)}) \mathcal{A}_\varepsilon (u^{(k+1)} - u^{(k)}) \, \mathrm{d}\mathcal{L}^n \le \mathcal{P}_\varepsilon(u_{0,\varepsilon}). \tag{6.1.18}$$

This yields bounds independent from j

$$\int_{\Omega} \left|\nabla u^{(j)}\right|^2 \,\mathrm{d}\mathcal{L}^n \leq \frac{2}{\varepsilon}\mathcal{P}_\varepsilon(u_{0,\varepsilon}) \quad \text{and} \quad \int_{\Omega} W(u^{(j)}) \,\mathrm{d}\mathcal{L}^n \leq \varepsilon\mathcal{P}_\varepsilon(u_{0,\varepsilon}).$$

By the same argument as in the proof of Lemma 6.1.3 we can find a bound for $\|u^{(j)}\|_{L^4(\Omega)}$ independent from j

$$\|u^{(j)}\|_{L^4(\Omega)} \leq C(\Omega, u_{0,\varepsilon}, \varepsilon).$$

Thus $(u^{(j)})_{j\in\mathbb{N}}$ is uniformly bounded in $H^1(\Omega) \hookrightarrow H^1(\Omega)'$ and since $\mathcal{A}_\varepsilon$ is continuous we conclude by (6.1.8) that $(H^{(j)})_{j\in\mathbb{N}}$ is uniformly bounded in $H^1(\Omega)$ as well and thus $H_h, \overline{H}_h \in L^\infty(0,T;H^1(\Omega))$.

The uniform bounds for $u^{(j)}$ also yield bounds for $\overline{u}_h$ independent from h, t. We obtain that

$$\|\overline{u}_h\|_{L^\infty(0,T;H^1(\Omega))}, \|\overline{u}_h\|_{L^\infty(0,T;L^4(\Omega))} \leq C(\Omega, T, u_{0,\varepsilon}, \varepsilon). \tag{6.1.19}$$

Thus $\overline{u}_h$ is uniformly bounded in $L^\infty(0,T;H^1(\Omega))$ and in $L^\infty(0,T;L^4(\Omega))$. We observe

$$\|u_h\|^4_{L^\infty(0,T;L^4(\Omega))} = \underset{\substack{t\in[hk,h(k+1))\\ k\in\mathbb{N}}}{\text{ess-sup}} \int_\Omega \left|\left(1 - \frac{t-hk}{h}\right)u^{(k)}(x) + \frac{t-hk}{h}u^{(k+1)}(x)\right|^4 \mathrm{d}x$$

Noting that $t \in [hk, h(k+1))$ is equivalent to $\frac{t-hk}{h} \in [0,1)$ and using the convexity of the function $(0,\infty) \ni r \longmapsto |r|^4$ we get that

$$\begin{aligned}
\|u_h\|^4_{L^\infty(0,T;L^4(\Omega))} &\leq \underset{s\in[0,1)}{\text{ess-sup}} \int_\Omega \left|(1-s)u^{(k)}(x) + su^{(k+1)}(x)\right|^4 \mathrm{d}x \\
&\leq \underset{s\in[0,1)}{\text{ess-sup}} \int_\Omega \left((1-s)\left|u^{(k)}(x)\right|^4 + s\left|u^{(k+1)}(x)\right|^4\right) \mathrm{d}x \\
&= \underset{s\in[0,1)}{\text{ess-sup}} \left((1-s)\|u^{(k)}\|^4_{L^4(\Omega)} + s\|u^{(k+1)}\|^4_{L^4(\Omega)}\right) \leq C(\Omega, u_{0,\varepsilon}, \varepsilon).
\end{aligned}$$

We can use the technique similarly on ∇u_h for a bound in $L^\infty(0,T;L^2(\Omega))$. We conclude that $(u_h)_{h>0}$ is uniformly bounded in $L^\infty(0,T;H^1(\Omega)) \hookrightarrow L^3(0,T;H^1(\Omega))$ and in $L^\infty(0,T;L^4(\Omega)) \hookrightarrow L^4(0,T;L^4(\Omega))$.

Returning to (6.1.18) we rewrite the estimate for $\overline{u}_h$. For $t \in [hj, h(j+1))$ we have $\mathcal{P}_\varepsilon(\overline{u}_h(t,\cdot)) = \mathcal{P}_\varepsilon(u^{(j)})$ and

$$\frac{1}{2}\sum_{k=0}^{j-1} h \int_\Omega \varepsilon \frac{u^{(k+1)} - u^{(k)}}{h} \mathcal{A}_\varepsilon \frac{u^{(k+1)} - u^{(k)}}{h} \,\mathrm{d}\mathcal{L}^n \geq \frac{1}{2}\int_0^{t-h}\int_\Omega \varepsilon \partial_t^h \overline{u}_h \mathcal{A}_\varepsilon \partial_t^h \overline{u}_h \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1.$$

We rewrite (6.1.18) and obtain that for all $t \in [h, T]$

$$\mathcal{P}_\varepsilon(\overline{u}_h(t,\cdot)) + \frac{\varepsilon}{2}\int_0^{t-h}\int_\Omega \partial_t^h \overline{u}_h \mathcal{A}_\varepsilon \partial_t^h \overline{u}_h \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 \leq \mathcal{P}_\varepsilon(u_{0,\varepsilon}). \tag{6.1.20}$$

We take the supremum over t and get

$$\sup_{t\in[0,T]} \mathcal{P}_\varepsilon(u_h(t,\cdot)) + \frac{1}{2}\int_0^{T-h}\int_\Omega \varepsilon\partial_t^h\overline{u}_h\mathcal{A}_\varepsilon\partial_t^h\overline{u}_h \,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \leq 2\mathcal{P}_\varepsilon(u_{0,\varepsilon}). \tag{6.1.21}$$

To improve this estimate we establish a bound for $(\partial_t^h\overline{u}_h)_{h>0}$ in $L^2(0,T;H^1(\Omega)')$. Let $t\in(0,T)$, $\eta\in H^1(\Omega)$, we have $\eta=\mathcal{A}_\varepsilon(-\varepsilon^2\Delta+\mathrm{Id})\eta$ because $(-\varepsilon^2\Delta+\mathrm{Id})\eta\in H^1(\Omega)'$. We estimate

$$\begin{aligned}\int_\Omega \partial_t u_h(t,\cdot)\eta\,\mathrm{d}\mathcal{L}^n &= \int_\Omega \partial_t^h\overline{u}_h(t,\cdot)\mathcal{A}_\varepsilon(-\varepsilon^2\Delta+\mathrm{Id})\eta\,\mathrm{d}\mathcal{L}^n \\ &= \left\langle \mathcal{A}_\varepsilon\partial_t^h\overline{u}_h(t,\cdot),(-\varepsilon^2\Delta+\mathrm{Id})\eta\right\rangle_{H^1(\Omega)'} \\ &= \int_\Omega \left(\varepsilon^2\nabla\mathcal{A}_\varepsilon\partial_t^h\overline{u}_h(t,\cdot)\cdot\nabla\eta+\mathcal{A}_\varepsilon\partial_t^h\overline{u}_h(t,\cdot)\eta\right)\mathrm{d}\mathcal{L}^n.\end{aligned} \tag{6.1.22}$$

Now we apply the Cauchy-Schwarz estimate for the $H^1(\Omega)$-scalar product and obtain that

$$\begin{aligned}\int_\Omega \partial_t u_h(t,\cdot)\eta\,\mathrm{d}\mathcal{L}^n &\leq \Bigg[\int_\Omega \Big[\varepsilon^2\big|\nabla\mathcal{A}_\varepsilon\partial_t^h\overline{u}_h(t,\cdot)\big|^2+\big|\mathcal{A}_\varepsilon\partial_t^h\overline{u}_h(t,\cdot)\big|^2\Big]\,\mathrm{d}\mathcal{L}^n \\ &\qquad\cdot\int_\Omega\Big[\varepsilon^2\big|\nabla\eta\big|^2+|\eta|^2\Big]\,\mathrm{d}\mathcal{L}^n\Bigg]^{\frac{1}{2}} \\ &\leq \left(\int_\Omega \mathcal{A}_\varepsilon\partial_t^h\overline{u}_h(t,\cdot)(-\varepsilon^2\Delta+\mathrm{Id})\mathcal{A}_\varepsilon\partial_t^h\overline{u}_h(t,\cdot)\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{2}}\|\eta\|_{H^1(\Omega)} \\ &\leq \left(\int_\Omega \partial_t^h\overline{u}_h(t,\cdot)\mathcal{A}_\varepsilon\partial_t^h\overline{u}_h(t,\cdot)\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{2}}\|\eta\|_{H^1(\Omega)}.\end{aligned}$$

It follows that the mapping $\eta\longmapsto\int_\Omega\partial_t^h\overline{u}_h\eta\,\mathrm{d}\mathcal{L}^n$ lies in $H^1(\Omega)'$. Slightly abusing notation we call this mapping $\partial_t^h\overline{u}_h$ without denoting the embedding into $H^1(\Omega)'$. By taking the supremum over all $\eta\in H^1(\Omega)$ with $\|\eta\|_{H^1(\Omega)}\leq 1$ we get

$$\|\partial_t^h\overline{u}_h(t,\cdot)\|_{H^1(\Omega)'}\leq\left(\int_\Omega\partial_t^h\overline{u}_h(t,\cdot)\mathcal{A}_\varepsilon\partial_t^h\overline{u}_h(t,\cdot)\,\mathrm{d}\mathcal{L}^n\right)^{\frac{1}{2}}.$$

Squaring and integrating in time results in

$$\|\partial_t^h\overline{u}_h\|^2_{L^2(0,T-h;H^1(\Omega)')}\leq\int_0^{T-h}\int_\Omega\partial_t^h\overline{u}_h\mathcal{A}_\varepsilon\partial_t^h\overline{u}_h\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1\leq\frac{4}{\varepsilon}\mathcal{P}_\varepsilon(u_{0,\varepsilon}).$$

Since $\partial_t^h\overline{u}_h(t,\cdot)=\partial_t u_h(t,\cdot)$ we also get that $\partial_t u_h$ is uniformly bounded in $L^2(0,T;H^1(\Omega)')$.

We have

$$\begin{aligned}(u_h)_{h>0}&\quad\text{is bounded in}\quad L^3(0,T;H^1(\Omega))\cap L^4(0,T;L^4(\Omega)),\\ (\partial_t u_h)_{h>0}&\quad\text{is bounded in}\quad L^2(0,T;H^1(\Omega)'),\quad\text{and}\\ H^1(\Omega)\cap L^4(\Omega)&\overset{c}{\hookrightarrow}L^3(\Omega)\hookrightarrow H^1(\Omega)'.\end{aligned}$$

We apply a generalized Aubin-Lion-Dubinskii's Theorem; see Lemma 7.7 in [Rou05]. It follows that there exists a limit function $u_\varepsilon \in L^3(0,T;L^3(\Omega))$ such that up to a subsequence we have

$$u_h \longrightarrow u_\varepsilon \quad \text{in} \quad L^3(0,T;L^3(\Omega)). \tag{6.1.23}$$

Lemma 11.3 from [Sch13] implies $\overline{u}_h \longrightarrow u_\varepsilon$ in $L^2(0,T;L^2(\Omega))$.

We proved that $(\overline{u}_h)_{h>0}$ is bounded in $L^\infty(0,T;L^4(\Omega))$, $L^\infty(0,T;H^1(\Omega))$, and we have a bound for $(\partial_t u_h)_{h>0}$ in $L^2(0,T;H^1(\Omega)')$. Since $L^1(0,T;L^{\frac{4}{3}}(\Omega))$, $L^1(0,T;H^1(\Omega)')$, and $L^2(0,T;H^1(\Omega))$ are separable and

$$\begin{aligned} &L^\infty(0,T;L^4(\Omega)) \cong L^1(0,T;L^{\frac{4}{3}}(\Omega))', \quad L^\infty(0,T;H^1(\Omega)) \cong L^1(0,T;H^1(\Omega)')', \\ \text{and} \quad &L^2(0,T;H^1(\Omega)') \cong L^2(0,T;H^1(\Omega))' \end{aligned}$$

there exists a subsequence $h \to 0$ such that

$$\overline{u}_h \xrightarrow{w^*} u_\varepsilon \quad \text{in} \quad L^1(0,T;L^{\frac{4}{3}}(\Omega))' \cong L^\infty(0,T;L^4(\Omega)).$$

The limit function is u_ε because $L^1(0,T;L^{\frac{4}{3}}(\Omega))' \hookrightarrow L^2(0,T;L^2(\Omega))'$ and in the latter space weak*-convergence is equivalent to w-convergence of the embedded objects in $L^2(0,T;L^2(\Omega))$. We proceed similar for $L^\infty(0,T;H^1(\Omega))$ and conclude

$$\overline{u}_h \xrightarrow{w^*} u_\varepsilon \quad \text{in} \quad L^1(0,T;H^1(\Omega)')' \cong L^\infty(0,T;H^1(\Omega)).$$

Next we show $\partial_t u_\varepsilon \in L^2(0,T;H^1(\Omega)')$. We know that $\partial_t u_h$ has a weak*-cluster point v_ε in $L^2(0,T;H^1(\Omega)')$ by the previous argument. On the other hand we have $u_h \longrightarrow u_\varepsilon$ in $L^2(0,T;L^2(\Omega))$ and thus $\partial_t u_h \longrightarrow \partial_t u_\varepsilon$ in $H^{-1}(0,T;L^2(\Omega))$. Both imply weak*-convergence in $H^{-1}(0,T;H^1(\Omega)')$ and thus we conclude by uniqueness of weak*-limits $\partial_t u_\varepsilon = v_\varepsilon \in L^2(0,T;H^1(\Omega)')$ and thus $u_\varepsilon \in H^1(0,T;H^1(\Omega)')$.

Since $\overline{u}_h \longrightarrow u_\varepsilon$ in $L^2(0,T;L^2(\Omega))$ we also have $\overline{u}_h^2 \longrightarrow u_\varepsilon^2$ in $L^1(0,T;L^1(\Omega))$. Furthermore $(\overline{u}_h^2)_{h>0}$ is bounded in $L^2(0,T;L^2(\Omega))$ because of the bound in $L^\infty(0,T;L^4(\Omega))$. Thus there exists a subsequence and a limit function $w \in L^2(0,T;L^2(\Omega))$ such that $\overline{u}_h^2 \xrightarrow{w} w$ in $L^2(0,T;L^2(\Omega)) \hookrightarrow L^1(0,T;L^1(\Omega))$. By uniqueness of weak limits in $L^1(0,T;L^1(\Omega))$ we conclude $w = u_\varepsilon^2$.

Lastly we consider $\overline{H}_h$. We rewrite (6.1.8) and get for a.e. $t \in (h,T)$ that

$$-\varepsilon \mathcal{A}_\varepsilon \partial_t u_h(t,\cdot) = \overline{H}_h(t,\cdot).$$

This identity holds in $H^1(\Omega)$. We apply $\partial_t u_h(t,\cdot) \in H^1(\Omega)'$ to both sides plug both sides and get

$$\begin{aligned} \Big\langle \varepsilon \mathcal{A}_\varepsilon \partial_t u_h(t,\cdot), \partial_t u_h(t,\cdot) \Big\rangle_{H^1(\Omega)'} &= -\Big\langle \overline{H}_h(t,\cdot), \partial_t u_h(t,\cdot) \Big\rangle_{H^1(\Omega)'} \\ &= -\Big\langle \mathcal{A}_\varepsilon(-\varepsilon^2\Delta + \mathrm{Id})\overline{H}_h(t,\cdot), \partial_t u_h(t,\cdot) \Big\rangle_{H^1(\Omega)'}. \end{aligned}$$

Since $\overline{H}_h(t,\cdot) \in H^1(\Omega)$ we have $(-\varepsilon^2\Delta + \mathrm{Id})\overline{H}_h(t,\cdot) \in H^1(\Omega)'$ and thus we get with (6.1.15)

$$\begin{aligned}\Big\langle -\mathcal{A}_\varepsilon\partial_t u_h(t,\cdot), (-\varepsilon^2\Delta + \mathrm{Id})\overline{H}_h(t,\cdot)\Big\rangle_{H^1(\Omega)'} &= \frac{1}{\varepsilon}\Big\langle \overline{H}_h(t,\cdot), (-\varepsilon^2\Delta + \mathrm{Id})\overline{H}_h(t,\cdot)\Big\rangle_{H^1(\Omega)'} \\ &= \int_\Omega \frac{1}{\varepsilon}\Big(|\overline{H}_h(t,\cdot)|^2 + \varepsilon^2|\nabla\overline{H}_h(t,\cdot)|^2\Big)\,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

We plug this into (6.1.20) and get for all $t \in (h,T]$

$$\mathcal{P}_\varepsilon(u_h(t,\cdot)) + \frac{1}{2}\int_0^{t-h}\int_\Omega \frac{1}{\varepsilon}\Big(|\overline{H}_h|^2 + \varepsilon^2|\nabla\overline{H}_h|^2\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \le \mathcal{P}_\varepsilon(u_{0,\varepsilon}). \tag{6.1.24}$$

Since ε is fixed choosing $t = T$ implies that $(\overline{H}_h)_{h>0}$ is uniformly bounded in $L^2(0,T;H^1(\Omega))$. Thus there exists a limit function $H_\varepsilon \in L^2(0,T;H^1(\Omega))$ such that up to a subsequence we have as $h \to 0$

$$\overline{H}_h \xrightarrow{w} H_\varepsilon \quad \text{in} \quad L^2(0,T;H^1(\Omega)).$$

We can apply the limit $h \to 0$ to (6.1.24), use $\overline{u}_h \xrightarrow{w^*} u_\varepsilon$ in $L^\infty(0,T;H^1(\Omega))$ and in $L^\infty(0,T;L^4(\Omega))$, $\overline{H}_h \xrightarrow{w} H_\varepsilon$ in $L^2(0,T;H^1(\Omega))$, and the weakly and weakly* lower semi-continuity of the norm to obtain for a.e. $t \in (0,T)$

$$\mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) + \frac{1}{2}\int_0^{t}\int_\Omega \frac{1}{\varepsilon}\Big(|H_\varepsilon|^2 + \varepsilon^2|\nabla H_\varepsilon|^2\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \le \mathcal{P}_\varepsilon(u_{0,\varepsilon}). \tag{6.1.25}$$

We can identify H_ε by combining previous convergences. Let $\eta \in C_c^1(\Omega_T)$ then we have by $\overline{H}_h \xrightarrow{w} H_\varepsilon$ in $L^2(0,T;L^2(\Omega))$

$$\begin{aligned}\int_0^T\int_\Omega H_\varepsilon\eta\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \xleftarrow{h\to 0} &\int_h^T\int_\Omega \overline{H}_h\eta\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \\ &= \int_h^T\int_\Omega \Big(\varepsilon\nabla\overline{u}_h\cdot\nabla\eta + \frac{1}{\varepsilon}W'(\overline{u}_h)\eta\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \\ &= \int_h^T\int_\Omega \Big(\varepsilon\nabla\overline{u}_h\cdot\nabla\eta - \frac{4}{\varepsilon}(\overline{u}_h - \overline{u}_h^3)\eta\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1.\end{aligned}$$

Since η is bounded we get $\eta\overline{u}_h \longrightarrow \eta u_\varepsilon$ in $L^2(0,T;L^2(\Omega))$. Furthermore we have $\overline{u}_h^2 \xrightarrow{w} u_\varepsilon^2$ in $L^2(0,T;L^2(\Omega))$. Combining these we get the convergence of $\langle \overline{u}_h^2|\eta\overline{u}_h\rangle_{L^2(0,T;L^2(\Omega))}$. Using this and $\overline{u}_h \xrightarrow{w} u_\varepsilon$ in $L^2(0,T;H^1(\Omega))$ we get

$$\begin{aligned}&\int_h^T\int_\Omega \Big(\varepsilon\nabla\overline{u}_h\cdot\nabla\eta - \frac{4}{\varepsilon}(\overline{u}_h - \overline{u}_h^3)\eta\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \\ &\qquad\qquad \xrightarrow{h\to 0} \int_0^T\int_\Omega \Big(\varepsilon\nabla u_\varepsilon\cdot\nabla\eta - \frac{4}{\varepsilon}(u_\varepsilon - u_\varepsilon^3)\eta\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \\ &\qquad\qquad = \Big\langle \eta, -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big\rangle_{L^2(0,T;H^1(\Omega))'}.\end{aligned}$$

Since $C_c^1(\Omega_T)$ is dense in $L^2(0,T;H^1(\Omega))$ we conclude $H_\varepsilon = -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)$ in $L^2(0,T;H^1(\Omega)')$. $\square$

At last we can prove Theorem 6.1.2.

Proof of Theorem 6.1.2. We start by showing that the limit function u_ε constructed in Lemma 6.1.5 is a weak solution to (6.1.1)-(6.1.4). Take any $\phi \in C_c^\infty([0,T)\times\Omega)$. For fixed $t \in [0,T)$ we apply (6.1.14) and get

$$\begin{aligned}
-\int_\Omega \varepsilon\partial_t u_h \mathcal{A}_\varepsilon \phi(t,\cdot)\,\mathrm{d}\mathcal{L}^n &= \int_\Omega \Big(\varepsilon\nabla\overline{u}_h\cdot\nabla\phi(t,\cdot) + \frac{1}{\varepsilon}W'(\overline{u}_h)\phi(t,\cdot)\Big)\,\mathrm{d}\mathcal{L}^n \\
&= \int_\Omega \phi(t,\cdot)\overline{H}_h(t,\cdot)\,\mathrm{d}\mathcal{L}^n.
\end{aligned}$$

We integrate in time and use Fubini's Theorem and a partial integration in time on the left-hand side

$$\begin{aligned}
-\int_\Omega\int_0^T \varepsilon\partial_t u_h\mathcal{A}_\varepsilon\phi\,\mathrm{d}\mathcal{L}^1\,\mathrm{d}\mathcal{L}^n &= -\int_\Omega \varepsilon u_h\mathcal{A}_\varepsilon\phi\Big|_0^T\,\mathrm{d}\mathcal{L}^n + \int_\Omega\int_0^T \varepsilon u_h\partial_t\mathcal{A}_\varepsilon\phi\,\mathrm{d}\mathcal{L}^1\,\mathrm{d}\mathcal{L}^n \\
&= \int_\Omega \varepsilon\phi(0,\cdot)\mathcal{A}_\varepsilon u_{0,\varepsilon}\,\mathrm{d}\mathcal{L}^n + \int_0^T\int_\Omega \varepsilon u_h\partial_t\mathcal{A}_\varepsilon\phi\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1.
\end{aligned}$$

Thus we have

$$\int_\Omega \varepsilon\phi(0,\cdot)\mathcal{A}_\varepsilon u_{0,\varepsilon}\,\mathrm{d}\mathcal{L}^n + \int_0^T\int_\Omega \varepsilon u_h\partial_t\mathcal{A}_\varepsilon\phi\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 = \int_0^T\int_\Omega \phi\overline{H}_h\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1.$$

Now we consider the limit $h\to 0$. We apply (6.1.16) on the second term on the left-hand side and (6.1.17) on the right-hand side. We get

$$\begin{aligned}
&\int_\Omega \varepsilon\phi(0,\cdot)\mathcal{A}_\varepsilon u_{0,\varepsilon}\,\mathrm{d}\mathcal{L}^n + \int_0^T\int_\Omega \varepsilon\mathcal{A}_\varepsilon u_\varepsilon\partial_t\phi\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \\
&\qquad = \int_0^T\int_\Omega \Big(\varepsilon\nabla u_\varepsilon\cdot\nabla\phi + \frac{1}{\varepsilon}W'(u_\varepsilon)\phi\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1
\end{aligned}$$

which is the weak formulation introduced in (6.1.5). Next we prove the higher regularity. We have $H_\varepsilon \in L^2(0,T;L^2(\Omega))$ and thus

$$\begin{aligned}
\|H_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T\int_\Omega \Big|-\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)\Big|^2\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \qquad (6.1.26)\\
&= \int_0^T\int_\Omega \Big(|\varepsilon\Delta u_\varepsilon|^2 - 2\Delta u_\varepsilon W'(u_\varepsilon) + \frac{1}{\varepsilon^2}|W'(u_\varepsilon)|^2\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \\
&= \int_0^T\int_\Omega \Big(|\varepsilon\Delta u_\varepsilon|^2 + 2|\nabla u_\varepsilon|^2 W''(u_\varepsilon) + \frac{1}{\varepsilon^2}|W'(u_\varepsilon)|^2\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1.
\end{aligned}$$

In the last step we applied Theorem 8.3.6. The boundary integral vanishes because of the Neumann boundary conditions of u_ε. Using $W''(r) = 12r^2 - 4$ for $r\in\mathbb{R}$ we get the estimate

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(\varepsilon^2|\Delta u_\varepsilon|^2 + 24|u_\varepsilon|^2|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^2}|W'(u_\varepsilon)|^2\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \\
&\qquad \le \|H_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} + 8\|u_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))} \qquad (6.1.27)
\end{aligned}$$

and conclude $\Delta u_\varepsilon \in L^2(0,T;L^2(\Omega))$. Since $\partial\Omega$ is C^3 we conclude by elliptic regularity theory that $u_\varepsilon \in L^2(0,T;H^2(\Omega))$. Furthermore we conclude $u_\varepsilon \in L^6(0,T;L^6(\Omega))$ because W' is a polynomial of degree three. In fact we can use a similar argument as in the proof of Lemma 6.1.3, where we extracted a uniform bound for $(u_j)_{j\in\mathbb{N}}$ in $L^4(\Omega)$ from a bound on $(W(u_j))_{j\in\mathbb{N}}$ in $L^1(\Omega)$.

We have $u_\varepsilon \in L^2(0,T;H^1(\Omega))$ and $\partial_t u_\varepsilon \in L^2(0,T;H^1(\Omega)')$. By Lemma 7.3 from [Rou05] we conclude $u_\varepsilon \in C^0([0,T];L^2(\Omega))$ and thus the $\lim_{t\to 0} u_\varepsilon(t,\cdot)$ exists in $L^2(\Omega)$. From the initial conditions of the PDE We conclude $\lim_{t\to 0} u_\varepsilon(t,\cdot) = u_{0,\varepsilon}$ in $L^2(\Omega)$. It remains to prove the additional claims for $n \le 3$.

If $n \le 3$ we prove $\nabla[W'(u_\varepsilon)] \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$. With $W''(r) = 12r^2 - 4$ and thus $|W''(r)|^2 \le 288r^4 + 32$ in mind we estimate

$$\begin{aligned}\int_0^T \int_\Omega \left|\nabla W'(u_\varepsilon)\right|^2 \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 &= \int_0^T \int_\Omega |W''(u_\varepsilon)|^2 \left|\nabla u_\varepsilon\right|^2 \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 \\ &\le 32\|u_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))} + 288 \int_0^T \int_\Omega |u_\varepsilon|^4 \left|\nabla u_\varepsilon\right|^2 \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1.\end{aligned}$$

It is sufficient to control the last integral. Using a Hölder-estimate and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ we continue

$$\begin{aligned}\int_0^T \int_\Omega |u_\varepsilon|^4 \left|\nabla u_\varepsilon\right|^2 \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 &\le \int_0^T \left[\int_\Omega |u_\varepsilon|^6 \,\mathrm{d}\mathcal{L}^n\right]^{\frac{2}{3}} \left[\int_\Omega \left|\nabla u_\varepsilon\right|^6 \,\mathrm{d}\mathcal{L}^n\right]^{\frac{1}{3}} \mathrm{d}\mathcal{L}^1 \qquad (6.1.28)\\ &\le \int_0^T \|u_\varepsilon\|^4_{L^6(\Omega)} \|\nabla u_\varepsilon\|^2_{L^6(\Omega;\mathbb{R}^n)} \,\mathrm{d}\mathcal{L}^1 \\ &\le C(\Omega) \int_0^T \|u_\varepsilon\|^4_{H^1(\Omega)} \|u_\varepsilon\|^2_{H^2(\Omega)} \,\mathrm{d}\mathcal{L}^1 \\ &\le C(\Omega) \|u_\varepsilon\|^4_{L^\infty(0,T;H^1(\Omega))} \int_0^T \|u_\varepsilon\|^2_{H^2(\Omega)} \,\mathrm{d}\mathcal{L}^1 \\ &\le C(\Omega) \|u_\varepsilon\|^4_{L^\infty(0,T;H^1(\Omega))} \|u_\varepsilon\|^2_{L^2(0,T;H^2(\Omega))}.\end{aligned}$$

Since the right-hand side is finite it follows

$$\nabla W'(u_\varepsilon) \in L^2(0,T;L^2(\Omega;\mathbb{R}^n)) \quad \text{and thus} \quad W'(u_\varepsilon) \in L^2(0,T;H^1(\Omega)). \qquad (6.1.29)$$

Finally we have

$$-\varepsilon\Delta u_\varepsilon = H_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \in L^2(0,T;H^1(\Omega))$$

and thus $u_\varepsilon \in L^2(0,T;H^3(\Omega))$ by elliptic regularity theory. Furthermore we get with Lemma 7.3 from [Rou05] that $u_\varepsilon \in C^0([0,T];H^1(\Omega))$ because we have the regularity $u_\varepsilon \in L^2(0,T;H^3(\Omega))$, $\partial_t u_\varepsilon \in L^2(0,T;H^1(\Omega)')$, and $H^3(\Omega) \overset{c}{\hookrightarrow} H^1(\Omega) \hookrightarrow H^1(\Omega)' \hookrightarrow H^3(\Omega)'$.

It also follows $\lim_{t\to 0} u_\varepsilon(t,\cdot) = u_{0,\varepsilon}$ in $H^1(\Omega) \hookrightarrow L^4(\Omega)$. We conclude

$$\lim_{t\to 0} \mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) = \mathcal{P}_\varepsilon(u_{0,\varepsilon})$$

because the integrand in the definition of $\mathcal{P}_\varepsilon$ is controlled by the norms in $H^1(\Omega)$ and $L^4(\Omega)$.

We have $\mathcal{P}_\varepsilon(u_\varepsilon) \in L^1(0,T)$ because of $u_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap C^0(\overline{\Omega_T})$. The next step is to show the same for the weak derivative. Let $\psi \in C_c^1(0,T)$, we use Fubini's Theorem and partial integrations (in space and time)

$$
\begin{aligned}
-\int_0^T \mathcal{P}_\varepsilon(u_\varepsilon)\psi' \,\mathrm{d}\mathcal{L}^1 &= -\int_\Omega\int_0^T \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\Big)\psi' \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{L}^n \\
&= \int_0^T \psi(t)\Big(\varepsilon\Big\langle \nabla u_\varepsilon(t,\cdot), \nabla\partial_t u_\varepsilon(t,\cdot)\Big\rangle_{H^2(\Omega)'} \\
&\quad + \frac{1}{\varepsilon}\Big\langle W'(u_\varepsilon(t,\cdot)), \partial_t u_\varepsilon(t,\cdot)\Big\rangle_{H^1(\Omega)'}\Big)\,\mathrm{d}t \\
&= \int_0^T \psi(t)\Big\langle H_\varepsilon(t,\cdot), \partial_t u_\varepsilon(t,\cdot)\Big\rangle_{H^1(\Omega)'}\,\mathrm{d}t \\
&= \int_0^T \psi(t)\Big\langle \mathcal{A}_\varepsilon\partial_t u_\varepsilon(t,\cdot), (-\varepsilon^2\Delta + \mathrm{Id})H_\varepsilon(t,\cdot),\Big\rangle_{H^1(\Omega)'}\,\mathrm{d}t.
\end{aligned}
$$

We apply (6.1.1) and get

$$
\begin{aligned}
\int_0^T \psi(t)\Big\langle \mathcal{A}_\varepsilon\partial_t u_\varepsilon(t,\cdot), (-\varepsilon^2\Delta + \mathrm{Id})H_\varepsilon(t,\cdot)\Big\rangle_{H^1(\Omega)'}\,\mathrm{d}t \\
= -\int_0^T \psi(t)\int_\Omega \frac{1}{\varepsilon}\Big\langle H_\varepsilon, (-\varepsilon^2\Delta + \mathrm{Id})H_\varepsilon(t,\cdot),\Big\rangle_{H^1(\Omega)'}\,\mathrm{d}t \\
= -\int_0^T \psi\int_\Omega \Big(\frac{1}{\varepsilon}|H_\varepsilon|^2 + \varepsilon|\nabla H_\varepsilon|^2\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1.
\end{aligned}
$$

Since $H_\varepsilon \in L^2(0,T;H^1(\Omega))$ we conclude

$$
\partial_t\mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) = -\mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon(t,\cdot)) \in L^1(0,T)
$$

and thus $\mathcal{P}_\varepsilon(u_\varepsilon) \in W^{1,1}(0,T)$. Since $u_\varepsilon \in C^0([0,T];H^1(\Omega))$ we also have $\mathcal{P}_\varepsilon(u_\varepsilon) \in C^0[0,T]$. From Theorem 2.2.8 and the continuity of $\mathcal{P}_\varepsilon(u_\varepsilon)$ at $t=0$ we deduce that for a.e. $t \in (0,T)$

$$
\mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) + \int_0^t \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon(s,\cdot))\,\mathrm{d}s = \mathcal{P}_\varepsilon(u_{0,\varepsilon}),
$$

which concludes the proof. □

6.2 Construction of convergent subsequences as $\varepsilon \to 0$

In the last section $\varepsilon > 0$ was fixed and thus no assumption on the initial data with respect to ε was necessary. In this section we establish compactness results for the solutions and the induced measures as $\varepsilon \to 0$. To achieve this we need additional assumptions.

Assumption 6.2.1 (Well-prepared initial data)**.**
We use Notations 6.1.1. Let $n \le 3$, $u_{0,\varepsilon} \in H^1(\Omega) \cap L^4(\Omega)$, and assume there exist $\Lambda > 0$ and $\mu^0 \in C_0^0(\Omega)'$ such that for $\mu_\varepsilon^0 := \Big(\frac{\varepsilon}{2}|\nabla u_{0,\varepsilon}|^2 + \frac{1}{\varepsilon}W(u_{0,\varepsilon})\Big)\mathcal{L}^n \llcorner \Omega \in C_0^0(\Omega)'$ we have

$$\sup_{\varepsilon>0} \mu_\varepsilon^0(\Omega) \leq \frac{\Lambda}{2}, \tag{6.2.1}$$
$$\lim_{\varepsilon\to 0} \mu_\varepsilon^0(\Omega) = \mu^0(\Omega), \tag{6.2.2}$$
$$\text{and} \quad \mu_\varepsilon^0 \xrightarrow{w^*} \mu^0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad C_0^0(\Omega)'.$$

Let $u_\varepsilon \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^1(\Omega)') \cap L^3(0,T;L^3(\Omega))$ *be a weak solution to* (6.1.1)-(6.1.4) *such that* (6.1.6) *holds for a.e.* $t \in (0,T)$. *Furthermore we consider a sequence of positive numbers with* $\varepsilon \to 0$.

Remark.

- *Note* $\mu_\varepsilon^0(\Omega) = \mathcal{P}_\varepsilon(u_{0,\varepsilon})$.
- *The energy-dissipation* (6.1.6) *immediately implies*

$$u_\varepsilon \in L^\infty(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^4(\Omega)) \quad \text{and} \quad H_\varepsilon \in L^2(0,T;H^1(\Omega))$$

  with $H_\varepsilon := -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon}W'(u_\varepsilon)$ *as before.*
- *With the same calculation as in* (6.1.26) *and* (6.1.27) *we even get*

$$u_\varepsilon \in L^2(0,T;H^3(\Omega)) \cap C^0([0,T];H^1(\Omega)).$$

Corollary 6.2.2 (A priori bounds from the energy-dissipation)**.**
Let Assumptions 6.2.1 hold. Then we have

$$\operatorname*{ess\text{-}sup}_{t\in[0,T)} \mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) + \int_0^T \int_\Omega \Big(\frac{1}{\varepsilon}|H_\varepsilon|^2 + \varepsilon\big|\nabla H_\varepsilon\big|^2\Big) \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 \leq 2\mathcal{P}_\varepsilon(u_{0,\varepsilon}) \leq \Lambda. \tag{6.2.3}$$

Analogously to the proof of Theorem 6.1.2, higher regularity implies that

$$\lim_{t\to 0} u_\varepsilon(t,\cdot) = u_{0,\varepsilon} \quad \text{in} \quad H^1(\Omega),\ L^4(\Omega), \quad \text{and a.e. in } \Omega.$$

Proof. Let $\varepsilon > 0$ be arbitrary. The energy-dissipation follows from (6.1.6) and (6.2.1). The convergence of $u_\varepsilon(t,\cdot)$ as $t \to 0$ follows from $u_\varepsilon \in C^0([0,T];H^1(\Omega))$. This also implies the convergence a.e. in Ω for a subsequence. To prove that this holds for the entire sequence $t \to 0$ we assume that there exist $\varepsilon, \tau > 0$, a subsequence $(t_l)_{l\in\mathbb{N}}$, and $A \subseteq \Omega$ with $\mathcal{L}^n(A) > 0$ such that for all $x \in A$

$$|u_\varepsilon(t_l,x) - u_{0,\varepsilon}(x)| > \tau \quad \text{for all } l \in \mathbb{N}. \tag{6.2.4}$$

However we have $u_\varepsilon(t,\cdot) \longrightarrow u_{0,\varepsilon}$ in $L^4(\Omega)$ and thus there exists a subsequence $(t_{l_m})_{m\in\mathbb{N}}$ such that $u_\varepsilon(t_{l_m},\cdot) \longrightarrow u_{0,\varepsilon}$ as $m \to \infty$ a.e. in Ω, which is a contradiction to (6.2.4). □

We introduce the measures

$$\mu_\varepsilon^t := \Big(\frac{\varepsilon}{2}\big|\nabla u_\varepsilon(t,\cdot)\big|^2 + \frac{1}{\varepsilon}W(u_\varepsilon(t,\cdot))\Big)\mathcal{L}^n \llcorner \Omega \quad \text{for } t \in [0,T),$$
$$\xi_\varepsilon^t := \Big(\frac{\varepsilon}{2}\big|\nabla u_\varepsilon(t,\cdot)\big|^2 - \frac{1}{\varepsilon}W(u_\varepsilon(t,\cdot))\Big)\mathcal{L}^n \llcorner \Omega \quad \text{for } t \in [0,T),$$

$$\begin{aligned}
&\alpha_\varepsilon^t \coloneqq \frac{1}{\varepsilon}|H_\varepsilon(t,\cdot)|^2\mathcal{L}^n \llcorner \Omega \quad \text{for } t \in (0,T),\\
&\kappa_\varepsilon^t \coloneqq \Big(\frac{1}{\varepsilon}|H_\varepsilon(t,\cdot)|^2 + \varepsilon\big|\nabla H_\varepsilon(t,\cdot)\big|^2\Big)\mathcal{L}^n \llcorner \Omega \quad \text{for } t \in (0,T),\\
&\mu_\varepsilon \coloneqq \mathcal{L}^1 \otimes (\mu_\varepsilon^t)_{t\in[0,T)}, \quad \xi_\varepsilon \coloneqq \mathcal{L}^1 \otimes (\xi_\varepsilon^t)_{t\in[0,T)},\\
&\alpha_\varepsilon \coloneqq \mathcal{L}^1 \otimes (\alpha_\varepsilon^t)_{t\in(0,T)}, \quad \text{and} \quad \kappa_\varepsilon \coloneqq \mathcal{L}^1 \otimes (\kappa_\varepsilon^t)_{t\in(0,T)}.
\end{aligned}$$

The a priori estimate (6.2.3) is an excellent basis for compactness results.

Lemma 6.2.3.
Let Assumptions 6.2.1 hold. Then we have

$$\mu_\varepsilon(\Omega_T) + \kappa_\varepsilon(\Omega_T) \le C(\Lambda, T) \tag{6.2.5}$$

and for a.e. $t \in (0,T)$

$$\sup_{\varepsilon>0} \mu_\varepsilon^t(\Omega) + \sup_{\varepsilon>0} \kappa_\varepsilon(\Omega_t) \le C(\Lambda, T) \tag{6.2.6}$$

$$\liminf_{\varepsilon\to 0}\, [\mu_\varepsilon^t(\Omega) + \kappa_\varepsilon^t(\Omega)] < \infty. \tag{6.2.7}$$

Proof. From (6.2.3) we immediately get good bounds for μ_ε and κ_ε. Let $\varepsilon > 0$ then we have for a.e. $t \in (0,T)$

$$\mu_\varepsilon^t(\Omega) + \kappa_\varepsilon(\Omega_t) \le \operatorname*{ess\text{-}sup}_{(0,T)} \mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) + \kappa_\varepsilon(\Omega_t) \le \Lambda$$

$$\mu_\varepsilon(\Omega_T) + \kappa_\varepsilon(\Omega_T) \le T \operatorname*{ess\text{-}sup}_{(0,T)} \mathcal{P}_\varepsilon(u_\varepsilon) + \int_0^T \mathcal{W}_\varepsilon^{\mathrm{KK}}(u_\varepsilon)\, \mathrm{d}\mathcal{L}^1 \le \Lambda(T+1).$$

Thus (6.2.5) and (6.2.6) are proven. Using Fatou's Lemma we deduce that

$$\int_0^T \liminf_{\varepsilon\to 0} \kappa_\varepsilon^t(\Omega)\, \mathrm{d}t \le \liminf_{\varepsilon\to 0} \int_0^T \kappa_\varepsilon^t(\Omega)\, \mathrm{d}t \overset{(6.2.3)}{\le} \Lambda.$$

we conclude that for a.e. $t \in (0,T)$ we have

$$\liminf_{\varepsilon\to 0} \kappa_\varepsilon^t(\Omega) < \infty.$$

Thus we get from (6.2.3) for a.e. $t \in (0,T)$

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\, [\mu_\varepsilon^t(\Omega) + \kappa_\varepsilon^t(\Omega)] &\le \limsup_{\varepsilon\to 0} \mu_\varepsilon^t(\Omega) + \liminf_{\varepsilon\to 0} \kappa_\varepsilon^t(\Omega)\\
&\le \sup_{\varepsilon>0} \mathcal{P}_\varepsilon(u_0) + \liminf_{\varepsilon\to 0} \kappa_\varepsilon^t(\Omega) < \infty
\end{aligned}$$

and (6.2.7) is proven. □

It follows that for a.e. $t \in (0,T)$ there exists a t-dependent subsequence $\varepsilon \to 0$ such that the assumptions of the Γ–liminf estimate in Theorem 5.2.5 are satisfied. However the subsequences chosen in the proof of Theorem 5.2.5 will depend on t. Next we will deduce suitable uniform a priori bounds for $(u_\varepsilon)_{\varepsilon>0}$.

Lemma 6.2.4.
Let Assumptions 6.2.1 hold. Then

(*i*) $(u_\varepsilon)_{\varepsilon>0}$ *is bounded in* $L^\infty(0,T;L^4(\Omega))$.

(*ii*) $(\varepsilon^{-\frac{1}{2}}W'(u_\varepsilon))_{\varepsilon>0}$ *is bounded in* $L^2(0,T;L^2(\Omega))$.

(*iii*) $(\varepsilon^{\frac{1}{2}}u_\varepsilon\nabla u_\varepsilon)_{\varepsilon>0}$ *is bounded in* $L^2(0,T;L^2(\Omega))$.

(*iv*) $(\varepsilon^{\frac{3}{2}}\Delta u_\varepsilon)_{\varepsilon>0}$ *is bounded in* $L^2(0,T;L^2(\Omega))$.

(*v*) $(u_\varepsilon)_{\varepsilon>0}$ *is bounded in* $L^6(0,T;L^6(\Omega))$.

(*vi*) *For all* $\phi\in L^2(0,T;H^1(\Omega))$ *we have*

$$\left|\int_0^T\left\langle\phi,\sqrt{\varepsilon}\partial_t u_\varepsilon\right\rangle_{H^1(\Omega)'}\,\mathrm{d}\mathcal{L}^1\right|\le\sqrt{\Lambda}\left(\int_{\Omega_T}\left(\varepsilon^2|\nabla\phi|^2+|\phi|^2\right)\mathrm{d}\mathcal{L}^{n+1}\right)^{\frac{1}{2}}. \tag{6.2.8}$$

Proof. From (6.2.3) we get

$$\operatorname*{ess-sup}_{[0,T)}\int_\Omega\frac{1}{\varepsilon}W(u_\varepsilon)\,\mathrm{d}\mathcal{L}^n\le\Lambda.$$

With the notations from the proof of Lemma 6.1.3 we have

$$\begin{aligned}\operatorname*{ess-sup}_{[0,T)}\int_\Omega|u_\varepsilon|^4\,\mathrm{d}\mathcal{L}^n&=\operatorname*{ess-sup}_{[0,T)}\int_{\{|u_\varepsilon|\le R_0\}}|u_\varepsilon|^4\,\mathrm{d}\mathcal{L}^n+\operatorname*{ess-sup}_{[0,T)}\int_{\{|u_\varepsilon|>R_0\}}|u_\varepsilon|^4\,\mathrm{d}\mathcal{L}^n\\&\le R_0^4\mathcal{L}^n(\Omega)+2\varepsilon\Lambda\le C(\Omega,\Lambda).\end{aligned}$$

Since $\sqrt[4]{r}\le r$ for all $r\in[1,\infty)$ we get $\sqrt[4]{r}\le 1+r$ for all $r\in[0,\infty)$ and thus

$$\begin{aligned}\|u_\varepsilon\|_{L^\infty(0,T;L^4(\Omega))}&=\operatorname*{ess-sup}_{[0,T)}\left[\int_\Omega|u_\varepsilon|^4\,\mathrm{d}\mathcal{L}^n\right]^{\frac{1}{4}}\le 1+\operatorname*{ess-sup}_{[0,T)}\int_\Omega|u_\varepsilon|^4\,\mathrm{d}\mathcal{L}^n\\&\le 1+\operatorname*{ess-sup}_{[0,T)}\int_{\{|u_\varepsilon|\le R_0\}}|u_\varepsilon|^4\,\mathrm{d}\mathcal{L}^n+\operatorname*{ess-sup}_{[0,T)}\int_{\{|u_\varepsilon|>R_0\}}|u_\varepsilon|^4\,\mathrm{d}\mathcal{L}^n\\&\le 1+R_0^4\mathcal{L}^n(\Omega)+2\varepsilon\Lambda\le C(\Omega,\Lambda),\end{aligned}$$

which proves (*i*). Next we prove (*ii*)-(*iv*). From (6.2.3) we conclude

$$\|H_\varepsilon\|_{L^2(0,T;L^2(\Omega))}\le\varepsilon\Lambda\quad\text{and}\quad\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\le\frac{\Lambda T}{\varepsilon}.$$

With the same estimates as in (6.1.26) and (6.1.27) we deduce that

$$\int_0^T\int_\Omega\left(\varepsilon^3|\Delta u_\varepsilon|^2+24\varepsilon|u_\varepsilon|^2|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon}|W'(u_\varepsilon)|^2\right)\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1\le\Lambda(8T+\varepsilon^2). \tag{6.2.9}$$

This proves (*ii*)-(*iv*). For (*v*) we argue as we did in the proof of (*i*). Since $W'(r)=4r^3-4r$ for $r\in\mathbb{R}$ there exists $R_1>0$ such that $|r|^3\le|W'(r)|$ for $|r|\ge R_1$. Thus we get

$$\int_0^T\int_\Omega|u_\varepsilon|^6\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1\le T\mathcal{L}^n(\Omega)R_1^6+\varepsilon\int_0^T\int_\Omega\frac{1}{\varepsilon}|W'(u_\varepsilon)|^2\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1\le C(\Omega,T)$$

which proves (v). To prove (vi) take $\phi \in L^2(0,T;H^1(\Omega))$, we estimate similarly as in (6.1.22), using $\phi = \mathcal{A}_\varepsilon(-\varepsilon\Delta + \mathrm{Id})\phi$ and the Cauchy-Schwarz estimate for the $H^1(\Omega)$ scalar product

$$
\begin{aligned}
&\Big| \int_0^T \Big\langle \phi, \sqrt{\varepsilon}\partial_t u_\varepsilon \Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 \Big| \\
&= \Big| \int_0^T \Big\langle \mathcal{A}_\varepsilon \sqrt{\varepsilon}\partial_t u_\varepsilon, (-\varepsilon^2\Delta + \mathrm{Id})\phi \Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 \Big| \\
&= \Big| \int_0^T \int_\Omega \Big(\varepsilon^2 \nabla\sqrt{\varepsilon}\mathcal{A}_\varepsilon\partial_t u_\varepsilon \cdot \nabla\phi + \sqrt{\varepsilon}\mathcal{A}_\varepsilon\partial_t u_\varepsilon \phi \Big) \,\mathrm{d}\mathcal{L}^1 \Big| \\
&\leq \int_0^T \bigg(\int_\Omega \Big(\varepsilon^2 \big|\nabla\sqrt{\varepsilon}\mathcal{A}_\varepsilon\partial_t u_\varepsilon\big|^2 + \big|\sqrt{\varepsilon}\mathcal{A}_\varepsilon\partial_t u_\varepsilon\big|^2 \Big) \,\mathrm{d}\mathcal{L}^n \int_\Omega \Big(\varepsilon^2|\nabla\phi|^2 + |\phi|^2 \Big) \,\mathrm{d}\mathcal{L}^n \bigg)^{\frac{1}{2}} \,\mathrm{d}\mathcal{L}^1 \\
&= \bigg(\int_0^T \varepsilon \Big\langle \mathcal{A}_\varepsilon\partial_t u_\varepsilon, (-\varepsilon^2\Delta + \mathrm{Id})\mathcal{A}_\varepsilon\partial_t u_\varepsilon \Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 \int_{\Omega_T} \Big(\varepsilon^2|\nabla\phi|^2 + |\phi|^2 \Big) \,\mathrm{d}\mathcal{L}^{n+1} \bigg)^{\frac{1}{2}} \\
&= \bigg(\int_0^T \varepsilon \Big\langle \mathcal{A}_\varepsilon\partial_t u_\varepsilon, \partial_t u_\varepsilon \Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 \int_{\Omega_T} \Big(\varepsilon^2|\nabla\phi|^2 + |\phi|^2 \Big) \,\mathrm{d}\mathcal{L}^{n+1} \bigg)^{\frac{1}{2}} \\
&= \bigg(\int_0^T \varepsilon \Big\langle H_\varepsilon, (-\varepsilon^2\Delta + \mathrm{Id})H_\varepsilon \Big\rangle_{H^1(\Omega)'} \int_{\Omega_T} \Big(\varepsilon^2|\nabla\phi|^2 + |\phi|^2 \Big) \,\mathrm{d}\mathcal{L}^{n+1} \bigg)^{\frac{1}{2}} \\
&\leq \sqrt{\Lambda} \bigg(\int_{\Omega_T} \Big(\varepsilon^2|\nabla\phi|^2 + |\phi|^2 \Big) \,\mathrm{d}\mathcal{L}^{n+1} \bigg)^{\frac{1}{2}}. \qquad \square
\end{aligned}
$$

This proves a uniform bound for $\sqrt{\varepsilon}\partial_t u_\varepsilon$ in $L^2(0,T;H^1(\Omega)')$. However in contrast to the construction in the first section we do not have a uniform bound for u_ε in $L^2(0,T;H^1(\Omega))$. Thus we can not directly apply an Aubin-Lion-Dubinskii type argument. We will work around this difficulty by considering a different function as a stepping stone first. We prove bounds for $(Z(u_\varepsilon))_{\varepsilon>0}$ where $Z(r) = \int_0^r \sqrt{2W(s)}\,\mathrm{d}s$ for $r \in \mathbb{R}$ as in Chapter 5.

Lemma 6.2.5.
Let Assumptions 6.2.1 hold. Then

(1) $(Z(u_\varepsilon))_{\varepsilon>0}$ *is bounded in* $L^\infty(0,T;L^{\frac{4}{3}}(\Omega))$.

(2) $(\partial_t Z(u_\varepsilon))_{\varepsilon>0}$ *is bounded in* $L^2(0,T;H^2(\Omega)')$.

(3) $(\nabla Z(u_\varepsilon))_{\varepsilon>0}$ *is bounded in* $L^\infty(0,T;L^1(\Omega))$.

Proof. From the particular form of W we get that there exists $C > 0$ such that for all $r \in \mathbb{R}$ we have $|Z(r)| \leq C(1 + |r|^3)$. Using the convexity of $(0,\infty) \ni r \longmapsto r^{\frac{4}{3}}$ we get for all $t \in (0,T)$

$$
\begin{aligned}
\|Z(u_\varepsilon(t,\cdot))\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{4}{3}} &\leq \int_\Omega \Big| C(1 + |u_\varepsilon(t,\cdot)|^3) \Big|^{\frac{4}{3}} \,\mathrm{d}\mathcal{L}^n \leq C \int_\Omega (1 + |u_\varepsilon(t,\cdot)|^4) \,\mathrm{d}\mathcal{L}^n \\
&\leq C(\Omega)\Big(1 + \|u_\varepsilon(t,\cdot)\|_{L^4(\Omega)}^4\Big) \leq C(\Omega)\Big(1 + \operatorname*{ess\text{-}sup}_{[0,T)} \|u_\varepsilon\|_{L^4(\Omega)}^4\Big).
\end{aligned}
$$

This yields that for all $0 < \varepsilon < 1$ we have

$$\|Z(u_\varepsilon)\|_{L^\infty(0,T;L^{\frac{4}{3}}(\Omega))} \leq C(\Omega, \Lambda)$$

and thus (1). To show (2) we start with (vi) from Lemma 6.2.4. Let $\eta \in C_c^2(\Omega)$ be arbitrary, we define $\phi := \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\eta$. We want to apply estimate (6.2.8) to this ϕ, therefore we need to confirm $\phi \in L^2(0,T;H^1(\Omega))$ first. We have

$$\begin{aligned} \|\phi\|^2_{L^2(0,T;L^2(\Omega))} &= \int_0^T \int_\Omega |\phi|^2 \, \mathrm{d}\mathcal{L}^n \, \mathrm{d}\mathcal{L}^1 = \int_0^T \int_\Omega \frac{2}{\varepsilon} W(u_\varepsilon)|\eta|^2 \, \mathrm{d}\mathcal{L}^n \, \mathrm{d}\mathcal{L}^1 \\ &\leq 2T\|\eta\|^2_{C^0(\Omega)} \operatorname*{ess-sup}_{t\in[0,T)} \mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) \leq 2T\|\eta\|^2_{C^0(\Omega)}\Lambda. \end{aligned} \tag{6.2.10}$$

For the gradient estimate, we use that $|W'(r)|^2 = 16r^2W(r)$ for all $r \in \mathbb{R}$ and get

$$\begin{aligned} \|\nabla\phi\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} &= \int_0^T \int_\Omega \frac{1}{\varepsilon}\Big|\eta \frac{W'(u_\varepsilon)}{\sqrt{2W(u_\varepsilon)}}\nabla u_\varepsilon + \sqrt{2W(u_\varepsilon)}\nabla\eta\Big|^2 \, \mathrm{d}\mathcal{L}^n \, \mathrm{d}\mathcal{L}^1 \\ &\leq \int_0^T \int_\Omega \frac{16}{\varepsilon}|\eta u_\varepsilon \nabla u_\varepsilon|^2 \, \mathrm{d}\mathcal{L}^n \, \mathrm{d}\mathcal{L}^1 + 4\int_0^T \int_\Omega W(u_\varepsilon)|\nabla\eta|^2 \, \mathrm{d}\mathcal{L}^n \, \mathrm{d}\mathcal{L}^1 \\ &\leq \frac{1}{\varepsilon^2}\|\eta\|^2_{C^0(\Omega)} \int_0^T \int_\Omega 16\varepsilon|u_\varepsilon\nabla u_\varepsilon|^2 \, \mathrm{d}\mathcal{L}^n \, \mathrm{d}\mathcal{L}^1 \\ &\quad + \int_0^T \Big[\int_\Omega |W(u_\varepsilon)|^{\frac{3}{2}} \, \mathrm{d}\mathcal{L}^n\Big]^{\frac{2}{3}} \mathrm{d}\mathcal{L}^1 \Big[\int_\Omega |\nabla\eta|^6 \, \mathrm{d}\mathcal{L}^n\Big]^{\frac{1}{3}}. \end{aligned}$$

In the last step we applied the Hölder-inequality for the last term. The first integral on the right-hand side is bounded because of (iii) from Lemma 6.2.4 up to a factor of ε^{-2}. For the last term we use $W(r) = (1-r^2)^2 \leq (1+r^4)$ for $r \in \mathbb{R}$. Thus we get

$$\begin{aligned} \int_0^T \int_\Omega |W(u_\varepsilon)|^{\frac{3}{2}} \, \mathrm{d}\mathcal{L}^n \, \mathrm{d}\mathcal{L}^1 &\leq \int_0^T \int_\Omega (1+|u_\varepsilon|^4)^{\frac{3}{2}} \, \mathrm{d}\mathcal{L}^n \, \mathrm{d}\mathcal{L}^1 \\ &\leq \int_0^T \int_\Omega (1+|u_\varepsilon|^6) \, \mathrm{d}\mathcal{L}^n \, \mathrm{d}\mathcal{L}^1 \leq C(\Omega, T). \end{aligned} \tag{6.2.11}$$

We used (v) from Lemma 6.2.4 and the convexity of $(0,\infty) \ni r \longmapsto r^{\frac{3}{2}}$. Back to the previous estimate we get

$$\|\nabla\phi\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} \leq \frac{C(\Omega,T,\Lambda)\|\eta\|^2_{C^0(\Omega)}}{\varepsilon^2} + C(\Omega,T)\|\eta\|^2_{W^{1,6}(\Omega)}. \tag{6.2.12}$$

For fixed $\varepsilon > 0$ this is finite and thus $\phi \in L^2(0,T;H^1(\Omega))$. Since $n \leq 3$ there exists $C(\Omega) > 0$ such that by the Sobolev and Sobolev-Morrey embedding we obtain that

$$\|\eta\|_{W^{1,6}(\Omega)} \leq C(\Omega)\|\eta\|_{H^2(\Omega)} \quad \text{and} \quad \|\eta\|_{C^0(\Omega)} \leq C(\Omega)\|\eta\|_{H^2(\Omega)}.$$

Now we apply (vi), (6.2.10), and (6.2.12) to this particular ϕ and get that

$$
\begin{aligned}
\Big| \int_0^T \Big\langle \eta, \partial_t Z(u_\varepsilon) \Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 \Big| &= \Big| \int_0^T \Big\langle \phi, \sqrt{\varepsilon}\partial_t u_\varepsilon \Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 \Big| \\
&\leq \sqrt{\Lambda}\Big(\|\phi\|^2_{L^2(0,T;L^2(\Omega))} + \varepsilon^2 \|\nabla\phi\|^2_{L^2(0,T;L^2(\Omega;\mathbb{R}^n))} \Big)^{\frac{1}{2}} \\
&\leq \sqrt{\Lambda}\Big(C(\Omega,T,\Lambda)\|\eta\|^2_{C^0(\Omega)} + \varepsilon^2 C(\Omega,T)\|\eta\|^2_{W^{1,6}(\Omega)} \Big)^{\frac{1}{2}} \\
&\leq C(\Omega,T,\Lambda)\|\eta\|^2_{H^2(\Omega)}.
\end{aligned}
$$

Taking the supremum over $\eta \in C_c^2(\Omega)$ with $\|\eta\|_{H^2(\Omega)} \leq 1$ yields (2). For (3) we calculate for $t \in (0,T)$

$$
\begin{aligned}
\int_\Omega |\nabla Z(u_\varepsilon(t,\cdot))| \,\mathrm{d}\mathcal{L}^n &= \int_\Omega \sqrt{2W(u_\varepsilon(t,\cdot))}|\nabla u_\varepsilon(t,\cdot)| \,\mathrm{d}\mathcal{L}^n \\
&\leq \int_\Omega \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon(t,\cdot)|^2 + \frac{1}{\varepsilon} W(u_\varepsilon(t,\cdot)) \Big) \,\mathrm{d}\mathcal{L}^n \\
&\leq \mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)).
\end{aligned}
$$

Taking the essential supremum over $t \in (0,T)$ directly yields that

$$
\|\nabla Z(u_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega;\mathbb{R}^n))} \leq \operatorname*{ess-sup}_{t\in(0,T)} \mathcal{P}_\varepsilon(u_\varepsilon(t,\cdot)) \leq \Lambda. \qquad \square
$$

Lemma 6.2.6 (Convergent subsequence of $(u_\varepsilon)_{\varepsilon>0}$)**.**
Let Assumptions 6.2.1 hold. There exists $u \in L^\infty(0,T;BV(\Omega;\{\pm 1\}))$ such that up to a (t-independent) subsequence we have as $\varepsilon \to 0$

- $Z(u_\varepsilon) \longrightarrow Z(u)$ *in* $L^1(\Omega_T)$,
- $u_\varepsilon \longrightarrow u$ *in* $L^1(\Omega)$.

In addition we get for a.e. $t \in (0,T)$ as $\varepsilon \to 0$ that

- $Z(u_\varepsilon(t,\cdot)) \longrightarrow Z(u(t,\cdot))$ *in* $L^1(\Omega)$,
- $u_\varepsilon(t,\cdot) \longrightarrow u(t,\cdot)$ *in* $L^1(\Omega)$,
- *For a.e. $x \in \Omega$: $u_\varepsilon(t,x) \longrightarrow u(t,x)$,*
- *There exist subsets $E(t) \subseteq \Omega$ with finite perimeter such that $u(t,\cdot) = 2\chi_{E(t)} - 1$.*

Proof. From Lemma 6.2.5 we get with $L^{\frac{4}{3}}(\Omega) \hookrightarrow H^2(\Omega)'$ and $L^{\frac{4}{3}}(\Omega) \hookrightarrow L^1(\Omega)$

$$
(Z(u_\varepsilon))_{0<\varepsilon<1} \quad \text{is bounded in} \quad L^\infty(0,T;W^{1,1}(\Omega)) \quad \text{and in} \quad H^1(0,T;H^2(\Omega)').
$$

With Lemma 8.3.2 we can extract a convergent subsequence from $(Z(u_\varepsilon))_{\varepsilon>0}$ as $\varepsilon \to 0$ in $L^1(0,T;L^1(\Omega))$. Thus there exists a further subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$ and $k \in L^1(0,T;L^1(\Omega))$ such that as $j \to \infty$

$$
\begin{aligned}
Z(u_{\varepsilon_j}) &\longrightarrow k && \text{in} \quad L^1(\Omega_T), \\
Z(u_{\varepsilon_j}(t,\cdot)) &\longrightarrow k(t,\cdot) && \text{in} \quad L^1(\Omega) \quad \text{for a.e.} \quad t \in (0,T), \quad \text{and thus}
\end{aligned}
$$

$$Z(u_{\varepsilon_j}(t,x)) \longrightarrow k(t,x) \qquad \text{for a.e.} \quad t \in (0,T) \quad \text{and a.e.} \quad x \in \Omega.$$

Since $Z' = \sqrt{2W} \geq 0$ on $\mathbb{R}$ with $\{W = 0\} = \{\pm 1\}$ we know that Z has a continuous inverse function Z^{-1}. This results in

$$u_{\varepsilon_j}(t,x) \longrightarrow u(t,x) \coloneqq Z^{-1}(k(t,x)) \quad \text{for a.e.} \quad t \in (0,T) \quad \text{and a.e.} \quad x \in \Omega.$$

To prove the convergence $u_{\varepsilon_j} \longrightarrow u$ in $L^1(\Omega_T)$ we use a well-known technique, which is for instance presented in the proof of Theorem 1.6 in [Leo13].

First we prove that $(u_{\varepsilon_j})_{j\in\mathbb{N}}$ is equi-integrable. We can find $R_2 > 0$ such that $|r| \leq W(r)$ for all $|r| > R_2$. Then we estimate for any measurable subset $A \subseteq \Omega_T$

$$\begin{aligned} \int_A |u_{\varepsilon_j}| \,\mathrm{d}\mathcal{L}^{n+1} &\leq \int_{A\cap\{|u_{\varepsilon_j}|\leq R_2\}} |u_{\varepsilon_j}| \,\mathrm{d}\mathcal{L}^{n+1} + \int_{A\cap\{|u_{\varepsilon_j}|> R_2\}} |u_{\varepsilon_j}| \,\mathrm{d}\mathcal{L}^{n+1} \\ &\leq R_2\mathcal{L}^{n+1}(A) + \varepsilon_j \int_{\Omega_T} \frac{1}{\varepsilon_j} W(u_{\varepsilon_j}) \,\mathrm{d}\mathcal{L}^{n+1} \\ &\leq R_2\mathcal{L}^{n+1}(A) + \varepsilon_j \operatorname*{ess-sup}_{t\in(0,T)} \int_{\Omega} \frac{1}{\varepsilon_j} W(u_{\varepsilon_j}(t,\cdot)) \,\mathrm{d}\mathcal{L}^{n+1} \leq R_2\mathcal{L}^{n+1}(A) + \varepsilon_j\Lambda. \end{aligned} \tag{6.2.13}$$

Setting $A = \Omega_T$ yields that there exists $\Lambda' > 0$ such that

$$\sup_{j\in\mathbb{N}} \int_{\Omega_T} |u_{\varepsilon_j}| \,\mathrm{d}\mathcal{L}^{n+1} \leq \Lambda' < \infty. \tag{6.2.14}$$

Next we prove that for all $\tau > 0$ there exists $\delta > 0$ such that for all measurable subsets $A \subseteq \Omega_T$ we have

$$\mathcal{L}^{n+1}(A) < \delta \Longrightarrow \sup_{j\in\mathbb{N}} \int_A |u_{\varepsilon_j}| \,\mathrm{d}\mathcal{L}^{n+1} < \tau. \tag{6.2.15}$$

We start by choosing $j_0 \in \mathbb{N}$ such that for all $j > j_0$ we have $\varepsilon_j\Lambda < \frac{\tau}{2}$ and define $\delta_0 \coloneqq \frac{\tau}{2R_2}$. Then we get for $j > j_0$ from (6.2.13) for any measurable set $A \subseteq \Omega_T$ with $\mathcal{L}^{n+1}(A) < \delta_0$

$$\sup_{j\geq j_1} \int_A |u_{\varepsilon_j}| \,\mathrm{d}\mathcal{L}^{n+1} < \frac{R_2\tau}{2R_2} + \frac{\tau}{2} = \tau.$$

By the absolute continuity of the Lebesgue measure and since $u_{\varepsilon_1}, \ldots, u_{\varepsilon_{j_0}} \in L^1(\Omega_T)$ there exist $\delta_1, \ldots, \delta_{j_0} > 0$ such that for all $l \in \{1, \ldots, j_0\}$ we have for measurable sets $A \subseteq \Omega_T$

$$\mathcal{L}^{n+1}(A) < \delta_l \Longrightarrow \int_A |u_{\varepsilon_l}| \,\mathrm{d}\mathcal{L}^{n+1} < \tau. \tag{6.2.16}$$

By choosing $\delta \coloneqq \min\{\delta_0, \delta_1, \ldots, \delta_{j_0}\}$ we conclude (6.2.15).

Now we prove $u_{\varepsilon_j} \longrightarrow u$ in $L^1(\Omega_T)$ using Egorov's Theorem. Let $\tau > 0$ be arbitrary, we choose $\delta > 0$ according to (6.2.15) such that

$$\mathcal{L}^{n+1}(A) < \delta \Longrightarrow \sup_{j\in\mathbb{N}} \int_A |u_{\varepsilon_j}| \,\mathrm{d}\mathcal{L}^{n+1} < \frac{\tau}{3}$$

and such that

$$\mathcal{L}^{n+1}(A) < \delta \Longrightarrow \int_A |u| \,\mathrm{d}\mathcal{L}^{n+1} < \frac{\tau}{3}. \tag{6.2.17}$$

This is possible by the same argument as before since $u \in L^1(\Omega_T)$. We have $u_{\varepsilon_j} \longrightarrow u$ a.e. in Ω_T and thus by Egorov's Theorem there exists $A \subseteq \Omega_T$ such that

$$\mathcal{L}^{n+1}(A) < \delta \quad \text{and} \quad u_{\varepsilon_j} \longrightarrow u \quad \text{uniformly on} \quad \Omega_T \setminus A.$$

We choose $j_1 \in \mathbb{N}$ such that for all $j \geq j_1$ we have

$$\sup_{\Omega_T \setminus A} |u_{\varepsilon_j} - u| < \frac{\tau}{3\mathcal{L}^{n+1}(\Omega_T)}$$

Combining these results we get

$$\begin{aligned}
\int_{\Omega_T} |u_{\varepsilon_j} - u| \,\mathrm{d}\mathcal{L}^{n+1} &= \int_A |u_{\varepsilon_j} - u| \,\mathrm{d}\mathcal{L}^{n+1} + \int_{\Omega_T \setminus A} |u_{\varepsilon_j} - u| \,\mathrm{d}\mathcal{L}^{n+1} \\
&\leq \int_A |u_{\varepsilon_j}| \,\mathrm{d}\mathcal{L}^{n+1} + \int_A |u| \,\mathrm{d}\mathcal{L}^{n+1} + \int_{\Omega_T \setminus A} |u_{\varepsilon_j} - u| \,\mathrm{d}\mathcal{L}^{n+1} \\
&< \frac{\tau}{3} + \frac{\tau}{3} + \frac{\tau}{3\mathcal{L}^{n+1}(\Omega_T)} \mathcal{L}^{n+1}(\Omega_T) = \tau.
\end{aligned}$$

Thus we have $u_{\varepsilon_j} \longrightarrow u$ in $L^1(\Omega_T)$ and also $u_{\varepsilon_j}(t,\cdot) \longrightarrow u(t,\cdot)$ for a.e. $t \in (0,T)$.

By the Modica-Mortola Theorem 2.4.2, the energy bound (6.2.3), and the assumption (6.2.1) we conclude for a.e. $t \in (0,T)$

$$\mathcal{P}(u(t,\cdot)) \leq \liminf_{\varepsilon \to 0} \mu_\varepsilon^t(\Omega) \leq \sup_{\varepsilon > 0} \mathcal{P}_\varepsilon(u_0) \leq \Lambda.$$

The fact that this expression is finite shows $u(t,\cdot) \in BV(\Omega;\{\pm 1\})$ for a.e. $t \in [0,T)$, hence we can write $u = 2\chi_{E(t)} - 1$ for some set $E(t) \subseteq \Omega$ with finite perimeter. ☐

The next goal is to find a subsequence independent from t such that $(\mu_\varepsilon^t)_{\varepsilon > 0}$ is convergent. For that we need uniform bounds, one of which is provided by the next lemma. Here we will again not label all subsequences.

Lemma 6.2.7.
Let Assumptions 6.2.1 hold and let $t \in (0,T)$ be arbitrary. We have for the finite Radon measure $\mu_\varepsilon^t \in C_0^0(\Omega)'$ from Lemma 6.2.3 that there exists $C = C(\Lambda, T) > 0$ such that we have for all $\varepsilon > 0$ and all $\eta \in C_0^2(\Omega)$ with $\eta \geq 0$

$$\|\langle \eta, \mu_\varepsilon^t \rangle_{C_0^0(\Omega)'}\|_{W^{1,1}(0,T)} \leq C(\Lambda, T) \|\eta\|_{C^2(\Omega)}, \tag{6.2.18}$$

which shows that the functions $(0,T) \ni t \longmapsto \langle \eta, \mu_\varepsilon^t \rangle_{C_0^0(\Omega)'}$ are uniformly bounded in $W^{1,1}(0,T)$ with respect to $\varepsilon > 0$.

Proof. We have

$$\int_0^T \left| \int_\Omega \eta \Big(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W'(u_\varepsilon) \Big) \,\mathrm{d}\mathcal{L}^n \right| \mathrm{d}\mathcal{L}^1 \leq \|\eta\|_{C^0(\Omega)} \Lambda T$$

and thus $\langle \eta, \mu_\varepsilon^t \rangle_{C_0^0(\Omega)'}$ is uniformly bounded in $L^1(0,T)$. Next we calculate the weak derivative. Let $\psi \in C_0^1(0,T)$. The following calculations are justified because of the regularity $u_\varepsilon \in H^1(0,T;H^1(\Omega)') \cap L^1(0,T;H^3(\Omega))$.

$$\begin{aligned}
-\int_0^T \psi' \langle \eta, \mu_\varepsilon^t \rangle_{C_0^0(\Omega)'} \,\mathrm{d}\mathcal{L}^1 &= -\int_\Omega \int_0^T \psi' \eta \Big(\frac{\varepsilon}{2} |\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon} W(u_\varepsilon) \Big) \,\mathrm{d}\mathcal{L}^1 \,\mathrm{d}\mathcal{L}^n \\
&= \int_0^T \psi \Big(\Big\langle \eta \varepsilon \nabla u_\varepsilon, \nabla \partial_t u_\varepsilon \Big\rangle_{H^2(\Omega;\mathbb{R}^n)'} + \frac{1}{\varepsilon} \langle W'(u_\varepsilon), \partial_t u_\varepsilon \rangle_{H^1(\Omega)'} \Big) \,\mathrm{d}\mathcal{L}^1 \\
&= \int_0^T \psi \Big\langle -\varepsilon \nabla \eta \cdot \nabla u_\varepsilon - \varepsilon \eta \Delta u_\varepsilon + \frac{1}{\varepsilon} \eta W'(u_\varepsilon), \partial_t u_\varepsilon \Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 \\
&= \int_0^T \psi \Big\langle \nabla \eta \cdot \nabla u_\varepsilon - \frac{1}{\varepsilon} \eta H_\varepsilon, (-\varepsilon^2 \Delta + \mathrm{Id}) H_\varepsilon \Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1.
\end{aligned}$$

From this we can identify the weak derivative, we have

$$\partial_t \langle \eta, \mu_\varepsilon^t \rangle_{C_0^0(\Omega)'} = \Big\langle \nabla \eta \cdot \nabla u_\varepsilon - \frac{1}{\varepsilon} \eta H_\varepsilon, (-\varepsilon^2 \Delta + \mathrm{Id}) H_\varepsilon \Big\rangle_{H^1(\Omega)'}.$$

In the next step we estimate the weak derivative in $L^1(0,T;L^1(\Omega))$ which would finish the proof. We split the integral into two terms, one of which is a localized version of $\partial_t \mathcal{P}_\varepsilon(u_\varepsilon)$ and the other term contains $\nabla\eta$. The latter one is called drift term. We have

$$\begin{aligned}
\int_0^T \Big| \partial_t \langle \eta, \mu_\varepsilon^t \rangle_{C_0^0(\Omega)'} \Big| \,\mathrm{d}\mathcal{L}^1 \le& \int_0^T \Big| \langle \nabla \eta \cdot \nabla u_\varepsilon, (-\varepsilon^2 \Delta + \mathrm{Id}) H_\varepsilon \rangle_{H^1(\Omega)'} \Big| \,\mathrm{d}\mathcal{L}^1 \\
&+ \int_0^T \frac{1}{\varepsilon} \Big| \langle \eta H_\varepsilon, (-\varepsilon^2 \Delta + \mathrm{Id}) H_\varepsilon \rangle_{H^1(\Omega)'} \Big| \,\mathrm{d}\mathcal{L}^1.
\end{aligned}$$

We split the integrals even further and get

$$\begin{aligned}
\int_0^T \Big| \partial_t \langle \eta, \mu_\varepsilon^t \rangle_{C_0^0(\Omega)'} \Big| \,\mathrm{d}\mathcal{L}^1 \le& \int_0^T \Big| \langle \nabla \eta \cdot \nabla u_\varepsilon, \varepsilon^2 \Delta H_\varepsilon \rangle_{H^1(\Omega)'} \Big| \,\mathrm{d}\mathcal{L}^1 && (6.2.19) \\
&+ \int_0^T \Big| \int_\Omega (\nabla \eta \cdot \nabla u_\varepsilon) H_\varepsilon \,\mathrm{d}\mathcal{L}^n \Big| \,\mathrm{d}\mathcal{L}^1 && (6.2.20) \\
&+ \int_0^T \Big| \int_\Omega \varepsilon (\nabla \eta \cdot \nabla H_\varepsilon) H_\varepsilon \,\mathrm{d}\mathcal{L}^n \Big| \,\mathrm{d}\mathcal{L}^1 && (6.2.21) \\
&+ \int_0^T \Big| \langle \eta, \kappa_\varepsilon^t \rangle_{C_0^0(\Omega)'} \Big| \,\mathrm{d}\mathcal{L}^1. && (6.2.22)
\end{aligned}$$

The integral in (6.2.20) can be controlled with a Young estimate and the bounds from the energy-dissipation inequality (6.2.3)

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{1}{\sqrt{\varepsilon}} |\nabla \eta| |\sqrt{\varepsilon} \nabla u_\varepsilon| |H_\varepsilon| \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 \\
&\qquad \le \frac{\|\eta\|_{C^1(\Omega)}}{2} \int_0^T \int_\Omega \Big(\frac{1}{\varepsilon} |H_\varepsilon|^2 + \varepsilon |\nabla u_\varepsilon|^2 \Big) \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 \le \|\eta\|_{C^1(\Omega)} C(\Lambda, T).
\end{aligned}$$

The integral in (6.2.21) can be controlled with a partial integration and the bounds from the energy-dissipation inequality

$$\int_0^T \left| \int_\Omega \varepsilon(\nabla\eta \cdot \nabla H_\varepsilon) H_\varepsilon \,\mathrm{d}\mathcal{L}^n \right| \mathrm{d}\mathcal{L}^1 = \int_0^T \left| \int_\Omega \frac{\varepsilon}{2} \nabla\eta \cdot \nabla |H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \right| \mathrm{d}\mathcal{L}^1 \tag{6.2.23}$$
$$= \int_0^T \left| \int_\Omega \frac{\varepsilon}{2} \Delta\eta |H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \right| \mathrm{d}\mathcal{L}^1$$
$$\leq \frac{\varepsilon^2 \|\eta\|_{C^2(\Omega)}}{2} \int_0^T \int_\Omega \frac{1}{\varepsilon} |H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 \leq \varepsilon^2 \frac{\|\eta\|_{C^2(\Omega)}\Lambda}{2}.$$

The integral in (6.2.22) is directly controlled by the bounds from the energy-dissipation inequality. Since $\eta \geq 0$ we have

$$\int_0^T \left| \langle \eta, \kappa_\varepsilon^t \rangle_{C_0^0(\Omega)'} \right| \mathrm{d}\mathcal{L}^1 = \langle \eta, \kappa_\varepsilon \rangle_{C_0^0(\Omega_T)'} \leq \|\eta\|_{C^0(\Omega)} \kappa_\varepsilon(\Omega_T) \leq \|\eta\|_{C^0(\Omega)} \Lambda.$$

The most difficult part is to estimate (6.2.19), which comes from the drift term. As a stepping stone we define for $t \in (0,T)$ and $x \in \Omega$ the function $K_t : \Omega \longrightarrow \mathbb{R}^n$

$$K_t(x) := [\nabla\eta(x) \cdot \nabla u_\varepsilon(t,x)] \nabla H_\varepsilon(t,x) - \nabla\eta(x) [\nabla u_\varepsilon(t,x) \cdot \nabla H_\varepsilon(t,x)]$$
$$+ [\nabla\eta(x) \cdot \nabla H_\varepsilon(t,x)] \nabla u_\varepsilon(t,x).$$

Since $\eta \in C^1(\Omega)$ and $u_\varepsilon(t,\cdot), H_\varepsilon(t,\cdot) \in H^1(\Omega)$ for a.e. $t \in (0,T)$ we have $K_t \in L^1(\Omega; \mathbb{R}^n)$ for a.e. $t \in (0,T)$. It follows that for a.e. $t \in (0,T)$ and all $\phi \in W^{1,\infty}(\Omega)$ we can define

$$\langle \phi, -\nabla \cdot K_t \rangle_{W^{1,\infty}(\Omega)'} := \int_\Omega \nabla\phi \cdot K_t \,\mathrm{d}\mathcal{L}^n.$$

In this sense we have $\nabla \cdot K_t \in W^{1,\infty}(\Omega)'$ for a.e. $t \in (0,T)$. We conclude that

$$0 = -\int_\Omega \nabla 1 \cdot K_t \,\mathrm{d}\mathcal{L}^n = \langle 1, \nabla \cdot K_t \rangle_{W^{1,\infty}(\Omega)'} \tag{6.2.24}$$
$$= \Big\langle 1, \nabla \cdot \Big([\nabla\eta \cdot \nabla u_\varepsilon] \nabla H_\varepsilon - \nabla\eta [\nabla u_\varepsilon \cdot \nabla H_\varepsilon] + [\nabla\eta \cdot \nabla H_\varepsilon] \nabla u_\varepsilon \Big) \Big\rangle_{W^{1,\infty}(\Omega)'}$$
$$= \int_\Omega \big(\nabla\eta \cdot D^2 u_\varepsilon \nabla H_\varepsilon + \nabla H_\varepsilon \cdot D^2\eta \nabla u_\varepsilon \big) \,\mathrm{d}\mathcal{L}^n + \big\langle \nabla\eta \cdot \nabla u_\varepsilon, \Delta H_\varepsilon \big\rangle_{H^1(\Omega)'}$$
$$- \int_\Omega \Big(\nabla\eta \cdot D^2 u_\varepsilon \nabla H_\varepsilon + [\nabla u_\varepsilon \cdot \nabla H_\varepsilon] \Delta\eta \Big) \,\mathrm{d}\mathcal{L}^n - \big\langle \nabla\eta \otimes \nabla u_\varepsilon, D^2 H_\varepsilon \big\rangle_{H^1(\Omega;\mathbb{R}^{n\times n})'}$$
$$+ \int_\Omega \Big(\nabla u_\varepsilon \cdot D^2\eta \nabla H_\varepsilon + [\nabla\eta \cdot \nabla H_\varepsilon] \Delta u_\varepsilon \Big) \,\mathrm{d}\mathcal{L}^n + \big\langle \nabla u_\varepsilon \otimes \nabla\eta, D^2 H_\varepsilon \big\rangle_{H^1(\Omega;\mathbb{R}^{n\times n})'}$$
$$= \int_\Omega \Big(2\nabla H_\varepsilon \cdot D^2\eta \nabla u_\varepsilon - [\nabla u_\varepsilon \cdot \nabla H_\varepsilon] \Delta\eta + [\nabla\eta \cdot \nabla H_\varepsilon] \Delta u_\varepsilon \Big) \,\mathrm{d}\mathcal{L}^n$$
$$+ \big\langle \nabla\eta \cdot \nabla u_\varepsilon, \Delta H_\varepsilon \big\rangle_{H^1(\Omega)'}.$$

We used the symmetry of the Hessian in the classical sense for $D^2\eta$, for $D^2 u_\varepsilon$ in the weak sense and for $D^2 H_\varepsilon$ in the sense of distributions. With this divergence we can rewrite the integral (6.2.19). We get

$$\varepsilon^2 \int_0^T \left|\langle \nabla\eta \cdot \nabla u_\varepsilon, \Delta H_\varepsilon \rangle_{H^1(\Omega)'}\right| \,\mathrm{d}\mathcal{L}^1 \tag{6.2.25}$$
$$= \varepsilon^2 \int_0^T \left| \int_\Omega \Big(\nabla u_\varepsilon \cdot \nabla H_\varepsilon \Delta\eta - \nabla\eta \cdot \nabla H_\varepsilon \Delta u_\varepsilon - 2\nabla u_\varepsilon \cdot D^2\eta \nabla H_\varepsilon \Big) \,\mathrm{d}\mathcal{L}^n \right| \mathrm{d}\mathcal{L}^1.$$

This creates three further integrals. We can estimate the last term by

$$\begin{aligned} 2\varepsilon^2 \int_0^T \left| \int_\Omega \nabla u_\varepsilon \cdot D^2\eta \nabla H_\varepsilon \,\mathrm{d}\mathcal{L}^n \right| \mathrm{d}\mathcal{L}^1 & \\ &\le \varepsilon \|\eta\|_{C^2(\Omega)} \int_0^T \left(\int_\Omega \varepsilon |\nabla u_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n + \int_\Omega \varepsilon |\nabla H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \right) \mathrm{d}\mathcal{L}^1 \\ &\le \varepsilon \|\eta\|_{C^2(\Omega)} C(\Lambda, T). \end{aligned} \tag{6.2.26}$$

The estimate is obtained analogously for the first term on the right-hand side of (6.2.25) since both $D^2\eta$ and $\Delta\eta$ can be estimated by $\|\eta\|_{C^2(\Omega)}$. The remaining second term can be estimated using that $-\varepsilon\Delta u_\varepsilon = H_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)$ and

$$\begin{aligned} \varepsilon \int_0^T \left| \int_\Omega -\varepsilon \Delta u_\varepsilon \nabla\eta \cdot \nabla H_\varepsilon \,\mathrm{d}\mathcal{L}^n \right| \mathrm{d}\mathcal{L}^1 &\le \varepsilon \int_0^T \left| \int_\Omega H_\varepsilon \nabla\eta \cdot \nabla H_\varepsilon \,\mathrm{d}\mathcal{L}^n \right| \mathrm{d}\mathcal{L}^1 \\ &\quad + \int_0^T \left| \int_\Omega W'(u_\varepsilon) \nabla\eta \cdot \nabla H_\varepsilon \,\mathrm{d}\mathcal{L}^n \right| \mathrm{d}\mathcal{L}^1. \end{aligned}$$

The first term is identical to (6.2.23) and we use its estimate again. For the remaining second term we apply (ii) from Lemma 6.2.4 and get that

$$\begin{aligned} \int_0^T \left| \int_\Omega W'(u_\varepsilon) \nabla\eta \cdot \nabla H_\varepsilon \,\mathrm{d}\mathcal{L}^n \right| \mathrm{d}\mathcal{L}^1 &\le \|\eta\|_{C^1(\Omega)} \int_0^T \int_\Omega \frac{1}{\sqrt{\varepsilon}} |W'(u_\varepsilon)| \sqrt{\varepsilon} |\nabla H_\varepsilon| \,\mathrm{d}\mathcal{L}^n \Big| \,\mathrm{d}\mathcal{L}^1 \\ &\le \frac{\|\eta\|_{C^1(\Omega)}}{2} \left(\int_0^T \int_\Omega \frac{1}{\varepsilon} |W'(u_\varepsilon)|^2 \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 + \int_0^T \int_\Omega \varepsilon |\nabla H_\varepsilon|^2 \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 \right) \\ &\le \|\eta\|_{C^1(\Omega)} C(\Lambda, T), \end{aligned}$$

which concludes the proof. □

With this preparation we can prove a compactness result. We follow the proofs by Mugnai and Röger presented in [MR08, MR11].

Proposition 6.2.8 (Convergent sequence of the measures)**.**
Let Assumptions 6.2.1 hold. For $t \in [0,T)$ we can find finite Radon measures $\mu^t \in C_0^0(\Omega)'$, $\mu \in C_0^0([0,T) \times \Omega)'$ such that up to a t-independent subsequence we have as $\varepsilon \to 0$

$$\begin{aligned} \textit{for all} \quad t \in [0,T) : \mu_\varepsilon^t &\xrightarrow{w^*} \mu^t \quad \textit{in} \quad C_0^0(\Omega)' \\ \textit{and} \quad \mu_\varepsilon &\xrightarrow{w^*} \mu \quad \textit{in} \quad C_0^0([0,T) \times \Omega)', \end{aligned}$$

with $\mu = \mathcal{L}^1 \otimes (\mu^t)_{t \in [0,T)}$.

Since a lot of subsequences appear in this proof we relabel the most important one. We will return to the standard notation of not relabeling the subsequence after this proof is completed.

Proof. From (6.2.3) we obtain that

$$\mu_\varepsilon(\Omega_T) = \int_0^T \mu_\varepsilon^t(\Omega)\,\mathrm{d}t \leq T \operatorname*{ess-sup}_{(0,T)} \mathcal{P}_\varepsilon(u_\varepsilon) \leq T\Lambda.$$

By Theorem 2.2.2 we can find a finite Radon measure $\mu \in C_0^0([0,T) \times \Omega)'$ such that up to a subsequence $\mu_\varepsilon \xrightarrow{w^*} \mu$ in $C_0^0([0,T) \times \Omega)'$. Next we want to show weak*-convergence of $(\mu_\varepsilon^t)_{\varepsilon>0}$ in $C_0^0(\Omega)'$ for a subsequence independent of $t \in [0,T)$. We choose a dense subset and countable subset $\{\phi_j\}_{j\in\mathbb{N}} \subseteq C_0^2(\Omega)$. From Lemma 6.2.7 we have that for every $j \in \mathbb{N}$ the function defined by

$$f_j^{(\varepsilon)}(t) \coloneqq \langle \phi_j, \mu_\varepsilon^t \rangle_{C_0^0(\Omega)'}, \quad \text{for} \quad t \in [0,T)$$

is uniformly bounded in $W^{1,1}(0,T)$. Owing to the embeddings

$$W^{1,1}(0,T) \hookrightarrow BV(0,T) \overset{c}{\hookrightarrow} L^1(0,T)$$

for each $j \in \mathbb{N}$ we can extract asubsequence from the subsequence which was chosen in the first step such that

$$f_j^{(\varepsilon)} \longrightarrow f_j \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad L^1(0,T) \tag{6.2.27}$$

for some $f_j \in BV(0,T)$. By choosing the subsequences iteratively with the standard technique of a diagonal sequence we find a subsequence $(\varepsilon_m)_{m\in\mathbb{N}}$ independent of $j \in \mathbb{N}$ such that (6.2.27) holds for all $j \in \mathbb{N}$. We also obtain pointwise convergence a.e. in $(0,T)$ for this subsequence, i.e.

$$f_j^{(\varepsilon_m)}(t) \longrightarrow f_j(t) \quad \text{as} \quad m \to \infty \quad \text{for all} \quad j \in \mathbb{N} \quad \text{and a.e.} \quad t \in [0,T). \tag{6.2.28}$$

Note that the exception set can be chosen independent of j since countable unions of null sets still have measure 0. Since $f_j \in BV(0,T)$ there exists $\nu_j \in C_0^0(0,T)'$ such that $\nu_j = \partial_t f_j$ in the sense of measures . Since ν_j is σ-finite as a Radon measure on $(0,T)$ the set of its singletons S_j is at most countable. This remains true for $S \coloneqq \cup_j S_j$. For $t \in (0,T) \setminus S$ we prove (6.2.28) in the following. Let $t \in (0,T) \setminus S$, then there exists a sequence $(t_l)_l$ in $(0,T) \setminus S$ with $t_l \nearrow t$ and such that (6.2.28) holds for all t_l. From basic measure theory we get that for all $j \in \mathbb{N}$

$$\lim_{l\to\infty} \nu_j([t_l,t]) = \nu_j\Big(\bigcap_{l\in\mathbb{N}} [t_j,t]\Big) = \nu_j(\{t\}) = 0.$$

Since $\nu_j(\{t,t_l\}) = 0$ for all $l \in \mathbb{N}$ we get with standard properties of Radon measure convergence (see Proposition 1.62 (b) in [AFP00]) that

$$\lim_{m\to\infty} \Big(\partial_t f_j^{(\varepsilon_m)} \mathcal{L}^1\Big)([t_l,t]) = \nu_j([t_l,t]).$$

Additionally from Theorem 2.2.8 we get that

$$f_j(t) - f_j(t_l) = \int_{t_l}^t \mathrm{d}\nu_j.$$

From the collected results we obtain that

$$\begin{aligned}|f_j^{(\varepsilon_m)}(t)-f_j(t)| &\le |f_j^{(\varepsilon_m)}(t)-f_j^{(\varepsilon_m)}(t_l)|+|f_j^{(\varepsilon_m)}(t_l)-f_j(t_l)|+|f_j(t_l)-f_j(t)|\\ &\le \left|\left(\partial_t f_j^{(\varepsilon_m)}\mathcal{L}^1\right)([t_l,t])\right|+|f_j^{(\varepsilon_m)}(t_l)-f_j(t_l)|+|\nu_j([t_l,t])|.\end{aligned}$$

Taking $\limsup_{m\to\infty}$ yields

$$\limsup_{m\to\infty}|f_j^{(\varepsilon_m)}(t)-f_j(t)|\le 2|\nu_j([t_l,t])|\longrightarrow 0 \quad \text{as} \quad t_l\nearrow t.$$

Thus we have shown (6.2.28) for all $t\in(0,T)\setminus S$.

In the next step we want to show that for all $t\in(0,T)\setminus S$ there exists $\mu^t\in C_0^0(\Omega)'$ such that

$$\mu_{\varepsilon_m}^t\overset{w^*}{\longrightarrow}\mu^t \quad \text{as} \quad m\to\infty \quad \text{in} \quad C_0^0(\Omega)'. \tag{6.2.29}$$

Take any $t\in(0,T)$ such that (6.2.28) holds. By (6.2.3) there exists a further (t-dependent) subsequence $(\varepsilon_{m_r})_r$ and a Radon measure $\mu^t\in C_0^0(\Omega)'$ such that

$$\mu_{\varepsilon_{m_r}}^t\overset{w^*}{\longrightarrow}\mu^t \quad \text{as} \quad r\to\infty \quad \text{in} \quad C_0^0(\Omega)'.$$

We know from (6.2.28) that for all $t\in(0,T)\setminus S$ and all $j\in\mathbb{N}$ we have

$$f_j(t)=\lim_{m\to\infty}\langle\phi_j,\mu_{\varepsilon_m}^t\rangle_{C_0^0(\Omega)'}.$$

For the subsequence $(\varepsilon_{m_r})_{r\in\mathbb{N}}$ we now have

$$\lim_{r\to\infty}\langle\phi_j,\mu_{\varepsilon_{m_r}}^t\rangle_{C_0^0(\Omega)'}=\langle\phi_j,\mu^t\rangle_{C_0^0(\Omega)'}.$$

By uniqueness it follows that

$$f_j(t)=\langle\phi_j,\mu^t\rangle_{C_0^0(\Omega)'}$$

and that the limit exists for the whole sequence $(\mu_{\varepsilon_m}^t)_{m\in\mathbb{N}}$. For the proof of (6.2.29) we take any $\eta\in C_0^0(\Omega)$. Since $\{\phi_j\}_j$ is dense in $C_0^2(\Omega)$ and thus dense in $C_0^0(\Omega)$ there exists a subsequence $(\phi_{j_l})_{l\in\mathbb{N}}$ such that $\phi_{j_l}\longrightarrow\eta$ as $l\to\infty$ in $C_0^0(\Omega)$. Given any $\tau>0$ there exists $l_0=l_0(\tau)\in\mathbb{N}$ such that for all $l\ge l_0$ we have

$$|\eta-\phi_{j_l}|\le\frac{\tau}{3\Lambda} \quad \text{on} \quad \Omega.$$

We then find $m_0=m_0(\tau,l_0)\in\mathbb{N}$ such that for all $m\ge m_0$ we have

$$\left|\left\langle\phi_{j_{l_0}},\mu_{\varepsilon_m}^t-\mu^t\right\rangle_{C_0^0(\Omega)'}\right|\le\frac{\tau}{3}.$$

We conclude

$$\begin{aligned}\left|\left\langle\eta,\mu_{\varepsilon_m}^t-\mu^t\right\rangle_{C_0^0(\Omega)'}\right| &\le \left|\left\langle\eta-\phi_{j_{l_0}},\mu_{\varepsilon_m}^t\right\rangle_{C_0^0(\Omega)'}\right|+\left|\left\langle\eta-\phi_{j_{l_0}},\mu^t\right\rangle_{C_0^0(\Omega)'}\right|\\ &\quad+\left|\left\langle\phi_{j_{l_0}},\mu^t-\mu_{\varepsilon_m}^t\right\rangle_{C_0^0(\Omega)'}\right|<\frac{\tau}{3\Lambda}\Lambda+\frac{\tau}{3\Lambda}\Lambda+\frac{\tau}{3}=\tau.\end{aligned}$$

Since the remaining set S is at most countable continuing this process with another diagonal sequence yields convergence for all $t \in (0,T)$.
All that remains to be shown is the connection between the limit measures, given by $\mu = \mathcal{L}^1 \otimes (\mu^t)_{t\in[0,T)}$. To obtain this we consider test functions $\psi \in C_0^0[0,T), \eta \in C_0^0(\Omega)$. Then $\psi\eta \in C_0^0([0,T)\times\Omega)$ and thus

$$\begin{aligned}\int_{\Omega_T} \psi\eta \,\mathrm{d}\mu \longleftarrow \int_{\Omega_T} \psi\eta \,\mathrm{d}\mu_{\varepsilon_m} &= \int_0^T \int_\Omega \psi\eta \,\mathrm{d}\mu_{\varepsilon_m}^t \,\mathrm{d}\mathcal{L}^1 \\ &= \int_0^T \psi \int_\Omega \eta \,\mathrm{d}\mu_{\varepsilon_m}^t \,\mathrm{d}\mathcal{L}^1 \longrightarrow \int_0^T \psi \int_\Omega \eta \,\mathrm{d}\mu^t \,\mathrm{d}\mathcal{L}^1 = \int_0^T \int_\Omega \psi\eta \,\mathrm{d}\mu^t \,\mathrm{d}\mathcal{L}^1.\end{aligned}$$

The last convergence follows from the Dominated Convergence Theorem since

$$\left|\psi \int_\Omega \eta \,\mathrm{d}\mu_{\varepsilon_m}^t\right| \le \|\psi\|_{C^0(0,T)} \|\eta\|_{C^0(\Omega)} \mu_{\varepsilon_m}^t(\Omega) \le \|\psi\|_{C^0[0,T)} \|\eta\|_{C^0(\Omega)} \Lambda.$$

It follows $\mu = \mathcal{L}^1 \otimes (\mu^t)_{t\in[0,T)}$ because the linear hull of tensor products $C_0^0[0,T) \otimes C_0^0(\Omega)$ is dense in $C_0^0([0,T)\times\Omega)$. ☐

6.3 De Giorgi type varifold solutions

Using the preparations from sections 6.1 and 6.2 we can prove that solutions of the diffuse equation (6.1.5) converge as $\varepsilon \to 0$ towards a De Giorgi type varifold solution for rescaled mean curvature flow; see Definition 2.5.3. First we construct oriented varifolds from the measures μ_ε^t.

Definition 6.3.1.
Let $u_\varepsilon \in L^2(0,T;H^3(\Omega))$ such that (6.1.6) holds for a.e. $t \in (0,T)$. We recall

$$H_\varepsilon = -\varepsilon\Delta u_\varepsilon + \frac{1}{\varepsilon} W'(u_\varepsilon) \in L^2(0,T;H^1(\Omega)), \tag{6.3.1}$$

$$\mu_\varepsilon^t = \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon(t,\cdot)|^2 + \frac{1}{\varepsilon} W(u_\varepsilon(t,\cdot))\Big)\mathcal{L}^n \llcorner \Omega \in C_0^0(\Omega)', \tag{6.3.2}$$

$$\xi_\varepsilon^t = \Big(\frac{\varepsilon}{2}|\nabla u_\varepsilon(t,\cdot)|^2 - \frac{1}{\varepsilon} W(u_\varepsilon(t,\cdot))\Big)\mathcal{L}^n \llcorner \Omega \in C_0^0(\Omega)', \tag{6.3.3}$$

$$\kappa_\varepsilon^t = \frac{1}{\varepsilon}\Big(|H_\varepsilon(t,\cdot)|^2 + \varepsilon^2|\nabla H_\varepsilon(t,\cdot)|^2\Big)\mathcal{L}^n \llcorner \Omega \in C_0^0(\Omega)', \tag{6.3.4}$$

$$\alpha_\varepsilon^t = \frac{1}{\varepsilon}|H_\varepsilon(t,\cdot)|^2 \mathcal{L}^n \llcorner \Omega \in C_0^0(\Omega)', \tag{6.3.5}$$

$$\mu_\varepsilon = \mathcal{L}^1 \otimes (\mu_\varepsilon^t)_{t\in[0,T)}, \quad \xi_\varepsilon = \mathcal{L}^1 \otimes (\xi_\varepsilon^t)_{t\in[0,T)} \in C_0^0([0,T)\times\Omega)', \tag{6.3.6}$$

$$\textit{and} \quad \alpha_\varepsilon = \mathcal{L}^1 \otimes (\alpha_\varepsilon^t)_{t\in(0,T)}, \quad \kappa_\varepsilon = \mathcal{L}^1 \otimes (\kappa_\varepsilon^t)_{t\in(0,T)} \in C_0^0(\Omega_T)'. \tag{6.3.7}$$

Note that for μ_ε and ξ_ε the time interval is closed at 0 while it is open for α and κ (since $\Omega_T = (0,T)\times\Omega$). Recall additionally that $u_{0,\varepsilon} \in H^1(\Omega)$ and $\mu_\varepsilon^0, \mu^0 \in C_0^0(\Omega)'$ are given by Assumptions 6.2.1. We define the oriented and unoriented varifolds

$${}^\oplus V_\varepsilon^0 := \mu_\varepsilon^0 \otimes \nu_{0,\varepsilon} \in {}^\oplus\mathbb{V}_{n-1}(\Omega), \tag{6.3.8}$$

$${}^\oplus V_\varepsilon^t := \mu_\varepsilon^t \otimes \nu_\varepsilon(t,\cdot) \in {}^\oplus\mathbb{V}_{n-1}(\Omega), \tag{6.3.9}$$

$$V_\varepsilon^t := \mu_\varepsilon^t \otimes \nu_\varepsilon(t,\cdot)^\perp \in \mathbb{V}_{n-1}(\Omega), \tag{6.3.10}$$

$$
\begin{aligned}
{}^{\oplus}V_\varepsilon &:= \mathcal{L}^1 \otimes ({}^{\oplus}V_\varepsilon^t)_{t\in[0,T)} \in {}^{\oplus}\mathbb{V}_{n-1}([0,T)\times\Omega), && (6.3.11)\\
\textit{and} \quad V_\varepsilon &:= \mathcal{L}^1 \otimes (V_\varepsilon^t)_{t\in[0,T)} \in \mathbb{V}_{n-1}([0,T)\times\Omega), && (6.3.12)
\end{aligned}
$$

where

$$
\begin{aligned}
\nu_\varepsilon(t,x) &:= \frac{\nabla u_\varepsilon(t,x)}{|\nabla u_\varepsilon(t,x)|} \quad \textit{for} \quad \nabla u_\varepsilon(t,x) \neq 0 \quad \textit{and} \quad \nu_\varepsilon(t,x) := e_1 \quad \textit{else}\\
\textit{and} \quad \nu_{0,\varepsilon}(x) &:= \frac{\nabla u_{0,\varepsilon}(x)}{|\nabla u_{0,\varepsilon}(x)|} \quad \textit{for} \quad \nabla u_{0,\varepsilon}(x) \neq 0 \quad \textit{and} \quad \nu_{0,\varepsilon}(x) := e_1 \quad \textit{else.}
\end{aligned}
$$

For the comfort of the reader we gather the compactness results from the last section and their immediate implications.

Lemma 6.3.2 (Compactness results from Section 6.2)**.**
Let Assumptions 6.2.1 hold. Then there exists $C(\Lambda,T)>0$ such that

$$
\mu_\varepsilon(\Omega_T) + \kappa_\varepsilon(\Omega_T) \leq C(\Lambda,T) \tag{6.3.13}
$$

and for a.e. $t\in(0,T)$:

$$
\begin{aligned}
\sup_{\varepsilon>0} \mu_\varepsilon^t(\Omega) + \sup_{\varepsilon>0} \kappa_\varepsilon(\Omega_t) &\leq C(\Lambda,T) && (6.3.14)\\
\textit{and} \quad \liminf_{\varepsilon\to 0} \Big(\mu_\varepsilon^t(\Omega) + \kappa_\varepsilon^t(\Omega)\Big) &< \infty. && (6.3.15)
\end{aligned}
$$

Furthermore there exists a subsequence $\varepsilon\to 0$ such that the following hold.

There exist limit functions $u\in L^\infty(0,T;BV(\Omega;\{\pm 1\}))$ and $u_0\in BV(\Omega;\{\pm1\})$, oriented varifolds ${}^{\oplus}V \in C_0^0([0,T)\times\Omega\times\mathbb{S}^{n-1})'$ and ${}^{\oplus}V^0 \in {}^{\oplus}\mathbb{V}_{n-1}(\Omega)$, finite Radon measures $\alpha,\kappa\in C_0^0(\Omega_T)'$, $\mu\in C_0^0([0,T)\times\Omega)'$, and for a.e. $t\in(0,T)$ there exists a Radon measure $\mu^t\in C_0^0(\Omega)'$ with $\mu=\mathcal{L}^1\otimes(\mu^t)_{t\in[0,T)}$ such that

$$
\begin{aligned}
u_\varepsilon(t,\cdot) &\xrightarrow{t\to 0} u_{0,\varepsilon} && \textit{a.e. in } \Omega, && (6.3.16)\\
u_{0,\varepsilon} &\longrightarrow u_0 && \textit{a.e. in } \Omega \textit{ and in } L^1(\Omega), && (6.3.17)\\
u_\varepsilon &\longrightarrow u && \textit{a.e. in } \Omega_T \textit{ and in } L^1(\Omega_T), && (6.3.18)\\
\mu_\varepsilon^t &\xrightarrow{w^*} \mu^t && \textit{in} \quad C_0^0(\Omega)' \quad \textit{for all } t\in[0,T), && (6.3.19)\\
\mu_\varepsilon &\xrightarrow{w^*} \mu && \textit{in} \quad C_0^0([0,T)\times\Omega)', && (6.3.20)\\
\alpha_\varepsilon &\xrightarrow{w^*} \alpha && \textit{in} \quad C_0^0(\Omega_T)', && (6.3.21)\\
\kappa_\varepsilon &\xrightarrow{w^*} \kappa && \textit{in} \quad C_0^0(\Omega_T)', && (6.3.22)\\
{}^{\oplus}V_\varepsilon^0 &\xrightarrow{w^*} {}^{\oplus}V^0 && \textit{in} \quad {}^{\oplus}\mathbb{V}_{n-1}(\Omega), && (6.3.23)\\
\textit{and} \quad {}^{\oplus}V_\varepsilon &\xrightarrow{w^*} {}^{\oplus}V && \textit{in} \quad C_0^0([0,T)\times\Omega\times\mathbb{S}^{n-1})'. && (6.3.24)
\end{aligned}
$$

Note that ${}^{\oplus}V$ is not only an oriented varifold on $[0,T)\times\Omega$, i.e. a Radon measure on $[0,T)\times\Omega\times\mathbb{S}^{n-1}$ but it is even a *finite* Radon measure on $[0,T)\times\Omega\times\mathbb{S}^{n-1}$. Also note that (6.3.18) yields in particular that for a.e. $t\in(0,T)$

$$
u_\varepsilon(t,\cdot) \longrightarrow u(t,\cdot) \quad \text{in} \quad L^1(\Omega). \tag{6.3.25}
$$

Proof. By Lemma 6.2.3 we can find a subsequence $\varepsilon \to 0$ such that (6.3.13) holds and that (6.3.14), (6.3.15) hold for a.e. $t \in (0,T)$. (6.3.16) follows from Corollary 6.2.2. By Assumptions 6.2.1 and Theorem 2.4.3 there exists a subsequence $\varepsilon \to 0$ and $u_0 \in BV(\Omega, \{\pm 1\})$ such that (6.3.17) holds.

The claims (6.3.18)-(6.3.20) have already been proven in Lemma 6.2.6 and Proposition 6.2.8. The convergences (6.3.21) and (6.3.22) follow immediately by applying Theorem 2.2.2 owing to the bound (6.3.13). By definitions of ${}^{\oplus}V_\varepsilon, {}^{\oplus}V_\varepsilon^0$ we have

$$\|{}^{\oplus}V_\varepsilon\|(\Omega_T) \leq \sup_{\varepsilon>0} \operatorname*{ess-sup}_{t\in(0,T)} \mu_\varepsilon^t(\Omega) \leq \Lambda \quad \text{and} \quad \|{}^{\oplus}V_\varepsilon^0\|(\Omega) \leq \sup_{\varepsilon>0} \mu_\varepsilon^0(\Omega) \leq \Lambda$$

and thus the sequences of varifolds $({}^{\oplus}V_\varepsilon)_{\varepsilon>0}, ({}^{\oplus}V_\varepsilon^0)_{\varepsilon>0}$ are uniformly bounded with respect to ε. With the compactness Theorem 2.2.2 we can find a subsequence $\varepsilon \to 0$ and oriented varifolds $V \in C_c^0([0,T) \times \Omega \times \mathbb{S}^{n-1})'$, $V^0 \in {}^{\oplus}\mathbb{V}_{n-1}(\Omega)$ such that (6.3.23) and (6.3.24) hold. ☐

Lemma 6.3.3 (Time dependent compactness).
Let Assumptions 6.2.1 hold. For a.e. $t \in (0,T)$ there exist finite Radon measures $\alpha^t, \kappa^t \in C_0^0(\Omega)'$ and an oriented varifold ${}^{\oplus}V^t \in {}^{\oplus}\mathbb{V}_{n-1}(\Omega)$ (writing $V^t \in \mathbb{V}_{n-1}(\Omega)$ for the unoriented varifold induced by ${}^{\oplus}V^t$) such that up to a (possibly t-dependent) subsequence we have as $\varepsilon \to 0$

$$\alpha_\varepsilon^t \xrightarrow{w^*} \alpha^t \quad \textit{in} \quad C_0^0(\Omega)', \tag{6.3.26}$$

$$\kappa_\varepsilon^t \xrightarrow{w^*} \kappa^t \quad \textit{in} \quad C_0^0(\Omega)', \tag{6.3.27}$$

$${}^{\oplus}V_\varepsilon^t \xrightarrow{w^*} {}^{\oplus}V^t \quad \textit{in} \quad {}^{\oplus}\mathbb{V}_{n-1}(\Omega), \tag{6.3.28}$$

$$V_\varepsilon^t \xrightarrow{w^*} V^t \quad \textit{in} \quad \mathbb{V}_{n-1}(\Omega), \tag{6.3.29}$$

$$\textit{and} \quad \|{}^{\oplus}V^t\| = \mu^t = \|V^t\|. \tag{6.3.30}$$

Proof. Let $t \in (0,T)$ such that (6.3.15) holds. With the same arguments as in the proof of Lemma 6.3.2 the claims (6.3.26)-(6.3.29) follow from (6.3.15) and Theorem 2.2.2. The claim (6.3.30) follows from $\|V_\varepsilon^t\| = \mu_\varepsilon^t = \|{}^{\oplus}V_\varepsilon^t\|$ and (6.3.19). Note that the mass measures even converge for a subsequence independent from t. ☐

Corollary 6.3.4 (Results from Chapter 5).
Let Assumptions 6.2.1 hold. For a.e. $t \in (0,T)$ the varifold $\frac{1}{c_0}V^t$ is integral, thus there exist a $(n-1)$-rectifiable set $\Gamma_t \subseteq \Omega$ and a $\mathcal{H}^{n-1}$-measurable multiplicity function $\theta_t : \Gamma_t \longrightarrow \mathbb{N}$ such that

$$\|V^t\| = c_0\theta_t\mathcal{H}^{n-1} \llcorner \Gamma_t$$

and for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma_t$ there exists $S_{t,x} \in G(n, n-1)$ with

$$T_x\Gamma_t = c_0\theta_t(x)S_{t,x}.$$

Additionally for a.e. $t \in (0,T)$ the varifold V^t has a weak a mean curvature vector $\vec{H}_t \in L^2(\Omega, \mu^t; \mathbb{R}^n)$ such that

$$\limsup_{\rho\to 0} \rho^{1-n}\mu^t(B_\rho(x)) < \infty \quad \textit{for all } x \in \Omega,$$

$$
\begin{aligned}
|\vec{H}_t|^2\mu^t &\le \alpha^t,\\
\sigma|\vec{H}_t|^2\mu^t &\le \kappa^t,\\
\textit{and}\quad \frac{c_0}{2}|\nabla u(t,\cdot)| &\le \mu^t,
\end{aligned}
$$

in the sense of Borel measures on Ω.

Proof. Since (6.3.15) holds for a.e. $t \in (0,T)$ by Lemma 6.3.2 all of the claims follow immediately from the Theorems 5.2.3 and 5.2.5. □

We highlight important results from [MR08] which will be used in this section.

Theorem 6.3.5 (Key results from [MR08])**.**
Let $u_\varepsilon \in L^2(0,T;H^3(\Omega))$ *and assume that* (6.3.13) *holds for some* $\Lambda > 0$. *Then there exists a subsequence such that as* $\varepsilon \to 0$

$$
|\xi_\varepsilon| \xrightarrow{w^*} 0 \quad \textit{in} \quad C_0^0(\overline{\Omega}_T)' \tag{6.3.31}
$$
$$
\textit{and} \quad H_\varepsilon \nabla u_\varepsilon \mathcal{L}^{n+1} \llcorner \Omega_T \xrightarrow{w^*} \vec{H}\mu \quad \textit{in} \quad C_c^0(\Omega_T;\mathbb{R}^n)', \tag{6.3.32}
$$

with $\vec{H} \in L^2(0,T;L^2(\Omega,\mu^t;\mathbb{R}^n))$ *for* $\vec{H}(t) \coloneqq \vec{H}_t$ *for a.e.* $t \in (0,T)$.

Proof. The claim (6.3.31) follows from Proposition 6.1 in [MR08] because in the proof no more than (6.3.13) is used.

For the proof of (6.3.32) we define $\hat{H}_\varepsilon \coloneqq \frac{H_\varepsilon}{\varepsilon|\nabla u_\varepsilon|^2}\nabla u_\varepsilon \chi_{\{\nabla u_\varepsilon \neq 0\}}$. By Lemma 7.1 in [MR08] we have

$$
(\mu_\varepsilon + \xi_\varepsilon, \hat{H}_\varepsilon) \xrightarrow{w^*} (\mu, \vec{H})
$$

in the sense of measure-function pair convergence; see Definition 2.2.13. Thus we have for all $\phi \in C_c^0(\Omega_T;\mathbb{R}^n)$

$$
\int_{\Omega_T} H_\varepsilon \phi \cdot \nabla u_\varepsilon \,\mathrm{d}\mathcal{L}^{n+1} = \int_{\Omega_T} \phi \cdot \hat{H}_\varepsilon \,\mathrm{d}\hat{\mu}_\varepsilon \longrightarrow \int_{\Omega_T} \vec{H} \cdot \phi \,\mathrm{d}\mu. \qquad \square
$$

We apply the results to our setting.

Corollary 6.3.6.
Let $u_\varepsilon \in L^2(0,T;H^3(\Omega))$ *and assume that* (6.3.13) *holds for some* $\Lambda > 0$. *Then there exists a subsequence such that as* $\varepsilon \to 0$

$$
\xi_\varepsilon \xrightarrow{w^*} 0 \quad \textit{in} \quad C_0^0([0,T)\times\Omega)' \tag{6.3.33}
$$
$$
\textit{and} \quad \xi_\varepsilon^t \xrightarrow{w^*} 0 \quad \textit{in} \quad C_0^0(\Omega)' \quad \textit{for a.e. } t \in [0,T). \tag{6.3.34}
$$

Proof. Since $\xi_\varepsilon = \mathcal{L}^1 \otimes (\xi_\varepsilon^t)_{t\in[0,T)}$ we have for all $\varepsilon > 0$ that $\xi_\varepsilon(\{0\}\times\Omega) = 0$ thus we can conclude (6.3.33) from (6.3.31). This implies

$$
\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon) \longrightarrow 0 \quad \text{in} \quad L^1(\Omega_T) \cong L^1(0,T;L^1(\Omega)).
$$

This yields that there exists a subsequence $\varepsilon \to 0$ such that for a.e. $t \in [0,T)$ we have (6.3.34). □

Next we probe properties of the disintegration of ${}^{\oplus}V$.

Lemma 6.3.7 (Disintegration of the limit varifold)**.**
Let Assumptions 6.2.1 hold. For a.e. $t \in (0,T)$ there exists ${}^{\oplus}\tilde{V}^t \in C_0^0(\Omega \times \mathbb{S}^{n-1})'$ (writing $\tilde{V}^t$ for the unoriented varifolds induced by ${}^{\oplus}\tilde{V}^t$) such that

- ${}^{\oplus}V = \mathcal{L}^1 \otimes ({}^{\oplus}\tilde{V}^t)_{t\in(0,T)}$, $(0,T) \ni t \mapsto {}^{\oplus}\tilde{V}^t$ *is Borel measurable.*
- *Given $\phi \in L^1(0,T; C_0^0({}^{\oplus}G_{n-1}(\Omega)))$ the mapping*

$$(0,T) \ni t \longmapsto \int_{{}^{\oplus}G_{n-1}(\Omega)} \phi(t,\cdot,\cdot)\,\mathrm{d}{}^{\oplus}\tilde{V}_\varepsilon^t$$

  is Borel measurable.
- *For a.e. $t \in (0,T)$ we have*

$$\|{}^{\oplus}\tilde{V}^t\| = \mu^t = \|\tilde{V}^t\| \tag{6.3.35}$$

- *For a.e. $t \in (0,T)$ the varifold $\tilde{V}^t$ has the same weak mean curvature vector $\vec{H}_t \in L^2(\Omega, \mu^t; \mathbb{R}^n)$ as V^t.*

Proof. By the Disintegration Theorem, see Theorem 9.1 in Ambrosio's paper in [ADD$^+$03], we conclude that for a.e. $t \in (0,T)$ there exists an oriented varifold ${}^{\oplus}\tilde{V}^t \in C_0^0(\Omega \times \mathbb{S}^{n-1})'$ such that $t \mapsto {}^{\oplus}\tilde{V}^t$ is Borel measurable and we have

$${}^{\oplus}V = \mathcal{L}^1 \otimes ({}^{\oplus}\tilde{V}^t)_{t\in(0,T)}. \tag{6.3.36}$$

It also follows that for $\phi \in L^1(0,T; C_c^0({}^{\oplus}G_{n-1}(\Omega)))$ the mapping

$$(0,T) \ni t \longmapsto \int_{{}^{\oplus}G_{n-1}(\Omega)} \phi(t,\cdot,\cdot)\,\mathrm{d}{}^{\oplus}\tilde{V}^t$$

is Borel measurable. As before ${}^{\oplus}\tilde{V}^t$ are not only oriented varifolds but even finite Radon measures on $\Omega \times \mathbb{S}^{n-1}$.

Next we prove $\mu^t = \|{}^{\oplus}\tilde{V}^t\|$. Let $\psi \in C_c^0(0,T)$ and $\phi \in C_c^0(\Omega)$ then we have

$$\begin{aligned}
\int_0^T \psi(t) \int_\Omega \phi\,\mathrm{d}\mu^t\,\mathrm{d}t = \int_{\Omega_T} \psi\phi\,\mathrm{d}\mu \longleftarrow \int_{\Omega_T} \psi\phi\,\mathrm{d}\mu_\varepsilon &= \int_0^T \psi(t) \int_\Omega \phi\,\mathrm{d}\mu_\varepsilon^t\,\mathrm{d}t \\
&= \int_0^T \psi(t) \int_\Omega \phi\,\mathrm{d}\|{}^{\oplus}V_\varepsilon^t\|\,\mathrm{d}t = \int_0^T \psi(t) \int_{{}^{\oplus}G_{n-1}(\Omega)} \phi\,\mathrm{d}{}^{\oplus}V_\varepsilon^t\,\mathrm{d}t \\
&= \int_{{}^{\oplus}G_{n-1}(\Omega_T)} \psi\phi\,\mathrm{d}{}^{\oplus}V_\varepsilon \longrightarrow \int_{{}^{\oplus}G_{n-1}(\Omega_T)} \psi\phi\,\mathrm{d}{}^{\oplus}V \\
&= \int_0^T \psi(t) \int_{{}^{\oplus}G_{n-1}(\Omega)} \phi\,\mathrm{d}{}^{\oplus}\tilde{V}^t\,\mathrm{d}t = \int_0^T \psi(t) \int_\Omega \phi\,\mathrm{d}\|{}^{\oplus}\tilde{V}^t\|\,\mathrm{d}t.
\end{aligned}$$

By localizing in time we conclude $\mu^t = \|{}^{\oplus}\tilde{V}^t\|$. Since $\tilde{V}^t$ is defined by the projection from ${}^{\oplus}\mathbb{V}_{n-1}(\Omega)$ onto $\mathbb{V}_{n-1}(\Omega)$ we have $\|\tilde{V}^t\| = \|{}^{\oplus}\tilde{V}^t\|$. Thus (6.3.35) follows for a.e. $t \in (0,T)$.

Lastly we prove that $\vec{H}_t$ is the weak mean curvature vector of $\tilde{V}^t$. Let $\psi \in C_c^0(0,T)$ and $\eta \in C_c^0(\Omega;\mathbb{R}^n)$. Then we use that the projection from oriented varifolds onto varifolds is continuous with respect to varifold convergence and (in order of appearance in the following calculation) (6.3.36), (6.3.24), Lemma 5.2.4, (6.3.32), and (6.3.33)

$$\begin{aligned}\int_0^T \psi\langle\eta,\delta\tilde{V}^t\rangle_{C_c^0(\Omega)'}\,\mathrm{d}t &= \langle\psi\eta,\delta V\rangle_{C_c^0([0,T)\times\Omega)'} \longleftarrow \langle\psi\eta,\delta V_\varepsilon\rangle_{C_c^0([0,T)\times\Omega)'}\\ &= \int_0^T \langle\psi\eta,\delta V_\varepsilon^t\rangle\rangle_{C_c^0(\Omega)'}\,\mathrm{d}t\\ &= -\int_{\Omega_T}\psi\eta\cdot\nabla u_\varepsilon H_\varepsilon\,\mathrm{d}\mathcal{L}^{n+1} + \int_{\Omega_T}\psi\nu_\varepsilon\cdot D\eta\nu_\varepsilon\,\mathrm{d}\xi_\varepsilon\\ &\longrightarrow -\int_{\Omega_T}\psi\eta\cdot\vec{H}_t\,\mathrm{d}\mu = -\int_0^T\psi\int_\Omega\eta\cdot\vec{H}_t\,\mathrm{d}\|\tilde{V}^t\|\,\mathrm{d}t.\end{aligned}$$

By localizing in time we conclude that for a.e. $t\in(0,T)$ we have

$$\langle\eta,\delta\tilde{V}^t\rangle_{C_c^0(\Omega)'} = -\int_\Omega \eta\cdot\vec{H}_t\,\mathrm{d}\|\tilde{V}^t\|, \tag{6.3.37}$$

which concludes the proof. $\square$

Before we are able to prove that ${}^{\oplus}V$ is a De Giorgi type varifold solution for rescaled mean curvature flow we need a few technical preparations. In Chapter 3 the function f with $\overline{q}_0 = f(q_0)$ was helpful in many ways. For the estimates in this section we also consider such a function. Recall the optimal profiles $q_0,\overline{q}_0$ from Assumptions 4.1.1 and the Lemmata 4.1.2, 4.1.3. We define

$$f:\mathbb{R}\longrightarrow[-1,1],\quad f:=\begin{cases}\overline{q}_0\circ q_0^{-1}, & \text{on}\quad(-1,1)\\ \operatorname{sgn}, & \text{on}\quad\mathbb{R}\setminus(-1,1)\end{cases} \tag{6.3.38}$$

and

$$G:\mathbb{R}\longrightarrow\mathbb{R},\quad G(r):=\int_0^r f'(s)\sqrt{2W(s)}\,\mathrm{d}s. \tag{6.3.39}$$

The functions $q_0,\overline{q}_0$ have been defined in 4.1.1. These functions have the following properties.

Lemma 6.3.8 (Properties of f and G)**.**
The functions f,G from (6.3.38) and (6.3.39) satisfy

(1) $f,G\in C^0(\mathbb{R})$ *and* $G'\in C_b^0(\mathbb{R})$,

(2) $f|_{(-1,1)}, G|_{(-1,1)}\in C^\infty(-1,1)$,

(3) $2f''W = -f'W' + f - \mathrm{Id}$ *in* $(-1,1)$,

(4) $G'(q_0) = \overline{q}_0'$,

(5) $\frac{c_0}{\sigma} = G(1) - G(-1) = 2G(1)$.

Proof. Both $q_0, \overline{q}_0 : (-1,1) \longrightarrow (-1,1)$ are C^∞, strictly increasing, $q_0(\pm 1) = \pm 1 = \overline{q}_0(\pm 1)$. Thus f is well-defined, $f \in C^0(\mathbb{R})$, G is well-defined on $(-1,1)$, and the restrictions to $(-1,1)$ are smooth, i.e. $f|_{(-1,1)}, G|_{(-1,1)} \in C^\infty(-1,1)$. To prove that the improper integral in the definition of $G(\pm 1)$ exists, $G \in C^0(\mathbb{R})$ and $G' \in C^0(\mathbb{R})$ we need to examine $\lim_{r\to\pm 1} f'(r)\sqrt{2W(r)}$. For $-1 < r < 1$ we calculate

$$\begin{aligned} G'(r) &= f'(r)\sqrt{2W(r)} = \overline{q}_0'(q_0^{-1}(r))(q_0^{-1})'(r)\sqrt{2W(r)} \\ &= \frac{\overline{q}_0'(q_0^{-1}(r))}{q_0'(q_0^{-1}(r))}\sqrt{2W(r)} = \overline{q}_0'(q_0^{-1}(r)) \overset{r\to\pm 1}{\longrightarrow} 0. \end{aligned}$$

In the last step we used the ODE $q_0' = \sqrt{2W(q_0)}$ of the profile. It follows that G is well-defined on $\mathbb{R}$, since $f'(r) = 0$ for all $|r| > 1$ and we have $G \in C^0(\mathbb{R})$. With the limit from above we conclude

$$\begin{aligned} G'(1) &= \lim_{h\to 0} \frac{\int_0^1 f'\sqrt{2W}\,\mathrm{d}\mathcal{L}^1 - \int_0^{1-h} f'\sqrt{2W}\,\mathrm{d}\mathcal{L}^1}{h} = \lim_{h\to 0}\frac{1}{h}\int_{1-h}^1 f'\sqrt{2W}\,\mathrm{d}\mathcal{L}^1 \\ &= \lim_{r\to 1} f'(r)\sqrt{2W(r)} = 0 \end{aligned}$$

using a version of the Fundamental Theorem of Calculus. We argue analogously for $G'(-1)$. It follows that $G' \in C^0(\mathbb{R})$. Furthermore we have

$$\frac{c_0}{\sigma} = \int_{\mathbb{R}} \overline{q}_0' q_0'\,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} f'(q_0)|q_0'|^2\,\mathrm{d}\mathcal{L}^1 = \int_{-1}^1 f'(r)\sqrt{2W(r)}\,\mathrm{d}r = G(1) - G(-1) = 2G(1).$$

In the last step we used that f', W are even functions. f' is even because f is odd. Thus G is well-defined. We also get $G'(q_0) = \overline{q}_0'$ from $\overline{q}_0' = f'(q_0)\sqrt{2W(q_0)}$. For all $r \in (-1,1)$ we have

$$|G'(r)| = |G'(q_0(q_0^{-1}(r)))| = |\overline{q}_0'(q_0^{-1}(r))| \le \|\overline{q}_0\|_{C^1(\mathbb{R})}.$$

Furthermore we have

$$\begin{aligned} 0 &= -\overline{q}_0'' + \overline{q}_0 - q_0 \\ &= -2f''(q_0)W(q_0) - f'(q_0)q_0'' + f(q_0) - q_0 \\ &= -2f''(q_0)W(q_0) - f'(q_0)W'(q_0) + f(q_0) - q_0. \end{aligned}$$

It follows for all $r \in (-1,1)$

$$0 = -2f''(r)W(r) - f'(r)W'(r) + f(r) - r.$$

This ODE characterizes f on $(-1,1)$. □

We additionally consider modified versions of the functions $f, G.$ and $\overline{q}_0$. Given $0 < \delta < \frac{1}{2}$ we choose an even test function $\chi_\delta \in C_c^\infty(\mathbb{R})$ with

- $0 \le \chi_\delta \le 1$
- $\chi_\delta(r) = 1$ for $|r| < 1 - 2\delta$
- $\chi_\delta(r) = 0$ for $|r| > 1 - \delta$

- $|\chi_\delta'(r)| \le \frac{C}{\delta}$, $|\chi_\delta''(r)| \le \frac{C}{\delta^2}$, and $|\chi_\delta'''(r)| \le \frac{C}{\delta^3}$ for some $C > 0$ independent from δ.

With this cut-off function we modify the functions f, G.

Definition 6.3.9.
Let $0 < \delta < \frac{1}{2}$ then we define

$$\begin{aligned} f_\delta(r) &:= \chi_\delta(r) f(r) + (1 - \chi_\delta(r)) \operatorname{sgn}(r) && \textit{for} \quad r \in \mathbb{R} \\ \overline{q}_\delta(r) &:= f_\delta(q_0(z)) && \textit{for} \quad r \in \mathbb{R} \\ G_\delta(r) &:= \int_0^r f_\delta'(s) \sqrt{2W(s)}\, \mathrm{d}s && \textit{for} \quad r \in \mathbb{R}. \end{aligned}$$

We also define $\sigma_\delta > 0$ by

$$\frac{c_0}{\sigma_\delta} := \int_{\mathbb{R}} (|\overline{q}_\delta'|^2 + |\overline{q}_\delta''|^2)\, \mathrm{d}\mathcal{L}^1. \tag{6.3.40}$$

The functions have the following properties.

Lemma 6.3.10 (Properties of the modified auxiliary functions f_δ, G_δ)**.**
The modified functions satisfy $\operatorname{supp}(f_\delta'), \operatorname{supp}(f_\delta''), \operatorname{supp}(f_\delta''') \subseteq [-1+\delta, 1-\delta]$,

$$\begin{aligned} f_\delta' &= \chi_\delta f' + (f - \operatorname{sgn})\chi_\delta', && (6.3.41) \\ f_\delta'' &= \chi_\delta f'' + 2\chi_\delta' f' + (f - \operatorname{sgn})\chi_\delta'', && (6.3.42) \\ f_\delta''' &= \chi_\delta f''' + 3\chi_\delta' f'' + 3\chi_\delta'' f' + (f - \operatorname{sgn})\chi_\delta''', && (6.3.43) \end{aligned}$$

and

- $\|f_\delta\|_{C^0(\mathbb{R})} \le 1$ *and* $\|f_\delta'\|_{C^0(\mathbb{R})}, \|f_\delta''\|_{C^0(\mathbb{R})}, \|f_\delta'''\|_{C^0(\mathbb{R})} \le C(\delta)$,
- $\|G_\delta'\|_{C^0(\mathbb{R})}, \|G_\delta''\|_{C^0(\mathbb{R})}, \|G_\delta'''\|_{C^0(\mathbb{R})} \le C(\delta)$,
- $G_\delta, G_\delta', G_\delta''$ *are Lipschitz continuous,*
- $G_\delta'(q_0) = \overline{q}_\delta'$,
- $G_\delta \longrightarrow G \quad \textit{as} \quad \delta \to 0 \quad \textit{in} \quad C_b^0(\mathbb{R})$,
- $\overline{q}_\delta' \longrightarrow \overline{q}_0' \quad \textit{as} \quad \delta \to 0 \quad \textit{in} \quad H^1(\mathbb{R})$,
- $\lim_{\delta \to 0} \sigma_\delta = \sigma$.

Proof. Let $0 < \delta < \frac{1}{2}$ then we have for all $r \in \mathbb{R}$

$$\begin{aligned} |f_\delta(r)| &\le \chi_\delta(r)|f(r)| + (1 - \chi_\delta(r))|\operatorname{sgn}(r)| \le \chi_\delta + 1 - \chi_\delta = 1, \\ |f_\delta'(r)| &\le \|f'\|_{C^0[-1+\delta,1-\delta]} + \frac{C}{\delta} \le C(\delta), \\ |f_\delta''(r)| &\le \|f''\|_{C^0[-1+\delta,1-\delta]} + \frac{2C}{\delta}\|f'\|_{C^0[-1+\delta,1-\delta]} + \frac{C}{\delta^2} \le C(\delta), \\ |f_\delta'''(r)| &\le C(\delta) \sum_{j=0}^{3} \|f^{(j)}\|_{C^0[-1+\delta,1-\delta]} \le C(\delta), \\ |G_\delta'(q_0(r))| &= f_\delta'(q_0(r))\sqrt{2W(q_0(r))} = f_\delta'(q_0(r)) q_0'(r) = \overline{q}_\delta'(r) \le C(\delta), \end{aligned}$$

$$
\begin{aligned}
|G_\delta''(r)| &= \left| f_\delta''(r)\sqrt{2W(r)} + f_\delta'(r)\frac{W'(r)}{\sqrt{2W(r)}} \right| = \left| f_\delta''(r)\sqrt{2W(r)} - 2\sqrt{2} r f_\delta'(r) \right| \\
&\le \sqrt{2}\|f_\delta''\|_{C^0(\mathbb{R})}\|\sqrt{W}\|_{C^0[-1,1]} + 2\sqrt{2}\|f_\delta'\|_{C^0(\mathbb{R})} \le C(\delta), \\
|G_\delta'''(r)| &= \left| f_\delta'''(r)\sqrt{2W(r)} - 4\sqrt{2} r f_\delta''(r) - 2\sqrt{2} f_\delta'(r) \right| \\
&\le \sqrt{2}\|f_\delta'''\|_{C^0(\mathbb{R})}\|W\|_{C^0[-1,1]} + 4\sqrt{2}\|f_\delta''\|_{C^0(\mathbb{R})} + 2\sqrt{2}\|f_\delta'\|_{C^0(\mathbb{R})} \le C(\delta).
\end{aligned}
$$

Since $G_\delta', G_\delta'', G_\delta'''$ are bounded on the whole space $\mathbb{R}$ we get that the functions $G_\delta, G_\delta', G_\delta''$ are Lipschitz continuous.

Next we show that G_δ converges uniformly to G_δ as $\delta \to 0$. For $r \in \mathbb{R}$ we estimate

$$
\begin{aligned}
|G_\delta(r) - G(r)| &= \left| \int_0^r (f_\delta - f)'(s)\sqrt{2W(s)}\,\mathrm{d}s \right| \\
&= \left| (f_\delta(s) - f(s))\sqrt{2W(s)}\Big|_0^r - \int_0^r (f_\delta(s) - f(s))\frac{W'(s)}{\sqrt{2W(s)}}\,\mathrm{d}s \right| \\
&\le \left| (f_\delta(r) - f(r))\sqrt{2W(r)} \right| + \left| 2\sqrt{2}\int_0^r \left| f_\delta(s) - f(s) \right| |s|\,\mathrm{d}s \right|.
\end{aligned}
$$

In the last step we used the explicit formula for W. For the following step we apply $f_\delta - f = (1-\chi_\delta)(\operatorname{sgn} - f)$ which vanishes outside $(-1,1)$:

$$
\begin{aligned}
|G_\delta(r) - G(r)| &\le \left| (1-\chi_\delta(r))(\operatorname{sgn}(r) - f(r))\sqrt{2W(r)} \right| \\
&\quad + \left| 2\sqrt{2}\int_0^r (1-\chi_\delta(s))(\operatorname{sgn}(s) - f(s)) s\,\mathrm{d}s \right| \\
&\le \left| (1-\chi_\delta(r))\sqrt{2W(r)} \right| + 2\sqrt{2}\int_0^1 (1-\chi_\delta(s))\,\mathrm{d}s \\
&\le \sqrt{2W(1-2\delta)} + 2\sqrt{2}(1-(1-2\delta)) = \sqrt{2W(1-2\delta)} + 4\sqrt{2}\delta.
\end{aligned}
$$

This holds uniformly for all $r \in \mathbb{R}$ and thus G_δ converges uniformly to G as $\delta \to 0$.

For the convergence of $\overline{q}_\delta' \longrightarrow \overline{q}_0'$ as $\delta \to 0$ in $H^1(\mathbb{R})$ we have to show

$$
\int_{\mathbb{R}} |\overline{q}_\delta' - \overline{q}_0'|^2\,\mathrm{d}\mathcal{L}^1 \longrightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}} |\overline{q}_\delta'' - \overline{q}_0''|^2\,\mathrm{d}\mathcal{L}^1 \longrightarrow 0 \quad \text{as} \quad \delta \to 0.
$$

We start with the first term. Using the already proven properties we obtain that

$$
\begin{aligned}
\int_{\mathbb{R}} \left|\overline{q}_\delta' - \overline{q}_0'\right|^2\,\mathrm{d}\mathcal{L}^1 &= \int_{\mathbb{R}} \left|G_\delta'(q_0) - G'(q_0)\right|^2\,\mathrm{d}\mathcal{L}^1 = \int_{-1}^1 2\left|f_\delta'(r) - f'(r)\right|^2 W(r)\,\mathrm{d}r \\
&= \int_0^1 4\left|\chi_\delta(r) f'(r) + (f(r) - \operatorname{sgn}(r))\chi_\delta'(r) - f'(r)\right|^2 W(r)\,\mathrm{d}r \\
&\le \int_0^1 8\left|1-\chi_\delta(r)\right|^2 \left|f'(r)\right|^2 W(r)\,\mathrm{d}r + \int_0^1 8\left|f(r) - 1\right|^2 \left|\chi_\delta'(r)\right|^2 W(r)\,\mathrm{d}r.
\end{aligned}
$$

From the definition and properties of χ_δ we get

$$\begin{aligned}\int_{\mathbb{R}} \left|\overline{q}_\delta' - \overline{q}_0'\right|^2 \mathrm{d}\mathcal{L}^1 &\leq \int_{1-2\delta}^{1} 8\left|1-\chi_\delta(r)\right|^2 \left|f'(r)\right|^2 W(r)\,\mathrm{d}r \\ &\quad + \int_{1-2\delta}^{1-\delta} 8\left|f(r)-1\right|^2 \left|\chi_\delta'(r)\right|^2 W(r)\,\mathrm{d}r \\ &\leq \int_{1-2\delta}^{1} 8\left|f'(r)\right|^2 W(r)\,\mathrm{d}r + 8\delta\left|f(1-2\delta)-1\right|^2 \cdot \frac{C}{\delta^2} W(1-2\delta).\end{aligned}$$

In the first term we can transform $r \longmapsto q_0(r)$, in the second we use $2W(1-2\delta) \leq C\delta^2$ for $0 < \delta < \frac{1}{2}$. We get

$$\int_{\mathbb{R}} \left|\overline{q}_\delta' - \overline{q}_0'\right|^2 \mathrm{d}\mathcal{L}^1 \leq 4\int_{q_0^{-1}(1-2\delta)}^{\infty} \left|\overline{q}_0'(r)\right|^2 \mathrm{d}r + C\delta\left|f(1-2\delta)-1\right|^2 \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

This follows from the continuity of f and $\overline{q}' \in L^2(\mathbb{R})$. For the second derivatives we have

$$\begin{aligned}\int_{\mathbb{R}} |\overline{q}_\delta'' - \overline{q}_0''|^2 \,\mathrm{d}\mathcal{L}^1 &= \int_{\mathbb{R}} \left|G_\delta''(q_0) - G''(q_0)\right|^2 |q_0'|^2 \,\mathrm{d}\mathcal{L}^1 = 2\int_0^1 \left|G_\delta'' - G''\right|^2 \sqrt{2W}\,\mathrm{d}\mathcal{L}^1 \\ &= 2\int_0^1 \left| f_\delta''\sqrt{2W} + \frac{f_\delta' W'}{\sqrt{2W}} - f''\sqrt{2W} - \frac{f'W'}{\sqrt{2W}} \right|^2 \sqrt{2W}\,\mathrm{d}\mathcal{L}^1 \\ &= 2\int_0^1 \frac{\left|2Wf_\delta'' + f_\delta'W' - 2Wf'' - f'W'\right|^2}{\sqrt{2W}}\,\mathrm{d}\mathcal{L}^1.\end{aligned}$$

We plug in (6.3.41)-(6.3.42) and afterwards (3) from Lemma 6.3.8 which results in

$$\begin{aligned}&\int_{\mathbb{R}} |\overline{q}_\delta'' - \overline{q}_0''|^2 \,\mathrm{d}\mathcal{L}^1 \\ &\quad = 2\int_0^1 \frac{\left|[\chi_\delta - 1][2Wf'' + f'W'] + 4\chi_\delta' f'W + [2\chi_\delta''W + W'\chi_\delta'][f - \operatorname{sgn}]\right|^2}{\sqrt{2W}}\,\mathrm{d}\mathcal{L}^1 \\ &\quad \leq 6\int_{1-2\delta}^1 \frac{|1-\chi_\delta|^2|f - \operatorname{Id}|^2 + 8|\chi_\delta' G'|^2 W + (4|\chi_\delta'' W|^2 + 2|W'\chi_\delta'|^2)|f-1|^2}{\sqrt{2W}}\,\mathrm{d}\mathcal{L}^1.\end{aligned}$$

Now we use $|G'| \leq C$, $|\chi_\delta'| \leq \frac{C}{\delta}$, $|\chi_\delta''| \leq \frac{C}{\delta^2}$, $W'(r) = -4r\sqrt{W(r)}$, $W(1-2\delta) \leq C\delta^2$ and get

$$\begin{aligned}&\int_{\mathbb{R}} |\overline{q}_\delta'' - \overline{q}_0''|^2 \,\mathrm{d}\mathcal{L}^1 \\ &\quad \leq C(\chi)\int_{1-2\delta}^1 \Big(\frac{|f(r) - r|^2}{\sqrt{2W(r)}} + \frac{1}{\delta^2}\cdot\delta^2 + \Big(\frac{1}{\delta^4}\delta^3 + r^2\delta\cdot\frac{1}{\delta^2}\Big)|f(r)-1|^2 \Big)\,\mathrm{d}r \\ &\quad \leq C(\chi)\int_{1-2\delta}^1 \frac{|f(r)-r|^2}{\sqrt{2W(r)}}\,\mathrm{d}r + C(\chi)\Big(\delta + 2|f(1-2\delta) - 1|^2\Big).\end{aligned}$$

Owing to the continuity of f the second term vanishes for $\delta \to 0$. It remains to prove that the integral goes to 0 as well. We can do so by transforming the expressions back as we did for the other integral as well. Since $\overline{q}_0'' \in L^2(\mathbb{R})$ we get

$$\int_{1-2\delta}^1 \frac{|f(r)-r|^2}{\sqrt{2W(r)}}\,\mathrm{d}r = \int_{q_0^{-1}(1-2\delta)}^{\infty} \left|\overline{q}_0''\right|^2 \mathrm{d}\mathcal{L}^1 \longrightarrow 0 \quad \text{as} \quad \delta \to 0.$$

From the $H^1(\mathbb{R})$-convergence of $\overline{q}_0'$ it follows that $\sigma_\delta \to \sigma$ as $\delta \to 0$. $\square$

Owing to the cut-off and the uniform convergence of G_δ we can consider $f'_\delta(u_\varepsilon)$ instead of $f'(u_\varepsilon)$. Similar to ϑ_ε from Chapter 5 we introduce a modified diffuse area measure $\beta_{\varepsilon,\delta}$. Here the modification is achieved using G_δ instead of a PDE.

Definition 6.3.11 (Modified diffuse area measures)**.**
Let $u_\varepsilon \in L^2(0,T;H^1(\Omega))$. We define for $\varepsilon > 0$, $t \in (0,T)$, and $0 < \delta < \frac{1}{2}$

$$\beta^t_{\varepsilon,\delta} := \Big(\frac{1}{\varepsilon}|G'_\delta(u_\varepsilon)|^2 + \varepsilon\big|G''_\delta(u_\varepsilon)\nabla u_\varepsilon\big|^2\Big)\mathcal{L}^n \llcorner \Omega \quad \textit{and} \quad \beta_{\varepsilon,\delta} := \mathcal{L}^1 \otimes (\beta^t_{\varepsilon,\delta})_{t\in(0,T)}. \tag{6.3.44}$$

We can extend the compactness properties of $(\mu_\varepsilon)_{\varepsilon>0}$ and $(\mu^t_\varepsilon)_{\varepsilon>0}$ to $(\beta_{\varepsilon,\delta})_{\varepsilon>0}$ and $(\beta^t_{\varepsilon,\delta})_{\varepsilon>0}$.

Lemma 6.3.12 (Compactness of the modified diffuse area measures)**.**
Let $u_\varepsilon \in L^2(0,T;H^3(\Omega))$, $0 < \delta < \frac{1}{2}$, and $t \in (0,T)$ then there exists $C(\delta) > 0$, such that for all $\varepsilon > 0$ we have

$$\beta^t_{\varepsilon,\delta}(\Omega) \le C(\delta)\mu^t_\varepsilon(\Omega). \tag{6.3.45}$$

Proof. Recall that $G'_\delta(u_\varepsilon) = f'_\delta(u_\varepsilon)\sqrt{2W(u_\varepsilon)}$ from Definition 6.3.9. Furthermore we have $\|f'_\delta\|_{C^0(\mathbb{R})}, \|G''_\delta\|_{C^0(\mathbb{R})} \le C(\delta)$ by Lemma 6.3.10. It follows

$$\begin{aligned}
\beta^t_{\varepsilon,\delta}(\Omega) &= \int_\Omega \Big(\frac{1}{\varepsilon}|G'_\delta(u_\varepsilon(t,\cdot))|^2 + \varepsilon\big|\nabla G'_\delta(u_\varepsilon(t,\cdot))\big|^2\Big)\,\mathrm{d}\mathcal{L}^n \\
&= \int_\Omega \Big(2|f'_\delta(u_\varepsilon(t,\cdot))|^2\frac{1}{\varepsilon}W(u_\varepsilon(t,\cdot)) + |G''_\delta(u_\varepsilon(t,\cdot))|^2\varepsilon\big|\nabla u_\varepsilon(t,\cdot)\big|^2\Big)\,\mathrm{d}\mathcal{L}^n \\
&\le C(\delta)\int_\Omega \Big(\frac{\varepsilon}{2}\big|\nabla u_\varepsilon(t,\cdot)\big|^2 + \frac{1}{\varepsilon}W(u_\varepsilon(t,\cdot))\Big)\,\mathrm{d}\mathcal{L}^n = C(\delta)\mu^t_\varepsilon(\Omega).
\end{aligned}$$ □

Next we want to calculate the weak*-limit of $\beta^t_{\varepsilon,\delta}$. As in Chapter 5 we use a blow-up argument. The result is essential for the existence proof of the generalized normal velocity.

Theorem 6.3.13 (Blow-up argument)**.**
Let $(u_\varepsilon)_{\varepsilon>0}$ in $L^2(0,T;H^3(\Omega))$, $0 < \delta < \frac{1}{2}$, $t \in (0,T)$, and any subsequence $\varepsilon \to 0$ (possibly depending on t) such that (6.3.19), (6.3.25), *and*

$$\limsup_{\varepsilon\to 0} \kappa^t_\varepsilon(\Omega) < \infty \tag{6.3.46}$$

are satisfied. Then for all $0 < \delta < \frac{1}{2}$ and the whole subsequence $\varepsilon \to 0$ it holds

$$\beta^t_{\varepsilon,\delta} \xrightarrow{w^*} \frac{1}{\sigma_\delta}\mu^t \quad \textit{in} \quad C^0_c(\Omega)', \tag{6.3.47}$$

with $\sigma_\delta > 0$ from (6.3.40)*.*

The proof for the first claim is done throughout Section 6.4.

This result enables us to prove convergence for $(\beta_{\varepsilon,\delta})_{\varepsilon>0}$.

Proposition 6.3.14.
There exists a subsequence $\varepsilon \to 0$ such that (6.3.19) *and* (6.3.25) *hold for almost all $t \in (0,T)$ and such that*

$$\beta_{\varepsilon,\delta} \xrightarrow{w^*} \frac{1}{\sigma_\delta}\mu \quad \textit{in} \quad C^0_c([0,T)\times\Omega)'. \tag{6.3.48}$$

Proof. First we restrict ourselves to a subsequence $\varepsilon \to 0$ such that (6.3.19) and (6.3.25) hold for almost all $t \in (0,T)$. For $\varepsilon > 0$, $k \in \mathbb{N}$, we define the sets

$$\mathcal{B}_{\varepsilon,k} \coloneqq \{t \in (0,T) \mid \kappa_\varepsilon^t(\Omega) > k\}. \tag{6.3.49}$$

We then obtain from (6.3.13) that

$$\Lambda \geq \int_0^T \kappa_\varepsilon^t(\Omega)\,\mathrm{d}t \geq \mathcal{L}^1(\mathcal{B}_{\varepsilon,k})k. \tag{6.3.50}$$

Next we define the Radon-measures $\beta^t_{\varepsilon,\delta,k}$ by

$$\beta^t_{\varepsilon,\delta,k} \coloneqq \begin{cases} \beta^t_{\varepsilon,\delta} & \text{for } t \in (0,T) \setminus \mathcal{B}_{\varepsilon,k}, \\ \frac{1}{\sigma_\delta}\mu^t & \text{for } t \in \mathcal{B}_{\varepsilon,k}. \end{cases} \tag{6.3.51}$$

Theorem 6.3.13 yields for any subsequence $\varepsilon_j \to 0$ $(j \to \infty)$ with

$$\limsup_{j\to\infty} \kappa^t_{\varepsilon_j}(\Omega) < \infty$$

that

$$\beta^t_{\varepsilon_j,\delta} \xrightarrow{w^*} \frac{1}{\sigma_\delta}\mu^t \quad \text{as} \quad j \to \infty \quad \text{in} \quad C^0_c(\Omega)'. \tag{6.3.52}$$

By (6.3.51), (6.3.52) we therefore obtain for any $\eta \in C^0_c([0,T)\times\Omega)$ with $\eta \geq 0$, $k \in \mathbb{N}$ and almost all $t \in (0,T)$

$$\langle \eta(t,\cdot), \beta^t_{\varepsilon,\delta,k}\rangle_{C^0_c(\Omega)'} \longrightarrow \frac{1}{\sigma_\delta}\langle \eta(t,\cdot), \mu^t\rangle_{C^0_c(\Omega)'} \quad \text{as} \quad \varepsilon \to 0. \tag{6.3.53}$$

Furthermore, (6.3.45) yields

$$\begin{aligned} |\langle \eta(t,\cdot), \beta^t_{\varepsilon,\delta,k}\rangle_{C^0_c(\Omega)'}| &= (1 - \chi_{\mathcal{B}_{\varepsilon,k}}(t))\langle \eta(t,\cdot), |\beta^t_{\varepsilon,\delta}|\rangle_{C^0_c(\Omega)'} + \frac{1}{\sigma_\delta}\chi_{\mathcal{B}_{\varepsilon,k}}(t)\langle \eta(t,\cdot), |\mu^t|\rangle_{C^0_c(\Omega)'} \\ &\leq C(\delta,\Lambda)\|\eta\|_{C^0(\overline{\Omega_T})}. \end{aligned} \tag{6.3.54}$$

The Dominated Convergence Theorem, (6.3.53), and (6.3.54) imply

$$\int_0^T \langle \eta(t,\cdot), \beta^t_{\varepsilon,\delta,k}\rangle_{C^0_c(\Omega)'}\,\mathrm{d}t \longrightarrow \frac{1}{\sigma_\delta}\int_0^T \langle \eta(t,\cdot), \mu^t\rangle_{C^0_c(\Omega)'}\,\mathrm{d}t \quad \text{as} \quad \varepsilon \to 0. \tag{6.3.55}$$

Further we obtain that

$$\begin{aligned} \left| \int_0^T \langle \eta(t,\cdot), \beta^t_{\varepsilon,\delta}\rangle_{C^0_c(\Omega)'}\,\mathrm{d}t - \int_0^T \langle \eta(t,\cdot)\beta^t_{\varepsilon,\delta,k}\rangle_{C^0_c(\Omega)'}\,\mathrm{d}t \right| &\leq \int_{\mathcal{B}_{\varepsilon,k}} \left|\left\langle \eta(t,\cdot), \beta^t_{\varepsilon,\delta} - \frac{1}{\sigma_\delta}\mu^t\right\rangle\right| \mathrm{d}t \\ &\leq C(\delta)\int_{\mathcal{B}_{\varepsilon,k}} \langle \eta(t,\cdot), \mu^t_\varepsilon\rangle_{C^0_c(\Omega)'}\,\mathrm{d}t. \end{aligned} \tag{6.3.56}$$

For $k \in \mathbb{N}$ fixed we deduce from (6.3.13) (6.3.50), (6.3.56) that

$$\limsup_{\varepsilon\to 0}\left|\int_0^T \langle \eta(t,\cdot),\beta^t_{\varepsilon,\delta}\rangle_{C^0_c(\Omega)'}\,\mathrm{d}t-\frac{1}{\sigma_\delta}\int_0^T \langle \eta(t,\cdot),\mu^t\rangle_{C^0_c(\Omega)'}\,\mathrm{d}t\right|$$
$$\leq \lim_{\varepsilon\to 0}\left|\int_0^T \langle \eta(t,\cdot),\beta^t_{\varepsilon,\delta,k}\rangle_{C^0_c(\Omega)'}\,\mathrm{d}t-\frac{1}{\sigma_\delta}\int_0^T \langle \eta(t,\cdot),\mu^t\rangle_{C^0_c(\Omega)'}\,\mathrm{d}t\right|+\frac{C(\delta,\Lambda,\eta)}{k}. \tag{6.3.57}$$

By (6.3.55) and since $k\in\mathbb{N}$ was arbitrary this proves the Proposition. □

Similar to Lemma 5.5.1 we want to ignore certain terms from the product rule in the limit $\varepsilon\to 0$. This is achieved in the following lemma.

Lemma 6.3.15.
Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^2(0,T;H^3(\Omega))$, let $\phi\in C^2_c([0,T)\times\Omega)$ and $\tau\in(0,T]$. If either of the limits in the following identity exists, the other limit exists as well and we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_\tau}\Big(\frac{1}{\varepsilon}|\phi G_\delta(u_\varepsilon)|^2+\varepsilon\big|\nabla[\phi G'_\delta(u_\varepsilon)]\big|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1} \tag{6.3.58}$$
$$=\lim_{\varepsilon\to 0}\int_{\Omega_\tau}|\phi|^2\Big(\frac{1}{\varepsilon}|G'_\delta(u_\varepsilon)|^2+\varepsilon\big|G''_\delta(u_\varepsilon)\nabla u_\varepsilon\big|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1}.$$

Proof. Since $\mathrm{supp}(\phi)\Subset\Omega_T$ we can find a bounded and open set with C^1-boundary between $\mathrm{supp}(\phi)$ and Ω_T such that we can do a partial integration. We calculate

$$\int_{\Omega_\tau}\Big(\varepsilon\big|\nabla[\phi G'_\delta(u_\varepsilon)]\big|^2+\frac{1}{\varepsilon}|\phi G_\delta(u_\varepsilon)|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1}$$
$$=\int_{\Omega_\tau}\Big(\varepsilon\big|G'_\delta(u_\varepsilon)\nabla\phi+\phi G''_\delta(u_\varepsilon)\nabla u_\varepsilon\big|^2+\frac{1}{\varepsilon}|\phi G'_\delta(u_\varepsilon)|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1}$$
$$=\int_{\Omega_\tau}\Big(\varepsilon\big|G'_\delta(u_\varepsilon)\nabla\phi\big|^2+\frac{\varepsilon}{2}\nabla[G'_\delta(u_\varepsilon)^2]\cdot\nabla[\phi^2]+\varepsilon\big|\phi G''_\delta(u_\varepsilon)\nabla u_\varepsilon\big|^2+\frac{1}{\varepsilon}|\phi G'_\delta(u_\varepsilon)|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1}.$$

The claim is that the first and second term in $(\,\cdot\,)$ vanish as $\varepsilon\to 0$ in $L^1(\Omega_\tau)$. We estimate with a partial integration and the boundedness of G'_δ

$$\left|\int_{\Omega_\tau}\varepsilon\Big(\big|G'_\delta(u_\varepsilon)\nabla\phi\big|^2+\frac{1}{2}\nabla[G'_\delta(u_\varepsilon)^2]\cdot\nabla[\phi^2]\Big)\,\mathrm{d}\mathcal{L}^{n+1}\right|$$
$$=\left|\int_{\Omega_\tau}\varepsilon|G'_\delta(u_\varepsilon)|^2\Big(\big|\nabla\phi\big|^2-\frac{1}{2}\Delta(\phi^2)\Big)\,\mathrm{d}\mathcal{L}^{n+1}\right|\leq\varepsilon C(\delta,\phi,\Omega_\tau). \qquad □$$

To show that V from Lemma 6.3.7 is a De Giorgi type varifold solution for rescaled mean curvature flow we mainly have to establish the existence of a generalized mean curvature vector $\vec{H}$, the existence of a generalized normal velocity $\mathcal{V}$ and the motion law which connects $\vec{H}$ with $\mathcal{V}$. For the existence of a generalized mean curvature vector we can use the results from Theorem 5.2.3. The following theorem proves the existence of a generalized normal velocity.

Theorem 6.3.16 (Existence of generalized normal velocity).
Let Assumptions 6.2.1 hold, we use the Notations from Lemma 6.3.7. Then there exists $\mathcal{V} \in L^2(0,T; L^2(\Omega, \mu^t))$ *such that for all* $\phi \in C_c^2([0,T) \times \Omega)$ *and a.e.* $\tau \in (0,T)$

$$\frac{c_0}{2} \int_{\Omega_\tau} u \partial_t \phi \,\mathrm{d}\mathcal{L}^{n+1} - \int_{\Omega_\tau} \mathcal{V} \phi \,\mathrm{d}\mu = \frac{c_0}{2} \int_\Omega u(\tau, \cdot) \phi(\tau, \cdot) \,\mathrm{d}\mathcal{L}^n - \frac{c_0}{2} \int_\Omega u_0 \phi(0, \cdot) \,\mathrm{d}\mathcal{L}^n. \quad (6.3.59)$$

For all $\tau \in (0,T]$ *the velocity* $\mathcal{V}$ *can be estimated by*

$$\frac{1}{\sigma} \int_{\Omega_\tau} |\mathcal{V}|^2 \,\mathrm{d}\mu \leq \liminf_{\varepsilon \to 0} \left\langle \varepsilon \mathcal{A}_\varepsilon \partial_t u_\varepsilon, \partial_t u_\varepsilon \right\rangle_{L^2(0,\tau; H^1(\Omega))'}. \quad (6.3.60)$$

In addition we have $u \in BV(\Omega_T)$ *and* $\partial_t u$ *is a finite Radon measure on* Ω_T. *We have for all* $\tau \in (0,T]$ *and all* $\phi \in C_c^2(\Omega_\tau)$

$$-\frac{c_0}{2} \int_{\Omega_\tau} \phi \,\mathrm{d}\partial_t u = \int_{\Omega_\tau} \mathcal{V} \phi \,\mathrm{d}\mu. \quad (6.3.61)$$

Proof. Let $\phi \in C_c^2([0,T) \times \Omega)$. First we prove that for a.e. $\tau \in (0,T)$ we have

$$\begin{aligned} \frac{c_0}{2\sigma} \int_{\Omega_\tau} u \partial_t \phi \,\mathrm{d}\mathcal{L}^{n+1} &= \frac{c_0}{2\sigma} \int_\Omega \phi(\tau, \cdot) u(\tau, \cdot) \,\mathrm{d}\mathcal{L}^n - \frac{c_0}{2\sigma} \int_\Omega \phi(0, \cdot) u_0 \,\mathrm{d}\mathcal{L}^n \\ &\quad - \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \left\langle \phi G_\delta'(u_\varepsilon), \partial_t u_\varepsilon \right\rangle_{L^2(0,\tau; H^1(\Omega))'}. \end{aligned} \quad (6.3.62)$$

Let $\tau \in (0,T)$. We start by using the auxiliary function G, $u(t, \cdot) \in BV(\Omega; \{\pm 1\})$ for $t \in (0,\tau)$, $\frac{c_0}{2\sigma} = G(1)$, and that G is odd

$$\frac{c_0}{2\sigma} \int_{\Omega_\tau} u \partial_t \phi \,\mathrm{d}\mathcal{L}^{n+1} = \int_{\Omega_\tau} G(u) \partial_t \phi \,\mathrm{d}\mathcal{L}^{n+1}.$$

Next we use $G_\delta \longrightarrow G$ in $C_b^0(\mathbb{R})$ by Lemma 6.3.10 and get

$$\int_{\Omega_\tau} G(u) \partial_t \phi \,\mathrm{d}\mathcal{L}^{n+1} = \lim_{\delta \to 0} \int_{\Omega_\tau} G_\delta(u) \partial_t \phi \,\mathrm{d}\mathcal{L}^{n+1}.$$

Applying that $u_\varepsilon \longrightarrow u$ a.e. in Ω_T, $G_\delta \in C_b^0(\mathbb{R})$, and the Dominated Convergence Theorem we obtain that

$$\lim_{\delta \to 0} \int_{\Omega_\tau} G_\delta(u) \partial_t \phi \,\mathrm{d}\mathcal{L}^{n+1} = \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \int_{\Omega_\tau} G(u_\varepsilon) \partial_t \phi \,\mathrm{d}\mathcal{L}^{n+1}.$$

We have $\lim_{t \to 0} u_\varepsilon(t, \cdot) = u_{0,\varepsilon}$ a.e. in Ω by (6.3.16), $G_\delta \in C_b^1(\mathbb{R})$ and thus $\lim_{t \to 0} G_\delta(u_\varepsilon(t, \cdot)) = G_\delta(u_{0,\varepsilon})$ a.e. in Ω. Also $u_\varepsilon \in H^1(0,T; H^1(\Omega)') \cap C^0([0,T]; H^1(\Omega))$ and $G_\delta \in C_b^2(\mathbb{R})$ imply $G_\delta(u_\varepsilon) \in H^1(0,T; H^1(\Omega)') \cap C^0([0,T]; L^\infty(\Omega))$ such that

$$\begin{aligned} \int_{\Omega_\tau} G_\delta(u_\varepsilon) \partial_t \phi \,\mathrm{d}\mathcal{L}^{n+1} &= \int_\Omega \phi(\tau, \cdot) G_\delta(u_\varepsilon(\tau, \cdot)) \,\mathrm{d}\mathcal{L}^n - \int_\Omega \phi(0, \cdot) G_\delta(u_{0,\varepsilon}) \,\mathrm{d}\mathcal{L}^n \\ &\quad - \left\langle \phi, \partial_t G_\delta(u_\varepsilon) \right\rangle_{L^2(0,\tau; H^1(\Omega))'} \\ &= \int_\Omega \phi(\tau, \cdot) G_\delta(u_\varepsilon(\tau, \cdot)) \,\mathrm{d}\mathcal{L}^n - \int_\Omega \phi(0, \cdot) G_\delta(u_{0,\varepsilon}) \,\mathrm{d}\mathcal{L}^n \\ &\quad - \left\langle \phi G_\delta'(u_\varepsilon), \partial_t u_\varepsilon \right\rangle_{L^2(0,\tau; H^1(\Omega))'} \end{aligned}$$

For the time-independent integrals we use (6.3.17), (6.3.18) and get for a.e. $\tau \in (0,T)$

$$\begin{aligned}
&\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\left(\int_\Omega \phi(\tau,\cdot)G_\delta(u_\varepsilon(\tau,\cdot))\,\mathrm{d}\mathcal{L}^n - \int_\Omega \phi(0,\cdot)G_\delta(u_{0,\varepsilon})\,\mathrm{d}\mathcal{L}^n\right)\\
&\qquad = \lim_{\delta\to 0}\left(\int_\Omega \phi(\tau,\cdot)G_\delta(u(\tau,\cdot))\,\mathrm{d}\mathcal{L}^n - \int_\Omega \phi(0,\cdot)G_\delta(u_0)\,\mathrm{d}\mathcal{L}^n\right)\\
&\qquad = \int_\Omega \phi(\tau,\cdot)G(u(\tau,\cdot))\,\mathrm{d}\mathcal{L}^n - \int_\Omega \phi(0,\cdot)G(u_0)\,\mathrm{d}\mathcal{L}^n\\
&\qquad = \frac{c_0}{2\sigma}\int_\Omega \phi(\tau,\cdot)u(\tau,\cdot)\,\mathrm{d}\mathcal{L}^n - \frac{c_0}{2\sigma}\int_\Omega \phi(0,\cdot)u_0\,\mathrm{d}\mathcal{L}^n.
\end{aligned}$$

We conclude (6.3.62). In the next step we estimate the double limit. Therefor we apply $\mathrm{Id} = (-\varepsilon^2\Delta + \mathrm{Id})\mathcal{A}_\varepsilon$ and get

$$\begin{aligned}
\langle \phi G_\delta'(u_\varepsilon), \partial_t u_\varepsilon\rangle_{L^2(0,\tau;H^1(\Omega))'} &= \langle \mathcal{A}_\varepsilon\partial_t u_\varepsilon, (-\varepsilon^2\Delta + \mathrm{Id})\phi G_\delta'(u_\varepsilon)\rangle_{L^2(0,\tau;H^1(\Omega))'}\\
&= \int_{\Omega_\tau}\Big(\varepsilon^2\nabla\mathcal{A}_\varepsilon\partial_t u_\varepsilon\cdot\nabla[\phi G_\delta'(u_\varepsilon)] + \phi G_\delta'(u_\varepsilon)\mathcal{A}_\varepsilon\partial_t u_\varepsilon\Big)\,\mathrm{d}\mathcal{L}^{n+1}.
\end{aligned}$$

We use the Cauchy-Schwarz estimate for the $H^1(\Omega)$ scalar product. We get

$$\begin{aligned}
&\left|\int_{\Omega_\tau}\Big(\varepsilon^2\nabla\mathcal{A}_\varepsilon\partial_t u_\varepsilon\cdot\nabla[\phi G_\delta'(u_\varepsilon)] + \phi G_\delta'(u_\varepsilon)\mathcal{A}_\varepsilon\partial_t u_\varepsilon\Big)\,\mathrm{d}\mathcal{L}^{n+1}\right|\\
&\qquad\le\Bigg[\int_{\Omega_\tau}\Big(\frac{1}{\varepsilon}|\phi G_\delta'(u_\varepsilon)|^2 + \varepsilon\big|\nabla[\phi G_\delta'(u_\varepsilon)]\big|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1}\\
&\qquad\quad\cdot\int_{\Omega_\tau}\varepsilon\Big(|\mathcal{A}_\varepsilon\partial_t u_\varepsilon|^2 + \varepsilon^2|\nabla\mathcal{A}_\varepsilon\partial_t u_\varepsilon|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1}\Bigg]^{\frac12}\\
&\qquad=\Bigg[\int_{\Omega_\tau}\Big(\frac{1}{\varepsilon}|\phi G_\delta'(u_\varepsilon)|^2 + \varepsilon\big|\nabla[\phi G_\delta'(u_\varepsilon)]\big|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1}\langle\varepsilon\mathcal{A}_\varepsilon\partial_t u_\varepsilon, \partial_t u_\varepsilon\rangle_{L^2(0,\tau;H^1(\Omega))'}\Bigg]^{\frac12}.
\end{aligned}$$

We show that the first factor on the right-hand side is convergent as $\varepsilon\to 0$. We get from Lemma 6.3.15

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_\tau}\Big(\frac{1}{\varepsilon}|\phi G_\delta'(u_\varepsilon)|^2 + \varepsilon\big|\nabla[\phi G_\delta'(u_\varepsilon)]\big|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1}\\
&\qquad=\lim_{\varepsilon\to 0}\int_{\Omega_\tau}|\phi|^2\Big(\frac{1}{\varepsilon}|G_\delta'(u_\varepsilon)|^2 + \varepsilon|G_\delta''(u_\varepsilon)\nabla u_\varepsilon|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1} = \lim_{\varepsilon\to 0}\int_{\Omega_\tau}|\phi|^2\,\mathrm{d}\beta_{\varepsilon,\delta}
\end{aligned}$$

where $\beta_{\varepsilon,\delta}$ is the Radon measure introduced in (6.3.44). In the next step we use the compactness result for $(\beta_{\varepsilon,\delta})_{\varepsilon>0}$. We have by Proposition 6.3.14

$$\lim_{\varepsilon\to 0}\int_{\Omega_\tau}|\phi|^2\,\mathrm{d}\beta_{\varepsilon,\delta} = \frac{1}{\sigma_\delta}\int_{\Omega_\tau}|\phi|^2\,\mathrm{d}\mu.$$

Thus we get

$$\lim_{\varepsilon\to 0}\int_{\Omega_\tau}\Big(\frac{1}{\varepsilon}|\phi G_\delta'(u_\varepsilon)|^2 + \varepsilon\big|\nabla[\phi G_\delta'(u_\varepsilon)]\big|^2\Big)\,\mathrm{d}\mathcal{L}^{n+1} \le \frac{1}{\sigma_\delta}\|\phi\|_{L^2(\Omega_\tau,\mu)}.$$

We conclude using $\sigma_\delta \to \sigma$ as $\delta \to 0$

$$\left| \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \left\langle \phi G_\delta'(u_\varepsilon), \partial_t u_\varepsilon \right\rangle_{L^2(0,\tau;H^1(\Omega))'} \right| \tag{6.3.63}$$
$$\leq \|\phi\|_{L^2(\Omega_\tau)} \sqrt{\frac{1}{\sigma} \liminf_{\varepsilon\to 0} \left\langle \varepsilon \mathcal{A}_\varepsilon \partial_t u_\varepsilon, \partial_t u_\varepsilon \right\rangle_{L^2(0,\tau;H^1(\Omega))'}}$$

The right-hand side is finite because of the PDE (6.1.1) and (6.3.13). By (6.3.63) the linear mapping $\tilde{L} : \left(C_c^2([0,T) \times \Omega),\ \| \ \cdot \ \|_{L^2(\Omega_T,\mu)} \right) \longrightarrow \mathbb{R}$ with

$$\tilde{L}(\phi) \coloneqq \sigma \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \left\langle \phi G_\delta'(u_\varepsilon), \partial_t u_\varepsilon \right\rangle_{L^2(0,T;H^1(\Omega))'} \tag{6.3.64}$$

is bounded. Owing to the Hahn-Banach Theorem we can extend this mapping, i.e. there exists $L \in L^2(\Omega_T, \mu)'$ such that

$$L : L^2(\Omega_T, \mu) \longrightarrow \mathbb{R} \quad \text{with} \quad \|L\|_{L^2(\Omega_T)'} = \|\tilde{L}\| \quad \text{and} \quad L|_{C_c^2([0,T)\times\Omega)} = \tilde{L}. \tag{6.3.65}$$

By Riesz's Representation Theorem for Hilbert spaces there exists a unique $\mathcal{V} \in L^2(\Omega_T)$ with $\|\mathcal{V}\|_{L^2(\Omega_T,\mu)} = \|L\|_{L^2(\Omega)'}$ such that for all $v \in L^2(\Omega_T)$ we have

$$Lv = \int_{\Omega_T} \mathcal{V} v \,\mathrm{d}\mu.$$

In particular we have for all $\phi \in C_c^2([0,T) \times \Omega)$ that

$$\sigma \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \left\langle \phi G_\delta'(u_\varepsilon), \partial_t u_\varepsilon \right\rangle_{L^2(0,T;H^1(\Omega))'} = \tilde{L}(\phi) = L(\phi) = \int_{\Omega_T} \mathcal{V}\phi \,\mathrm{d}\mu. \tag{6.3.66}$$

Combining this with (6.3.62) yields (6.3.68).

We have by (6.3.63)

$$\|\mathcal{V}\|_{L^2(\Omega_T,\mu)} = \|\tilde{L}\| = \sup_{\substack{\phi\in C_c^2([0,T)\times\Omega) \\ \|\phi\|_{L^2(\Omega_T)}\leq 1}} |\tilde{L}(\phi)| \leq \sqrt{\sigma \liminf_{\varepsilon\to 0} \left\langle \varepsilon \mathcal{A}_\varepsilon \partial_t u_\varepsilon, \partial_t u_\varepsilon \right\rangle_{L^2(0,T;H^1(\Omega))'}}.$$

Squaring and rearranging yields (6.3.60).

By (6.3.62) and (6.3.63) we conclude that for all $\hat{\phi} \in C_c^2(\Omega_\tau)$ (so no boundary integrals at $t = 0$ or $t = \tau$) we have

$$\left| \int_{\Omega_\tau} u \partial_t \hat{\phi} \,\mathrm{d}\mathcal{L}^{n+1} \right| \leq C(\Lambda, \tau, \sigma) \|\hat{\phi}\|_{C^0(\Omega_\tau)}. \tag{6.3.67}$$

Since we already knew $u(t, \cdot) \in BV(\Omega)$ from the Modica-Mortola Theorem 2.4.2 it follows that $u \in BV(\Omega_\tau)$. Now we also have this regularity with respect to time, and $\partial_t u$ is a finite Radon measure on Ω_τ. The measure a priori depends on τ. But since for all $0 < \tau_1 < \tau_2 < T$ we have $C_c^2([0, \tau_1) \times \Omega) \hookrightarrow C_c^2([0, \tau_2) \times \Omega)$ the Radon measure $\partial_t u \in C_c^0(\Omega_T)'$ is well-defined.

By Theorem 2.2.8 we deduce that for all $\phi \in C_c^2([0,T) \times \Omega)$

$$\frac{c_0}{2}\int_{\Omega_\tau} u\partial_t\phi \,\mathrm{d}\mathcal{L}^{n+1} + \frac{c_0}{2}\int_\Omega \phi(0,\cdot)u(0,\cdot)\,\mathrm{d}\mathcal{L}^n - \frac{c_0}{2}\int_\Omega \phi(\tau,\cdot)u(\tau,\cdot)\,\mathrm{d}\mathcal{L}^n = -\frac{c_0}{2}\int_{\Omega_\tau} \phi \,\mathrm{d}\partial_t u. \tag{6.3.68}$$

Here the pointwise evaluations $u(0,\cdot)$ and $u(\tau,\cdot)$ are traces in the sense of the Trace Theorem. Combining this with (6.3.68) shows that the initial data u_0 is attained in the sense of the Trace Theorem and that for a.e. $\tau \in (0,T)$ the function $u \in BV(\Omega_T)$ attains the value $u(\tau,\cdot)$ in the sense of the Trace Theorem.

Considering test functions $\phi \in C_c^2(\Omega_\tau)$ proves (6.3.61).

$\square$

Note that in the very first step of the proof we can use Z instead of G and apply $\frac{c_0}{2}u = Z(u)$. While we would have to deal with the term $\int_\Omega \varepsilon|Z''(u_\varepsilon)\nabla u_\varepsilon|^2\,\mathrm{d}\mathcal{L}^{n+1}$ it might work as well. Now the stage is set for the major theorem of this Chapter where we prove that ${}^\oplus V$ is a De Giorgi type varifold solution for rescaled mean curvature flow as in Definition 2.5.3.

Theorem 6.3.17 (De Giorgi type varifold solution for rescaled mean curvature flow)**.** *Let the Assumptions 6.2.1 hold and assume ${}^\oplus V$ is the family of varifolds constructed in Lemma 6.3.7. Then ${}^\oplus V$ is a De Giorgi type varifold solution for rescaled mean curvature flow with initial data (V^0, u_0). The rescaling parameter is given by $\sigma > 0$ from* (6.3.40).

Proof. From Lemma 6.3.7 we already know that ${}^\oplus V = \mathcal{L}^1 \otimes ({}^\oplus V^t)_{t\in(0,T)}$ and that the measurability condition from Definition 2.5.3 is satisfied for ${}^\oplus V^\cdot$. We show the conditions (a)-(d) from Definition 2.5.3. Firstly (a) follows from Theorem 6.3.16.

For (b) let $\eta \in C_c^0(\Omega;\mathbb{R}^n)$ and $\psi \in C_c^0[0,T)$. We consider (6.3.37) for η, multiply by ψ and integrate in time. We obtain

$$\int_{{}^\oplus G_{n-1}(\Omega_T)} \psi(t)D\eta(x) : (\mathrm{Id} - s\otimes s)\,\mathrm{d}^\oplus V(t,x,s) = \int_0^T \psi\langle \eta, \delta\tilde{V}^t\rangle\,\mathrm{d}t = -\int_{\Omega_T}\psi\eta\cdot\vec{H}\,\mathrm{d}\mu.$$

Since $C_c^0[0,T)\otimes C_c^0(\Omega;\mathbb{R}^n)$ is dense in $C_c^0([0,T)\times\Omega;\mathbb{R}^n)$ (b) follows.

For (c) we use that (6.3.15) holds for a.e. $\tau \in (0,T)$ and let $\eta \in C_c^0(\Omega)$ such that $0 \le \eta \le 1$. We estimate as a preparation

$$\liminf_{\varepsilon\to 0}\mu_\varepsilon^\tau(\Omega) \ge \liminf_{\varepsilon\to 0}\langle \eta, \mu_\varepsilon^\tau\rangle = \langle \eta, \mu^\tau\rangle.$$

We take the supremum over all $\eta \in C_c^0(\Omega)$ with $0 \le \eta \le 1$ and obtain

$$\liminf_{\varepsilon\to 0}\mu_\varepsilon^\tau(\Omega) \ge \mu^\tau(\Omega). \tag{6.3.69}$$

For κ_ε we estimate with Fatou's Lemma

$$\liminf_{\varepsilon\to 0}\kappa_\varepsilon(\Omega_\tau) = \liminf_{\varepsilon\to 0}\int_0^\tau \kappa_\varepsilon^t(\Omega)\,\mathrm{d}t \ge \int_0^\tau \liminf_{\varepsilon\to 0}\kappa_\varepsilon^t(\Omega)\,\mathrm{d}t. \tag{6.3.70}$$

Let $t \in (0,T)$ such that (6.3.14), (6.3.19), and (6.3.29) hold up to a possibly t-dependent subsequence, which is true for a.e. $t \in (0,T)$. Then we can apply Corollary 5.2.6 and get that

$$\liminf_{\varepsilon \to 0} \kappa_\varepsilon^t(\Omega) \geq \sigma \int_\Omega |\vec{H}_t|^2 \, \mathrm{d}\mu^t.$$

We conclude that

$$\liminf_{\varepsilon \to 0} \kappa_\varepsilon(\Omega_\tau) \geq \sigma \int_{\Omega_\tau} |\vec{H}_t|^2 \, \mathrm{d}\mu. \tag{6.3.71}$$

With these results we can prove (c). We rewrite (6.1.6)

$$\mu_\varepsilon^\tau(\Omega) + \frac{1}{2}\kappa_\varepsilon(\Omega_\tau) + \frac{1}{2}\langle \varepsilon \mathcal{A}_\varepsilon \partial_t u_\varepsilon, \partial_t u_\varepsilon \rangle_{L^2(0,\tau;H^1(\Omega))'} = \mu_\varepsilon^0(\Omega).$$

We apply the limes inferior to both sides and apply (6.3.69), (6.3.71), (6.3.60), and (6.2.2). It follows that

$$\mu^\tau(\Omega) + \frac{\sigma}{2}\int_{\Omega_\tau} |\vec{H}|^2 \, \mathrm{d}\mu + \frac{1}{2\sigma}\int_{\Omega_\tau} |\mathcal{V}|^2 \, \mathrm{d}\mu \leq \mu^0(\Omega).$$

This proves (c).

For (d) we use that as $\varepsilon \to 0$ we have $Z(u_\varepsilon(t,\cdot)) \longrightarrow Z(u(t,\cdot))$ for a.e. $t \in (0,T)$ as in Lemma 6.2.6. In the following we use in addition (6.3.18), (6.3.34), and (6.3.24). Let $\eta \in C^1(\Omega;\mathbb{R}^n)$ and $\psi \in C_c^1(0,T)$, we calculate

$$\begin{aligned}
\frac{c_0}{2}\int_0^T \psi \int_\Omega \eta \cdot \mathrm{d}\nabla u(t,\cdot) \, \mathrm{d}t &= -\int_{\Omega_T} \frac{c_0}{2}\psi u \nabla \cdot \eta \, \mathrm{d}\mathcal{L}^{n+1} = -\int_{\Omega_T} \psi Z(u) \nabla \cdot \eta \, \mathrm{d}\mathcal{L}^{n+1} \\
&= -\lim_{\varepsilon \to 0}\int_{\Omega_T} \psi Z(u_\varepsilon) \nabla \cdot \eta \, \mathrm{d}\mathcal{L}^{n+1}.
\end{aligned}$$

We apply Theorem 8.3.7 and get

$$\begin{aligned}
-\lim_{\varepsilon \to 0}\int_{\Omega_T} \psi Z(u_\varepsilon) \nabla \cdot \eta \, \mathrm{d}\mathcal{L}^{n+1} &= \lim_{\varepsilon \to 0}\int_{\Omega_T} \psi \eta \cdot \nabla u_\varepsilon \sqrt{2W(u_\varepsilon)} \, \mathrm{d}\mathcal{L}^{n+1} \\
&= \lim_{\varepsilon \to 0}\int_{\Omega_T} \psi \eta \cdot \nu_\varepsilon |\nabla u_\varepsilon| \sqrt{2W(u_\varepsilon)} \, \mathrm{d}\mathcal{L}^{n+1}.
\end{aligned}$$

We have

$$\begin{aligned}
|\nabla u_\varepsilon|\sqrt{2W(u_\varepsilon)} &= \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) \\
&\quad - \left(\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon| - \sqrt{\frac{1}{\varepsilon}W(u_\varepsilon)}\right)^2.
\end{aligned}$$

This yields

$$\frac{c_0}{2}\int_0^T\int_\Omega \psi\eta\cdot \mathrm{d}\nabla u(t,\cdot)\,\mathrm{d}t = \lim_{\varepsilon\to 0}\Bigg[\int_{\Omega_T}\psi\eta\cdot\nu_\varepsilon\,\mathrm{d}\mu_\varepsilon - \int_{\Omega_T}\psi\eta\cdot\nu_\varepsilon\Big(\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon| - \sqrt{\frac{1}{\varepsilon}W(u_\varepsilon)}\Big)^2\,\mathrm{d}\mathcal{L}^{n+1}\Bigg].$$

In the next step we show

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\psi\eta\cdot\nu_\varepsilon\Big(\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon| - \sqrt{\frac{1}{\varepsilon}W(u_\varepsilon)}\Big)^2\,\mathrm{d}\mathcal{L}^{n+1} = 0.$$

As an auxiliary identity we calculate for $a,b>0$ that

$$(a-b)^2 = |a-b||a-b| \le |a-b||a+b| = |a^2-b^2|.$$

We apply this with $a=\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon|$, $b=\sqrt{\frac{2W(u_\varepsilon)}{\varepsilon}}$ and estimate

$$\begin{aligned}
&\left|\int_{\Omega_T}\psi\eta\cdot\nu_\varepsilon\Big(\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon| - \sqrt{\frac{1}{\varepsilon}W(u_\varepsilon)}\Big)^2\,\mathrm{d}\mathcal{L}^{n+1}\right| \\
&\qquad\le \int_{\Omega_T}|\psi\eta|\left|\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon| - \sqrt{\frac{1}{\varepsilon}W(u_\varepsilon)}\right|^2\,\mathrm{d}\mathcal{L}^{n+1} \\
&\qquad\le \|\psi\eta\|_{C^0(\Omega)}\int_{\Omega_T}\left|\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 - \frac{1}{\varepsilon}W(u_\varepsilon)\right|\,\mathrm{d}\mathcal{L}^{n+1} \\
&\qquad= \|\psi\eta\|_{C^0(\Omega_T)}\;|\xi_\varepsilon|(\Omega_T)\longrightarrow 0.
\end{aligned}$$

So far we have proven that

$$\frac{c_0}{2}\int_0^T\psi\int_\Omega \eta\cdot \mathrm{d}\nabla u(t,\cdot)\,\mathrm{d}t = \lim_{\varepsilon\to 0}\int_{\Omega_T}\psi\eta\cdot\nu_\varepsilon\,\mathrm{d}\mu_\varepsilon.$$

We can rewrite the first integral in terms of the oriented varifold ${}^{\oplus}V_\varepsilon^t$ and get

$$\begin{aligned}
\frac{c_0}{2}\int_0^T\psi\int_\Omega \eta\cdot\,\mathrm{d}\nabla u(t,\cdot)\,\mathrm{d}t &= \lim_{\varepsilon\to 0}\int_{{}^{\oplus}G_{n-1}(\Omega_T)}\psi\eta\cdot s\,\mathrm{d}^{\oplus}V_\varepsilon(\cdot,\cdot,s) \\
&= \int_{{}^{\oplus}G_{n-1}(\Omega_T)}\psi\eta\cdot s\,\mathrm{d}^{\oplus}V(\cdot,\cdot,s) \\
&= \int_0^T\psi\int_{{}^{\oplus}G_{n-1}(\Omega)}\eta\cdot s\,\mathrm{d}^{\oplus}\tilde{V}^t(\cdot,s).
\end{aligned}$$

We localize in time and conclude (d). $\square$

6.4 Proof of Theorem 6.3.13

By (6.3.19), (6.3.45), and Theorem 2.2.2 it follows that $(\beta^t_{\varepsilon,\delta})_{\varepsilon>0}$ has a weak*-cluster point in $C^0_c(\Omega)'$. However we want to prove weak*-convergence of the entire sequence for which the assumptions of Theorem 6.3.13 are satisfied. Assume that there exists a subsequence which violates (6.3.47). Since the subsequence still satisfies (6.3.19) and (6.3.45) we conclude by Theorem 2.2.2 that there exist a subsequence of the subsequence $(\varepsilon \to 0)$ and $\beta^t_\delta \in C^0_c(\Omega)'$ such that

$$\beta^t_{\varepsilon,\delta} \xrightarrow{w^*} \beta^t_\delta \quad \text{in} \quad C^0_c(\Omega)'. \tag{6.4.1}$$

In the following we consider this subsequence and prove $\beta^t_\delta = \frac{1}{\sigma_\delta}\mu^t$ which is a contradiction.

First we introduce the notations for the proof and reduce the claims without loss of generality. As in Section 5.4 we use the concept of Lebesgue points and proceed similarly with a blow-up.

Lemma 6.4.1 (Good points).
Assume $(u_\varepsilon)_{\varepsilon>0}$ is a sequence in $L^2(0,T;H^3(\Omega))$ let Definition 6.3.1 hold. Let $0<\delta<\frac{1}{2}$ and assume that for $t \in (0,T)$ there exists a possibly t-dependent subsequence $\varepsilon \to 0$ such that the conditions (6.3.15), (6.3.19), (6.3.26)-(6.3.29), (6.3.34), *and* (6.4.1) *hold. Then we have for μ^t-a.e. $x \in \Omega$ (the exception set possibly depends on t, δ)*

- *$B_{2\rho_0}(x) \Subset \Omega$ for some $\rho_0 = \rho_0(x) > 0$,*
- *x is a μ^t-Lebesgue point of $D_{\mu^t}\beta^t_\delta$,*
- *$\limsup_{\rho\to 0} \rho^{1-n}\mu^t(B_\rho(x)) < \infty$,*
- *$\kappa^t(\{x\}) = 0$,*
- *the approximate tangent space $T_x\Gamma_t$ exists,*
- *there exist $\theta_t(x) \in \mathbb{N}$ and $S_{t,x} \in G(n, n-1)$ such that $T_x\mu^t = c_0\theta_t(x)S_{t,x}$.*

Proof. Since Ω is open there exists $\rho_0 : \Omega \longrightarrow (0,\infty)$ with $B_{2\rho_0(x)}(x) \Subset \Omega$. We know from from Lemma 6.3.12 and the Radon-Nikodym Theorem 8.2.5 that $D_\mu\beta^t_\delta \in L^1(\Omega,\mu^t)$ and $\beta^t_\delta = D_{\mu^t}\beta^t_\delta\mu^t$. In particular μ^t-a.e. $x \in \Omega$ is a μ^t-Lebesgue point of $D_{\mu^t}\beta^t_\delta$ by Theorem 8.3.5. Furthermore, we get from Corollary 6.3.4 that $\limsup_{\rho\to 0}\rho^{1-n}\mu^t(B_\rho(x)) < \infty$ for all $x \in \Omega$.

The fourth condition is true for a cocountable subset of Ω because κ^t is a finite Radon measure on Ω. It follows that κ^t can at most have a countable set of atoms.

By (6.3.15) the claims from Corollary 6.3.4 hold. Thus the fifth condition is satisfied by μ^t-a.e. $x \in \Omega$ because V^t is a rectifiable $(n-1)$-varifold and $\mu^t = c_0\theta_t\mathcal{H}^{n-1} \llcorner \Gamma_t$. The last point stems from the fact that $\frac{1}{c_0}V^t$ is integral; see Theorem 6.3.7. ☐

In the following we fix a good point $x \in \operatorname{supp}(\mu^t)$ and $\rho_0 > 0$ such that the properties in Lemma 6.4.1 hold. Set $\theta := \theta_t(x)$. In the following we consider the function $\zeta_{x,\rho}(y) = \frac{y-x}{\rho}$

for $\rho > 0$ and $y \in \mathbb{R}^n$. Recall also $\zeta^{\#}_{\rho,x}$ and $\zeta_{\rho,x,\#}$ from Definition 2.2.19.

Without loss of generality we can assume $x = 0$ and $S \coloneqq S_{0,x} = \mathbb{R}^{n-1} \times \{0\}$ for the proof of Theorem 6.3.13. In fact this is possible because ζ shifts x to 0 anyways. The assumption $x = 0$ translates into the simpler notation $\zeta_{\rho,0,\#}\mu^t$ instead of $\zeta_{\rho,x,\#}\mu^t$. We write $\zeta_{\rho,\#}\mu^t \coloneqq \zeta_{\rho,0,\#}\mu^t$. We can assume $S = \mathbb{R}^{n-1} \times \{0\}$ without loss of generality by using an orthogonal coordinate transformation. We write $\theta \coloneqq \theta_t(x)$.

Lemma 6.4.2.
Assume $(u_\varepsilon)_{\varepsilon>0}$ is a sequence in $L^2(0,T;H^3(\Omega))$. Let $0 < \delta < \frac{1}{2}$ and assume that for $t \in (0,T)$ the conditions (6.3.15), (6.3.19), (6.3.26)-(6.3.29), (6.3.34), and (6.4.1) hold. Then there exist sequences $(\rho_j)_{j\in\mathbb{N}}$ and $(\varepsilon_j)_{j\in\mathbb{N}}$ with $0 < \rho_j < \rho_0$ for all $j \in \mathbb{N}$ such that as $j \to \infty$ we have for all $0 < \delta < \frac{1}{2}$

$$\varepsilon_j \to 0, \quad \rho_j \to 0, \tag{6.4.2}$$

$$\frac{\varepsilon_j}{\rho_j} \to 0, \quad \frac{\varepsilon_j^2}{\rho_j^{n+1}} \to 0, \tag{6.4.3}$$

$$\rho_j^{1-n}\zeta_{\rho_j,\#}\mu^t_{\varepsilon_j} \xrightarrow{w^*} c_0\theta\mathcal{H}^{n-1} \llcorner S \quad \textit{in} \quad C^0_c(B_{16}(0))', \tag{6.4.4}$$

$$\rho_j^{1-n}\zeta_{\rho_j,\#}\beta^t_{\varepsilon_j,\delta} \xrightarrow{w^*} D_{\mu^t}\beta^t_\delta(0)c_0\theta\mathcal{H}^{n-1} \llcorner S \quad \textit{in} \quad C^0_c(B_{16}(0))', \tag{6.4.5}$$

$$\textit{and for all } j \in \mathbb{N}: \quad \kappa^t_{\varepsilon_j}(B_\rho(0)) \le \kappa^t(B_{2\rho}(0)) + \rho_j^{n-2} \quad \textit{for} \quad \rho_j \le \rho \le \rho_0. \tag{6.4.6}$$

Proof. Let $(\rho_j)_{j\in\mathbb{N}}$ be a decreasing sequence with $\rho_1 < \rho_0$ and $\rho_j \to 0$ as $j \to \infty$. By the definition of the approximate tangent space we have

$$\rho_j^{1-n}\zeta_{\rho_j,\#}\mu^t \xrightarrow{w^*} c_0\theta\mathcal{H}^{n-1} \llcorner S \quad \text{in} \quad C^0_c(B_{16}(0))'.$$

Since 0 is a μ^t-Lebesgue point of $D_{\mu^t}\beta^t_\delta$ we get by Lemma 8.2.7 that

$$\rho_j^{1-n}\zeta_{\rho_j,\#}\beta^t_\delta = \rho_j^{1-n}\zeta_{\rho_j,\#}D_{\mu^t}\beta^t_\delta\mu^t \xrightarrow{w^*} D_{\mu^t}\beta^t_\delta(0)c_0\theta\mathcal{H}^{n-1} \llcorner S \quad \text{in} \quad C^0_c(B_{16}(0))'.$$

Using that the weak*-topology on bounded subsets of $C^0_c(B_{16}(0))'$ is metrizable, (6.3.19), (6.4.1), and

$$\begin{aligned}
\rho_j^{1-n}\zeta_{\rho_j,\#}\mu^t_\varepsilon &= \rho_j^{1-n}\zeta_{\rho_j,\#}(\mu^t_\varepsilon - \mu^t) + \rho_j^{1-n}\zeta_{\rho_j,\#}\mu^t \\
\rho_j^{1-n}\zeta_{\rho_j,\#}\beta^t_{\varepsilon,\delta} &= \rho_j^{1-n}\zeta_{\rho_j,\#}(\beta^t_{\varepsilon,\delta} - \beta^t_\delta) + \rho_j^{1-n}\zeta_{\rho_j,\#}\beta^t_\delta
\end{aligned}$$

we can choose a subsequence $(\varepsilon_j)_{j\in\mathbb{N}}$ dependent on $(\rho_j)_{j\in\mathbb{N}}$ such that (6.4.2)-(6.4.5) hold.

Finally by possibly lowering the value of ε_j, we obtain for all $l \in \mathbb{N}_0$ with $2^{-l}\rho_0 > \rho_j$

$$\kappa^t_{\varepsilon_j}(B_{2^{-l}\rho_0}(0)) \le \kappa^t(\overline{B_{2^{-l}\rho_0}(0)}) + \rho_j^{n-2} \le \kappa^t(B_{2^{-l+1}\rho_0}(0)) + \rho_j^{n-2}.$$

We deduce for any $\rho_j \le \rho \le \rho_0$ and $l \in \mathbb{N}_0$ such that $\rho \in (2^{-l-1}\rho_0, 2^{-l}\rho_0)$

$$\kappa^t_{\varepsilon_j}(B_\rho(0)) \le \kappa^t_{\varepsilon_j}(B_{2^{-l}\rho_0}(0)) \le \kappa^t(\overline{B_{2^{-l}\rho_0}(0)}) + \rho_j^{n-2} \le \kappa^t(B_{2\rho}(0)) + \rho_j^{n-2}.$$

Thus (6.4.6) holds as well. ☐

The blow-up method will be applied only to the space variable while the time variable remains the fixed t which we chose at the start of the proof. We drop the time variable in the notation.

Proposition 6.4.3 (Properties of the rescaled functions and measures).
Let the assumptions from Theorem 6.3.13 hold. We use the notations from the Lemmata 6.4.1 and 6.4.2. We set $\tilde{\varepsilon}_j := \frac{\varepsilon_j}{\rho_j}$ and define for $x \in B_{\frac{\rho_0}{\rho_j}}(0)$

$$\begin{aligned} \tilde{u}_{\tilde{\varepsilon}_j}(x) &:= u_{\varepsilon_j}(t, \rho_j x), \qquad \tilde{H}_{\tilde{\varepsilon}_j}(x) := \rho_j H_{\varepsilon_j}(t, \rho_j x), \\ \text{and} \quad \tilde{\nu}_{\tilde{\varepsilon}_j}(x) &:= \frac{\nabla \tilde{u}_{\tilde{\varepsilon}_j}(x)}{|\nabla \tilde{u}_{\tilde{\varepsilon}_j}(x)|} \quad \text{for} \quad \nabla \tilde{u}_{\tilde{\varepsilon}_j}(x) \neq 0 \quad \text{and} \quad \tilde{\nu}_{\tilde{\varepsilon}_j}(x) := e_1 \quad \text{else.} \end{aligned}$$

Moreover we set

$$\tilde{\mu}^t_{\tilde{\varepsilon}_j} := \Big(\frac{\tilde{\varepsilon}_j}{2} |\nabla \tilde{u}_{\tilde{\varepsilon}_j}(t, \cdot)|^2 + \frac{1}{\tilde{\varepsilon}_j} W(\tilde{u}_{\tilde{\varepsilon}_j}(t, \cdot)) \Big) \mathcal{L}^n \llcorner B_{\frac{\rho_0}{\rho_j}}(0), \tag{6.4.7}$$

$$\tilde{\xi}^t_{\tilde{\varepsilon}_j} := \Big(\frac{\tilde{\varepsilon}_j}{2} |\nabla \tilde{u}_{\tilde{\varepsilon}_j}(t, \cdot)|^2 - \frac{1}{\tilde{\varepsilon}_j} W(\tilde{u}_{\tilde{\varepsilon}_j}(t, \cdot)) \Big) \mathcal{L}^n \llcorner B_{\frac{\rho_0}{\rho_j}}(0), \tag{6.4.8}$$

$$\tilde{\alpha}^t_{\tilde{\varepsilon}_j} := \frac{1}{\tilde{\varepsilon}_j} \tilde{H}^2_{\tilde{\varepsilon}_j} \mathcal{L}^n \llcorner B_{\frac{\rho_0}{\rho_j}}(0), \tag{6.4.9}$$

$$\tilde{\beta}^t_{\tilde{\varepsilon}_j, \delta} := \Big(\frac{1}{\tilde{\varepsilon}_j} |G'_\delta(\tilde{u}_{\tilde{\varepsilon}_j})|^2 + \tilde{\varepsilon}_j |\nabla G'_\delta(\tilde{u}_{\tilde{\varepsilon}_j})|^2 \Big) \mathcal{L}^n \llcorner B_{\frac{\rho_0}{\rho_j}}(0), \tag{6.4.10}$$

$$\tilde{\kappa}^t_{\tilde{\varepsilon}_j} := \Big(\frac{1}{\tilde{\varepsilon}_j} |\tilde{H}_{\tilde{\varepsilon}_j}|^2 + \tilde{\varepsilon}_j |\nabla \tilde{H}_{\tilde{\varepsilon}_j}|^2 \Big) \mathcal{L}^n \llcorner B_{\frac{\rho_0}{\rho_j}}(0), \tag{6.4.11}$$

$$\text{and} \quad \tilde{V}^t_{\tilde{\varepsilon}_j} := \tilde{\mu}^t_{\tilde{\varepsilon}_j} \otimes \tilde{\nu}^\perp_{\tilde{\varepsilon}_j} \in \mathbb{V}_{n-1}\Big(B_{\frac{\rho_0}{\rho_j}}(0) \Big). \tag{6.4.12}$$

Then it holds

$$\tilde{\varepsilon}_j \to 0 \quad \text{as} \quad j \to \infty, \tag{6.4.13}$$

$$-\tilde{\varepsilon}_j \Delta \tilde{u}_{\tilde{\varepsilon}_j} + \frac{1}{\tilde{\varepsilon}_j} W'(\tilde{u}_{\tilde{\varepsilon}_j}) = \tilde{H}_{\tilde{\varepsilon}_j} \quad \text{in} \quad B_{\frac{\rho_0}{\rho_j}}(0), \tag{6.4.14}$$

and with $j \to \infty$ we have

$$\rho_j^{1-n} \zeta_{\rho_j, \#} \mu^t_{\varepsilon_j} = \tilde{\mu}^t_{\tilde{\varepsilon}_j} \xrightarrow{w^*} c_0 \theta \mathcal{H}^{n-1} \llcorner S, \tag{6.4.15}$$

$$\rho_j^{1-n} \zeta_{\rho_j, \#} \beta^t_{\varepsilon_j, \delta} = \tilde{\beta}^t_{\tilde{\varepsilon}_j, \delta} \xrightarrow{w^*} c_0 \theta D_{\mu^t} \beta^t_\delta(0) \mathcal{H}^{n-1} \llcorner S, \tag{6.4.16}$$

$$\tilde{\alpha}^t_{\tilde{\varepsilon}_j} \xrightarrow{w^*} 0, \quad \text{and} \quad \tilde{\kappa}^t_{\tilde{\varepsilon}_j} \xrightarrow{w^*} 0 \tag{6.4.17}$$

in $C^0_c(B_{16}(0))'$. Furthermore, there exist $\tilde{V}^t \in \mathbb{V}_{n-1}(B_{15}(0))$ such that up to a subsequence we have as $j \to \infty$

$$\tilde{V}^t_{\tilde{\varepsilon}_j} \xrightarrow{w^*} \tilde{V}^t \quad \text{in} \quad \mathbb{V}_{n-1}(B_{15}(0)). \tag{6.4.18}$$

Proof. The claims (6.4.13), (6.4.14), (6.4.15), and (6.4.17) have analogous proofs in the proof of Proposition 6.4.3. For (6.4.16) we use (6.4.5) and calculate for any $\eta \in C^0_c(B_{16}(0))$ that

$$
\begin{aligned}
&\rho_j^{1-n}\langle\eta,\zeta_{\rho_j,\#}\beta^t_{\varepsilon_j,\delta}\rangle_{C_c^0(B_{16}(0))'}\\
&\quad=\rho_j^{1-n}\langle\zeta_{\rho_j}^{\#}\eta,\beta^t_{\varepsilon_j,\delta}\rangle_{C_c^0(\Omega)'}=\rho_j^{1-n}\int_\Omega\eta\circ\zeta_{\rho_j}\,\mathrm{d}\beta^t_{\varepsilon_j,\delta}\\
&\quad=\int_{B_{16\rho_j}(0)}\rho_j^{1-n}\eta\Big(\frac{x}{\rho_j}\Big)\Big(\frac{1}{\varepsilon_j}|G_\delta'(u_{\varepsilon_j}(t,x))|^2+\varepsilon_j\big|\nabla G_\delta'(u_{\varepsilon_j}(t,x))\big|^2\Big)\,\mathrm{d}x\\
&\quad=\int_{B_{16}(0)}\rho_j\eta(x)\Big(\frac{1}{\varepsilon_j}|G_\delta'(u_{\varepsilon_j}(t,\rho_jx))|^2+\varepsilon_j\big|\nabla G_\delta'(u_{\varepsilon_j}(t,\rho_jx))\big|^2\Big)\,\mathrm{d}x\\
&\quad=\int_{B_{16}(0)}\eta(x)\Big(\frac{1}{\frac{\varepsilon_j}{\rho_j}}|G_\delta'(u_{\varepsilon_j}(t,\rho_jx))|^2+\frac{\varepsilon_j}{\rho_j}\big|\rho_j\nabla G_\delta'(u_{\varepsilon_j}(t,\rho_jx))\big|^2\Big)\,\mathrm{d}x\\
&\quad=\int_{B_{16}(0)}\eta(x)\Big(\frac{1}{\tilde\varepsilon_j}|G_\delta'(\tilde u_{\tilde\varepsilon_j})|^2+\tilde\varepsilon_j\big|\nabla G_\delta'(\tilde u_{\tilde\varepsilon_j})\big|^2\Big)\,\mathrm{d}x=\langle\eta,\tilde\beta^t_{\tilde\varepsilon_j,\delta}\rangle_{C_c^0(B_{16}(0))'}.
\end{aligned}
$$

The claim (6.4.18) follows from

$$\|\tilde V^t_{\tilde\varepsilon_j}\|(B_{15}(0))=\tilde\mu^t_{\tilde\varepsilon_j}(B_{15}(0))\le C\quad\text{for all } j\in\mathbb{N}$$

because $(\tilde\mu^t_{\tilde\varepsilon_j})_{j\in\mathbb{N}}$ is weakly*-convergent in $C_c^0(B_{16}(0))'$ and Theorem 2.2.2. $\square$

In order to prove Theorem 6.3.13 it is therefore sufficient to prove the following statement and apply it with $\Omega,(u_{\varepsilon_j})_{j\in\mathbb{N}}$ replaced with $B_8(0),(\tilde u_{\tilde\varepsilon_j})_{j\in\mathbb{N}}$ (the rescaled functions and measures also satisfy the assumptions of Theorem 6.3.13).

Proposition 6.4.4.
Let $(u_\varepsilon)_{\varepsilon>0}$ as in Theorem 6.3.13 with $B_4(0)\Subset\Omega$. Consider Radon measures $\mu^t_\varepsilon,\xi^t_\varepsilon,\alpha^t_\varepsilon,\kappa^t_\varepsilon,\beta^t_{\varepsilon,\delta}\in C_c^0(\Omega)'$ with (6.3.2)-(6.3.5) and (6.3.44). Consider additionally varifolds $V^t_\varepsilon\in\mathbb{V}_{n-1}(\Omega)$ with (6.3.10), a subsequence $\varepsilon\to0$, limit measures $\mu^t,\alpha^t,\kappa^t,\beta^t_\delta$, and a limit varifold V^t such that (6.3.19), (6.3.26), (6.3.27), (6.3.29), (6.3.34), and (6.4.1) hold on $B_4(0)$. In addition, assume that

$$\mu^t=c_0\theta\mathcal{H}^{n-1}\llcorner S\quad\textit{for some}\quad\theta\in\mathbb{N},\ S\in G(n,n-1),\quad\textit{and}\quad\alpha^t=0=\kappa^t.$$

Then

$$\beta^t_\delta=\frac{1}{\sigma_\delta}\mu^t$$

holds.

We prepare the proof of Proposition 6.4.4 with the following generalization of Proposition 5.5 in [RS06].

Proposition 6.4.5.
For all $\tau,\gamma,\delta\in(0,1)$ and $\Lambda>0$ there exist $\omega=\omega(\delta,\tau,\gamma,\Lambda)>0$ and $L=L(\gamma,\tau,\delta)>1$ such that the following holds: Let the assumptions from Proposition 6.4.4 be satisfied with $\Omega=B_{4L\varepsilon}(0)$ and further assume

$$|u_\varepsilon(0)|\le1-\tau,\tag{6.4.19}$$

$$|\xi_\varepsilon^t|(B_{4L\varepsilon}(0)) + \int_{B_{4L\varepsilon}(0)} \varepsilon|\nabla u_\varepsilon|^2 \sqrt{1-|\nu_{\varepsilon,n}|^2}\,\mathrm{d}\mathcal{L}^n \le \omega(4L\varepsilon)^{n-1}, \tag{6.4.20}$$

with $\nu_{\varepsilon,n} \coloneqq e_n \cdot \nu_\varepsilon$ *and*

$$\mu_\varepsilon^t(B_{4L\varepsilon}(0)) \le \Lambda(4L\varepsilon)^{n-1}, \tag{6.4.21}$$

$$\kappa_\varepsilon^t(B_{4L\varepsilon}(0)) \le \Lambda(4L\varepsilon)^{n-3}. \tag{6.4.22}$$

Then we also have, writing $(0,s) \in \mathbb{R}^{n-1}\times\mathbb{R}$

$$|u_\varepsilon(0,s)| \ge 1 - \frac{\tau}{2} \quad \textit{for all} \quad L\varepsilon \le |s| \le 3L\varepsilon, \tag{6.4.23}$$

$$\left|\frac{1}{\omega_{n-1}(L\varepsilon)^{n-1}}\mu_\varepsilon^t(B_{L\varepsilon}(0)) - c_0\right| \le \gamma, \tag{6.4.24}$$

$$\left|\int_{-L\varepsilon}^{L\varepsilon} \frac{1}{\varepsilon} W(u_\varepsilon(0,s))\,\mathrm{d}s - \frac{c_0}{2}\right| \le \gamma, \tag{6.4.25}$$

$$\left|\int_{-L\varepsilon}^{L\varepsilon}\Big(\Big(\frac{1}{\varepsilon}|G_\delta'(u_\varepsilon)|^2 + \varepsilon\big|\nabla G_\delta'(u_\varepsilon)\big|^2\Big) - \frac{2}{\varepsilon\sigma_\delta}W(u_\varepsilon)\Big)(0,s)\,\mathrm{d}s\right| \le \gamma. \tag{6.4.26}$$

Here ω_m *is defined by* $\mathcal{L}^m(B_1(0)) = \omega_m$ *for* $m \in \mathbb{N}$.

Proof. We follow the proof of Proposition 5.5 in [RS06]. From it follows the existence of ω, L satisfying (6.4.23)-(6.4.25). In the following we possibly lower the value of ω and increase the value of L, which maintains (6.4.23)-(6.4.25).

We prove in the following that we can assume $\varepsilon = 1$ without loss of generality. In fact since ε is fixed, by rescaling $x \longmapsto \varepsilon x$ and defining $\mathbf{u}(x) \coloneqq u_\varepsilon(\varepsilon x)$ for $x \in B_L(0)$ we prove the statements independent of ε. For the claims (6.4.23)-(6.4.25) this has already been done in [RS06], we prove it for the remaining expression in (6.4.26)

$$\begin{aligned}
&\int_{-L\varepsilon}^{L\varepsilon}\Big(\Big(\frac{1}{\varepsilon}|G_\delta'(u_\varepsilon)|^2 + \varepsilon\big|G_\delta''(u_\varepsilon)\nabla u_\varepsilon\big|^2\Big) - \frac{2}{\varepsilon\sigma_\delta}W(u_\varepsilon)\Big)(0,s)\,\mathrm{d}s \\
&\qquad = \int_{-L}^{L}\Big(\Big(\frac{1}{\varepsilon}|G_\delta'(u_\varepsilon)|^2 + \frac{1}{\varepsilon}\big|G_\delta''(u_\varepsilon)\nabla u_\varepsilon\big|^2 - \frac{2}{\varepsilon\sigma_\delta}W(u_\varepsilon)\Big)\Big)(0,\varepsilon s)\varepsilon\,\mathrm{d}s \\
&\qquad = \int_{-L}^{L}\Big(\Big(|G_\delta'(\mathbf{u})|^2 + \big|G_\delta''(\mathbf{u})\nabla\mathbf{u}\big|^2\Big) - \frac{2}{\sigma_\delta}W(\mathbf{u})\Big)(0,s)\,\mathrm{d}s.
\end{aligned}$$

We recall that by Lemma 4.1.2 and the definitions of c_0, σ in Assumptions 4.1.1 we have

- $|q_0| < 1$ and $q_0' > 0$,
- $\lim\limits_{z\to\pm\infty} q_0(z) = \pm 1$,
- $\displaystyle\int_{\mathbb{R}} \frac{1}{2}|q_0'|^2\,\mathrm{d}\mathcal{L}^1 = \int_{\mathbb{R}} W(q_0)\,\mathrm{d}\mathcal{L}^1 = \frac{c_0}{2}$,
- $\displaystyle\int_{\mathbb{R}} \Big(|G_\delta'(q_0)|^2 + |G_\delta''(q_0)q_0'|^2\Big)\,\mathrm{d}\mathcal{L}^1 = \frac{c_0}{\sigma_\delta}$.

For a given $a \in \mathbb{R}$ we define

$$q_a(s) \coloneqq q(s+a) \quad \text{and} \quad Q_a(x) \coloneqq q_a(x_n) \quad \text{for} \quad s \in \mathbb{R},\ x \in \mathbb{R}^n$$

and claim that we can choose $L(\tau, \delta)$ sufficiently large such that: If

$$|q_0(a)| \leq 1 - \tau,$$

then

$$|Q_a(0,s)| \geq 1 - \frac{\tau}{3} \quad \text{for all} \quad L \leq |s| \leq 3L, \tag{6.4.27}$$

$$\left| \frac{1}{\omega_{n-1} L^{n-1}} \int_{B_L(0)} \left(\frac{1}{2} |\nabla Q_a|^2 + W(Q_a) \right) \mathrm{d}\mathcal{L}^n - c_0 \right| \leq \frac{\gamma}{2}, \tag{6.4.28}$$

$$\left| \int_{-L}^{L} W(Q_a(0,s))\,\mathrm{d}s - \frac{c_0}{2} \right| \leq \frac{\gamma \sigma_\delta}{8}, \tag{6.4.29}$$

$$\left| \int_{-L}^{L} \left(|G'_\delta(q_a)|^2 + |G''_\delta(q_a) q'_a|^2 \right) \mathrm{d}\mathcal{L}^1 - \frac{c_0}{\sigma_\delta} \right| \leq \frac{\gamma}{4}. \tag{6.4.30}$$

The first three properties are guaranteed by [RS06]. For the fourth identity we use $|q_0(a)| \leq 1 - \tau$, thus $|a| \leq q_0^{-1}(1-\tau)$ and conclude

$$\int_{-L}^{L} \left(|G'_\delta(q_a)|^2 + |G''_\delta(q_a) q'_a|^2 \right) \mathrm{d}\mathcal{L}^1 = \int_{-L-a}^{L-a} \left(|G'_\delta(q_0)|^2 + |G''_\delta(q_0) q'_0|^2 \right) \mathrm{d}\mathcal{L}^1 \longrightarrow \frac{c_0}{\sigma_\delta}.$$

Since we have a uniform bound on $|a|$ only dependent on τ we can choose $L(\tau, \delta) > 1$ independent from a such that (6.4.30) holds.

Since H is bounded in $H^1(B_{4L}(0))$ by (6.4.22) we conclude by inner elliptic regularity theory similar as in [RS06]

$$\|u\|_{H^3(B_{\frac{7L}{2}}(0))} \leq C(\Lambda, L). \tag{6.4.31}$$

We proceed by a contradiction argument, adapting [RS06]. Assume that the claim is wrong, then there exists a sequence $(u_k)_{k \in \mathbb{N}}$ with $\omega_k \to 0$ as $k \to \infty$ and for any $k \in \mathbb{N}$ there exist functions $u_k, \overline{u}_k, H_k$ satisfying the assumptions of Proposition 6.4.4 with $\varepsilon = 1$, $\Omega = B_{4L}(0)$ and satisfying the properties (6.4.19)-(6.4.22) but violating (6.4.26).

Because of (6.4.31) there exists $u \in H^3(B_{3L}(0))$ such that up to a subsequence we have as $k \to \infty$

$$u_k \overset{w}{\longrightarrow} u \quad \text{in} \quad H^3(B_{3L}(0)). \tag{6.4.32}$$

By the compact Sobolev embedding $H^3(B_{3L}(0)) \overset{c}{\hookrightarrow} C^1(\overline{B_{3L}(0)})$ as $n \leq 3$ hence

$$u_k \longrightarrow u \quad \text{and} \quad \nabla u_k \longrightarrow \nabla u \quad \text{uniformly in} \quad B_{3L}(0). \tag{6.4.33}$$

As in the proof of Proposition 5.5 in [RS06], writing $x = (y, s) \in \mathbb{R}^{n-1} \times \mathbb{R}$ we get

$$u(y,s) = u_0(s) \quad \text{for all} \quad (y,s) \in B_{3L}(0),$$

where $u_0 = \pm q_{s_0}$ with s_0 determined by $u(0)$. Since a reflection $(y, x_n) \mapsto (y, -x_n)$ does neither affect the assumptions nor the conclusions of the proposition we can assume $u_0 = +q_{s_0}$ without loss of generality. By uniform convergence we have $|u(0)| \leq 1 - \tau$ which implies that (6.4.30) is satisfied. We use that to prove (6.4.26) for large k; in fact we have

$$\begin{aligned}
&\left| \int_{-L}^{L} \Big(\big(|G_\delta'(u_k)|^2 + |\nabla G_\delta'(u_k)|^2 \big) - \frac{2}{\sigma_\delta} W(u_k) \Big)(0,s) \,\mathrm{d}s \right| \\
&\quad \leq \left| \int_{-L}^{L} \big(|G_\delta'(u_k)|^2 + |\nabla G_\delta'(u_k)|^2 \big)(0,s) \,\mathrm{d}s - \frac{c_0}{\sigma_\delta} \right| + \left| \frac{c_0}{\sigma_\delta} - \int_{-L}^{L} \frac{2}{\sigma_\delta} W(u_k)(0,s) \,\mathrm{d}s \right| \\
&\quad \leq \left| \int_{-L}^{L} \big(|G_\delta'(u_k)|^2 + |\nabla G_\delta'(u_k)|^2 \big)(0,s) \,\mathrm{d}s - \int_{-L}^{L} \big(|G_\delta'(u)|^2 + |\nabla G_\delta'(u)|^2 \big)(0,s) \,\mathrm{d}s \right| \\
&\qquad + \left| \int_{-L}^{L} \big(|G_\delta'(u)|^2 + |\nabla G_\delta'(u)|^2 \big)(0,s) \,\mathrm{d}s - \frac{c_0}{\sigma_\delta} \right| + \left| \frac{c_0}{\sigma_\delta} - \int_{-L}^{L} \frac{2}{\sigma_\delta} W(u)(0,s) \,\mathrm{d}s \right| \\
&\qquad + \left| \int_{-L}^{L} \frac{2}{\sigma_\delta} W(u)(0,s) \,\mathrm{d}s - \int_{-L}^{L} \frac{2}{\sigma_\delta} W(u_k)(0,s) \,\mathrm{d}s \right|. \qquad (6.4.34)
\end{aligned}$$

The second term on the right-hand side is estimated by (6.4.30), the third term by (6.4.29). For the first term we use that $G_\delta \in C_b^3(\mathbb{R})$ which implies $|G_\delta'|^2, |G_\delta''|^2 \in C_b^1(\mathbb{R})$ and thus we can apply the Mean-Value Theorem and assuming $\|u_k - u\|_{C^0(B_{3L}(0))} \leq 1$ we get that

$$\begin{aligned}
&\left| \int_{-L}^{L} \Big(|G_\delta'(u_k)|^2 - |G_\delta'(u)|^2 + \big|G_\delta''(u_k)\nabla u_k\big|^2 - \big|G_\delta''(u)\nabla u\big|^2 \Big)(0,s) \,\mathrm{d}s \right| \\
&\quad = \Bigg| \int_{-L}^{L} \Big(|G_\delta'(u_k)|^2 - |G_\delta'(u)|^2 + \big|G_\delta''(u_k)\nabla u_k\big|^2 - \big|G_\delta''(u)\nabla u_k\big|^2 \\
&\qquad + \big|G_\delta''(u)\nabla u_k\big|^2 - \big|G_\delta''(u)\nabla u\big|^2 \Big)(0,s) \,\mathrm{d}s \Bigg| \\
&\quad \leq 4L \|u_k - u\|_{C^1(B_{3L}(0))} \Big(\|G_\delta' G_\delta''\|_{C^0(\mathbb{R})} + \|u_k\|_{C^1(B_{3L}(0))} \|G_\delta'' G_\delta'''\|_{C^0(\mathbb{R})} + \|G_\delta''\|_{C^0(\mathbb{R})}^2 \Big)
\end{aligned}$$

Since $u_k \longrightarrow u$ in $C^1(B_{3L}(0))$ there exists $R > 0$ such that $\|u_k\|_{C^1(B_{3L}(0))} \leq R$ for all $k \in \mathbb{N}$. We choose $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have

$$\|u_k - u\|_{C^1(\mathbb{R})} \leq \frac{\gamma}{16L \Big(\|G_\delta' G_\delta''\|_{C^0(\mathbb{R})} + R \|G_\delta'' G_\delta'''\|_{C^0(\mathbb{R})} + \|G_\delta''\|_{C^0(\mathbb{R})}^2 \Big) + \gamma}.$$

We estimate the last term on the right-hand side of (6.4.34) by

$$\left| \int_{-L}^{L} \frac{2}{\sigma_\delta} W(u)(0,s) \,\mathrm{d}s - \int_{-L}^{L} \frac{2}{\sigma_\delta} W(u_k)(0,s) \,\mathrm{d}s \right| \leq \frac{4L}{\sigma_\delta} \|W'\|_{C^1[-R,R]} \|u_k - u\|_{C^0(B_{3L}(0))}.$$

Then we choose $\mathbb{N} \ni k_1 \geq k_0$ such that it holds for all $k \geq k_1$ that

$$\|u_k - u\|_{C^0(B_{3L}(0))} \leq \frac{\gamma \sigma_\delta}{16L \|W'\|_{C^1[-R,R]}}.$$

By applying these estimates into (6.4.34) we get for all $k \geq k_1$ that

$$\left| \int_{-L}^{L} \Big(\big(|G'_\delta(u_k)|^2 + |\nabla G'_\delta(u_k)|^2 \big) - \frac{2}{\sigma_\delta} W(u_k) \Big)(0,s) \,\mathrm{d}s \right| \leq \frac{\gamma}{4} + \frac{\gamma}{4} + \frac{2}{\sigma_\delta} \cdot \frac{\gamma \sigma_\delta}{8} + \frac{\gamma}{4} = \gamma.$$

Thus for $k \geq k_1$ (6.4.26) holds, a contradiction to our assumption. □

In the following we prove 6.4.4 by proceeding similarly to the proof of Proposition 5.4.5.

Proof of Proposition 6.4.4. We assume that $x = 0$ is a good point in the sense of Lemma 6.4.1 and $S = \mathbb{R}^{n-1} \times \{0\}$. Let $\Pi : \mathbb{R}^n \longrightarrow S$ be the orthogonal projection. We use the representation $x = (y, s) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and denote by $\nabla' = \nabla_y$ the horizontal gradient. By Theorem 5.2.3 the limit of V^t of V^t_ε is given by $V^t = c_0 \theta \mathcal{H}^{n-1} \llcorner S \otimes \delta_S$. Convergence as varifolds yields in particular

$$\lim_{\varepsilon \to 0} \int_{B_4(0)} \varepsilon |\nabla u_\varepsilon(t,\cdot)|^2 \sqrt{1 - \nu_{\varepsilon,n}(t,\cdot)^2} \,\mathrm{d}\mathcal{L}^n = 0.$$

By the proof of Proposition 5.2 in [RS06] for any $\gamma > 0$ there exist $\omega_0, \varepsilon_0, \tau_0 > 0$, all depending on γ, δ, t such that for any $0 < \omega < \omega_0$, any $0 < \tau < \tau_0$, and any $0 < \varepsilon < \varepsilon_0$ the following two properties hold:

(1)

$$\int\limits_{\{|u_\varepsilon(t,\cdot)| \geq 1-\tau\} \cap B_4(0)} \frac{1}{\varepsilon} W'(u_\varepsilon(t,\cdot))^2 \,\mathrm{d}\mathcal{L}^n \leq \gamma \quad \text{and} \quad \mu^t_\varepsilon\Big(\{|u_\varepsilon(t,\cdot)| \geq 1 - \tau\} \cap B_4(0) \Big) \leq 3\gamma. \tag{6.4.35}$$

(2) For the set

$$\begin{aligned} A_\varepsilon := \Big\{ x \in B_1(0) \;\Big|\; & |u_\varepsilon(t,x)| \leq 1 - \tau, \\ & \forall \varepsilon \leq \rho \leq 3 : |\xi^t_\varepsilon|(B_\rho(x)) + \int\limits_{B_\rho(x)} \varepsilon |\nabla u_\varepsilon(t,\cdot)|^2 \sqrt{1 - \nu_{\varepsilon,n}(t,\cdot)^2} \leq \omega \rho^{n-1}, \\ & \text{and} \quad \alpha^t_\varepsilon(B_\rho(x)) \leq \omega \rho^{\frac{1}{2}} \Big\} \end{aligned}$$

we have

$$\mu^t_\varepsilon(B_1(0) \setminus A_\varepsilon) \leq 4\gamma. \tag{6.4.36}$$

We now define a subset of A_ε with additional "good properties",

$$A'_\varepsilon := A_\varepsilon \cap \Big\{ x \in B_1(0) \mid \forall \rho \in [\varepsilon, 3] : \kappa^t_\varepsilon(B_\rho(x)) \leq \omega \rho^{\frac{1}{2}} \Big\}.$$

We show that $A_\varepsilon \setminus A'_\varepsilon$ is "small"in a suitable sense. For all $x \in A_\varepsilon \setminus A'_\varepsilon$ there exists $\rho_x \in (0, \frac{1}{2})$ such that $B_{2\rho_x}(x) \subseteq B_1(0)$. It follows that

$$A_\varepsilon \setminus A'_\varepsilon \subseteq \bigcup_{x \in A_\varepsilon \setminus A'_\varepsilon} \overline{B_{\rho_x}(x)}.$$

By Besicovitch's covering Theorem there exist $N \in \mathbb{N}$ only dependent on the dimension n and sets $D_1, \dots, D_N \subseteq B_1(0)$ such that for all $k \in \{1, \dots, N\}$ the collection

$$\left\{ \overline{B_{\rho_x}(x)} \mid x \in D_k \right\}$$

is disjoint and

$$A_\varepsilon \setminus A'_\varepsilon \subseteq \bigcup_{k=1}^{N} \dot{\bigcup_{x \in D_k}} \overline{B_{\rho_x}(x)}.$$

Since for all $k \in \{1, \dots, N\}$ the union $\bigcup_{x \in D_k} \overline{B_{\rho_x}(x)} \subseteq B_1(0)$ is disjoint it follows that

$$\omega_n \sum_{x \in D_k} \rho_x^n = \sum_{x \in D_k} \mathcal{L}^n\left(\overline{B_{\rho_x}(x)}\right) = \mathcal{L}^n\left(\dot{\bigcup_{x \in D_k}} \overline{B_{\rho_x}(x)} \right) \leq \mathcal{L}^n(B_1(0)) = \omega_n < \infty.$$

The sum is convergent and thus D_k has to be at most countable. We conclude

$$A_\varepsilon \setminus A'_\varepsilon \subseteq \bigcup_{k=1}^{N} \dot{\bigcup_{j \in \mathbb{N}}} \overline{B_{\rho_{k,j}}(x_{k,j})}. \tag{6.4.37}$$

Since $x_{k,j} \in A_\varepsilon \setminus A'_\varepsilon$ we have for all k, j that there exists $\varepsilon \leq \rho_{k,j} \leq 3$ such that

$$\kappa^t_\varepsilon(B_{\rho_{k,j}}(x_{k,j})) > \omega \rho_{k,j}^{\frac{1}{2}}.$$

Since $x_{k,j} \in A_\varepsilon$ we can use $\alpha^t_\varepsilon(B_\rho(x_{k,j})) \leq \omega \rho^{\frac{1}{2}}$ for all $\varepsilon \leq \rho \leq 3$ and (6.4.35). We deduce from Proposition 4.7 in [RS06] that

$$\mu^t_\varepsilon(\overline{B_{\rho_{k,j}}(x_{k,j})}) \leq C \rho_{k,j}^{n-1}.$$

We then obtain by (6.4.37) that

$$\begin{aligned} \mu_\varepsilon(A_\varepsilon \setminus A'_\varepsilon) &\leq C \sum_{k=1}^{N} \sum_{j \in \mathbb{N}} \rho_{k,j}^{n-1} \\ &\leq C \omega^{2(1-n)} \kappa^t_\varepsilon(B_4(0))^{2(n-1)-1} \sum_{k=1}^{N} \sum_{j \in \mathbb{N}} \kappa^t_\varepsilon(B_{\rho_{k,j}}(x_{k,j})). \end{aligned}$$

For ε sufficiently small we conclude by using $n \in \{2, 3\}$ and $\kappa^t_\varepsilon \xrightarrow{w^*} 0$ in $C^0_c(\Omega)'$ with $B_4(0) \Subset \Omega$ that

$$\mu_\varepsilon(A_\varepsilon \setminus A'_\varepsilon) \leq \omega^{2(1-n)} N \kappa^t_\varepsilon(B_4(0))^{2(n-1)} \leq \gamma. \tag{6.4.38}$$

By the definition of A_ε for all $x \in A'_\varepsilon$ we can apply Proposition 5.4 from [RS06] with $N = 1$ and deduce (6.4.21) (with 0 replaced by x). Together with the definition of A'_ε we obtain that we can apply Proposition 6.4.5 for all $x \in A'_\varepsilon$. By page 713 in [RS06] this yields that for all $y \in S \cap B_1(0)$ there exist $\mathbb{N} \ni K = K(y) \leq \theta$ and $s_1(y), \dots, s_K(y) \in \mathbb{R}$ with

$$A_\varepsilon \cap \Pi^{-1}(y) \subseteq \{y\} \times \bigcup_{l=1}^{K(y)} (s_l(y) - L\varepsilon, s_l(y) + L\varepsilon).$$

We now fix an arbitrary $\eta \in C_c^1(B_1(0))$ and deduce from $\xi_\varepsilon^t \xrightarrow{w^*} 0$, (6.4.36) and (6.3.45)

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\left|\int_{B_1(0)} \eta \,\mathrm{d}\beta_{\varepsilon,\delta}^t - \frac{1}{\sigma_\delta}\int_{B_1(0)} \eta \,\mathrm{d}\mu_\varepsilon^t\right| \\
&\qquad\le \limsup_{\varepsilon\to 0}\left|\int_{A'_\varepsilon} \eta \,\mathrm{d}\beta_{\varepsilon,\delta}^t - \int_{A'_\varepsilon} \frac{2}{\sigma_\delta\varepsilon}\eta W(u_\varepsilon(t,\cdot))\,\mathrm{d}\mathcal{L}^n\right| + C\gamma\|\eta\|_{C^0(\overline{B_1(0)})}
\end{aligned}
\tag{6.4.39}
$$

for some $C>0$. Furthermore we obtain

$$
\begin{aligned}
&\left|\int_{A'_\varepsilon} \eta \,\mathrm{d}\beta_{\varepsilon,\delta}^t - \int_{A'_\varepsilon} \frac{2}{\sigma_\delta\varepsilon}\eta W(u_\varepsilon(t,\cdot))\,\mathrm{d}\mathcal{L}^n\right| \\
&= \left|\int_{\Pi(A'_\varepsilon)} \sum_{l=1}^{K(y)} \int_{s_l(y)-L\varepsilon}^{s_l(y)+L\varepsilon} \eta(y,s)\Big(\frac{1}{\varepsilon}|G'_\delta(u_\varepsilon)|^2 + \varepsilon\big|\nabla G'_\delta(u_\varepsilon)\big|^2 - \frac{2}{\sigma_\delta\varepsilon}W(u_\varepsilon)\Big)(t,y,s)\,\mathrm{d}s\,\mathrm{d}y\right| \\
&\le \int_{\Pi(A'_\varepsilon)} \sum_{l=1}^{K(y)} |\eta(y,s_j)| \int_{s_l(y)-L\varepsilon}^{s_l(y)+L\varepsilon} \Big(\frac{1}{\varepsilon}|G'_\delta(u_\varepsilon)|^2 + \varepsilon\big|\nabla G'_\delta(u_\varepsilon)\big|^2 - \frac{2}{\sigma_\delta\varepsilon}W(u_\varepsilon)\Big)(t,y,s)\,\mathrm{d}s\,\mathrm{d}y \\
&\quad + C \sup_{\substack{(y,s),(y,r)\in B_1(0)\\ |r-s|<L\varepsilon}} |\eta(y,r)-\eta(y,s)|\Big(\beta_{\varepsilon,\delta}^t(B_1(0)) + \mu_\varepsilon^t(B_1(0))\Big)
\end{aligned}
$$

For the first term we apply (6.4.26). For the second term we use that $\eta \in C_c^1(B_1(0))$ is uniformly continuous, thus for ε sufficiently small we have for all s,r with $|r-s|<L\varepsilon$ that $|\eta(y,r)-\eta(y,s)|<\gamma$. We conclude that

$$
\left|\int_{A'_\varepsilon} \eta \,\mathrm{d}\beta_{\varepsilon,\delta}^t - \int_{A'_\varepsilon} \frac{2}{\sigma_\delta\varepsilon}\eta W(u_\varepsilon(t,\cdot))\,\mathrm{d}\mathcal{L}^n\right| \le \|\eta\|_{C^0(B_1(0))}\int_{\Pi(A'_\varepsilon)} \theta\gamma\,\mathrm{d}y + \gamma C(\Lambda).
$$

Hence, we conclude with (6.4.39) that

$$
\limsup_{\varepsilon\to 0}\left|\int_{B_1(0)} \eta \,\mathrm{d}\beta_{\varepsilon,\delta}^t - \frac{1}{\sigma_\delta}\int_{B_1(0)} \eta \,\mathrm{d}\mu_\varepsilon^t\right| \le C(\Lambda,\eta,\theta)\gamma.
$$

Since $\gamma>0$ and $\eta \in C_c^1(B_1(0))$ were arbitrary we deduce that

$$
\beta_\delta^t = \lim_{\varepsilon\to 0}\beta_{\varepsilon,\delta}^t = \lim_{\varepsilon\to 0}\frac{1}{\sigma_\delta}\mu_\varepsilon^t = \frac{1}{\sigma_\delta}\mu^t. \qquad \square
$$

6.5 Brakke's formulation

Owing to the results from Section 6.3 $(V^t)_{t\in[0,T)}$ satisfies Assumptions 2.5.1. Thus to prove that $(V^t)_{t\in[0,T)}$ evolves by mean curvature flow in the sense of Brakke's formulation it is sufficient to prove (2.5.1).

Theorem 6.5.1 (Partial result for convergence towards mean curvature flow in the sense of Brakke in the KK model)**.**
Let Assumptions 6.2.1 hold and assume additionally that $|u_\varepsilon| \le 1$. Then there exists

$\vec{H}_* \in L^2(\Omega_T, \mu; \mathbb{R}^n)$ such that for all non-negative test functions $\psi \in C_c^1[0,T)$, $\eta \in C_c^2(\Omega)$ and for all $0 \le t_1 < t_2 < T$

$$\psi(t) \int_\Omega \eta \,\mathrm{d}\mu^t \Big|_{t_1}^{t_2} \le -\sigma \int_{t_1}^{t_2} \psi \int_\Omega \eta |\vec{H}_t|^2 \,\mathrm{d}\mu^t \,\mathrm{d}t + \int_{t_1}^{t_2} \psi \int_\Omega \nabla\eta \cdot \left(\vec{H}_* + \vec{H}_t\right) \mathrm{d}\mu^t \,\mathrm{d}t + \int_{t_1}^{t_2} \psi' \int_\Omega \eta \,\mathrm{d}\mu^t \,\mathrm{d}t.$$

Where $(\mu, \vec{H}_*)$ is characterized as

$$\left(\mu_\varepsilon + \xi_\varepsilon, \frac{\nabla H_\varepsilon W'(u_\varepsilon)}{\frac{2}{\varepsilon} W(u_\varepsilon)}\right) \xrightarrow{w} (\mu, \vec{H}_*)$$

in the sense of weak measure-function pair convergence; see Definition 2.2.13.

If $\vec{H}_* + \vec{H}_t = \sigma \vec{H}_t$ then the family of varifolds $(V^t)_{t\in[0,T)}$ constructed in Lemma 6.3.7 evolves by mean curvature flow in the sense of Brakke.

Proof. We begin by calculating a diffuse version of (2.5.1). Some calculations are similar to those from the proof of Lemma 6.2.7. Let $\eta \in C_c^2(\Omega)$, $\psi \in C_c^1[0,T)$ be non-negative and $0 \le t_1 < t_2 < T$ be arbitrary.

$$\begin{aligned}
\psi(t)\langle \eta, \mu_\varepsilon^t\rangle_{C_c^0(\Omega)'}\Big|_{t_1}^{t_2} &= \int_{t_1}^{t_2} \partial_t[\psi\langle \eta, \mu_\varepsilon^t\rangle_{C_c^0(\Omega)'}] \,\mathrm{d}t = \int_{t_1}^{t_2} \psi'\langle \eta, \mu_\varepsilon^t\rangle_{C_c^0(\Omega)'} \,\mathrm{d}\mathcal{L}^1 \\
&\quad + \int_{t_1}^{t_2} \psi\Big(\Big\langle \varepsilon\eta\nabla u_\varepsilon, \nabla\partial_t u_\varepsilon\Big\rangle_{H^2(\Omega;\mathbb{R}^n)'} + \Big\langle \frac{1}{\varepsilon}\eta W'(u_\varepsilon), \partial_t u_\varepsilon\Big\rangle_{H^1(\Omega)'}\Big) \,\mathrm{d}\mathcal{L}^1 \\
&= \int_{t_1}^{t_2} \psi'\langle \eta, \mu_\varepsilon^t\rangle_{C_c^0(\Omega)'} \,\mathrm{d}\mathcal{L}^1 + \int_{t_1}^{t_2} \psi\Big\langle \eta H_\varepsilon - \varepsilon\nabla\eta\cdot\nabla u_\varepsilon, \partial_t u_\varepsilon\Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1.
\end{aligned}$$

Now we apply in the PDE (6.1.1) on the last term on the right-hand side.

$$\begin{aligned}
&\int_{t_1}^{t_2} \psi\Big\langle \eta H_\varepsilon - \varepsilon\nabla\eta\cdot\nabla u_\varepsilon, \partial_t u_\varepsilon\Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 \\
&\qquad = \int_{t_1}^{t_2} \psi\Big\langle -\frac{1}{\varepsilon}\eta H_\varepsilon + \nabla\eta\cdot\nabla u_\varepsilon, (-\varepsilon^2\Delta + \mathrm{Id})H_\varepsilon\Big\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1.
\end{aligned}$$

Next we apply the weak definition of Δ and get a diffuse version of Brakke's inequality.

$$\psi(t)\langle \eta, \mu_\varepsilon^t\rangle_{C_c^0(\Omega)'}\Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \psi'\langle \eta, \mu_\varepsilon^t\rangle_{C_c^0(\Omega)'} \,\mathrm{d}\mathcal{L}^1 - \int_{t_1}^{t_2} \psi(t)\langle \eta, \kappa_\varepsilon^t\rangle_{C_c^0(\Omega)'} \,\mathrm{d}t \tag{6.5.1}$$

$$- \int_{t_1}^{t_2} \psi \int_\Omega \varepsilon\nabla\eta\cdot\nabla H_\varepsilon H_\varepsilon \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 + \int_{t_1}^{t_2} \psi \int_\Omega \nabla\eta\cdot\nabla u_\varepsilon H_\varepsilon \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1 \tag{6.5.2}$$

$$- \int_{t_1}^{t_2} \varepsilon^2\psi\langle \nabla\eta\cdot\nabla u_\varepsilon, \Delta H_\varepsilon\rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 \tag{6.5.3}$$

We examine each of the terms separately. For the term on the left-hand side of (6.5.1) we have $\mu_\varepsilon^t \xrightarrow{w^*} \mu^t$ for all $t \in [0,T)$ by (6.3.19) and thus

$$\lim_{\varepsilon\to 0} \psi(t)\langle \eta, \mu_\varepsilon^t\rangle_{C_c^0(\Omega)'}\Big|_{t_1}^{t_2} = \psi(t)\langle \eta, \mu^t\rangle_{C_c^0(\Omega)'}\Big|_{t_1}^{t_2}. \tag{6.5.4}$$

Furthermore, from (6.3.14) we have the bound $|\psi'\langle\eta,\mu_\varepsilon^t\rangle_{C_c^0(\Omega)'}| \leq \|\psi\|_{C^1[0,T)}\|\eta\|_{C^0(\Omega)}\Lambda$. So we can apply the Dominated Convergence Theorem on the first term on the right-hand side of (6.5.1) and get

$$\lim_{\varepsilon\to 0}\int_{t_1}^{t_2}\psi'\langle\eta,\mu_\varepsilon^t\rangle_{C_c^0(\Omega)'}\,\mathrm{d}t = \int_{t_1}^{t_2}\psi'\int_\Omega \eta\,\mathrm{d}\mu^t\,\mathrm{d}t. \tag{6.5.5}$$

For the second term on the right-hand side of (6.5.1) we use Fatou's Lemma, (6.3.27), and $\sigma|\vec{H}_t|^2\mu^t \leq \kappa^t$ from Corollary 6.3.4, which yield that

$$\begin{aligned}
\limsup_{\varepsilon\to 0}\left[-\int_{t_1}^{t_2}\psi\langle\eta,\kappa_\varepsilon^t\rangle_{C_c^0(\Omega)'}\,\mathrm{d}t\right] &= -\liminf_{\varepsilon\to 0}\int_{t_1}^{t_2}\psi\langle\eta,\kappa_\varepsilon^t\rangle_{C_c^0(\Omega)'}\,\mathrm{d}t\\
&\leq -\int_{t_1}^{t_2}\psi\liminf_{\varepsilon\to 0}\langle\eta,\kappa_\varepsilon^t\rangle_{C_c^0(\Omega)'}\,\mathrm{d}t\\
&= -\int_{t_1}^{t_2}\psi\langle\eta,\kappa^t\rangle_{C_c^0(\Omega)'}\,\mathrm{d}t\\
&\leq -\sigma\int_0^T\psi\int_\Omega\eta|\vec{H}_t|^2\,\mathrm{d}\mu^t\,\mathrm{d}t. \qquad (6.5.6)
\end{aligned}$$

In (6.2.23) we already estimated the first term on the right-hand side in (6.5.2) it vanishes as $\varepsilon\to 0$, i.e.

$$\lim_{\varepsilon\to 0}\int_{t_1}^{t_2}\psi\int_\Omega\varepsilon\nabla\eta\cdot\nabla H_\varepsilon H_\varepsilon\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 = 0 \tag{6.5.7}$$

We use (6.3.32) on the second term on the right-hand side of (6.5.2) and get

$$\lim_{\varepsilon\to 0}\int_{t_1}^{t_2}\psi\int_\Omega H_\varepsilon\nabla\eta\cdot\nabla u_\varepsilon\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 = \int_{t_1}^{t_2}\psi\int_\Omega\vec{H}_t\cdot\nabla\eta\,\mathrm{d}\mu^t\,\mathrm{d}\mathcal{L}^1. \tag{6.5.8}$$

For the remaining term in (6.5.3) we use the calculations from (6.2.24). As in (6.2.25) we conclude that

$$\begin{aligned}
&\varepsilon^2\int_{t_1}^{t_2}\big\langle\nabla\eta\cdot\nabla u_\varepsilon,\Delta H_\varepsilon\big\rangle_{H^1(\Omega)'}\,\mathrm{d}\mathcal{L}^1\\
&\qquad = \varepsilon^2\int_{t_1}^{t_2}\int_\Omega\Big(\nabla u_\varepsilon\cdot\nabla H_\varepsilon\Delta\eta - \nabla\eta\cdot\nabla H_\varepsilon\Delta u_\varepsilon - 2\nabla u_\varepsilon\cdot D^2\eta\nabla H_\varepsilon\Big)\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1.
\end{aligned}$$

Since $|\psi|\leq\|\psi\|_{C^0(0,T)}$ the error estimate from (6.2.26) holds for

$$\int_{t_1}^{t_2}\psi\int_\Omega\varepsilon^2\nabla H_\varepsilon\cdot D^2\eta\nabla u_\varepsilon\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1 \quad\text{and}\quad \int_{t_1}^{t_2}\psi\int_\Omega\varepsilon^2\nabla u_\varepsilon\cdot\nabla H_\varepsilon\Delta\eta\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1.$$

Thus we have

$$-\lim_{\varepsilon\to 0}\int_{t_1}^{t_2}\psi\big\langle\varepsilon^2\nabla\eta\cdot\nabla u_\varepsilon,\Delta H_\varepsilon\big\rangle_{H^1(\Omega)'}\,\mathrm{d}\mathcal{L}^1 = \lim_{\varepsilon\to 0}\int_{t_1}^{t_2}\psi\int_\Omega\varepsilon^2\nabla\eta\cdot\nabla H_\varepsilon\Delta u_\varepsilon\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1.$$

We calculate further

$$\int_{t_1}^{t_2}\psi\int_\Omega\varepsilon^2\nabla\eta\cdot\nabla H_\varepsilon\Delta u_\varepsilon\,\mathrm{d}\mathcal{L}^n\,\mathrm{d}\mathcal{L}^1$$

$$= -\int_{t_1}^{t_2} \psi \int_\Omega \varepsilon \nabla\eta \cdot \nabla H_\varepsilon \Big(H_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \Big) \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1$$

$$= -\int_{t_1}^{t_2} \psi \int_\Omega \varepsilon \nabla\eta \cdot \nabla H_\varepsilon H_\varepsilon + \int_{t_1}^{t_2} \psi \int_\Omega \nabla\eta \cdot \nabla H_\varepsilon W'(u_\varepsilon) \,\mathrm{d}\mathcal{L}^n \,\mathrm{d}\mathcal{L}^1.$$

The first term on the right-hand side vanishes as $\varepsilon \to 0$ because of (6.2.23). Thus we have

$$-\lim_{\varepsilon\to 0} \int_{t_1}^{t_2} \psi \langle \varepsilon^2 \nabla\eta \cdot \nabla u_\varepsilon, \Delta H_\varepsilon \rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 = \lim_{\varepsilon\to 0} \int_{\Omega_T} \psi \nabla\eta \cdot \nabla H_\varepsilon W'(u_\varepsilon) \,\mathrm{d}\mathcal{L}^{n+1}.$$

To deal with this term we write

$$\int_{\Omega_T} \psi \nabla\eta \cdot \nabla H_\varepsilon W'(u_\varepsilon) \,\mathrm{d}\mathcal{L}^{n+1} = \int_{\Omega_T} \psi \nabla\eta \cdot \frac{\nabla H_\varepsilon W'(u_\varepsilon)}{\frac{2}{\varepsilon} W(u_\varepsilon)} \frac{2}{\varepsilon} W(u_\varepsilon) \,\mathrm{d}\mathcal{L}^{n+1}.$$

This is well-defined because of $W'(u_\varepsilon) = 0$ whenever $W(u_\varepsilon) = 0$. We use the theory of measure-function pairs by Hutchinson to show the convergence of this term. We have

$$\frac{2}{\varepsilon} W(u_\varepsilon) \mathcal{L}^{n+1} \llcorner \Omega_T = \mu_\varepsilon - \xi_\varepsilon \xrightarrow{w^*} \mu \quad \text{as} \quad \varepsilon \to 0 \quad \text{in} \quad C_c^0(\Omega_T)'.$$

Furthermore we estimate with the assumption $|u_\varepsilon| \le 1$ and $W'(r)^2 = 16 r^2 W(r)$ for $r \in \mathbb{R}$ that

$$\int_{\Omega_T} \left| \frac{\nabla H_\varepsilon W'(u_\varepsilon)}{\frac{2}{\varepsilon} W(u_\varepsilon)} \right|^2 \frac{2}{\varepsilon} W(u_\varepsilon) \,\mathrm{d}\mathcal{L}^{n+1} = \int_{\Omega_T} 8\varepsilon \big|\nabla H_\varepsilon\big|^2 u_\varepsilon^2 \,\mathrm{d}\mathcal{L}^{n+1} \le 8\Lambda.$$

By Theorem 2.2.14 there exists $\vec{H}_* \in L^2(\Omega_T, \mu; \mathbb{R}^n)$ such that for all $\phi \in C_c^0(\Omega_T; \mathbb{R}^n)$ we have

$$\int_{\Omega_T} \phi \cdot \nabla H_\varepsilon W'(u_\varepsilon) \,\mathrm{d}\mathcal{L}^{n+1} \longrightarrow \int_{\Omega_T} \phi \cdot \vec{H}_* \,\mathrm{d}\mu.$$

This holds in particular for $\phi = \psi \nabla \eta$ and thus we get

$$-\lim_{\varepsilon\to 0} \int_{t_1}^{t_2} \psi \langle \varepsilon^2 \nabla\eta \cdot \nabla u_\varepsilon, \Delta H_\varepsilon \rangle_{H^1(\Omega)'} \,\mathrm{d}\mathcal{L}^1 = \int_{\Omega_T} \psi \nabla\eta \cdot \vec{H}_* \,\mathrm{d}\mu. \tag{6.5.9}$$

Now we dealt with each of the terms from (6.5.1)-(6.5.3). We apply the limes superior onto the identity and apply (6.5.4)-(6.5.9). Since the $\limsup$ is subadditive we can estimate each of the terms separately and obtain that

$$\psi(t) \langle \eta, \mu^t \rangle_{C_c^0(\Omega)'} \Big|_{t_1}^{t_2} \le -\sigma \int_0^T \psi \int_\Omega \eta |\vec{H}_t|^2 \,\mathrm{d}\mu^t \,\mathrm{d}t + \int_{t_1}^{t_2} \psi \int_\Omega \nabla\eta \cdot \Big(\vec{H}_t + \vec{H}_* \Big) \,\mathrm{d}\mu^t \,\mathrm{d}\mathcal{L}^1$$
$$+ \int_{t_1}^{t_2} \psi' \int_\Omega \eta \,\mathrm{d}\mu^t \,\mathrm{d}t. \qquad \square$$

7 Summary and Outlook

We analyzed two new diffuse curvature models and modified the techniques and results from the standard diffuse curvature model such that the could be applied to the new models. In Chapter 3 we considered a gradient-free approximation of the Willmore energy in the Amstutz-Van Goethem model and proved the Γ–$\limsup$ property by developing techniques based on those presented in [BP93]. The biggest hurdle here was to prove that an asymptotic expansion of u_ε leads to a similar expansion for $\overline{u}_\varepsilon = (-\varepsilon^2\Delta + \mathrm{Id})^{-1}u_\varepsilon$. We considered function classes with a suitable exponential control for that proof. Unfortunately the Γ–$\liminf$ estimate remains open, we could not identify an ansatz for an approximation in the sense of varifolds with suitable properties like it was done in [RS06]. To prove the full Γ–convergence would be of high mathematical interest. We also proved convergence of the gradient flow of the diffuse perimeter in the Amstutz-Van Goethem model towards the mean curvature flow and of the diffuse Willmore energy towards the Willmore flow respectively by asymptotic methods under strict assumptions. For the proof we adapted methods from [LM00] and [Wan08].

We also considered a higher order approximation of the Willmore energy in the Karali-Katsoulakis model. We proved Γ–convergence towards a multiple of the Willmore energy in smooth points and small dimensions. At first glance it seems surprising that the higher order term contributes on the same scale as the standard terms. However this makes sense as it is a consequence from the distribution as a quasi one dimensional profile in the $\limsup$–construction. For the construction of the recovery sequence in Chapter 4 we proceeded similarly as in the construction of the recovery sequence in the AG model. The proof of the $\liminf$–estimate in Chapter 5 however was much more difficult. The proof builds on the results from [RS06]. We introduced a modified diffuse area measure and the central part of our proof is to identify its weak*-limit. Here we apply and adapt the blow up method from [HT00, RS06].

We also considered the convergence of the gradient flow of the standard diffuse perimeter with respect to the inner product induced by $(-\varepsilon^2\Delta + \mathrm{Id})^{-1}$ towards a rescaled mean curvature flow. In Chapter 4 we proved this in the asymptotic setting with strict assumptions using similar strategies as in Chapter 3. The same is true for the convergence of the gradient flow of the diffuse Willmore energy in the Karali-Katsoulakis model, we proved convergence towards a rescaled Willmore flow under strict assumptions with the methods from Chapter 3.

In Chapter 6 we firstly constructed weak solutions for the gradient flow of the diffuse perimeter with respect to the metric induced by $(-\varepsilon^2\Delta + \mathrm{Id})^{-1}$, which was the original equation considered in [KK07]. We gave suitable compactness results and proved that there exists a limit varifold which satisfies the conditions for a De Giorgi type solution for rescaled mean curvature flow. This is a new type of varifold solutions introduced in [HL21]. We made use of another blowup in the process. The proof has a similar structure as the blow up in Chapter 5 but there are also differences as there is an additional parameter and all of the functions and measures are time dependent. It remains open to prove that the limit varifold is a Brakke solution to mean curvature flow. The challenge is to identify the limit of the drift term, a partial result is given. This remains open and is a question for future research.

It would be interesting to generalize the different diffuse approximations of the perimeter and the Willmore energy, as many of the techniques repeat themselves from the standard approximation to the gradient-free approximations and the higher order approximation. Examples for reoccurring patterns are the reduction to quasi-one dimensional functions in the construction of the recovery sequences, the appearing of a Fredholm operator with one dimensional kernel and the need for exponential decay of the profile functions.

8 Appendix

8.1 Basic terms of differential geometry

We follow [Jos17] for the entire section.

Definition 8.1.1 (Hypersurface).
A topological space $(\Gamma, \mathcal{T})$ *with* $\Gamma \subseteq \mathbb{R}^n$ *is called hypersurface, or* $(n-1)$*-dimensional submanifold, if it is a connected and compact Hausdorff space for which every point* $p \in \Gamma$ *has a neighborhood* $U = U_p$ *that is homeomorphic to an open subset* Σ *of* $\mathbb{R}^{n-1}$. *Such a homeomorphism* $x = x_p : U \longrightarrow \Sigma$ *is called a (coordinate) chart. An atlas is a family* $\{U_p, x_p\}_{p\in\Gamma}$.

Remark.

- *A point* $q \in U_p$ *is determined by its image* $x_p(q)$*, hence they are often identified. Often the index* p *is omitted and the components of* $x(q) \in \mathbb{R}^{n-1}$ *are called local coordinates of* q.

- *It is customary to write the Euclidean coordinates* $x = (x_1, \dots, x_{n-1})$ *and these then are considered as local coordinates on* Γ *when* $x : U \longrightarrow \Sigma$ *is a chart.*

Definition 8.1.2 (C^m-regularity of hypersurfaces and functions).
An atlas on a hypersurface is called C^m*-differentiable,* C^m*-regular or just* C^m*-atlas for* $m \in \mathbb{N} \cup \{+\infty\}$ *if all chart transitions*

$$x_p \circ x_q^{-1} : x_q(U_q \cap U_p) \longrightarrow x_p(U_q \cap U_p)$$

have C^m *regularity for* $U_q \cap U_p \neq \emptyset$. *A* C^m*-hypersurface is a hypersurface with a maximal* C^m*-atlas. The chart transitions are diffeomorphisms.*

Let Γ *be a* C^m*-hypersurface. For* $k \in \mathbb{N}$ *with* $k \leq m$ *a function* $f : \Gamma \longrightarrow \mathbb{R}$ *is called* k*-times continuously differentiable on* Γ *or* $f \in C^k(\Gamma)$ *if for all charts* $x : U \longrightarrow \Sigma$ *the function* $f \circ x : U \longrightarrow \Sigma$ *is differentiable in the classical sense, i.e.,* $f \circ x \in C^k(U; \Sigma)$.

Definition 8.1.3 (Orientation of hypersurfaces).
An atlas for a C^m*-hypersurface is called oriented if all chart transitions have positive functional determinant. A* C^m*-hypersurface is called orientable if it possesses an oriented atlas. For an oriented hypersurface there exists a continuous normal, i.e., there exists* $\nu \in C^0(\Gamma; \mathbb{S}^{n-1})$.

Definition 8.1.4 (Tangent space)**.**
Let Γ be C^m-hypersurface and $p \in \Gamma$. We define

$$T_p\Gamma := \Big\{ v \in \mathbb{R}^n \,\big|\, \exists \varepsilon > 0 \quad \text{and} \quad c : (-\varepsilon, \varepsilon) \longrightarrow \Gamma \quad \text{with } c(0) = 0 \text{ and } c'(0) = v \Big\}.$$

This is well-defined and in addition we get

$$T_p\Gamma = \operatorname{span}\Big([\partial_1 x^{-1}](x(p)), \dots, [\partial_{n-1} x^{-1}](x(p)) \Big) \tag{8.1.1}$$

for each chart x of p. As usual we will denote $X = \sum_j X_j e_j$ for a vector $X \in T_p\Gamma$ with components X_j and a vector basis $\{e_j\}_j$ of T_pX.

Definition 8.1.5 (Riemannian metric and Riemannian surface)**.**
Let Γ be a hypersurface. A Riemannian metric is a function g which maps $p \in \Gamma$ smoothly onto a scalar product $g_p(\cdot, \cdot)$ on $T_p\Gamma$. We write $g = (g_p)_{p \in \Gamma}$ for the Riemannian metric. (Γ, g) is called a Riemannian surface.

In local coordinates we can express the scalar product on $T_p\Gamma$ with a positive definite and symmetric matrix $g_{jk}(x)$ with coefficients that depend smoothly on x. This property is independent from the choice of coordinates. Often the dependency on p is omitted in the notation of g. Let $v, w \in T_p\Gamma$ with coordinates $v = \sum_{j=1}^{n-1} v_j e_j$ and $w = \sum_{j=1}^{n-1} w_j e_j$ then we have

$$\langle v | w \rangle_{T_p\Gamma} := \sum_{j,k=1}^{n-1} v_j g_{jk} w_k \quad \text{and} \quad \langle e_j | e_k \rangle_{T_p\Gamma} = g_{jk}.$$

It is standard to denote the coefficients of the inverse matrix of g as upper indices

$$g^{jk} := (g^{-1})_{jk}.$$

As usual we define the induced norm for $v \in T_p\Gamma$

$$\|v\|_{T_p\Gamma} := \sqrt{\langle v | v \rangle_{T_p\Gamma}}.$$

The standard surface measure on Γ is defined by

$$\mu_\Gamma := \sqrt{g}\mathcal{H}^{n-1} \llcorner \Gamma,$$

where $\sqrt{g} := \sqrt{\det(g)}$ and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

Lemma 8.1.6 (Dircetional derivative)**.**
Let Γ be a C^2-hypersurface, $p \in \Gamma$, U a neighborhood of p, and let $X, Y \in C^1(U; \mathbb{R}^n)$. Then the directional derivative

$$D_X Y(p) := \lim_{r \to 0} \frac{Y(p + rX(p)) - Y(p)}{r}$$

exists and is only dependent on the values of X, Y at p. In fact we have for any $\varepsilon > 0$ and any C^1-curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^n$ with $\gamma(0) = p$ and $\gamma'(0) = X(p)$

$$D_X Y(p) = \lim_{r \to 0} \frac{Y(\gamma(s)) - Y(\gamma(r))}{r}.$$

8.2 Radon measures

We follow [AFP00] until other sources are cited.

Definition 8.2.1 (Borel and Radon measure)**.**
Let Ω be a locally compact and separable metric space. We denote by $\mathcal{B}(\Omega)$ its Borel σ-algebra. A positive measure on $(\Omega, \mathcal{B}(\Omega))$ is called Borel measure. If a Borel measure has the property that each compact set has finite measure, it is called a positive Radon measure.

Let $m \in \mathbb{N}$ and assume that for every $K \Subset \Omega$ there exists a $\mathbb{R}^m$-valued measure μ_K on $(K, \mathcal{B}(K))$ such that if $K_1, K_2 \Subset \Omega$ and $A \in \mathcal{B}(K_1) \cap \mathcal{B}(K_2)$ it holds $\mu_{K_1}(A) = \mu_{K_2}(A)$. Then the family $(\mu_K)_{K \Subset \Omega}$ is called a ($\mathbb{R}^m$-valued) Radon measure on Ω and is denoted by μ.

If $\mu : \mathcal{B}(\Omega) \longrightarrow \mathbb{R}^m$ is a $\mathbb{R}^m$-valued measure then μ is called a finite ($\mathbb{R}^m$-valued) Radon measure on $(\Omega, \mathcal{B}(\Omega))$. Often it is referred to as a Radon measure on Ω.

Note that if $m = 1$ the $\mathbb{R}$-valued measure is not necessarily non-negative. Also note that real-valued positive Radon measures are Borel measures and that every finite Radon measure is a Radon measure. Usually Ω will be an open set of $\mathbb{R}^n$.

Borel measures are regular as described by the next proposition.

Proposition 8.2.2 (Inner and Outer regularity of Borel measures)**.**
Let Ω be a locally compact and separable metric space, μ a Borel measure on Ω, and let $E \subseteq \Omega$ be μ-measurable.

- *If μ is σ-finite then*
$$\mu(E) = \sup\{\mu(K) \mid K \Subset E\}.$$
- *Assume that a sequence $(\Omega_j)_{j \in \mathbb{N}}$ of open sets in Ω exists such that $\mu(\Omega_j) < \infty$ for all $j \in \mathbb{N}$ and $\Omega = \bigcup_{j \in \mathbb{N}} \Omega_j$; then*
$$\mu(E) = \inf\{\mu(A) \mid E \subseteq A \text{ and } A \text{ is open in } \Omega\}.$$

If $\Omega \subseteq \mathbb{R}^n$ and μ is a Radon measure then both of the additional conditions are satisfied.

For every measure there exists the total variation measure, which counts every volume non-negatively. We denote disjoint unions by $\dot\cup$.

Definition 8.2.3 (Total variation measure)**.**
Let Ω be a locally compact and separable metric space and μ a $\mathbb{R}^m$-valued measure on Ω. Then we define for all $E \in \mathcal{B}(\Omega)$
$$|\mu|(E) := \sup\left\{ \sum_{j \in \mathbb{N}} |\mu(E_j)| \;:\; E = \dot{\bigcup_{j \in \mathbb{N}}} E_j \right\}.$$
If μ is a finite $\mathbb{R}^m$-valued Radon measure on Ω then it holds for all open sets $A \subseteq \Omega$
$$|\mu|(A) = \sup\left\{ \int_\Omega \eta \cdot \,\mathrm{d}\mu \;\middle|\; \eta \in C_c^0(A; \mathbb{R}^m) \text{ and } \|\eta\|_{C^0(A;\mathbb{R}^m)} \le 1 \right\}.$$

The space of Radon measures can be represented as a dual space. Let Ω be a locally compact and separable metric space and Y a normed vector space, then we consider the function spaces

$$C_c^0(\Omega;Y) \coloneqq \{f:\Omega \longrightarrow Y \mid f \text{ is continuous and } \operatorname{supp}(f) \Subset \Omega\}.$$

If $Y = \mathbb{R}$ we simply write $C_c^0(\Omega) \coloneqq C_c^0(\Omega;\mathbb{R})$. Endowed with the supremum norm $\|\cdot\|_{C^0(\Omega;Y)}$ the space $C_c^0(\Omega;Y)$ is a normed vector space and we define

$$C_0^0(\Omega;Y) \coloneqq \overline{C_c^0(\Omega;Y)}^{\|\cdot\|_{C^0(\Omega;Y)}} \quad \text{in} \quad C^0(\Omega;Y),$$

which is a representation of the completion of $\left(C_c^0(\Omega;Y), \|\cdot\|_{C^0(\Omega;Y)}\right)$. Again we write $C_c^0(\Omega) \coloneqq C_c^0(\Omega;\mathbb{R})$. We will consider the space of Y-valued Radon measures and finite Y-valued Radon measure as the dual spaces of $C_c^0(\Omega;Y)$ and $C_0^0(\Omega;Y)$ however therefore we need to endow these spaces with suitable topologies. We consider $C_0^0(\Omega;Y)$ with the standard norm of uniform convergence $\|\cdot\|_{C^0(\Omega;Y)}$). For $C_c^0(\Omega;Y)'$ we need a topology which acknowledges the structure of the functions with compact support. For all $K \Subset \Omega$ we define the seminorm

$$p_K : C_c^0(\Omega;Y) \longrightarrow Y, \quad p_K(f) \coloneqq \|f\|_{C^0(K;Y)}.$$

Then we endow $C_c^0(\Omega;Y)$ with topology induced by the family of seminorms $\{p_K\}_{K\Subset\Omega}$, we refer to it as the *natural topology* on $C_c^0(\Omega;Y)$. If not specified otherwise we will always consider $C_c^0(\Omega;Y)$ with the natural topology, in particular the dual space $C_c^0(\Omega;Y)'$ is the space of all linear functionals on $C_c^0(\Omega;Y)$ that are continuous with respect to the natural topology. This can be characterized by the following criterion: let $L : C_c^0(\Omega;Y) \longrightarrow \mathbb{R}$ be linear, then $L \in C_c^0(\Omega;Y)'$ if and only if for all $K \Subset \Omega$

$$\sup\left\{L(\eta) \mid \eta \in C^0(K;Y) \text{ and } \|\eta\|_{C^0(K;Y)} \le 1\right\} < \infty.$$

Definition 8.2.4 (Absolute continuity of measures)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be open and μ, ν Borel measures on Ω. We say that ν is absolute continuous with respect to μ, i.e. $\nu \ll \mu$ if for all Borel sets $A \subseteq \Omega$ we have $\mu(A) = 0 \Longrightarrow \nu(A) = 0$.

The following result can be found as Theorem 1.30 in Section 1.6 of [EG15].

Theorem 8.2.5 (Radon-Nikodym)**.**
Let $\mu, \nu \in C_c^0(\mathbb{R}^n)'$ with $\nu \ll \mu$. Then there exists $f \in L^1(\mathbb{R}^n;\mu)$ such that

$$\nu = f\mu.$$

We call $D_\mu \nu \coloneqq f$ the measure derivative.

The theorem remains true if $\mu, \nu \in C_c^0(\Omega)'$ for some open set $\Omega \subseteq \mathbb{R}^n$. We get this by defining $\tilde{\mu} \in C_c^0(\mathbb{R}^n)'$ with $\tilde{\mu}(A) \coloneqq \mu(\Omega \cap A)$ for all Borel sets $A \subseteq \mathbb{R}^n$, same for ν. If $\nu \ll \mu$ then we get $\tilde{\nu} \ll \tilde{\mu}$ and thus $\tilde{\nu} = f\tilde{\mu}$ as Radon measures on $\mathbb{R}^n$ by Theorem 8.2.5. It follows $\nu = f\mu$ as Radon measures on Ω and thus $D_\mu \nu = D_{\tilde{\mu}} \tilde{\nu}$.

Lemma 8.2.6 (Absolute continuity of measures)**.**
Let $\Omega_0 \subseteq \mathbb{R}^n$ be open and bounded, let μ, ϑ be Radon measures on Ω_0 with the property: For all $\phi \in C_c^\infty(\Omega_0)$ with $\phi \ge 0$ we have

$$\int_{\Omega_0} \phi \,\mathrm{d}\vartheta \le \int_{\Omega_0} \phi \,\mathrm{d}\mu.$$

Then we get $\vartheta(U) \le \mu(U)$ for all Borel-measurable $U \subseteq \Omega_0$, in particular $\vartheta \ll \mu$.

Proof. Assume there is a Borel-measurable set $U \subseteq \Omega_0$ with $\mu(U) < \vartheta(U)$. Owing to the inner regularity of Radon measures we find a compact set $K \subseteq M$ such that

$$\mu(K) < \vartheta(K).$$

We choose a test function $\psi \in C_c^\infty(B_1(0))$ with $0 \leq \psi \leq 1$ and $\int_{B_1(0)} \psi \, \mathrm{d}\vartheta = 1$. Define for $\beta > 0$ and $x \in \mathbb{R}^n$

$$\phi_\beta(x) := \int_{\mathbb{R}^n} \beta^{-n} \psi\Big(\frac{x-y}{\beta}\Big) \chi_K(y) \, \mathrm{d}\vartheta(y). \tag{8.2.1}$$

ϕ_β has the properties $0 \leq \phi_\beta \leq 1$ and $\phi_\beta(x) \longrightarrow 0$ as $\beta \to 0$ if $x \notin K$. Hence we get for all $x \in \Omega_0$

$$\limsup_{\beta \to 0} \phi_\beta(x) \leq \chi_K(x)$$

For $\beta < \operatorname{dist}(K, \partial\Omega)$ we get $\phi_\beta \in C_c^\infty(\Omega_0)$, $\int_{\Omega_0} \phi_\beta \, \mathrm{d}\vartheta = \int_{\Omega_0} \chi_K \, \mathrm{d}\vartheta$ and thus

$$\mu(K) < \vartheta(K) = \int_{\Omega_0} \phi_\beta \, \mathrm{d}\vartheta \leq \int_{\Omega_0} \phi_\beta \, \mathrm{d}\mu \overset{\beta \to 0}{\longrightarrow} \mu(K),$$

by dominated convergence which is a contradiction. Hence $\vartheta \leq \mu$ and $\vartheta \ll \mu$. $\square$

In the following we consider the function $\zeta_{\rho,x} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for $\rho > 0$ and $x \in \mathbb{R}^n$ with

$$\zeta_{\rho,x}(y) := \frac{y-x}{\rho}$$

and the push-forward measure introduced in Definition 2.2.18.

Lemma 8.2.7.
Let $\Omega \subseteq \mathbb{R}^n$ be open, μ a Radon measure on Ω, and $f \in L^1(\Omega, \mu)$. Let $x \in \Omega$ be a μ-Lebesgue point of f such that that there exists $C > 0$ with $\limsup_{\rho \to 0} \rho^{1-n} \mu(B_\rho(x)) \leq C$. Then we have

$$\lim_{\rho \to 0} \rho^{1-n} \zeta_{x,\rho,\#}(f\mu) = f(x) \lim_{\rho \to 0} \rho^{1-n} \zeta_{x,\rho,\#}\mu \quad \text{in} \quad C_c^0(B_1(0))'. \tag{8.2.2}$$

Proof. Let $x \in \Omega$ be a μ-Lebesgue point of f, we can assume $x = 0$ because $\zeta_{x,\rho}$ would shift x to 0 anyways. Take $\eta \in C_c^0(\Omega)$ with $\eta(0) \neq 0$, we can assume $\eta(0) = 1$, otherwise we can rescale η. For $R > 0$ large enough we get $\operatorname{supp}(\eta) \subseteq B_R(0)$, we can assume $R = 1$, because if not we could scale the variable ρ by the factor of R which does not affect $\rho \to 0$. Testing the measures gives

$$\begin{aligned}
&\rho^{1-n} \langle \eta, \zeta_{\rho,\#}(f\mu) \rangle_{C_c^0(B_1(0))'} \\
&\quad = \rho^{1-n} \int_{B_1(0)} \eta \, \mathrm{d}\zeta_{\rho,\#}(f\mu) = \rho^{1-n} \int_{B_\rho(0)} \eta \circ \zeta_\rho \cdot f \, \mathrm{d}\mu \\
&\quad = \rho^{1-n} \int_{B_\rho(0)} \eta\Big(\frac{y}{\rho}\Big) f(y) \, \mathrm{d}\mu(y) \\
&\quad = \rho^{1-n} \int_{B_\rho(0)} \eta\Big(\frac{y}{\rho}\Big) f(0) \, \mathrm{d}\mu(y) + \rho^{1-n} \int_{B_\rho(0)} \eta\Big(\frac{y}{\rho}\Big) (f(y) - f(0)) \, \mathrm{d}\mu(y)
\end{aligned}$$

$$
\begin{aligned}
&= f(0)\rho^{1-n}\int_{B_1(0)} \eta \,\mathrm{d}\zeta_{\rho,\#}\mu + \rho^{1-n}\int_{B_\rho(0)} \eta\Big(\frac{y}{\rho}\Big)(f(y)-f(0))\,\mathrm{d}\mu(y)\\
&= \rho^{1-n}\langle \eta, f(0)\zeta_{\rho,\#}\mu\rangle_{C_c^0(B_1(0))'} + \rho^{1-n}\int_{B_\rho(0)} \eta\Big(\frac{y}{\rho}\Big)(f(y)-f(0))\,\mathrm{d}\mu(y).
\end{aligned}
$$

This yields the desired result if we can show that the second term vanishes as $\rho \to 0$. This is true because of

$$
\begin{aligned}
&\left|\int_{B_\rho(0)} \rho^{1-n}\eta\Big(\frac{y}{\rho}\Big)(f(y)-f(0))\,\mathrm{d}\mu(y)\right| \le \|\eta\|_{C^0(\Omega)}\rho^{1-n}\int_{B_\rho(0)} |f(y)-f(0)|\,\mathrm{d}\mu(y)\\
&\le \|\eta\|_{C^0(\Omega)}\underbrace{\rho^{1-n}\mu(B_\rho(0))}_{\le C}\fint_{B_\rho(0)} |f(y)-f(0)|\,\mathrm{d}\mu(y) \longrightarrow 0
\end{aligned}
$$

because 0 is a μ-Lebesgue point of f. ☐

We used the notation $\fint_\Omega f\,\mathrm{d}\mu := \frac{1}{\mu(\Omega)}\int_\Omega f\,\mathrm{d}\mu$.

8.3 Some results from analysis

Lemma 8.3.1 (Standard Aubin-Lions-Dubinskii Lemma).
Let X, Y, Z be Banach spaces with X, Z reflexive and

$$
X \overset{c}{\hookrightarrow} Y \hookrightarrow Z.
$$

The the embedding

$$
L^2(0,T;X) \cap H^1(0,T;Z) \hookrightarrow L^2(0,T;Y)
$$

is compact, meaning that if a sequence is bounded in both spaces on the left-hand side it has a convergent subsequence in the space on the right-hand side.

Based on the standard result we can prove a specialized version.

Lemma 8.3.2 (Aubin-Lions-Dubinskii type embedding).
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with C^1-boundary. For $s \in [1, 3/2)$ the embedding

$$
L^2(0,T;W^{1,1}(\Omega)) \cap H^1(0,T;H^2(\Omega)') \hookrightarrow L^2(0,T;L^s(\Omega))
$$

is compact.

Proof. Given $s \in [1, 3/2)$ we can choose $\gamma, \eta > 0$ small such that

$$
W^{1,1}(\Omega) \hookrightarrow W^{1-\gamma,1+\eta}(\Omega) \overset{c}{\hookrightarrow} L^s(\Omega) \hookrightarrow H^2(\Omega)'.
$$

$W^{1-\gamma,1+\eta}(\Omega)$ has the advantage of being reflexive whereas $W^{1,1}(\Omega)$ is not. This also makes the Bochner-space reflexive. After this setup it is sufficient to consider

$$
L^2(0,T;W^{1-\gamma,1+\eta}(\Omega)) \cap H^1(0,T;H^2(\Omega)') \hookrightarrow L^2(0,T;L^s(\Omega)).
$$

This embedding is compact owing to Lemma 8.3.1. ☐

Remark. *More versions of the Aubin-Lions-Dubinskii works [Aub63, Dub65] can be found in [Sim87, CJL14, Mou16]. For the comfort of the reader we gave a proof based on a standard version of the lemma.*

Lemma 8.3.3 (Continuation of monotone functions)**.**
Let $D \subseteq \mathbb{R}$ be dense and $\tilde{F} : D \longrightarrow \mathbb{R}$ decreasing. Then the function

$$F : \mathbb{R} \longrightarrow \mathbb{R}, \quad F(x) := \sup\{\tilde{F}(y) \mid y \in D \cap [x, \infty)\}$$

is well-defined, decreasing, right-continuous with limits to the left and $F|_D = \tilde{F}$.

Proof. We start by showing, that $F(x)$ is well-defined for $x \in \mathbb{R}$. Since $D \subseteq \mathbb{R}$ is dense we can find $\xi \in D \cap [x-1, x)$. Since $\tilde{F}$ is decreasing we have

$$\forall y \in D \cap (x, \infty) : \quad \tilde{F}(y) \leq \tilde{F}(\xi) \in \mathbb{R}.$$

So the supremum exists and is finite. Next we prove $F = \tilde{F}$ on D. Take any $x \in D$ and $z \in D \cap [x, \infty)$ then we have

$$F(x) = \sup\{\tilde{F}(y) \mid y \in D \cap [x, \infty)\} \geq \tilde{F}(x) \geq \tilde{F}(z).$$

Now we take the supremum over $z \in D \cap [x, \infty)$ and get

$$F(x) \geq \tilde{F}(x) \geq \sup\{\tilde{F}(z) \mid z \in D \cap [x, \infty)\} = F(x).$$

Thus $F(x) = \tilde{F}(x)$. For the monotonicity take any $x, y \in \mathbb{R}$ with $x < y$. Then we can find $z \in D \cap (x, y)$. Thus we get

$$F(x) = \sup\{\tilde{F}(a) \mid a \in D \cap [x, \infty)\} \overset{\tilde{F}\searrow}{=} \sup\{\tilde{F}(a) \mid a \in D \cap [x, z)\} \geq \tilde{F}(z) \geq \tilde{F}(b)$$

for all $b \in D \cap [z, \infty)$, in particular for all $b \in D \cap [y, \infty)$. Taking the supremum over $b \in D \cap [y, \infty)$ we get

$$F(x) \geq \tilde{F}(z) \geq \sup\{\tilde{F}(b) \mid b \in D \cap [z, \infty)\} = F(z),$$

so F is decreasing as well. We proceed to the right-continuity. Let $x \in \mathbb{R}$ be arbitrary and $(x_j)_{j \in \mathbb{N}}$ a sequence in (x, ∞) with $x_j \longrightarrow x$ as $j \to \infty$. For a given $k \in \mathbb{N}$ we can find $y_k \in D \cap (x, \infty)$ such that

$$F(x) \geq \tilde{F}(y_k) > F(x) - \frac{1}{k}$$

owing to the properties of the supremum. Since $x_j \longrightarrow x$ we find $j_k \in \mathbb{N}$ with

$$x \leq x_{j_k} \leq y_k.$$

Since F is decreasing we get

$$F(x) \geq F(x_{j_k}) \geq F(y_k) = \tilde{F}(y_k) > F(x) - \frac{1}{k}.$$

Thus we get

$$\lim_{k \to \infty} F(x_{j_k}) = F(x).$$

If $(x_j)_{j\in\mathbb{N}}$ is decreasing then $(F(x_j))_{j\in\mathbb{N}}$ is decreasing and has a convergent subsequence, forcing $\lim_{j\to\infty} F(x_j) = F(x)$.
Assume not $F(x) = \lim_{j\to\infty} F(x_j)$. Then there is $\tau > 0$ and a subsequence $(F(x_{j_m}))_{m\in\mathbb{N}}$ such that

$$|F(x) - F(x_{j_m})| \geq \tau \quad \text{for all } m \in \mathbb{N}.$$

We choose a decreasing subsequence $(x_{j_{m_q}})_{q\in\mathbb{N}}$ thus $F(x_{j_{m_q}})$ is convergent towards $F(x)$ by the previous argument, which is a contradiction. This shows that F is right continuous.
The last step is to show that at every point $x \in \mathbb{R}$ the limit $\lim_{y\nearrow x} F(y)$ exists. Let $(z_j)_{j\in\mathbb{N}}$ be a sequence in $(-\infty, x)$ with $z_j \longrightarrow x$ as $j \to \infty$. Owing to the last argument from the right-continuity proof it is sufficient to consider an increasing sequence. From this we get

$$F(x) \leq F(z_{j+1}) \leq F(x_j) \quad \text{for all } j \in \mathbb{N}.$$

So $(F(z_j))_{j\in\mathbb{N}}$ is convergent. We still need to prove that the limit is independant from the sequence $(z_j)_{j\in\mathbb{N}}$. Let $(w_n)_{n\in\mathbb{N}}$ be another increasing sequence in $(-\infty, x)$ with $w_n \longrightarrow x$. Because of $w_n \nearrow x$ we can find $n_j \in \mathbb{N}$ such that $z_j \leq w_{n_j}$ and we find $j_n \in \mathbb{N}$ with $w_n \leq z_{j_n}$. Since F is decreasing we get

$$\lim_{j\to\infty} F(z_j) \geq \lim_{j\to\infty} F(w_{n_j}) = \lim_{n\to\infty} F(w_n) \geq \lim_{n\to\infty} F(z_{j_n}) = \lim_{j\to\infty} F(z_j).$$

Thus the limit are the same and is independant from the chosen sequence. □

Definition 8.3.4 (Lebesgue point)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be open, μ a Borel measure on Ω and $f \in L^1(\Omega, \mu)$. $x \in \Omega$ is called a μ-Lebesgue point of f if

$$\lim_{r\to 0} \fint_{B_r(x)} f \,\mathrm{d}\mu = f(x).$$

In the following we state the main result on Lebesgue points, which can be found as Theorem 5.16 in [Mag12].

Theorem 8.3.5 (On Lebesgue points)**.**
Let μ be a Radon measure on $\mathbb{R}^n$, $p \in [1, \infty)$, and $f \in L^p_{\text{loc}}(\mathbb{R}^n, \mu)$. Then μ-a.e. $x \in \Omega$ is a μ-Lebesgue point of f.

Theorem 8.3.6 (Gauß' Divergence Theorem)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded with C^1-boundary and $f \in C^1(\overline{\Omega}; \mathbb{R}^n)$. Then we have

$$\int_\Omega \nabla \cdot u \,\mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} u \cdot \nu \,\mathrm{d}\mathcal{H}^{n-1}$$

where ν denotes the outer normal of $\partial\Omega$.

Theorem 8.3.7 (Theorem of Partial Integration)**.**
Let $\Omega \subseteq \mathbb{R}^n$ be open.
(i) Let Ω be bounded with C^1-boundary and let $u, v \in C^1(\overline{\Omega})$. Then we have for all $j \in \{1, \ldots, n\}$

$$\int_\Omega u\partial_j v \,\mathrm{d}\mathcal{L}^n = \int_{\partial\Omega} uv\nu_j \,\mathrm{d}\mathcal{H}^{n-1} - \int_\Omega v\partial_j u \,\mathrm{d}\mathcal{L}^n$$

where ν denotes the outer normal of $\partial\Omega$.

(ii) Let $u, \phi \in C^1(\Omega)$ and let $\operatorname{supp}(\phi) \Subset \Omega$. *Then we have for all $j \in \{1, \dots, n\}$*

$$\int_\Omega u \partial_j \phi \, \mathrm{d}\mathcal{L}^n = -\int_\Omega \phi \partial_j u \, \mathrm{d}\mathcal{L}^n.$$

The second claim follows from the first because we can find an open and bounded set U with C^1-boundary such that $\operatorname{supp}(\phi) \Subset U \subseteq \Omega$.

Calculation of constants for the Amstutz-Van Goethem model

We consider $W(x) \coloneqq \frac{1}{4}(1-x^2)^2$ for $x \in \mathbb{R}$. We can calculate the constants for the approximations and flow equations.

$$\begin{aligned}
\|\overline{q}_0'\|^2_{L^2(\mathbb{R})} &= \int\limits_{\mathbb{R}} \overline{q}_0' \sqrt{W_*(\overline{q}_0)} \, \mathrm{d}\mathcal{L}^1 = \int\limits_{(-1,1)} \sqrt{W_*} \, \mathrm{d}\mathcal{L}^1 = \int\limits_{(-1,1)} \sqrt{W \circ f^{-1} + \frac{1}{4}(W' \circ f^{-1})^2} \, \mathrm{d}\mathcal{L}^1 \\
&= \int\limits_{(-1,1)} f' \sqrt{W + \frac{1}{4}(W')^2} \, \mathrm{d}\mathcal{L}^1 = \int\limits_{(-1,1)} \left(1 + \frac{1}{2}W''\right) \sqrt{W + \frac{1}{4}(W')^2} \, \mathrm{d}\mathcal{L}^1.
\end{aligned}$$

Now we need to plug in the concrete double-well potential and substitute $x = \sinh(\theta)$ with $a \coloneqq \operatorname{Arsinh}(1) = \log(1+\sqrt{2})$

$$\begin{aligned}
\|\overline{q}_0'\|^2_{L^2(\mathbb{R})} &= \frac{1}{2}\int_{(0,1)} (3x^2+1)(1-x^2)\sqrt{1+x^2} \, \mathrm{d}x \\
&= \frac{1}{2}\int_{(0,a)} (3\sinh^2(\theta)+1)(1-\sinh^4(\theta)) \, \mathrm{d}\theta \\
&= \frac{1}{128}(4a + 19\sinh(2a) + 7\sinh(4a) - \sinh(6a)) \\
&= \frac{1}{32}(13\sqrt{2} + \log(1+\sqrt{2})).
\end{aligned}$$

Next we have

$$\begin{aligned}
\|q_0'\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left| \frac{\overline{q}_0'}{f'(f^{-1}(\overline{q}_0))} \right|^2 \mathrm{d}\mathcal{L}^1 \\
&= \int_{(-1,1)} \frac{\sqrt{W_*}}{(f' \circ f^{-1})^2} \, \mathrm{d}\mathcal{L}^1 = \int_{(-1,1)} \frac{\sqrt{W + \frac{1}{4}(W')^2}}{1 + \frac{1}{2}W''} \, \mathrm{d}\mathcal{L}^1.
\end{aligned}$$

Now we need to plug in the concrete double-well potential and substitute $x = \sinh(\theta)$

$$\|q_0'\|^2_{L^2(\mathbb{R})} = \int_{(0,1)} \frac{2(1-x^2)\sqrt{1+x^2}}{3x^2+1} \, \mathrm{d}x = \int_{(0,a)} \frac{2(1-\sinh^2(\theta))(1+\sinh^2(\theta))}{3\sinh^2(\theta)+1} \, \mathrm{d}\theta.$$

We shorten the fraction by a factor of $\sinh^6(\theta)$ and apply $(\coth^2(x)-1)\sinh^2(x) = 1$

$$\|q_0'\|^2_{L^2(\mathbb{R})} = \int_{(0,a)} \frac{2\left(\frac{1}{\sinh^2(\theta)} - 1\right)\left(\frac{1}{\sinh^2(\theta)} + 1\right)\frac{1}{\sinh^2(\theta)}}{\frac{3}{\sinh^4(\theta)} + \frac{1}{\sinh^6(\theta)}} \,\mathrm{d}\theta$$
$$= \int_{(0,a)} \frac{2\coth^2(x)(\coth^2(x) - 2)\frac{1}{\sinh^2(\theta)}}{\coth^6(\theta) - 3\coth^2(\theta) + 2} \,\mathrm{d}\theta.$$

Having prepared the substitution we can now transform $w = \coth(\theta)$

$$\|q_0'\|^2_{L^2(\mathbb{R})} = \int\limits_{(\sqrt{2},\infty)} \frac{2w^2(w^2-2)}{w^6 - 3w^2 + 2} \,\mathrm{d}w = \int\limits_{(\sqrt{2},\infty)} \frac{2w^2(w^2-2)}{(w-1)^2(w+1)^2(w^2+2)} \,\mathrm{d}w$$
$$= \int\limits_{(\sqrt{2},\infty)} \left(\frac{16}{9(w^2+2)} + \frac{5}{18(w-1)} - \frac{5}{18(w+1)} - \frac{1}{6(w+1)^2} - \frac{1}{6(w-1)^2}\right) \mathrm{d}w$$
$$= \left[\frac{8\sqrt{2}}{9}\arctan\left(\frac{w}{\sqrt{2}}\right) + \frac{5}{18}\log\left|\frac{w-1}{w+1}\right| + \frac{w}{3(w^2-1)}\right]_{\sqrt{2}}^{\infty}$$
$$= \frac{1}{9}\big(2\sqrt{2}\pi - 5\log(\sqrt{2}-1) - 3\sqrt{2}\big).$$

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"'Day 31: I finally succeeded in my time reversal experiment!
'Day 30: I might have a problem here.'
—Journal of the Prime Izmagnus"[1]

[1]quoted from the *Magic: The Gathering* card *Inspiration* from the set *Return to Ravnica* created by Wizards of the Coast.