Refining Mortality Projections at Advanced Ages: Evaluating the Significance of Wittstein's Mortality Law[1]

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Abstract: Age-specific mortality rates for semi-supercentenarians and supercentenarians play a pivotal role in comprehending longevity and population dynamics at advanced ages. In this study, we introduce a modified Wittstein Model, offering an alternative to the conventional S-shaped curve models used in mortality forecasting. The Wittstein Model, originally formulated by Theodor Wittstein, has been adapted to suit contemporary contexts. Utilizing life table data for German women from 2019/2021, we project age-specific mortality rates, construct life tables commencing from age 100, and conduct a sensitivity analysis to assess the impact of model parameters on mortality patterns.
The sensitivity analysis unveils the influence of parameter values on the shape of age-specific mortality rates. This study contributes to research in mortality forecasting, with a specific focus on semi-supercentenarians and supercentenarians, shedding light on an understudied population segment. Accurate projections carry profound implications for public health, healthcare planning, and social policy. Further research should explore the model's applicability in different contexts, providing a deeper understanding of mortality patterns at advanced ages. As the empirical database of centenarians expands, the model is expected to enhance its precision and reliability in forecasting age-specific mortality rates at advanced ages.

Keywords: Mortality Projections, Wittstein's Mortality Law, Death Probabilities, Gompertz Model, Supercentenarians, Maximum Age.

1. Introduction

Age-specific mortality rates for semi-supercentenarians and supercentenarians[2] play a crucial role in understanding longevity and population dynamics at advanced ages. However, due to limited and unreliable data in these age classes, projecting mortality rates becomes essential for gaining insights into mortality patterns and making informed decisions. In this article, we propose a modified Wittstein Model, as an alternative to traditional S-shaped curve models commonly used in mortality forecasting.
The Wittstein Model, initially developed by Theodor Wittstein in the 19th century, provides a foundation for our analysis. We have transformed the traditional Wittstein Model to enhance interpretability and applicability in contemporary settings. To demonstrate the effectiveness of our approach, we apply it to life table data for women from the German life table of 2019/2021.
From the projected age-specific mortality rates, we construct a life table starting at age x=100, enabling us to calculate life expectancy at age x. Our analysis includes presenting the projection results and performing a sensitivity analysis to assess the influence of different model parameters on the curve of age-specific mortality rates.

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[1] This paper is the written version of a poster to be presented at the 15th International Seminar on Supercentenarians, organized by the Institut national d'études démographiques (INED) at the Campus Condorcet in Aubervilliers, France, on November 16-17, 2023.
[2] Supercentenarians are individuals who have reached the age of 110 years or older. The term "semi-supercentenarians" refers to individuals who have reached an age between 105 and 109 years.

Through the sensitivity analysis, we investigate the impact of varying key parameters on the shape and trajectory of age-specific mortality rates. By systematically varying parameters such as M (the parameter associated with the maximum age), n (the shape parameter), and y (the threshold parameter or median), we can better understand how changes in these parameters affect the projected mortality rates. This analysis provides valuable insights into the robustness and flexibility of the modified Wittstein Model, allowing for a comprehensive assessment of its performance and applicability in different scenarios.
Importantly, we emphasize that the parameter M, conventionally associated with the age where the death probability reaches 1, should not be considered as the maximum age. This is because the probability for an individual to reach M is infinitesimally small. Instead, the maximum age should be understood as the age at which the last remaining individual in a population of size N would reach. We discuss the implications of this finding and emphasize the significance of forecasting age-specific mortality rates for individuals aged 100 and above.
By addressing these key points, including the sensitivity analysis, and presenting our findings, we contribute to the growing body of research on mortality forecasting. Our focus on semi-supercentenarians and supercentenarians sheds light on an understudied population segment and underscores the importance of accurate and robust projections for informed decision-making in various fields, including public health, healthcare planning, and social policy.

2. Wittstein's Mortality Formula

Wittstein's complete formula is expressed as:

$$q(x) = \frac{1}{m} \cdot \exp\left(-k \cdot \left(m \cdot x\right)^n\right) + \exp\left(-k \cdot \left(M - x\right)^n\right)$$

where:

q(x) represents the death probability at age x, ranging from 0 to M; k, m, and n are parameters with values greater than 0. The significance and interpretation of these parameters will be further explained in this chapter.

It is important to note that q(M) = 1, and q(0) is approximately equal to 1/m (exact value: $\frac{1}{m} + \exp\left(-k \cdot M^n\right)$) The function exhibits a minimum at age x = M/(m+1), resembling the shape of a bathtub in terms of the death probability function. Starting with high infant mortality, the function decreases until the age x = M/(m+1), and then it gradually increases up to a death probability of 1 at age x = M. It should be emphasized that the parameter M should not be regarded as the maximum age, as it is highly unlikely to be reached.
The first part of the formula represents a decreasing function of x, eventually approaching zero, while the second part is an increasing function of x, reaching its maximum value of 1 at age x = M.
In current research, typically only the second part of the formula is utilized to model mortality at advanced ages. When n = 1, the formula becomes a special case of the Gompertz formula. Therefore, Wittstein's formula offers greater flexibility than the Gompertz formula.

In the following sections, our focus will solely be on the second part of Wittstein's formula. This part has been widely employed, such as by the US Bureau of the Census for calculating the life table of 1910, where the death probability was assumed to be 1 at the age of 115 (see US Bureau of the Census, 1916, p.12).

Now we transform the second part of the above formula for a better interpretation:

By setting q(y) = 0.5, we can determine the "median" value $y = M - \left(\frac{\ln 2}{k}\right)^{\frac{1}{n}}$.

Solving the median formula for k yields:

$k = (M - y)^{-n} \cdot \ln 2$.

Substituting k into the formula leads to an easier-to-interpret form of Wittstein's mortality law for old ages:

$$\boxed{q(x) = 2^{-((M-x)/(M-y))^n}}.$$

Here, q(x) depends on the values of n and the difference between M and y. The parameter n is a shape parameter that influences the growth rate of the death probability function. The growth rate, denoted by r(x), is given by:

$$\boxed{r(x) = n \cdot (M-x)^{n-1} \cdot (M-y)^{n-1} \cdot \ln 2}$$

For values of n greater than 1, q(x) increases with decreasing growth rates. When n equals 1, q(x) exponentially increases with a constant growth rate: $r(x) = \frac{\ln 2}{M-y}$. For values of n between 0 and 1, q(x) increases with increasing growth rates.

If n equals 1, Wittstein's model becomes a special case of the Gompertz model: $q(x) = A \cdot e^{k \cdot x}$, where $A = \exp\left(-M \cdot \frac{\ln 2}{M-y}\right)$ and $k = \frac{\ln 2}{M-y}$.

Additionally, if n is greater than 1, the function exhibits an S-shaped curve and has a turning point at $x_{TP} = M - \left((M-y)^n \cdot \frac{n-1}{n \cdot \ln 2}\right)^{\frac{1}{n}}$. Specifically, when $n = \frac{1}{1-\ln 2} \approx 3.26$, the turning point x_{TP} is equal to y. For values of n between 1 and $n = \frac{1}{1-\ln 2}$ x_{TP} is greater than y, and for n greater than $n = \frac{1}{1-\ln 2}$, x_{TP} is less than y.

Using the R package *nlstools*, we conducted parameter estimations for the Wittstein model with data on the death probabilities of the female population from the life table 2019/2021 in Germany between ages 60 and 100. The corresponding estimators are shown in Table 1. (Refer to Appendix 1 for the estimation results of life tables for both females and males from various years.)

From the projected q(x), we can calculate the life table with l(100) = 1 and the life expectancy at age x, denoted by e(x), from age 100 onwards. At very high ages, the life expectancy can be approximated by the inverse of the death probability, 1/q(x) (see Table 2). The results are illustrated in Figure 1. The upper-left corner shows the R plotfit function, displaying the actual values (depicted as circles) and the fitted ones (represented by solid lines). The upper-right corner presents the q(x) values of the life table from age 60 to age 100, followed by the predicted/forecast values. The lower-left corner shows the life table function from age x = 100 with l(100) = 1. Notably, the life table rapidly decreases between ages 100 and 105, with only

5.07% of centenarians reaching the age of 105 and 0.043% reaching the age of 110. The life expectancy is illustrated in the lower right corner of Figure 1.

Table 1: Estimation Results

```
Formula: q ~ 2^(-((M - x)^n)/(M - y)^n); x=60,61,......

Parameters:
  Estimate Std. Error t value      Pr(>|t|)
n   2.1873     0.2854   7.663 0.00000000378
y 103.6403     0.3171 326.831       < 2e-16
M 126.6124     4.6308  27.341       < 2e-16
```

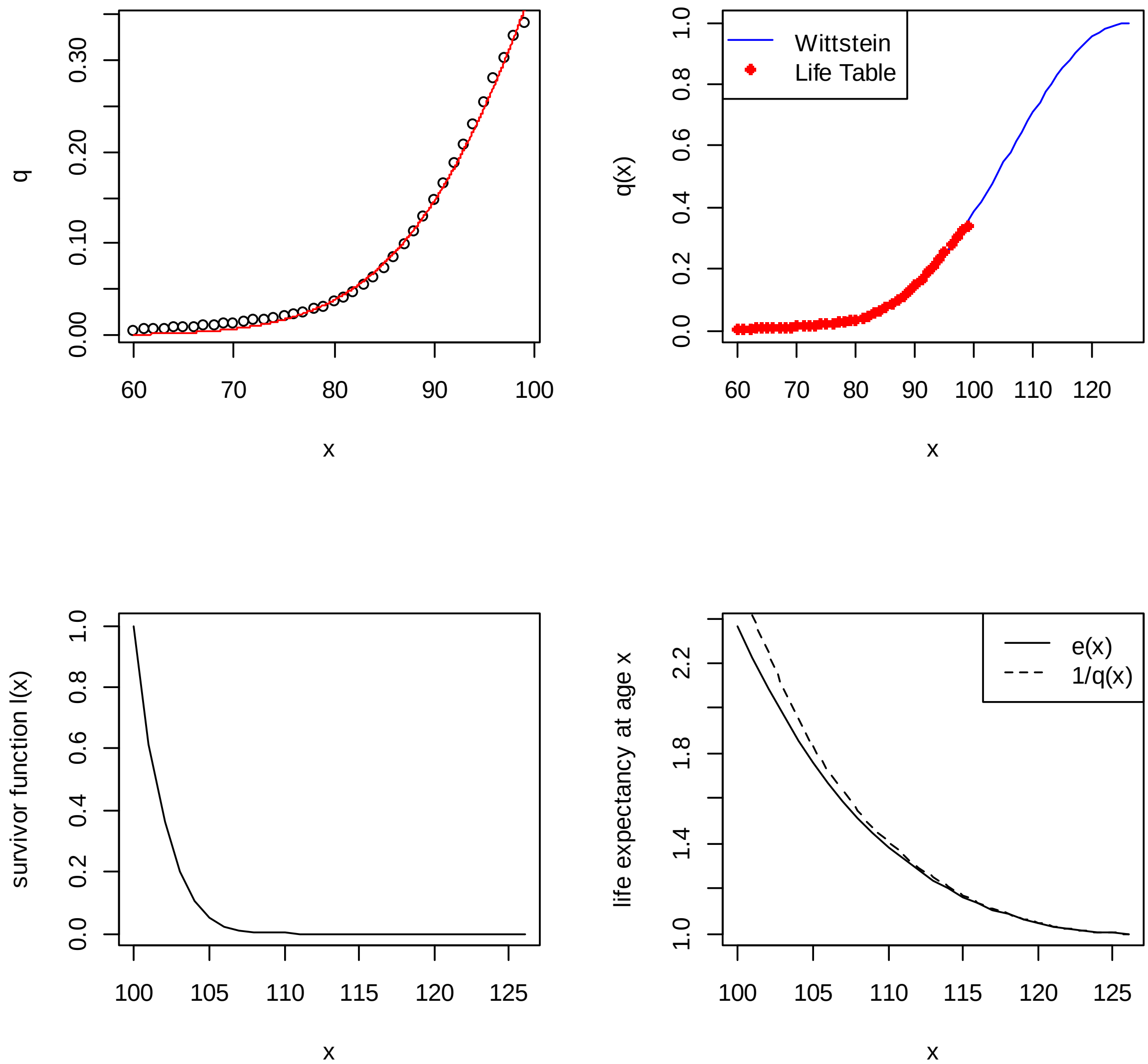

Figure 1: Analysis of Results: Survivor Function, Life Expectancy and Death Probability Projections

Table 2: Projected Life Table with l(100)=1

x	qx	lx	ex	1/qx
100	0.3843	1.00E+00	2.37	2.60
101	0.4150	6.16E-01	2.22	2.41
102	0.4466	3.60E-01	2.09	2.24
103	0.4790	1.99E-01	1.97	2.09
104	0.5119	1.04E-01	1.86	1.95
105	0.5452	5.07E-02	1.76	1.83
106	0.5788	2.30E-02	1.67	1.73
107	0.6123	9.71E-03	1.59	1.63
108	0.6457	3.76E-03	1.51	1.55
109	0.6786	1.33E-03	1.44	1.47
110	0.7110	4.29E-04	1.38	1.41
111	0.7424	1.24E-04	1.33	1.35
112	0.7728	3.19E-05	1.28	1.29
113	0.8020	7.25E-06	1.24	1.25
114	0.8297	1.43E-06	1.20	1.21
115	0.8557	2.44E-07	1.16	1.17
116	0.8798	3.53E-08	1.13	1.14
117	0.9020	4.24E-09	1.11	1.11
118	0.9221	4.15E-10	1.08	1.08
119	0.9400	3.23E-11	1.06	1.06
120	0.9555	1.94E-12	1.05	1.05
121	0.9687	8.63E-14	1.03	1.03
122	0.9795	2.70E-15	1.02	1.02
123	0.9880	5.53E-17	1.01	1.01
124	0.9941	6.66E-19	1.01	1.01
125	0.9979	3.96E-21	1.00	1.00
126	0.9998	8.22E-24		

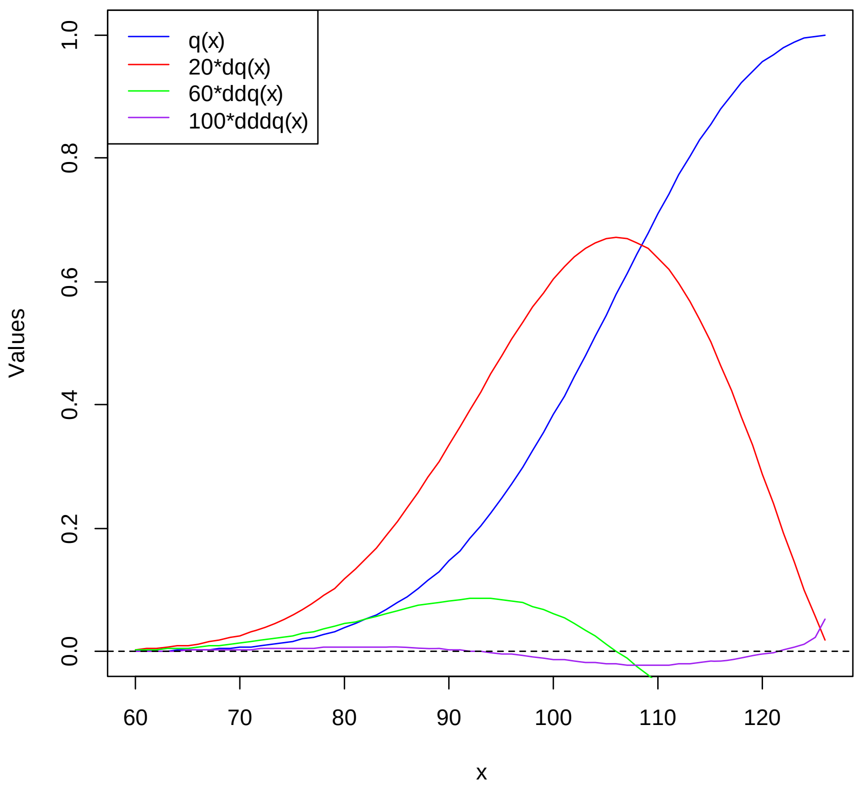

Figure 2: Wittstein Function and Its Derivatives

R codes for functions in Figure 2

```
# Function
q <- 2^(-((M - x)^n)/(M - y)^n)

# First derivative
dq <- 2^(-((M - x)^n)/(M - y)^n) * n * (M - x)^(n - 1) * (M - y)^(-n) * log(2)

# Second derivative
ddq <- 2^(-((M - x)^n)/(M - y)^n) * (n^2 * (M - x)^(2 * (n - 1)) * (M - y)^(-2 * n) * log(2)^2
   +      n * (1 - n) * (M - x)^(n - 2) * (M - y)^(-n) * log(2))

# Third derivative
dddq <- -2^(-((M - x)^n)/(M - y)^n) * (-n^3 * (M - x)^(3 * (n - 1)) * (M - y)^(-3 * n) * log(2)^3 +   3 * n^2 * (M
- x)^(2 * n - 3) * (M - y)^(-2 * n) * (n - 1) * log(2)^2 +
         n * (1 - n) * (M - x)^(n - 3) * (M - y)^(-n) * (n - 2) * log(2))
```

Figure 2 provides a visual representation of the Wittstein function and its derivatives. By examining these derivatives, we gain additional insights into the characteristics of the model. The first derivative represents the rate of change of death probabilities. It exhibits an increasing trend, reaching its modal value of approximately 106. This modal value coincides with the turning point of the death probability function, which is the crossing point of the second derivative (green line) with the x-axis.
The turning points of the first derivatives, calculated numerically, occur at ages 92.94 and 121.34. These turning points are graphically illustrated as the intersections of the olive curve with the x-axis, denoting the 3rd derivatives.

3. Graphical Sensitivity Analysis

We perform a graphical sensitivity analysis to illustrate the impact of different parameter values on the Wittstein formula (see Figures 3a and 3b). In Figure 3a, we observe that the slope or gradient at the turning point increases with higher values of n. This can be interpreted as the death probability before the median age (y) decreasing, while the death probability after the median age increases.
For values of n greater than 1, we observe S-shaped curves, while n = 1 represents an exponential curve (Gompertz curve). Increasing the median age from 103.64 to 113.64 (or reducing the difference between M and y) leads to a delayed increase in death probabilities with a steeper slope.
In Figure 3b, we observe curves with increasing growth rates ($0 < n < 1$). Smaller values of n result in death probabilities approaching 0.5 at an earlier age, followed by a steep increase shortly before reaching M. Additionally, Figure 3b demonstrates that a constant death probability of 0.5 is observed when n = 0. For values of n less than 0, the death probability decreases with age."

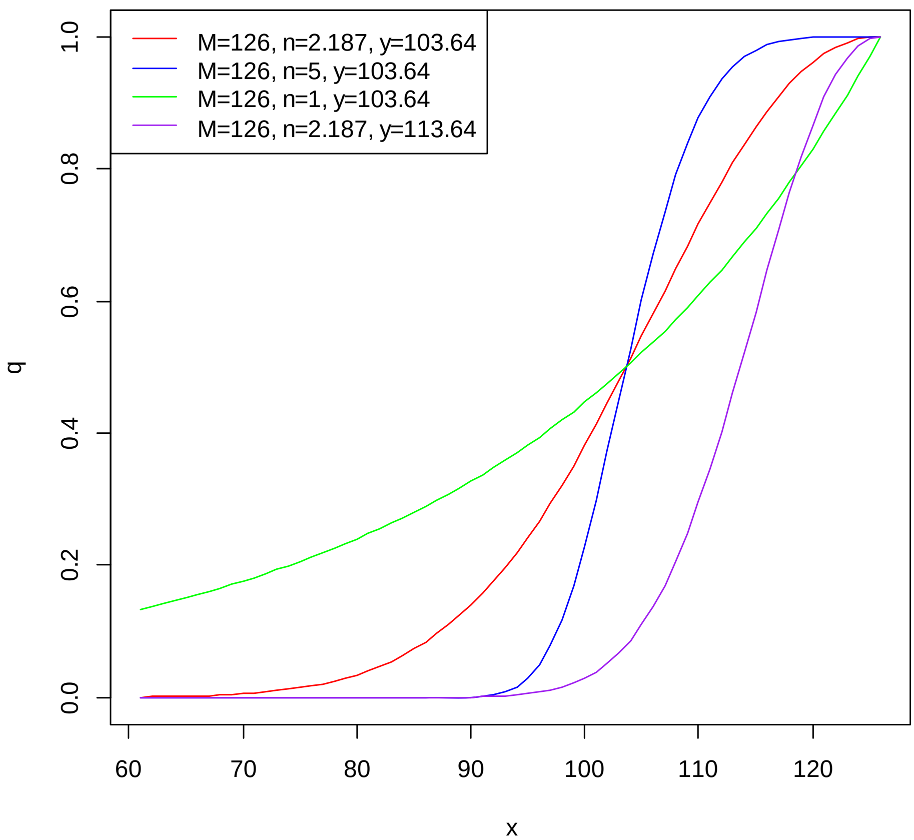

Figure 3 a: Wittstein-Mortality Formulas with Varying Parameters ($n \geq 1$)

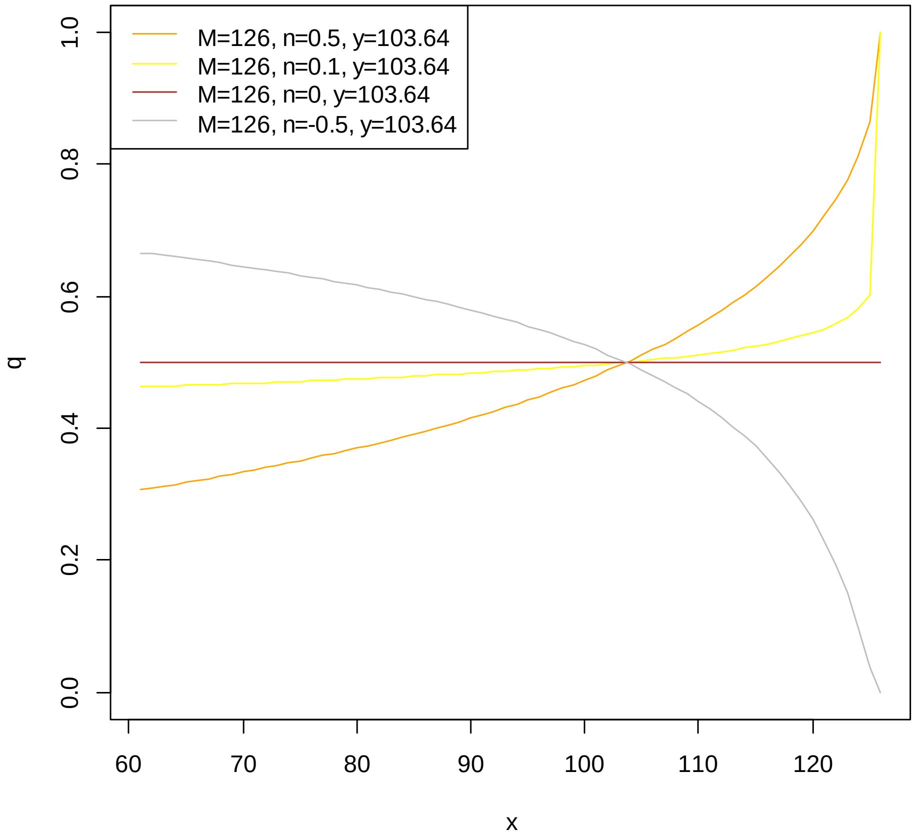

Figure 3 b: Wittstein-Mortality Formulas with Varying Parameters ($n < 1$)

Figure 4 displays the curves depicting the estimated death probabilities using the Wittstein model for selected life tables of Germany for females since 1871. The medians, represented by the values of y, exhibit a similar pattern across the years. Notably, the estimators of M exhibit exceptionally high values in the years 1901 and 1949. This observation reinforces the notion that M should not be solely interpreted as the maximum age or life span. However, to ensure a realistic representation of the death probabilities and account for an increased spread between M and y, an augmentation of the shape parameter n is required. This relationship is further explored in the sensitivity analysis depicted in Figure 5.

We observe a mortality crossover phenomenon: earlier life tables exhibit higher age-specific mortality rates between ages 60 and approximately 105, but they show lower rates at very high ages. However, when considering the overall effect, newborns have a lower probability of reaching the supercentenarian age in the case of earlier life tables. The pattern observed in the earlier life tables may lend support to the hypothesis that age-specific mortality will never reach one.

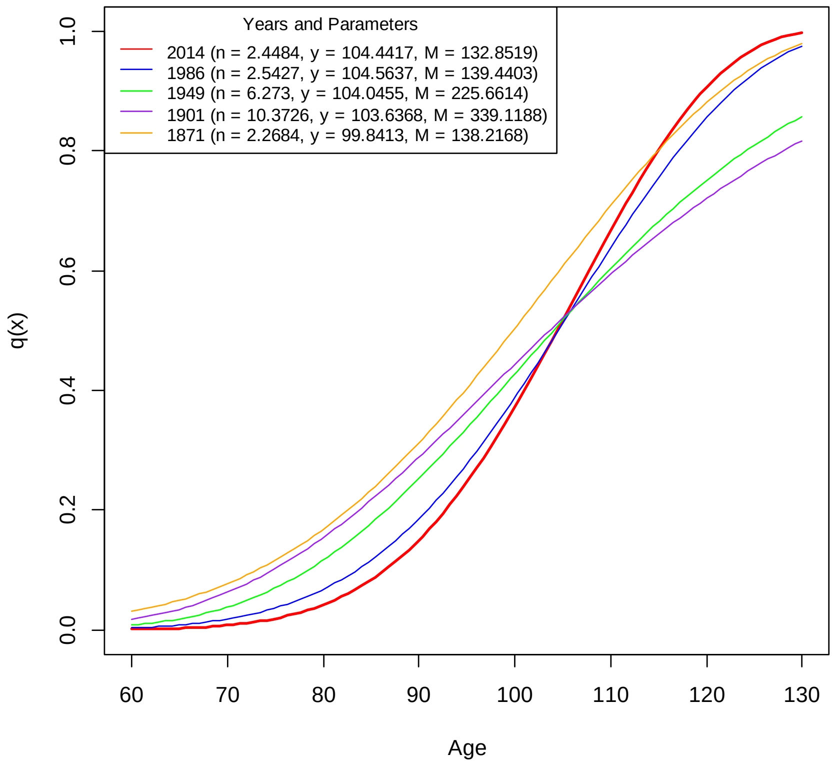

Figure 4: Wittstein Models for Selected Life Tables in Germany (females) since 1871 (see also Appendix 2)

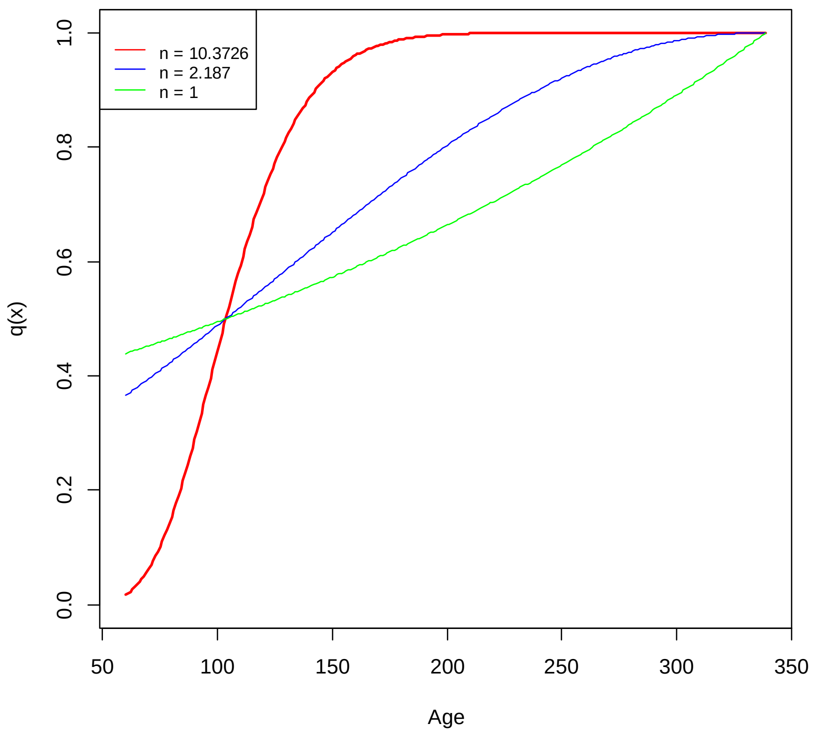

Figure 5: Wittstein Model for the Life Table 1901 (Female) Assuming Different Shape Parameters n

Finally, let's consider the interpretation of M. It undoubtedly represents the age at which the death probability reaches one, as mentioned earlier. However, we need to examine whether it can be considered the maximum age or lifespan. To evaluate this, we refer to Table 3, which presents the survivor function starting at age x=100 with an initial value of l(100)=1. The survivor function decreases rapidly as age increases. To assess the number of individuals at age 100 required for one individual to survive at a specific age, we compute the inverse of the survivor function.

For instance, at age x=110, the inverse is 2334, indicating that we need 2334 individuals at age 100 to observe one survivor at age 110. At age x=120, the inverse is 515,234,717,752 or 5.152E+11. At age x=126, the inverse is 1.217E+23. These results suggest that M may be a theoretical value for the maximum age or lifespan, but it is not a realistic value considering the number of individuals required for survival at advanced ages.

Therefore, while M represents the age at which the death probability reaches one, it is important to recognize that it may not be a feasible or attainable maximum age in practice.

Table 3: Maximum Age

x	lx	lx	1/lx (rounded)	1/lx
100	1.000000	1.00E+00	1	1.000E+00
101	0.615654	1.62E+00	2	1.624E+00
102	0.360131	2.78E+00	3	2.777E+00
103	0.199283	5.02E+00	5	5.018E+00
104	0.103831	9.63E+00	10	9.631E+00
105	0.050680	1.97E+01	20	1.973E+01
106	0.023048	4.34E+01	43	4.339E+01
107	0.009708	1.03E+02	103	1.030E+02
108	0.003764	2.66E+02	266	2.657E+02
109	0.001334	7.50E+02	750	7.499E+02
110	0.000429	2.33E+03	2,334	2.334E+03
111	0.000124	8.07E+03	8,074	8.074E+03
112	0.000032	3.13E+04	31,346	3.135E+04
113	0.000007	1.38E+05	137,993	1.380E+05
114	0.000001	6.97E+05	696,884	6.969E+05
115	0.000000	4.09E+06	4,090,975	4.091E+06
116	0.000000	2.83E+07	28,343,354	2.834E+07
117	0.000000	2.36E+08	235,889,002	2.359E+08
118	0.000000	2.41E+09	2,408,076,496	2.408E+09
119	0.000000	3.09E+10	30,922,453,907	3.092E+10
120	0.000000	5.15E+11	515,234,717,752	5.152E+11
121	0.000000	1.16E+13	11,587,332,450,070	1.159E+13
122	0.000000	3.70E+14	370,476,299,149,238	3.705E+14
123	0.000000	1.81E+16	18,094,304,256,811,700	1.809E+16
124	0.000000	1.50E+18	1,501,774,571,922,910,000	1.502E+18
125	0.000000	2.52E+20	252,465,897,537,725,000,000	2.525E+20
126	0.000000	1.22E+23	121,711,522,028,142,000,000,000	1.217E+23

4. Conclusion

In this article, we have explored the potential of the Wittstein Mortality Formula as an alternative approach to forecasting age-specific mortality rates at advanced ages. By modifying and adapting the traditional Wittstein Model, we have enhanced its interpretability and applicability in contemporary settings.

Through our analysis, we have demonstrated the effectiveness of the modified Wittstein Model by applying it to life table data for women from the German life table of 2019/2021. By constructing a life table starting at age x=100, we were able to calculate life expectancy at different ages and project age-specific mortality rates.

Our sensitivity analysis provided valuable insights into how different parameter values influence the shape and trajectory of age-specific mortality rates. We observed that varying parameters such as M (the maximum age parameter), n (the shape parameter), and y (the threshold parameter or median) had a significant impact on the projected mortality rates. By systematically exploring these parameter values, we gained a comprehensive understanding of the model's performance and flexibility in different scenarios.

Importantly, we highlight that the parameter M, conventionally associated with the age where the death probability reaches one, should not be considered as the maximum age. Our examination of the survivor function starting at age x=100 revealed that the number of individuals required for survival at advanced ages becomes astronomically high. While M represents the age at which the death probability reaches one, it is crucial to acknowledge that it may not be a feasible or attainable maximum age in practice.

In conclusion, our exploration of the Wittstein Mortality Formula as an alternative approach to forecasting age-specific mortality rates at advanced ages contributes to the growing body of research in mortality forecasting. By focusing on semi-supercentenarians and super-centenarians, we shed light on an understudied population segment and emphasize the importance of accurate and robust projections for informed decision-making in various fields, including public health, healthcare planning, and social policy. Our findings highlight the need for continued research and refinement of mortality models to better understand longevity and population dynamics at advanced ages.

Further research is warranted to expand the application of the modified Wittstein Model to different countries, time periods and age ranges. In this study, the model was applied to the age range from 60 to 100 (data range of the regression). However, it would be valuable to investigate its applicability to broader age ranges and explore its performance in diverse demographic contexts. By examining diverse populations, we can gain a deeper understanding of the model's performance and its ability to capture variations in mortality patterns. Moreover, as the empirical database of centenarians continues to grow, the accuracy and reliability of mortality projections using the Wittstein Model are expected to improve, making it an invaluable tool for forecasting age-specific mortality rates at advanced ages.

Literature

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Appendix

1. Estimation Results (qxm: for males; qxw: for females)

```
Formula: qxm2014 ~ 2^(-((M - x)^n)/(M - y)^n)

Parameters:
  Estimate Std. Error t value Pr(>|t|)
n   1.6125     0.1121   14.38   <2e-16 ***
y 102.5132     0.2073  494.61   <2e-16 ***
M 119.3180     2.0297   58.78   <2e-16 ***


Formula: qxw2014 ~ 2^(-((M - x)^n)/(M - y)^n)

Parameters:
  Estimate Std. Error t value Pr(>|t|)
n   2.4484     0.2978   8.223 7.12e-10 ***
y 104.4417     0.2985 349.903  < 2e-16 ***
M 132.8519     5.1314  25.890  < 2e-16 ***

Formula: qxm1986 ~ 2^(-((M - x)^n)/(M - y)^n)

Parameters:
   Estimate Std. Error t value Pr(>|t|)
n   2.40627    0.09718   24.76   <2e-16 ***
y 104.30765    0.11286  924.23   <2e-16 ***
M 142.56368    2.23183   63.88   <2e-16 ***

Formula: qxw1986 ~ 2^(-((M - x)^n)/(M - y)^n)

Parameters:
  Estimate Std. Error t value Pr(>|t|)
n   2.5427     0.1449   17.55   <2e-16 ***
y 104.5637     0.1463  714.58   <2e-16 ***
M 139.4403     2.8706   48.57   <2e-16 ***

Formula: qxm1949 ~ 2^(-((M - x)^n)/(M - y)^n)

Parameters:
  Estimate Std. Error t value Pr(>|t|)
n   3.8550     0.6825   5.648 1.88e-06 ***
y 101.7995     0.2548 399.591  < 2e-16 ***
M 168.6082    14.5447  11.592 7.05e-14 ***

Formula: qxw1949 ~ 2^(-((M - x)^n)/(M - y)^n)

Parameters:
  Estimate Std. Error t value     Pr(>|t|)
n   6.2730     1.4544   4.313     0.000115 ***
y 104.0455     0.2596 400.750      < 2e-16 ***
M 225.6614    32.3262   6.981 0.0000000301 ***

Formula: qxm1901 ~ 2^(-((M - x)^n)/(M - y)^n)

Parameters:
  Estimate Std. Error t value Pr(>|t|)
n   2.9971     0.7255   4.131 0.000198 ***
y  99.8355     0.3708 269.252  < 2e-16 ***
M 152.5698    16.2047   9.415 2.31e-11 ***
```

```
Formula: qxw1901 ~ 2^(-((M - x)^n)/(M - y)^n)

Parameters:
  Estimate Std. Error t value Pr(>|t|)
n  10.3726     5.2133    1.99   0.0541 .
y 103.6368     0.3468  298.85   <2e-16 ***
M 339.1188   127.9898    2.65   0.0118 *


Formula: qxm1871 ~ 2^(-((M - x)^n)/(M - y)^n)

Parameters:
  Estimate Std. Error t value      Pr(>|t|)
n   2.3881     0.3182   7.505 0.00000000609 ***
y  99.9092     0.2515 397.214       < 2e-16 ***
M 142.1713     7.5498  18.831       < 2e-16 ***

Formula: qxw1871 ~ 2^(-((M - x)^n)/(M - y)^n)

Parameters:
  Estimate Std. Error t value Pr(>|t|)
n   2.2684     0.1672   13.56 6.23e-16 ***
y  99.8413     0.1430  698.22  < 2e-16 ***
M 138.2168     3.8595   35.81  < 2e-16 ***
```

2. Death probability projections from age 100 to 130 using the Wittstein Model for selected life tables of females in Germany