

# Analyzing Pension Payout Risks with Continuous Gompertz Mortality Models: A Data-Driven Approach\*

Karl-Heinz Jöckel<sup>1</sup> and Peter Pflaumer<sup>2</sup>

<sup>1</sup>Medical Faculty, University of Duisburg-Essen, Germany,

`karl-heinz.joeckel@uk-essen.de`

<sup>2</sup>Department of Statistics, Technical University of Dortmund, Germany,

`peter.pflaumer@tu-dortmund.de`

## Abstract

This paper analyzes lifetime pension payouts under the German statutory system using continuous Gompertz mortality models. We compare pay-as-you-go benefits with a scenario in which contributions are invested in the capital market, yielding a benchmark capital stock of approximately €604,000 over a 45-year career.

We derive the full distribution of individual pension payouts, including expected values and quantiles, and extend traditional life-table calculations to a continuous-time framework. Present-value distributions account for differences between pension growth rates and discount rates, and a sensitivity analysis quantifies the impact of key Gompertz parameters on both expected payouts and their dispersion.

Results indicate that higher discount rates reduce present values and tighten payout distributions, while higher pension growth increases risk due to heavier tails. The framework provides a transparent, extendable method for evaluating pension schemes, assessing demographic and financial risks, and supporting actuarial decision-making.

**Keywords:** Pension payouts, Gompertz mortality model, Lifetime distribution, Actuarial analysis, Pay-as-you-go system, Present-value risk, Sensitivity analysis, Data-driven modeling, Financial Mathematics

## 1 Introduction

Demographic projections have long highlighted the tension between population aging and the sustainability of pay-as-you-go systems. Pflaumer (1986), for example, developed a stochastic Leslie matrix model to analyze how demographic processes affect social security systems. The present study, however, examines this tension on an individual level, focusing on how lifetime pension payouts can be quantified using continuous Gompertz models.

Using continuous mortality models, we derive the full distribution of individual pension payouts, capturing both expected values and quantiles. Our approach extends traditional life-table calculations to a continuous-time framework, allowing us to assess the uncertainty of lifetime benefits. We also incorporate present-value calculations to account for differences between pension growth rates and discount rates, and we conduct a sensitivity analysis to quantify the impact of key mortality parameters on both the expected value and dispersion of payouts. This framework provides a robust and transparent method for comparing pension schemes, evaluating demographic risks, and supporting actuarial decision-making. To

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illustrate the practical application of this framework, we consider a representative worker in Germany and evaluate the distribution of statutory pension payouts.

An average worker in Germany with a full contribution history of 45 years can expect a statutory pension of roughly 1,500 EUR per month in retirement.<sup>1</sup> This benchmark follows directly from the German pension-point system and will serve only as a reference in what follows. Our analysis instead focuses on the counterfactual scenario that the worker does not participate in the pay-as-you-go system, but instead invests the *entire contribution* (employee and employer share combined) on the capital market. Using the annual earnings and contribution rates reported in Table ?? in the Appendix, the first year’s total contribution equals

$$0.17 \times 22,860 \text{ DM} \approx 3,886 \text{ DM}.$$

We then assume that each year’s contribution is invested in a fund yielding the historical average yield of German Federal Bonds (“daily yields of current Federal securities”, *Umlaufrendite*) given in Table 9 the Appendix. Accumulating these annual deposits over the period 1980–2025 leads to a final capital stock of approximately € 604,000.

In the next step, we compare this capital value with the present value of the statutory pension. The reference case is an average pension of €1,500 per month or €18,000 per year. Assuming that the increase of pensions exactly offsets the inflation rate, no discounting is necessary.

Let  $Y$  denote the total lifetime pension payout. If the retiree survives  $T$  years after retirement, then

$$Y = pens \cdot T, \quad pens = 18,000.$$

The discrete distribution of  $Y$  thus follows directly from the survival probabilities: one year of retirement yields €18,000, two years €36,000, and so on. To approximate this discrete distribution in a mathematically tractable way, we use the *Gompertz distribution*, which provides a continuous mortality law. In our framework, this allows us to move from the stepwise payout profile to a smooth continuous distribution of lifetime pension values, enabling the computation of expectations, quantiles, and sensitivity measures.

## 2 Gompertz Survival and Mortality

Let  $X$  denote the age at death. In the Gompertz model, mortality increases approximately exponentially with age. The model is characterized by two parameters: the modal age  $m$ , which determines the location of the distribution, and the slope parameter  $k$ , which governs how rapidly mortality rises.

The survival function, i.e. the probability of surviving beyond age  $x$ , is

$$S(x) = \exp\left(\exp(-km) - \exp(k(x - m))\right), \quad x \geq 0.$$

The associated force of mortality (also called hazard rate) is given by

$$\mu(x) = k \exp(k(x - m)),$$

which describes the instantaneous risk of death at age  $x$  conditional on being alive at that age. In the Gompertz framework, this hazard rate grows exponentially with age, reflecting the empirically observed acceleration of mortality in adult life.

When conditioning on survival to a fixed retirement age  $a_0$  (for example  $a_0 = 65$ ), the density of the remaining lifetime distribution is

$$f_{X|a_0}(x) = \frac{S(x) \mu(x)}{S(a_0)}, \quad x \geq a_0.$$

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<sup>1</sup>For details on statutory pension calculations, as well as related data and time series, see <https://www.deutsche-rentenversicherung.de/> and <https://statistik-rente.de/drv/extern/>.

This conditional density describes the distribution of death ages for individuals who have already reached age  $a_0$ . It serves as the probabilistic foundation for modeling lifetime pension payouts in the continuous framework.

### 3 Discrete and Continuous Pension Payout Models

#### 3.1 Discrete Life Table

Let  $l_x$  be the number of survivors at age  $x$  from a life table, and  $d_x = l_x - l_{x+1}$  the number of deaths between ages  $x$  and  $x + 1$ . Conditional on survival to age  $a_0$  (here  $a_0 = 65$ ), the probability mass function of the total pension payout is given by

$$P(Y = k \cdot \text{pens}) = \frac{d_{a_0+k-1}}{l_{a_0}}, \quad k = 1, 2, \dots$$

where  $Y$  denotes the total lifetime pension payout and  $\text{pens} = 18,000$  is the annual pension payment.

- The pmf sums to 1:  $\sum_k P(Y = k \cdot \text{pens}) = 1$ .
- The cumulative distribution function is

$$F_Y(y) = \sum_{j=1}^k P(Y = j \cdot \text{pens}), \quad k = \left\lfloor \frac{y}{\text{pens}} \right\rfloor.$$

Using the official German life table 2019/21, we obtain the distribution of total payouts  $Y$  for both females and males. Table 1 reports selected quantiles and the mean. The results show the strong dispersion of outcomes: while some retirees receive only a few multiples of the annual pension, others live long enough to accumulate payouts exceeding half a million euros.

For males, the distribution of payouts is shifted downward because of their higher mortality. The mean total payout of €329,760 is about 15% lower than the corresponding female mean of €388,020.

Table 1: Quantiles of the total lifetime pension payout  $Y$  for females and males, based on the German life table 2019/21. Annual pension amount is fixed at €18,000.

Quantile	Female (€)	Male (€)
5%	108,000	54,000
10%	162,000	108,000
25%	288,000	216,000
50% (Median)	414,000	342,000
75%	504,000	450,000
90%	558,000	522,000
95%	594,000	558,000
Mean	388,020	329,760

#### 3.2 Historical Foundations: Ulpian’s Table and de Witt’s Actuarial Methodology

Pflaumer (2018, pp. 113–ff.) explored the use of Ulpian’s table to estimate Roman life expectancy, considering whether the figures represented life expectancies, durations of annuities certain, or maximum legal valuations of annuities. Using continuous mortality models and approximation formulas for life annuities, it was demonstrated that these figures could

be interpreted as present values of life-contingent payments, laying an early foundation for actuarial analysis of lifetime payouts.

Building upon this conceptual framework, the present study applies a modern continuous Gompertz approach to quantify individual pension distributions in a contemporary pay-as-you-go system. Unlike the historical analysis, which focused on approximating life annuities from Ulpian's data, the current work employs de Witt's methodology to model stochastic present values and distributions of pension benefits under varying discount and growth rates, using the **death density function** to weight each future payment by the probability of death at a given age.

In the 17th century, Jan de Witt introduced a method for calculating life annuities that combined mortality probabilities and interest rates, specifying the value of a life annuity as the expected present value of future payments based on the remaining lifetime (Ciecka 2008). De Witt's approach was mathematically correct, although his mortality assumptions were simplified, and it laid the theoretical foundation for modern actuarial methods. Edmond Halley (1693) later developed the first empirically based life table to improve annuity valuations, but Ciecka (2008) emphasizes that de Witt published the correct mathematical formula for expected present value first.

### 3.3 Continuous Gompertz Model

Let age at death  $X$  follow a Gompertz distribution with parameters  $m$  and  $k$ :

$$S(x) = \exp\left(\exp(-km) - \exp(k(x - m))\right), \quad \mu(x) = k \exp(k(x - m)).$$

Conditional on survival to  $a_0$ :

$$f_{X|a_0}(x) = \frac{S(x)\mu(x)}{S(a_0)}, \quad x \geq a_0.$$

Total payout for a constant yearly pension pens:

$$Y = \text{pens} \cdot (X - a_0).$$

Density of total payout:

$$f_Y(y) = \frac{1}{\text{pens}} f_{X|a_0}\left(a_0 + \frac{y}{\text{pens}}\right), \quad y \geq 0.$$

Cumulative distribution function:

$$F_Y(y) = \Pr(Y \leq y) = 1 - \frac{S(a_0 + y/\text{pens})}{S(a_0)}.$$

Quantiles of total payout:

$$y_q = \text{pens} \cdot (x_q - a_0), \quad x_q = S^{-1}((1 - q)S(a_0)).$$

Figure 1 illustrates the fit of the Gompertz model for the male population retiring at age 65. The top-left panel shows the observed life table survival function compared with the fitted Gompertz survival curve. The top-right panel displays the corresponding death density. The bottom-left panel presents the discrete pension payout distribution derived from the life table, while the bottom-right panel shows the continuous pension density computed from the Gompertz approximation.

The discrete life-table probabilities and the continuous Gompertz density describe the same underlying distribution of payouts; the continuous model provides a smooth approximation to the stepwise structure of the life table. In both settings, the expected total payout is computed in the usual way:  $\mathbb{E}[Y] = \sum_k (k \cdot \text{pens}) P(Y = k)$  in the discrete case,

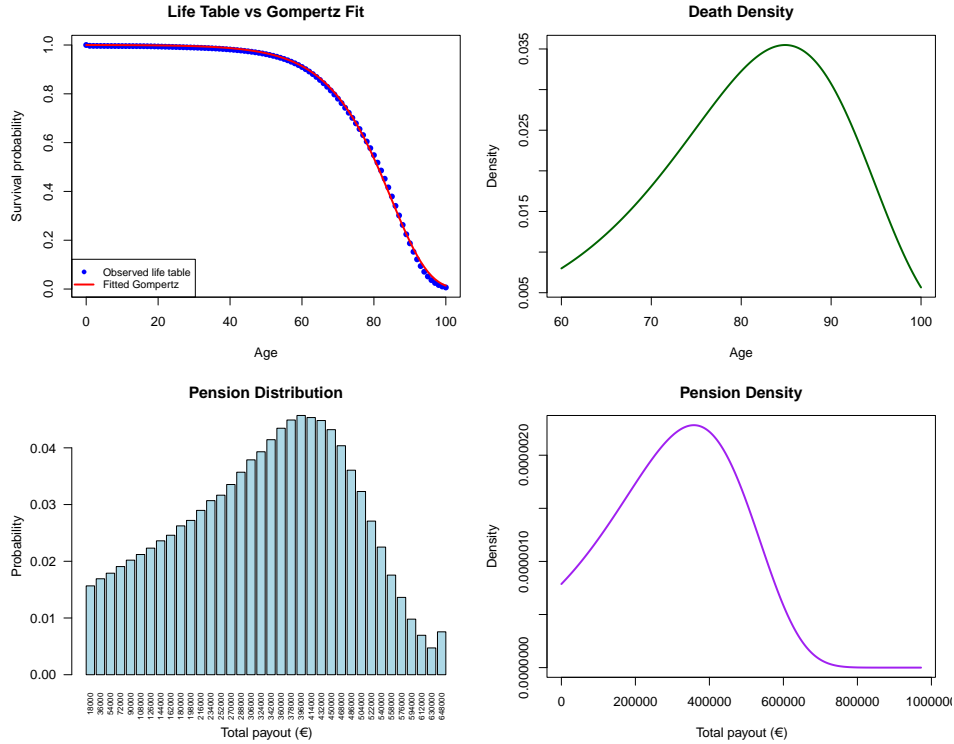


Figure 1: 2x2 diagnostic plots for the male population

or  $E[Y] = \int_0^\infty y f_Y(y) dy$  in the continuous case. Quantiles of the payout distribution can be obtained either directly from the life table or, in the continuous formulation, by evaluating the Gompertz quantile function. This makes it possible to interpolate smoothly between the discrete outcomes and to compute percentiles consistently.

Table 2: Quantiles of the total lifetime pension payout  $Y$  for females, based on the German life table 2019/21 and the fitted Gompertz model ( $m = 89.04$ ,  $k = 0.1143$ ). Assumptions: retirement age  $a_0 = 65$ , annual pension €18,000.

Quantile	Discrete [€]	Continuous [€]
5%	108,000	92,615
10%	162,000	153,144
25%	288,000	268,180
50%	414,000	388,924
75%	504,000	491,274
90%	558,000	568,386
95%	594,000	608,838
Mean	388,020	372,800

The table 2 reports selected quantiles of total pension payouts for females retiring at age 65. The discrete distribution is derived directly from the German female life table 2019/21, which also serves as the basis for estimating the Gompertz parameters. The fitted Gompertz law yields parameter estimates of  $m = 89.04$  and  $k = 0.1143$  for females, while for males the corresponding values are  $m = 84.89$  and  $k = 0.0964$ , reflecting their higher mortality. The continuous quantiles in Table 2 and Table 3 have been calculated using the Gompertz quantile formulas derived in Section 3.3.

Figure 2 compares the cumulative distribution of total lifetime pension payouts for males retiring at age 65. The step function represents the discrete distribution derived from the German male life table 2019/21, while the smooth curve corresponds to the continuous Gompertz approximation ( $m=84.89$ ,  $k=0.0964$ ). The close alignment between the two curves confirms that the Gompertz model captures both central tendencies and tail behavior of the pension payout distribution.

The continuous Gompertz approximation captures the central part of the distribution reasonably well, with the largest deviations occurring at the lower and upper tails. For females, the median payout in the Gompertz model is slightly lower than in the discrete case (389k vs. 414k), and the mean differs by about 4%. This illustrates that the continuous model smooths the life table data, preserving the general shape of the distribution while slightly underestimating payouts in the mid-range. For males, the lower modal age and smaller shape parameter imply reduced expected payouts, with the mean about 15% lower compared to females.

Table 3 shows that female pension payouts are systematically higher across all quantiles compared to males. This reflects the higher modal age and steeper slope of the female Gompertz law, implying lower mortality at most ages. For example, the female median payout amounts to about 389k, whereas the male median is only 325k, a difference of roughly 16%. The gap persists at both lower and upper quantiles, confirming that longer female life expectancy translates into consistently higher lifetime pension values.

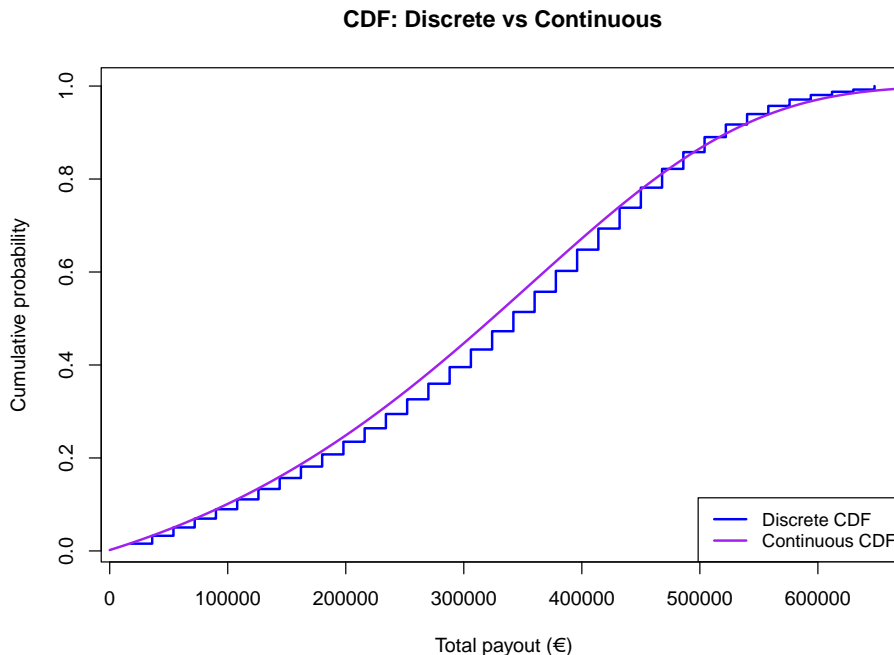


Figure 2: Cumulative distribution of total lifetime pension payouts for males retiring at age 65

### 3.4 Quantiles of the Total Payout

A natural benchmark is the median of the total payout distribution, corresponding to the point where survival probability is reduced by one half:

$$y_{0.5} = \text{pens} \cdot (x_{0.5} - a_0), \quad x_{0.5} = S^{-1}(0.5 S(a_0)).$$

Table 3: Quantiles of the total lifetime pension payout  $Y$  for females and males, based on the Gompertz model (female:  $m = 89.04$ ,  $k = 0.1143$ ; male:  $m = 84.89$ ,  $k = 0.0964$ ). Assumptions: retirement age  $a_0 = 65$ , annual pension €18,000.

Quantile	Female [€]	Male [€]
5%	92,615	55,892
10%	153,144	100,915
25%	268,180	202,450
50%	388,924	325,495
75%	491,274	437,827
90%	568,386	525,307
95%	608,838	571,833

More generally, the  $q$ -th quantile of the total payout can be obtained by solving the survival function for  $x_q$  and scaling by the annual pension:

$$y_q = \text{pens} \cdot (x_q - a_0), \quad x_q = S^{-1}((1 - q)S(a_0)).$$

In words, the quantile  $y_q$  is the payout that will not be exceeded with probability  $q$ .

## Median and General Quantiles (Gompertz)

With Gompertz survival

$$S(x) = \exp(e^{-km} - e^{k(x-m)}), \quad x \geq a_0,$$

and hazard function  $\mu(x) = ke^{k(x-m)}$ , the conditional density of the age at death given survival to  $a_0$  is

$$f_{X|a_0}(x) = \frac{S(x)\mu(x)}{S(a_0)}, \quad x \geq a_0.$$

**Median (special case).** For  $q = \frac{1}{2}$ , solving  $S(x_{0.5}) = 0.5S(a_0)$  gives the median age

$$x_{0.5} = m + \frac{1}{k} \ln(e^{k(a_0-m)} + \ln 2)$$

and the median payout

$$y_{0.5} = \text{pens} \left( m - a_0 + \frac{1}{k} \ln(e^{k(a_0-m)} + \ln 2) \right).$$

**General  $q$ -quantile.** In the general case,

$$x_q = m + \frac{1}{k} \ln(e^{k(a_0-m)} - \ln(1 - q)), \quad 0 < q < 1,$$

so that

$$y_q = \text{pens} \left( m - a_0 + \frac{1}{k} \ln(e^{k(a_0-m)} - \ln(1 - q)) \right).$$

**Remark.** The formulas are valid provided

$$e^{k(a_0-m)} - \ln(1-q) > 0,$$

which holds for typical Gompertz parameters and realistic quantiles. For numerical work, it is advisable to evaluate the expressions directly to avoid rounding issues.

Figure 3 illustrates the comparison between female and male Gompertz-based pension models. The panels show (1) survival functions, (2) death densities conditional on survival to age 65, (3) pension payout densities in Euro, and (4) cumulative distribution functions (CDFs) of total pension payouts. Female curves (red) lie consistently to the right of male curves (blue), reflecting higher life expectancy and, consequently, higher expected lifetime pension payouts. The pension density is a linear transformation of the death density, starting at retirement age 65.

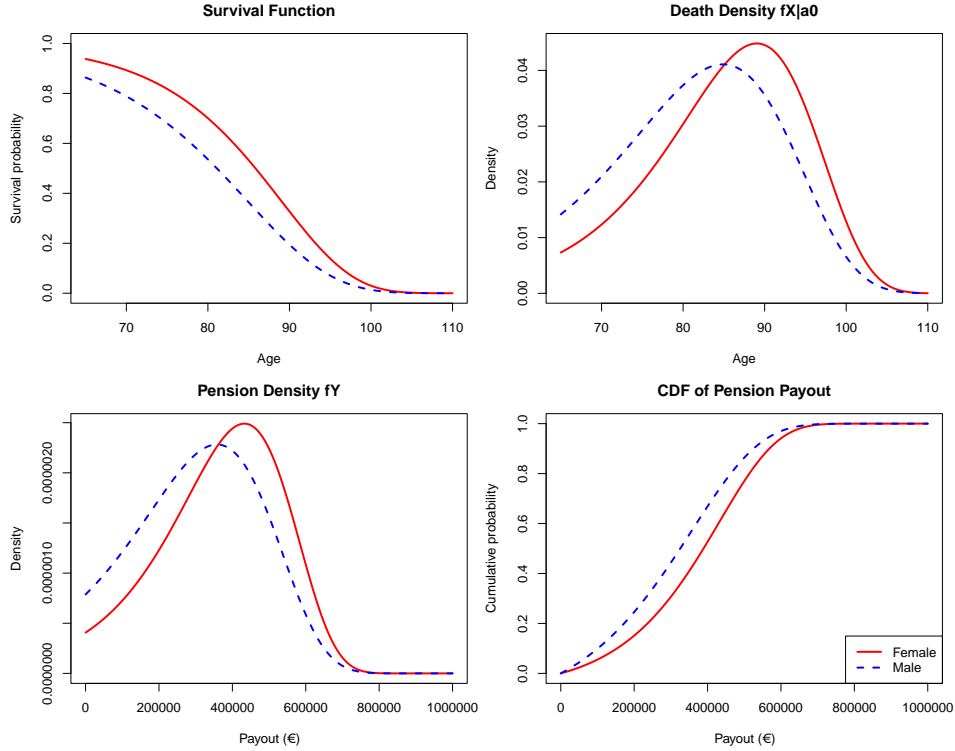


Figure 3: Comparison of female and male Gompertz-based pension models

## 4 Sensitivity Analysis

### 4.1 Sensitivity of Pension Quantiles to Modal Age $m$

Differentiating the quantile formula with respect to  $m$  gives

$$\frac{dy_q}{dm} = \text{pens} \cdot \frac{\ln(1-q)}{\ln(1-q) - e^{k(a_0-m)}}, \quad 0 < q < 1.$$

This derivative measures the change in payout quantiles when life expectancy (rising with  $m$ ) increases. Table 4 reports the effect of a one-year increase in  $m$ .

**Interpretation:** The derivative is positive for all quantiles. Longer life expectancy always increases payouts, and the effect is stronger for higher quantiles since these correspond to longer-lived individuals.

Table 4: Sensitivity of selected pension quantiles to a one-year increase in modal age  $m$  (pens = 18,000,  $a_0 = 65$ ,  $m = 89.04$ ,  $k = 0.1143$ ).

Quantile $q$	$\frac{dy_q}{dm}$ [€]
0.1	11,193
0.2	13,985
0.3	15,259
0.4	15,994
0.5	16,477
0.6	16,824
0.7	17,091
0.8	17,311
0.9	17,513

## 4.2 Sensitivity of Pension Quantiles to the Gompertz Shape Parameter $k$

The derivative of  $y_q$  with respect to  $k$  is

$$\frac{dy_q}{dk} = \text{pens} \left( -\frac{1}{k^2} \ln(e^{k(a_0-m)} - \ln(1-q)) + \frac{(a_0-m)e^{k(a_0-m)}}{k(e^{k(a_0-m)} - \ln(1-q))} \right), \quad 0 < q < 1.$$

Table 5: Sensitivity of selected pension quantiles to changes in  $k$  (pens = 18,000,  $a_0 = 65$ ,  $m = 89.04$ ,  $k = 0.1143$ ).

Quantile $q$	$\frac{dy_q}{dk}$ [€]
0.1	1,014,374
0.2	874,294
0.3	616,283
0.4	340,773
0.5	62,837
0.6	-220,091
0.7	-518,475
0.8	-854,391
0.9	-1,289,418

**Interpretation:** Increasing  $k$  (mortality rectangularization) raises payouts for lower quantiles but reduces them for higher quantiles. This reflects a redistribution: with more concentrated lifetimes around the modal age, short-lived individuals gain, while long-lived individuals lose. For example, if  $k$  rises by 0.01, the median payout increases by about 628 €, while the 90% quantile decreases by about 12,894 €.

## 5 Present Value Distribution and Quantile Analysis under Growth and Discounting Differentials

This section generalizes the valuation formulas for pension payouts by allowing the annual pension growth rate  $g$  to differ from the annual discount rate  $r$ . When  $r = g$ , the present value (PV) of the pension stream grows linearly with the remaining lifetime. When  $r \neq g$ , discounting and benefit growth interact, producing either bounded or unbounded payout distributions. The following derivations yield general expressions for the probability density function and quantiles of the present value under Gompertz survival.

**Notation.** Let  $a_0$  denote the retirement age,  $X$  the random age at death, and  $T = X - a_0$  the remaining lifetime ( $T \geq 0$ ). Let pens be the annual pension payable at retirement. The annual pension growth rate is  $g$ , and the market discount rate is  $r$ . Define  $\alpha := r - g$  and, for discrete payments,  $v = \frac{1+g}{1+r}$ . Let  $S(x)$  be the (unconditional) survival function, with

$$S_T(t) = \Pr(T > t) = \frac{S(a_0 + t)}{S(a_0)}, \quad f_T(t) = \frac{S(a_0 + t)\mu(a_0 + t)}{S(a_0)},$$

where  $\mu(x)$  is the force of mortality.

## 5.1 Discrete annual payments

If pensions are paid once per year at amounts growing by  $(1+g)^{k-1}$  and discounted by  $(1+r)^{k-1}$ , the present value (PV) received when surviving  $T \in \{0, 1, 2, \dots\}$  full years after retirement is

$$Y = \text{pens} \sum_{k=0}^{T-1} v^k = \text{pens} \frac{1-v^T}{1-v}, \quad v = \frac{1+g}{1+r}.$$

Special case  $r = g$  ( $v = 1$ ):  $Y = \text{pens} \cdot T$ .

The discrete distribution is the mixture induced by  $T$ :

$$\Pr\left(Y = \text{pens} \frac{1-v^t}{1-v}\right) = \Pr(T = t) = p_T(t),$$

and the  $q$ -quantile  $y_q$  is obtained from the  $q$ -quantile  $T_q$  of  $T$  (use integer rounding for exact discrete quantiles):

$$y_q = \text{pens} \frac{1-v^{T_q^{\text{int}}}}{1-v}, \quad T_q^{\text{int}} = \min\{t : \Pr(T \leq t) \geq q\}.$$

## 5.2 Continuous (instantaneous) payments

Assume continuous payments at instantaneous rate pens that grow continuously at rate  $g$  and are discounted continuously at rate  $r$ . For realized remaining lifetime  $T$ , the PV is

$$Y = \int_0^T \text{pens} e^{gs} e^{-rs} ds = \text{pens} \int_0^T e^{-\alpha s} ds = \begin{cases} \frac{\text{pens}}{\alpha} (1 - e^{-\alpha T}), & \alpha \neq 0, \\ \text{pens} T, & \alpha = 0. \end{cases}$$

**Monotonicity and inversion.** For  $\alpha \neq 0$ , the map  $T \mapsto Y$  is strictly increasing on  $[0, \infty)$  with inverse

$$T(Y) = -\frac{1}{\alpha} \ln\left(1 - \frac{\alpha Y}{\text{pens}}\right), \quad \text{domain: } 1 - \frac{\alpha Y}{\text{pens}} > 0.$$

Hence:

- If  $\alpha > 0$ , then  $0 \leq Y < \text{pens}/\alpha$  (finite upper bound).
- If  $\alpha < 0$ , then  $Y \geq 0$  (no finite upper bound).

For  $\alpha = 0$ , the inverse is  $T(Y) = Y/\text{pens}$ .

**Density of  $Y$  (continuous case).** By change of variables ( $y \mapsto t = T(y)$ ),

$$f_Y(y) = f_T(T(y)) \frac{dT}{dy} = \frac{S(a_0 + T(y))\mu(a_0 + T(y))}{S(a_0)} \cdot \frac{1}{\text{pens} - \alpha y},$$

because  $\frac{dT}{dy} = \frac{1}{\text{pens} - \alpha y}$  for  $\alpha \neq 0$ . The CDF is

$$F_Y(y) = \Pr(Y \leq y) = \Pr(T \leq T(y)) = 1 - \frac{S(a_0 + T(y))}{S(a_0)}.$$

**Quantiles (continuous case).** Let  $y_q$  denote the  $q$ -quantile of  $Y$  ( $0 < q < 1$ ). Since  $T \mapsto Y$  is increasing,

$$\Pr(Y \leq y_q) = q \iff \frac{S(a_0 + T(y_q))}{S(a_0)} = 1 - q.$$

Thus

$$T_q = S^{-1}((1 - q)S(a_0)) - a_0, \quad y_q = \begin{cases} \frac{\text{pens}}{\alpha}(1 - e^{-\alpha T_q}), & \alpha \neq 0, \\ \text{pens } T_q, & \alpha = 0. \end{cases}$$

### 5.3 Gompertz model and explicit quantiles

Under the Gompertz law with modal age  $m$  and slope  $k$ ,

$$S(x) = \exp(e^{-km} - e^{k(x-m)}), \quad \mu(x) = ke^{k(x-m)}.$$

Solving  $S(x_q) = (1 - q)S(a_0)$  yields

$$x_q = m + \frac{1}{k} \ln(e^{k(a_0-m)} - \ln(1 - q)), \quad T_q = x_q - a_0.$$

Hence

$$y_q = \begin{cases} \frac{\text{pens}}{\alpha}(1 - e^{-\alpha(x_q - a_0)}), & \alpha \neq 0, \\ \text{pens}(x_q - a_0), & \alpha = 0. \end{cases}$$

Table 6: Comparison of payout quantiles  $y_q$  (EUR) under different scenarios. Parameters: pens = 18,000, retirement age  $a_0 = 65$ , female Gompertz fit  $m = 89.04$ ,  $k = 0.1143$ .

Quantile	A	B	C	D
5%	92,615	88,009	89,009	97,548
10%	153,144	140,823	145,295	166,945
25%	268,180	231,914	249,159	312,419
50%	388,924	315,792	349,776	486,493
75%	491,274	378,591	429,937	653,482
90%	568,386	421,405	487,391	792,454
95%	608,838	442,440	516,561	870,260

*Scenario legend:*

- A:  $r = g = 0.02$  ( $\alpha = 0$ )
- B:  $r = 0.03, g = 0.01$  ( $\alpha = 0.02$ )
- C:  $r = 0.02, g = 0.01$  ( $\alpha = 0.01$ )
- D:  $r = 0.01, g = 0.03$  ( $\alpha = -0.02$ )

**Remarks:** Scenario A is the baseline  $r = g = 0.02$  case (see Table 3).

Continuous-time PV formula:  $y_q = \frac{\text{pens}}{\alpha}(1 - e^{-\alpha T_q})$  for  $\alpha = r - g \neq 0$ , and  $y_q = \text{pens } T_q$  for  $\alpha = 0$ , where  $T_q = x_q - a_0$  and  $x_q$  is the Gompertz age quantile.

### Interpretation of Results

Table 6 illustrates how differences between the discount rate  $r$  and the pension growth rate  $g$  affect the distribution of total lifetime payouts.

When  $r = g$  (Scenario A), the present value increases linearly with lifetime, and the quantiles reflect the survival distribution alone. Higher discount rates relative to growth ( $r > g$ ; Scenarios B and C) compress the payout distribution and reduce all quantiles, particularly in the upper tail. Conversely, if the pension grows faster than it is discounted ( $g > r$ ; Scenario D), the PV rises sharply with age and produces a much heavier upper tail. This effect captures the substantial valuation risk borne by the pension provider when benefit indexation exceeds the discount rate used in funding or valuation.

## Conclusion

This study introduced a continuous Gompertz approach to modeling the distribution of lifetime pension payouts and extended it to include present-value effects under varying discount and growth rates. Compared with the benchmark capital stock of approximately €604,000 accumulated from invested contributions, the expected lifetime pension values based on the German 2019/21 life tables are lower—about €388,000 for females and €330,000 for males. Only at very high ages do payouts exceed this benchmark.

When the discount rate  $r$  exceeds the pension growth rate  $g$ , the distribution of present values becomes more compressed, reducing both the mean and upper quantiles. Conversely, when  $g > r$ , the present value of pensions becomes highly sensitive to longevity, producing heavy-tailed distributions and increasing valuation uncertainty. This interaction between demographic longevity risk and financial discounting reveals how macroeconomic conditions can amplify or mitigate pension system risks.

Beyond numerical comparisons, the pay-as-you-go system retains essential social and redistributive roles not captured by purely financial evaluation. The continuous Gompertz model, complemented by the growth–discounting extension, provides an analytically tractable and empirically grounded tool for quantifying pension risks, facilitating transparent policy discussions, and linking demographic dynamics with actuarial and financial assessments.

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## Appendix

### Life Table Statistics

Based on the official German life table 2019/21, females have a life expectancy at birth of approximately 83 years, a median age at death of about 89 years, and a conditional life expectancy of around 22 years at retirement age 65. The corresponding figures for males are about 79, 85, and 19 years, respectively. The fitted Gompertz parameters are consistent with these statistics, yielding modal ages of roughly 89 years for females and 85 years for males.

### Tables

Table 7: Average annual earnings in DM and statutory pension insurance contribution rates (1980–2001). Source: Deutsche Rentenversicherung.

Year	Average Earnings (DM)	Total Rate (%)	Employee Share (%)
1980	22,860	17.0	8.50
1981	24,360	18.0	9.00
1982	25,440	18.5	9.25
1983	26,280	18.0 / 18.5	9.00 / 9.25
1984	27,600	18.5	9.25
1985	28,560	18.7 / 19.2	9.35 / 9.60
1986	29,280	19.2	9.60
1987	30,120	19.2	9.60
1988	30,960	19.2	9.60
1989	32,160	19.2	9.60
1990	33,720	18.7	9.35
1991	35,280	17.7	8.85
1992	37,140	17.5	8.75
1993	38,304	17.5	8.75
1994	38,976	19.2	9.60
1995	41,200	17.5	8.75
1996	42,480	17.4	8.70
1997	43,104	20.3	10.15
1998	43,200	20.3	10.15
1999	43,104	19.5 / 19.3	9.75 / 9.65
2000	43,104	19.3	9.65
2001	43,104	19.1	9.55

Table 8: Average annual earnings in € and statutory pension insurance contribution rates (2002–2025). Source: Deutsche Rentenversicherung.

Year	Average Earnings (€)	Total Rate (%)	Employee Share (%)
2002	22,022	19.5	9.75
2003	22,256	19.5	9.75
2004	22,427	19.5	9.75
2005	22,753	19.5	9.75
2006	23,018	19.5	9.75
2007	23,337	19.9	9.95
2008	23,639	19.9	9.95
2009	24,088	19.9	9.95
2010	24,320	19.9	9.95
2011	25,105	19.9	9.95
2012	25,174	19.6	9.80
2013	25,590	18.9	9.45
2014	26,155	18.9	9.45
2015	26,653	18.7	9.35
2016	27,279	18.7	9.35
2017	27,760	18.7	9.35
2018	28,164	18.6	9.30
2019	28,608	18.6	9.30
2020	29,072	18.6	9.30
2021	29,848	18.6	9.30
2022	38,901	18.6	9.30
2023	43,142	18.6	9.30
2024	45,358	18.6	9.30
2025	50,493	18.6	9.30

Table 9: Daily yields of current Federal securities (%). Source: Deutsche Bundesbank.

Year	Yield	Year	Yield	Year	Yield	Year	Yield
1980	8.82	1981	9.87	1982	8.97	1983	8.02
1984	8.05	1985	7.63	1986	6.43	1987	6.32
1988	6.44	1989	7.16	1990	8.11	1991	8.24
1992	8.07	1993	6.29	1994	6.41	1995	6.56
1996	6.05	1997	5.61	1998	4.68	1999	4.31
2000	5.30	2001	5.09	2002	4.68	2003	4.20
2004	4.02	2005	3.36	2006	3.92	2007	4.25
2008	4.11	2009	3.25	2010	2.75	2011	2.43
2012	1.44	2013	1.51	2014	1.06	2015	0.55
2016	0.22	2017	0.39	2018	0.43	2019	-0.26
2020	-0.52	2021	-0.23	2022	1.60	2023	2.82
2024	2.67	2025	2.69				